On regularity properties of solutions to the hysteresis-type problem

Darya E. Apushkinskaya

Department of Mathematics, Saarland University, P.O. Box 151150,
66041 Saarbrücken, Germany
Faculty of Mathematics and Mechanics, St. Petersburg State University,
Universitetskii pr. 28, St. Petersburg 198504, Russia
E-mail: *darya@math.uni-sb.de*

Nina N. Uraltseva

Faculty of Mathematics and Mechanics, St. Petersburg State University,
Universitetskii pr. 28, St. Petersburg 198504, Russia
E-mail: *uraltsev@pdmi.ras.ru*

We consider equations with the simplest hysteresis operator at the right-hand side. Such equations describe the so-called processes "with memory" in which various substances interact according to the hysteresis law.

We restrict our consideration on the so-called "strong solutions" belonging to the Sobolev class $W_q^{2,1}$ with sufficiently large q and prove that in fact $q = \infty$. In other words, we establish the optimal regularity of solutions. Our arguments are based on quadratic growth estimates for solutions near the free boundary.

1. Introduction.

In this paper we study the regularity properties of solutions of the following parabolic equation:
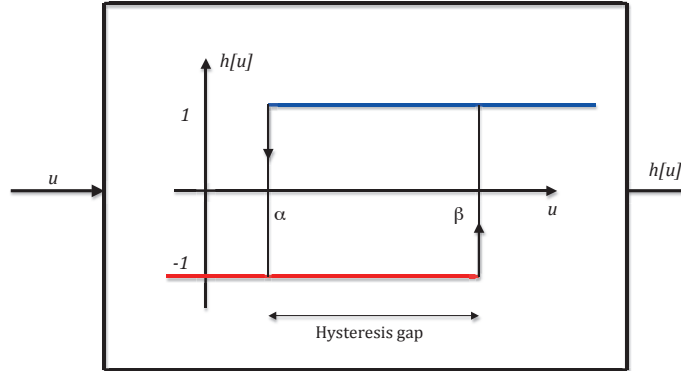
$$H[u] = h[u] \quad \text{in} \quad Q = \mathcal{U} \times]0, T]. \tag{1}$$

Here $H = \Delta - \partial_t$ is the heat operator, $\mathcal{U}$ is a domain in $\mathbb{R}^n$, and h is a hysteresis-type operator acting from $C(\overline{Q})$ to $\{\pm 1\}$ which is defined as follows.

We fix two numbers α and β ($\alpha < \beta$) and consider a *multivalued* function

$$f(s) = \begin{cases} -1, & \text{for} \quad s \in]-\infty, \alpha], \\ 1, & \text{for} \quad s \in [\beta, +\infty[, \\ -1 \text{ or } 1, & \text{for} \quad s \in]\alpha, \beta[. \end{cases}$$

For $u \in C(\overline{Q})$ we suppose that on the bottom of the cylinder Q the initial values of u as well as of $h[u](x, 0) := f(u(x, 0))$ are prescribed.

FIG. 1. The hysteresis operator h

After that for every point $z = (x, t) \in Q$ the corresponding value of $h[u](z)$ is uniquely defined in the following manner. Let us denote by E a set of points

$$E := \{z \in Q : u(z) \leqslant \alpha\} \cup \{z \in Q : u(z) \geqslant \beta\} \cup \{\mathcal{U} \times \{0\}\}.$$

In other words, E is a set where $f(u(z))$ is well-defined.

If $z \in E$ then $h[u](z) = f(u(z))$. Otherwise, for $z = (x, t) \in Q$ such that $\alpha < u(z) < \beta$ we set

$$h[u](x, t) = h[u]\big(x, \hat{t}(x)\big). \tag{2}$$

Here

$$\hat{t}(x) = \max \big\{s : (x, s) \in E; \ s \leqslant t\big\}.$$

Roughly speaking, condition (2) means that the hysteresis function $h[u](x, t)$ takes for $u(x, t) \in (\alpha, \beta)$ the same value as "at the previous moment" (see Figure 1).

Let us emphasize that for fixed x a jump of $h[u](x, \cdot)$ can happen only on thresholds $\{u(x, t) = \alpha\}$ and $\{u(x, t) = \beta\}$. Moreover, "jump down" (from $h = 1$ to $h = -1$) is possible on $\{u(x, t) = \alpha\}$ only, whereas "jump up" (from $h = -1$ to $h = 1$) is possible on $\{u(x, t) = \beta\}$ only.

We say that u is a (strong) solution of Eq. (1) if $u \in W_q^{2,1}(Q)$, $q > n + 2$, and u satisfies (1) a.e. in Q. In particular, it implies that the $(n + 1)$-dimensional Lebesgue measure of the sets $\{u = \alpha\}$ and $\{u = \beta\}$ equals zero.

Thus, the cylinder Q consists of two disjoint regions where $h[u]$ assumes the values $+1$ and -1, respectively. If u is a solution of (1) then the interface between these two regions is apriori unknown and, therefore, may be considered as the free boundary.

Equation of type (1) arises in various biological and chemical processes in which diffusive and nondiffusive substances interact according to hysteresis law (see, for instance, [8–10], and references therein).

Difficulties in study of challenging hysteresis phenomenon include the discontinuous nonlinearity and the multivalence of corresponding operator as well. A first attempt to create a mathematical theory of hysteresis was made in the monograph [11]. We mention also the fundamental books [4, 17] and [12] where the hysteretic effects in spatial-distributed systems are

described. The above-listed monographs are mainly devoted to the existence results as well as to investigations of qualitative properties of solutions.

The solvability of initial-boundary value problems for equation (1) was studied in papers [1] and [16] in one-(space)-dimensional case and in multi-(space)-dimensional case, respectively. The global existence in a specially defined classes of weak solutions were established there. Moreover, in [1] the nonuniqueness and nonstability of such weak solutions were discussed in several examples. Recently, in papers [6] and [7] the strong transversal solutions, belonging to the Sobolev space $W_q^{2,1}$ with sufficiently large q, were studied in the one-(space)-dimensional case. This transversality property roughly speaking means that the solution has a nonvanishing spatial gradient on the free boundary. In the paper [6] the authors proved the local existence of strong transversal solutions and showed that such solutions depend continuously on initial data. A theorem on the uniqueness of strong transversal solutions was established in [7].

In this paper we are interested in local L^∞-estimates for the derivatives D^2u and $\partial_t u$ of the strong solutions of Eq. (1). We do not suppose that our solutions have the transversality property.

We assume that

$$\sup_Q |u| \leqslant M \quad \text{with} \quad M > 1. \tag{3}$$

Since the right-hand side of (1) is bounded, the general parabolic theory (see, e.g. [13]) implies for any $\epsilon > 0$ the estimates

$$\|\partial_t u\|_{q,Q^\epsilon} + \|D^2u\|_{q,Q^\epsilon} \leqslant N_1(\epsilon, q, M) \quad \forall q < \infty, \tag{4}$$

where $Q^\epsilon = \mathcal{U}^\epsilon \times]\epsilon^2, T]$, $\mathcal{U}^\epsilon \subset \mathcal{U}$ and $dist\{\mathcal{U}^\epsilon, \partial\mathcal{U}\} \geqslant \epsilon$.

In particular, (4) implies that functions u and Du are Hölder continuous in Q.

We note that if $\partial\mathcal{U}$ as well as the values of u on the parabolic boundary of Q are smooth then the corresponding estimates of L^q-norm for $\partial_t u$ and D^2u are true in the whole cylinder Q.

The paper is organized as follows. In Section 2 we introduce notations used in this paper, describe the different components of the free boundary and formulate the main result of the paper: Theorem 2.3. In Section 3 we show the continuity of the time-derivative $\partial_t u$ across the special part of the free boundary where the spatial gradient Du does not vanish, and estimate $|\partial_t u|$ on this part by a constant depending only on given quantities. Further, in Section 4 we verify that positive and negative parts of the space directional derivatives $D_e u$ for any direction $e \in \mathbb{R}^n$ are subcaloric outside some "pathological" part of the free boundary. We use this information in Section 5 for proving the quadratic growth estimates which are crucial for the final estimates of the higher order derivatives. The uniform L^∞-estimates of $\partial_t u$ and D^2u depending on given quantities and on the distance to the "pathological" part of the free boundary are obtained in Section 6. Finally, in Section 7 we state and prove some preliminary facts which are used intensively for proving of almost all results in the previous sections.

2. Notation and preliminaries

Throughout this article we use the following notation:

$z = (x, t)$ are points in $\mathbb{R}^{n+1}_{x,t}$, where $x \in \mathbb{R}^n$, $n \geqslant 1$, and $t \in \mathbb{R}^1$;

$x = (x_1, x') = (x_1, x_2, \ldots, x_n)$, if $n \geqslant 2$;

$|x|$ is the Euclidean norm of x;

$B_r(x^0)$ denotes the open ball in $\mathbb{R}^n$ with center x^0 and radius r;

$Q_r(z^0) = Q_r(x^0, t^0) = B_r(x^0) \times]t^0 - r^2, t^0 + r^2[$;

$Q_r^-(z^0) = Q_r(z^0) \cap \{t < t^0\}$.

When omitted, x^0 (or $z^0 = (x^0, t^0)$, respectively) is assumed to be the origin.

$\partial' Q_r(z^0)$ or $\partial' Q_r^-(z^0)$ denote the parabolic boundary of the corresponding cylinder, i.e., the topological boundary minus the top of the cylinder.

