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Mean curvature flow with obstacles: Existence, uniqueness and regularity of solutions*

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We show short time existence and uniqueness of $C^{1,1}$ solutions to the mean curvature flow with obstacles, when the obstacles are of class $C^{1,1}$. If the initial interface is a periodic graph we show long time existence of the evolution and convergence to a minimal constrained hypersurface.

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1. Introduction and main results

Mean curvature flow is a prototypical geometric evolution, arising in many models from Physics, Biology and Material Science, as well as in a variety of mathematical problems. For such a reason, this flow has been widely studied in the past years, starting from the pioneering work of K. Brakke [3] (we refer to [4, 8, 11, 14, 15] for a far from complete list of references).

In some models, one needs to include the presence of hard obstacles, which the evolving surface cannot penetrate (see for instance [12] and references therein). This leads to a double obstacle problem for the mean curvature flow, which reads

$$v = H \qquad \text{on } M_t \cap U, \tag{1}$$

with constraint

$$M_t \subset \overline{U} \qquad \text{for all } t, \tag{2}$$

where v, H denote respectively the normal velocity and d times the mean curvature of the interface $M_t \subset \mathbb{R}^{d+1}$, and the closed set U^c represents the obstacle. Notice that, due to the presence of obstacles, the evolving interface is in general only of class $C^{1,1}$ in the space variable, differently from the unconstrained case where it is analytic (see [17]). While the regularity of parabolic obstacle problems is relatively well understood (see [23] and references therein), a satisfactory existence and uniqueness theory for solutions is still missing.

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In [1] (see also [24]) the authors approximate such an obstacle problem with an implicit variational scheme introduced in [2, 18]. As a byproduct, they prove global existence of weak (variational) solutions, and short time existence and uniqueness of regular solutions in the two-dimensional case. In [20] the first author adapts to this setting the theory of viscosity solutions introduced in [4, 5], and constructs globally defined continuous (viscosity) solutions.

Let us now state the main results of this paper.

THEOREM 1 Let $M_0 \subset U$ be an initial hypersurface, and assume that both M_0 and ∂U are uniformly of class $C^{1,1}$, with $\mathrm{dist}(M_0, \partial U) > 0$. Then there exists $T > 0$ and a unique solution M_t to (1), (2) on $[0, T)$, such that M_t is of class $C^{1,1}$ for all $t \in [0, T)$.

Notice that Theorem 1 extends a result in [1] to dimensions greater than two.

When the hypersurface M_t can be written as the graph of a function $u(\cdot, t) : \mathbb{R}^d \to \mathbb{R}$, equation (1) reads

$$u_t = \sqrt{1 + |\nabla u|^2}\, \mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right). \tag{3}$$

If the obstacles are also graphs, the constraint (2) can be written as

$$\psi^- \leqslant u \leqslant \psi^+, \tag{4}$$

where the functions $\psi^\pm : \mathbb{R}^d \to \mathbb{R}$ denote the obstacles.

THEOREM 2 Assume that $\psi^\pm \in C^{1,1}(\mathbb{R}^d)$, and let $u_0 \in C^{1,1}(\mathbb{R}^d)$ satisfy (4). Then there exists a unique (viscosity) solution u of (3), (4) on $\mathbb{R}^d \times [0, +\infty)$, such that

$$\|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leqslant \max\left(\|\nabla u_0\|_{L^\infty(\mathbb{R}^d)}, \|\nabla \psi^\pm\|_{L^\infty(\mathbb{R}^d)}\right)$$

$$\|u_t(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leqslant \left\| \sqrt{1 + |\nabla u_0|^2}\, \mathrm{div}\left(\frac{\nabla u_0}{\sqrt{1 + |\nabla u_0|^2}}\right)\right\|_{L^\infty(\mathbb{R}^d)}$$

for all $t > 0$. Moreover u is also of class $C^{1,1}$ uniformly on $[0, +\infty)$.

We observe that Theorem 2 extends previous results by Ecker and Huisken [8] in the unconstrained case (see also [6]).

THEOREM 3 Assume that u_0 and $\psi^\pm$ are Q-periodic, with periodicity cell $Q = [0, L]^d$, for some $L > 0$. Then the solution $u(\cdot, t)$ of (3), (4) is also Q-periodic. Moreover there exists a sequence $t_n \to +\infty$ such that $u(\cdot, t_n)$ converges uniformly as $n \to +\infty$ to a stationary solution to (3), (4).

Our strategy of proof will be to approximate the obstacles with “soft obstacles” modeled by a sequence of uniformly bounded forcing terms. Differently from [1], where the existence of regular solution is derived from variational estimates on the approximating scheme, we obtain estimates on the evolving interface, in the spirit of [7, 9, 10], which are uniform in the forcing terms.

The plan of the paper is the following: in Section 2 we adapt some well known results on mean curvature flow, such as Huisken’s monotonicity formula, to the case of forced mean curvature flow. In Section 3 we prove Theorem 1. In Section 4 we prove Theorem 2. In Section 5 we prove Theorem 3. Eventually, in the Appendix at the end of the paper we adapt the concept of viscosity solution in order to treat the case of mean curvature flow of graphs in the presence of obstacles.

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2. Mean curvature flow with a forcing term

2.1 *Evolution of geometric quantities*

Let M be a complete orientable d-dimensional Riemannian manifold without boundary, let $F(\cdot,t) : M \to \mathbb{R}^{d+1}$ be a smooth family of immersions, and denote by M_t the image $F(M,t)$. Since M_t is orientable, we can write $M_t = \partial E(t)$ where $E(t)$ is a family of open subsets of $\mathbb{R}^{d+1}$ depending smoothly on t. We say that M_t evolves by mean curvature with forcing term k if

$$\frac{d}{dt}F(p,t) = -\big(H(p,t) + k(F(p,t))\big)\,\nu(p,t), \tag{5}$$

where $k : \mathbb{R}^{d+1} \to \mathbb{R}$ is a smooth forcing term, ν is the unit normal to M_t pointing outside $E(t)$, and H is (d times) the mean curvature of M_t, with the convention that H is positive whenever $E(t)$ is convex.

We shall compute the evolution of some relevant geometric quantities under the law (5). We denote by ∇^S, Δ^S respectively the covariant derivative and the Laplace-Beltrami operator on M. As in [15], the metric on M_t is denoted by $g_{ij}(t)$, it inverse is $g^{ij}(t)$, the scalar product (or any tensors contraction using the metric) on M_t is denoted by $\langle\cdot\,,\,\cdot\rangle$ whereas the ambient scalar product is $(\cdot\,,\,\cdot)$, the volume element is μ_t, and the second fundamental form is A. In particular we have $A\left(\partial_i, \partial_j\right) = h_{ij}$, where we set for simplicity $\partial_i = \frac{\partial}{\partial x_i}$, and $H = g^{ij}h_{ij}$, using the Einstein notation (we implicitly sum over every index which appears twice). We also denote by $\lambda_1, \ldots, \lambda_d$ the eigenvalues of A.

Notice that, in terms of the parametrization F, we have

$$g_{ij} = \left(\partial_i F\,,\,\partial_j F\right), \qquad h_{ij} = -\left(\partial^2_{ij} F\,,\,\nu\right) \qquad \text{for all } i,j \in \{1,\ldots,d\}. \tag{6}$$

PROPOSITION 1 The following equalities hold:

$$\frac{d}{dt}g_{ij} = -2(H+k)h_{ij} \tag{7}$$

$$\frac{d}{dt}\nu = \nabla^S(H+k) \tag{8}$$

$$\frac{d}{dt}\mu_t = -H(H+k)\mu_t \tag{9}$$

$$\frac{d}{dt}h_{ij} = \Delta^S h_{ij} + \nabla^S_i\nabla^S_j k - 2Hh_{il}g^{lm}h_{mj} - kg^{ml}h_{im}h_{jl} + |A|^2 h_{ij} \tag{10}$$

$$\frac{d}{dt}H = \Delta^S(H+k) + (H+k)|A|^2 \tag{11}$$

$$\frac{d}{dt}|A|^2 = \Delta^S|A|^2 + 2kg^{ij}g^{sl}g^{mn}h_{is}h_{lm}h_{nj} + 2|A|^4 - 2|\nabla^S A|^2 + 2\left\langle A\,,\,(\nabla^S)^2 k\right\rangle. \tag{12}$$

Proof. The proof follows by direct computations as in [10, 15]. Recalling (6), we get

$$\frac{d}{dt}g_{ij} = \frac{d}{dt}\left(\partial_i F\,,\,\partial_j F\right) = -(H+k)\big(\left(\partial_i \nu\,,\,\partial_j F\right) + \left(\partial_i F\,,\,\partial_j \nu\right)\big) = -2(H+k)h_{ij},$$

$$\begin{aligned}\frac{d}{dt}\nu &= \left(\frac{d}{dt}\nu\,,\,\partial_i F\right)g^{ij}\partial_j F = -\left(\nu\,,\,\frac{d}{dt}\partial_i F\right)g^{ij}\partial_j F\\ &= (\nu\,,\,\partial_i((H+k)\nu))\,g^{ij}\partial_j F = \partial_i(H+k)g^{ij}\partial_j F = \nabla^S(H+k).\end{aligned}$$

The evolution of the measure on M_t

$$\mu_t = \sqrt{\det[g]}$$

is given by

$$\begin{aligned}\frac{d}{dt}\sqrt{\det[g]} = \frac{\frac{d}{dt}\det[g]}{2\sqrt{\det[g]}} &= \frac{\det[g]\cdot \mathrm{Tr}\left(g^{ij}\,\frac{d}{dt}g_{ij}\right)}{2\sqrt{\det[g]}}\\ &= -\sqrt{\det[g]}\cdot(H+k)g^{ij}h_{ji} = -\mu_t H(H+k).\end{aligned}$$

In order to prove (10) we compute (as usual, we denote the Christoffel symbols by Γ_{ij}^k)

$$\begin{aligned}\frac{d}{dt}h_{ij} &= -\frac{d}{dt}\left(\nu\,,\,\partial_{ij}^2 F\right)\\ &= -\left(\nabla^S(H+k)\,,\,\partial_{ij}^2 F\right) + \left(\partial_{ij}^2(H+k)\nu\,,\,\nu\right)\\ &= -\left(g^{kl}\partial_k(H+k)\partial_l F\,,\,\Gamma_{ij}^k\partial_k F - h_{ij}\nu\right)\\ &\qquad + \partial_{ij}^2(H+k) + (H+k)\left(\partial_j\left(h_{im}g^{ml}\partial_l F\right),\,\nu\right)\\ &= \partial_{ij}^2(H+k) - \Gamma_{ij}^k\partial_k(H+k) + (H+k)h_{im}g^{ml}\left(\Gamma_{lj}^k\partial_k F - h_{lj}\nu\,,\,\nu\right)\\ &= \nabla_i^S\nabla_j^S(H+k) - (H+k)h_{il}g^{lm}h_{mj}.\end{aligned} \tag{13}$$

Using Codazzi's equations, one can show that

$$\Delta^S h_{ij} = \nabla_i^S\nabla_j^S H + H h_{il}g^{lm}h_{mj} - |A|^2 h_{ij}, \tag{14}$$

so that (10) follows from (14) and (13). From (10) we deduce

$$\begin{aligned}\frac{d}{dt}H &= \frac{d}{dt}g^{ij}h_{ij}\\ &= 2(H+k)g^{is}h_{sl}g^{lj}h_{ij} + g^{ij}\left(\nabla_i^S\nabla_j^S(H+k) - (H+k)h_{il}g^{lm}h_{mj}\right)\\ &= \Delta^S(H+k) + (H+k)|A|^2,\end{aligned}$$

which gives (11). In addition, we get

$$
\begin{aligned}
\frac{d}{dt}|A|^2 &= \frac{d}{dt}\left(g^{ik}g^{jl}h_{ij}h_{kl}\right) \\
&= 2\frac{d}{dt}g^{jl}h_{ij}h_{kl} + 2g^{ik}g^{jl}\frac{d}{dt}h_{ij}h_{kl} \\
&= 2\left(2(H+k)g^{js}h_{st}g^{tl}\right)g^{jl}h_{ij}h_{kl} \\
&\quad + 2g^{ik}g^{jl}\left(\Delta^S h_{ij} + \nabla_i^S\nabla_j^S k - 2Hh_{il}g^{lm}h_{mj} - kg^{ml}h_{im}h_{jl} + |A|^2 h_{ij}\right)h_{kl} \\
&= 2kg^{js}h_{st}g^{tl}g^{jl}h_{ij}h_{kl} + 2g^{ik}g^{jl}\Delta^S h_{ij}h_{kl} + 2|A|^4 + 2\left\langle A\,,\,(\nabla^S)^2 k\right\rangle.
\end{aligned}
\tag{15}
$$

On the other hand, one has

$$
\Delta^S|A|^2 = 2\left\langle \Delta^S A\,,\, A\right\rangle + 2|\nabla^S A|^2 = 2g^{pq}g^{mn}h_{pm}\Delta^S h_{qn} + 2|\nabla^S A|^2. \tag{16}
$$

so that (12) follows from (16) and (15). □

2.2 *The monotonicity formula*

We extend Huisken's monotonicity formula [16] to the forced mean curvature flow (5) (see also [7, Section 2.2]).

