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The limit as $p \to \infty$ in a two-phase free boundary problem for the p-Laplacian

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In this paper, we study the limit as p goes to infinity of a minimizer of a variational problem that is a two-phase free boundary problem of phase transition for the p-Laplacian. Under a geometric compatibility condition, we prove that this limit is a solution of a free boundary problem for the ∞-Laplacian. When the compatibility condition does not hold, we prove that there still exists a uniform limit that is a solution of a minimization problem for the Lipschitz constant. Moreover, we provide, in the latter case, an example that shows that the free boundary condition can be lost in the limit.

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1. Introduction

In this paper, we study a two-phase free boundary problem of phase transition for the p-Laplacian. More precisely, given a bounded Lipschitz domain Ω in $\mathbb{R}^n$, we minimize the functional

$$J_p(u) = \int_\Omega \frac{1}{p}|\nabla u(x)|^p + Q^p(x)\lambda\big(u(x)\big)\,dx, \tag{1}$$

subject to the boundary condition $u - \sigma \in W_0^{1,p}(\Omega)$, where an indicator function

$$\lambda(s) = \begin{cases} \lambda_1^p & \text{if } s > 0, \\ \lambda_2^p & \text{if } s \leqslant 0, \end{cases}$$

with $\lambda_1 > \lambda_2 > 0$, a continuous weight function $Q(x) > 0$, and boundary data $\sigma \in Lip(\partial\Omega)$ are given. Here u is allowed to take both positive and negative values. Note that taking the McShane or Whitney extension of σ, we can consider that σ is in fact the restriction to $\partial\Omega$ of a function in $W^{1,\infty}(\Omega)$, that we will call again σ with the abuse of notation. We denote by $Lip(\sigma)$ the Lipschitz constant of σ and we assume without the loss of generality that $Lip(\sigma) = Lip(\sigma\,|_{\partial\Omega})$, as we can just take σ as the absolute minimizing Lipschitz extension of its boundary data (see [2] for the existence of such an absolute minimizing Lipchitz extension). Alt, Caffarelli and Friedman studied this two-phase free boundary problem for $p = 2$ in [3]. They proved the existence of a minimizer, the free boundary condition, non-degeneracy and Lipschitz continuity of a minimizer, and the measure theoretic regularity and differentiability of the free boundary there. In an earlier manuscript [1], Alt and Caffarelli solved a similar problem for the variational one phase problem when $p = 2$. In fact, their work is the inspiration of the authors' current and related research projects. The corresponding one-phase problem, i.e. when $u \geqslant 0$, has been studied in [6] where most properties for $p = 2$ are proved to hold. In [14], these properties have been recovered in Orlicz spaces for an one-phase problem. However, it is still open whether a minimizer of a two-phase problem for a general p is Lipschitz continuous.

In the general case, there is a minimizer of (1), which is proved in Lemma 2.1 in the next section. A minimizer is a weak solution to the p-Laplace equation in the positive and negative domains, namely

$$-\Delta_p u_p = -\mathrm{div}(|\nabla u_p|^{p-2}\nabla u_p) = 0, \qquad \text{in } \{u_p > 0\} \cup \{u_p < 0\},$$

satisfying the Dirichlet boundary condition $u\ |_{\partial\Omega} = \sigma$, and, under the assumption that the "flat region" where $u_p = 0$ is of measure zero, the minimizer satisfies the free boundary condition

$$(u^+_{p,\nu})^p - (u^-_{p,\nu})^p = \frac{p}{p-1}(\lambda_1^p - \lambda_2^p)$$

at every regular point in a weak sense, as stated in Lemma 2.4. For study on free boundary problems involving quasilinear equations like the one considered here, either two-phase or one-phase, there is a long list of references, among which we would like to refer the reader to [4, 6–9, 12–15] and [16].

Our main concern in this paper is to study the limit as $p \to \infty$ of the minimizers.

First, to clarify the statements and the discussion, we assume that $Q(x) = 1$. Let us consider the three terms that appear in (1),

$$\frac{1}{p}\int_\Omega |\nabla u|^p, \quad \lambda_1^p|\{u > 0\}| \qquad \text{and} \quad \lambda_2^p|\{u < 0\}|. \tag{2}$$

As $\lambda_1 > \lambda_2$, the third term is not the leading one as $p \to \infty$. Between the first two, the one that dominates as $p \to \infty$ depends on the relation between $Lip(\sigma)$ and λ_1. When $\lambda_1 \geqslant Lip(\sigma)$, it is the second term that dominates, and this implies that when we take $p \to \infty$ we get a limit function whose gradient, or equivalently its Lipschitz constant, is not greater than λ_1, and that minimizes the measure of its positive set. Therefore, we are led to consider the following two-phase minimization problem:

$$\begin{cases} \text{Minimize } |\{u(x) > 0\}| & \text{subject to } Lip(u) \leqslant \lambda_1,\ u = \sigma \text{ on } \partial\Omega, \text{ with} \\ \Delta_\infty u = 0 & \text{in } \{u > 0\} \cup \{u < 0\}, \\ u = 0, \quad u^+_\nu = \lambda_1 & \text{on } \partial\{u > 0\} \cap \Omega, \end{cases} \tag{3}$$

where ν is the normal to the free boundary $\partial\{u > 0\} \cap \Omega$ pointing inward of the positive set $\{u > 0\}$.

