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Optimal regularity at the free boundary for the infinity obstacle problem*

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This paper deals with the obstacle problem for the infinity Laplacian. The main results are a characterization of the solution through comparison with cones that lie above the obstacle and the sharp $C^{1,\frac{1}{3}}$-regularity of the solutions at the free boundary.

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1. Introduction

The regularity of infinity harmonic functions is an outstanding issue in the theory of nonlinear partial differential equations. The belief that viscosity solutions of $\Delta_\infty u = 0$ are of class $C^{1,\frac{1}{3}}$ has hitherto remained unproven despite some recent exciting developments. The flatland example of Aronsson

$$u(x, y) = |x|^{\frac{4}{3}} - |y|^{\frac{4}{3}}$$

sets the framework to what can be expected: the first derivatives of u are Hölder continuous with exponent $1/3$, whereas its second derivatives do not exist on the lines $x = 0$ and $y = 0$. The sharpest results to date are due to Evans and Savin, who prove in [10] that infinity harmonic functions in the plane are of class $C^{1,\alpha}$, building upon Savin's breakthrough in [23] (the optimal α remains unknown even in 2-D), and to Evans and Smart, who recently obtained in [11] the everywhere differentiability, irrespective of the dimension.

This paper addresses the obstacle problem for the infinity Laplacian (see [14, 18]) and its most striking results concern the behavior at the free boundary. We prove, for the zero obstacle problem, that the solution leaves the ground as a $C^{1,\frac{1}{3}}$–function and that this regularity is optimal. The sharp

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estimates we derive are yet another conspicuous hint towards the optimal regularity for infinity harmonic functions.

As in a number of subfields in the analysis of PDEs, the study of obstacle problems evolved through two parallel paths, namely the variational and the non-variational theories. The former takes into account energy considerations and is driven by elliptic operators in divergence form, while the latter deals with operators in non-divergence form. In turn, the infinity Laplacian

$$\Delta_\infty u = \sum_{i,j=1}^{d} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}$$

enjoys a sort of duality character. On the one hand, it is a genuine degenerate elliptic operator in non-divergence form but, on the other hand, ∞-harmonic functions can be obtained as limits of p-harmonic functions, which are solutions to a divergence form equation. This operator is connected with the optimal Lipschitz extension problem [13], random tug-of-war games [2, 22], mass transportation problems [12] and several other applications [8, 20].

The variational approach to the obstacle problem for elliptic operators has been extensively studied. The classical setting amounts at minimizing the energy

$$E(u) = \int_\Omega |Du|^2$$

among the functions that coincide with a given function F at the boundary of $\Omega \subset \mathbb{R}^d$ and remain above a prescribed obstacle Ψ. Such a problem is motivated by the description of the equilibrium position of a membrane (the graph of the solution) attached at level F along the boundary of Ω and that is forced to remain above the obstacle in the interior of Ω. The same mathematical framework appears in many other contexts: fluid filtration in porous media, elasto-plasticity, optimal control or financial mathematics, to name just a few. In the section 2, we explore the “limiting divergence structure” of the infinity Laplacian to introduce the infinity obstacle problem and obtain a solution u_∞, passing to the limit, as $p \to \infty$, in a sequence of solutions u_p to the obstacle problem for the p-Laplacian. With the aim of gaining some insight on the problem, a radially symmetric explicit example is studied in an appendix. We then deal with characterizations of the limit. We first show that u_∞ is the smallest infinity superharmonic function in Ω that is above the obstacle and equals F on the boundary, a result that implies its uniqueness. Then we establish a sort of comparison with cones that lie above the obstacle. This characterization is interesting in its own right but it also implies a regularity result at the free boundary, a warm-up for what will come later. The section closes with the analysis of the behavior at infinity of the coincidence sets for the p-obstacle problem and its relation with the coincidence set of the limiting problem.

The heart of the paper is section 3, where the zero-obstacle type problem that views Δ_∞ as a degenerate elliptic operator in non-divergence form is studied. We establish the optimal asymptotic profile near the free boundary, showing the solution behaves as a $C^{1,\frac{1}{3}}$-function. We use this sharp information to deduce the uniform positive density of the non-coincidence set. In particular, the free boundary does not develop cusps pointing inwards to the coincidence set.

2. The variational ∞–obstacle problem and characterizations of the limit

Let $\Omega \subset \mathbb{R}^d$ be a bounded smooth domain, F a Lipschitz function on $\partial\Omega$ and $1 < p < \infty$. Given an obstacle $\Psi: \overline{\Omega} \to \mathbb{R}$, with

$$\sup_{\partial\Omega} \Psi < \inf_{\partial\Omega} F, \tag{2.1}$$

the p-degenerate obstacle problem for Ψ refers to the minimization problem

$$\text{Min} \left\{ \int_\Omega |Dv(x)|^p dx \;\Big|\; v \in W_F^{1,p} \text{ and } v \geqslant \Psi \right\}. \tag{2.2}$$

Here $W_F^{1,p}$ means the set of functions in $W^{1,p}(\Omega)$ with trace F on $\partial\Omega$.