Given a vector field $\omega : M_t \to \mathbb{R}^{d+1}$, we let

$$
\omega^\perp = (\omega\,,\,\nu)\,\nu, \qquad \omega^T = \omega - \omega^\perp\,.
$$

Letting $X_0 \in \mathbb{R}^{d+1}$ and $t_0 \in \mathbb{R}$, for $(x,t) \in \mathbb{R}^{d+1} \times [t_0, +\infty)$ we define the kernel

$$
\rho(x,t) = \frac{1}{(4\pi(t_0-t))^{d/2}}\exp\left(\frac{-|x-x_0|^2}{4(t_0-t)}\right).
$$

A direct computation gives

$$
\frac{d\rho}{dt} = -\Delta^S\rho + \rho\left(\frac{(x_0-x\,,\,(H+k)\nu)}{t_0-t} - \frac{|(x_0-x)^\perp|^2}{4(t_0-t)^2}\right). \tag{17}
$$

PROPOSITION 2 (Monotonicity formula)

$$
\frac{d}{dt}\int_{M_t}\rho = -\int_{M_t}\rho\left(\left|H + \frac{k}{2} + \frac{(x-x_0\,,\,\nu)}{2(t_0-t)}\right|^2 - \frac{k^2}{4}\right).
$$

Proof. Recalling (9), we compute

$$\frac{d}{dt}\int_{M_t}\rho = \int_{M_t}\frac{d}{dt}\rho - H(H+k)\rho$$
$$= \int_{M_t}\rho\left(-\frac{|x-x_0|^2}{4(t_0-t)^2}+\frac{d}{2(t_0-t)}-\frac{(x-x_0\,,\,\nu)}{2(t_0-t)}(H+k)-H(H+k)\right)$$
$$= -\int_{M_t}\rho\left(\left|H\nu+\frac{x-x_0}{2(t_0-t)}+\frac{k\nu}{2}\right|^2-\frac{k^2}{4}\right)+\int_{M_t}\frac{d}{2(t_0-t)}\rho$$
$$+\int_{M_t}\rho\frac{(x-x_0\,,\,\nu)\,H}{2(t_0-t)}$$

We use the first variation formula: for all vector field Y on M_t, we have

$$\int_{M_t}\operatorname{div}_{M_t}Y = \int_{M_t}\langle H\nu\,,\,Y\rangle\,.$$

As a result, with $Y = \frac{\rho(x-x_0)}{2(t-t_0)}$, we get

$$\frac{d}{dt}\int_{M_t}\rho = -\int_{M_t}\rho\left(\left|H\nu+\frac{x-x_0}{2(t_0-t)}+\frac{k\nu}{2}\right|^2-\frac{k^2}{4}-\frac{|(x-x_0)^T|^2}{4(t_0-t)^2}\right)$$
$$= -\int_{M_t}\rho\left(\left|H+\frac{(x-x_0\,,\,\nu)}{2(t_0-t)}+\frac{k}{2}\right|^2-\frac{k^2}{4}\right).$$

□

In a similar way (see [8]) one can prove that for all functions $f(X,t)$ defined on M_t, one has

$$\partial_t\int_{M_t}\rho f = \int_{M_t}\left(\frac{df}{dt}-\Delta^S f\right)\rho-\int_{M_t}f\rho\left(\left|H+\frac{(x-x_0\,,\,\nu)}{2(t_0-t)}+\frac{k}{2}\right|^2-\frac{k^2}{4}\right). \tag{18}$$

Indeed, using (17)

$$\frac{d}{dt}\int_{M_t}\rho f = \int_{M_t}f\frac{d\rho}{dt}+\frac{df}{dt}\rho-H(H+k)f\rho$$
$$= \int_{M_t}f\left(\frac{d\rho}{dt}-H(H+k)\rho\right)+\frac{df}{dt}\rho$$
$$= \int_{M_t}f\left(-\Delta^S\rho+\rho\left(\frac{(X_0-X\,,\,(H+k)\nu)}{t_0-t}-\frac{1}{4}\frac{|(X_0-X)^{\perp}|^2}{(t_0-t)^2}\right)-H(H+k)\rho\right)+\frac{df}{dt}\rho$$
$$= \int_{M_t}-\Delta^S f\rho+\left(\rho\left(\frac{(X_0-X\,,\,(H+k)\nu)}{t_0-t}-\frac{1}{4}\frac{|(X_0-X)^{\perp}|^2}{(t_0-t)^2}\right)-H(H+k)\rho\right)+\frac{df}{dt}\rho$$
$$= \int_{M_t}\rho\left(\frac{d}{dt}f-\Delta^S f\right)-\int f\rho\left(\left|H+\frac{(x-x_0\,,\,\nu)}{2(t_0-t)}+\frac{k}{2}\right|^2-\frac{k^2}{4}\right).$$

LEMMA 1 Let f be defined on M_t and satisfy

$$\frac{d}{dt} f - \Delta^S f \leqslant a \cdot \nabla^S f \quad \text{on } M_t \tag{19}$$

for some vector field a bounded on $[0, t_1]$. Then,

$$\sup_{M_t, t \in [0,t_1]} f \leqslant \sup_{M_0} f.$$

Proof. Denote by a_0 the bound on a, $k := \sup_{M_0} f$ and define $f_l = \max(f - l, 0)$. Assumption (19) implies

$$\left(\frac{d}{dt} - \Delta^S\right) f_l^2 \leqslant 2 f_l a \cdot \nabla^S f_l - 2|\nabla^S f_l|^2$$

which, thanks to Young's inequality, gives

$$\left(\frac{d}{dt} - \Delta^S\right) f_l^2 \leqslant \frac{1}{2} a_0^2 f_l^2.$$

Applying (18) to f_l^2, we get

$$\frac{d}{dt} \int f_l^2 \rho \leqslant \frac{1}{2}(a_0^2 + \|k\|_\infty^2) \int f_l^2 \rho. \tag{20}$$

Letting $l = \sup_{M_0} f$, so that $f_l \equiv 0$ on M_0, from (20) and the Gronwall's Lemma we obtain that $f_l \equiv 0$ on M_t for all $t \in (0, t_1]$, which gives thesis. □

3. Proof of Theorem 1

We now prove short time existence for the mean curvature flow with obstacles (1), (2). Let $M_0 = \partial E(0) \subset U$, where we assume that U, $E(0)$ are open sets with boundary uniformly of class $C^{1,1}$. In particular, M_0 satisfies a uniform exterior and interior ball condition, that is, there is $R > 0$ such that, for every $x \in M_0$, one can find two open balls B^+ and B^- of radius R which are tangent to M_0 at x and such that $B^+ \subset E(0)^c$ and $B^- \subset E(0)$. Let also $\Omega^- := E(0) \setminus \overline{U}$, and $\Omega^+ := E(0) \cup U$. Notice that $\Omega^\pm$ are open sets with $C^{1,1}$ boundaries, with $\text{dist}(\Omega^-, \partial\Omega^+) > 0$. Note that the condition $M_t \subset \overline{U}$ can be rewritten as

$$\Omega^- \subset E(t) \subset \Omega^+.$$

Let also

$$k := 2N(1 - \chi_{\Omega^+} - \chi_{\Omega^-})$$

where N is bigger than (d times) the mean curvature of ∂U.

We want to show that equation (5), with k as above, has a solution in an interval $[0, T)$. To this purpose, letting ρ_ε be a standard mollifier supported in the ball of radius ε centered at 0, we introduce a smooth regularization $k_\varepsilon = k * \rho_\varepsilon$ of k. Notice that $\|k_\varepsilon\|_\infty = 2N$, $k_\varepsilon(x) = -2N$ (resp. $k_\varepsilon(x) = 2N$) at every $x \in \Omega^-$ (resp. $x \notin \Omega^+$) such that $\text{dist}(x, \partial U) \geqslant \varepsilon$, and $k_\varepsilon(x) = 0$ at every $x \in U$ such that $\text{dist}(x, \partial U) \geqslant \varepsilon$.

Using standard arguments (see for instance [10, Theorem 4.1] and [9, Prop. 4.1]) one can show existence of a smooth solution M_t^ε of (5), with k replaced by k_ε, on a maximal time interval $[0, T_\varepsilon)$.

Let now

$$\Omega_\varepsilon^- := \{x \in \Omega^- : \operatorname{dist}(x, \partial\Omega^-) > \varepsilon\}$$

and

$$\Omega_\varepsilon^+ := \{x \in \mathbb{R}^{d+1} : \operatorname{dist}(x, \Omega^+) < \varepsilon\}.$$

The following result follows directly from the definition of k_ε.

PROPOSITION 3 The hypersurfaces $\partial\Omega_\varepsilon^-$ and $\partial(\Omega_\varepsilon^+)^c$ are respectively a super and a subsolution of (5), with k replaced with k_ε. In particular, by the parabolic comparison principle M_t^ε cannot intersect $\partial\Omega_\varepsilon^\pm$.

We will show that we can find a time $T > 0$ such that for every ε, there exists a smooth solution of (5) (with k replaced with k_ε) on $[0, T)$.

The following result will be useful in the sequel. We omit the proof which is a simple ODE argument.

LEMMA 2 Let $M_0 = \partial B_R(x_0)$ be a ball of radius $R \leqslant 1$ centered at x_0. Then, the evolution M_t by (5), with constant forcing term $k = 2N$, is given by $M_t = B_{R(t)}(x_0)$ with $R(t) \geqslant \sqrt{R^2 - (4N + 2d)t}$. In particular, the solution exists at least on $[0, \frac{R^2}{4N+2d})$.

PROPOSITION 4 There exists $r > 0$, a collection of balls $B_i = B_r(x_i)$ of radius r, and a positive time T_0 such that $M_t^\varepsilon \subset \bigcup_i B_i$ for every $t \in [0, \min(T_0, T_\varepsilon))$. In addition, we can choose the balls B_i in such a way that, for every i, there exists $\omega_i \in \mathbb{R}^{d+1}$ such that the sets $\partial\Omega^\pm \cap B_{4r}(x_i)$, if nonempty, are graphs of some functions $\psi_i^\pm : \mathbb{R}^d \to \mathbb{R}$ over $\omega_i^\perp$.
In particular, one has

$$(\nabla k_\varepsilon \, , \, \omega_i) \geqslant |\nabla k_\varepsilon|/2 \quad \text{on } B_{2r}(x_i).$$

Most of this notation is summarized in Figure 1.

