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The porous media equation in an infinite cylinder, between two infinite parallel plates, and like spatial domains

BRIAN H. GILDING
Department of Mathematics, College of Science, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait
E-mail: gilding@sci.kuniv.edu.kw

JAN GONCERZEWICZ
Faculty of Pure and Applied Mathematics, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50 – 370 Wrocław, Poland
E-mail: jan.goncerzewicz@pwr.edu.pl

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The porous media equation has played a prominent role in the current development of the mathematical theory of interfaces and free boundaries. One occurs whenever the equation is solved in an unbounded spatial domain with initial data that have bounded support, and its appearance is of relevance to the physical and biological phenomena that the equation models. For a number of commonly studied spatial domains, the large-time behaviour of a solution of the porous media equation and of the solution's free boundary is known. The present paper is concerned with this topic for a class of spatial domains which includes an infinite and a semi-infinite strip in two-dimensional space, an infinite and a semi-infinite cylinder of arbitrary cross-section in three-dimensional space, certain subdomains of these domains, and, their higher dimensional analogues. The homogeneous Cauchy–Dirichlet problem with initial data that are locally integrable is considered. Dependent upon the dimensionality, it is shown that there is a universal pattern of convergence to a self-similar solution. Moreover, the large-time behaviour of the free boundary in every solution mimics that of the self-similar one. The results rely on the establishment of an invariance principle for solutions of the problem.

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1. Introduction

If there is one mathematical equation that has been inexorably associated with the study of interfaces and free boundaries during the past fifty or more years, then that equation is the porous media equation. This equation arises in diverse areas of application in which the unknown represents a density, concentration, thickness, or similar nonnegative quantity that is to be found as a function of position and time; and, in such a setting, it is the occurrence of interfaces between that part of the problem domain where a solution is positive and that part where it is zero that has been of so much interest. Besides being of relevance for the application, such an interface is of mathematical significance because a perfectly acceptable solution may fail to classically solve the equation on it. Questions that have attracted attention and indeed continue to be the object of investigation are those of the smoothness of these interfaces, their initial behaviour, their behaviour immediately prior to any instant at which they may disappear, and their behaviour for large time [19].

The typical question of large-time interfacial behaviour pertains to the solution of an initial-value problem in an unbounded spatial domain with a smooth boundary, when the initial-data function is positive just in a bounded set of positive measure. In such a situation, the positivity set of the solution with respect to the spatial variable does not shrink as a function of time, and at some moment becomes a connected set. Furthermore, given any bounded subset of the spatial domain, there is a moment at which the positivity set will contain this subset. The question is then how fast or slow the positivity set will expand into the remaining unbounded components of the spatial domain as time becomes increasingly and increasingly larger. This question has been answered when the spatial domain comprises the whole space, a half space with homogeneous Dirichlet boundary conditions, and, the complement of a compact region with homogeneous or with time-independent inhomogeneous Dirichlet boundary conditions [19]. In the present paper we consider the problem with homogeneous Dirichlet boundary conditions with a spatial domain which, in terms of the familiar three-dimensional real world, is an infinite cylinder, part of such a cylinder containing a semi-infinite cylinder of the same cross-section, the void between two parallel plates of infinite extent, or such a void with the exclusion of a bounded portion.

The Cauchy–Dirichlet problem for the porous media equation with homogeneous boundary conditions reads

$$\begin{cases} \partial_t u = \Delta u^m & \text{for } (x,t) \in \Omega \times \mathbb{R}_+, \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1.1}$$

in which u is the unknown, ∂_t denotes partial differentiation with respect to time t, Δ denotes the Laplace operator with respect to the spatial variable $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$ for some natural number N,

$$m > 1,$$

Ω is a connected open subset of $\mathbb{R}^N$ with boundary $\partial\Omega$, $\mathbb{R}_+$ denotes the set of positive real numbers, and, u_0 is a given function. It is known that if $\partial\Omega$ is locally Lipschitz continuous and u_0 is nonnegative and integrable then this problem has a unique solution defined in some suitable sense. This solution u is continuous in $\overline{\Omega} \times \mathbb{R}_+$. Moreover, if u_0 is nontrivial and has bounded support, then the positivity set

$$P(t) = \{x \in \Omega : u(x,t) > 0\}, \tag{1.2}$$

is bounded for all $t > 0$. It has been further established that $P(t)$ is connected for large enough t, and will contain any given bounded subset of Ω for sufficiently large t, when Ω satisfies a uniform interior ball condition [19].

When the spatial domain Ω is the whole space $\mathbb{R}^N$, the half-space $\mathbb{R}^{N-1} \times \mathbb{R}_+$, or the complement of a compact subset of $\mathbb{R}^N$ with the lateral boundary condition in (1.1) replaced by

$$u(x,t) = \varphi(x) \quad \text{for } (x,t) \in \partial\Omega \times \mathbb{R}_+$$

for a given nonnegative function $\varphi \in C(\partial\Omega)$ which may be identically zero, the typical behaviour as $t \to \infty$ is that the solution u of problem (1.1) converges to a member of a family of self-similar solutions of the generic type

$$\mathfrak{U}(x,t;a) = t^{(2q-1)/(m-1)} a^{2/(m-1)} \Phi(x/at^q) \tag{1.3}$$

in which q and a are positive numbers. The number q is dictated by an invariance principle for the problem under consideration. The value of a is determined subsidiarily by the given initial-data function u_0. Simultaneously, the interface $\partial P(t)$ converges as $t \to \infty$ to the corresponding interface possessed by the self-similar solution.

For the problem where the spatial domain is the complement of a compact set and the lateral boundary-data function $\varphi \not\equiv 0$ [14, 15],

$$q = \begin{cases} 1/2 & \text{for } N \leqslant 2, \\ m/\{(m-1)N+2\} & \text{for } N \geqslant 2, \end{cases}$$

subject to the caveat described below. For the Cauchy problem *pur sang* [8, 13, 16],

$$q = 1/\{(m-1)N+2\}.$$

For the problem where the spatial domain is the complement of a compact set and $\varphi \equiv 0$ [5, 9],

$$q = \begin{cases} 1/2m & \text{for } N \leqslant 2, \\ 1/\{(m-1)N+2\} & \text{for } N \geqslant 2, \end{cases}$$

subject to the previously mentioned caveat. For the half-space problem [11],

$$q = 1/\{(m-1)N+m+1\}.$$

The caveat to the first and third of these problems relates to the case $N = 2$. In this exceptional case, the free boundary $\partial P(t)$ actually expands asymptotically proportionally to $t^q (\ln t)^{(m-1)/2m}$ as $t \to \infty$. Correspondingly, the convergence of the respective solution u to the self-similar solution $\mathcal{U}$ is contingent upon a time-dependent logarithmic correction to the parameter a which amounts to multiplying it by $(\ln t)^{(m-1)/2m}$ [9, 15]. Note nevertheless how the magnitude of q decreases successively from one problem to the next in reflection of the relative influence of the imposed boundary condition. For further details, especially regarding the precise manner in which the convergence of the solution and its free boundary is to be interpreted, we defer to the cited references.

It is also known [4] that when $\Omega = \mathbb{R}^{N-k} \times \mathbb{R}^k_+$ for some $2 \leqslant k \leqslant N$ the variable $\sup\{|x| : x \in \partial P(t)\}$ associated with the solution u of problem (1.1) grows at a rate t^q where

$$q = 1/\{(m-1)(N+k)+2\}.$$

We conjecture that in this situation too there is a self-similar solution of the form (1.3) to which u converges, and consequently that asymptotically the whole free boundary $\partial P(t)$ expands proportionally to t^q as $t \to \infty$.

The aforementioned power-law growth of the interface is drastically lost when the spatial domain

$$\Omega = D \times \mathbb{R}$$

and D is a bounded connected open subset of $\mathbb{R}^n$ with $n = N - 1 \geqslant 1$. This represents an infinite strip in the case $N = 2$ and an infinite cylinder in the case $N = 3$. For such a domain, Vázquez [18, 19] has shown that there is a self-similar solution of the type

$$U(x, t; a) = t^{-1/(m-1)} f(x_1, x_2, \ldots, x_n, x_{n+1} - c \ln t - a) \tag{1.4}$$

for a unique number $c > 0$ which depends only on D and m, while the function f has the property that

$$f(\xi) > 0 \quad \text{if and only if} \quad \xi_{n+1} < \sigma(\xi_1, \xi_2, \dots, \xi_n) \tag{1.5}$$

for some $\sigma \in C(D)$. Furthermore, the solution u of problem (1.1) satisfies

$$u(\cdot, t) \leqslant U(\cdot, t; A) \quad \text{for all } t > T \tag{1.6}$$

for some numbers A and $T \geqslant 0$, for every $(x_1, x_2, \dots, x_n) \in D$ there is a $T \geqslant 0$ such that

$$\{x_{n+1} \in \mathbb{R} : x \in P(t)\} = \big(-\Upsilon^-(t; x_1, x_2, \dots, x_n), \Upsilon^+(t; x_1, x_2, \dots, x_n)\big) \quad \text{for all } t > T$$

for some functions $\Upsilon^\pm(\cdot; x_1, x_2, \dots, x_n) : (T, \infty) \to \mathbb{R}_+$, and,

$$\Upsilon^\pm(t; \cdot) - c \ln t = \mathrm{o}(\ln t) \quad \text{as } t \to \infty$$

pointwise in D. If the initial-data function u_0 vanishes in some set $D \times (\ell, \infty)$ and u_0 satisfies a certain minimal growth criterion as $x_{n+1} \to -\infty$, then there exists a second number A such that (1.6) holds with the inequality sign reversed, and hence

$$\Upsilon^+(t; \cdot) - c \ln t = \mathrm{O}(1) \quad \text{as } t \to \infty$$

uniformly in D.

We shall complete the above results, merely considering an initial-data function u_0 satisfying a mild integrability condition in Ω with support contained in $D \times (-\infty, \ell)$ for some ℓ. Without any minimal growth condition as $x_{n+1} \to -\infty$, we shall establish the following.

- First, that the critical number

$$c = \frac{1}{(m-1)\sqrt{\lambda}}, \tag{1.7}$$

  where λ is the first eigenvalue of the eigenvalue problem for the Laplacian with homogeneous boundary conditions in D. (In terms of the nomenclature employed in [18, 19], this means that the critical "wave speed" c_* actually only depends on D and is independent of m.)
- Second, the function f is unique modulo translation.
- Third, identifying one member f of this translation class, there is a number a for which

$$t^{1/(m-1)} u(x, t) \to f(x_1, x_2, \dots, x_n, x_{n+1} - c \ln t - a) \quad \text{as } t \to \infty \tag{1.8}$$

  uniformly with respect to $x \in D \times \mathbb{R}_+$ in an appropriate reference frame.
- Fourth, taking σ to be the corresponding function in (1.5),

$$\liminf_{t \to \infty} \Upsilon^+(t; \cdot) - c \ln t \geqslant \sigma + a \tag{1.9}$$

  at every point in D.
- Fifth,

$$\limsup_{t \to \infty} \Upsilon^+(t; \cdot) - c \ln t \leqslant \bar{\sigma} + a, \tag{1.10}$$

  where $\bar{\sigma}$ denotes the concave envelope of σ, uniformly in D.

- Sixth, the results extend to any domain Ω such that

$$D \times \mathbb{R}_+ \subseteq \Omega \subseteq D \times \mathbb{R}. \tag{1.11}$$

- Finally, for all the domains considered, there is a formula to determine the appropriate number a from the initial-data function u_0.

By a symmetry argument, the results may be applied to the convergence of $t^{1/(m-1)}u(\cdot, t)$ in $D \times (-\infty, 0)$ and to the asymptotic behaviour of Υ^- as $t \to \infty$ when the support of u_0 is contained in $D \times (\ell, \infty)$ for some ℓ. Similarly, the results for domains of the type (1.11) may be transferred to any domain Ω such that $D \times (-\infty, 0) \subseteq \Omega \subset D \times \mathbb{R}$.

The key to our results is an invariance principle for problem (1.1) in a domain of the form (1.11) analogous to those known for the problem in $\mathbb{R}^N$, in $\mathbb{R}^{N-1} \times \mathbb{R}_+$, and in a domain $\Omega \subset \mathbb{R}^N$ that is the complement of a compact subset of $\mathbb{R}^N$.

After completing the analysis for a domain of the type (1.11), we turn our attention to the equivalent invariance principle for problem (1.1) with a domain

$$D \times \{z \in \mathbb{R}^k : |z| > \varrho\} \subseteq \Omega \subseteq D \times \mathbb{R}^k \quad \text{for some } \varrho > 0, \tag{1.12}$$

where D is as aforesaid (i.e. a bounded connected open subset of $\mathbb{R}^n$ for some $n \geqslant 1$), and $k = N - n \geqslant 2$. In mundane three-dimensional space, this means that Ω is the void between two parallel plates of infinite extent, possibly with a bounded portion removed.

We establish the generalization of the invariance principle for solutions of problem (1.1) in a domain of the type (1.11) to that for a domain of the type (1.12). Formal calculations based on this invariance principle lead to the conclusion that if the initial-data function is nontrivial and vanishes outside a bounded set then the asymptotic behaviour of the solution u is given by

$$t^{1/(m-1)}u(x, t) \to f(x_1, x_2, \ldots, x_n, r - c \ln t + \gamma \ln|\ln t| - a) \quad \text{as } t \to \infty, \tag{1.13}$$

where f is as above,

$$r = \sqrt{x_{n+1}^2 + x_{n+2}^2 + \cdots + x_{n+k}^2},$$

c is as above, the number γ is also invariant and quantifiable as

$$\gamma = \frac{k-1}{2\sqrt{\lambda}} \tag{1.14}$$

with λ as above denoting the first eigenvalue of the eigenvalue problem for the Laplacian with homogeneous boundary conditions in D, and, the number a is dependent on the specific initial-data function. Contemporaneously, setting

$$\Upsilon_{\mathrm{i}}(x_1, x_2, \ldots, x_n, t) = \min\{r > \varrho : x \in \partial P(t)\}$$

and

$$\Upsilon_{\mathrm{s}}(x_1, x_2, \ldots, x_n, t) = \max\{r > \varrho : x \in \partial P(t)\},$$

there holds

$$\liminf_{t\to\infty} \Upsilon_{\mathrm{i}}(\cdot, t) - c \ln t + \gamma \ln|\ln t| \geqslant \sigma + a \tag{1.15}$$

and

$$\limsup_{t\to\infty} \Upsilon_s(\cdot, t) - c \ln t + \gamma \ln|\ln t| \leqslant \bar{\sigma} + a \tag{1.16}$$

at every point in D.