That the ruling equation for the limit configuration is the infinity Laplace equation $-\Delta_\infty u = -\langle D^2 u Du, Du\rangle = 0$ is due to the fact that infinity harmonic functions, the viscosity solutions to the equation $-\Delta_\infty u = 0$, appear naturally as the limit of p-harmonic functions, the viscosity solutions to the p-Laplace equation $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$ (see [5] and the survey [2]).

This discussion leads us to believe that when $Lip(\sigma) \leqslant \lambda_1$ the limit as $p \to \infty$ of the minimizers of (1) is a solution to (3), which constitutes the first part of the next theorem.

The case $Lip(\sigma) > \lambda_1$ is different, since in this case the leading term of the three in (2) is the first one. Here we can also prove that there is a uniform limit, but it could happen that this limit is just the absolute minimizing Lipschitz extension of σ to the inside of Ω and hence there is no free boundary that survives in the limit. This is exactly what happens in a one-dimensional example, Example 2.16.

We summarize the results mentioned above in the following theorem.

Theorem 1.1 *Assume that $Q = 1$. Let u_p be a minimizer of (1), then there exists a continuous function u_∞ such that, for a subsequence denoted still by $\{u_p\}$,*

$$\lim_{p\to\infty} u_p = u_\infty,$$

uniformly in $\overline{\Omega}$. In addition:

(i) *If $Lip(\sigma) \leqslant \lambda_1$, let*

$$P = \bigcup_{z\in\partial\Omega,\sigma(z)>0} B_{\sigma(z)/\lambda_1}(z)\cap\Omega$$

be a reference set, then the limit u_∞ is a solution to (3) and its positivity set $\{u_\infty > 0\}$ verifies

$$|\{u_\infty > 0\}| = |P|,\ \{u_\infty > 0\} \supseteq P, \quad \textit{and} \quad \partial\{u_\infty > 0\} \cap \Omega \subset \partial P \cap \Omega. \tag{4}$$

Moreover, in this case, the limit u_∞ satisfies the free boundary condition $u_\nu^+ = \lambda_1$ along the free boundary $\partial\{u_\infty > 0\} \cap \Omega$ in the sense that, if $x_0 \in \partial\{u_\infty > 0\} \cap \Omega$ is a regular free boundary point, then

$$u_\nu^+(x_0) = \lim_{\epsilon\downarrow 0} \frac{u_\infty(x_0 - \epsilon\nu) - u_\infty(x_0)}{\epsilon} = \lambda_1.$$

where ν is a external normal vector to the set $\{u_\infty > 0\}$ at x_0.

(ii) *If $Lip(\sigma) > \lambda_1$, then u_∞ is a minimal Lipschitz extension of σ. That is, it minimizes the Lipschitz constant in Ω subject to the boundary data σ, or equivalently,*

$$Lip(u_\infty) = \min_{v=\sigma \text{ on } \partial\Omega} Lip(v).$$

Moreover, in this case, it can happen that the free boundary condition is lost in the limit. That is, the limit u_∞ may be independent of λ_1 and λ_2 as shown by the one-dimensional example (2.16).

In both cases, the limit u_∞ is also a viscosity solution to the infinity Laplace equation $\triangle_\infty u = 0$ in $\{u > 0\} \cup \{u < 0\}$.

The case $Q \neq 1$ is different since we have again three terms that in this case are the following

$$\frac{1}{p}\int_\Omega |\nabla u|^p, \quad \lambda_1^p \int_{\{u>0\}} Q^p(x)\,dx \quad \text{and} \quad \lambda_2^p \int_{\{u\leqslant 0\}} Q^p(x)\,dx.$$

Note that now the third term can be dominant depending on the size of Q even if $\lambda_1 > \lambda_2$.

In this case we can also show uniform convergence and that the limit is a solution to a minimization problem as stated below.

Theorem 1.2 *Let u_p be a minimizer of* (1) *for each p, then, for a subsequence $\{u_{p_k}\}$ of $\{u_p\}$, it holds that*

$$\lim_{k\to\infty} u_{p_k} = u_\infty$$

uniformly in $\overline{\Omega}$. In addition, the limit u_∞ is a solution to the minimization problem of minimizing

$$\max\Big\{Lip(u), \lambda_1\|Q\|_{L^\infty(u>0)}, \lambda_2\|Q\|_{L^\infty(u\leqslant 0)}\Big\},$$

subject to $u \in A$ and $u|_{\partial\Omega} = \sigma$, where the admissible set

$$A = \Big\{u \ : \ Lip(u) \leqslant \max\big\{Lip(\sigma), \lambda_1\|Q\|_{L^\infty(\sigma>0)}, \lambda_2\|Q\|_{L^\infty(\sigma\leqslant 0)}\big\}\Big\}.$$

As in Theorem 1.1, the free boundary may be lost in the limit.