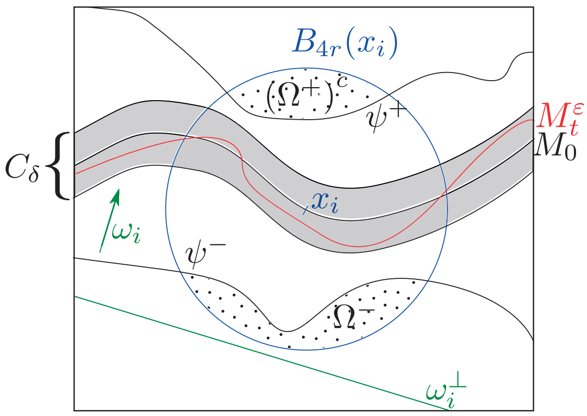

FIG. 1. Notation in Proposition 4

Proof. By assumption, for every $\mathbf{x} \in M_0$ there exist interior and exterior balls $B_x^{\pm}$ of fixed radius $R \leqslant 1$. Let $B_x^{\pm}(t)$ be the evolution of $B_x^{\pm}$ by (5) with forcing term $k = 2N$. By comparison, for every $t \in [0, T_\varepsilon)$, $B_x^{+}(t) \subset \Omega(t)^c$ and $B_x^{-}(t) \subset \overline{\Omega}(t)$. Recalling Lemma 2, there exists $\delta > 0$ and $T_0 > 0$, independent of ε, such that $M_t \subset \{d_{M_0} \leqslant \delta\} =: C_\delta$, for all $t \in [0, \min(T_\varepsilon, T_0))$.
We eventually reduce δ, T_0 such that C_δ can be covered with a collection of balls $B_i = B_r(x_i)$, centered at $x_i \in M_0$ and with a radius r such that, for every i, there exists a unit vector $\omega_i \in \mathbb{R}^{d+1}$ satisfying

$$\left(\omega_i\, ,\, \nu^+(x)\right) \geqslant \frac{1}{2} \quad \text{and} \quad \left(\omega_i\, ,\, \nu^-(y)\right) \geqslant \frac{1}{2}$$

for every $x \in \partial\Omega^+ \cap B_{4r}(x_i)$ and $y \in \partial\Omega^- \cap B_{4r}(x_i)$, where $\nu^{\pm}$ is the outer normal to $\Omega^{\pm}$.

As a result, $\partial\Omega^{\pm} \cap B_{4r}(x_i)$ are graphs of some functions $\psi_i^{\pm} : \mathbb{R}^d \to \mathbb{R}$ over $\omega_i^{\perp}$ (see Figure 1).

Notice also that k is a BV function and Dk is a Radon measure concentrated on ∂U such that

$$(Dk\, ,\, \omega_i) \geqslant \frac{|Dk|}{2} \text{ on } B_{4r}(x_i).$$

Then, for every $x \in B_{2r}(x_i)$ and ε sufficiently small (such that $\rho_\varepsilon(x) = 0$ as soon as $|x| \geqslant 2r$), we have

$$\begin{aligned}
(\nabla k_\varepsilon\, ,\, \omega_i) &= \left(\nabla \int_{\mathbb{R}^{d+1}} k(x-y)\rho_\varepsilon(y)dy\, ,\, \omega_i\right)\\
&= \int_{\mathbb{R}^{d+1}} (Dk(x-y)\, ,\, \omega_i)\, \rho_\varepsilon(y)dy\\
&\geqslant \int_{\mathbb{R}^{d+1}} \frac{|Dk|(x-y)}{2}\rho_\varepsilon(y)dy\\
&\geqslant \frac{|Dk| * \rho_\varepsilon}{2} \geqslant \frac{|\nabla k_\varepsilon|}{2}.
\end{aligned}$$

□

In what follows, we will control the geometric quantities of M_t^ε inside each ball B_i. As in [9], we introduce a localization function ϕ_i as follows: let $\eta_i(x,t) = |x - x_i|^2 + (2d + \Lambda)t$ (Λ be a positive constant that will be fixed later) and, for $R = 2r$, $\phi_i(x,t) = (R^2 - \eta_i(x,t))^+$. We denote by ϕ_i the quantity $\phi_i(\mathbf{x}, t)$, where $\mathbf{x} = \mathbf{x}(p,t)$ will be a generic point in M_t. Notice that there exists $T_1 = \frac{r^2}{2d+\Lambda}$ such that for all $t \in [0, \min(T_1, T_\varepsilon))$,

$$M_t^\varepsilon \subset \bigcup_i \{\phi_i > r^2\}. \tag{21}$$

As a result, we have the following

LEMMA 3 Let f be a smooth function defined on M_t^ε. Assume that there is a $C > 0$ such that

$$\phi_i f \leqslant C \text{ on } M_t^\varepsilon \qquad \forall t \leqslant \min(T_\varepsilon, T_1) \text{ and } \forall i \in \mathbb{N}.$$

Then,

$$f \leqslant \alpha C \text{ on } M_t^\varepsilon \qquad \forall t \leqslant \min(T_\varepsilon, T_1),$$

where α depends only on the $C^{1,1}$ norm of M_0.

LEMMA 4 Let $v := (\nu\,,\,\omega)^{-1}$. The quantity $v^2\phi^2$ satisfies

$$\left(\frac{d}{dt} - \Delta^S\right)\left(\frac{v^2\phi^2}{2}\right) \leqslant \frac{1}{2}\left(\nabla^S(v^2\phi^2)\,,\,\frac{\nabla^S\phi^2}{\phi^2}\right) - \phi^2 v^3\left(\nabla^S k_\varepsilon\,,\,\omega\right) + v^2\phi(2k_\varepsilon(\mathbf{x}\,,\,\nu) - \Lambda). \tag{22}$$

Proof. In this proof and the proofs further, we use normal coordinates: we assume that $g_{ij} = \delta_{ij}$ (Kronecker symbol) and that the Christoffel symbols Γ_{ij}^k vanish at the computation point.

We expand the derivatives

$$\left(\frac{d}{dt} - \Delta^S\right)\left(\frac{v^2\phi^2}{2}\right) = v^2\left(\frac{d}{dt} - \Delta^S\right)\frac{\phi^2}{2} + \phi^2\left(\frac{d}{dt} - \Delta^S\right)\frac{v^2}{2} - 2\left\langle\nabla^S\frac{\phi^2}{2}\,,\,\nabla^S\frac{v^2}{2}\right\rangle.$$

First term. We start computing

$$\left(\frac{d}{dt} - \Delta^S\right)|\mathbf{x}|^2 = -2k_\varepsilon(\mathbf{x}\,,\,\nu) - 2d.$$

Then,

$$\left(\frac{d}{dt} - \Delta^S\right)\phi^2 = 2\phi(2k_\varepsilon(\mathbf{x} - x_i\,,\,\nu) - \Lambda) - 2|\nabla^S|\mathbf{x}|^2|^2.$$

Second term. We are interested in

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\omega\,,\,\nu)^2 &= (\omega\,,\,\nu)\left(\frac{d}{dt}\nu\,,\,\omega\right)\\ &= (\omega\,,\,\nu)\left(\nabla^S(H + k_\varepsilon)\,,\,\omega\right).\end{aligned}$$

So,

$$\frac{1}{2}\frac{d}{dt}(\omega\,,\,\nu)^{-2} = -(\omega\,,\,\nu)^{-3}\left(\nabla^S(H + k_\varepsilon)\,,\,\omega\right). \tag{23}$$

On the other hand,

$$\frac{1}{2}\Delta^S\left((\omega\,,\,\nu)^{-2}\right) = (\omega\,,\,\nu)^{-1}\,\Delta^S(\omega\,,\,\nu)^{-1} - \left\langle\nabla^S(\omega\,,\,\nu)^{-1}\,,\,\nabla^S(\omega\,,\,\nu)^{-1}\right\rangle. \tag{24}$$

Let us note that

$$\partial_{ij}\nu = \partial_i\left(h_{jl}g^{lm}\partial_m F\right) = \partial_i(h_{jl})\delta_{lm}\partial_m F - h_{jl}\delta_{lm}(-h_{im}\nu) = \partial_i(h_{jl})\partial_l F - \lambda_i^2\delta_{ij}\nu.$$

We then get

$$\begin{aligned}\Delta^S(\omega\,,\,\nu)^{-1} &= \partial_{ii}(\omega\,,\,\nu)^{-1} = \partial_i\left(-(\omega\,,\,\partial_i\nu)(\omega\,,\,\nu)^{-2}\right)\\ &= -(\omega\,,\,\partial_{ii}\nu)(\omega\,,\,\nu)^{-2} + 2(\omega\,,\,\partial_i\nu)^2(\omega\,,\,\nu)^{-3}\\ &= -(\omega\,,\,\nu)^{-2}\left(\partial_i h_{il}\partial_l F - \lambda_i^2\nu\,,\,\omega\right) + 2(\omega\,,\,\nu)^{-3}(\omega\,,\,\lambda_i\partial_i F)^2.\\ &= -(\omega\,,\,\nu)^{-2}(\partial_l h_{ii}\partial_l F\,,\,\omega) + |A|^2(\nu\,,\,\omega)^{-1} + 2(\omega\,,\,\nu)^{-3}(\omega\,,\,\lambda_i\partial_i F)^2.\end{aligned}$$

We also have

$$\left\langle \nabla^S (\omega\,,\,\nu)^{-1}\,,\,\nabla^S (\omega\,,\,\nu)^{-1} \right\rangle = (\omega\,,\,\nu)^{-4}\,(\omega\,,\,\partial_k \nu)\,(\omega\,,\,\partial_k \nu)$$
$$= (\omega\,,\,\nu)^{-4}\,(\omega\,,\,h_{ku} g^{uv} \partial_v F)^2 = (\omega\,,\,\nu)^{-4}\,(\omega\,,\,\lambda_k \partial_k F)^2\,,$$

which leads to

$$\left(\frac{d}{dt} - \Delta^S\right)\frac{v^2}{2} = -v^3\left(\nabla^S (H + k_\varepsilon)\,,\,\omega\right) + v^3 \partial_m (h_{ii})\,(\omega\,,\,\partial_m F)$$
$$- |A|^2 v^2 - 2v^4 \lambda_k^2\,(\omega\,,\,\partial_k F)^2 - v^4\,(\omega\,,\,\lambda_k \partial_k F)^2\,.$$

Third term. We notice, as in [9] that $|\nabla^S \phi^2|^2 = 4\phi^2 |\nabla^S (|\mathbf{x}|^2)|^2$ and

$$-\left(\nabla^S (v^2)\,,\,\nabla^S \phi^2\right) = -3\left(v \nabla^S (v)\,,\,\nabla^S \phi^2\right) + \frac{1}{2}\left(\left(\nabla^S (v^2 \phi^2)\,,\,\frac{\nabla^S \phi^2}{\phi^2}\right) - v^2 \frac{|\nabla^S \phi^2|^2}{\phi^2}\right).$$

Then, Young's inequality gives

$$2\left|v\left(\nabla^S v\,,\,\nabla^S \phi^2\right)\right| \leqslant 2\phi^2 |\nabla^S v^2|^2 + \frac{1}{2\phi^2} |\nabla^S \phi^2|^2$$
$$\leqslant 2\phi^2 |\nabla^S v^2|^2 + 2v^2 |\nabla^S |\mathbf{x}|^2|^2.$$

Hence,

$$\left(\nabla^S (v^2)\,,\,\nabla^S \phi^2\right) \leqslant -3\phi^2 |\nabla^S v^2|^2 - 3v^2 |\nabla^S |\mathbf{x}|^2|^2 + \frac{1}{2}\left(\left(\nabla^S (v^2 \phi^2)\,,\,\frac{\nabla^S \phi^2}{\phi^2}\right) - v^2 \frac{|\nabla^S \phi^2|^2}{\phi^2}\right).$$

Summing the three terms, we get

$$\left(\frac{d}{dt} - \Delta^S\right)\left(\frac{v^2 \phi^2}{2}\right) \leqslant \frac{1}{2}\left(\nabla^S (v^2 \phi^2)\,,\,\frac{\nabla^S \phi^2}{\phi^2}\right) - \phi^2 v^3\left(\nabla^S k_\varepsilon\,,\,\omega\right) + v^2 \phi\big(2k_\varepsilon\,(\mathbf{x}\,,\,\nu) - \Lambda\big).$$

□

For $\gamma > 0$, we let

$$\psi(v^2) := \frac{\gamma v^2}{1 - \gamma v^2}.$$

LEMMA 5 For $\varepsilon \leqslant r$, we have

$$\left(\frac{d}{dt} - \Delta^S\right)\frac{\phi^2 |A|^2 \psi(v^2)}{2} \leqslant \phi^2 \psi(v^2)\Big(-\gamma |A|^4 - 2k_\varepsilon \sum_i \lambda_i^3 - 2\big\langle A\,,\,(\nabla^S)^2 k_\varepsilon \big\rangle\Big)$$
$$- \phi^2 |A|^2 v^3 \psi'(v^2)\left(\nabla^S k_\varepsilon\,,\,\omega\right) - \phi^2 |A|^2 \sum_i (\lambda_i \omega^i)^2 \frac{2v^4 + \gamma v^6}{(1 - \gamma v^2)^3}.$$