Simple soft functional analysis arguments assure that (2.2) has a unique solution u_p. Let z be a Lipschitz extension of F such that $z \geqslant \Psi$ (for the proof of the existence of such z see Proposition 2.3). Since z competes in the minimization problem (2.2) for every p, we have

$$\left(\int_\Omega |Du_p|^p \right)^{1/p} \leqslant L|\Omega|^{1/p},$$

where $L := \|Dz\|_{L^\infty(\Omega)}$. For a fixed q and $p \geqslant q$, we can write

$$\left(\int_\Omega |Du_p|^q \right)^{1/q} \leqslant \left(\int_\Omega |Du_p|^p \right)^{1/p} |\Omega|^{\frac{p-q}{pq}} \leqslant L|\Omega|^{1/p}|\Omega|^{\frac{p-q}{pq}} = L|\Omega|^{1/q}.$$

Hence, we have a uniform bound for the sequence (u_p) in every $W^{1,q}(\Omega)$. Taking the limit as $p \to \infty$, we conclude that there exists a function u_∞ such that, up to a subsequence, $u_p \to u_\infty$, locally uniformly in $\overline{\Omega}$ and weakly in every $W^{1,q}(\Omega)$. Clearly, $u_\infty \geqslant \Psi$ pointwise. Also,

$$\left(\int_\Omega |Du_\infty|^q \right)^{1/q} \leqslant L|\Omega|^{\frac{1}{q}} \qquad \forall q > 1.$$

We then conclude that u_∞ is a Lipschitz function, with

$$\|Du_\infty\|_{L^\infty(\Omega)} \leqslant L.$$

Since this holds being L the L^∞-norm of the gradient of any extension of F that is above Ψ, we conclude that u_∞ is a solution of the minimization problem (cf. [14])

$$\min_{w|_{\partial\Omega}=F;\; w \geqslant \Psi \text{ in } \Omega} \|Dw\|_{L^\infty(\Omega)}. \tag{2.3}$$

The minimizers u_p are weak, and hence viscosity, solutions (see [15]) of the following obstacle problem:

$$\begin{cases} u_p(x) & = & F(x) & \text{on } \partial\Omega, \\ u_p(x) & \geqslant & \Psi(x) & \text{in } \Omega, \\ -\Delta_p u_p & = & 0 & \text{in } \Omega \setminus A_p := \{u_p > \Psi\}, \\ -\Delta_p u_p & \geqslant & 0 & \text{in } \Omega. \end{cases}$$

Concerning the PDE problem satisfied by u_∞, we verify that it is a viscosity solution to the obstacle problem for the infinity Laplacian:

$$\begin{cases} u_\infty(x) & = & F(x) & \text{on } \partial\Omega, \\ u_\infty(x) & \geqslant & \Psi(x) & \text{in } \Omega, \\ -\Delta_\infty u_\infty & = & 0 & \text{in } \Omega \setminus A_\infty = \{u_\infty > \Psi\}, \\ -\Delta_\infty u_\infty & \geqslant & 0 & \text{in } \Omega. \end{cases}$$

Indeed, fix a point y in the set $\{u_\infty > \Psi\}$. From the uniform convergence, $u_p > \Psi$ in a neighborhood of y, provided $p \gg 1$. Hence, taking the limit as $p \to \infty$ in the viscosity sense, we obtain

$$-\Delta_\infty u_\infty = 0 \quad \text{in } \{u_\infty > \Psi\}.$$

Moreover, a uniform limit u_∞ verifies

$$-\Delta_\infty u_\infty \geqslant 0 \quad \text{in } \Omega$$

in the viscosity sense, since this holds for every u_p. Let us remark that the limit obtained here does not necessarily coincide with the solution of the infinity obstacle problem obtained by direct methods in [6].

A crucial issue, with striking implications, is to characterize the limit u_∞. We give two characterizations, one involving supersolutions of the infinity Laplacian, the other making use of appropriately defined cones. From both we will derive important properties of the limit.

THEOREM 2.1 The limit u_∞ is the smallest continuous infinity superharmonic function in Ω that is above the obstacle and equals F on the boundary.

Proof. Let $\mathfrak{F}$ be the set of all continuous functions v that are infinity superharmonic in Ω and satisfy $v \geqslant \Psi$ in Ω and $v = F$ on $\partial\Omega$. This set is not empty because $u_\infty \in \mathfrak{F}$. Let

$$v_\infty := \inf_{v \in \mathfrak{F}} v,$$

which is upper semicontinuous (as it is the infimum of continuous functions) and infinity superharmonic in Ω. Since $u_\infty \in \mathfrak{F}$, it is obvious that

$$u_\infty \geqslant v_\infty \ \text{ in } \ \overline{\Omega}.$$

Now, define the open set

$$W = \{x \in \Omega : u_\infty(x) > v_\infty(x)\}.$$

On $\partial W \subset \overline{\Omega}$, we have $v_\infty = u_\infty$. Moreover,

$$u_\infty > v_\infty \geqslant \Psi \ \text{ in } W,$$

so $W \subset \{u_\infty > \Psi\}$ and u_∞ is infinity harmonic in W. Thus, by the comparison principle,

$$u_\infty \leqslant v_\infty \ \text{ in } \ W,$$

a contradiction that shows that $W = \emptyset$. Consequently, $u_\infty \equiv v_\infty$. □

COROLLARY 2.2 The limit u_∞ is unique.

Proof. Suppose we have two limits, say $u_{1,\infty}$ and $u_{2,\infty}$. Then

$$v = u_{1,\infty} \wedge u_{2,\infty}$$

is also an infinity superharmonic function in Ω that is above the obstacle and equals F on the boundary. By the theorem, we have

$$u_{i,\infty} \leqslant v, \quad i = 1, 2$$

and since, trivially, $v \leqslant u_{i,\infty}$, $i = 1, 2$, we conclude that

$$u_{1,\infty} = v = u_{2,\infty}.$$

□

Let's now turn to our second characterization of the limit. For this, consider the family of cones with vertex at a boundary point and positive opening, which lie above both the obstacle and the boundary data. For more on comparison with cones and the characterization of infinity harmonic functions see [9].