For a cylinder $Q = \mathcal{U} \times]0, T[$ and any $\epsilon > 0$ we define the corresponding cylinder Q^ϵ as

$$Q^\epsilon = \mathcal{U}^\epsilon \times]\epsilon^2, T[,$$

where $\mathcal{U}^\epsilon \subset \mathcal{U}$ and $dist\{\mathcal{U}^\epsilon, \partial\mathcal{U}\} \geq \varepsilon$.

$u_+ = \max\{u, 0\}; \qquad u_- = \max\{-u, 0\}$;

D_i denotes the differential operator with respect to x_i;

$D = (D_1, D') = (D_1, D_2, \ldots, D_n)$ denotes the spatial gradient;

$D^2 u = D(Du)$ denotes the Hessian of u;

$\partial_t u = \dfrac{\partial u}{\partial t}$.

D_ν stands for the operator of differentiation along a direction $\nu \in \mathbb{R}^n$, i.e., $|\nu| = 1$ and $D_\nu u = \sum\limits_{i=1}^{n} \nu_i D_i u$.

We adopt the convention that the indices i, j, l always vary from 1 to n. We also adopt the convention regarding summation with respect to repeated indices.

$\| \cdot \|_{p, \mathfrak{D}}$ denotes the norm in $L^p(\mathfrak{D})$, $1 < p \leq \infty$;

$W_p^{2,1}(\mathfrak{D})$ and $W_p^{1,0}(\mathfrak{D})$ are anisotropic Sobolev spaces with the norms

$$\|u\|_{W_p^{2,1}(\mathfrak{D})} = \|\partial_t u\|_{p, \mathfrak{D}} + \|D^2 u\|_{p, \mathfrak{D}} + \|u\|_{p, \mathfrak{D}},$$

$$\|u\|_{W_p^{1,0}(\mathfrak{D})} = \|Du\|_{p, \mathfrak{D}} + \|u\|_{p, \mathfrak{D}},$$
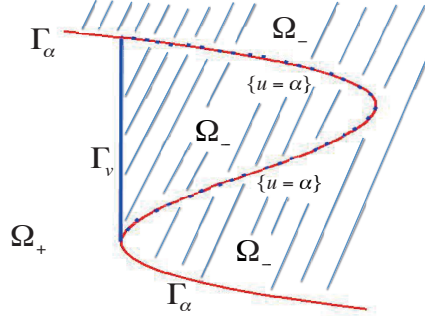
respectively.

For a cylinder $\mathcal{Q} = \mathcal{U} \times]T_1, T_2[\subset \mathbb{R}_x^n \times \mathbb{R}_t^1$ we denote by $V_2(\mathcal{Q})$ the Banach space consisting of all elements of $W_2^{1,0}(\mathcal{Q})$ with a finite norm

$$\|u\|_{V_2(\mathcal{Q})} = \sup_{T_1 < t < T_2} \|u\|_{2, \mathcal{U}} + \|Du\|_{2, \mathcal{Q}}.$$

$\fint_{\mathfrak{D}} \ldots$ stands for the average integral over the set $\mathfrak{D}$, i.e.,

$$\fint_{\mathfrak{D}} \cdots = \frac{1}{\text{meas}\{\mathfrak{D}\}} \int_{\mathfrak{D}} \cdots.$$

FIG. 2. Structure of the free boundary for $n = 1$

We say that $\xi = \xi(x, t)$ is a cut-off function for a cylinder $Q_r(\hat{z})$ if

$$\xi(x, t) = \xi_1(x)\xi_2(t),$$

where $\xi_i \geqslant 0, i = 1, 2,$

$$\xi_1 \in C_0^\infty(B_r(\hat{x})), \qquad \xi_1 \equiv 1 \quad \text{in} \quad B_{r/2}(\hat{x}),$$

while $\xi_2 \in C^\infty([\hat{t} - r^2, \hat{t}])$, $\xi_2(\hat{t} - r^2) = 0$ and $\xi_2(t) \equiv 1$ for $t \geqslant \hat{t} - r^2/4$.

We define the parabolic distance $dist_p$ from a point $z = (x, t)$ to a set $\mathfrak{D} \subset \mathbb{R}^{n+1}$ by

$$dist_p(z, \mathfrak{D}) := \sup\{r > 0 : Q_r^-(z) \cap \mathfrak{D} = \emptyset\}.$$

We use letters M, N, C and c (with or without sub-indices) to denote various constants. To indicate that, say, C depends on some parameters, we list them in the parentheses: $C(\dots)$. We do not indicate the dependence of constants on n. In addition, we will write *sup* instead of *ess sup* and *inf* instead of *ess inf*.

We denote

$$\Omega_\pm(u) := \{z \in Q, \text{ where } h[u](z) = \pm 1\},$$
$$\Gamma(u) := \partial\Omega_+ \cap \partial\Omega_- \text{ is the free boundary.}$$

The latter means that $\Gamma(u)$ is the set where the function $h[u](z)$ has a jump.

We also introduce special notation for the different parts of $\Gamma(u)$

$$\Gamma_\alpha(u) := \Gamma(u) \cap \{u = \alpha\},$$
$$\Gamma_\beta(u) := \Gamma(u) \cap \{u = \beta\}.$$

By definition,

$$\{u \leqslant \alpha\} \subset \Omega_- \quad \text{and} \quad \{u \geqslant \beta\} \subset \Omega_+.$$

It is also easy to see that the sets $\{u = \alpha\}$ and $\{u = \beta\}$ are separated from each other.

REMARK 2.1 In any cylinder Q^ϵ the distance from the level set $\{u = \alpha\}$ to the level set $\{u = \beta\}$ is estimated from below by a positive constant depending on M, ϵ and $\beta - \alpha$ only.

Observe that the level sets $\{u = \alpha\}$ and $\{u = \beta\}$ are not always the parts of the free boundary $\Gamma(u)$. Indeed, if the level set $\{u = \alpha\}$ is locally not a t-graph, then a part of $\{u = \alpha\}$ may occur inside Ω_-. In this case $\Gamma(u)$ may contain several components of Γ_α connected by parts of cylindrical surfaces with generatrices parallel to t-axis (see Figure 2). Similar statement is true for the level set $\{u = \beta\}$. We will denote by Γ_v the set of all points z lying in such vertical parts of $\Gamma(u)$. It should be noted that Γ_v is, in general, not the level set $\{u = \alpha\}$ as well as not the level set $\{u = \beta\}$. This Γ_v is just the "pathological" part of the free boundary that we have mentioned in Introduction. Thus, we have

$$\Gamma(u) = \Gamma_\alpha(u) \cup \Gamma_\beta(u) \cup \Gamma_v.$$

We will also distinguish the following parts of Γ:

$$\Gamma_\alpha^0(u) = \Gamma_\alpha(u) \cap \{|Du| = 0\}, \qquad \Gamma_\alpha^*(u) = \Gamma_\alpha(u) \setminus \Gamma_\alpha^0(u).$$

The sets Γ_β^0 and Γ_β^* are defined analogously. In addition, we set

$$\Gamma^0(u) := \Gamma_\alpha^0(u) \cup \Gamma_\beta^0(u), \quad \Gamma^*(u) := \Gamma_\alpha^*(u) \cup \Gamma_\beta^*(u).$$

REMARK 2.2 It is obvious that $u \in C^\infty$ in the interior of the sets $\Omega_\pm$.

Now we formulate the main result of the paper.

THEOREM 2.3 Let u be a (strong) solution of Eq. (1), and let $z \in Q \setminus \Gamma(u)$. Then

$$|\partial_t u(z)| + |D^2 u(z)| \leqslant C(\rho_0, \varepsilon, M, \beta - \alpha).$$

Here $\rho_0 := dist_p\{z, \Gamma_v\}$ and $\epsilon := dist_p\{z, \partial' Q\}$.

Proof. The proof of this statement follows from Lemmas 6.1 and 6.2. □

3. Estimates of $\partial_t u$ on $\Gamma^*(u)$

LEMMA 3.1 Let u be a solution of Eq. (1), and let $Q_{3\rho}^-(z^*)$ be an arbitrary cylinder contained in Q. Then we have the estimates

$$\inf_{Q_\rho^-(z^*)} \partial_t u \geqslant -N, \quad \text{provided that} \quad Q_{3\rho}^-(z^*) \cap \Gamma_\beta = \emptyset, \tag{5}$$

$$\sup_{Q_\rho^-(z^*)} \partial_t u \leqslant N, \quad \text{provided that} \quad Q_{3\rho}^-(z^*) \cap \Gamma_\alpha = \emptyset. \tag{6}$$

Here $N = N(M, \rho)$.

Proof. Assume for the definiteness that z^* lies in a neighborhood of Γ_β. Consider in $Q_{2\rho}^-(z^*)$ the difference quotient of u in the t-direction, i.e.,

$$u^{(\tau)}(x, t) = \frac{u(x, t) - u(x, t - \tau)}{\tau}$$

with some small positive τ. To prove (6) it is sufficient to get the corresponding estimate for $u^{(\tau)}$ uniformly with respect to τ.