Proof. We denote $V = \frac{\phi^2|A|^2\psi(v^2)}{2}$ and compute

$$\left(\frac{d}{dt} - \Delta^S\right)\frac{\phi^2|A|^2\psi(v^2)}{2} = |A|^2\psi(v^2)\left(\frac{d}{dt} - \Delta^S\right)\frac{1}{2}\phi^2 + \phi^2\psi(v^2)\left(\frac{d}{dt} - \Delta^S\right)\frac{1}{2}|A|^2$$
$$+ \phi^2|A|^2\left(\frac{d}{dt} - \Delta^S\right)\frac{1}{2}\psi(v^2) - 2\left\langle 1/2\nabla^S|A|^2\,,\, 1/2\nabla^S\phi^2\right\rangle$$
$$- 2\left\langle 1/2\nabla^S|A|^2\,,\, 1/2\nabla^S\psi(v^2)\right\rangle - 2\left\langle 1/2\nabla^S\phi^2\,,\, 1/2\nabla^S\psi(v^2)\right\rangle.$$

The two first terms have already been computed. Let us consider the third one.

$$\frac{1}{2}\frac{d}{dt}\psi(v^2) = v\frac{dv}{dt}\psi'(v^2) = -v^3\psi'(v^2)\left(\nabla^S(H + k_\varepsilon)\,,\, \omega\right),$$

$$\begin{aligned}\frac{1}{2}\Delta^S\psi(v^2) &= \frac{1}{2}\partial_{ii}\psi(v^2) = \partial_i\left(v\partial_i v\psi'(v^2)\right)\\ &= v\Delta^S v\psi'(v^2) + 2v^2|\nabla^S v|^2\psi''(v^2) + |\nabla^S v|^2\psi'(v^2)\\ &= \Big(3|\nabla^S v|^2 - v^3\big(\partial_l(h_{kk})w^l\big) + v^2|A|^2\Big)\psi'(v^2) + 2|\nabla^S v|^2\psi''(v^2).\end{aligned}$$

Hence

$$\left(\frac{d}{dt} - \Delta^S\right)\frac{1}{2}\psi(v^2) = -v^3\psi'(v^2)\left(\nabla^S k_\varepsilon\,,\, \omega\right) - (3|\nabla^S v|^2 + v^2|A|^2)\psi'(v^2) - 2v^2|\nabla^S v|^2\psi''(v^2).$$

As above, we want to conclude the proof using the weak maximum principle. So, we want to rewrite the last terms (which are gradient terms) using the gradient of V. Let us expand $\nabla^S V$.

$$\nabla^S\frac{\phi^2|A|^2\psi(v^2)}{2} = \phi^2|A|^2\frac{1}{2}\nabla^S\psi(v^2) + |A|^2\psi(v^2)\frac{1}{2}\nabla^S\phi^2 + \phi^2\psi(v^2)\frac{1}{2}\nabla^S|A|^2.$$

So,

$$\left|\nabla^S\frac{\phi^2|A|^2\psi(v^2)}{2}\right|^2 = \phi^4|A|^4\frac{|\nabla^S\psi(v^2)|^2}{4} + |A|^4\psi^2(v^2)\frac{|\nabla^S\phi^2|^2}{4} + \phi^4\psi^2(v^2)\frac{|\nabla^S|A|^2|^2}{4}$$
$$+ \phi^2|A|^4\psi(v^2)\left\langle\nabla^S\psi(v^2)\,,\, \nabla^S\phi^2\right\rangle + \phi^4|A|^2\psi(v^2)\left\langle\nabla^S\psi(v^2)\,,\, \nabla^S|A|^2\right\rangle$$
$$+ |A|^2\psi^2(v^2)\phi^2\left\langle\nabla^S\phi^2\,,\, \nabla^S|A|^2\right\rangle.$$

As a matter of fact,

$$\frac{1}{\phi^2|A|^2\psi(v^2)}\left|\nabla^S\frac{\phi^2|A|^2\psi(v^2)}{2}\right|^2 = \phi^2|A|^2\frac{|\nabla^S\psi(v^2)|^2}{4\psi(v^2)} + |A|^2\psi(v^2)\frac{|\nabla^S\phi^2|^2}{4\phi^2}$$
$$+ \phi^2\psi(v^2)\frac{|\nabla^S|A|^2|^2}{4|A|^2} + 2|A|^2\left\langle\nabla^S\psi(v^2)/2\,,\, \nabla^S\phi^2/2\right\rangle$$
$$+ 2\phi^2\left\langle\nabla^S\psi(v^2)/2\,,\, \nabla^S|A|^2/2\right\rangle + 2\psi(v^2)\left\langle\nabla^S\phi^2/2\,,\, \nabla^S|A|^2/2\right\rangle.$$

We use the last equality to rewrite

$$\begin{aligned}
&\Big(\frac{d}{dt}-\Delta^S\Big)\frac{\phi^2|A|^2\psi(v^2)}{2}\\
&\quad=|A|^2\psi(v^2)\left(\phi(2k_\varepsilon(\mathbf{x},\nu)-\Lambda)-|\nabla^S|x|^2|^2\right)\\
&\qquad+\phi^2\psi(v^2)\left(-\langle\nabla^SA,\nabla^SA\rangle+|A|^4-2k_\varepsilon g^{js}h_{st}g^{tl}g^{jl}h_{ij}h_{kl}-2\langle A,(\nabla^S)^2k_\varepsilon\rangle\right)\\
&\qquad+\phi^2|A|^2\left(-v^3\psi'(v^2)\left(\nabla^Sk_\varepsilon,\omega\right)-(3|\nabla^Sv|^2+v^2|A|^2)\psi'(v^2)-2v^2|\nabla^Sv|^2\psi''(v^2)\right)\\
&\qquad-\frac{1}{\phi^2|A|^2\psi(v^2)}\left|\nabla^S\frac{\phi^2|A|^2\psi(v^2)}{2}\right|^2\\
&\qquad+\phi^2|A|^2\frac{|\nabla^S\psi(v^2)|^2}{4\psi(v^2)}+|A|^2\psi(v^2)\frac{|\nabla^S\phi^2|^2}{4\phi^2}+\phi^2\psi(v^2)\frac{|\nabla^S|A|^2|^2}{4|A|^2}.
\end{aligned}\tag{25}$$

Let us rewrite some terms as follows:

$$\begin{aligned}
|\nabla^S\phi^2|^2&=4\phi^2\cdot|-2\mathbf{x}^T|^2=4\phi^2(4|\mathbf{x}|^2-4(\mathbf{x},\nu)),\\
|\nabla^S\psi(v^2)|^2&=\psi'(v^2)^2|\nabla^Sv^2|^2=4\psi'(v^2)^2v^6\sum_k(\lambda_k\omega^k)^2,\\
|\nabla^S|A|^2|^2&=4\sum_i(\partial_i(h_{ll})\lambda_l)^2,\\
|\nabla^SA|^2&=\sum_{i,k,l}(\partial_i(h_{km}))^2.
\end{aligned}$$

In addition, we have the obvious estimate

$$|\nabla^S|A|^2|^2\leqslant 4|A|^2|\nabla^SA|^2.$$

So,

$$\begin{aligned}
&\phi^2|A|^2\frac{|\nabla^S\psi(v^2)|^2}{4\psi(v^2)}+|A|^2\psi(v^2)\frac{|\nabla^S\phi^2|^2}{4\phi^2}+\phi^2\psi(v^2)\frac{|\nabla^S|A|^2|^2}{4|A|^2}\\
&\qquad\leqslant\phi^2|A|^2\frac{\psi'(v^2)^2v^6\sum_k(\lambda_k\omega^k)^2}{\psi(v^2)}+4|A|^2\psi(v^2)(|\mathbf{x}|^2-(\mathbf{x},\nu)^2)+\phi^2\psi(v^2)|\nabla^SA|^2.
\end{aligned}$$

We plug this inequality into (25) and obtain

$$\begin{aligned}
&\Big(\frac{d}{dt}-\Delta^S\Big)\frac{\phi^2|A|^2\psi(v^2)}{2}\leqslant|A|^2\psi(v^2)\left(\phi(2k_\varepsilon(\mathbf{x},\nu)-\Lambda)-|\nabla^S|x|^2|^2\right)\\
&\qquad+\phi^2\psi(v^2)\left(-\langle\nabla^SA,\nabla^SA\rangle+|A|^4-2k_\varepsilon g^{js}h_{st}g^{tl}g^{jl}h_{ij}h_{kl}-2\langle A,(\nabla^S)^2k_\varepsilon\rangle\right)\\
&\qquad+\phi^2|A|^2\left(-v^3\psi'(v^2)\left(\nabla^Sk_\varepsilon,\omega\right)-(3|\nabla^Sv|^2+v^2|A|^2)\psi'(v^2)-2v^2|\nabla^Sv|^2\psi''(v^2)\right)\\
&\qquad-\frac{1}{\phi^2|A|^2\psi(v^2)}\left|\nabla\frac{\phi^2|A|^2\psi(v^2)}{2}\right|^2\\
&\qquad+\phi^2|A|^2\frac{\psi'(v^2)^2v^6\sum_k(\lambda_k\omega^k)^2}{\psi(v^2)}+4|A|^2\psi(v^2)(|\mathbf{x}|^2-(\mathbf{x},\nu)^2)+\phi^2\psi(v^2)|\nabla^SA|^2.
\end{aligned}$$

Let us regroup some terms (noting that $|\nabla^S v|^2 = v^4 \sum_i (\lambda_i \omega^i)^2$), we get

$$\begin{aligned}
&\Big(\frac{d}{dt} - \Delta^S\Big) \frac{\phi^2 |A|^2 \psi(v^2)}{2} \\
&\quad \leqslant |A|^2 \psi(v^2)\,(\phi(2k_\varepsilon(\mathbf{x}\,,\,\nu) - \Lambda)) \\
&\quad + \phi^2 |A|^4 (\psi(v^2) - v^2 \psi'(v^2)) - 2\phi^2 \psi(v^2) k_\varepsilon g^{js} h_{st} g^{tl} g^{jl} h_{ij} h_{kl} - 2\phi^2 \psi(v^2) \big\langle A\,,\,(\nabla^S)^2 k_\varepsilon \big\rangle \\
&\quad - \phi^2 |A|^2 v^3 \psi'(v^2)\,\big(\nabla^S k_\varepsilon\,,\,\omega\big) - \frac{1}{\phi^2 |A|^2 \psi(v^2)} \left| \nabla^S \frac{\phi^2 |A|^2 \psi(v^2)}{2} \right|^2 \\
&\quad + \phi^2 |A|^2 \sum_i (\lambda_i \omega^i)^2 \left(\frac{v^6 \psi'(v^2)^2}{\psi(v^2)} - 3v^4 \psi'(v^2) - 2v^6 \psi''(v^2) \right).
\end{aligned}$$

Then, we note that

$$\frac{v^6 \psi'(v^2)^2}{\psi(v^2)} - 3v^4 \psi'(v^2) - 2v^6 \psi''(v^2) = -\frac{2v^4 + \gamma v^6}{(1-\gamma v^2)^3} \leqslant 0$$

and

$$\psi(v^2) - v^2 \psi'(v^2) = -\gamma \psi^2(v^2) \leqslant 0.$$

So,

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta^S\right) \frac{\phi^2 |A|^2 \psi(v^2)}{2} &\leqslant \phi^2 \psi(v^2)(-\gamma |A|^4 - 2k_\varepsilon \sum_i \lambda_i^3 - 2\big\langle A\,,\,(\nabla^S)^2 k_\varepsilon \big\rangle) \\
&\quad - \phi^2 |A|^2 v^3 \psi'(v^2)\,\big(\nabla^S k_\varepsilon\,,\,\omega\big) - \phi^2 |A|^2 \sum_i (\lambda_i \omega^i)^2 \frac{2v^4 + \gamma v^6}{(1-\gamma v^2)^3},
\end{aligned}$$

what was expected. □

We now show that M_t can be locally written as a Lipschitz graph, with Lipschitz constant independent of ε.