To be concrete, for $y \in \partial\Omega$ and $b = (b_1, b_2)$, with $b_1 \geqslant 0$, we consider the cones

$$K_y^b(x) = b_1|x - y| + b_2$$

such that

$$K_y^b(x) \geqslant F(x), \qquad x \in \partial\Omega$$

and

$$K_y^b(x) \geqslant \Psi(x), \qquad x \in \Omega.$$

Note that, since the vertex of the cone is at the boundary of Ω, these cones are infinity harmonic in Ω, that is, $-\Delta_\infty K_y^b = 0$ in Ω. We denote by $\mathcal{K}$ the family of all such cones.

Now, we define

$$K_\infty(x) := \inf_{\mathcal{K}} K_y^b(x), \quad x \in \overline{\Omega}.$$

It is obvious that

$$K_\infty(x) \geqslant F(x), \quad x \in \partial\Omega$$

and

$$K_\infty(x) \geqslant \Psi(x), \quad x \in \Omega.$$

PROPOSITION 2.3 The function K_∞ is Lipschitz continuous in $\overline{\Omega}$ and infinity superharmonic in Ω. Moreover,

$$K_\infty(y) = F(y), \quad y \in \partial\Omega.$$

Proof. Since we assume that F is Lipschitz, we have that for every point $y \in \partial\Omega$, there exists a constant L such that, for every $b_1 > L$ and every $b_2 > L$,

$$K_y^b(x) \geqslant F(x) \quad \text{and} \quad K_y^b(x) \geqslant \Psi(x).$$

Hence, when computing the infimum that defines $K_\infty(x)$, we can restrict to cones with $b = (b_1, b_2)$ in a compact set and since $y \in \partial\Omega$ (which is also compact), we conclude that the infimum is in fact a minimum. This means that, for every $x \in \overline{\Omega}$, there exists a $y \in \partial\Omega$ and a $b = (b_1, b_2)$, with $|b_i| \leqslant L$, depending on x, such that

$$K_\infty(x) = K_{y(x)}^{b(x)}(x).$$

From this fact, it follows that K_∞ is Lipschitz continuous in $\overline{\Omega}$. Let's show why. Take any two points $\hat{x}, \tilde{x} \in \overline{\Omega}$; we have

$$K_\infty(\hat{x}) = K_{y(\hat{x})}^{b(\hat{x})}(\hat{x}) \quad \text{and} \quad K_\infty(\tilde{x}) = K_{y(\tilde{x})}^{b(\tilde{x})}(\tilde{x}).$$

From the definition, it is clear that $K_\infty(\hat{x}) \leqslant K_{y(\tilde{x})}^{b(\tilde{x})}(\hat{x})$ and thus

$$\begin{aligned} K_\infty(\hat{x}) - K_\infty(\tilde{x}) &\leqslant K_{y(\tilde{x})}^{b(\tilde{x})}(\hat{x}) - K_{y(\tilde{x})}^{b(\tilde{x})}(\tilde{x}) \\ &= b_1(\tilde{x}) \left(|\hat{x} - y(\tilde{x})| - |\tilde{x} - y(\tilde{x})|\right) \\ &\leqslant L\, |\hat{x} - \tilde{x}| \, . \end{aligned}$$

Reversing the role of $\hat{x}$ and $\tilde{x}$ gives the desired Lipschitz regularity.

Moreover, as the infimum of infinity harmonic functions, K_∞ is infinity superharmonic, *i.e.*,

$$-\Delta_\infty K_\infty \geqslant 0 \ \text{ in } \Omega. \tag{2.4}$$

Finally, by taking b_1 large enough and $b_2 = F(y)$, we also have, recalling (2.1),

$$F(y) \leqslant K_\infty(y) \leqslant K_y^b(y) = F(y)$$

and, hence, $K_\infty(y) = F(y)$, for $y \in \partial\Omega$. □

THEOREM 2.4 The limit u_∞ is such that

$$u_\infty(x) \leqslant K_\infty(x), \quad x \in \overline{\Omega}. \tag{2.5}$$

Equality holds if, and only if, $K_\infty(x)$ is infinity harmonic outside of its coincidence set $\{K_\infty = \Psi\}$.

Proof. Inequality (2.5) follows immediately from Proposition 2.3 and Theorem 2.1. If we have an equality it is also immediate that $K_\infty(x)$ is infinity harmonic outside of its coincidence set $\{K_\infty = \Psi\}$ So we are left to prove the other implication.

Arguing by contradiction, assume that

$$W = \{x \in \Omega : K_\infty(x) > u_\infty(x)\} \neq \emptyset.$$

Note that W is open because u_∞ and K_∞ are continuous functions. Since $W \subset \{K_\infty > \Psi\}$, we deduce that $-\Delta_\infty K_\infty = 0$ in W. But $-\Delta_\infty u_\infty \geqslant 0$ in Ω (thus in W) and $u_\infty = K_\infty$ on ∂W so, by the comparison principle for the infinity Laplacian, we conclude that

$$u_\infty \geqslant K_\infty \ \text{ in } W,$$

a contradiction that shows that $W = \emptyset$ and completes the proof. □

REMARK 2.5 The condition that $K_\infty(x)$ is infinity harmonic outside of its coincidence set $\{K_\infty = \Psi\}$ strongly depends on the geometry of the problem. In the radial example explicitly computed in the appendix, the condition holds. However, in general, this is not the case, as the following example shows. Consider Ω to be the union of two disjoints balls connected by a narrow tube of width δ, an obstacle placed in one of the balls and boundary data $F = 0$. It can be readily checked that, as $\delta \to 0$, $u_\infty \to 0$ in the ball without obstacle. But K_∞ is uniformly bounded below inside this ball since the opening of the corresponding cones is uniformly bounded below (as these cones have to be above the obstacle).