2. Proof of the main theorems

2.1 *The two-phase problem for the p-Laplacian for finite p*

First we prove the existence of a minimizer of (1) for a fixed p in $[1, \infty)$.

Lemma 2.1 *There exists a minimizer u_p of the variational problem* (1). *The minimizer verifies that*

$$\|u_p\|_{L^\infty(\Omega)} \leqslant \|\sigma\|_{L^\infty(\Omega)}.$$

Proof. Without the loss of generality, one may assume the domain Ω is bounded. Take a minimizing sequence $\{u^k\}$ of J_p. Then

$$\lim_{k\to\infty} J_p(u^k) \leqslant J_p(\sigma).$$

So $\{u^k\}$ is a bounded sequence in $W^{1,p}(\Omega)$, since $\int_\Omega |\nabla u^k|^p \leqslant pJ_p(u^k)$. As a result, one may conclude that, for a subsequence denoted still by $\{u^k\}$,

$$u^k \to v \text{ weakly in } W^{1,p}(\Omega),$$
$$u^k \to v \text{ a. e. in } \Omega$$

and

$$Q^p(x)\lambda^p(u^k) \to q(x) \text{ weakly star in } L^\infty_{loc}(\Omega),$$

where

$$q(x)\begin{cases} = Q^p(x)\lambda^p(v) & \text{if } v \neq 0,\\ \geqslant Q^p(x)\lambda^p(v) & \text{if } v = 0.\end{cases}$$

Then Fatou's Lemma implies that

$$\begin{aligned} J_p(v) &= \frac{1}{p}\int_\Omega |\nabla v|^p + Q^p(x)\lambda^p(v)\\ &\leqslant \liminf_{k\to\infty}\frac{1}{p}\int_\Omega |\nabla u^k|^p + Q^p(x)\lambda^p(u^k)\\ &= \liminf_{k\to\infty} J_p(u^k).\end{aligned}$$

So v is a minimizer of J_p, since clearly $v - \sigma \in W_0^{1,p}(\Omega)$.

For the bound for $\|u_p\|_{L^\infty(\Omega)}$ we refer to [10], Proposition 1. ☐

REMARK 2.2 The previous proof also works if Ω is unbounded, one may simply replace Ω by $\Omega \cap B_R$ for all large balls B_R in the above argument and send R to ∞.

REMARK 2.3 The uniqueness of a minimizer of the variational problem does not hold. In fact, one may take $\Omega = B$, the unit ball of $\mathbb{R}^n$, and take the simplest boundary data $\sigma = 1$ on $\partial\Omega$.

Next, we take $u_0 \equiv 1$ on Ω. Then $J_p(u_0) = \frac{1}{p}\lambda_2^p\omega_n$, where ω_n is the volume of the unit ball.

Suppose there is a unique minimizer u_1 of the functional J_p. Then, since the problem is invariant under rotations, u_1 is radially symmetric.

Now assume that there is an $s \in (0,1)$ such that $u_1 \equiv 0$ on B_s, and $\triangle_p u_1 \equiv 0$ in $B\backslash B_s$. A simple computation gives that

$$u_1(x) = \begin{cases} a|x|^{\frac{p-n}{p-1}} + b, & \text{if } s \leqslant |x| \leqslant 1, \\ 0, & \text{if } |x| < s, \end{cases}$$

where a and b satisfy $a + b = 1$ and $as^{\frac{p-n}{p-1}} + b = 0$. Then

$$J_p(u_0) - J_p(u_1) = \frac{1}{p}(\lambda_2^p - \lambda_1^p)\omega_n s^n - \frac{1}{p}|a|^p \left|\frac{p-n}{p-1}\right|^p \frac{p-1}{p-n}(1 - s^{\frac{p-n}{p-1}})n\omega_n.$$

If one carefully chooses the values of λ_1 and λ_2, one can make this difference equal to 0. The details are very similar to those in the computation contained in [11] and hence we omit the details. So one ends up with two distinct minimizers u_0 and u_1.

Lemma 2.4 *Let $Q = 1$. Suppose that u_p is a minimizer of J_p, and that*

$$\left|\{x : u_p(x) = 0\}\right| = 0.$$

Then u_p satisfies the free boundary condition

$$(u_{p,\nu}^+)^p - (u_{p,\nu}^-)^p = \frac{p}{p-1}(\lambda_1^p - \lambda_2^p)$$

in the weak sense, that is,

$$\lim_{\epsilon\downarrow 0}\int_{\partial\{u_p>\epsilon\}}\left(\frac{p-1}{p}|\nabla u_p|^p - \lambda_1^p\right)\eta\cdot\nu + \lim_{\delta\downarrow 0}\int_{\partial\{u_p<-\delta\}}\left(\frac{p-1}{p}|\nabla u_p|^p - \lambda_2^p\right)\eta\cdot\nu = 0$$

for any smooth function $\eta \in C_0^2(\Omega;\mathbb{R}^n)$. Here ν denotes the normal to a surface pointing to the positive domain $\{u_p > 0\}$.