PROPOSITION 5 Let $\varepsilon \leqslant r$. Then, for every $t \in [0, \min(T_\varepsilon, T_1))$, $M_t \cap B_i$ can be written as a Lipschitz graph over $\omega_i^\perp$, with Lipschitz constant independent of ε.

Proof. We want to show that the quantity $(\nu\,,\,\omega_i)$ is bounded from below, or, equivalently, that $v := (\nu\,,\,\omega_i)^{-1}$ is bounded from above on every ball B_i. We want to estimate the quantity $v^2\phi^2$ (we drop the explicit dependence on the index i) using Lemma 4.

We choose Λ such that the last term in (22) is nonpositive (take for instance $\Lambda = 2NR$). We also have to control

$$v\,\big(\nabla^S k_\varepsilon\,,\,\omega\big) = (\nu\,,\,\omega)^{-1}\,\big((\nabla k_\varepsilon\,,\,\omega) - (\nabla k_\varepsilon\,,\,\nu)\,(\nu\,,\,\omega)\big) = (\nu\,,\,\omega)^{-1}\,(\nabla k_\varepsilon\,,\,\omega) - (\nabla k_\varepsilon\,,\,\nu)\,.$$

Proposition 4 provides immediately

$$(\nu\,,\,\omega)^{-1}\,(\nabla k_\varepsilon\,,\,\omega) - (\nabla k_\varepsilon\,,\,\nu) \geqslant (\nu\,,\,\omega)^{-1}\,\frac{|\nabla k_\varepsilon|}{2} - |\nabla k_\varepsilon|$$

which is nonnegative as soon as $(\omega\,,\,\nu) \leqslant \frac{1}{2}$. From Lemma 4 and the weak maximum principle (see [22]), we obtain that $\|v^2\phi^2\|_\infty(t) \leqslant \max(\|v^2\phi^2\|_\infty(0), 4R^2)$. Thanks to Lemma 3, this provides a uniform Lipschitz bound on the whole M_t, for $t \leqslant T_1$. □

Recalling Theorem 8.1 in [15], from Proposition 5 it follows that, if $T_\varepsilon < T_1$, the second fundamental form of M_t blows up as $t \to T_\varepsilon$. Let us show that it does not happen.

PROPOSITION 6 For every $\varepsilon \leqslant r$, there exists $C_\varepsilon > 0$ such that

$$\|A\|_{L^\infty(M_t)} \leqslant C_\varepsilon \qquad \text{for all } t \in [0, \min(T_\varepsilon, T_1)).$$

Proof. As in [9], we are interested in the evolution of the quantity

$$\frac{\phi^2|A|^2\psi(v^2)}{2}$$

and use the estimates of Lemma 5. Notice that

$$|\lambda_i|^3 = |\lambda_i||\lambda_i|^2 \leqslant \frac{1}{2\alpha}\lambda_i^4 + \frac{\alpha}{2}\lambda_i^2.$$

Choosing α such that $\frac{2N}{\alpha} \leqslant \frac{\gamma}{2}$, one can write

$$\left|-2k_\varepsilon\phi^2\psi(v^2)\sum_i \lambda_i^3\right| \leqslant \phi^2\psi(v^2)\left(\frac{\gamma}{2}|A|^4 + N\alpha|A|^2\right).$$

In addition, as soon as $|A|^2 \geqslant 1$, one has $\langle A\,,\, (\nabla^S)^2 k_\varepsilon\rangle \leqslant |A|^2|\nabla^2 k_\varepsilon|$. One can also notice that as above, $v\left(\nabla^S k_\varepsilon\,,\,\omega\right) \geqslant 0$ as soon as $v \geqslant 2$. On the other hand, if $v \leqslant 2$, one has $v^3\psi'(v^2) = \frac{\psi(v)v}{1-\gamma v^2} \leqslant 4\psi(v)$ for γ sufficiently small.

So, anyway, if $|A| \geqslant 1$,

$$\left(\frac{d}{dt} - \Delta^S\right)\frac{\phi^2|A|^2\psi(v^2)}{2} \leqslant 2N\alpha\frac{\phi^2|A|^2\psi(v^2)}{2} + 4|\nabla^2 k_\varepsilon|\frac{\phi^2|A|^2\psi(v^2)}{2} + 8\frac{\phi^2|A|^2\psi(v^2)}{2}|\nabla^S k_\varepsilon|.$$

Finally, we apply the maximum principle to the quantity

$$\tilde{A} := e^{-(2N\alpha + 4\|\nabla^2 k_\varepsilon\|_\infty + 8\|\nabla k_\varepsilon\|_\infty)t}\,\frac{\phi^2|A|^2\psi(v^2)}{2}$$

which satisfies

$$\left(\frac{d}{dt} - \Delta^S\right)\tilde{A} \leqslant 0.$$

It provides

$$\forall t \leqslant \min(T_\varepsilon, T_1), \quad \|\tilde{A}\|_\infty(t) \leqslant \|\tilde{A}\|_\infty(0)$$

which shows that $\frac{\phi^2|A|^2\psi(v^2)}{2}$ does not blow up.

Using Lemma 3 and choosing γ such that $\psi(v^2)$ is bounded and remains far from zero, we know that $|A|$ does not blow up for $t \leqslant T_1$. □

COROLLARY 1 There exists T_1, depending only on the dimension, $\|k\|_\infty$ and the radius in the ball condition for M_0, such that there exists a solution M_t^ε of the mean curvature flow with forcing term k_ε on $[0, T_1)$.

The surfaces M_t^ε are uniformly Lipschitz and every $M_t^\varepsilon \cap B_i$ can be written as the graph of some function $u_i^\varepsilon(x,t)$. All the u_i^ε are Lipchitz (in space) with a constant which depends neither on i nor in ε. We want to show that they are also equicontinuous in time.

PROPOSITION 7 The functions u_i^ε are Lipschitz continuous in x and $1/2$-Hölder continuous in t on $B_i \times [0, T_1)$, uniformly with respect to ε and i.
In addition, they are (classical) solutions of the equation

$$\partial_t u_i^\varepsilon = \sqrt{1+|\nabla u_i^\varepsilon|^2}\,\mathrm{div}\left(\frac{\nabla u_i^\varepsilon}{\sqrt{1+|\nabla u_i^\varepsilon|^2}}\right) - \sqrt{1+|\nabla u_i^\varepsilon|^2}\,k_\varepsilon(x, u_i^\varepsilon). \tag{26}$$

Proof. Let δ be fixed (we drop the index ε in what follows), and let $t_0 \in [0, T_1)$. Let $x_0 \in M_t$ and i such that $x_0 \in B_i$. Then, $(\nu(x_0)\,,\,\omega_i)^{-1}$ is bounded above and M_t is the graph of a function u over $\omega_i^\perp$. Then, let $x_1 = x_0 + \delta\omega_i$. Thanks to the Lipschitz condition, there is a ball $B_{1/C\delta}(x_1)$ that does not touch M_t. Evolving by mean curvature with forcing term k_ε, this ball vanishes in a positive time $T_\delta \geqslant \omega(\delta) := \frac{\delta^2}{C^2(2d+1)}$ (note that T_δ does not depend on ε). By comparison principle, for $t \in [t_0, t_0 + \omega(\delta))$, M_t does not go beyond x_1. That is equivalent to say that u is $1/2$-Hölder continuous in time, with a constant independent of ε.

The equation satisfied by u_i^ε is usual. One just has to notice that with the definitions above,

$$\mathrm{div}\left(\frac{\nabla u_i^\varepsilon}{\sqrt{1+|\nabla u_i^\varepsilon|^2}}\right) = -H.$$

□

We now pass to the limit as ε goes to zero. By Proposition 7, the family (u_i^ε) is equi-Lipschitz in space and equi-continuous in time on $B_i \times [0, T_1)$. Therefore, by Arzelà–Ascoli's Theorem one can find a sequence $\varepsilon_n \to 0$ and continuous functions u_i such that, for every i, $u_i^{\varepsilon_n} \underset{n\to\infty}{\longrightarrow} u_i$ locally uniformly on $B_i \times [0, T_1)$.

PROPOSITION 8 The functions u_i are viscosity solutions of (3) on $B_i \times [0, T_1)$, with obstacles $U \cap B_i$ (see Appendix 5).

Proof. Thanks to Proposition 4, every $x \in B_i$ can be decomposed as $x = x' + z\omega_i$ with $z = (x\,,\,\omega_i)$. Then, there exists functions $\psi_i^\pm$ of class $C^{1,1}$ such that

$$U \cap B_i = \{(x', z) \in B_i \;:\; \psi_i^-(x') \leqslant z \leqslant \psi_i^+(x')\}.$$

For simplicity we shall drop the explicit dependence on the index i. Since $u^\varepsilon(x,0) = u_0(x)$ for all ε, and u^{ε_n} converges uniformly to u as $n \to +\infty$, it is clear that $u(x,0) = u_0(x)$.

Condition (A3) immediately follows from Proposition 3.

We now check that u is a subsolution of (3). Let $(x_0, t_0) \in \mathbb{R}^d \times \mathbb{R}$ and $\varphi \in C^2$ such that $\psi^-(x_0, t_0) < u(x_0, t_0)$ and

$$(u-\varphi)(x_0, t_0) = \max_{|(x,t)-(x_0,t_0)|\leqslant r} (u-\varphi)(x,t).$$

One can change φ so that (x_0, t_0) is a strict maximum point, and $u(x_0, t_0) = \varphi(x_0, t_0)$. Let $2\delta := u(x_0, t_0) - \psi^-(x_0, t_0)$. Thanks to the definition of k_ε, for all $\varepsilon \leqslant \delta$, we have $k_\varepsilon(x, \varphi(x,t)) \geqslant 0$ in a small neighborhood V of (x_0, t_0). Hence, for ε sufficiently small $u^\varepsilon - \varphi$ attains its maximum in V at $(x_\varepsilon, t_\varepsilon)$, with $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$ as $\varepsilon \to 0$. Since u^ε is a classical solution of (26), it is also a viscosity solution, therefore

$$\varphi_t - \sqrt{1+|\nabla\varphi|^2}\,\mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right) \leqslant -\sqrt{1+|\nabla\varphi|^2}\,k_\varepsilon(x,\varphi) \leqslant 0 \qquad \text{at}(x_\varepsilon, t_\varepsilon).$$

Letting $\varepsilon \to 0$ we obtain that u is a subsolution of (3). A similar argument shows that u is also a supersolution of (3), and this concludes the proof. □

Conclusion of the proof of Theorem 1. The result in [21, Theorem 4.1] (see also Section A.4) applies, showing that the functions u_i are of class $C^{1,1}$. As the uniform convergence $u_i^{\varepsilon_n}$ implies the Hausdorff convergence of $M_t^{\varepsilon_n}$ to a limit M_t such that $M_t \cap B_i = \mathrm{graph}(u_i(t))$, we built a $C^{1,1}$ evolution to the mean curvature motion with obstacles on the time interval $[0, T_1)$. Thanks to [1, Theorem 4.8 and Corollary 4.9] this evolution is also unique. This concludes the proof of Theorem 1. □

4. Proof of Theorem 2

Let $\psi_\varepsilon^\pm$ be smooth functions such that $\psi_\varepsilon^\pm \to \psi^\pm$ as $\varepsilon \to 0$, uniformly in $C^{1,1}(\mathbb{R}^d)$, and let $N > 0$ be such that

$$N \geqslant \left\| \sqrt{1+|\psi_\varepsilon^\pm|^2}\,\mathrm{div}\left(\frac{\psi_\varepsilon^\pm}{\sqrt{1+|\psi_\varepsilon^\pm|^2}}\right)\right\|_{L^\infty(\mathbb{R}^d)} \qquad \text{for all } \varepsilon > 0. \tag{27}$$

We proceed as in Section 3 and we approximate (3), (4) with the forced mean curvature equation

$$u_t = \sqrt{1+|\nabla u|^2}\left[\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + k_\varepsilon(x,u)\right], \tag{28}$$

where

$$k_\varepsilon(x,u) = 2N\left(\chi\left(\frac{\psi_\varepsilon^-(x) - u}{\varepsilon}\right) - \chi\left(\frac{u - \psi_\varepsilon^+(x)}{\varepsilon}\right)\right),$$

and χ is a smooth increasing function such that $\chi(s) \equiv 0$ for all $s \in (-\infty, 0]$, and $\chi(s) \equiv 1$ for all $s \in [1, \infty)$. In particular $\partial_u k_\varepsilon(x,u) \leqslant 0$ for all (x,u).