COROLLARY 2.6 Assume the obstacle Ψ is differentiable and equality holds in (2.5). Then u_∞ is differentiable at the free boundary and

$$Du_\infty(x_0) = D\Psi(x_0), \quad \forall x_0 \in \partial\{u_\infty = \Psi\}.$$

Proof. Let $x_0 \in \partial\{u_\infty = \Psi\}$. It follows from the previous results that there exists a cone $K^b_{y_0}$ such that

$$K^b_{y_0}(x_0) = K_\infty(x_0) = u_\infty(x_0) = \Psi(x_0) \tag{2.6}$$

and

$$K^b_{y_0}(x) \geqslant K_\infty(x) = u_\infty(x) \geqslant \Psi(x), \quad \forall x \in \Omega. \tag{2.7}$$

Hence, $K^b_{y_0}(x) - \Psi(x)$ attains a minimum at x_0 and, since it is differentiable,

$$DK^b_{y_0}(x_0) = D\Psi(x_0).$$

From (2.6) and (2.7), we conclude that u_∞ is also differentiable at x_0, with

$$Du_\infty(x_0) = D\Psi(x_0),$$

as claimed. □

REMARK 2.7 As a consequence of this corollary, we conclude that u_∞ is differentiable everywhere in Ω. In fact, in the interior of the coincidence set, it coincides with the differentiable obstacle and, in the interior of the non-coincidence set, it is infinity harmonic, thus differentiable everywhere by the results of [11]. Also note that the radial solution constructed in the appendix is a C^1–solution that can be characterized by the equality in (2.5).

We close this section with the analysis of the behavior at infinity of the coincidence sets for the p-obstacle problem and relate it with the coincidence set of the limiting problem. We recall that

$$\limsup_{p\to\infty} A_p = \bigcap_{p=1}^{\infty} \bigcup_{n\geqslant p} A_n \qquad \text{and} \qquad \liminf_{p\to\infty} A_p = \bigcup_{p=1}^{\infty} \bigcap_{n\geqslant p} A_n.$$

THEOREM 2.8 Assume Ψ is smooth and verifies $-\Delta_\infty \Psi > 0$. Let $A_p = \{u_p = \Psi\}$ be the coincidence sets of the p-obstacle problems and $A_\infty = \{u_\infty = \Psi\}$ be the coincidence set of the limiting problem. Then

$$\overline{\mathrm{int}(A_\infty)} \subset \liminf_{p\to\infty} A_p \subset \limsup_{p\to\infty} A_p \subset A_\infty. \tag{2.8}$$

Proof. Given a neighborhood V of A_∞, $\Omega \setminus V$ is a closed set contained in $\{u_\infty > \Psi\}$. Thus, the continuity of $u_\infty - \Psi$ gives us a $\eta > 0$ such that $u_\infty - \Psi > \eta$ in $\Omega \setminus V$. Using the uniform convergence of u_p to u_∞, we conclude that, for p large enough, we also have $u_p - \Psi > \eta$ in $\Omega \setminus V$. Therefore, we conclude that $\Omega \setminus V \subset \{u_p > \Psi\}$ and, consequently, that

$$A_p \subset V,$$

for every large enough p. This shows that

$$\limsup_{p\to\infty} A_p \subset V,$$

for any neighborhood V of A_∞, and since A_∞ is compact, we also obtain

$$\limsup_{p\to\infty} A_p \subset A_\infty.$$

Next, given $x_0 \in \mathrm{int}(A_\infty)$, if we have

$$u_{p_j}(x_0) > \Psi(x_0),$$

for a subsequence $p_j \to \infty$, then

$$-\Delta_{p_j} u_{p_j}(x_0) = 0$$

and, passing to the limit in the viscosity sense, we conclude that

$$-\Delta_\infty \Psi(x_0) = -\Delta_\infty u_\infty(x_0) = 0,$$

a contradiction with $-\Delta_\infty \Psi > 0$. Therefore, we conclude that for every $x_0 \in \mathrm{int}(A_\infty)$, there exists $p_0 = p_0(x_0)$ such that

$$u_n(x_0) = \Psi(x_0),$$

for every $n \geqslant p_0$. This means that

$$x_0 \in \bigcap_{n \geqslant p_0} A_n$$

and consequently

$$\mathrm{int}(A_\infty) \subset \liminf_{p\to\infty} A_p.$$

Since the larger set is closed, we also obtain

$$\overline{\mathrm{int}(A_\infty)} \subset \liminf_{p\to\infty} A_p$$

and the proof is complete. □

3. ∞-obstacle type problems and $C^{1,\frac{1}{3}}$-behavior at the free boundary

In this section we turn our analysis towards optimal regularity estimates along the free boundary. We shall consider here the zero obstacle type problem that consists in the analysis of a function satisfying:

$$u \geqslant 0 \text{in} B_1 \tag{3.1}$$

$$\Delta_\infty u = f(x) \text{in} \{u > 0\}. \tag{3.2}$$

Equivalently, we can write the zero obstacle type problem as

$$\min\{\Delta_\infty u - f(x), u\} = 0, \tag{3.3}$$

which is understood in the viscosity sense. We comment that the limiting variational problem studied in section 2 can be transformed into a zero obstacle-type problem by defining $v = u - \Psi$, under the assumption that $\Delta_\infty v$ is bounded. Thus, the results proven in this section apply to the variational setting, provided this extra assumption is verified.