In a sense, the paper has two mathematical flavours. The analysis for problem (1.1) with a domain of the type (1.11) is entirely rigorous and thus can be considered ‘pure’. Contrastingly, barring the derivation of the invariance principle itself, that for the solution of the problem with a domain of the form (1.12) for $k \geqslant 2$ is intuitive, and may be viewed ‘applied’.

Just for the record, our results answer two of the open problems – one of which is starred – in [19, Chapter 20], and broach a third.

The remainder of the paper comprises five sections. In the next, we review the requisite theory of solutions of problem (1.1) and that of the self-similar solution U. In the section thereafter, we establish the invariance principle (Theorem 3.1) for solutions of problem (1.1) with domains of the form (1.11). Formula (1.7) is a consequence (Theorem 3.2). The subsequent section is devoted to the proof of (1.8) and the accompanying formula to determine the number a from the initial data (Theorems 4.1 and 4.2). From this, the uniqueness of the function f follows (Theorem 4.3). A section is then devoted to the proof of (1.9) and (1.10) (Theorems 5.1 and 5.2 respectively). In the final section, we present the corresponding invariance principle for a solution u of problem (1.1) in a domain satisfying (1.12) for some $k \geqslant 2$ (Theorem 6.1), and use this to justify the proposition that (1.13), (1.15) and (1.16) describe the large-time behaviour of u and its free boundary when the initial data u_0 have bounded support.

2. Preliminaries

Throughout the paper, we shall assume the following.

HYPOTHESIS 2.1 The domain D is a bounded connected open subset of $\mathbb{R}^n$ for some natural number n whose boundary satisfies a uniform interior ball condition.

With $N = n + k$, let us introduce the notation

$$x = (y, z) \in \mathbb{R}^N \qquad \text{for} \quad y \in \mathbb{R}^n \quad \text{and} \quad z \in \mathbb{R}^k. \tag{2.1}$$

We shall use ∇_y, ∇_z, and ∇_x, and, Δ_y, Δ_z, and Δ_x to denote the del and Laplace operators with respect to $y \in \mathbb{R}^n$, $z \in \mathbb{R}^k$, and $x \in \mathbb{R}^N$ respectively, as the need arises. Where there is no confusion, we may drop the subscript. As an alternative to (2.1), we employ

$$\xi = (\eta, \zeta) \in \mathbb{R}^N \qquad \text{for} \quad \eta \in \mathbb{R}^n \quad \text{and} \quad \zeta \in \mathbb{R}^k. \tag{2.2}$$

Because it appears frequently let us define

$$\frac{1}{m-1} = \mu.$$

From [2, 6, 17] the porous media equation in the domain $D \times \mathbb{R}_+$ admits a self-similar solution nicknamed the friendly giant,

$$t^{-\mu} F(y),$$

where F is the unique weak solution of

$$\begin{cases} \Delta F^m + \mu F = 0 & \text{in } D, \\ F = 0 & \text{on } \partial D. \end{cases}$$

In fact, $F \in C(\overline{D}) \cap C^\infty(D)$ is a classical solution of this problem, and, $F > 0$ in D.

From Vázquez's analysis [18] we also know that there is a further self-similar solution with domain $D \times \mathbb{R} \times \mathbb{R}_+$ given by

$$t^{-\mu} f(y, z - c \ln t),$$

where $c > 0$, and, defining the constant vector $\boldsymbol{c}$ in $\mathbb{R}^{n+1}$ by

$$\boldsymbol{c} = \langle 0, \ldots, 0, c \rangle, \tag{2.3}$$

$f \in C(\overline{D} \times \mathbb{R})$ is a weak solution of the equation

$$\nabla \cdot (\nabla f^m + f \boldsymbol{c}) + \mu f = 0 \quad \text{in } D \times \mathbb{R} \tag{2.4}$$

satisfying the auxiliary conditions

$$f = 0 \quad \text{on } \partial D \times \mathbb{R}, \tag{2.5}$$

$$f(y, z) \to F(y) \quad \text{as } z \to -\infty \quad \text{for all } y \in D, \tag{2.6}$$

and

$$f(y, z) \to 0 \quad \text{as } z \to \infty \quad \text{for all } y \in D. \tag{2.7}$$

Moreover,

$$f = 0 \quad \text{in } D \times (L, \infty) \quad \text{for some } L \in \mathbb{R}, \tag{2.8}$$

and there is a unique number $c > 0$ for which problem (2.4)–(2.7) has such a solution.

As it has not yet been proven that f is in any way unique, we shall take f to be that solution of problem (2.4)–(2.7) constructed in [18] and c correspondingly. This particular solution has the added properties that there is a function $\sigma \in C(D) \cap L^\infty(D)$ such that

$$f(y, z) > 0 \quad \text{if and only if } z < \sigma(y) \text{ for all } y \in D; \tag{2.9}$$

$$z \mapsto f(y, z) \text{ is non-increasing for every } y \in D; \tag{2.10}$$

$$f(y, z)/F(y) \to 1 \quad \text{as } z \to -\infty \quad \text{uniformly with respect to } y \in D; \tag{2.11}$$

and f is infinitely continuously differentiable in its positivity set and a classical solution of (2.4) there. Furthermore, if D is a ball with centre 0, then f is symmetric about the line $y = 0$, and $|y| \mapsto f(y, z)$ is non-increasing for every $z \in \mathbb{R}$. It follows that in this particular case, $|y| \mapsto \sigma(y)$ is non-increasing, and, hence, $\sigma \in C(\overline{D})$.

Regarding the whole problem domain, besides either (1.11) or (1.12) with $k \geqslant 2$, we shall assume the following.

HYPOTHESIS 2.2 The domain Ω is connected, open, and has a boundary that is locally Lipschitz continuous.

DEFINITION 2.1 A weak solution of the equation

$$\partial_t u = \nabla \cdot (\nabla u^m + u\boldsymbol{v}) + \nu u \quad \text{for } (x,t) \in Q, \tag{2.12}$$

where $\boldsymbol{v}$ is a fixed vector in $\mathbb{R}^N$, ν is a fixed real number, and Q is an open subset of $\mathbb{R}^{N+1}$, is a nonnegative function $u \in L^1_{\text{loc}}(Q)$ for which every component of ∇u^m exists as a weak derivative in $L^1_{\text{loc}}(Q)$, and

$$\iint_Q \{(\nabla u^m + u\boldsymbol{v}) \cdot \nabla\psi - \nu u\psi - u\partial_t\psi\}\, dx\, dt = 0 \quad \text{for all } \psi \in C_0^1(Q).$$

DEFINITION 2.2 A strong solution of equation (2.12) is a weak solution u possessing the property that $\partial_t u \in L^1_{\text{loc}}(Q)$.

DEFINITION 2.3 A solution of problem (1.1) is a strong solution u of (2.12) with $\boldsymbol{v} = \mathbf{0}$, $\nu = 0$, and $Q = \Omega \times \mathbb{R}_+$, such that the trace of $u^m(\cdot,t)$ on $\partial\Omega$ is defined and equal to zero for almost all $t \in \mathbb{R}_+$, and $u(\cdot,t) \to u_0$ as $t \downarrow 0$ in $L^1_{\text{loc}}(\overline{\Omega})$.

Lemma 2.1 *For any nonnegative function $u_0 \in L^1_{\text{loc}}(\overline{\Omega})$, problem* (1.1) *has a unique solution u. Furthermore, $u \in C(\overline{\Omega} \times \mathbb{R}_+)$, $u^m \in L^2_{\text{loc}}(\mathbb{R}_+; W^{1,2}_{\text{loc}}(\overline{\Omega}))$,*

$$u(x,t) < t^{-\mu} F(y) \quad \textit{for all } (x,t) \in \Omega \times \mathbb{R}_+, \tag{2.13}$$

$$t\,(\partial_t u)(x,t) \geqslant -\mu\, u(x,t) \quad \textit{for almost all } (x,t) \in \Omega \times \mathbb{R}_+, \tag{2.14}$$

and u is infinitely continuously differentiable and solves the porous media equation classically in that subset of $\Omega \times \mathbb{R}_+$ where it is positive. Moreover, if $u_0 \in L^\infty(\Omega)$ then $u \in L^\infty(\Omega \times \mathbb{R}_+)$, if $u_0 \in C(\Omega)$ then $u \in C(\Omega \times [0,\infty))$, and, if $u_0 \in C(\overline{\Omega})$ is such that $u_0 = 0$ on $\partial\Omega$ then $u \in C(\overline{\Omega} \times [0,\infty))$.

Proof. The existence of a unique solution u satisfying the *a priori* estimate (2.13) without strict inequality is provided by extension of the development for the case $\Omega = D \times \mathbb{R}$ in [19, Subsection 12.8.2]. Moreover, this gives $u \in C([0,\infty); L^1_{\text{loc}}(\overline{\Omega}))$ and $u^m \in L^2_{\text{loc}}(\mathbb{R}_+; W^{1,2}_{\text{loc}}(\overline{\Omega}))$. Following the further general theory of the porous media equation in [19], the bound (2.13) without its strictness is sufficient to establish the continuity of u in $\overline{\Omega} \times \mathbb{R}_+$. Whence it can be deduced that u is infinitely continuously differentiable and a classical solution of $\partial_t u = \Delta u^m$ in $\{(x,t) \in \Omega \times \mathbb{R}_+ : u(x,t) > 0\}$, and possesses the conditional properties mentioned at the end of the statement of the lemma. Subsequently, recalling that $F > 0$ in D, the strict inequality in (2.13) can be obtained from the strong maximum principle. The inequality (2.14) is deducible using either of the arguments substantiating [19, Lemma 8.1]. □

Lemma 2.2 *Let u_i denote a solution of problem* (1.1) *with spatial domain Ω_i and initial data $u_{0,i}$, for $i = 1, 2$. If $u_{0,1} \leqslant u_{0,2}$ almost everywhere in $\Omega_1 \subseteq \Omega_2$, then $u_1 \leqslant u_2$ in $\Omega_1 \times \mathbb{R}_+$.*

Proof. This comparison principle follows from the arguments leading to Lemma 2.1 similarly to the corresponding principle for other problems treated in the existent theory of the porous media equation [19]. □

In the statement of each of the remaining lemmata of this section, u denotes the solution of problem (1.1) given by Lemma 2.1.

Lemma 2.3 *Let $\tilde{u}_0 = u_0$ in Ω, and $\tilde{u}_0 = 0$ elsewhere in $\mathbb{R}^N$. Suppose that*

$$\limsup_{R\to\infty} R^{-2\mu-N} \int_{\{x\in\Omega:|x|<R\}} u_0(x)\,dx < \infty. \tag{2.15}$$

Then there exists a $\Upsilon \in \mathbb{R}_+$ such that the Cauchy problem for porous media equation with initial data $\tilde{u}_0$ has a suitably defined unique nonnegative solution $\tilde{u} \in C(\mathbb{R}^N \times (0,\Upsilon))$, and, $\tilde{u} \geqslant u$ in $\Omega \times (0,\Upsilon)$. Moreover, when the left-hand side of (2.15) is zero, one can take $\Upsilon = \infty$.

Proof. See [19, Chapter 12]. □

Lemma 2.4 *Let $\Omega = D \times \mathbb{R}$, and v be a nonnegative, uniformly continuous, bounded function in $\overline{\Omega} \times [0,\infty)$ that vanishes on $\partial\Omega \times \mathbb{R}_+$, and is a classical solution of the porous media equation in that subset of $\Omega \times \mathbb{R}_+$ where it is positive. Suppose that $u_0 \in C(\overline{\Omega}) \cap L^\infty(\Omega)$, and $u_0 \geqslant v(\cdot,0)$ in Ω. Then $u \geqslant v$ in $\Omega \times \mathbb{R}_+$.*

Proof. If it were hypothesized that v was a strong solution of the porous media equation in $\Omega \times \mathbb{R}_+$, then the present lemma would be a restatement of Lemma 2.2. To obtain the lemma without this hypothesis, we start at the foundation of comparison principles for solutions of the porous media equation.