Further, we observe that equation (1) and integration by parts provide for all functions $\eta \in W_2^{1,0}(Q_{2\rho}^-(z^*))$ vanishing on $\partial B_{2\rho}(x^*) \times [t^* - 4\rho^2, t^*]$ the validity of the following integral identity

$$\int_{Q_{2\rho}^-(z^*)} (\partial_t u\eta + DuD\eta)\, dxdt = -\int_{Q_{2\rho}^-(z^*)} h[u]\eta\, dxdt. \qquad (7)$$

Using the same reasonings as in deriving of (7) we get for all functions $\widetilde{\eta} \in W_2^{1,0}(Q_{2\rho}^-(x^*, t^* + \tau))$ that are equal to zero on $\partial' Q_{2\rho}^-(x^*, t^* + \tau)$ the integral identity

$$\int_{Q_{2\rho}^-(x^*,t^*+\tau)} (\partial_t u\widetilde{\eta} + DuD\widetilde{\eta})\, dxdt = -\int_{Q_{2\rho}^-(x^*,t^*+\tau)} h[u]\widetilde{\eta}\, dxdt. \qquad (8)$$

Putting in (8) $\widetilde{\eta}(x, t) = \eta(x, t + \tau)$ we obtain after elementary change of variables the relation

$$\int_{Q_{2\rho}^-(z^*)} \left[\partial_t u(x, t - \tau)\eta(x, t) + Du(x, t - \tau)D\eta(x, t)\right] dxdt$$
$$= -\int_{Q_{2\rho}^-(z^*)} h[u](x, t - \tau)\eta(x, t) dxdt. \quad (9)$$

Now, we substract (9) from (7), divide the result by τ and integrate by parts. After these transformations we arrive at the equality

$$\int_{Q_{2\rho}^-(z^*)} \left[\partial_t u^{(\tau)}\eta + Du^{(\tau)}D\eta\right] dxdt = -\frac{1}{\tau}\int_{Q_{2\rho}^-(z^*)} \left(h[u](x, t) - h[u](x, t - \tau)\right)\eta\, dxdt. \quad (10)$$

Setting in (10)

$$\eta(x, t) = \left(u^{(\tau)} - k\right)_+ \xi^2(x, t),$$

where ξ is a standard cut-off function for a cylinder $Q_{2\rho}^-(z^*)$ (see Notation), we can rewrite (10) in the form

$$\int_{Q_{2\rho}^-(z^*)} \left\{\partial_t u^{(\tau)}\left(u^{(\tau)} - k\right)_+ \xi^2 + Du^{(\tau)}D\left[(u^{(\tau)} - k)_+\xi^2\right]\right\} dxdt$$
$$= -\frac{1}{\tau}\int_{Q_{2\rho}^-(z^*)} \left(h[u](x, t) - h[u](x, t - \tau)\right)(u^{(\tau)} - k)_+\xi^2 dxdt. \quad (11)$$

We claim that $h[u](x, t) - h[u](x, t - \tau) \geqslant 0$ in $Q_{2\rho}^-(z^*)$. Indeed, we have the relation

$$Q_{2\rho}^-(z^*) \cap \Gamma_\alpha = \emptyset.$$

Recall that by definition $h[u](x,t)$ may decrease in t only in a neighborhood of Γ_α. Therefore, in $Q^-_{2\rho}(z^*)$ the function $h[u]$ is either constant or increasing one. The latter means that for we have instead of (11) the inequality

$$\int\limits_{Q^-_{2\rho}(z^*)} \left\{ \partial_t u^{(\tau)} (u^{(\tau)} - k)_+ \xi^2 + Du^{(\tau)} D\left[(u^{(\tau)} - k)_+ \xi^2\right] \right\} dxdt \leqslant 0. \tag{12}$$

Observe that we may take in (12) the cut-off function ξ multiplied by the characteristic function of an interval $[t^* - 4\rho^2, t]$ with an arbitrary $t \in]t^* - 4\rho^2, t^*]$ instead of ξ. This leads to the inequalities

$$\int\limits_{t^*-4\rho^2}^{t} \int\limits_{B_{2\rho}(x^*)} \left\{ \partial_t u^{(\tau)} (u^{(\tau)} - k)_+ \xi^2 + Du^{(\tau)} D\left[(u^{(\tau)} - k)_+ \xi^2\right] \right\} dxdt \leqslant 0,$$
$$\forall t \in]t^* - 4\rho^2, t^*].$$

Further arguments are rather standard. We leave the trivially nonnegative terms in the left-hand side of the above inequalities, while the rest terms are transferred to the right-hand side and estimated from above with the help of Young's inequality. As a consequence, we get

$$\sup_{t^*-4\rho^2 < t \leqslant t^*} \int\limits_{B_{2\rho}(x^*)} (u^{(\tau)} - k)_+^2 \xi^2 dx \bigg|^t + \int\limits_{Q^-_{2\rho}(z^*)} \left[D\left((u^{(\tau)} - k)_+\right) \right]^2 \xi^2 dxdt$$
$$\leqslant \int\limits_{Q^-_{2\rho}(z^*)} \left(u^{(\tau)} - k \right)_+^2 \left[4|D\xi|^2 + 2\xi|\partial_t \xi| \right] dxdt. \tag{13}$$

With inequalities (13) for an arbitrary $k \geqslant 0$ at hands we may apply successively Fact 7.1 with $v = u^{(\tau)}$ and inequalities (4) with $q = 2$ which immediately imply the desired estimate (6).

It remains only to observe that the case of z^* lying near Γ_α is treated almost similarly. The only differences are that we should choose in (10)

$$\eta(x,t) = \left(u^{(\tau)} - k\right)_- \xi^2(x,t), \quad k \leqslant 0,$$

and then check the validity of the inequality $h[u](x,t) - h[u](x, t-\tau) \leqslant 0$ in the cylinder $Q^-_{2\rho}(z^*)$.
□

LEMMA 3.2 Let u be a solution of Eq. (1) and let $z^* \in \Gamma^* \setminus \Gamma_v$.
   Then $\Gamma^* \setminus \Gamma_v$ is locally a C^1-surface and $\partial_t u$ is a continuous function in a neighborhood of z^*.

Proof. Continuity of $\partial_t u$ across Γ^* can be proved by using the same arguments as in (the proof of) Lemma 7.1 [15]. For the readers convenience we sketch the details.

Suppose for the definiteness that $z^* \in \Gamma^*_\alpha \setminus \Gamma_v$. Without restriction it may be assumed that $D_1 u(z^*) > 0$. Then, in a sufficiently small cylinder $Q_\rho(z^*)$ satisfying $Q_\rho(z^*) \cap \Gamma_v$ the function u is strictly increasing in x_1-direction.

Further, using the von Mises transformation, we introduce the new variables

$$(x_1, x', t) \to (y, x', t),$$

where $y := u(x, t) - \alpha$. We also introduce the function v such that

$$x_1 = v(y, x', t).$$

Transforming in $Q_\rho(z^*)$ Eq. (1) for u into terms of v we obtain the uniformly parabolic equation

$$\partial_t v - a^{ij}(\partial v) \partial_i(\partial_j v) = g(y)\partial_1 v,$$

where $\partial_1 v := \dfrac{\partial v}{\partial y} = \dfrac{1}{D_1 u} > 0$, $\partial_m v := \dfrac{\partial v}{\partial x_m} = D_m v = -\dfrac{D_m u}{D_1 u}$,

$$\partial v = (\partial_1 v, \partial' v) = \left(\frac{1}{D_1 u}, -\frac{D' u}{D_1 u} \right), \qquad \partial_t v := \frac{\partial v}{\partial t} = -\frac{\partial_t u}{D_1 u}, \tag{14}$$

$$g(y) = \begin{cases} 1, & \text{if } y > 0 \\ -1, & \text{if } y < 0 \end{cases},$$

and the coefficients a^{ij} are defined as follows

$$a^{11}(p) = \frac{1 + |p'|^2}{p_1^2}, \quad a^{mm}(p) = 1, \quad a^{1m}(p) = a^{m1}(p) = -\frac{p_m}{p_1}, \tag{15}$$

$$a^{m\widetilde{m}}(p) = 0 \quad \text{if} \quad m \neq \widetilde{m}$$

(here the indices m and $\widetilde{m}$ vary from 2 to n, and $p \in \mathbb{R}^n$).

Elementary calculation shows that for the difference quotient in the t-direction

$$v^{(\tau)}(y, x', t) := \frac{v(y, x', t) - v(y, x', t - \tau)}{\tau}$$

we have

$$\partial_t v^{(\tau)} - a^{ij}(\partial v) \partial_i(\partial_j v^{(\tau)}) - b^k \partial_k v^{(\tau)} = g(y)\partial_1 v^{(\tau)}, \tag{16}$$

where $b^k := \dfrac{\partial a^{ij}(Z_\tau)}{\partial p_k} \partial_i \left(\partial_j v(y, x', t - \tau) \right)$,

$$Z_\tau = \vartheta(y, x', t)\partial v(y, x', t - \tau) - \left[1 - \vartheta(y, x', t) \right] \partial v(y, x', t)$$

and $\vartheta(y, x', t) \in [0, 1]$.

Observe that for the second derivatives of v we have the relations

$$\partial_1(\partial_1 v) = -\frac{D_{11} u}{(D_1 u)^3}, \quad \partial_1(\partial_m v) = \frac{D_{11} u D_m u}{|D_1 u|^2} - \frac{D_{1m} u}{D_1 u},$$

$$\partial_m(\partial_{\widetilde{m}} v) = \frac{D_{11} u D_m u D_{\widetilde{m}} u}{|D_1 u|^2} \left(\frac{1}{D_1 u} - 2 \right) \tag{17}$$

$$+ \frac{D_{1m} u D_{\widetilde{m}} u}{D_1 u} + \frac{D_{1\widetilde{m}} u D_m u}{D_1 u} - \frac{D_{m\widetilde{m}} u}{D_1 u}.$$

According to estimates (4) and formulas (14)-(15) and (17) we may conclude that in Eq. (16) the coefficients a^{ij} are Hölder continuous functions satisfying the ellipticity condition, whereas the

coefficients b^k are elements of L^q with an arbitrary $q < \infty$. Therefore, the parabolic theory implies that $v^{(\tau)} \in C^\sigma$ for some $\sigma \in (0,1)$. We note also that all the estimates of corresponding norms are uniformly bounded in τ. Hence we immediately conclude that $\partial_t u$ is also Hölder continuous with some exponent σ' satisfying $0 < \sigma' < \sigma$. It is also evident that near z^* the free boundary Γ_α is a C^1-surface .