The ultimate goal is to show that a solution to (3.3) grows precisely as

$$[\mathrm{dist}(x, \partial\{u > 0\})]^{4/3}$$

away from the free boundary. Throughout this section, we work under the assumption that $f(x)$ is continuous and bounded away from zero and infinity, *i.e.*,

$$0 < \nu \leqslant f(x) \leqslant M < \infty \tag{3.4}$$

will be enforced from this point on. Such a condition is natural in the context of obstacle-type problems and allows us to prove existence and uniqueness for problem (3.3) by means of a Perron's type method.

THEOREM 3.1 Given a function $g \in C(\partial B_1)$, with $g > 0$, and f satisfying (3.4), there exists a unique function $u \in C(\bar{B}_1)$, satisfying

$$\begin{cases} \min\{\Delta_\infty u - f(x), u\} & = & 0 \text{ in } B_1 \\ u & = & g \text{ on } \partial B_1 \end{cases} \tag{3.5}$$

in the viscosity sense. Assuming further that f is uniformly Lipschitz continuous in B_1, then u is locally Lipschitz continuous in B_1.

Proof. The proof of existence goes along the lines of the proof of [19, Theorem 1]. Define

$$\mathfrak{A}^+_{f,g} := \left\{ v \in C(\overline{B_1}) \;\middle|\; v \geqslant 0,\ \Delta_\infty v \leqslant f(x) \text{ in } B_1, \text{ and } v \geqslant g \text{ on } \partial B_1 \right\}. \tag{3.6}$$

In the sequel, take

$$u(x) := \inf_{v \in \mathfrak{A}^+_{f,g}} v(x), \quad \text{for } x \in \overline{B_1}. \tag{3.7}$$

Clearly $u \geqslant 0$ in B_1 and $\Delta_\infty u \leqslant f(x)$ in the viscosity sense. It is also an upper-semicontinuous function and thus the set $\{u > 0\}$ is open. Given an open ball B_δ, with $\overline{B_\delta} \subset \{u > 0\}$, we can perform the same Perron's argument implemented in the proof of [19, Theorem 1] to show that $\Delta_\infty u \geqslant f(x)$ in B_δ. Hence, the function u defined in (3.7) does solve the obstacle-type PDE

$$\min\{\Delta_\infty u - f(x), u\} = 0 \text{ in } B_1.$$

Continuity of u up to the boundary follows precisely as in [19, Theorem 1] and uniqueness is proven as in [19, Theorem 3].

Let us now turn our attention towards the local Lipschitz regularity of the solution u. Locally in $\{u > 0\}$, u satisfies $\Delta_\infty u \in L^\infty$ in the viscosity sense, thus u is locally Lipschitz continuous in the non-coincidence set (see, for instance [16, Corollary 2]). Hence, such an estimate needs only to be proven near the free boundary. By continuity of u and the fact that $g > 0$ on ∂B_1, there exists a small number $\tau_0 > 0$ such that $u > 0$ in $B_1 \setminus B_{1-\tau_0}$. From our previous argument, there exists a constant $\Sigma > 0$, depending on M and τ_0, such that

$$|\nabla u(x)| < \Sigma, \quad \forall x \in B_{1-\frac{\tau_0}{5}} \setminus B_{1-\frac{\tau_0}{10}}. \tag{3.8}$$

For any vector ν, with $|\nu| < \frac{\tau_0}{100}$, define σ_ν by

$$\sigma_\nu^3 := \inf_{B_{1-\frac{\tau_0}{100}}} \frac{f(x)}{f(x+\nu)}.$$

Since f is strictly positive and Lipchitz continuous, it follows that

$$|1-\sigma_\nu| + |1-\sigma_\nu^3| \leqslant K_0|\nu|.$$

In the sequel, let us label $r_0 := 1 - \frac{3}{20}\tau_0$ and define $u_\nu\colon B_{r_0} \to \mathbb{R}$ by

$$u_\nu(x) := \sigma_\nu \cdot u(x+\nu) + \Big(\Sigma + K_0 \sup_{B_1} u\Big) \cdot |\nu|.$$

We now apply the analysis from the beginning of this proof to the domain B_{r_0}. One simply verifies that u_ν belongs to the set

$$\tilde{\mathfrak{A}}^+_{f,g} := \left\{ v \in C(\overline{B_{r_0}}) \;\middle|\; v \geqslant 0,\ \Delta_\infty v \leqslant f(x) \text{ in } B_{r_0}, \text{ and } v \geqslant u \text{ on } \partial B_{r_0} \right\}.$$

By uniqueness, $u|_{B_{r_0}}$ is the infimum among all functions in $\tilde{\mathfrak{A}}^+_{f,g}$. Thus, we can write, for any $x \in B_{r_0}$,

$$u_\nu(x) \geqslant u(x),$$

which immediately yields

$$u(x+\nu) - u(x) \geqslant -\Big(\Sigma + 2K_0 \sup_{B_1} u\Big) \cdot |\nu|$$

and the local Lipschitz estimate for u follows. □

We remark that, assuming only the boundedness of $f(x)$, the local Lipschitz continuity of the solution to the infinity obstacle problem is a consequence of the next lemma.