Proof. There is some subtle issue in this seemly parallel computation to the special case when $p = 2$, mainly due to the absence of regular second derivatives of a minimizer, as in general a p-harmonic function possesses only Hölder continuous first derivatives. For this reason the details are given below.

Take $x_\epsilon = \tau_\epsilon(x) = x + \epsilon\eta$ for $x \in \Omega$, and define $u_\epsilon(x_\epsilon) = u_p(x)$. So

$$u_\epsilon(x) = u_p(\tau_\epsilon^{-1}x),$$
$$\nabla u_\epsilon(x) = \big(D\tau_\epsilon^{-1}(x)\big)\nabla u_p(\tau_\epsilon^{-1}x),$$

and

$$(D\tau_\epsilon^{-1})(x) = (D\tau_\epsilon)^{-1}(\tau_\epsilon^{-1}x) = (I + \epsilon\nabla\eta)^{-1}(\tau_\epsilon^{-1}x) = I - \epsilon D\eta(\tau_\epsilon^{-1}x) + O(\epsilon^2).$$

We will also use the following identities

$$|(I - \epsilon D\eta + O(\epsilon^2))\nabla u_p|^p = |\nabla u_p|^p - \epsilon p|\nabla u_p|^{p-2} < D\eta\nabla u_p, \nabla u_p > +O(\epsilon^2)$$

and

$$\det(I + \epsilon D\eta) = 1 + \epsilon\, tr(D\eta) + O(\epsilon^2),$$

where $tr(D\eta) = \nabla \cdot \eta$.

The minimality of $J_p(u_p)$ then implies

$$\begin{aligned}
0 &\leqslant J_p(u_\epsilon) - J_p(u_p) \\
&= \int_\Omega \frac{1}{p}\Big|D\tau_\epsilon^{-1}(x)\nabla u_p\big(\tau_\epsilon^{-1}(x)\big)\Big|^p + \lambda\Big(u_p\big(\tau_\epsilon^{-1}(x)\big)\Big)dx - \int_\Omega \frac{1}{p}|\nabla u(x)|^p + \lambda\big(u(x)\big) \\
&= \int_\Omega \frac{1}{p}\Big|(D\tau_\epsilon)^{-1}(\tau_\epsilon^{-1}x)\nabla u_p(\tau_\epsilon^{-1}x)\Big|^p + \lambda\big(u_p(\tau_\epsilon^{-1}x)\big)dx - \int_\Omega \frac{1}{p}|\nabla u|^p + \lambda(u) \\
&= \int_\Omega \left\{\frac{1}{p}\Big|(D\tau_\epsilon)^{-1}(x)\nabla u_p(x)\Big|^p + \lambda\big(u_p(x)\big)\right\}\det(D\tau_\epsilon)dx - \int_\Omega \frac{1}{p}|\nabla u|^p + \lambda(u) \\
&= \int_\Omega \frac{1}{p}\Big|\big(I - \epsilon D\eta + O(\epsilon^2)\big)\nabla u_p\Big|^p \det(I + \epsilon\nabla\eta) + \lambda\big(u_p(x)\big)\det(I + \epsilon D\eta)dx \\
&\qquad\qquad - \int_\Omega \frac{1}{p}|\nabla u|^p + \lambda(u) \\
&= \int_\Omega \frac{1}{p}\Big\{|\nabla u_p|^p - \epsilon p|\nabla u_p|^{p-2} < D\eta\nabla u_p, \nabla u_p > +O(\epsilon^2)\Big\} \\
&\qquad \Big\{1 + \epsilon\, tr(D\eta) + O(\epsilon^2)\Big\}dx + \int_\Omega \lambda(u_p)\big(1 + \epsilon\, tr(D\eta) + O(\epsilon^2)\big)dx \\
&\qquad\qquad - \int_\Omega \frac{1}{p}|\nabla u_p|^p + \lambda(u_p)dx.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
0 &\leqslant J_p(u_\epsilon) - J_p(u_p) \\
&= \epsilon\int_\Omega \frac{1}{p}|\nabla u_p|^p\, tr(D\eta) - |\nabla u_p|^{p-2}\langle D\eta\nabla u_p, \nabla u_p\rangle \\
&\qquad\qquad + \lambda(u_p)\, tr(D\eta)dx + O(\epsilon^2) \\
&= \epsilon\int_\Omega \left\{\frac{1}{p}|\nabla u_p|^p + \lambda(u_p)\right\} tr(D\eta) - |\nabla u_p|^{p-2}\langle D\eta\nabla u_p, \nabla u_p\rangle dx + O(\epsilon^2).
\end{aligned}$$