It remains only to observe that in the case $z^* \in \Gamma_\beta^* \setminus \Gamma_v$ we should choose the new variable y in von Mises transformation as $y := u(x,t) - \beta$ and repeat the above steps. $\qquad \square$

COROLLARY 3.3 Let u satisfy Eq. (1). Then for any cylinder $Q^\epsilon \subset Q$ we have

$$\sup_{(\Gamma^* \setminus \Gamma_v) \cap Q^\epsilon} |\partial_t u| \leqslant N_*(M, \epsilon, \beta - \alpha). \tag{18}$$

In addition, the mixed second derivatives $D_i(\partial_t u)$ are L^2_{loc}-functions in $Q \setminus (\Gamma^0 \cup \Gamma_v)$.

Proof. Consider for the definiteness the case $z^* \in (\Gamma_\alpha^* \setminus \Gamma_v) \cap Q^\epsilon$. Due to Lemma 3.2 a function $\partial_t u$ is continuous in a neighborhood of z^*.

Recall that by definition of Γ_α the function $h[u]$ has a jump in t-direction from $+1$ to -1 there. The latter means that if we cross the free boundary Γ_α^* in positive t-direction then the corresponding phases change from Ω_+ to Ω_-. Since $u(z^*) = \alpha$ and $u(x^*, t^* - \varepsilon) > \alpha$ for any small $\varepsilon > 0$ we conclude that $\partial_t u(z^*) \leqslant 0$. Hence the inequality

$$\partial_t u \leqslant 0 \qquad \text{on} \quad \Gamma_\alpha^* \setminus \Gamma_v \tag{19}$$

is valid.

Now , taking into account Remark 2.1, one may combine (19) with one-sided inequality (5). It gives the desired estimate (18) with Γ_α^* instead of the whole Γ^*.

The other case, i.e., $z^* \in \Gamma_\beta^* \setminus \Gamma_v$ is treated in a similar manner. It is necessary only to observe that if we cross the free boundary Γ_β^* in positive t-direction then the phases will change from Ω_- to Ω_+ and, consequently, $\partial_t u(z^*) \geqslant 0$ and the inequality

$$\partial_t u \geqslant 0 \qquad \text{on} \quad \Gamma_\beta^* \setminus \Gamma_v \tag{20}$$

holds true. In view of Remark 2.1, the combination of (20) with one-sided estimate (6) finishes the proof of (18).

Finally, using the same arguments as in the proof of Lemma 3.1 we may get inequality (12) with sufficiently small ρ and any $k \geqslant -1$ which permits us to conclude that the mixed derivatives $D_i(\partial_t u)$ belong locally to a class of L^2-functions. $\qquad \square$

4. Sub-caloricity of $D_e u$

LEMMA 4.1 Let $w \in C(\mathfrak{D}) \cap W^{1,0}_{2,loc}(\mathfrak{D})$ with $\mathfrak{D}$ being a domain in $\mathbb{R}^{n+1}$, and let the inequality

$$\int_{\mathfrak{D}} (-w\partial_t \eta + Dw D\eta)\, dz \leqslant 0 \tag{21}$$

hold for any nonnegative function $\eta \in C_0^\infty(\mathfrak{D})$ with $supp\, \eta \subset \{w > 0\}$.

Then the function w_+ is sub-caloric in $\mathfrak{D}$.

Proof. First, we take in (21) nonnegative functions $\eta \in C_0^\infty(\mathfrak{D})$ with

$$supp\,\eta \subset \left\{w \geqslant \frac{\delta}{2} > 0\right\}. \tag{22}$$

Without loss of generality we may consider instead of w in (21) its mollifier w_ρ with sufficiently small parameter ρ. After integration by parts we arrive at

$$\int\limits_{\mathfrak{D}} \left[\partial_t w_\rho \eta + D w_\rho D\eta\right] dz \leqslant 0. \tag{23}$$

We set in (23) $\eta = \psi_\delta(w_\rho)\varphi$, where $\varphi \in C_0^\infty(\mathfrak{D})$ is an arbitrary nonnegative test function, while

$$\psi_\delta(s) = \begin{cases} 0, & \text{if } s \leqslant \delta \\ \dfrac{(s-\delta)}{\delta}, & \text{if } \delta < s < 2\delta \\ 1, & \text{if } s \geqslant 2\delta \end{cases}.$$

Observe that such a choice of η is not restrictive, since due to definition of ψ_δ we have for sufficiently small ρ the evident inclusions

$$supp\,\eta \subset \{w_\rho \geqslant \delta\} \subset \left\{w > \frac{\delta}{2}\right\}.$$

After substitution of η inequality (23) takes the form

$$\int\limits_{\mathfrak{D}} \left[\partial_t w_\rho \psi_\delta(w_\rho)\varphi + |Dw_\rho|^2 \psi_\delta'(w_\rho)\varphi + Dw_\rho \psi_\delta(w_\rho)D\varphi\right] dz \leqslant 0. \tag{24}$$

Elementary calculation shows that $\partial_t w_\rho \psi_\delta(w_\rho) = \frac{d}{dt} F_\delta(w_\rho)$ where the function F_δ is defined as

$$F_\delta(s) = \int\limits_0^s \psi_\delta(\tau)d\tau = \begin{cases} 0, & \text{if } s \leqslant \delta \\ \dfrac{(s-\delta)^2}{2\delta}, & \text{if } \delta < s < 2\delta \\ s - (3/2)\delta, & \text{if } s \geqslant 2\delta \end{cases}.$$

So, again integrating by parts and taking into account that the second term in (24) is nonnegative we get the inequality

$$\int\limits_{\mathfrak{D}} \left[-F_\delta(w_\rho)\partial_t \varphi + Dw_\rho \psi_\delta(w_\rho)D\varphi\right] dz \leqslant 0. \tag{25}$$

Tending in (25) $\rho \to 0$ and taking into account the definitions of ψ_δ and F_δ we arrive at

$$\int\limits_{\{w>2\delta\}} \left[-w\partial_t \varphi + DwD\varphi\right] dz \leqslant \int\limits_{\{\delta<w<2\delta\}} |DwD\varphi|dz + C\delta.$$

Letting $\delta \to 0$ in the above inequality provides the inequality

$$\int\limits_{\{w>0\}} \left[-w\partial_t \varphi + DwD\varphi\right] dz \leqslant 0. \tag{26}$$

It remains only to recall that φ in (26) is an arbitrary nonnegative test-function. This completes the proof. $\square$

LEMMA 4.2 Let u be a solution of Eq. (1). Then for any direction $e \in \mathbb{R}^n$ functions $(D_e u)_\pm$ are sub-caloric in $Q \setminus \Gamma_v$.

Proof. Due to Lemma 4.1 it suffices to check that for $w = D_e u$ inequality (21) holds true for any nonnegative function $\eta \in C_0^\infty(Q \setminus \Gamma_v)$ with $supp\, \eta \subset \{D_e u > 0\}$.

It follows from Eq. (1) that functions $D_e u$ satisfy in Q the equation

$$H[D_e u] = D_e(h[u]) \tag{27}$$

in the weak (distributional) sense. Hence we obtain

$$\int_Q D_e u\,(\partial_t \eta + \Delta \eta)\,dz = -\int_Q h[u]D_e \eta\,dz = -\int_{\Omega_+} D_e \eta\,dz + \int_{\Omega_-} D_e \eta\,dz$$

$$= 2\int_{\Gamma^*} \eta \cos\left(\widehat{\mathbf{n},\mathbf{e}}\right)d\mathcal{H}^n,$$

where $\mathbf{n} = \mathbf{n}(z)$ is the unit normal vector to Γ^* directed into Ω_+, $\mathbf{e} := (e,0)$, and $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure.

It is easy to see that the normal vector $\mathbf{n}$ has on Γ^* the following representation

$$\mathbf{n}(z) = \left(\frac{Du(z)}{\sqrt{|Du(z)|^2 + \left(\partial_t u(z)\right)^2}},\ \frac{\partial_t u(z)}{\sqrt{|Du(z)|^2 + \left(\partial_t u(z)\right)^2}}\right). \tag{28}$$

Indeed, since $u > \alpha$ in Ω_+ and $\Gamma_\alpha \subset \{u = \alpha\}$, the vector $Du(z)$ at $z \in \Gamma_\alpha^*$ is directed into Ω_+. In addition, we recall (see (19)) that $\partial_t u \leqslant 0$ on Γ_α^*. Therefore, the projection of $\mathbf{n}$ from formula (28) on the t-axis is also nonpositive. Because of Ω_+ is locally a subgraph of Γ_α in t-direction, we conclude that on Γ_α^* the whole vector $\mathbf{n}$ defined by (28) is directed into Ω_+. Similarly, we have $\{u < \beta\}$ in Ω_- and $\Gamma_\beta \subset \{u = \beta\}$. Therefore, the spatial gradient $Du(z)$ at $z \in \Gamma_\beta^*$ is directed into Ω_+. Moreover, on Γ_β^* we have $\partial_t u \geqslant 0$ (see (20)) and Ω_+ is a t-epigraph of Γ_β^*. So, the vector $\mathbf{n}$ from formula (28) is again directed into Ω_+.