LEMMA 3.2 Let (3.4) be in force and let u be the viscosity solution to the obstacle problem (3.5). Then

$$|\Delta_\infty u| \leqslant M.$$

Proof. The idea of the proof is to perform a singular approximation of the obstacle problem. Let ζ be a nonnegative real C^1 function satisfying $\operatorname{supp}\zeta = [0,1]$ and $\int \zeta(t)dt = 1$. For each $\epsilon > 0$, consider the boundary value problem

$$\begin{cases} \Delta_\infty u_\epsilon = f(x) \cdot \displaystyle\int_0^{u_\epsilon/\epsilon} \zeta(t)\,dt \text{ in } B_1 \\ u_\epsilon = g \text{ on } \partial B_1. \end{cases} \tag{3.9}$$

Notice that the reaction term

$$f(x) \cdot \int_0^{u_\epsilon/\epsilon} \zeta(t)\, dt =: \beta(x, u_\epsilon),$$

is monotone non-decreasing with respect to u_ϵ. Hence, as before, by means of a Perron's type method (see [4, 5]), the Dirichlet problem (3.9) can be uniquely solved. Clearly,

$$|\Delta_\infty u_\epsilon| \leqslant M.$$

Thus, it follows from Lipschitz estimates and uniform continuity up to the boundary (cf., for example, [16, Corollary 2]), that the family $\{u_\epsilon\}_{\epsilon>0}$ is equicontinuous in B_1. Up to a subsequence, u_ϵ converges uniformly to a function v. The limiting function v is nonnegative, agrees with g on the boundary, and satisfies $|\Delta_\infty v| \leqslant M$, in the viscosity sense. In particular, v is locally Lipschitz continuous in B_1. Now, given a point $z \in \{v > 0\} \cap B_1$, by the triangular inequality, one easily checks that

$$B := B_{\frac{v(z)}{2L}}(z) \subset \left\{ v > \frac{v(z)}{2} > 0 \right\},$$

where L is the Lipschitz norm of v on $B_{1-|z|}$. In particular

$$\Delta_\infty u_\epsilon = f(x) \text{ in } B,$$

for all $\epsilon < \frac{v(z)}{2}$. By stability, we deduce that $\Delta_\infty v = f(x)$ in B as well. Since $z \in \{v > 0\}$ was taken arbitrary, it follows that v satisfies $\Delta_\infty v = f(x)$ in $\{v > 0\}$. We have verified that v solves the same boundary value problem as u. Thus, by uniqueness, $u = v$ and the lemma is proven. □

As commented earlier, it remains unknown, up to now, whether a generic infinity harmonic functions is more regular than differentiable. Hence the gradient estimate given by Theorem 3.1 is the best we can reach at this point. Surprisingly enough, at the free boundary, there is more. We are now ready for our main result, which gives the optimal $C^{1,\frac{1}{3}}$-regularity estimate for solutions of the infinity obstacle problem along the free boundary.

THEOREM 3.3 (Sharp $C^{1,\frac{1}{3}}$–regularity at the free boundary) Let u be a solution to (3.3) and $x_0 \in \partial\{u > 0\}$ be a generic free boundary point. Then

$$\sup_{y \in B_r(x_0)} u(y) \leqslant C\, r^{4/3}, \tag{3.10}$$

for a constant C that depends only upon the data of the problem.

Proof. For simplicity, and without loss of generality, assume $x_0 = 0$. By combining discrete iterative techniques and a continuous reasoning (see, for instance, [7]), it is well established that proving estimate (3.10) is equivalent to verifying the existence of a constant $C > 0$, such that

$$\mathfrak{s}_{j+1} \leqslant \max \left\{ C\, 2^{-4/3 \cdot (j+1)},\ 2^{-4/3} \mathfrak{s}_j \right\}, \quad \forall\, j \in \mathbb{N}, \tag{3.11}$$

where

$$\mathfrak{s}_j = \sup_{B_{2^{-j}}} u.$$

Let us suppose, for the sake of contradiction, that (3.11) fails to hold, *i.e.*, that for each $k \in \mathbb{N}$, there exists $j_k \in \mathbb{N}$ such that

$$\mathfrak{s}_{j_k+1} > \max\left\{k\, 2^{-4/3\cdot(j_k+1)},\ 2^{-4/3}\mathfrak{s}_{j_k}\right\}. \tag{3.12}$$

Now, for each k, define the rescaled function $v_k\colon B_1 \to \mathbb{R}$ by

$$v_k(x) := \frac{u(2^{-j_k}x)}{\mathfrak{s}_{j_k+1}}.$$

One easily verifies that

$$0 \leqslant v_k(x) \leqslant \sqrt[3]{16}, \quad \forall x \in B_1; \tag{3.13}$$

$$v_k(0) = 0; \tag{3.14}$$

$$\sup_{B_{\frac{1}{2}}} v_k = 1. \tag{3.15}$$

Moreover, we formally have

$$\begin{aligned} \Delta_\infty v_k(x) &= \frac{2^{-j_k}}{\mathfrak{s}_{j_k+1}} Du(2^{-j_k}x)\cdot\left(\frac{2^{-2j_k}}{\mathfrak{s}_{j_k+1}} D^2u(2^{-j_k}x)\right)\cdot \frac{2^{-j_k}}{\mathfrak{s}_{j_k+1}} Du(2^{-j_k}x) \\ &= \frac{2^{-4j_k}}{\mathfrak{s}^3_{j_k+1}} \Delta_\infty u(2^{-j_k}x) =: f_k. \end{aligned}$$

It is a matter of routine to rigorously justify the above calculations using the language of viscosity solutions (see, *e.g.*, [25, section 2]). We estimate

$$|f_k| \leqslant \frac{2^{-4j_k}}{2^{-4(j_k+1)}\,k^3} M = \frac{16M}{k^3} \leqslant 16M, \tag{3.16}$$

using Lemma 3.2 and (3.12).