Note that the signs between (28) and (5) are reversed.

Notice that $k_\varepsilon \to k$ as $\varepsilon \to 0$, with

$$k(x,u) = \begin{cases} 2N \text{ if } u < \psi^-(x) \\ -2N \text{ if } u > \psi^+(x) \\ 0 \text{ elsewhere} \end{cases}.$$

We denote by u_ε the solution of the approximate problem (28), which exists and is smooth for short times.

PROPOSITION 9 The solution u_ε is defined for $t \in [0, +\infty)$, and satisfies the estimates

$$\|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\mathbb{R}^d)} \leqslant C \qquad \text{for all } t \in [0, +\infty), \tag{29}$$

$$\|u_\varepsilon(\cdot,t)\|_{W^{2,\infty}(\mathbb{R}^d)} \leqslant C(T) \qquad \text{for all } t \in [0, T]. \tag{30}$$

Proof. Estimate (29) follows from Proposition 5, choosing $B_i = \mathbb{R}^{d+1}$, $\omega_i = e_{d+1}$ and $\phi \equiv 1$. Estimate (30) follows from (29) and Proposition 6. □

In what follows, we use intrinsic derivatives on the graph $M_t := \{(x, u_\varepsilon(x,t))\}$, which will be denoted as above by an exponent S. The metric on M_t is

$$g_{ij} = \delta_{ij} + \partial_i u_\varepsilon \partial_j u$$

with inverse

$$g^{ij} = \delta_{ij} - \frac{\partial_i u_\varepsilon \partial_j u_\varepsilon}{1 + |\nabla u_\varepsilon|^2}.$$

The tangential gradient of a function f defined on M_t is given by

$$(\nabla^S f)^i = g^{ij}\partial_j f = \partial_i f - \frac{\partial_i u_\varepsilon \partial_j u_\varepsilon}{1 + |\nabla u_\varepsilon|^2}\partial_j f\,,$$

so that

$$\left(\nabla^S f\,,\, \nabla u_\varepsilon\right) = (\nabla f\,,\, \nabla u_\varepsilon) - \frac{|\nabla u_\varepsilon|^2}{1 + |\nabla u_\varepsilon|^2}(\nabla f\,,\, \nabla u_\varepsilon) = \frac{1}{1 + |\nabla u_\varepsilon|^2}(\nabla f\,,\, \nabla u_\varepsilon)\,, \tag{31}$$

and

$$\begin{aligned} |\nabla^S f|^2 &= \left(f_i - (u_\varepsilon)_i \sum_j \frac{(u_\varepsilon)_j f_j}{1 + |\nabla u_\varepsilon|^2}\right)^2 \\ &= |\nabla f|^2 + (u_\varepsilon)_i^2 \left(\frac{(\nabla u_\varepsilon\,,\, \nabla f)}{1 + |\nabla u_\varepsilon|^2}\right)^2 - 2\frac{(u_\varepsilon)_i (u_\varepsilon)_j f_i f_j}{1 + |\nabla u_\varepsilon|^2} \\ &= |\nabla f|^2 + \frac{|\nabla u_\varepsilon|^2}{1 + |\nabla u_\varepsilon|^2}\frac{(\nabla u_\varepsilon\,,\, \nabla f)^2}{1 + |\nabla u_\varepsilon|^2} - 2\frac{(\nabla u_\varepsilon\,,\, \nabla f)^2}{1 + |\nabla u_\varepsilon|^2} \\ &= |\nabla f|^2 - \frac{(\nabla u_\varepsilon\,,\, \nabla f)^2}{1 + |\nabla u_\varepsilon|^2} - \frac{(\nabla u_\varepsilon\,,\, \nabla f)^2}{(1 + |\nabla u_\varepsilon|^2)^2}. \end{aligned} \tag{32}$$

In addition, the Laplace-Beltrami operator applied to f is

$$\Delta^S f = g^{ij} f_{ij} = \Delta f - \frac{\partial_i u_\varepsilon \partial_j u_\varepsilon}{1 + |\nabla u_\varepsilon|^2} f_{ij} = \Delta f - \frac{\left(\nabla u_\varepsilon \nabla^2 f\,,\, \nabla u_\varepsilon\right)}{1 + |\nabla u_\varepsilon|^2}.$$

PROPOSITION 10 The quantity $\|(u_\varepsilon)_t^2\|_\infty(t)$ is nonincreasing in time. In particular,

$$\|(u_\varepsilon)_t(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leqslant \left\| \sqrt{1 + |\nabla u_0|^2}\, \mathrm{div}\left(\frac{\nabla u_0}{\sqrt{1 + |\nabla u_0|^2}}\right)\right\|_{L^\infty(\mathbb{R}^d)}.$$

Proof. We compute

$$\frac{d}{dt}\frac{(u_\varepsilon)_t^2}{2} = (u_\varepsilon)_t \left[\sqrt{1 + |\nabla u_\varepsilon|^2}\left(\mathrm{div}\left(\frac{\nabla u_\varepsilon}{\sqrt{1 + |\nabla u_\varepsilon|^2}}\right) + k_\varepsilon\right)\right]_t.$$

Expanding this expression, we get

$$\frac{d}{dt}\frac{(u_\varepsilon)_t^2}{2} = (u_\varepsilon)_t\left[\frac{(\nabla(u_\varepsilon)_t,\nabla u_\varepsilon)}{\sqrt{1+|\nabla u_\varepsilon|^2}}\left(\operatorname{div}\left(\frac{\nabla u_\varepsilon}{\sqrt{1+|\nabla u_\varepsilon|^2}}\right)+k_\varepsilon\right)\right.$$
$$\left.+\sqrt{1+|\nabla u_\varepsilon|^2}\left(\operatorname{div}\left(\frac{(\nabla u_\varepsilon)_t}{\sqrt{1+|\nabla u_\varepsilon|^2}}-\frac{((\nabla u_\varepsilon)_t,\nabla u_\varepsilon)\nabla u_\varepsilon}{(1+|\nabla u_\varepsilon|^2)^{3/2}}\right)+(u_\varepsilon)_t\,\partial_u k_\varepsilon\right)\right].$$

Let us compute more explicitly the three terms of the expression above:

$$(u_\varepsilon)_t\frac{((\nabla u_\varepsilon)_t,\nabla u_\varepsilon)}{\sqrt{1+|\nabla u_\varepsilon|^2}}\left(\operatorname{div}\left(\frac{\nabla u_\varepsilon}{\sqrt{1+|\nabla u_\varepsilon|^2}}\right)+k_\varepsilon\right)$$
$$=\frac{\left(\nabla(\frac{(u_\varepsilon)_t^2}{2}),\nabla u_\varepsilon\right)}{\sqrt{1+|\nabla u_\varepsilon|^2}}\left(\frac{\Delta u}{\sqrt{1+|\nabla u_\varepsilon|^2}}-\frac{(u_\varepsilon)_i\,(\nabla u_\varepsilon\,,\,(\nabla u_\varepsilon)_i)}{(1+|\nabla u_\varepsilon|^2)^{3/2}}+k_\varepsilon\right)$$
$$=\left(\nabla(\frac{(u_\varepsilon)_t^2}{2}),\nabla u_\varepsilon\right)\left(\frac{\Delta u_\varepsilon}{1+|\nabla u_\varepsilon|^2}-\frac{\left(\nabla u_\varepsilon,\nabla(\frac{|\nabla u_\varepsilon|^2}{2})\right)}{(1+|\nabla u_\varepsilon|^2)^2}+k_\varepsilon\right),$$

$$(u_\varepsilon)_t\operatorname{div}\left(\frac{\nabla(u_\varepsilon)_t}{\sqrt{1+|\nabla u_\varepsilon|^2}}\right)=(u_\varepsilon)_t\partial_i\left(\frac{(u_\varepsilon)_{ti}}{\sqrt{1+|\nabla u_\varepsilon|^2}}\right)$$
$$=\frac{(u_\varepsilon)_t(u_\varepsilon)_{tii}}{\sqrt{1+|\nabla u_\varepsilon|^2}}-\frac{1}{(1+|\nabla u_\varepsilon|^2)^{3/2}}(u_\varepsilon)_t(u_\varepsilon)_{ti}(\nabla u_\varepsilon,(\nabla u_\varepsilon)_i)$$
$$=\frac{(u_\varepsilon)_t\Delta(u_\varepsilon)_t}{\sqrt{1+\nabla u_\varepsilon^2}}-\frac{1}{(1+|\nabla u_\varepsilon|^2)^{3/2}}(u_\varepsilon)_t(u_\varepsilon)_{ti}\partial_i(\frac{|\nabla u_\varepsilon|^2}{2})$$
$$=\frac{(u_\varepsilon)_t\Delta(u_\varepsilon)_t}{\sqrt{1+\nabla u_\varepsilon^2}}-\frac{1}{(1+|\nabla u_\varepsilon|^2)^{3/2}}\left(\nabla\left(\frac{(u_\varepsilon)_t^2}{2}\right),\nabla\left(\frac{|\nabla u_\varepsilon|^2}{2}\right)\right)$$

and

$$(u_\varepsilon)_t\operatorname{div}\left(\frac{((\nabla u_\varepsilon)_t,\nabla u_\varepsilon)\nabla u_\varepsilon}{(1+|\nabla u_\varepsilon|^2)^{3/2}}\right)$$
$$=\Delta u_\varepsilon\frac{(\nabla u_\varepsilon\,,\,(u_\varepsilon)_t\nabla(u_\varepsilon)_t)}{(1+|\nabla u_\varepsilon|^2)^{3/2}}+\frac{(u_\varepsilon)_t(u_\varepsilon)_{tij}(u_\varepsilon)_j(u_\varepsilon)_i}{(1+|\nabla u_\varepsilon|^2)^{3/2}}+\frac{((u_\varepsilon)_i\nabla(u_\varepsilon)_i\,,\,(u_\varepsilon)_t\nabla(u_\varepsilon)_t)}{(1+|\nabla u_\varepsilon|^2)^{3/2}}$$
$$-3(u_\varepsilon)_i\frac{((u_\varepsilon)_t\nabla(u_\varepsilon)_t\,,\,\nabla u_\varepsilon)\,(\nabla(u_\varepsilon)_i\,,\,\nabla u_\varepsilon)}{(1+|\nabla u_\varepsilon|^2)^{5/2}}$$
$$=\Delta u_\varepsilon\frac{\left(\nabla u_\varepsilon\,,\,\nabla(\frac{(u_\varepsilon)_t^2}{2})\right)}{(1+|\nabla u_\varepsilon|^2)^{3/2}}+\frac{(u_\varepsilon)_t(u_\varepsilon)_{tij}(u_\varepsilon)_j(u_\varepsilon)_i}{(1+|\nabla u_\varepsilon|^2)^{3/2}}+\frac{\left(\nabla(\frac{|\nabla u_\varepsilon|^2}{2})\,,\,\nabla(\frac{(u_\varepsilon)_t^2}{2})\right)}{(1+|\nabla u_\varepsilon|^2)^{3/2}}$$
$$-3\frac{\left(\nabla(\frac{(u_\varepsilon)_t^2}{2})\,,\,\nabla u_\varepsilon\right)\left(\nabla(\frac{|\nabla u_\varepsilon|^2}{2})\,,\,\nabla u_\varepsilon\right)}{(1+|\nabla u_\varepsilon|^2)^{5/2}}.$$