Now, taking into account the inclusion $supp\, \eta \subset \{D_e u > 0\}$ and representation (28) we conclude that

$$\eta \cos\left(\widehat{\mathbf{n}(z),\mathbf{e}}\right) \geqslant 0 \qquad \forall z \in \Gamma^*$$

and complete the proof. $\qquad\square$

REMARK 4.3 We emphasize that $(D_e u)_\pm$ are, in general, not sub-caloric near Γ_v.

5. Quadratic growth estimates

LEMMA 5.1 Let u satisfy (1), let $z^0 \in \Gamma^0$, and let

$$dist_p\{z^0, \Gamma_v\} \geqslant \rho_0 > 0, \qquad dist_p\{z^0, \partial'Q\} \geqslant \rho_0.$$

There exists a positive constant C_0 completely defined by the values of ρ_0 and M such that

$$\operatorname*{osc}_{Q_r^-(z^0)} u \leqslant C_0 r^2 \qquad \text{for all} \quad r \leqslant \rho_0. \tag{29}$$

Proof. We verify inequality (29) for $z^0 \in \Gamma_\alpha^0$. The other case, i.e., $z^0 \in \Gamma_\beta^0$ can be proved by using similar arguments.

We argue by contradiction. Suppose (29) fails. Then there exist a sequence $r_k > 0$ as well as sequences u_k of solutions to (1) satisfying (3), and points $z^k \in \Gamma_\alpha^0(u_k)$ such that for all $k \in \mathbb{N}$ we have

$$dist_p\left(z^k, \Gamma_v(u_k)\right) \geq \rho_0, \qquad dist_p\left(z^k, \partial' Q\right) \geq \rho_0$$

and

$$\sup_{Q_{r_k}^-(z^k)} |u_k - \alpha| \geq k r_k^2. \tag{30}$$

Thanks to assumption (3) the left-hand side of (30) is bounded by $2M$ and, consequently, $r_k \to 0$ as $k \to \infty$. It is evident that we can choose r_k as the maximal value of r for which

$$\sup_{Q_r^-(z^k)} |u_k - \alpha| \geq k r^2.$$

In other words, we have the relations

$$\begin{cases} \mathfrak{M}_r(z^k, u_k) := \sup_{Q_r^-(z^k)} |u_k - \alpha| < k r^2 \quad \text{for all } r \in (r_k, \rho_0], \\ \mathfrak{M}_{r_k}(z^k, u_k) = k r_k^2. \end{cases} \tag{31}$$

Next, we define a scaling $\tilde{u}_k$ as

$$\tilde{u}_k(x,t) = \frac{u_k(x^k + r_k x, t^k + r_k^2 t) - \alpha}{\mathfrak{M}_{r_k}(z^k, u_k)}$$

for $(x,t) \in Q_{\rho_0/r_k}^-$. Then $\tilde{u}_k$ has the following properties

$$\sup_{Q_1^-} |\tilde{u}_k| = 1, \tag{32}$$

$$\tilde{u}_k(0,0) = 0, \qquad |D\tilde{u}_k(0,0)| = 0, \tag{33}$$

$$\|H[\tilde{u}_k]\|_{\infty, Q_{1/r_k}^-} \leq \frac{r_k^2}{\mathfrak{M}_{r_k}(z^k, u_k)} = \frac{1}{k} \to 0 \text{ as } k \to \infty. \tag{34}$$

In addition, due to (31) we have for $R \in (1, \rho_0/r_k]$ the inequality

$$\sup_{Q_R^-} |\tilde{u}_k| = \frac{\mathfrak{M}_{r_k R}(z^k, u_k)}{\mathfrak{M}_{r_k}(z^k, u_k)} < \frac{k(r_k R)^2}{k r_k^2} = R^2. \tag{35}$$

Now, by (32)–(35) we will have a subsequence of $\tilde{u}_k$ weakly converging in $W_{q,loc}^{2,1}\left(\mathbb{R}_{x,t}^{n+1} \cap \{t \leq 0\}\right), q < \infty$, to a caloric function u_0 satisfying

$$\sup_{Q_R^-} |u_0| \leq R^2 \qquad \forall R \geq 1,$$

$$u_0(0,0) = |Du_0(0,0)| = 0,$$

$$\sup_{Q_1^-} |u_0| = 1. \tag{36}$$

According to the Liouville theorem (see, for example, Lemma 2.1 [2]), there exist constants a^{ij} such that

$$u_0(x,t) = a^{ij} x_i x_j + 2\Big(\sum_{i=1}^{n} a^{ii}\Big)t \quad \text{in } \mathbb{R}^{n+1}_{x,t} \cap \{t \leqslant 0\}. \tag{37}$$

On the other hand, due to inequalities (4), Lemma 4.2 and Fact 7.3 we may conclude that for any direction $e \in \mathbb{R}^n$ and for all $k \in \mathbb{N}$ such that $r_k \leqslant \rho_0$

$$\Phi(r_k, (D_e u_k)_+, (D_e u_k)_-, \xi_{\rho_0, z^k}, z^k) \leqslant c(\rho_0), \tag{38}$$

where $c(\rho_0)$ is defined completely by the values of ρ_0 and M. More precisely, by $c(\rho_0)$ we may take a majorant of the right-hand side of inequality (52) calculated for $\theta_1 = (D_e u_k)_+$ and $\theta_2 = (D_e u_k)_-$. After simple rescaling (38) takes the form

$$\Phi(1, (D_e \tilde{u}_k)_+, (D_e \tilde{u}_k)_-, \zeta^k, 0, 0) \leqslant c(\rho_0)\Big(\frac{r_k^2}{\mathfrak{M}_{r_k}(z^k, u_k)}\Big)^4 = \frac{c(\rho_0)}{k^4}, \tag{39}$$

where for brevity we denote the corresponding cut-off function $\xi_{\rho_0/r_k,(0,0)}$ by ζ^k. Observe that $\zeta^k \equiv 1$ in $B_{\rho_0/(2r_k)}$. In addition, $B_{\rho_0/(2r_k)} \supset B_1$ if k is big enough, while for $\varepsilon > 0$ (small and fixed) we have

$$G(x,-t) \geqslant N(n,\varepsilon) > 0 \quad \text{for} \quad -1 < t < -\varepsilon, \quad x \in B_1.$$

Hence,

$$N(n,\varepsilon) \int_{-1}^{-\varepsilon} \int_{B_1} |(D_e \tilde{u}_k)_\pm|^2 dxdt \leqslant \int_{-1}^{0} \int_{\mathbb{R}^n} \big|D_e\big((\tilde{u}_k)_\pm \zeta^k\big)\big|^2 G(x,-t) dxdt. \tag{40}$$

Next, using (40) and invoking the Poincare inequality we may reduce (39) to

$$\int_{-1}^{-\varepsilon} \int_{B_1} |(D_e \tilde{u}_k)_+ - m_+^k(t)|^2 dxdt \int_{-1}^{-\varepsilon} \int_{B_1} |(D_e \tilde{u}_k)_- - m_-^k(t)|^2 dxdt \leqslant N^{-2}(n,\varepsilon)\frac{c(\rho_0)}{k^4},$$

where $m_\pm^k(t)$ denotes the corresponding average of $(D_e \tilde{u}_k)_\pm$ on t-sections over B_1.

Letting k tend to infinity (and then ε tend to zero), we obtain

$$\int_{Q_1^-} |(D_e u_0)_+ - m^+|^2 dxdt \int_{Q_1^-} |(D_e u_0)_- - m^-|^2 dxdt = 0, \tag{41}$$

where $m^\pm$ is the corresponding average of $(D_e u_0)_\pm$ over B_1. Observe that, due to representation (37), $m^\pm$ do not depend on t.

Obviously, (41) implies that $D_e u_0$ does not change its sign in Q_1^-. Recall that e is an arbitrary direction in $\mathbb{R}^n$ and u_0 is a polinomial of the form (37). It means, in particularly, that $u_0 \equiv 0$ in Q_1^-. The latter contradicts (36) and complete the proof of (29). $\qquad\square$

We will need the extension of Lemma 5.1 to the "upper half-cylinders" $Q_r(z^0) \cap [t^0, t^0 + r^2]$ as well.

LEMMA 5.2 Let all the assumptions of Lemma 5.1 be valid. Then

$$\underset{Q_r(z^0)}{\text{osc}} \; u \leqslant C_1 r^2 \qquad \text{for all} \quad r \leqslant \rho_0, \tag{42}$$

where ρ_0 is the same constant as in Lemma 5.1 and $C_1 = C_1(\rho_0, M)$.

Proof. To obtain estimate (42) for $\{t > t^0\}$ we consider the barrier function

$$w(x,t) = C'(\rho_0, M) \left\{ |x - x^0|^2 + 2n(t - t^0) \right\} + (t - t^0),$$

where $C'(\rho_0, M) = \max \left\{ C_0, M\rho_0^{-2} \right\}$ and $C_0 = C_0(\rho_0, M)$ is the constant from Lemma 5.1. Using (29) for $t = t^0$ and the comparison principle one can easily verify that

$$|u(x,t)| \leqslant w(x,t) \quad \text{in} \quad B_{\rho_0}(x^0) \times]t^0, t^0 + r^2]. \tag{43}$$

Combination of (29) and (43) finishes the proof of (42). □

LEMMA 5.3 Let all the assumptions of Lemma 5.1 be valid. Then

$$\underset{Q_r(z^0)}{\text{sup}} \; |Du| \leqslant C_2 r \quad \text{for all} \quad r \leqslant \rho_0, \tag{44}$$

where $\rho_0 > 0$ is just the same as in Lemma 5.1, while C_2 is a positive constant completely defined by the values of M and ρ_0.