As ϵ could be any small number, positive as well as negative, the linear term in ϵ must be zero in the preceding inequality. Hence

$$\int_\Omega \left\{\frac{1}{p}|\nabla u_p|^p + \lambda(u_p)\right\}\nabla\cdot\eta - |\nabla u_p|^{p-2}\langle D\eta\nabla u_p, \nabla u_p\rangle = 0.$$

The left-hand-side of the preceding equation is given by, on account of the assumption that $|\{u_p = 0\}| = 0$,

$$\lim_{\epsilon\downarrow 0,\delta\downarrow 0}\int_{\Omega\setminus\{-\delta<u_p<\epsilon\}}\left\{\frac{1}{p}|\nabla u_p|^p+\lambda(u_p)\right\}\nabla\cdot\eta-|\nabla u_p|^{p-2}\langle D\eta\nabla u_p,\nabla u_p\rangle.$$

If u_p is of class C^2, then the preceding left-hand-side is equal to

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0,\delta\downarrow 0}\int_{\Omega\setminus\{-\delta<u_p<\epsilon\}}\nabla\cdot\left\{\Big(\frac{1}{p}|\nabla u_p|^p+\lambda(u_p)\Big)\eta-\eta\cdot\nabla u_p|\nabla u_p|^{p-2}\nabla u_p\right\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\eta\cdot\nabla u_p\nabla\cdot(|\nabla u_p|^{p-2}\nabla u_p)\\
&\quad=\lim_{\epsilon\downarrow 0,\delta\downarrow 0}\int_{\Omega\setminus\{-\delta<u_p<\epsilon\}}\nabla\cdot\left\{\Big(\frac{1}{p}|\nabla u_p|^p+\lambda(u_p)\Big)\eta-\eta\cdot\nabla u_p|\nabla u_p|^{p-2}\nabla u_p\right\}\\
&\quad=\lim_{\epsilon\downarrow 0}\int_{\partial\{u_p>\epsilon\}}\Big(\frac{1}{p}|\nabla u_p|^p+\lambda_1^p\Big)\eta\cdot\nu-\eta\cdot\nabla u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nu dH^{n-1}\\
&\qquad+\lim_{\delta\downarrow 0}\int_{\partial\{u_p<-\delta\}}\Big(\frac{1}{p}|\nabla u_p|^p+\lambda_2^p\Big)\eta\cdot\nu-\eta\cdot\nabla u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nu dH^{n-1}\\
&\quad=-\lim_{\epsilon\downarrow 0}\int_{\partial\{u_p>\epsilon\}}\Big(\frac{p-1}{p}|\nabla u_p|^p-\lambda_1^p\Big)\eta\cdot\nu dH^{n-1}\\
&\qquad\qquad\qquad\qquad-\lim_{\delta\downarrow 0}\int_{\partial\{u_p<-\delta\}}\Big(\frac{p-1}{p}|\nabla u_p|^p-\lambda_2^p\Big)\eta\cdot\nu dH^{n-1},
\end{aligned}$$

the second equation being the application of the divergence theorem, and ν denotes the outer normal to a domain.

In general, u_p is not of class C^2. However, it is of class $C^{1,\alpha}$ in both $\{u_p > \varepsilon\}$ and $\{u_p < -\delta\}$ locally in Ω. Let D be either $\{u_p > \varepsilon\}$ or $\{u_p < -\delta\}$ in the following. Set

$$D_0 = \big\{x \in D\colon |\nabla u_p(x)| \neq 0\big\}$$

and

$$D_\delta = \big\{x \in D\colon |\nabla u_p(x)| > \delta\big\}$$

for $\delta > 0$. Then D_0 and D_δ are both open, and $D_0 = \cup_{\delta>0}D_\delta$. The classical theory of uniformly elliptic equations implies u_p is smooth in D_δ and hence in D_0. As a consequence, u_p is a strong solution of $\triangle_p u = 0$ in D_0 and even on the part of $\{u_p = \varepsilon\}$ and $\{u_p = -\delta\}$ where $\nabla u \neq 0$. On the other hand, $|\nabla u_p| = 0$ on $D\setminus D_0$ and hence in the classical sense the following identities hold

$$\frac{1}{p}|\nabla u_p|^p\nabla\cdot\eta-|\nabla u_p|^{p-2} < D\eta\nabla u_p,\nabla u_p >= 0,$$

$$\frac{1}{p}|\nabla u_p|^p\eta-\eta\cdot\nabla u_p|\nabla u_p|^{p-2}\nabla u_p = 0,$$

and

$$\nabla\cdot\Big(\frac{1}{p}|\nabla u_p|^p\eta-\eta\cdot\nabla u_p|\nabla u_p|^{p-2}\nabla u_p\Big) = 0$$

on $D\backslash D_0$.