Combining the uniform bounds (3.13), (3.16), and local Lipschitz regularity results for the inhomogeneous infinity Laplace equation (cf., for example, [16, Corollary 2]), we obtain both the equiboundedness and the equicontinuity of the sequence $(v_k)_k$. By Ascoli's theorem, and passing to a subsequence if need be, we conclude that v_k converges locally uniformly to a infinity harmonic function v_∞ in B_1 (observe that $f_k \to 0$) such that

$$0 \leqslant v_\infty \leqslant \sqrt[3]{16} \quad \text{and} \quad v_\infty(0) = 0.$$

We now use Harnack's inequality for infinity harmonic functions (see [17, Corollary 2]) to obtain the bound

$$v_\infty(x) \leqslant e^{2|x|}\, v_\infty(0) = 0, \quad \forall\, x \in B_{1/2}.$$

It follows that $v_\infty \equiv 0$ in $B_{1/2}$, which contradicts (3.15). The theorem is proven. □

As a first consequence we improve the local Lipschitz regularity estimate provided by Theorem 3.1, where f needs only to satisfy (3.4). Indeed we obtain a finer gradient control near the free boundary.

COROLLARY 3.4 Let u be a solution to (3.3) in B_1. Then u is locally Lipschitz continuous and for any point $z \in \{u > 0\} \cap B_1$, there holds

$$|\nabla u(z)| \leqslant C \operatorname{dist}(z, \partial\{u > 0\})^{1/3}.$$

Proof. Fix $z \in \{u > 0\} \cap B_{1/2}$ and label $d := \operatorname{dist}(z, \partial\{u > 0\})$. Let $\zeta \in \partial\{u > 0\}$ be a free boundary point satisfying

$$|\zeta - z| = d.$$

From the $C^{1,\frac{1}{3}}$-smoothness of u at ζ, we know

$$\sup_{B_d(z)} u \leqslant \sup_{B_{2d}(\zeta)} u \leqslant C \cdot d^{4/3}. \tag{3.17}$$

We now define the auxiliary function $v \colon B_1 \to \mathbb{R}_+$, by

$$v(x) := \frac{u(z + dx)}{d^{4/3}}.$$

As argued before, v satisfies

$$\Delta_\infty v = f(z + dx), \quad \text{ in } B_1. \tag{3.18}$$

From (3.17) we can estimate

$$\sup_{B_1} v \leqslant C. \tag{3.19}$$

Finally, applying the gradient estimate for bounded solutions to (3.18), we conclude

$$|\nabla v(0)| = d^{-1/3} |\nabla u(z)| \leqslant C_2,$$

and the Corollary is proven. □

Our next theorem establishes a $C^{1,\frac{1}{3}}$–estimate from below, which implies that u leaves the zero-obstacle trapped by the graph of two functions of the order $\operatorname{dist}^{4/3}(x, \partial\{u > 0\})$.

THEOREM 3.5 Let u be a viscosity solution to (3.3) and $y_0 \in \overline{\{u > 0\}}$ be a generic point in the closure of the non-coincidence set. Then

$$\sup_{B_r(y_0)} u \geqslant c\, r^{4/3},$$

for a constant $c > 0$ that depends only upon ν.

Proof. By continuity arguments, it is enough to prove the result for points in the non-coincidence set. For simplicity, and without loss of generality, take $y_0 = 0$. Define the barrier

$$\mathfrak{B}_\infty(x) := \frac{3}{4} \sqrt[3]{3\nu}\, |x|^{4/3},$$

which satisfies, by direct computation,

$$\Delta_\infty \mathfrak{B}_\infty = \nu.$$

Hence,

$$\Delta_\infty u = f(x) \geqslant \nu = \Delta_\infty \mathfrak{B}_\infty, \quad \text{in } \{u > 0\},$$

in the viscosity sense. On the other hand,

$$u \equiv 0 < \mathfrak{B}_\infty \quad \text{on } \partial\{u > 0\} \cap B_r.$$

Therefore, for some point $y^\star \in \partial B_r \cap \{u > 0\}$, there must hold

$$u(y^\star) > \mathfrak{B}_\infty(y^\star); \tag{3.20}$$

otherwise, by Jensen's comparison principle for infinity harmonic functions [13], we would have, in particular,

$$0 < u(0) \leqslant \mathfrak{B}_\infty(0) = 0.$$

Estimate (3.20) implies the thesis of the theorem. □

As usual, as soon as we establish the precise sharp asymptotic behavior for a given free boundary problem, it becomes possible to obtain certain weak geometric properties of the phases. We conclude this section by proving that the region where the membrane is above the obstacle has uniform positive density along the free boundary, which is then inhibited to develop cusps pointing inwards to the coincidence set.

COROLLARY 3.6 Let u be a solution to (3.3) and $x_0 \in \partial\{u > 0\}$ be a free boundary point. Then

$$\mathscr{L}^n\left(B_\rho(x_0) \cap \{u > 0\}\right) \geqslant \delta_\star \rho^n,$$

for a constant $\delta_\star > 0$ that depends only upon the data of the problem.

Proof. It follows from Theorem 3.5 that there exists a point

$$z \in \partial B_\rho(x_0) \cap \{u > 0\}$$

such that $u(z) \geqslant c\,\rho^{4/3}$. By $C^{1,\frac{1}{3}}$–bounds along the free boundary, Theorem 3.3, it follows that

$$B_{\lambda\rho}(z) \subset \{u > 0\},$$

where the constant

$$\lambda := \sqrt[4]{\left(\frac{c}{2C}\right)^3}$$

depends only on the data of the problem. In fact, if this were not true, there would exist a free boundary point $y \in B_{\lambda\rho}(z)$. From (3.10), we would reach

$$c\,\rho^{4/3} \leqslant u(z) \leqslant \sup_{B_{\lambda\rho}(y)} u \leqslant C\,(\lambda\rho)^{4/3} = \frac{1}{2} c\,\rho^{4/3},$$

which is a contradiction. Thus,

$$B_\rho(x_0) \cap B_{\lambda\rho}(z) \subset B_\rho(x_0) \cap \{u > 0\}$$

and, finally,

$$\mathscr{L}^n\left(B_\rho(x_0) \cap \{u > 0\}\right) \geqslant \mathscr{L}^n\left(B_\rho(x_0) \cap B_{\lambda\rho}(z)\right) \geqslant \delta_\star \rho^n,$$

and the corollary is proven. □

We conclude by remarking that the thesis of Corollary 3.6 implies that the free boundary $\partial\{u > 0\}$ is porous, with porosity constant $\tau > 0$ that depends only on the data of the problem. In particular, the Hausdorff dimension of the free boundary is strictly less than n and hence it has Lebesgue measure zero.