Proof. We verify (44) for $z^0 \in \Gamma_\alpha^0$. The case $z^0 \in \Gamma_\beta^0$ is treated in a similar manner.

Let us choose an arbitrary $r \leqslant \rho_0/2$ and consider a point $\tilde{z} \in Q_r(z^0)$. Further, we take identity (7) with $Q_{2\rho}^-(z^*)$ replaced by $Q_r^-(\tilde{z}))$ and plug in this identity a test-function

$$\eta(x,t) = \left(u(x,t) - \alpha \right) \xi^2(x)$$

where $\xi \in C_0^\infty(B_r(\tilde{x}))$ satisfying $0 \leqslant \xi \leqslant 1$ and $|D\xi| \leqslant cr^{-1}$. After standard transformations we get the inequality

$$\int_{B_r(\tilde{x})} (u-\alpha)^2 \xi^2 dx \Big|^{\tilde{t}} + \int_{Q_r^-(\tilde{z})} |Du|^2 \xi^2 dx dt$$
$$\leqslant \int_{B_r(\tilde{x})} (u-\alpha)^2 \xi^2 dx \Big|^{\tilde{t}-r^2} + c \int_{Q_r^-(\tilde{z})} (u-\alpha)^2 |D\xi|^2 dx dt + c \int_{Q_r^-(\tilde{z})} |u - \alpha| \xi^2 dx dt, \tag{45}$$

where c stands for an absolute constant.

In view of (42) the right-hand side of (31) can be estimated from above by $2c\, C_1(\rho_0, M) r^{n+4}$ which guarantees

$$\int_{Q_r^-(\tilde{z})} |Du|^2 \xi^2 dx dt \leqslant 2c\, C_1 r^{n+4}.$$

It remains only to observe that combination of the latter inequality with Eq. (27) and Fact 7.2 implies the estimate

$$|Du(\tilde{z})| \leqslant \tilde{c}\, r$$

which completes the proof.   □

6. Estimates of $\partial_t u$ and $D^2 u$ beyond Γ_v

In this section we obtain the estimates of $|\partial_t u(\hat{z})|$ and $|D^2 u(\hat{z})|$ in any $\hat{z}$ being a point of smoothness for u. We emphasize that these bounds do not depend on the parabolic distance from $\hat{z}$ to Γ^0 as well as to Γ^*. Unfortunately, we cannot remove the dependence of both bounds on the parabolic distance from $\hat{z}$ to Γ_v.

LEMMA 6.1 Let u satisfy (1), let $\hat{z} \in Q \setminus \Gamma(u)$, and let

$$dist_p\{\hat{z}, \Gamma_v\} \geqslant \rho_0 > 0, \qquad dist_p\{\hat{z}, \partial'Q\} \geqslant \epsilon > 0.$$

There exists a positive constant C_3 depending only on ρ_0, ϵ, M and $\beta - \alpha$ such that

$$|\partial_t u(\hat{z})| \leqslant C_3. \tag{46}$$

Proof. Define $d_0 = d_0(\hat{z}) := \min\{dist_p\{\hat{z}, \Gamma^0\}, \rho_0, \epsilon/2\}$. Without loss of generality we may suppose that $Q^-_{d_0}(\hat{z}) \cap \Gamma_\beta = \emptyset$. Due to Lemma 3.1 we need only to estimate $\partial_t u(\hat{z})$ from above. It is obvious that for any small $\delta > 0$

$$Q^-_{d_0/2}(\hat{x}, \hat{t} - \delta) \cap \{\Gamma^0 \cup \Gamma_v \cup \partial'Q\} = \emptyset.$$

However, $Q^-_{d_0/2}(\hat{x}, \hat{t} - \delta)$ may contain the points of $\Gamma^*_\alpha \setminus \Gamma_v$.

1. First, we consider the case $d_0 = dist_p\{\hat{z}, \Gamma^0\}$.

   Using the same arguments as in the derivation of (10) in the proof of Lemma 3.1 we get for all functions $\eta \in W_2^{1,1}(Q^-_{d_0/2}(\tilde{x}, \tilde{t} - \delta))$ vanishing on $\partial'Q^-_{d_0/2}(\tilde{x}, \tilde{t} - \delta)$ the equality

$$\int\limits_{Q^-_{d_0/2}(\hat{x}, \hat{t} - \delta)} \left[\partial_t u^{(\tau)}\eta + Du^{(\tau)}D\eta\right] dxdt$$
$$= -\frac{1}{\tau} \int\limits_{Q^-_{d_0/2}(\hat{x}, \hat{t} - \delta)} \left(h[u](x, t) - h[u](x, t - \tau)\right)\eta \, dxdt, \quad (47)$$

   where $u^{(\tau)}$ denotes the difference quotient of u in the t-direction.
   Plugging in (47)

$$\eta(x, t) = (\partial_t u(x, t) - k)_+ \xi^2(x, t), \qquad k \geqslant 2N_*,$$

   where ξ is a standard cut-off function for a cylinder $Q^-_{d_0/2}(\hat{x}, \hat{t} - \delta)$ (see Notation), and N_* is the constant from Corollary 3.3, we arrive at the relation

$$\int\limits_{Q^-_{d_0/2}(\hat{x}, \hat{t} - \delta)} \left\{\partial_t u^{(\tau)}(\partial_t u - k)_+ \xi^2 + Du^{(\tau)}D\left[(\partial_t u - k)_+ \xi^2\right]\right\} dxdt$$
$$= -\frac{1}{\tau} \int\limits_{Q^-_{d_0/2}(\hat{x}, \hat{t} - \delta)} \left\{h[u](x, t) - h[u](x, t - \tau)\right\}(\partial_t u - k)_+ \xi^2 dxdt. \quad (48)$$

Observe that due to Corollary 3.3 the parabolic distance from the set $\{supp\,\eta\}$ to $\Gamma(u)$ is positive. Therefore, $\partial_t u$ is smooth on $\{supp\,\eta\}$ and the right-hand side of (48) vanishes if τ is small enough. In addition, we make take in (48) the cut-off function ξ multiplied by the characteristic function of an interval $[\hat{t} - \delta - d_0^2/4, t]$ with an arbitrary $t \in]\hat{t} - \delta - d_0^2/4, \hat{t} - \delta]$. This leads for sufficiently small τ to the inequalities

$$\int\limits_{\hat{t}-\delta-d_0^2/4}^{t} \int\limits_{B_{d_0/2}(\hat{x})} \left\{ \partial_t u^{(\tau)} (\partial_t u - k)_+ \xi^2 + Du^{(\tau)} D \left[(\partial_t u - k)_+ \xi^2 \right] \right\} dxdt \leqslant 0$$

$$\forall t \in]\hat{t} - \delta - d_0^2/4, \hat{t} - \delta].$$

Now, we let in the latter inequalities $\tau \to 0$ and then leave the nonnegative terms in the left-hand side, transfer the rest terms to the right-hand side and estimate these rest terms from above via Young's inequality. As a consequence, for $k \geqslant 2N_*$ we get the inequalities

$$\sup_{\hat{t}-d_0^2/4<t<\hat{t}-\delta} \int\limits_{B_{d_0/2}(\hat{x})} (\partial_t u - k)_+ \, dx \Big|^t + \int\limits_{Q^-_{d_0/2}(\hat{x},\hat{t}-\delta)\cap\{\partial_t u>k\}} |D\,(\partial_t u)\,|^2\xi^2 dxdt$$

$$\leqslant c \int\limits_{Q^-_{d_0/2}(\hat{x},\hat{t}-\delta)} (\partial_t u - k)_+ \left[|D\xi|^2 + 2\xi|\partial_t\xi| \right] dxdt.$$

Application of Fact 7.1 with $v = \partial_t u$ implies the estimate

$$\partial_t u(\hat{x}, \hat{t} - \delta) \leqslant 2N_* + N_0 \sqrt{\fint_{Q^-_{d_0/2}(\hat{x},\hat{t}-\delta)} |\partial_t u|^2 dxdt}. \tag{49}$$

In order to obtain a bound for the integral term on the right-hand side of (49) we take identity (7) with $Q^-_{2\rho}(z^*)$ replaced by $Q^-_{d_0}(\hat{x}, \hat{t} - \delta)$ and plug in this identity a test-function

$$\eta(x, t) = \partial_t u(x, t)\zeta^2(x),$$

where ζ is a smooth cut-off function in $B_{d_0}(\hat{x})$ that equals 1 in $B_{d_0/2}(\hat{z})$ and vanishes outside of $B_{3d_0/4}(\hat{x})$. After standard manipulations and taking into account Lemma 5.3 we end up with

$$\int\limits_{Q^-_{3d_0/4}(\hat{x},\hat{t}-\delta)} |\partial_t u|^2\zeta^2 dxdt \leqslant c \int\limits_{Q^-_{3d_0/4}(\hat{x},\hat{t}-\delta)} \left(h^2[u]\zeta^2 + |Du|^2|D\zeta|^2 \right) dxdt + cd_0^{n+2}$$

$$\leqslant \tilde{c}\,(d_0)^{n+2} + \tilde{c}\,(d_0)^{-2} \int\limits_{Q^-_{3d_0/4}(\hat{x},\hat{t}-\delta)} |Du|^2 dxdt \tag{50}$$

$$\leqslant \tilde{c} \left\{ 1 + C_1^2 \right\} (d_0)^{n+2}.$$

Thus, combination of (49) and (50) provides the estimate

$$\partial_t u(\hat{x}, \hat{t} - \delta) \leqslant 2N_* + N_0 \sqrt{\tilde{c} \left\{ 1 + C_1^2(\rho_0, \epsilon, M) \right\}}.$$