Let us first assume that $u \in C(\overline{\Omega} \times [0,\infty))$ is a classical solution of the porous media equation that is uniformly continuous and bounded away from zero in $\Omega \times \mathbb{R}_+$, and $u(\cdot,0) \geqslant v(\cdot,0) + \iota$ on Ω for some $\iota > 0$. In this case, there exists a $\delta > 0$ such that $u > v$ in $\big(\overline{\Omega} \times [0,\infty)\big) \setminus (D_\delta \times \mathbb{R} \times (\delta,\infty))$, where

$$D_\delta = \big\{y \in D : |\eta - y| \geqslant \delta \text{ for all } \eta \in \partial D\big\}. \tag{2.16}$$

On the other hand, by estimates of classical solutions of the porous media equation written as a linear equation [12, Theorems III.10.1 and III.12.1], there exists an $\alpha \in (0,1)$ such that $u \in C^{2+\alpha,1+\alpha/2}(\overline{D_\delta} \times \mathbb{R} \times [\delta,\infty))$. Let

$$V = \sup\big\{v(x,t) : (x,t) \in \Omega \times \mathbb{R}_+\big\},$$
$$b = (m-1)\left[\int_0^1 \big\{\omega u^m + (1-\omega)v^m\big\}^{-1/m}\,d\omega\right]\partial_t u,$$

and

$$\beta = \sup\big\{\,|b(x,t)| : (x,t) \in D_\delta \times \mathbb{R} \times (\delta,\infty)\big\} + 1.$$

The nonnegativity of v and the assumptions on u ensure that β is finite. Let $T \in (\delta,\infty)$ and $\varepsilon > 0$ be arbitrary, and consider the function

$$w(x,t) = \Big\{v^m(x,t) - u^m(x,t)\Big\}e^{-\beta t} - \varepsilon\Big\{mNV^{m-1}t + \big(1+|x|^2\big)^{1/2}\Big\}.$$

It is such that $w \leqslant -\varepsilon$ on $\big(\overline{D_\delta} \times \mathbb{R} \times \{\delta\}\big) \cup (\partial D_\delta \times \mathbb{R} \times [\delta,T])$, and $w(x,t) \to -\infty$ as $|z| \to \infty$ uniformly with respect to $(y,t) \in \overline{D_\delta} \times [\delta,T]$. Thus, if w is positive anywhere in $D_\delta \times \mathbb{R} \times (\delta,T]$, it must have a positive maximum in this set. This places such a maximum in the positivity set of v. So both u and v are classical solutions of the porous media equation there, and besides $w > 0$, necessarily $\nabla w = 0$, $\Delta w \leqslant 0$, and $\partial_t w \geqslant 0$ at this point. However, it can also be computed that

$$\begin{aligned} m v^{m-1} \Delta w - (\beta - b)\, w - \partial_t w \\ = \varepsilon m \Big[N V^{m-1} - \Big\{ N + (N-1)\,|x|^2 \Big\} \Big(1 + |x|^2\Big)^{-3/2} v^{m-1} \Big] \\ + \ \varepsilon \Big\{ m N V^{m-1} t + \Big(1 + |x|^2\Big)^{1/2} \Big\} (\beta - b) \end{aligned}$$

at such a point. So, at a positive maximum of w, we have an identity for which the left-hand side is negative and the right-hand side is positive. This absurdity rules out the positivity of w anywhere in $D_\delta \times \mathbb{R} \times (\delta, T]$. Hence, $w \leqslant 0$ in $\Omega \times (0, T]$. Passing to the limits $\varepsilon \downarrow 0$ and $T \to \infty$, we obtain $u \geqslant v$ in $\Omega \times \mathbb{R}_+$.

Now, we can find a sequence of functions $\{u_{0,j}\}_{j \in \mathbb{N}} \subset C(\overline{\Omega})$ such that $u_{0,j} = 1/j$ on $\partial\Omega$, $u_{0,j} \geqslant v(\cdot, 0) + 1/j$ and $u_{0,j} \geqslant u_{0,j+1} \geqslant u_0$ in Ω for each $j \in \mathbb{N}$, and $u_{0,j} \to u_0$ as $j \to \infty$ in $C(\overline{\Omega})$. Let u_j denote the classical solution of the porous media equation in $\Omega \times \mathbb{R}_+$ satisfying $u_j \geqslant 1/j$ in $\Omega \times \mathbb{R}_+$, $u_j(\cdot, 0) = u_{0,j}$ on Ω, and $u_j = 1/j$ on $\partial\Omega \times \mathbb{R}_+$, whose existence is given by the arguments of [19, Theorem 5.5 and Proposition 7.21]. By the arguments of [19, Theorem 5.14], $u_j \geqslant u_{j+1}$ in $\Omega \times \mathbb{R}_+$. Subsequently, employing the continuity estimates in [19, Theorem 7.1], we can ascertain that the sequence $\{u_j\}_{j \in \mathbb{N}}$ converges monotonically to u uniformly on compact subsets of $\Omega \times \mathbb{R}_+$.

The lemma follows from applying the argument for a classical solution of the porous media equation that is bounded away from zero to u_j and then letting $j \to \infty$. □

Lemma 2.5 *Define $P(t)$ for $t \in \mathbb{R}_+$ by (1.2).*
(i) *There holds $P(\theta) \subseteq P(T)$ for all $0 < \theta < T$.*
(ii) *Given any bounded open set $\Omega' \subseteq \Omega$ that satisfies a uniform interior ball condition, there exists a $T > 0$ such that $\Omega' \subseteq P(T)$.*

Proof. The first assertion follows from the continuity of u in $\Omega \times \mathbb{R}_+$ and (2.14). The second can be found in [19, Theorems 14.3 and 14.4]. □

Lemma 2.6 *Let $\Omega = D \times \mathbb{R}$, $Q = \{(x, t) \in \Omega \times (0, T] : z > \nu \cdot y + \Psi(t)\}$ where*

$$\nu \cdot y = \nu_1 y_1 + \cdots + \nu_n y_n$$

for some $\nu \in \mathbb{R}^n$, $\Psi \in C([0, T])$, and $T \in \mathbb{R}_+$, and let $v \in C(\overline{Q}) \cap L^\infty(Q)$ be a strong solution of the porous media equation in Q. Suppose that $u_0 \in C(\overline{\Omega}) \cap L^\infty(\Omega)$, $u \leqslant v$ on $\{(x, t) \in \overline{Q} : t = 0$ or $x \in \partial\Omega\}$, and $u < v$ on $\mathcal{S} = \{(x, t) \in \overline{Q} : z = \nu \cdot y + \Psi(t)\}$. Then $u \leqslant v$ in Q.

Proof. When Ψ is constant, so that Q is a cylinder, this lemma can be proven like Lemma 2.2. The result when Ψ is not constant follows by replacing Q with the union of a finite number of cylinders $Q_j = \Omega_j \times (t_{j-1}, t_j] \subset Q$ for $j = 1, \ldots, \mathsf{N}$, with $t_0 = 0$ and $t_{\mathsf{N}} = T$, such that $v - u$ is bounded away from zero on $\overline{Q} \setminus \left(\cup_{j=1}^{\mathsf{N}} \overline{Q_j} \right)$. The smoothness of $\mathcal{S}$, the continuity of u and v in $\overline{Q}$, and the condition $u < v$ on $\mathcal{S}$, provide the freedom to be able to do this. The previously established result can then be applied in each of the cylinders Q_j by induction on j. □

Lemma 2.7 *The solution u of problem* (1.1) *is such that $t^{\mu}\,|u(x,t)-\mathfrak{F}(x)| \to 0$ as $t \to \infty$ uniformly with respect to x in compact subsets of $\overline{\Omega}$, where $\mathfrak{F}$ is the unique weak solution of*

$$\begin{cases} \Delta\mathfrak{F}^m + \mu\mathfrak{F} = 0 & \text{in } \Omega, \\ \mathfrak{F} = 0 & \text{on } \partial\Omega. \end{cases}$$

This function $\mathfrak{F} \in C(\overline{\Omega}) \cap C^{\infty}(\Omega)$, and is such that $0 < \mathfrak{F}(x) \leqslant F(y)$ for all $x \in \Omega$ with equality on the right-hand side if $\Omega = D \times \mathbb{R}^k$.

Proof. Extension of the argument used to prove [17, Theorem 3.2] gives the result. □

3. The invariance principle

In this and the coming two sections

$$N = n + 1$$

and Ω is a domain of the type (1.11), exclusively.

Let λ denote the first eigenvalue of the eigenvalue problem for the Laplacian with homogeneous Dirichlet boundary conditions in D, and Y the corresponding eigenfunction. So,

$$\begin{cases} -\Delta Y = \lambda Y & \text{in } D, \\ Y = 0 & \text{on } \partial D. \end{cases}$$

By standard theory, $\lambda > 0$, and $Y \in C(\overline{D}) \cap C^{\infty}(D)$ is unique modulo multiplication with a constant. We shall normalize this constant, by henceforth assuming that

$$\int_D Y(y)\,dy = 1.$$

In this case $Y > 0$ in D. Consequently the function

$$G(x) = Y(y)e^{\sqrt{\lambda}\,z} \quad \text{for } x \in \overline{D} \times \mathbb{R} \tag{3.1}$$

is a positive classical solution of the problem

$$\begin{cases} \Delta G = 0 & \text{in } D \times \mathbb{R}, \\ G = 0 & \text{on } \partial D \times \mathbb{R}. \end{cases} \tag{3.2}$$

Let

$$h(x) = \begin{cases} \ln|x| & \text{if } N = 2, \\ 1 & \text{if } N \geqslant 3. \end{cases}$$

Lemma 3.1 *There exists a unique function $K \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

$$\begin{cases} \Delta K = 0 & \text{in } \Omega, \\ K = 0 & \text{on } \partial\Omega, \\ K(x) = G(x) + \mathrm{o}(h(x)) & \text{as } |z| \to \infty \end{cases} \tag{3.3}$$

uniformly with respect to $y \in D$. Moreover,

$$0 < K \leqslant G \quad \text{in } \Omega \tag{3.4}$$

with equality if and only if $\Omega = D \times \mathbb{R}$, and,

$$K(x)/G(x) \to 1 \quad \text{as } z \to \infty \tag{3.5}$$

uniformly with respect to $y \in D$.

Proof. Since $\partial\Omega$ is locally Lipschitz continuous, every point on $\partial\Omega$ satisfies the exterior segment condition in the case $N = 2$ and the exterior cone condition in the case $N \geqslant 3$, and hence is a regular boundary point [3, Examples 4.8 and 4.9, pages 337–338]. Furthermore, $\max\{G(x)/h(x) : x \in \partial\Omega \text{ and } |x| = R\} \to 0$ as $R \to \infty$, since $G = 0$ on $\partial D \times \mathbb{R}_+ = \{x \in \partial\Omega : z > 0\}$ and $G(x) \to 0$ as $z \to -\infty$ uniformly with respect to $y \in D$. So, by [3, Theorem 4.2, pages 363–364], the problem

$$\begin{cases} \Delta J = 0 & \text{in } \Omega, \\ J = G & \text{on } \partial\Omega, \\ J(x)/h(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

admits a unique solution $J \in C^2(\Omega) \cap C(\overline{\Omega})$. Subsequently, it is easily verified that $K = G - J$ is the unique solution of problem (3.3).

To prove the left-hand inequality in (3.4) for $N \geqslant 3$, consider the function $v = K + \varepsilon h$ for $\varepsilon > 0$. The boundary condition in (3.3) implies that $v \geqslant 0$ on $\partial\Omega$, while the last condition implies that $\liminf_{|x|\to\infty} v(x) \geqslant 0$. Furthermore, $\Delta v = 0$ in Ω. Hence [3, Corollary 2.4, page 246], $v \geqslant 0$ in Ω. In view of the arbitrariness of ε, this gives $K \geqslant 0$ in Ω. Strict inequality follows from the strong maximum principle [3, Proposition 2.8, page 245]. The proof of the right-hand inequality in (3.4) for $N \geqslant 3$ is analogous, using $v = G - K + \varepsilon h$. In the case $N = 2$ the inequalities in (3.4) may be proven similarly upon replacing $h(x)$ with $\ln(|y - y^*|^2 + z^2)$ for any fixed $y^* \in \mathbb{R}^n$ such that $|y^* - \eta| > 1$ for all $\eta \in \overline{D}$.

To obtain (3.5), we note that $H(x) = 2Y(y)\sinh(\sqrt{\lambda}z)$ is a solution of (3.3) with Ω replaced by $D \times \mathbb{R}_+$. Furthermore, $K = H$ on $\partial D \times \mathbb{R}_+$ and $K \geqslant 0 = H$ on $\overline{D} \times \{0\}$. A repetition of the preceding comparison principle argument leads to the conclusion that $K \geqslant H$ in $D \times \mathbb{R}_+$. Given (3.4), (3.5) follows. □

We are now in a position to formulate the following invariance principle, which is the key to all our results.

Theorem 3.1 *Let Ω be a domain of the type (1.11) fulfilling Hypotheses 2.1 and 2.2, and, let $u_0 \in L^1_{\text{loc}}(\overline{\Omega})$ be nonnegative. Then the solution u of problem (1.1) satisfies*

$$\int_\Omega K(x)u(x,t)\,dx = \int_\Omega K(x)u_0(x)\,dx \quad \text{for all } t > 0. \tag{3.6}$$

Proof. Let us first assume that $u_0(x)$ vanishes for large values of $|z|$ uniformly with respect to y. Define $\tilde{u}_0$ as in Lemma 2.3, and then $\bar{u}_0(x) = \tilde{u}_0(y,z) + \tilde{u}_0(y,-z)$ for all $x \in D \times \mathbb{R}$. Let $\bar{u}$ denote the solution of problem (1.1) with $\Omega = D \times \mathbb{R}$ and initial data $\bar{u}_0$. By part (i) of Lemma 4.1 below, given any $T > 0$ there exists an $L > 0$ such that $\bar{u}$ vanishes in $D \times [L, \infty) \times (0, T]$. However, since

$\bar{u}_0$ is symmetric about the hyperplane $z = 0$, the uniqueness of solutions of problem (1.1) implies that $\bar{u}(\cdot, t)$ preserves this symmetry for $t \in \mathbb{R}_+$. So $\bar{u}$ vanishes in $\{x \in D \times \mathbb{R} : |z| \geqslant L\} \times (0, T]$. Since $\bar{u}_0 \geqslant u_0$ in Ω, Lemma 2.2 subsequently implies that u vanishes in $\{x \in \Omega : |z| \geqslant L\} \times (0, T]$. Now, let $\phi \in C_0^2(\mathbb{R})$ be such that $\phi = 1$ on $[-L, L]$. Noting that $\partial_t u = \Delta u^m$ almost everywhere in $\Omega \times \mathbb{R}_+$, multiplying this identity with $K(x)\phi(z)$, integrating over $\Omega \times (\theta, T)$ for $\theta \in (0, T)$, applying integration by parts, and passing to the limit $\theta \downarrow 0$, we obtain (3.6) with $t = T$.