Now, our goal is to prove

$$\int_D \left\{\frac{1}{p}|\nabla u_p|^p + \lambda(u_p)\right\} \nabla\cdot\eta - |\nabla u_p|^{p-2} < D\eta\nabla u_p, \nabla u_p >$$
$$= \int_{\partial D} \Big(\frac{1}{p}|\nabla u_p|^p + \lambda(u_p)\Big)\eta\cdot\nu - \eta\cdot\nabla u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nu\, dH^{n-1}$$

so that the preceding computation for a C^2 function passes successfully in the general case. In fact, one needs only to prove

$$\int_D \frac{1}{p}|\nabla u_p|^p\nabla\cdot\eta - |\nabla u_p|^{p-2} < D\eta\nabla u_p, \nabla u_p >$$
$$= \int_{\partial D} \frac{1}{p}|\nabla u_p|^p\eta\cdot\nu - \eta\cdot\nabla u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nu\, dH^{n-1}.$$

For this purpose, one does the following computation, writing u for u_p.

$$\begin{aligned}
&\int_D \frac{1}{p}|\nabla u|^p\nabla\cdot\eta - |\nabla u|^{p-2} < D\eta\nabla u, \nabla u >\\
&= \int_{D_0} \frac{1}{p}|\nabla u|^p\nabla\cdot\eta - |\nabla u|^{p-2} < D\eta\nabla u, \nabla u >\\
&= \int_{D_0} \nabla\cdot\left\{\frac{1}{p}|\nabla u|^p\eta - \eta\cdot\nabla u|\nabla u|^{p-2}\nabla u\right\} + \eta\cdot\nabla u\nabla\cdot\big(\nabla u|^{p-2}\nabla u\big)\\
&= \int_{D_0} \nabla\cdot\left\{\frac{1}{p}|\nabla u|^p\eta - \eta\cdot\nabla u|\nabla u|^{p-2}\nabla u\right\}\\
&= \int_{\partial D_0} \left\{\frac{1}{p}|\nabla u|^p\eta - \eta\cdot\nabla u|\nabla u|^{p-2}\nabla u\right\}\cdot\nu\, dH^{n-1}\\
&= \int_{\partial D_0} \left\{\frac{1}{p}|\nabla u|^p\eta - \eta\cdot\nabla u|\nabla u|^{p-2}\nabla u\right\}\cdot\nu\, dH^{n-1}\\
&\quad + \int_{\partial(D\backslash D_0)} \left\{\frac{1}{p}|\nabla u|^p\eta - \eta\cdot\nabla u|\nabla u|^{p-2}\nabla u\right\}\cdot\nu\, dH^{n-1}\\
&= \int_{\partial D} \left\{\frac{1}{p}|\nabla u|^p\eta - \eta\cdot\nabla u|\nabla u|^{p-2}\nabla u\right\}\cdot\nu\, dH^{n-1}.
\end{aligned}$$

The fifth equality holds as the latter integral is equal to zero.

Therefore, one obtains

$$\int_\Omega \left\{\frac{1}{p}|\nabla u_p|^p + \lambda(u_p)\right\}\nabla\cdot\eta - |\nabla u_p|^{p-2} < D\eta\nabla u_p, \nabla u_p >$$
$$= -\lim_{\epsilon\downarrow 0}\int_{\partial\{u_p>\epsilon\}} \Big(\frac{p-1}{p}|\nabla u_p|^p - \lambda_1^p\Big)\eta\cdot\nu dH^{n-1}$$
$$- \lim_{\delta\downarrow 0}\int_{\partial\{u_p<-\delta\}} \Big(\frac{p-1}{p}|\nabla u_p|^p - \lambda_2^p\Big)\eta\cdot\nu dH^{n-1}.$$

The proof is finished. $\square$

REMARK 2.5 The above lemma does not imply that the conditions

$$u_{p,\nu}^{+} = \left(\frac{p}{p-1}\right)^{\frac{1}{p}}\lambda_1 \quad \text{and} \quad u_{p,\nu}^{-} = \left(\frac{p}{p-1}\right)^{\frac{1}{p}}\lambda_2$$

hold along the free boundary $\partial\{u_p > 0\}$ in any sense. In fact, if one defines a new functional

$$\tilde{J}_p(u) = \int_{\Omega} \frac{1}{p}|\nabla u|^p + \tilde{\lambda}(u)dx,$$

where

$$\tilde{\lambda}(s) = \begin{cases} \mu_1^p & \text{if } s > 0; \\ \mu_2^p & \text{if } s \leqslant 0, \end{cases}$$

and $\mu_1^p - \mu_2^p = \lambda_1^p - \lambda_2^p$. Then $\tilde{J}_p(u) = J_p(u) + (\mu_1^p - \lambda_1^p)|\Omega|$, and hence a minimizer of J_p is also a minimizer of $\tilde{J}_p$. Clearly, $u_{p,\nu}^{+} = (\frac{p}{p-1})^{\frac{1}{p}}\lambda_1$ and $u_{p,\nu}^{+} = (\frac{p}{p-1})^{\frac{1}{p}}\mu_1$ cannot both hold at the same time unless $\lambda_1 = \mu_1$.

REMARK 2.6 Note that the assumption

$$\big|\{u_p(x) = 0\}\big| = 0$$

is needed here. As the one-dimensional example, namely Example 2.16, shows, there are configurations of data, Ω, σ, λ_1 and λ_2, such that a zero flat region occurs.