Observe that the constant on the right-hand side of the above inequality depends neither on d_0 nor on δ. Thus, we get

$$\partial_t u(\hat{z}) \leqslant 2N_* + N_0 \sqrt{\tilde{c} \left\{ 1 + C_1^2(\rho_0, \epsilon, M) \right\}}.$$

2. Suppose now that $d_0 = \min\{\rho_0, \epsilon/2\}$. In this case we repeat all the above arguments up to deriving (49). Then we estimate the integral term on the right-hand side of (49) with the help of inequalities (4) with $q = 2$. This gives us the bound

$$\int\limits_{Q^-_{d_0/2}(\hat{x}, \hat{t} - \delta)} |\partial_t u|^2 dxdt \leqslant N_1(\epsilon, 2, M)$$

which together with (49) implies

$$\partial_t u(\hat{x}, \hat{t} - \delta) \leqslant 2N_* + N_0 N_1^{1/2} \big(\min\{\rho_0, \epsilon\} \big)^{-1-n/2}.$$

Again, the right-hand side of the latter bound is independent of δ as well as of the parabolic distance from $\hat{z}$ to Γ^0.

The proof is complete. $\qquad \square$

LEMMA 6.2 Let u satisfy the same assumptions as in Lemma 6.1. Then there exists a positive constant C_4 depending only on ρ_0, ϵ, M and $\beta - \alpha$ such that

$$\left| D^2 u(\hat{z}) \right| \leqslant C_4. \tag{51}$$

Proof. Let $\hat{z} \in Q \setminus \Gamma(u)$ be fixed, and let $v = Du(\hat{z})/|Du(\hat{z})|$. Suppose that e is an arbitrary direction in $\mathbb{R}^n$ if $|Du(\hat{z})| = 0$ and $e \perp v$ otherwise. We also define $d_0 = d_0(\hat{z}) := \min \left\{ dist_p \left\{ \hat{z}, \Gamma^0 \right\}, \rho_0, \epsilon/2 \right\}$.

In view of our choice of e we have $D_e u(\hat{z}) = 0$ and, consequently, we may apply Fact 7.4 to the sub-caloric functions $v = (D_e u)_\pm$ in $Q^-_{d_0}(\hat{z})$. From here, taking into account Lemma 5.3, we obtain the estimate

$$|D(D_e u)(\hat{z})| \leqslant C_4(\rho_0, \epsilon, M, \beta - \alpha),$$

where C_4 does not depend on d_0. Since e is an arbitrary direction in $\mathbb{R}^n$ satisfying $e \perp v$, the derivative $D_v(D_v u(\hat{z}))$ can now be estimated from Eq. (1). Thus, we proved the desired inequality (51). $\qquad \square$

7. Appendix

For the readers convenience and for the references, we recall and explain several facts. Most of these auxiliary results are known, but probably not well known in the context used in this paper.

FACT 7.1 Let $r_0 \in (0, 1)$, and let $v \in V_2 \left(Q^-_{r_0}(z^*) \right)$ satisfy the inequalities

$$\sup_{t^* - r_0^2 < t < t^*} \int\limits_{B_{r_0}(x^*)} (v - k)_+^2 \, \xi^2 dx \bigg|^t + \int\limits_{Q^-_{r_0}(z^*)} \left[D \left((v - k)_+ \right) \right]^2 \xi^2 dz$$

$$\leqslant c \int\limits_{Q^-_{r_0}(z^*)} (v - k)_+^2 \left[|D\xi|^2 + \xi |\partial_t \xi| \right] dz$$

for all $k \geqslant k_0$ and all cut-off functions $\xi = \xi(x,t)$ for the cylinder $Q_{r_0}^-(z^*)$ (see Notation). Here c stands for a positive constant.

Then there exists a positive constant $N_0 = N_0(c)$ such that

$$\sup_{Q_{r_0/2}^-(z^*)} v \leqslant k_0 + N_0 \sqrt{\fint_{Q_{r_0}^-(z^*)} (v - k_0)_+^2 (z)dz}.$$

Proof. For the proof of this assertion we refer the reader to (the proof of) Theorem 6.2, Chapter II [13]. □

FACT 7.2 Let $\mathfrak{D}$ be a domain in $\mathbb{R}^{n+1}$, and let $g^i \in L^\infty(\mathfrak{D})$, $i = 0, 1, \ldots, n$. Then if $v \in V_2(\mathfrak{D})$ is a solution of the equation

$$H[v] = \operatorname{div} \vec{g} + g^0, \qquad \vec{g} = (g^1, \ldots, g^n)$$

in $\mathfrak{D}$, we have, for any cylinder $Q_{2R}^-(z^0) \subset \mathfrak{D}$,

$$\sup_{Q_R^-(z^0)} |v| \leqslant \hat{N}_0 \sqrt{\fint_{Q_{2R}^-(z^0)} v^2 dxdt} + \hat{N}_1 R \|\vec{g}\|_{\infty, Q_{2R}^-(z^0)} + \hat{N}_2 R^2 \|g^0\|_{\infty, Q_{2R}^-(z^0)}.$$

Proof. The validity of Fact 7.2 follows from results of §6 Chapter II and §8 Chapter III [13] (see also Theorem 6.17 in [14]). □

We denote

$$I(r, v, z^*) = \int_{t^*-r^2}^{t^*} \int_{\mathbb{R}^n} |Dv(x,t)|^2 G(x - x^*, t^* - t)dxdt,$$

where $r \in]0, \rho_0]$, $z^* = (x^*, t^*)$ is a point in $\mathbb{R}^{n+1}$, a function v is defined n the strip $\mathbb{R}^n \times [t^* - \rho_0^2, t^*]$, and the heat kernel $G(x,t)$ is defined by

$$G(x,t) = \frac{\exp\left(-|x|^2/4t\right)}{(4\pi t)^{n/2}} \text{ for } t > 0 \text{ and } G(x,t) = 0 \text{ for } t \leqslant 0.$$

To prove the quadratic growth estimate for solutions of (1), we need the following local version of the famous Caffarelli monotonicity formula (see [5]) for pairs of disjointly supported subsolutions of the heat equation.

FACT 7.3 Let $z^* = (x^*, t^*)$ be a point in $\mathbb{R}^{n+1}$, let $\xi_{\rho_0, x^*} := \xi_{\rho_0, x^*}(x)$ be a standard time-independent cut-off function belonging $C^2(\overline{B}_{\rho_0}(x^*))$, having support in $B_{\rho_0}(x^*)$, and satisfying $\xi_{\rho_0, x^*} \equiv 1$ in $B_{\rho_0/2}(x^*)$, and let θ_1, θ_2 be nonnegative, sub-caloric functions in $Q_{\rho_0}^-(z^*)$, belonging to $C\left(Q_{\rho_0}^-(z^*)\right) \cap W_{2,loc}^{1,0}\left(Q_{\rho_0}^-(z^*)\right)$ satisfying

$$\theta_1(x^*, t^*) = \theta_2(x^*, t^*) = 0, \qquad \theta_1(x,t) \cdot \theta_2(x,t) = 0 \quad \text{in} \quad Q_{\rho_0}^-(z^*).$$

Then, for $0 < r < \rho_0$ the functional

$$\Phi(r, \xi_{\rho_0, z^*}) := \Phi(r, \theta_1, \theta_2, \xi_{\rho_0, z^*}, z^*) = \frac{1}{r^4} I(r, \theta_1 \xi_{\rho_0, z^*}, z^*) I(r, \theta_2 \xi_{\rho_0, z^*}, z^*)$$

satisfies the inequality

$$\Phi(r, \xi_{\rho_0, z^*}) \leqslant \frac{\tilde{N}}{\rho_0^{2n+8}} \|\theta_1\|^2_{2, Q_{\bar{\rho}_0}^-(z^*)} \|\theta_2\|^2_{2, Q_{\bar{\rho}_0}^-(z^*)} \tag{52}$$

with an absolute positive constant $\tilde{N}$.

Proof. Using the same arguments as in the proof of Lemma 2.4 and Remark after that in [2] (see also Fact 1.6 and Remark 1.7 in [3]) one can get the inequality

$$\Phi(r, \xi_{\rho_0, z^*}) \leqslant \Phi(\rho_0/2, \xi_{\rho_0, z^*}) + \frac{N'}{\rho_0^{2n+8}} \|\theta_1\|^2_{2, Q_{\bar{\rho}_0}(z^*)} \|\theta_2\|^2_{2, Q_{\bar{\rho}_0}(z^*)}, \tag{53}$$

where N' is an absolute positive constant.

We claim that the first term on the right-hand side of (53) can be estimated via the second term. Indeed, it is evident that

$$\Phi(\rho_0/2, \xi_{\rho_0, z^*}) \leqslant \frac{c}{\rho_0^4} I(\rho_0, \theta_1 \zeta_0, z^*) I(\rho_0, \theta_2 \zeta_0, z^*), \tag{54}$$

where $\zeta_0 = \zeta_0(x, t) = \xi_{\rho_0, z^*}(x) \varsigma_{\rho_0, z^*}(t)$, while ς_{ρ_0, z^*} stands for a nonnegative function belonging $C^2\left([t^* - \rho_0^2, t^*]\right)$, having support in $[t^* - 3\rho_0^2/4, t^*]$ and satisfying $\varsigma_{\rho_0, z^*}(t) \equiv 1$ in $[t^* - \rho_0^2/4, t^*]$.