Suppose next that $u_0 \geqslant \underline{u}_0 > 0$ in Ω for some $\underline{u}_0 \in C(\overline{\Omega})$. Let $\phi_i \in C(\mathbb{R})$ be a cutoff function with maximum value $\phi_i(z) = 1$ for $|z| \leqslant i$, and minimum value $\phi_i(z) = 0$ for $|z| \geqslant i + 1$, for $i \in \mathbb{N}$. Define $u_{0,i}(x) = u_0(x)\phi_i(z)$ and $\underline{u}_{0,i}(x) = \underline{u}_0(x)\phi_i(z)$ for $x \in \Omega$. Denote the solution of problem (1.1) with initial data $u_{0,i}$ by u_i and that of the problem with initial data $\underline{u}_{0,i}$ by $\underline{u}_i$. By Lemma 2.2, $\{u_i\}_{i\in\mathbb{N}}$ is non-decreasing and bounded above by u in $\Omega \times \mathbb{R}_+$. Consequently, it has a limit function u_∞ such that $0 \leqslant u_\infty \leqslant u$ in this set. On the other hand, $u_i \geqslant \underline{u}_j$ for all $i \geqslant j$, while $\underline{u}_j$ is positive and continuous in $\{x \in \Omega : |z| \leqslant j\} \times \mathbb{R}_+$. Thus, u_i is a positive classical solution of the porous media equation in $\{x \in \Omega : |z| \leqslant j\} \times \mathbb{R}_+$ for every $i \geqslant j$. By estimates of such solutions of the porous media equation written as a linear equation [12, Theorems III.10.1 and III.12.1], $\{u_i\}_{i=j}^{\infty}$ has a uniform bound in the norm of the space $C^{2+\alpha,1+\alpha/2}(\mathcal{K} \times [\theta, \Theta])$ for some $\alpha \in (0, 1)$ for every compact set $\mathcal{K} \subset \{x \in \Omega : |z| \leqslant j\}$, $0 < \theta < \Theta < \infty$, and $j \in \mathbb{N}$. So, u_∞ belongs to this space too. We conclude that u_∞ is a positive classical solution of the porous media equation in $\Omega \times \mathbb{R}_+$. The fact that $0 \leqslant u_\infty \leqslant u$ in $\Omega \times \mathbb{R}_+$ subsequently implies that $u_\infty \in C(\overline{\Omega} \times \mathbb{R}_+)$ and $u_\infty = 0$ on $\partial\Omega \times \mathbb{R}_+$. Finally, we note that

$$\int_{\{x\in\Omega:|z|<L\}} \left|u_\infty(x,t) - u_0(x)\right| dx \leqslant \int_{\{x\in\Omega:|z|<L\}} \max\left\{ |u(x,t) - u_0(x)|, |u_i(x,t) - u_0(x)| \right\} dx$$

for all $L > 0$, $t > 0$ and $i \in \mathbb{N}$. Consequently,

$$\limsup_{t\downarrow 0} \int_{\{x\in\Omega:|z|<L\}} |u_\infty(x,t) - u_0(x)|\, dx \leqslant \int_{\{x\in\Omega:|z|<L\}} |u_{0,i}(x) - u_0(x)|\, dx.$$

Passing to the limit $i \to \infty$, it follows that $u_\infty(\cdot, t) \to u_0$ as $t \downarrow 0$ in $L^1_{\mathrm{loc}}(\overline{\Omega})$. Altogether, this confirms that u_∞ is a solution of problem (1.1). Hence, because solutions of this problem are unique, $u_\infty = u$. Therefore, substituting u_i in (3.6), passing to the limit $i \to \infty$, and invoking the Monotone Convergence Theorem, we obtain (3.6) as desired.

The proof for a general initial-data function u_0 is now easily accomplished. It can be sandwiched between an increasing sequence of functions $\{\underline{u}_{0,i}\}_{i\in\mathbb{N}}$ of the first type considered, and a decreasing sequence $\{\bar{u}_{0,i}\}_{i\in\mathbb{N}}$ of the second type considered, both of which converge to u_0 in $L^1_{\mathrm{loc}}(\Omega)$. Moreover, if $Ku_0 \in L^1(\Omega)$ then the latter sequence can be chosen so that $K\bar{u}_{0,i} \in L^1(\Omega)$ for all $i \in \mathbb{N}$. Letting $\underline{u}_i$ and $\bar{u}_i$ denote the corresponding solutions of problem (1.1) for $i \in \mathbb{N}$, by Lemma 2.2 and what we have already proven,

$$\int_\Omega K(x)u(x,t)\, dx \geqslant \int_\Omega K(x)\underline{u}_i(x,t)\, dx = \int_\Omega K(x)\underline{u}_{0,i}(x)\, dx$$

and

$$\int_\Omega K(x)u(x,t)\, dx \leqslant \int_\Omega K(x)\bar{u}_i(x,t)\, dx = \int_\Omega K(x)\bar{u}_{0,i}(x)\, dx$$

for all $i \in \mathbb{N}$ and $t \in \mathbb{R}_+$. Taking the limit $i \to \infty$ completes the proof. □

Now, recall the definition (1.4) of the self-similar solution

$$U(x,t;a) = t^{-\mu} f(y, z - c \ln t - a). \tag{3.7}$$

Lemma 3.2 *Let a be any real number. For all $t > 0$ there holds*

$$\int_{D\times\mathbb{R}} G(x)U(x,t;a)\,dx = \kappa e^{\sqrt{\lambda}\,a}\, t^{\sqrt{\lambda}\,c-\mu}, \tag{3.8}$$

where

$$\kappa = \int_{D\times\mathbb{R}} Y(y)e^{\sqrt{\lambda}\,z} f(x)\,dx > 0. \tag{3.9}$$

Proof. By a straightforward change of variables,

$$\begin{aligned}\int_{D\times\mathbb{R}} G(x)U(x,t;a)\,dx &= \int_D \int_{\mathbb{R}} Y(y)e^{\sqrt{\lambda}\,z} t^{-\mu} f(y, z - c\ln t - a)\,dz\,dy \\ &= t^{-\mu} e^{\sqrt{\lambda}\,(c\ln t + a)} \int_D \int_{\mathbb{R}} Y(y)e^{\sqrt{\lambda}\,\zeta} f(y,\zeta)\,d\zeta\,dy.\end{aligned}$$

This gives (3.8). Noting that Y is bounded in D, f is bounded in $D \times \mathbb{R}$ and vanishes uniformly with respect to $y \in D$ for large z, while $z \mapsto e^{\sqrt{\lambda}\,z}$ is integrable on any interval that is bounded above, it can be verified that κ is finite. □

Since $(x,t) \mapsto U(x,t+1;0)$ is a *bona fide* solution of problem (1.1) with initial data $U(\cdot,1;0)$ and $K = G$ when $\Omega = D \times \mathbb{R}$, combining Theorem 3.1 and Lemma 3.2 immediately gives the following.

Theorem 3.2 *The unique number $c > 0$ for which problem (2.4), (2.5), (2.6), (2.8) admits a solution is given by (1.7) where λ is the first eigenvalue of the eigenvalue problem for the Laplacian with homogeneous Dirichlet boundary conditions in D.*

With c given by (1.7), (3.8) can be reformulated as

$$\int_{D\times\mathbb{R}} G(x)U(x,t;a)\,dx = \kappa e^{\sqrt{\lambda}\,a} = \kappa e^{\mu a/c} \quad \text{for all } t > 0. \tag{3.10}$$

To close this section, we record the following consequence of the invariance principle for future use.

Lemma 3.3 *Let u be a solution of problem (1.1) with initial data satisfying Hypothesis 4.1 below, and a be any real number. Then there is a number Λ for which*

$$\int_\Omega K(x)\big|u(x,t) - U(x,t;a)\big|\,dx \to \Lambda \quad \text{as } t \to \infty. \tag{3.11}$$

Proof. Let us drop a from the notation of U, and define

$$w^+ = \max\{u, U\} \quad \text{and} \quad w^- = \min\{u, U\}$$

in $\Omega \times \mathbb{R}_+$. Because $|u - U| = w^+ - w^-$, to verify (3.11) it suffices to show that there are numbers $\Lambda^+ \geqslant \Lambda^- \geqslant 0$ such that

$$\int_\Omega K(x) w^\pm(x,t)\,dx \to \Lambda^\pm \qquad \text{as } t \to \infty. \tag{3.12}$$

Fix $\varepsilon > 0$, and let $T > \theta > 0$ be arbitrary. By (3.5) there exists an $L > 0$ such that $G \leqslant (1+\varepsilon)K$ in $D \times (L, \infty)$. Hence,

$$\begin{aligned}\int_\Omega G(x) w^+(x,\theta)\,dx &\leqslant \int_{\{x\in\Omega: z<L\}} G(x) w^+(x,\theta)\,dx + (1+\varepsilon)\int_{D\times(L,\infty)} K(x) w^+(x,\theta)\,dx \\ &\leqslant \theta^{-\mu} \int_{\{x\in\Omega: z<L\}} G(x)F(y)\,dx + (1+\varepsilon)\int_\Omega K(x) w^+(x,\theta)\,dx, \end{aligned} \tag{3.13}$$

by (2.6), (2.10), (2.13), and (3.7). Now, define

$$v_0(x) = \begin{cases} w^+(x,\theta) & \text{for } x \in \Omega, \\ U(x,\theta) & \text{for } x \in (D\times\mathbb{R}) \setminus \Omega, \end{cases}$$

and consider the solution v of problem (1.1) with Ω replaced by $D \times \mathbb{R}$ and initial data v_0. By Theorem 3.1,

$$\int_{D\times\mathbb{R}} G(x) v(x, T-\theta)\,dx = \int_{D\times\mathbb{R}} G(x) v_0(x)\,dx \tag{3.14}$$

where

$$\int_{D\times\mathbb{R}} G(x) v_0(x)\,dx = \int_\Omega G(x) w^+(x,\theta)\,dx + \int_{(D\times\mathbb{R})\setminus\Omega} G(x) U(x,\theta)\,dx \tag{3.15}$$

and

$$\int_{(D\times\mathbb{R})\setminus\Omega} G(x) U(x,\theta)\,dx \leqslant \theta^{-\mu} \int_{(D\times\mathbb{R})\setminus\Omega} G(x) F(y)\,dx \tag{3.16}$$

by (2.6), (2.10), and (3.7). However, by Lemma 2.2, $U(\cdot,T) \leqslant v(\cdot, T-\theta)$ in $D\times\mathbb{R}$ and $u(\cdot,T) \leqslant v(\cdot, T-\theta)$ in Ω. Hence,

$$\int_\Omega G(x) w^+(x,T)\,dx \leqslant \int_\Omega G(x) v(x,T-\theta)\,dx \leqslant \int_{D\times\mathbb{R}} G(x) v(x, T-\theta)\,dx. \tag{3.17}$$

Recalling (3.4), and combining (3.13)–(3.17), we deduce that

$$\int_\Omega K(x) w^+(x,T)\,dx \leqslant (1+\varepsilon)\int_\Omega K(x) w^+(x,\theta)\,dx + 2\theta^{-\mu}\int_{D\times(-\infty,L)} G(x)F(y)\,dx.$$

Taking the limit supremum as $T \to \infty$, followed by the limit infimum as $\theta \to \infty$, and finally the limit $\varepsilon \downarrow 0$, we obtain that part of (3.12) pertaining to w^+ for some finite $\Lambda^+ \geqslant 0$.

That part of (3.12) pertaining to w^- is obtained in a similar but more direct fashion. Considering the solution v of problem (1.1) with initial data $w^-(\cdot,\theta)$,

$$\int_\Omega K(x) v(x, T-\theta)\,dx = \int_\Omega K(x) w^-(x,\theta)\,dx$$

by Theorem 3.1. Simultaneously, Lemma 2.2 implies that $v(\cdot, T-\theta) \leqslant U(\cdot, T; a)$ and $v(\cdot, T-\theta) \leqslant u(\cdot, T)$ in Ω. So we derive

$$\int_\Omega K(x)w^-(x,T)\,dx \geqslant \int_\Omega K(x)w^-(x,\theta)\,dx$$

for all $T > \theta > 0$. This yields the desired conclusion with $0 \leqslant \Lambda^- \leqslant \infty$. By the definition of $w^\pm$, necessarily $\Lambda^- \leqslant \Lambda^+$. □

4. Convergence of the solution

Supplementarily to Ω being as in the previous section, in this section and the next we shall assume the following.

HYPOTHESIS 4.1 The function $u_0 \in L^1_{\mathrm{loc}}(\overline{\Omega})$ is nonnegative, positive on a subset of Ω of positive measure, and such that $Ku_0 \in L^1(\Omega)$.

Furthermore, on occasion, we shall invoke the next assumption as well.

HYPOTHESIS 4.2 The function u_0 satisfies (2.15) and is such that $u_0 = 0$ in $\{x \in \Omega : z > \ell\}$ for some $\ell \in \mathbb{R}$.

Recalling (3.9), the first of the above hypotheses allows us to define

$$a = \frac{1}{\sqrt{\lambda}} \ln\left(\frac{1}{\kappa}\int_\Omega K(x)u_0(x)\,dx\right). \tag{4.1}$$

The second is sufficient for the existence of the free boundary whose large-time behaviour we wish to investigate, in accordance with the lemma below.

Lemma 4.1 *Under Hypothesis 4.2, the solution u of problem (1.1) has the following properties for any $T > \vartheta > 0$.*

(i) *There exists an $L > \ell$ such that $u = 0$ in $\{x \in \Omega : z \geqslant L\} \times (0, T]$.*
(ii) *There exists an A such that $u(x,t) \leqslant (1-\vartheta/t)^{-\mu}\, U(x,t;A)$ for all $(x,t) \in \Omega \times [T,\infty)$.*

Proof. Let us first assume that part (i) is true for some $T > 0$ and let $\vartheta \in (0, T)$ be arbitrary. By this assumption, (2.11), and (2.13) with $t = T$, we can find an A^* so large that $f(y, z - c\ln(T-\vartheta) - A^*) \geqslant (T-\vartheta)^\mu u(x,T)$ for all $x \in \Omega$. Hence, by Lemma 2.2, $u(x,t) \leqslant U(x, t-\vartheta; A^*)$ for all $(x,t) \in \Omega \times [T,\infty)$. Recalling (2.10) and (3.7), noting that $\ln(t-\vartheta) - \ln t = \ln(1-\vartheta/t) \geqslant \ln(1-\vartheta/T)$ for all $t \geqslant T$, and setting $A = A^* - c\ln(1-\vartheta/T)$, there holds $U(x, t-\vartheta; A^*) \leqslant U(x,t;A)$ for all such T. Thus if part (i) is true, then part (ii) is true too.