REMARK 2.7 In principle, one expects the p-problem to converge to a limiting problem as $p \to \infty$ and the limiting problem still to bear the characteristic of a phase transition problem, namely there is a jump in the normal derivative across the free boundary. Under such assumptions, in symbol, if one takes limit of the the free boundary condition $(u_{p,\nu}^{+})^p - (u_{p,\nu}^{-})^p = \frac{p}{p-1}(\lambda_1^p - \lambda_2^p)$ as p tends to infinity, one gets the free boundary condition $u_\nu^{+} = \lambda_1$ for a possible limit function u_∞. It is surprising that the limiting free boundary condition is essentially a one-phase condition, and whether this free boundary condition holds depends on the Lipschitz constant of the boundary data. On the other hand, the limit function u_∞ verifies more than just the infinity Laplace equation and the free boundary condition. It is a solution of a minimization problem on the measure of the positive set, which will be stated in the proof of Theorem 1.1.

REMARK 2.8 This problem can be scaled as follows: if u is a minimizer of J_p with constants λ_1, λ_2 and boundary data σ, then $u_k(x) = u(x)/k$, for $k > 0$, is a minimizer for J_p with constants λ_1/k, λ_2/k and boundary data $\sigma_k(x) = \sigma(x)/k$. Moreover if $0 \in \Omega$ and if we let $u_k(x) = u(x/k)$ then we obtain a minimizer for J_p in the domain $\Omega_k = k\Omega$ with constants λ_1/k, λ_2/k and boundary data $\sigma_k(x) = \sigma(x/k)$. Note that in the latter case, the Lipschitz constant of σ_k is equal to the Lipschitz constant of σ over k.

2.2 *The limit as $p \to \infty$ for $Q = 1$*

Here we remark that we are considering the Lipschitz constant of a function u to be $Lip(u) = \sup_{x,y\in\Omega} |u(x) - u(y)|/d(x, y)$ where $d(x, y)$ is the distance from x to y in Ω. Note that when Ω is convex we have $d(x, y) = |x - y|$.

Our next result shows that there is a precise bound for the L^p-norm of the gradient of a minimizer.

Lemma 2.9 *Assume that $Q = 1$. Let u_p be a minimizer of J_p. Then*

$$\left(\int_\Omega |\nabla u_p|^p\right)^{\frac{1}{p}} \leqslant C(p,\sigma,\Omega,\lambda_1),$$

where $C(p,\sigma,\Omega,\lambda_1)$ is a constant depending on p, σ, Ω, and λ_1, and such that

$$\lim_{p\to\infty} C(p,\sigma,\Omega,\lambda_1) = \begin{cases} \lambda_1 & \text{if } Lip(\sigma) \leqslant \lambda_1; \\ Lip(\sigma) & \text{if } Lip(\sigma) > \lambda_1. \end{cases}$$

Proof. One easily gets from $J_p(u_p) \leqslant J_p(\sigma)$ that

$$\int_\Omega |\nabla u_p|^p \leqslant \int_\Omega |\nabla \sigma|^p + p\int_\Omega \lambda(\sigma) \leqslant (Lip(\sigma))^p|\Omega| + p\lambda_1^p|\Omega|.$$

The result follows from this inequality by taking the constant to be

$$C(p,\sigma,\Omega,\lambda_1) = \left[\left(Lip(\sigma)\right)^p|\Omega| + p\lambda_1^p|\Omega|\right]^{\frac{1}{p}}.$$

□

Lemma 2.10 *Assume that $Q = 1$. There is a uniform limit u_∞ of a subsequence of $\{u_p\}_p$, as $p \to \infty$. Moreover, the limit u_∞ satisfies*

$$u_\infty = \sigma \text{ on } \partial\Omega,$$

and $u_\infty \in W^{1,\infty}(\Omega)$ with

$$\|\nabla u_\infty\|_{L^\infty(\Omega)} \leqslant \max\{\lambda_1, Lip(\sigma)\}$$

Proof. Fix q and let $p > q$. Using Hölder's inequality and Lemma 2.9, one gets

$$\left(\int_\Omega |\nabla u_p|^q\right)^{\frac{1}{q}} \leqslant |\Omega|^{\frac{p-q}{qp}} \left(\int_\Omega |\nabla u_p|^p\right)^{\frac{1}{p}} \leqslant |\Omega|^{\frac{p-q}{qp}} C(p,\sigma,\Omega,\lambda_1). \tag{5}$$

Hence $\{u_p\}_{p>q}$ is bounded in $W^{1,q}(\Omega)$ and hence there is a weakly convergent subsequence, still denoted by $\{u_p\}$, such that

$$u_p \to u_\infty \text{ weakly in } W^{1,q}(\Omega) \text{ and uniformly on } \overline{\Omega}.$$

Using a diagonal procedure one can assume that this convergence is verified for all integer q.