Notice that

$$\Delta^S \frac{(u_\varepsilon)_t^2}{2} = \Delta \frac{(u_\varepsilon)_t^2}{2} - \frac{\left(\nabla u_\varepsilon\,,\, \nabla^2 \frac{(u_\varepsilon)_t^2}{2} \nabla u_\varepsilon\right)}{1+|\nabla u_\varepsilon|^2}$$

$$= (u_\varepsilon)_t \Delta (u_\varepsilon)_t + |(\nabla u_\varepsilon)_t|^2 - \frac{(u_\varepsilon)_i (u_\varepsilon)_j (u_\varepsilon)_t (u_\varepsilon)_{tij} + (u_\varepsilon)_i (u_\varepsilon)_j (u_\varepsilon)_{ti} (u_\varepsilon)_{tj}}{1+|\nabla u_\varepsilon|^2}.$$

We then get

$$\frac{d}{dt}\frac{(u_\varepsilon)_t^2}{2} = \frac{\left(\nabla (\frac{(u_\varepsilon)_t^2}{2})\,,\, \nabla u_\varepsilon\right)}{\sqrt{1+|\nabla u_\varepsilon|^2}} k_\varepsilon + \Delta^S \left(\frac{(u_\varepsilon)_t^2}{2}\right) - 2\frac{\left(\nabla \left(\frac{(u_\varepsilon)_t^2}{2}\right)\,,\, \nabla \left(\frac{|\nabla u_\varepsilon|^2}{2}\right)\right)}{1+|\nabla u_\varepsilon|^2}$$

$$+ 2\frac{\left(\nabla (\frac{(u_\varepsilon)_t^2}{2})\,,\, \nabla u_\varepsilon\right)\left(\nabla (\frac{|\nabla u_\varepsilon|^2}{2})\,,\, \nabla u_\varepsilon\right)}{(1+|\nabla u_\varepsilon|^2)^2} + \frac{(\nabla u_\varepsilon\,,\,(\nabla u_\varepsilon)_t)^2}{1+|\nabla u_\varepsilon|^2} - |(\nabla u_\varepsilon)_t|^2 + (u_\varepsilon)_t^2 \partial_u k_\varepsilon.$$

Note that the last term is nonpositive by definition of k_ε.

In order to apply Lemma 1, we have to show the inequality

$$-\frac{(\nabla u_\varepsilon\,,\,(\nabla u_\varepsilon)_t)^2}{1+|\nabla u_\varepsilon|^2} + |(\nabla u_\varepsilon)_t|^2 \geqslant 0.$$

It is enough to note that, since the solution exists for all times and it is smooth, the term $\nabla(\frac{|\nabla u_\varepsilon|^2}{2})$ is bounded on each $[0,T]$ (the bound depends on T and ε but is enough to apply the lemma). In addition, every factor containing $\nabla((u_\varepsilon)_t^2/2)$ also contains ∇u_ε, hence the assumptions of Lemma 1 are satisfied for every $T>0$, and this concludes the proof. □

From Propositions 9 and 10, we deduce the following result.

PROPOSITION 11 If u_0 is C-Lipschitz in space for some $C>0$, and has bounded mean curvature, then the solution u_ε of the approximate problem (28) is C-Lipschitz in space and Lipschitz in time with constant

$$\left\| \sqrt{1+|\nabla u_0|^2}\, \operatorname{div}\left(\frac{\nabla u_0}{\sqrt{1+|\nabla u_0|^2}}\right)\right\|_{L^\infty(\mathbb{R}^d)}.$$

Moreover, the following inequalities hold

$$\psi_\varepsilon^-(x) - \varepsilon \leqslant u_\varepsilon(x,t) \leqslant \psi_\varepsilon^+(x) + \varepsilon. \tag{33}$$

Proof. The Lipschitz bounds of the solution are clear (it is Proposition 9 and 10).

In order to prove the second assertion, let us notice that by (27) and the definition of k_ε, we have

$$k_\varepsilon(x, \psi_\varepsilon^- - \varepsilon) = 2N \geqslant \left\| \sqrt{1+|\psi_\varepsilon^-|^2}\, \operatorname{div}\left(\frac{\psi_\varepsilon^-}{\sqrt{1+|\psi_\varepsilon^-|^2}}\right)\right\|_{L^\infty(\mathbb{R}^d)},$$

so that $\psi_\varepsilon^- - \varepsilon$ is a subsolution of (28). By the parabolic comparison principle (as in Proposition 3), we deduce that

$$\psi_\varepsilon^- - \varepsilon \leqslant u_\varepsilon.$$

The same argument shows the other inequality in (33). □

Conclusion of the proof of Theorem 2. Since the solutions u_ε are equi-Lipschitz in space and time, they converge uniformly, as $\varepsilon \to 0$, to a limit function u which is also Lipschitz continuous on $\mathbb{R}^d \times [0, +\infty)$.

Equation (33) yields

$$\psi^- \leqslant u \leqslant \psi^+,$$

and Proposition 8 gives that u is a viscosity solution of (A1).

Concerning the regularity of u, we proved that $(u_\varepsilon)_t$ and ∇u_ε are bounded on $[0, T]$, for any T in the approximate problem. This gives a bound on the mean curvature of the approximate solution. This bound does not depend on ε and remains true for the viscosity solution. As a result, the exact solution has bounded mean curvature and bounded gradient, which shows that Δu is L^∞ and, by elliptic regularity theory, u is also in $W^{2,p}$ for any $p > 1$, and so $C^{1,\alpha}$ for every $\alpha < 1$ (see [19] for details).

We can also directly apply to the solution u a regularity result by Shahgholian (see [21, 23] and Theorem 4 below), which implies that u is in fact of class $\mathcal{C}^{1,1}$. This concludes the proof of Theorem 2. □

5. Proof of Theorem 3

We compute the evolution of the area of the graph of u:

$$\frac{d}{dt}\int_Q \sqrt{1+|\nabla u|^2} = \int_Q \frac{(\nabla u_t\,,\, \nabla u)}{\sqrt{1+|\nabla u|^2}} = -\int_Q u_t \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{34}$$

Notice that, for almost every $t > 0$, $u_t(t, x) = 0$ almost everywhere on the contact set. Indeed, for almost every t, u_t exists for almost every $x \in Q$. If $u(x, t) = \psi^\pm(x)$, then $u - \psi^\pm$ reaches an extremum in (x, t), which gives, $u_t(x, t) = 0$. In particular, from (34) we get

$$\frac{d}{dt}\int_Q \sqrt{1+|\nabla u|^2} = -\int_Q u_t \left(\frac{u_t}{\sqrt{1+|\nabla u|^2}}\right).$$

Integrating this equality in time, we obtain

$$\int_Q \sqrt{1+|\nabla u|^2}\,\bigg|_0^T = \int_0^T \int_Q -\frac{u_t^2}{\sqrt{1+|\nabla u|^2}}.$$

which shows that

$$\int_0^T \int_Q u_t^2$$

is uniformly bounded in T. As a result $u_t \in L^2(\mathbb{R}^+, Q)$ so u is in $H^1(Q, B_R)$.

Since $\|u_t\|_{L^2(Q)}$ is $L^2(\mathbb{R}^+)$, there exists a sequence $t_n \to \infty$ such that

$$\|u_t\|_{L^2(Q)}(t_n) \underset{n\to\infty}{\longrightarrow} 0.$$

In addition, $u(t_n)$ is equi Lipschitz and converges uniformly on compact sets to some u_∞ which therefore satisfies in the viscosity sense

$$\sqrt{1+|\nabla u|^2} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+\nabla u^2}}\right) = 0$$

with obstacles $\psi^{\pm}$ (see Appendix 5). □

REMARK By [17], $u_{\min}$ is analytic out of the (closed) contact set $\{u_{\min} = \psi^{\pm}\}$.

Appendix A. Viscosity solutions with obstacles

A.1 *Definition of viscosity solution*

Given an open subset B of $\mathbb{R}^d$, let u_0, ψ^+ and ψ^- be three Lipschitz functions $B \to \mathbb{R}$ such that

$$\psi^-(x,0) \leqslant u_0(x) \leqslant \psi^+(x,0).$$

We are interested in the viscosity solutions of the equation

$$u_t = \sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right), \qquad u(x,0) = u_0(x), \tag{A1}$$

with the constraint

$$\psi^-(x) \leqslant u(x,t) \leqslant \psi^+(x). \tag{A2}$$

DEFINITION 1 (see [5, 20]) We say that a function $u : B \times [0,T) \to \mathbb{R}$ is a *viscosity subsolution* of (A1) if u satisfies the following conditions:

- u is upper semicontinuous;
- $u(x,0) \leqslant u_0(x)$;
- $$\psi^-(x) \leqslant u(x,t) \leqslant \psi^+(x); \tag{A3}$$
- for any $(x_0,t_0) \in \mathbb{R}^d \times \mathbb{R}^+$ and $\varphi \in C^2$ such that $u-\varphi$ has a maximum at (x_0,t_0) and $u(x_0,t_0) > \psi^-(x_0)$,
$$u_t \leqslant \sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{A4}$$

Similarly, u is a *viscosity supersolution* of (A1) if:

- u is lower semicontinuous;
- $u(x,0) \geqslant u_0(x)$;
- (A3) holds;
- for any $(x_0,t_0) \in \mathbb{R}^d \times \mathbb{R}^+$ and $\varphi \in C^2$ such that $u-\varphi$ has a minimum at (x_0,t_0) and $u(x_0,t_0) < \psi^+(x_0)$,
$$u_t \geqslant \sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

We say that u is a *viscosity solution* of (A1) if it is both a super and a subsolution.

A.2 *Comparison principle*

In order to prove uniqueness of continuous viscosity solutions of (A1), we shall prove a comparison principle between solutions following [13, Theorem 4] (see also [4]).

PROPOSITION 12 If u is a viscosity subsolution of (A1) on $[0, T)$, v is a viscosity supersolution, if $\psi^{\pm}$ are Lipschitz in space and if $u(x, 0) \leqslant v(x, 0)$, then $u(x, t) \leqslant v(x, t)$ for all $(x, t) \in \mathbb{R}^n \times [0, T)$.

Proof. We will check that the proof of [13, Theorem 2.1] can be extended to the obstacle case. Notice first that the assumptions $(A.1) - (A.3)$ of [13, Theorem 2.1] are satisfied also in our case. Indeed, $(A.1)$ comes directly from the Lipschitz bound on $\psi^{\pm}$ and the constraint $\psi^- \leqslant u, v \leqslant \psi^+$ whereas $(A.2)$ and $(A.3)$ result from the assumed time zero comparison.

Let us show that [13, Proposition 2.3] also holds. Indeed, up to Equation (2.9) nothing changes. To continue the proof, using the same notation of [13, Proposition 2.3], we have to check that if

$$\sup_V (w - \Psi) > 0,$$

then the supremum is reached in the complementary of the contact set $\{u = \psi^-\} \cup \{v = \psi^+\}$.

Indeed, notice that if $u(x, t) = \psi^-(x)$, then, for all x, y, t, s,

$$u(x, t) - v(y, s) = \psi^-(x) - v(y, s) \leqslant \psi^-(y) + L(|x - y|) - v(y, s) \leqslant L(|x - y|)$$

since $v \geqslant \psi^-$. Hence, if $u(x, t) = \psi^-(x)$, with $K' > L$, we must have $w - \Psi \leqslant 0$, so the supremum of $w - \Psi$ is attained in the complementary of $\{u = \psi^-\}$. One can show similarly that the supremum is reached in the complementary of $\{v = \psi^+\}$. Hence Proposition 2.3 of [13] holds.

From Proposition 2.4 to Lemma 2.7 of [13], every result holds without changes.