Now, fix $L > \ell$, and, let $\tilde{u}$ be the solution of the Cauchy problem for the porous media equation in $\mathbb{R}^{n+1} \times (0, \Upsilon)$ given by Lemma 2.3. By [19, Theorem 14.13] there exists a $\theta \in (0, \Upsilon)$ such that $\tilde{u} = 0$ in $\{x \in \mathbb{R}^N : z \geqslant L\} \times (0, \theta]$. Since $\tilde{u} \geqslant u$ in $\Omega \times (0, \Upsilon)$, this yields part (i) for $T = \theta$. Consequently, the conclusion of part (ii) is true for $\vartheta = \theta/2$ and $T = \theta$. In turn, taken together with part (i) for $T = \theta$, this deduction implies that part (i) actually holds for every $T > 0$. Hence, pulling ourselves up by our bootstraps as it were, we have parts (i) and (ii) with the desired generality. □

Let us next recall the conventions (2.1)–(2.3) and introduce the change of variables

$$\eta = y, \quad \zeta = z - c\ln t, \quad \tau = \ln t, \quad \text{and} \quad \hat{u} = t^\mu u.$$

Lemma 4.2 *If u is a solution of problem* (1.1) *then*

$$\hat{u}(\xi,\tau) = e^{\mu\tau}u(\eta,\zeta + c\tau, e^{\tau}) \tag{4.2}$$

is a weak solution of the equation

$$\partial_\tau \hat{u} = \nabla\cdot(\nabla\hat{u}^m + \hat{u}\boldsymbol{c}) + \mu\hat{u} \tag{4.3}$$

in $Q = \{(\xi,\tau) \in D\times\mathbb{R}^2 : (\eta,\zeta + c\tau) \in \Omega\}$ which is a classical solution of the equation at any point in Q where it is positive, and continuous in $\overline{Q}$. Conversely, if $\hat{u}$ denotes a solution of (4.3) *in $D\times\mathbb{R}^2$ in the sense of distributions that is continuous in $\overline{D}\times\mathbb{R}^2$, then*

$$u(x,t) = t^{-\mu}\hat{u}(y, z - c\ln t, \ln t). \tag{4.4}$$

defines a solution of the porous media equation in $D\times\mathbb{R}\times\mathbb{R}_+$ in the sense of distributions which is continuous in $\overline{D}\times\mathbb{R}\times\mathbb{R}_+$.

Note that if $\hat{U}$ is to U what $\hat{u}$ is to u, then

$$\hat{U}(\xi,\tau;a) = U(\xi,1;a) \quad \text{for all } (\xi,\tau)\in D\times\mathbb{R}^2.$$

Now, let u be an arbitrary solution of problem (1.1) whose initial data satisfy Hypothesis 4.1. Define a by (4.1), and $\hat{u}$ in $\{(\xi,\tau)\in D\times\mathbb{R}^2 : (\eta,\zeta + c\tau)\in\Omega\}$ by (4.2). Next, for $s > 0$, set

$$\hat{u}_s(\xi,\tau) = \hat{u}(\xi, s+\tau) \quad \text{for } (\xi,\tau)\in Q_s, \tag{4.5}$$

where

$$Q_s = \{(\xi,\tau)\in D\times\mathbb{R}^2 : \zeta + c(s+\tau) > 0\}.$$

By (1.11) and Lemma 4.2, $\hat{u}_s \in C(\overline{Q_s})$ is a weak solution of (4.3) in Q_s and a classical solution of that equation in $\{(\xi,\tau)\in Q_s : \hat{u}_s(\xi,\tau) > 0\}$.

Lemma 4.3 (i) *There holds $Q_s \supset Q_\varsigma$ for all $s > \varsigma > 0$, and $\cup_{s>0}Q_s = D\times\mathbb{R}^2$.*
(ii) *There holds $\hat{u}_s(\xi,\tau) \leqslant F(\eta)$ for all $(\xi,\tau)\in Q_s$ and $s > 0$.*
(iii) *The family $\{\hat{u}_s\}_{s\geqslant\varsigma}$ is equicontinuous in $\overline{Q_\varsigma}$ for all $\varsigma > 0$.*
(iv) *Given any $(\xi^*,\tau^*)\in Q_\varsigma$ such that $\hat{u}_s(\xi^*,\tau^*) > \delta > 0$ for all $s\geqslant\varsigma > 0$, there exists an open set $\mathbb{Q}\subset Q_\varsigma$, a number $\alpha\in(0,1)$, and a number $\mathfrak{C} > 0$ such that $(\xi^*,\tau^*)\in\mathbb{Q}$ and $\|\hat{u}_s\|_{C^{2+\alpha}(\mathbb{Q})}\leqslant\mathfrak{C}$ for all $s\geqslant\varsigma$.*
(v) *Hypothesis 4.2 implies that given any $\vartheta\in(0,1)$ there exists an A such that $\hat{u}_s(\xi,\tau)\leqslant(1-\vartheta e^{-s})^{-\mu}U(\xi,1;A)$ for all $(\xi,\tau)\in Q_s$ and $s\geqslant-\tau$.*

Proof. Property (i) is given by the definition of Q_s. Property (ii) is a corollary of (2.13). Property (iii) subsequently follows from regularity estimates of DiBenedetto [7]. Note that to utilize these, we have to write equation (4.3) in the form

$$\partial_\tau v^{1/m} = \nabla\cdot\boldsymbol{a}(v,\nabla v) + \mu b(v)$$

where

$$\boldsymbol{a}(v,\boldsymbol{p}) = \boldsymbol{p} + b(v)\boldsymbol{c}$$

and $b \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is such that $b(v) = v^{1/m}$ for $0 \leqslant v \leqslant \max\{F^m(\eta) : \eta \in D\}$. Then, using Young's inequality, we can satisfy the condition

$$\boldsymbol{a}(v, \boldsymbol{p}) \cdot \boldsymbol{p} = |\boldsymbol{p}|^2 + b(v)\boldsymbol{c} \cdot \boldsymbol{p} \geqslant (1-\varepsilon)\,|\boldsymbol{p}|^2 - \frac{|\boldsymbol{c}|^2}{4\varepsilon}\,\|b\|^2_{L^\infty(\mathbb{R})}$$

for any $\varepsilon \in (0, 1)$. Property (iv) is a consequence of the classical theory of parabolic equations [12, Theorems III.10.1 and III.12.1]. With regard to property (v), Lemma 4.1(ii) says that given any $T > \vartheta > 0$ there exists an A such that each $\hat{u}_s(\xi, \tau) \leqslant \left(1 - \vartheta e^{-(s+\tau)}\right)^{-\mu} U(\xi, 1; A)$ for all $(\xi, \tau) \in Q_s$ with $\tau \geqslant \ln T - s$. Choosing $T = 1$ gives the property. □

From the above lemma, it follows that any unbounded increasing sequence $\{s_i\}_{i\in\mathbb{N}} \subset \mathbb{R}_+$ has a like subsequence which we again denote by $\{s_i\}_{i\in\mathbb{N}}$ such that $\hat{u}_{s_i}$ converges to a nonnegative function $\hat{u}_\infty$ as $i \to \infty$ uniformly on compact subsets of $\overline{D} \times \mathbb{R}^2$. Furthermore, $\hat{u}_\infty$ is uniformly continuous in $\overline{D} \times \mathbb{R}^2$,

$$\hat{u}_\infty(\xi, \tau) \leqslant F(\eta) \quad \text{for all } (\xi, \tau) \in D \times \mathbb{R}^2, \tag{4.6}$$

$\hat{u}_\infty = 0$ on $\partial D \times \mathbb{R}^2$, $\hat{u}_\infty$ is a solution of equation (4.3) in $D \times \mathbb{R}^2$ in the sense of distributions, and, $\hat{u}_\infty \in C^{2+\alpha}(\mathfrak{Q})$ for some $\alpha \in (0, 1)$ in some neighbourhood $\mathfrak{Q}$ of any point in $D \times \mathbb{R}^2$ where it is positive. Thus, $\hat{u}_\infty$ is a classical solution of equation (4.3) in $\{(\xi, \tau) \in D \times \mathbb{R}^2 : \hat{u}_\infty(\xi, \tau) > 0\}$. Moreover, under the additional Hypothesis 4.2, the convergence is uniform on all sets of the form $\overline{D} \times [\alpha, \infty) \times [\tau_0, \tau_1]$.

Our further considerations comprise the analysis of the Lyapunov function

$$\mathfrak{L}(\tau) = \int_{D\times\mathbb{R}} G(\xi)\left|\hat{u}_\infty(\xi, \tau) - U(\xi, 1; a)\right| d\xi$$

for $\tau \in \mathbb{R}$. We split our argument into three lemmata.

Lemma 4.4 *There holds*

$$\int_{D\times\mathbb{R}} G(\xi)\hat{u}_\infty(\xi, \tau)\, d\xi = \kappa e^{\mu a/c} \quad \textit{for all } \tau \in \mathbb{R}. \tag{4.7}$$

Proof. Using (2.13), (3.1), and (3.4), it can be shown that

$$\int_{\{x\in\Omega:z<0\}} K(x)u(x, t)\, dx \to 0 \quad \text{as } t \to \infty.$$

Hence, by (3.6) and (4.1),

$$\int_{D\times\mathbb{R}_+} K(x)u(x, t)\, dx \to \kappa e^{\mu a/c} \quad \text{as } t \to \infty.$$

Substituting (4.4) and $t = s + \tau$ for fixed $\tau \in \mathbb{R}$ in the above, it follows that

$$\int_{D\times(-c(s+\tau),\infty)} e^{-\mu(s+\tau)} K(\eta, \zeta + c(s+\tau))\hat{u}_s(\xi, \tau)\, d\xi \to \kappa e^{\mu a/c} \quad \text{as } s \to \infty. \tag{4.8}$$

Now, (3.1) and (3.5) imply that $e^{-\mu L} K(\eta, \zeta + cL) \to G(\xi)$ as $L \to \infty$ for each $\xi \in D \times \mathbb{R}$. Thus, setting $s = s_i$ in (4.8), passing to the limit $i \to \infty$, and applying Fatou's Lemma, we obtain (4.7) with "$\leqslant$" in lieu of "$=$". Moreover when Hypothesis 4.2 holds, applying the Dominated Convergence Theorem rather than Fatou's Lemma, which is justified by Lemma 4.3 part (v), we obtain (4.7) with the asserted equality.

It remains to establish (4.7) with "$\geqslant$" instead of "$=$" when Hypothesis 4.2 does not hold. For this purpose, consider an auxiliary initial-data function $u_0^* \leqslant u_0$ that satisfies both Hypotheses 4.1 and 4.2. Denote the corresponding solution of problem (1.1) by u^*, the corresponding number (4.1) by $a^* \leqslant a$, and the corresponding function defined via (4.2) and (4.5) by $\hat{u}_s^*$. By Lemma 2.2, $u^* \leqslant u$ in $\Omega \times \mathbb{R}_+$. The preceding analysis implies that $\{s_i\}_{i\in\mathbb{N}}$ contains a subsequence $\{s_{i_j}\}_{j\in\mathbb{N}}$ such that $\hat{u}_{s_{i_j}}^*$ converges to a function $\hat{u}_\infty^*$ analogous to $\hat{u}_\infty$ uniformly on all sets of the form $\overline{D} \times [\alpha, \infty) \times [\tau_0, \tau_1]$. For this limit function, *mutatis mutandis* (4.7) does hold with equality. Moreover, $\hat{u}_\infty^* \leqslant \hat{u}_\infty$ in $D \times \mathbb{R}^2$. Thus, the actual left-hand side of (4.7) is bounded below by $\kappa e^{\mu a^*/c}$. In view of the arbitrariness of u_0^* and therewith a^*, this yields (4.7) with "$\geqslant$" where we already had the reverse inequality. □

Lemma 4.5 *Let Λ be the number in Lemma 3.3. Then $\mathfrak{l}(\tau) = \Lambda$ for all $\tau \in \mathbb{R}$.*

Proof. Reapplying the argument employed in the proof of the previous lemma to (3.11) yields $\mathfrak{l}(\tau) \leqslant \Lambda$ for every $\tau \in \mathbb{R}$, with equality if Hypothesis 4.2 holds. To deduce equality without Hypothesis 4.2, let u^*, a^*, and $\hat{u}_\infty^*$ be as in the proof of the previous lemma. Let Λ^* be the number given by Lemma 3.3 with u replaced by u^* and a not replaced. For every $(x, t) \in \Omega \times \mathbb{R}_+$,

$$\big|u(x,t) - U(x,t;a)\big| \leqslant u(x,t) - u^*(x,t) + \big|u^*(x,t) - U(x,t;a)\big|.$$

Hence, multiplying by $K(x)$, integrating with respect to x over Ω, recalling (3.6) and (4.1), and letting $t \to \infty$,

$$\Lambda \leqslant \kappa(e^{\mu a/c} - e^{\mu a^*/c}) + \Lambda^*. \tag{4.9}$$

On the other hand, for every $(\xi, \tau) \in D \times \mathbb{R}^2$,

$$\big|\hat{u}_\infty^*(\xi,\tau) - U(\xi,1;a)\big| \leqslant \hat{u}_\infty(\xi,\tau) - \hat{u}_\infty^*(\xi,\tau) + \big|\hat{u}_\infty(\xi,\tau) - U(\xi,1;a)\big|.$$

Therefore, multiplying by $G(\xi)$, integrating with respect to ξ over $D \times \mathbb{R}$, and recalling (4.7),

$$\Lambda^* \leqslant \kappa(e^{\mu a/c} - e^{\mu a^*/c}) + \mathfrak{l}(\tau). \tag{4.10}$$

Together (4.9) and (4.10) imply that $\mathfrak{l}(\tau) \geqslant \Lambda - 2\kappa(e^{\mu a/c} - e^{\mu a^*/c})$ whatever $\tau \in \mathbb{R}$. Consequently, passing to the limit $a^* \uparrow a$ and noting that we have already established that $\mathfrak{l}(\tau) \leqslant \Lambda$, we obtain the desired result. □

Lemma 4.6 *There holds $\mathfrak{l}(\tau) = 0$ for all $\tau \in \mathbb{R}$.*

Proof. We adapt an argument previously used in [9, 11, 16]. Care is needed because we know less about the function $\hat{u}_\infty$ than its counterpart in each of the cited papers.