On the other hand, functions θ_i, $i = 1, 2$, are sub-caloric in $Q_{\rho_0}^-(z^*)$, i.e., $H[\theta_i] \geqslant 0$ in the sense of distributions. Since

$$|D\theta_i|^2 + \theta_i H[\theta_i] = \frac{1}{2} H[\theta_i^2]$$

we have

$$\int_{t^*-\rho_0^2}^{t^*} \int_{\mathbb{R}^n} |D\theta_i(x, t)|^2 \zeta_0^2(x, t) G(x - x^*, t^* - t) \, dx \, dt$$

$$\leqslant \frac{1}{2} \int_{t^*-\rho_0^2}^{t^*} \int_{\mathbb{R}^n} H[\theta_i^2(x, t)] \zeta_0^2(x, t) G(x - x^*, t^* - t) \, dx \, dt. \tag{55}$$

After successive integration the right-hand side of (55) by parts we get

$$\int_{t^*-\rho_0^2}^{t^*} \int_{\mathbb{R}^n} |D\theta_i|^2 \zeta_0^2 G \, dx \, dt = \int_{t^*-\rho_0^2}^{t^*} \int_{B_{\rho_0}(x^*)} |D\theta_i|^2 \zeta_0^2 G \, dx \, dt$$

$$\leqslant - \int_{B_{\rho_0}(x^*)} \left(\frac{\theta_i^2}{2} \zeta_0^2 G \right) dx \Big|_{t^*-\rho_0^2}^{t^*}$$

$$+ \int_{t^*-\rho_0^2}^{t^*} \int_{B_{\rho_0}(x^*)} \frac{\theta_i^2}{2} \zeta_0^2 \left[\partial_t G + \Delta G \right] dx \, dt$$

$$+ \int_{t^*-\rho_0^2}^{t^*} \int_{B_{\rho_0}(x^*)} \theta_i^2 \left[2\zeta_0 D\zeta_0 DG + G|D\zeta_0|^2 + G\zeta_0\Delta\zeta_0\right] dxdt$$

$$+ \int_{t^*-\rho_0^2}^{t^*} \int_{B_{\rho_0}(x^*)} \theta_i^2 G\zeta_0|\partial_t\zeta_0|dxdt$$

$$=: J_1 + J_2 + J_3 + J_4.$$

It is evident that due to our choice of ζ_0 we have $J_1 \leqslant 0$.

Further, taking into account the relation

$$\partial_t G + \Delta G = \partial_t G(x - x^*, t^* - t) + \Delta G(x - x^*, t^* - t) = 0 \quad \text{for} \quad t < t^*,$$

we conclude that $J_2 = 0$.

Finally, we observe that the integral in J_3 is really taken over the set $\mathcal{E} =]t^*-\rho_0^2, t^*] \times \{B_{\rho_0}(x^*) \setminus B_{\rho_0/2}(x^*)\}$, while the integral in J_4 is taken over the set $\mathcal{E}' = [t^* - \rho_0^2, t^* - \rho_0^2/4] \times B_{\rho_0}(x^*)$. Therefore, in $\mathcal{E}$ we have the following estimates for functions involved into J_3

$$|G(x - x^*, t^* - t)| \leqslant \hat{c} \frac{e^{-\frac{\rho_0^2}{16(t^*-t)}}}{(t^* - t)^{n/2}} \leqslant \hat{c}\rho_0^{-n};$$

$$|DG(x - x^*, t^* - t)D\zeta_0(x, t)| \leqslant \hat{c}|G(x - x^*, t^* - t)|\frac{|x - x^*|}{\rho_0(t^* - t)}$$

$$\leqslant \hat{c} \frac{e^{-\frac{\rho_0^2}{16(t^*-t)}}}{(t^* - t)^{1+n/2}} \leqslant \hat{c}\rho_0^{-n-2}.$$

Similarly, in $\mathcal{E}'$ we have

$$|G(x - x^*, t^* - t)| \leqslant \hat{c}\rho_0^{-n},$$

and, consequently,

$$J_3 + J_4 \leqslant \tilde{c}\rho_0^{-n-2} \iint_{Q_{\rho_0}^-(z^*)} \theta_i^2 dxdt \leqslant \tilde{c}\rho_0^{-n-2}\|\theta_i\|_{2,Q_{\rho_0}^-(z^*)}^2.$$

Thus, collecting all inequalities we get

$$I(\rho_0, \theta_i\zeta_0, z^*) \leqslant 2 \int_{t^*-\rho_0^2}^{t^*} \int_{B_{\rho_0}(x^*)} \left[|D\zeta_0|^2\theta_i^2 + |D\theta_i|^2\zeta_0^2\right] Gdxdt$$

$$\leqslant N''\rho_0^{-n-2}\|\theta_i\|_{2,Q_{\rho_0}^-(z^*)}^2, \tag{56}$$

where N'' denotes a positive absolute constant.

Now, combination of (53), (54) and (56) finishes the proof of (52). $\qquad\square$

FACT 7.4 Let a function $v \in C\left(Q_R^-(z^0)\right) \cap W_{2,loc}^{1,0}\left(Q_R^-(z^0)\right)$ satisfy the following conditions:

$$v(z^0) = 0;$$

v is differentiable at z^0;

$v_\pm$ are subcaloric in $Q_R^-(z^0)$.

Then

$$|Dv(z^0)| \leqslant \tilde{N}' \sqrt{R^{-2} \fint_{Q_R^-(z^0)} v^2 dx dt}.$$

Proof. The above inequality follows directly from Fact 7.3. $\qquad\square$

REFERENCES

1. ALT, H. W., On the thermostat problem, *Control Cybernet.* **14** (1985), 171–193. MR0839520

2. APUSHKINSKAYA, D. E., SHAHGHOLIAN, H., & URALTSEVA, N. N., Boundary estimates for solutions of a parabolic free boundary problem, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **271** (2000), 39–55. Zbl1031.35150 MR1810607

3. APUSHKINSKAYA, D. E., & URALTSEVA, N. N., Uniform estimates near the initial state for solutions of the two-phase parabolic problem, *Algebra i Analiz* **25** (2013), 63–74. Zbl1304.35144 MR3114850

4. BROKATE, M., & SPREKELS, J., *Hysteresis and phase transitions*, vol. 121 of *Applied Mathematical Sciences*, Springer-Verlag, New York, 1996. Zbl0951.74002 MR1411908

5. CAFFARELLI, L., & SALSA, S., *A geometric approach to free boundary problems*, vol. 68 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, 2005. Zbl1083.35001 MR2145284

6. GUREVICH, P., SHAMIN, R., & TIKHOMIROV, S., Reaction-diffusion equations with spatially distributed hysteresis, *SIAM J. Math. Anal.* **45** (2013), 1328–1355. Zbl1276.35107 MR3054588

7. GUREVICH, P., & TIKHOMIROV, S., Uniqueness of transverse solutions for reaction-diffusion equations with spatially distributed hysteresis, *Nonlinear Anal.* **75** (2012), 6610–6619. Zbl1252.35009 MR2965244

8. HOPPENSTEADT, F. C., & JÄGER, W., Pattern formation by bacteria, In *Biological growth and spread (Proc. Conf., Heidelberg, 1979)* (1980), vol. 38 of *Lecture Notes in Biomath.*, Springer, Berlin-New York, 68–81. Zbl0437.92023 MR0609347

9. HOPPENSTEADT, F. C., JÄGER, W., & PÖPPE, C., A hysteresis model for bacterial growth patterns, In *Modelling of patterns in space and time (Heidelberg, 1983)* (1984), vol. 55 of *Lecture Notes in Biomath.*, Springer, Berlin, 123–134. MR0813709

10. KOPFOVA, J., Hysteresis and biological models, *J. Phys. Conference Series* **55** (2006), 130–134.

11. KRASNOSEL'SKIĬ, M. A., & POKROVSKIĬ, A. V., *Systems with Hysteresis*, (Translated from Russian: "Sistemy s Gisterezisom", Nauka, Moscow, 1983). Springer-Verlag, Berlin, 1989.

12. KREJČÍ, P., *Hysteresis, convexity and dissipation in hyperbolic equations*, GAKUTO International Series. Mathematical Sciences and Applications, 8. Gakkōtosho Co., Ltd., Tokyo, 1996. Zbl1187.35003 MR2466538

13. LADYŽENSKAJA, O. A., SOLONNIKOV, V. A., & URAL'CEVA, N. N., *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967. Zbl0164.12302 MR0241822

14. LIEBERMAN, G. M., *Second order parabolic differential equations*, World Scientific Publishing Co. Inc., River Edge, NJ, 1996. Zbl0884.35001 MR1465184

15. SHAHGHOLIAN, H., URALTSEVA, N., & WEISS, G. S., A parabolic two-phase obstacle-like equation, *Adv. Math.* **221** (2009), 861–881. Zbl1168.35452 MR2511041

16. VISINTIN, A., Evolution problems with hysteresis in the source term, *SIAM J. Math. Anal.* **17** (1986), 1113–1138. Zbl0618.35053 MR0853520

17. VISINTIN, A., *Differential models of hysteresis*, vol. 111 of *Applied Mathematical Sciences*, Springer-Verlag, Berlin, 1994. Zbl0820.35004 MR1329094