Concerning the proof of Theorem 2.1 of [13], the first assumption is

$$\alpha = \limsup_{\theta \to 0} \{w(t, x, y), \mid |x - y| \leqslant \theta\} > 0.$$

Then, Proposition 2.4 gives constants δ_0 and γ_0 such that for all $\delta \leqslant \delta_0$, $\gamma \leqslant \gamma_0$ and $\varepsilon > 0$, there holds

$$\Phi(\hat{x}, \hat{y}, \hat{t}) := \sup_{\mathbb{R}^d \times \mathbb{R}^d \times [0,T)} \Phi(x, y, t) > \frac{\alpha}{2}$$

with

$$\Phi(t, x, y) = u(x, t) - v(y, t) - \frac{|x - y|^4}{4\varepsilon} - \delta(|x|^2 + |y|^2) - \frac{\gamma}{T - t}$$

To conclude the proof, we only have to show that the maximum of Φ is once again attained on the complementary of $\{u = \psi^-\} \cup \{v = \psi^+\}$. In the same way as for Proposition 2.3, if $u(x, t) = \psi^-(x)$, we can write

$$\begin{aligned} \Phi(t, x, y) &= u(x, t) - v(y, t) - \frac{|x - y|^4}{4\varepsilon} - \delta(|x|^2 + |y|^2) - \frac{\gamma}{T - t} \\ &\leqslant \psi^-(y) + L|x - y| - v(y, t) \leqslant L|x - y|. \end{aligned}$$

Thanks to Proposition 2.5, $|\hat{x} - \hat{y}| \underset{\varepsilon \to 0}{\longrightarrow} 0$. So, with ε sufficiently small (one can reduce the quantity ε_0 given by Proposition 2.6), Φ has its maximum out of $\{u = \psi^-\}$ (and similarly out of $\{v = \psi^+\}$), which enables the application of Lemma 2.7 and gives a contradiction as in [13]. □

A.3 *Existence*

In this subsection, we prove the following result:

PROPOSITION 13 There exists a continuous viscosity solution to (A1).

We follow [5] to build a solution by means of the Perron's method. Let us state an obvious but useful proposition and a key lemma for applying Perron's method.

PROPOSITION 14 Let u be a subsolution of the mean curvature motion for graphs (without obstacles) which satisfies $u \leqslant u^+$. Then, $u_{ob} := u \vee u^-$ is a subsolution of (A1) with obstacles (the same happens for v supersolution and $v_{ob} = v \wedge u^+$).

In the sequel, we shall denote by u^* (resp. u_*) the upper (resp. lower) semicontinuous envelope of a function u.

LEMMA 6 Let $\mathfrak{F}$ be a family of subsolutions of (A1). We define

$$U(x,t) = \sup \{u(x,t) \mid u \in \mathfrak{F}\}.$$

Then, U^* is a subsolution of (A1).

The proof of the proposition and the lemma can be found in [5], Lemma 4.2 (with obvious changes due to the parabolic situation and obstacles).

A.3.1 Construction of barriers.. In the sequel, to claim that the initial condition is taken by the viscosity solution, we need to build barriers around the solution u. More precisely, we want to build a subsolution w^- such that $(w^-)^*(x,0) = u_0(x)$ and a supersolution w^+ such that $(w^+)_*(x,0) = u_0(x)$. To show this claim, let us begin by a simple fact.

Let

$$g^a_{\alpha,b}(x) = -\sum \alpha_i \frac{(x-a)_i^2}{\sqrt{1+(x-a)_i^2}} + b \tag{A5}$$

for some $(a,b) \in \mathbb{R}^d \times \mathbb{R}$ and $\alpha_i \geqslant 0$ such that $g(x) \leqslant u_0(x)$. Note in particular that

$$g^a_{\alpha,b}(x) \geqslant -\sum \alpha_i (x-a)_i^2 + b \quad \text{and} \quad H(g^a_{\alpha,b}) \geqslant H(g^a_{\alpha,b})|_{t=0} = -2\sum \alpha_i. \tag{A6}$$

Then, it is easy to show (using Proposition 14) that the function

$$v(x,t) = \left(g^a_{\alpha,b}(x) + \Big(2\sum_{i=1}^n \alpha_i + 3M\Big)t\right) \vee \psi^-$$

is a subsolution of (A1). Indeed, the curvature of $g^a_{\alpha,b}$ is smaller than $2\sum_i \alpha_i$ and its gradient is bounded by 2 (so $\sqrt{1+|\nabla g|^2} \leqslant 3$).

Thanks to Lemma 6, the function

$$w^-(x,t) = \left(\sup_{\substack{(\alpha_i),c \\ g^c_{\alpha,b} \leqslant u_0}} \Big(g^a_{\alpha,b}(x) - 2\sum_{i=1}^n \alpha_i t - 3Mt\Big) \vee \psi^-\right)^*$$

is a subsolution of (A1) (with obstacles).

It remains to show that $(w^-)^*(x,0) = u_0(x)$. To see this, notice that since u_0 is Lipschitz and $u_0 \geqslant \psi^-$, $u_0(x) = w^-(x,0)$, yielding $u_0(x) \leqslant (w^-)^*(x,0)$. But for all $t \geqslant 0$, $v(x,t) \leqslant u_0(x)$ so $w^-(x,t) \leqslant u_0(x)$. By continuity of u_0, $(w^-)^*(x,t) \leqslant u_0(x)$, which shows that $(w^-)^*(x,0) = u_0(x)$, and w^- is a low barrier for solutions of (A1).

We build w^+ in the same way.

A.3.2 Perron's method.. We use the classical Perron's method to build a solution of (A1) on $[0,T)$ for every $t > 0$. Let us define

$$W(x,t) = \sup\{u(x) \,:\, u \text{ is a subsolution of (A1) on } [0,T)\}.$$

Since ψ^- is a subsolution, this set in non empty and W is well defined. Every subsolution is less that ψ^+, so is W.

Thanks to Lemma 6, W^* is a subsolution of (A1) regardless the initial conditions. Applying the comparison principle (Proposition 12) to every subsolution u and w^+ gives

$$\forall x,t,\ W(x,t) \leqslant w^+(x,t).$$

Considering the upper-semi-continuous envelopes, we get

$$\forall x,t,\ W^*(x,t) \leqslant (w^+)^*(x,t)$$

which immediately yields to

$$W^*(x,0) = u_0(x).$$

Then, W^* is a subsolution of (A1), hence $W^* = W$ which shows the upper semi-continuity of W.

We want to prove that W is actually a solution of (A1). In order to do this, let us prove the following

LEMMA 7 Let u be a subsolution of (A1). If u_* fails to be a supersolution (regardless initial conditions) at some point $(\hat{x},\hat{t})$ then there exists a subsolution u_κ (regardless initial conditions) satisfying $u_\kappa \geqslant u$ and $\sup u_\kappa - u > 0$ and such that $u(x,t) = u_\kappa(x,t)$ for $|x-\hat{x}|, |t-\hat{t}| \leqslant \kappa$.

Proof. Let us assume that u_* fails to be a supersolution at $(0,1)$. Then there exists $(a,p,X) \in \mathcal{J}^{2,-}u_*(0,1)$ with

$$a + F(p,X) + k(0)\sqrt{1+p^2} < 0.$$

Let us then define

$$u_{\delta,\gamma}(x,t) = u_*(0,1) + \delta + (p\,,\,x) + a(t-1) + \frac{1}{2}(Xx\,,\,x) - \gamma(|x|^2 + t - 1).$$

Thanks to the continuity of F and k, $u_{\delta,\gamma}$ is a classical subsolution on $B_r(0,1)$ of $u_t + F(Du, D^2u) + k(x)\sqrt{1+|\nabla u|^2} = 0$ for δ, γ, r sufficiently small. By assumption,

$$u(x,t) \geqslant u_*(x,t) \geqslant u_*(0,1) + a(t-1) + (p\,,\,x) + \frac{1}{2}(Xx, x\,,\,+)\,o(|x|^2 + |t-1|).$$

With $\delta = \gamma \frac{r^2+r}{8}$, we get $u(x,t) > u_{\delta,\gamma}(x,t)$ for small r and $|x|, |t-1| \in [\frac{r}{2}, r]$. Reducing again r, we can assume that $u_{\delta,\gamma} < \psi^+$ on B_r. Thanks to Lemma 6,

$$\tilde{u}(x,t) = \begin{cases} \max(u(x,t), u_{\delta,\gamma}(x,t)) \text{ if } |x, t-1| < r \\ u(x) \text{ otherwise} \end{cases}$$

is a subsolution of (A1) (with no initial conditions). □

Finally, this lemma combined with the definition of W proves that W is in fact a solution of (A1) (the initial conditions were already checked).

A.4 *Regularity*

PROPOSITION 15 The unique solution u of (A1) is Lipschitz in space, with the same constant as $u_0, \psi^\pm$.

Proof. We will prove that $u_z(x,t) = u(x+z,t) - L|z|$ is in fact a subsolution of (A1). The Lipschitz bound is then straightforward (using the comparison principle).

To begin, we notice that $u(x+z,t) - L(|z|) \leqslant u^+(x,t)$ and $u(x+z,0) - L|z| \leqslant u_0(x+z) - L|z| \leqslant u_0(x)$.

Assume now that φ is any smooth function which is greater than u_z with equality at $(\hat{x},\hat{t})$. Then, either, $u_z(\hat{x},\hat{t}) = \psi^-(\hat{x},\hat{t})$ and nothing has to be done, or $u_z(\hat{x},\hat{t}) > \psi^-(\hat{x},\hat{t})$. In the second alternative, one can write

$$u(\hat{x}+t,\hat{t}) > \psi^-(\hat{x}) = \psi^-(\hat{x}+z) + (\psi^-(\hat{x}) - \psi^-(\hat{x}+z),$$

so

$$u(\hat{x}+z,\hat{t}) > \psi^-(\hat{x}+z) + \underbrace{\psi^-(\hat{x}) - \psi^-(\hat{x}+z) + L|z|}_{\geqslant 0} \geqslant u^-(\hat{x}+z,\hat{t}).$$

As u is a subsolution at $(\hat{x}+z,\hat{t})$ and $u(x+z,t) \leqslant \varphi(x,t) + L|z|$ with equality at $(\hat{x}+z,\hat{t})$, one can write with $y = x+z$, $s = t$,

$$u(y,t) \leqslant \varphi(y-z,s) + L|z| := \phi(y,s),$$

with equality at $(\hat{y},\hat{s})$ which gives

$$\phi_t + F\big(D\phi(\hat{x},\hat{t}), D^2\phi(\hat{x},\hat{t})\big) \leqslant 0.$$

Since the derivatives of ϕ and φ are the same, we deduce

$$\varphi_t + F(D\varphi, D^2\varphi) \leqslant 0,$$

what was expected. □

REMARK With the same arguments, one can prove that

$$\forall \delta > 0, \quad \forall x,t, \quad |u(x,t+\delta) - u(x,t)| \leqslant \sup_x |u(x,\delta) - u(x,0)|.$$

We now present a general regularity result by Shahgholian [23] which applies to viscosity solutions for parabolic equations with obstacles.

THEOREM 4 ([21], Th. 4.1) Let $Q^+ := \{(x,t) \in \mathbb{R}^d \times \mathbb{R} \,:\, |x| < 1, t \in [0,1)\}$ and $H(u) = F(D^2u, Du) - u_t$ where F is uniformly elliptic. Let u be a continuous viscosity solution of

$$\begin{aligned} (u-\psi)H(u) &= 0, \\ H(u) &\leqslant 0, \\ u &\geqslant \psi, \end{aligned} \tag{A7}$$

in Q^+, with boundary data

$$u(x,t) = g(x,t) \geqslant \psi(x,t) \qquad \text{on } \{|x| = 1\} \cup \{t = 0\}. \tag{A8}$$

Assume that $\psi \in C^{1,1}(Q^+)$ and g is continuous. Then, $u \in C^{1,1}$ on every compact subset of Q^+.

It has to be noticed $H = F - \partial_t$ where $F(D^2u, Du) = -\sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$ satisfies all the assumptions of [23], 1.3. Indeed, the uniform ellipticity is provided by the Lipschitz bound obtained in previous subsection.

Moreover, the viscosity solution u of (A1) satisfies (A7) and (A8) on every cylinder $Q_r^+(x_0) := \{|x - x_0| \leqslant r,\ t \in [t_0, t_0 + r)\}$ such that r is chosen sufficiently small in order to have either $Q_r^+(x_0) \cap \{u = \psi^+\} = \emptyset$ or $Q_r^+(x_0) \cap \{u = \psi^-\} = \emptyset$. In the second alternative, change every sign in the equations.

Applying Theorem 4 we get a $C^{1,1}$ bound for u on every compact subset of $Q_r^+(x_0)$. To show that u is $C^{1,1}$ in the whole space, just cover $\mathbb{R}^d \times \mathbb{R}^+$ with such $Q_r^+(x_i)$.

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