Let u_∞ be the function defined from $\hat{u}_\infty$ through (4.4). By Lemma 4.2, it is a solution of the porous media equation in $D \times \mathbb{R} \times \mathbb{R}_+$ in the sense of distributions, a classical solution of the equation in a neighbourhood of any point where it is positive, and vanishes on $\partial D \times \mathbb{R} \times \mathbb{R}_+$. Furthermore, by (4.6) and the uniform continuity of $\hat{u}_\infty$ in $\overline{D} \times \mathbb{R}^2$, u_∞ is bounded and uniformly continuous in $\overline{D} \times \mathbb{R} \times (\theta, \infty)$ for all $\theta \in \mathbb{R}_+$. Via the transformation (4.4), the identity (4.7) becomes

$$\int_{D\times\mathbb{R}} G(x) u_\infty(x,t)\, dx = \kappa e^{\mu a/c} \qquad \text{for all } t > 0, \tag{4.11}$$

while the outcome of Lemma 4.5 can be formulated as

$$\int_{D\times\mathbb{R}} G(x)\big|u_\infty(x,t) - U(x,t;a)\big|\,dx = \Lambda \quad \text{for all } t > 0. \tag{4.12}$$

For $(x,t) \in D \times \mathbb{R} \times \mathbb{R}_+$, we set

$$w(x,t) = \max\big\{u_\infty(x,t), U(x,t;a)\big\}. \tag{4.13}$$

Noting that $\max\{\alpha,\beta\} = (\alpha+\beta+|\alpha-\beta|)/2$ for any numbers α and β, and that (3.10), (4.11) and (4.12) hold,

$$\int_{D\times\mathbb{R}} G(x)w(x,t)\,dx = \kappa e^{\mu a/c} + \Lambda/2 \quad \text{for all } t > 0. \tag{4.14}$$

Now, suppose that there exist points $(x^\pm,\theta) \in D \times \mathbb{R} \times \mathbb{R}_+$ such that

$$u_\infty(x^+,\theta) > U(x^+,\theta;a) \quad \text{and} \quad u_\infty(x^-,\theta) < U(x^-,\theta;a). \tag{4.15}$$

In such an event, let v denote the solution of problem (1.1) with $\Omega = D\times\mathbb{R}$ and initial data $w(\cdot,\theta)$. By (4.13) and Lemma 2.2, $v(x,t) \geqslant U(x,t+\theta;a)$ for all $(x,t) \in D\times\mathbb{R}\times\mathbb{R}_+$, while by (4.13) and Lemma 2.4, $v(x,t) \geqslant u_\infty(x,t+\theta)$ for all such (x,t). Therefore,

$$v(x,t) \geqslant w(x,t+\theta) \quad \text{for all } (x,t) \in D\times\mathbb{R}\times\mathbb{R}_+. \tag{4.16}$$

On the other hand, because of the invariance of solutions of problem (1.1) with $\Omega = D \times \mathbb{R}_+$ dictated by Theorem 3.1, and, (4.14) for $t = \theta$,

$$\int_{D\times\mathbb{R}} G(x)v(x,t)\,dx = \kappa e^{\mu a/c} + \Lambda/2 \quad \text{for all } t > 0. \tag{4.17}$$

Taken together, (4.14), (4.16), and (4.17) imply that (4.16) holds with equality.

Next, recalling that D is connected, let $\mathcal{K}$ be a closed path in $D\times\mathbb{R}$ with endpoints $x^\pm$. Since $\mathcal{K}$ can be contained within a bounded open subset of $D\times\mathbb{R}$ that satisfies a uniform interior ball condition, part (ii) of Lemma 2.5 implies that there is a $T > 0$ such that $v > 0$ on $\mathcal{K}\times\{T\}$. Take $\mathcal{P}$ to be the largest connected component of $\{(x,t) \in D\times\mathbb{R}\times[0,T] : v(x,t) > 0\}$ that contains $\mathcal{K}\times\{T\}$. Consequently, if $v(x,T) = u_\infty(x,T+\theta)$ for some $x \in \mathcal{K}$ then the Strong Maximum Principle implies that $v(x,t) = u_\infty(x,t+\theta)$ for every $(x,t) \in \mathcal{P}$. Likewise, if $v(x,T) = U(x,T+\theta;a)$ for some $x \in \mathcal{K}$ then $v(x,t) = U(x,t+\theta;a)$ for every $(x,t) \in \mathcal{P}$. Whichever, since $\{x^\pm\}\times[0,T] \subset \mathcal{P}$ by part (i) of Lemma 2.5, we have a contradiction of (4.15).

The exclusion of (4.15) for any $\theta \in \mathbb{R}_+$ implies that for every $t \in \mathbb{R}_+$ either $u_\infty(\cdot,t) \geqslant U(\cdot,t;a)$ everywhere in $D\times\mathbb{R}$ or the reverse inequality holds. Either way, from (3.10), (4.11), (4.13), and (4.14) it follows that $\Lambda = 0$; in the light of which, the present lemma is the ultimate restatement of the previous one. □

From the above lemma, we deduce that $\hat{u}_\infty = U(\cdot,1;a)$ in $D\times\mathbb{R}^2$. We thus conclude that the whole family $\{\hat{u}_s\}_{s>0}$ converges to $U(\cdot,1;a)$ as $s \to \infty$ uniformly on compact subsets of $\overline{D}\times\mathbb{R}^2$. Therefore, $\hat{u}(\cdot,\tau)$ converges to $U(\cdot,1;a)$ as $\tau \to \infty$ uniformly on all compact subsets of $\overline{D}\times\mathbb{R}$. Moreover, under Hypothesis 4.2, the convergence is uniform on all sets of the form $\overline{D}\times(\alpha,\infty)$. In view of Lemma 4.3 parts (ii) and (v), the afore-stated uniform convergence converts to L^p-convergence on the said sets for every $p \geqslant 1$.

Transposing the above conclusions to the original variables gives the first of the two theorems below.

Theorem 4.1 *Let Ω be a domain of the type (1.11) fulfilling Hypotheses 2.1 and 2.2, and u_0 be a function satisfying Hypothesis 4.1. Define a by (4.1). Then given any $\alpha \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$, the solution u of problem (1.1) is such that*

$$t^{\mu}\big\|u(\cdot,t)-U(\cdot,t;a)\big\|_{L^p(D\times(\alpha+c\ln t,\beta+c\ln t))} \to 0 \quad \text{as } t\to\infty$$

for every $\beta > \alpha$. Moreover, if Hypotheses 4.2 holds, then

$$t^{\mu}\big\|u(\cdot,t)-U(\cdot,t;a)\big\|_{L^p(D\times(\alpha+c\ln t,\infty))} \to 0 \quad \text{as } t\to\infty.$$

Theorem 4.2 *Suppose that $\Omega = D\times\mathbb{R}$ and further to the hypotheses in Theorem 4.1 that* $\liminf_{z\to-\infty} u_0(x)/F(y) > 0$ *uniformly with respect to $y \in D$. Then*

$$t^{\mu}\big\|u(\cdot,t)-U(\cdot,t;a)\big\|_{L^\infty(D\times(-\infty,\beta+c\ln t))} \to 0 \quad \text{as } t\to\infty$$

for all $\beta\in\mathbb{R}$, and hence when Hypothesis 4.2 holds,

$$t^{\mu}\big\|u(\cdot,t)-U(\cdot,t;a)\big\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t\to\infty.$$

Proof. Fix $\beta\in\mathbb{R}$ and $\varepsilon > 0$. The extra assumption in this theorem implies that there exists a $\delta > 0$ and a $z^* \in \mathbb{R}$ such that $u_0(x) \geqslant \delta F(y)$ for all $x \in D\times(-\infty,z^*)$. In the light of (2.8), (2.11), and (3.7), this in turn implies that there exist a $\theta > 0$ and an $\mathsf{a} < a$ such that $u_0 \geqslant U(\cdot,\theta;\mathsf{a})$ in Ω. Hence, by Lemma 2.2, $u(x,t) \geqslant U(x,t+\theta;\mathsf{a})$ for all $(x,t)\in\Omega\times\mathbb{R}_+$. Recalling (2.10), this gives $u(x,t) \geqslant (1+\theta/t)^{-\mu}U(x,t;\mathsf{a})$, which converts into $\hat{u}(\xi,\tau) \geqslant (1+\theta e^{-\tau})^{-\mu}U(\xi,1;\mathsf{a})$ for all $(\xi,\tau)\in\Omega\times\mathbb{R}$. So, $|\hat{u}(\xi,\tau)-U(\xi,1;a)| \leqslant \max\{F(\eta)-U(\xi,1;a), |U(\xi,1;a)-(1+\theta e^{-\tau})^{-\mu}U(\xi,1;\mathsf{a})|\}$ for all such (ξ,τ). Utilizing (2.11) once more, we can subsequently find a Θ and $\alpha < \beta$ such that $|\hat{u}(\xi,\tau)-U(\xi,1;a)| \leqslant \varepsilon F(\eta)$ for all $(\xi,\tau)\in D\times(-\infty,\alpha]\times[\Theta,\infty)$. However, from the proof Theorem 4.1, we know that there then exists a $T \geqslant \Theta$ such that $|\hat{u}(\xi,\tau)-U(\xi,1;a)| \leqslant \varepsilon$ for all $(\xi,\tau)\in D\times(\alpha,\beta)\times[T,\infty)$. Thus,

$$\big|\hat{u}(\xi,\tau)-U(\xi,1;a)\big| \leqslant \varepsilon\{1+F(\eta)\} \qquad \text{for all } (\xi,\tau)\in D\times(-\infty,\beta)\times[T,\infty).$$

In view of the arbitrariness of ε and the boundedness of F, this yields $\hat{u}(\cdot,\tau)\to U(\cdot,1;a)$ as $\tau\to\infty$ uniformly on $D\times(-\infty,\beta)$. Transferring this deduction back to the original coordinate system gives the primary conclusion of the theorem. The secondary one follows from the observation that under Hypothesis 4.2 one may choose $\beta > A+\sup\{\sigma(y) : y\in D\}$ where A is prescribed by Lemma 4.1 part (ii). □

We are now in a position to prove the uniqueness of the function f.

Theorem 4.3 *Modulo translation with respect to z, problem (2.4)–(2.7) has at most one nonnegative solution f such that $Gf \in L^1(D\times\mathbb{R})$.*

Proof. Suppose that next to the constructed solution f, that has been fixed in Section 2, there is another nonnegative solution, g say. Define

$$v(x,t) = (t+1)^{-\mu} g\big(y, z-c\ln(t+1)\big).$$

Then v is a strong solution of the porous media equation in $D\times\mathbb{R}\times\mathbb{R}_+$ that vanishes on $\partial D\times\mathbb{R}\times\mathbb{R}_+$. Define a^* by (4.1) with $D\times\mathbb{R}$ and $Gv(\cdot,0)$ in lieu of Ω and Ku_0. By Theorem 4.1,

$t^\mu \sup\{|v(x,t) - U(x,t;a^*)| : x \in D \times (\alpha + c\ln t, \beta + c\ln t)\} \to 0$ as $t \to \infty$ for every $\alpha < \beta$. Hence, by the transformation of variables $(x,t) = (\eta, \zeta + c\tau, e^\tau - 1)$,

$$(1 - e^{-\tau})^\mu g(\xi) - f(\eta, \zeta - c\ln(1 - e^{-\tau}) - a^*) \to 0 \quad \text{as } \tau \to \infty$$

uniformly on compact subsets of $\overline{D} \times \mathbb{R}$. Thus, $g(\xi) = f(\eta, \zeta - a^*)$ for every $\xi \in D \times \mathbb{R}$. □

Any ambiguity regarding the function f can now be avoided. The particular solution of problem (2.4)–(2.7) constructed in [18] generates a one parameter family of solutions by translation with respect to z. We can select a unique f by any criterion that isolates a single member of this translation class. For instance, we could use (2.10) to select f to be that member for which $\max\{f(y,z) : y \in D\} < \max\{F(y)/2 : y \in D\}$ if and only if $z \in \mathbb{R}_+$. Alternatively, we could require $\inf\{\sigma(y) : y \in D\} = 0$ or $\int_D \sigma(y)\,dy = 0$. A further option would be to choose f so that (3.9) prescribes $\kappa = 1$.

5. Interfacial behaviour

In this section, we study the large-time behaviour of the free boundary in the solution u of problem (1.1).

Our first theorem gives a lower bound on the growth of the positivity set of $u(\cdot,t)$ for large t, and is independent of any assumptions on the support of the initial data u_0.