Appendix: A radial explicit example

In this appendix we construct a radially symmetric explicit solution to a (variational) obstacle problem, by means of a limiting process, namely, taking $p \to \infty$. For that, let us consider the p-obstacle problem in $B_2 \subset \mathbb{R}^d$, with zero boundary data and the spherical cap $\psi(x) = 1 - |x|^2$ as the obstacle. It is formulated as the following minimization problem:

$$\text{Min}\left\{\int_{B_2} |Dv(x)|^p dx \;\Big|\; v \in W_0^{1,p}(B_2) \text{ and } v(x) \geqslant \psi(x)\right\}.$$

As mentioned in section 2, the problem admits a unique minimizer u_p. By symmetry, we conclude u_p is radially symmetric, *i.e.*, $u_p(x) = u_p(|x|)$. By the geometry of the obstacle problem, as well as its regularity theory, we know that there exists an $h = h(p,d)$, that depends on p and dimension, such that

$$\begin{cases} u_p(x) = \psi(x) & \text{in} \quad |x| \leqslant h, \\ \Delta_p u_p = 0 & \text{in} \quad 2 > |x| > h, \\ u_p \in C^{1,\alpha_p} & \text{in} \quad B_2, \\ \|Du_p\|_{L^\infty(B_\rho)} \leqslant C(\rho, d), & \end{cases}$$

for a constant $C(\rho, d)$, which is independent of p. In particular, as observed in the main text, up to a subsequence, u_p converges locally uniformly to a function u_∞. Furthermore, u_∞ solves $\Delta_\infty u_\infty = 0$ within $\{u_\infty > \psi\}$ in the viscosity sense.

Our goal is to solve the p-obstacle problem explicitly and then analyze the limiting function u_∞. We are initially led to search for p-harmonic radially symmetric functions. If $g(x) = f(r)$, then

$$\Delta_p g = |f'(r)|^{p-2}\left\{(p-1)f''(r) + \frac{d-1}{r} f'(r)\right\}.$$

Solving the homogeneous ODE, we obtain

$$f(r) = \begin{cases} a + b \cdot r^{\frac{1-d}{p-1}+1} & \text{if} \quad p \neq d, \\ a + b \cdot \ln r & \text{if} \quad p = d, \end{cases}$$

for any constants $a, b \in \mathbb{R}$. Returning to the obstacle problem (we will only deal with the case, $p \neq d > 1$, as we are interested in the limiting problem as $p \to \infty$), by regularity considerations, we end up with the following system of equations:

$$a + b \cdot h^{-\alpha+1} = 1 - h^2 \quad \text{and} \quad b \cdot (-\alpha + 1)h^{\alpha} = -2h, \tag{1}$$

where the exponent $\alpha = \alpha(p)$ is given by

$$\alpha(p) = \frac{d-1}{p-1} \longrightarrow 0 \quad \text{as} \quad p \to \infty. \tag{2}$$

The first equation in (1) comes from continuity and the second from C^1–estimates. By the boundary condition, we have

$$a + b \cdot 2^{-\alpha+1} = 0.$$

Subtracting the first equality from the above equation, we obtain

$$b \cdot (2^{-\alpha+1} - h^{-\alpha+1}) = -1 + h^2,$$

which simplifies out to

$$(-\alpha + 1)b \cdot h^{-\alpha} = -2h.$$

Combining the above with the second equation in (1), we end up with

$$\frac{2}{1-\alpha}(2^{-\alpha+1}h^{1+\alpha} - h^2) = 1 - h^2,$$

that is,

$$\Big(\frac{2}{1-\alpha} - 1\Big)h^2 - 4\Big(\frac{2^{-\alpha}}{1-\alpha}\Big)h^{1+\alpha} + 1 = 0.$$

Now, we observe that, from (2), this equation converges to $h^2 - 4h + 1 = 0$, which has as solution in $(0,1)$ (the free boundary must lie in this interval) $h_\infty = 2 - \sqrt{3}$. With this limit, we can also compute the limit of

$$f_p(r) = a_p + b_p r^{-\frac{d-1}{p-1}+1} = a_p + b_p r^{-\alpha(p)+1}$$

that is given by

$$f_\infty(r) = a_\infty + b_\infty r,$$

with $a_\infty = 4h_\infty$ and $b_\infty = -2h_\infty$. Note that $f_\infty(r)$ is infinity harmonic in $B_2 \setminus B_{h_\infty}$ and verifies

$$f_\infty(h_\infty) = 1 - h_\infty^2 \qquad \text{and} \qquad f'_\infty(h_\infty) = -2h_\infty.$$

It is the solution of the limit obstacle problem.

To conclude, it might be interesting to observe that the solution constructed here behaves linearly along the free boundary – and not as a $C^{1,1/3}$ graph. This fact elucidates as to why condition (3.4) ought to be enforced so that solutions do leave the obstacle precisely as $\mathrm{dist}^{4/3}$.

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