Theorem 5.1 *Let Ω, u_0, and a be as in Theorem 4.1. Let σ be defined by (2.9) and D_δ by (2.16). Then given any $\delta > 0$ and $\varepsilon > 0$ there exists a $T > 0$ such that $u(x,t) > 0$ for all $0 < z \leqslant c\ln t + \sigma(y) + a - \varepsilon$, $y \in D_\delta$, and $t \geqslant T$.*

Proof. Fix $\delta > 0$ and $\varepsilon > 0$. Without loss of generality we may suppose that δ is so small that D_δ is not empty. Following the notation introduced in (4.2), define $\mathcal{K} = \{(\eta, \sigma(\eta) + a - \varepsilon) : \eta \in D_\delta\}$ and $\iota = \min\{U(\xi, 1; a) : \xi \in \mathcal{K}\} > 0$. In the course of the analysis in the previous section, we proved that $\hat{u}(\cdot,\tau)$ converges to $U(\cdot,1;a)$ as $\tau \to \infty$ uniformly on all sets of the form $D \times (\alpha,\beta)$ with $\alpha < \beta$. Hence, there exists a $\Theta > 0$ so large that $|\hat{u}(\xi,\tau) - U(\xi,1;a)| \leqslant \iota/2$ for all $\xi \in \mathcal{K}$ and $\tau \geqslant \ln\Theta$. In terms of the solution of problem (1.1), this means that $u(x,t) \geqslant \iota/2$ for all $z = c\ln t + \sigma(y) + a - \varepsilon$, $y \in D_\delta$, and $t \geqslant \Theta$. Thus, by Lemma 2.5(i), given any $t \geqslant \Theta$ and $y \in D_\delta$ we have $u(x,t) > 0$ for all $z \in [c\ln\Theta + \sigma(y) + a - \varepsilon, c\ln t + \sigma(y) + a - \varepsilon]$.

Now, without loss of generality, we may suppose that Θ is so large that $c\ln\Theta + \sigma + a - \varepsilon > \delta$ in D_δ. Subsequently defining Ω' as the set of $x' \in \Omega$ for which $|x' - x| < \delta$ for some $y \in D_\delta$ and $z \in [\delta, c\ln\Theta + \sigma(y) + a - \varepsilon]$, Lemma 2.5(ii) says that $\Omega' \subset P(T)$ for large enough $T \geqslant \Theta$. In the light of Lemma 2.5(i), this gives $u(x,t) > 0$ for all $z \in (0, c\ln\Theta + \sigma(y) + a - \varepsilon]$, $y \in D_\delta$, and $t \geqslant T$. Combining this conclusion with that ending the preceding paragraph, the proof of the theorem is complete. □

We turn next to an upper bound on the growth of the positivity set of $u(\cdot,t)$ for large t. We deduce a counterpart to the preceding theorem in two steps. First, we obtain a pointwise upper bound of $\{z \in \mathbb{R} : x \in P(t)\}$ for fixed $y \in D$. This is the content of the lemma below. Thereafter, we convert the pointwise bound into a uniform one.

Lemma 5.1 *Suppose that Hypothesis 4.2 holds. Then given any $y \in D$ and $\epsilon > 0$ there exists a $T > 0$ such that $u(x,t) = 0$ for all $z \geqslant c\ln t + \bar{\sigma}(y) + a + \epsilon$ and $t \geqslant T$, where $\bar{\sigma}$ is the concave envelope of σ.*

Proof. Fix $y \in D$. Let H be any hyperplane in $\mathbb{R}^{n+1}$ with the property that $(y, \bar{\sigma}(y)) \in H$ and the intersection of H with the positivity set of f is empty. Let ν be the unique point in $\mathbb{R}^n$, for which, with the notation introduced in the statement of Lemma 2.6, the hyperplane H is given by $\zeta = \bar{\sigma}(y) + \nu \cdot (\eta - y)$. Define

$$\mathfrak{N} = \left(|\nu|^2 + 1\right)^{1/2}.$$

We shall use a comparison argument involving the function $\hat{u}$ defined by (4.2) and a suitably constructed travelling-wave solution of equation (4.3) on a set $Q \subset D \times \mathbb{R}^2$. Let us fix $\omega \in (0, c/\mathfrak{N})$. From the analysis of travelling-wave solutions of reaction-diffusion equations in [10, Theorem 7.1], there exists a $\delta > 0$ such that the ordinary differential equation

$$(\phi^m)'' + \omega\phi' + \mu\phi = 0$$

has a weak solution ϕ in $(-\delta, \infty)$ for some $\delta > 0$ with the properties that $\phi^m \in C^1(-\delta, \infty) \cap C^\infty(-\delta, 0)$, $(\phi^m)' < 0$ on $(-\delta, 0)$, and $\phi = 0$ on $[0, \infty)$. This corresponds to a travelling-wave solution of the equation $\partial_\tau u = \partial_\zeta^2 u^m + \mu u$ of the self-similar form $u = \phi(\zeta - \omega\tau)$. Consequently for any number b the nonnegative function

$$w(\xi, \tau; b) = \phi\big(\{\zeta - \nu \cdot \eta + (c - \omega\mathfrak{N})\,\tau - b\}/\mathfrak{N}\big) \tag{5.1}$$

is a travelling-wave solution of equation (4.3) in the domain $\{(\xi, \tau) \in \mathbb{R}^{n+2} : \zeta > \nu \cdot \eta - (c - \omega\mathfrak{N})\,\tau + b - \mathfrak{N}\delta\}$. Moreover, $w(\cdot, \cdot; b)$ is of class $L^\infty \cap W^{1,1}_{\mathrm{loc}}$ in this domain, of class C^∞ everywhere except on the hypersurface $\zeta = \nu \cdot \eta - (c - \omega\mathfrak{N})\,\tau + b$, while $w^m(\cdot, \cdot; b)$ is of class C^1 throughout. Let us now fix $0 < \epsilon < \delta$ and

$$0 < \iota < \phi(-\epsilon/\mathfrak{N}). \tag{5.2}$$

In view of the choice of H and the uniform convergence of $\hat{u}(\cdot, \tau) \to U(\cdot, 1; a)$ on sets of the form $\overline{D} \times [\alpha, \infty)$, there exists a $T_0 \in \mathbb{R}_+$ such that $\hat{u}$ is defined and

$$\hat{u} \leqslant \iota \quad \text{in} \quad Q_0 = \big\{\xi \in D \times \mathbb{R} : \zeta > \nu \cdot (\eta - y) + \bar{\sigma}(y) + a\big\} \times (T_0, \infty). \tag{5.3}$$

Subsequently, Lemma 4.1(ii) says that there is an $\mathsf{A} > a$ such that

$$\hat{u} = 0 \quad \text{in} \quad \big\{\xi \in D \times \mathbb{R} : \zeta \geqslant \nu \cdot (\eta - y) + \bar{\sigma}(y) + \mathsf{A}\big\} \times [T_0, \infty). \tag{5.4}$$

Define

$$\theta = (\mathsf{A} - a)\,/\,(c - \omega\mathfrak{N})\,, \tag{5.5}$$

let $T_1 \geqslant T_0$ be arbitrary, and set

$$b = -\nu \cdot y + (c - \omega\mathfrak{N})\,T_1 + \bar{\sigma}(y) + \mathsf{A} \tag{5.6}$$

and

$$Q = \big\{(\xi, \tau) \in D \times \mathbb{R} \times (T_1, T_1 + \theta] : \zeta > \nu \cdot \eta - (c - \omega\mathfrak{N})\,\tau + b\big\}. \tag{5.7}$$

The function $w(\cdot, \cdot; b + \epsilon)$ is defined in $\overline{Q}$, and $Q \subset Q_0$ by (5.3) and (5.5)–(5.7). There holds $\hat{u} = 0$ on $\{(\xi, \tau) \in \overline{Q} : \tau = T_1\}$ by (5.4), (5.6), and (5.7), and, $\hat{u} = 0$ on $\{(\xi, \tau) \in \overline{Q} : \eta \in \partial D\}$ by virtue of the boundary condition in problem (1.1). Moreover, $\hat{u} \leqslant \iota < w(\cdot, \cdot; b + \epsilon)$ on $\{(\xi, \tau) \in \overline{Q} :$

$\zeta = \nu \cdot \eta - (c - \omega \mathfrak{N})\,\tau + b\}$ by (5.1)–(5.3) and (5.5)–(5.7). Applying the transfer of variables in Lemma 4.2 to Q, $\hat{u}$, and w, in cognizance of the regularity of the latter, and then the comparison principle Lemma 2.6 to the outcome, we deduce that $\hat{u} \leqslant w(\cdot,\cdot;b+\epsilon)$ in Q. Hence, $\hat{u} = 0$ in $\{(\xi,\tau) \in Q : \zeta \geqslant \nu \cdot \eta - (c - \omega \mathfrak{N})\,\tau + b + \epsilon\}$. In particular, recalling (5.5) and (5.6), this implies that $\hat{u}(y,\zeta,T_1+\theta) = 0$ for all $\zeta \geqslant \bar{\sigma}(y) + a + \epsilon$. Thus, in view of the arbitrariness of $T_1 \geqslant T_0$, $\hat{u}(y,\cdot,\cdot) = 0$ in $[\bar{\sigma}(y)+a+\epsilon,\infty)\times[T_0+\theta,\infty)$. Transforming back to the original variables yields the desired conclusion with $T = \exp(T_0+\theta)$. □

Theorem 5.2 *Let Ω, u_0, and a be as in Theorem 4.1, and u_0 satisfy Hypothesis 4.2. Let σ be defined by* (2.9)*, and $\bar{\sigma}$ denote the concave envelope of σ in D. Then given any $\varepsilon > 0$ there exists a $T > 0$ such that $u(x,t) = 0$ for all $z \geqslant c\ln t + \bar{\sigma}(y) + a + \varepsilon$, $y \in D$, and $t \geqslant T$.*

Proof. We begin with an approximation of $\bar{\sigma}$ by a greater smooth concave function. Fix $\varepsilon > 0$. With no loss of generality we may take $\bar{\sigma}$ to be defined and concave in D, where D is the convex hull of D in $\mathbb{R}^n$. Since σ is bounded, $\bar{\sigma} \in C(\overline{\mathsf{D}})$. By a standard mollifying procedure, we can subsequently construct a function $\sigma_\varepsilon \in C^\infty(\mathbb{R}^n)$ such that $\bar{\sigma} \leqslant \sigma_\varepsilon \leqslant \bar{\sigma} + \varepsilon/2$ in D. Set $\mathfrak{N}_\varepsilon = (\sup\{|\nabla\sigma_\epsilon|^2\,(y) : y \in \mathsf{D}\} + 1)^{1/2}$, and, let $\bar{\sigma}_\varepsilon$ be the concave envelope of σ_ϵ in $\overline{\mathsf{D}}$. From the concavity of $\bar{\sigma}$ and the bounds on σ_ε, it follows that $\bar{\sigma} \leqslant \bar{\sigma}_\varepsilon \leqslant \bar{\sigma} + \varepsilon/2$ in D.

Let us now retrace the proof of Lemma 5.1 for arbitrary $y \in D$, with $\bar{\sigma}_\varepsilon$ instead of $\bar{\sigma}$. The point $\nu \in \mathbb{R}^n$ will depend on y. Hence, so too will $\mathfrak{N}$. However, $\mathfrak{N}$ is necessarily bounded above by $\mathfrak{N}_\varepsilon$. So $\omega \in (0, c/\mathfrak{N}_\varepsilon)$ can be chosen independently of y, and therefore δ likewise. Consequently, given any $\epsilon \in (0,\delta)$ we can fix ι in (5.2), T_0 in (5.3), and A in (5.4) independently of y. The remainder of the argument then leads to the conclusion that $u(x,t) = 0$ for all $z \geqslant c\ln t + \bar{\sigma}_\varepsilon(y) + a + \epsilon$ and $t \geqslant \exp\{T_0 + (\mathsf{A} - a)\,/\,(c - \omega\mathfrak{N})\}$. Hence, taking $\epsilon \leqslant \varepsilon/2$ and $T = \exp\{T_0 + (\mathsf{A} - a)\,/\,(c - \omega\mathfrak{N}_\varepsilon)\}$, we have $u(x,t) = 0$ for all $(x,t) \in \Omega \times [T,\infty)$ such that $z \geqslant c\ln t + \bar{\sigma}_\varepsilon(y) + a + \varepsilon/2$. This yields $u(x,t) = 0$ for all $(x,t) \in \Omega \times [T,\infty)$ such that $z \geqslant c\ln t + \bar{\sigma}(y) + a + \varepsilon$, where T does not depend on $y \in D$. □

Theorem 5.1 corroborates (1.9). In fact, it establishes that (1.9) holds uniformly on compact subsets of D. Theorem 5.2 correspondingly confirms (1.10) uniformly on D. It is to be noted that (1.9) and (1.10) are complementary at those points in D where $\sigma = \bar{\sigma}$. Consequently, at such points, the inequalities become equality, and the limit infimum and the limit supremum necessarily coincide.

6. The higher dimensional problem

In this section

$$N = n + k \qquad \text{where} \qquad k \geqslant 2,$$

and Ω is a domain of the type (1.12). We use the shorthand

$$r = |z|\,.$$

For $\nu \geqslant 0$, let I_ν denote the Modified Bessel Function of the First Kind of order ν. Define

$$\phi(s) = \begin{cases} s^{-(k-2)/2} I_{(k-2)/2}(s) & \text{for } s > 0, \\ 2^{-(k-2)/2}/\Gamma(k/2) & \text{for } s = 0, \end{cases} \tag{6.1}$$

where Γ denotes the Gamma Function. Next, set

$$Z(z) = \omega_k{}^{-1}(2\pi)^{1/2}\lambda^{(k-1)/2}\phi(\lambda^{1/2}r) \tag{6.2}$$

and

$$G(x) = Y(y)Z(z), \tag{6.3}$$

where $\omega_k = 2\pi^{k/2}/\Gamma(k/2)$ is the surface area of the sphere of radius 1 in $\mathbb{R}^k$, and, λ and Y are as in Section 3. Utilizing the properties of the Modified Bessel Functions of the First Kind [1, Section 9.6], it can be verified that $\phi \in C^2([0,\infty))$, $\phi''(s) + (k-1)s^{-1}\phi'(s) = \phi(s)$ for $s > 0$, and, $\phi'(0) = 0$. Hence, $\Delta Z = \lambda Z$ in $\mathbb{R}^k$. It follows that G is a classical solution of problem (3.2) with $\mathbb{R}^k$ in lieu of $\mathbb{R}$.

Lemma 6.1 *There exists a unique function $K \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

$$\begin{cases} \Delta K = 0 & \text{in } \Omega, \\ K = 0 & \text{on } \partial\Omega, \\ K(x) = G(x) + \mathrm{o}(1) & \text{as } r \to \infty \end{cases} \tag{6.4}$$

uniformly with respect to $y \in D$. Moreover, (3.4) *holds with equality if and only if $\Omega = D \times \mathbb{R}^k$, and,*

$$K(x)/G(x) \to 1 \quad \text{as } r \to \infty \tag{6.5}$$

uniformly with respect to $y \in D$.

Proof. The proof of this result is entirely analogous to that of Lemma 3.1. The only part that requires special attention is identifying the counterpart to the function H in a suitable subdomain of Ω. This is $H = G - \mathfrak{C}G_c$ in $D \times \{z \in \mathbb{R}^k : r > \varrho\}$, where the number $\varrho > 0$ is provided by (1.12), G_c denotes the function defined via (6.1)–(6.3) with the Modified Bessel Function of the Second Kind $K_{(k-2)/2}$ instead of $I_{(k-2)/2}$, and the number $\mathfrak{C} = (I_{(k-2)/2}/K_{(k-2)/2})(\lambda^{1/2}\varrho)$. □

Given the above solution K of problem (6.4), the following can be proven similarly to Theorem 3.1. This is the invariance principle, which will hopefully lead to the determination of the asymptotic behaviour of the solution of problem (1.1).

Theorem 6.1 *Let Ω be a domain of the type* (1.12) *for some $k \geqslant 2$ fulfilling Hypotheses 2.1 and 2.2, and, let $u_0 \in L^1_{\mathrm{loc}}(\overline{\Omega})$ be nonnegative. Then the solution u of problem* (1.1) *satisfies* (3.6).

The following is a corollary of Theorem 6.1, which will be of benefit in due course.

Lemma 6.2 *Let u be a solution of problem* (1.1) *with initial data satisfying Hypothesis 6.1 below, and let $\tilde{u}$ be any extension of u to $D \times \mathbb{R}^k \times \mathbb{R}_+$ satisfying $0 \leqslant \tilde{u}(x,t) \leqslant t^{-\mu}F(y)$ for all $(x,t) \in D \times \mathbb{R}^k \times \mathbb{R}_+$. Then*

$$\int_{D\times\mathbb{R}^k} G(x)\tilde{u}(x,t)\,dx \to \int_{\Omega} K(x)u_0(x)\,dx \quad \text{as } t \to \infty.$$

Proof. By (3.4) and (6.5), given any $\varepsilon > 0$ there exists an $R > \varrho$ such that $(1-\varepsilon)G \leqslant K \leqslant G$ in $\Omega' = D \times \{z \in \mathbb{R}^k : r > R\}$. Simultaneously, by (2.13), $u(\cdot,t)$ tends to 0 as $t \to \infty$ uniformly with respect to $x \in \Omega \setminus \Omega'$, while $\tilde{u}(\cdot,t)$ does likewise with respect to $x \in (D \times \mathbb{R}^k) \setminus \Omega'$. Combining these estimates with the conclusion of Theorem 6.1 leads to the result. □

Let us henceforth suppose the following.

HYPOTHESIS 6.1 The function $u_0 \in L^1(\Omega)$ is nonnegative, positive on a subset of Ω of positive measure, and such that $u_0 = 0$ in $\{x \in \Omega : r > \ell\}$ for some $\ell > \varrho$.

Lemma 6.3 *Under Hypothesis 6.1, the solution u of problem (1.1) has the following properties for any* $T > \vartheta > 0$.

(i) *There exists an* $R > \ell$ *such that* $u = 0$ *in* $\{x \in \Omega : r \geqslant R\} \times (0, T]$.

(ii) *There exists an A such that* $u(x,t) \leqslant (1-\vartheta/t)^{-\mu}\, U(y,r,t;A)$ *for all* $(x,t) \in \Omega \times [T,\infty)$.

Proof. Part (i) is a corollary of Lemma 2.3, since the solution $\tilde{u}$ of the Cauchy problem for the porous media equation in $\mathbb{R}^N \times \mathbb{R}_+$ given by said lemma is such that $\{x \in \mathbb{R}^N : \tilde{u}(x,t) > 0\}$ is bounded for every $t \in \mathbb{R}_+$ [19, Proposition 9.18]. To verify part (ii), let us first define $u^*(x,t) = u(x,t+T)$ for $(x,t) \in \Omega \times [0,\infty)$, next, $\tilde{u}_0 = u^*(\cdot,0)$ in Ω and $\tilde{u}_0 = 0$ elsewhere in $D \times \mathbb{R}^k$, and, thereafter $\bar{u}_0(y,s) = \max\{\tilde{u}_0(y,z) : z \in \mathbb{R}^k \text{ and } |z| = |s|\}$ for every $(y,s) \in D \times \mathbb{R}$. As in the proof of Lemma 4.1(ii), we can find an A^* so large that $\bar{u}_0 \leqslant U(\cdot, T-\vartheta; A^*)$ in $D \times \mathbb{R}$. We subsequently take an arbitrary $e \in \mathbb{R}^k$ with $|e| = 1$, and set $v(x,t) = U(y, z_1e_1 + z_2e_2 + \cdots + z_ke_k, t+T-\vartheta; A^*)$ for $(x,t) \in D \times \mathbb{R}^k \times [0,\infty)$. By construction, $u^*(\cdot,0) \leqslant v(\cdot,0)$ in Ω. Furthermore, u^* is a solution of problem (1.1) with initial data $u^*(\cdot,0)$, while v is a solution of problem (1.1) with Ω replaced by $D \times \mathbb{R}^k$ and initial data $v(\cdot,0)$ in this set. Hence, by Lemma 2.2, $u^* \leqslant v$ in $\Omega \times \mathbb{R}_+$. In view of the arbitrariness of e, this implies that $u(x,t) \leqslant U(y,r,t-\vartheta;A^*)$ for all $(x,t) \in \Omega \times [T,\infty)$. Setting $A = A^* - c\ln(1-\vartheta/T)$, the proof of part (ii) of the present lemma can be completed along the lines of the proof of Lemma 4.1 part (ii). □

With $P(t)$ given by (1.2), it follows from Lemma 6.3(i) that we can define

$$\Upsilon_{\rm i}(y,t) = \inf\{r > \ell : x \notin P(t)\} \quad \text{and} \quad \Upsilon_{\rm s}(y,t) = \sup\{r > \ell : x \in P(t)\} \tag{6.6}$$

with the convention that each of these is equal to ℓ should the respective set be empty, as functions $D \times \mathbb{R}_+ \to [\ell,\infty)$. By definition, $\Upsilon_{\rm i} \leqslant \Upsilon_{\rm s}$ on $D \times \mathbb{R}_+$. Furthermore, by Lemma 2.5, $\Upsilon_{\rm i}(y,t) \to \infty$ as $t \to \infty$ uniformly with respect to $y \in D$.

Let us now endeavour to determine the large-time behaviour of u, $\Upsilon_{\rm i}$, and $\Upsilon_{\rm s}$ in the prototypical spatial domain $\Omega = D \times \mathbb{R}^k$. From Lemma 2.7 we know that the pointwise asymptotic behaviour of u is given by $t^{\mu}u(x,t) \to F(y)$ as $t \to \infty$, as in the case $k = 1$.

Given that as t increases, it is to be expected that the free boundary of u becomes more and more symmetric about the line $z = 0$, it makes sense that the large-time behaviour of u is described by a solution of the porous media equation with symmetry about this line, i.e.

$$\partial_t u = \Delta_y u^m + \partial_r^2 u^m + (k-1)r^{-1}\partial_r u^m. \tag{6.7}$$

Moreover, given the estimate in Lemma 6.3(ii), and the fact that pointwise $t^{\mu}u(x,t) \to F(y)$ as $t \to \infty$, one could anticipate that the large-time behaviour be represented by $t^{-\mu}\Phi(y, r-g(t))$ for some functions Φ and g in analogy to $t^{-\mu}f(y, z - c\ln t - a)$ in the case $k = 1$. Let us therefore look at the analogue to (4.4) and substitute

$$u(x,t) = t^{-\mu}\hat{u}\big(y, r-g(t), \ln t\big)$$

in (6.7). Setting $\eta = y$, $r - g(t) = \rho$, and $t = e^{\tau}$, we obtain

$$\partial_\tau \hat{u} = \Delta_\eta \hat{u}^m + \partial_\rho^2 \hat{u}^m + e^{\tau}g'(e^{\tau})\partial_\rho \hat{u} + (k-1)\{\rho + g(e^{\tau})\}^{-1}\partial_\rho \hat{u}^m + \mu\hat{u}.$$

Thus formally assuming that $g(t) \to \infty$ and $tg'(t) \to c$ as $t \to \infty$ for some as yet unspecified number c, we arrive at equation (4.3) with $N = n + 1$. The stationary solutions of this equation satisfy (2.4). At the same time, the pointwise behaviour of $u(\cdot, t)$ as $t \to \infty$ would indicate that (2.6) should apply to such a stationary solution f, while Lemma 6.3(ii) would indicate that (2.7) should apply likewise. Naturally, (2.5) should hold too. However, we know that in this case the unspecified number c can only have one value, namely (1.7). Moreover, modulo translation, there is then only one possibility for the stationary solution of (4.3), viz. the unique solution of problem (2.4)–(2.7). Thus we are led to the conclusion that $g(t) = c \ln t + h(t)$ for some function h such that $h'(t) = o(t^{-1})$ as $t \to \infty$, and the large-time behaviour of u is given by

$$w(x,t) = t^{-\mu} f\big(y, r - c \ln t - h(t)\big) \tag{6.8}$$

where c and f are as in the preceding sections of this paper.

It remains to find h. To do this, we compute

$$\begin{aligned}\int_{\mathbb{R}^k} Z(z) w(x,t)\, dz &= t^{-\mu} \int_{\mathbb{R}^k} Z(z) f\big(y, r - c \ln t - h(t)\big)\, dz \\ &= (2\pi)^{1/2} \lambda^{k/4} t^{-\mu} \int_{\mathbb{R}_+} r^{k/2} I_{(k-2)/2}(\lambda^{1/2} r) f\big(y, r - c \ln t - h(t)\big)\, dr\end{aligned}$$

with the aid of (6.1) and (6.2). Hence,

$$\int_{\mathbb{R}^k} Z(z) w(x,t)\, dz = \lambda^{(k-1)/4} t^{-\mu} \int_{\mathbb{R}_+} r^{(k-1)/2} \psi(\lambda^{1/2} r) e^{\sqrt{\lambda} r} f\big(y, r - c \ln t - h(t)\big)\, dr,$$

where

$$\psi(s) = (2\pi s)^{1/2} e^{-s} I_{(k-2)/2}(s).$$

Substituting

$$r = \rho + c \ln t + h(t)$$

and simplifying yields

$$\int_{\mathbb{R}^k} Z(z) w(x,t)\, dz = (\mu \ln t)^{(k-1)/2} e^{\sqrt{\lambda} h(t)} \mathfrak{J}(y,t)$$

where

$$\mathfrak{J}(y,t) = \int_{-c \ln t - h(t)}^{\sigma(y)} \left\{1 + \frac{\rho + h(t)}{c \ln t}\right\}^{(k-1)/2} \psi\big(\lambda^{1/2}\{\rho + c \ln t + h(t)\}\big) e^{\sqrt{\lambda}\rho} f(y,\rho)\, d\rho.$$

Now, $\rho \mapsto e^{\sqrt{\lambda}\rho}$ is integrable on $(-\infty, \sigma(y))$; the above expression in large curly parentheses is uniformly bounded with respect to the limits of integration for large t, and converges to unity pointwise as $t \to \infty$; while by the properties of $I_{(k-2)/2}$, the function ψ is bounded on $\mathbb{R}_+$, and, $\psi(s) \to 1$ as $s \to \infty$. Consequently, the Dominated Convergence Theorem may be applied, yielding

$$\mathfrak{J}(y,t) \to \int_{-\infty}^{\sigma(y)} e^{\sqrt{\lambda}\rho} f(y,\rho)\, d\rho \quad \text{as } t \to \infty.$$

Replacing Z by G and integration with respect to z over $\mathbb{R}^k$ by integration with respect to x over $D \times \mathbb{R}^k$, and noting that $\sigma \in L^\infty(D)$, extension of the above argument delivers

$$\int_{D\times\mathbb{R}^k} G(x)w(x,t)\,dx = (\mu \ln t)^{(k-1)/2} e^{\sqrt{\lambda}h(t)}\bar{\mathfrak{J}}(t),$$

where

$$\bar{\mathfrak{J}} \to \kappa \quad \text{as } t \to \infty$$

and κ is defined by (3.9). It subsequently follows from Theorem 6.1 that if the asymptotic behaviour of u is described by w then necessarily

$$h(t) = -\gamma \ln|\ln t| - \gamma \ln \mu + \frac{1}{\sqrt{\lambda}} \ln\left(\frac{1}{\kappa}\int_{D\times\mathbb{R}^k} G(x)u_0(x)\,dx\right) + \mathrm{o}(1) \quad \text{as } t \to \infty \tag{6.9}$$

where γ is given by (1.14).

Thus, our expectation is that the large-time behaviour of a solution of problem (1.1) with $\Omega = D \times \mathbb{R}^k$ is described by (6.8) with h satisfying (6.9). Seeing that we also expect that for large times the discrepancy between the solution u in an arbitrary domain Ω satisfying (1.12) and one with similar initial values in $D \times \mathbb{R}^k$ will disappear – as it were, the solution away from the line $z = 0$ will be oblivious to the circumstances near that line – we can couple Lemma 6.2 with this observation to formulate the following.

Proposition 6.1 *Let u be the solution of problem* (1.1) *in which Ω is of the form* (1.12) *for some $k \geqslant 2$, u_0 satisfies Hypothesis 6.1, and, Υ_{i} and Υ_{s} are given by* (1.2) *and* (6.6)*. Then, in some sense yet to be made more precise,* (1.13)*,* (1.15)*, and* (1.16) *all hold for*

$$a = \gamma \ln(m-1) + \frac{1}{\sqrt{\lambda}} \ln\left(\frac{1}{\kappa}\int_{\Omega} K(x)u_0(x)\,dx\right).$$

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