Clearly, $u_\infty = \sigma$ on $\partial\Omega$. In addition, if one sends p to ∞ in the estimate (5), one gets

$$\left(\int_\Omega |\nabla u_p|^q\right)^{\frac{1}{q}} \leqslant |\Omega|^{\frac{1}{q}} \lim_{p\to\infty} C(p,\sigma,\Omega,\lambda_1).$$

The result follows from here by sending q to ∞. □

Lemma 2.11 *The limit u_∞ is a viscosity solution to $-\triangle_\infty u_\infty = 0$ in the set $\{u_\infty > 0\} \cup \{u_\infty < 0\}$.*

Proof. In a ball $B \subseteq \{u_\infty > 0\}$, $u_p > 0$ for all sufficiently large p thanks to the uniform convergence of the subsequence. So $-\triangle_p u_p = 0$ in B, which implies, by passing to limit uniformly, $-\triangle_\infty u_\infty = 0$ in the viscosity sense in B. The case in $\{u_\infty < 0\}$ follows similarly. □

Now we are ready to prove our result concerning the limit as $p \to \infty$ when $Q \equiv 1$.

Proof of Theorem 1.1. First, we assume that $Lip(\sigma) \leqslant \lambda_1$. Our goal is to show that u_∞ is a solution to (3) and that its positivity set is given by

$$\{u_\infty > 0\} = \bigcup_{z \in \partial\Omega, \sigma(z)>0} B_{\sigma(z)/\lambda_1}(z) \cup Z,$$

for a set Z of measure zero.

Let us consider

$$v_\infty(x) = \max_{z \in \partial\Omega, \sigma(z)>0} \big(\sigma(z) - \lambda_1 |x - z|\big)_+.$$

Note that we have that

$$\|\nabla v_\infty\|_{L^\infty(\Omega \cap \{v_\infty > 0\})} = \lambda_1.$$

In fact, let

$$a_z(x) = \big(\sigma(z) - \lambda_1 |x - z|\big)_+,$$

and remark that, for each $x \in \Omega$, the maximum defining $v_\infty(x)$ is attained at some $z_x \in \partial\Omega, \sigma(z) > 0$ (in case $v_\infty(x) = 0$ just take any point $z_x \in \partial\Omega$ with $\sigma(z_x) > 0$). Then we have, for any $x, y \in \Omega$ (we can assume here that $v_\infty(x) \geqslant v_\infty(y)$, since the argument is symmetric),

$$\big|v_\infty(x) - v_\infty(y)\big| = v_\infty(x) - v_\infty(y) \leqslant a_{z_x}(x) - a_{z_x}(y) \leqslant \lambda_1 |x - y|.$$

Hence,

$$\|\nabla v_\infty\|_{L^\infty(\Omega \cap \{v_\infty > 0\})} \leqslant \lambda_1.$$

Now, let $x \in \Omega$ be any point such that $v_\infty(x) > 0$ and take y in the segment that joins x with z_x. We claim that $z_y = z_x$ for that y. In fact, suppose there is a point $z^* \in \partial\Omega$ such that

$$\sigma(z^*) - \lambda_1 |y - z^*| > \sigma(z_x) - \lambda_1 |y - z_x| \tag{6}$$

or equivalently

$$\sigma(z^*) - \sigma(z_x) > -\lambda_1 (|y - z_x| - |y - z^*|) \geqslant -\lambda_1 |z_x - z^*| \tag{7}$$

a contradiction with the fact that $Lip(\sigma) \leqslant \lambda_1$. Therefore, we get that on the segment joining x with z_x we have

$$v_\infty(y) = a_{z_x}(y) = (\sigma(z_x) - \lambda_1 |y - z_x|)_+$$

and we conclude that the derivative of v_∞ in the direction of the vector $(x - z_x)/|x - z_x|$ at the point x is exactly λ_1. Therefore we conclude that

$$\|\nabla v_\infty\|_{L^\infty(\Omega \cap \{v_\infty > 0\})} = \lambda_1.$$

It follows that $u_\infty \geqslant v_\infty$ in the set $\{v_\infty > 0\}$, since $\|\nabla u_\infty\|_{L^\infty(\Omega)} \leqslant \lambda_1$ and $u_\infty = v_\infty$ on $\partial\Omega$. Indeed, if this is not the case, there is a point $x_0 \in \{v_\infty > 0\}$ such that $u_\infty(x_0) < v_\infty(x_0)$. Then, as we stated before, from the definition of v_∞, we conclude the existence of a point $z_{x_0} \in \partial\Omega$ (that we will call z_0 in the sequel) with $\sigma(z_0) > 0$ such that

$$v_\infty(x_0) = \max_{z\in\partial\Omega,\sigma(z)>0} \big(\sigma(z) - \lambda_1|x_0 - z|\big)_+ = \big(\sigma(z_0) - \lambda_1|x_0 - z_0|\big)_+.$$

Now, note that, as $u_\infty = v_\infty = \sigma$ on $\partial\Omega$ we get

$$u_\infty(z_0) - u_\infty(x_0) > v_\infty(z_0) - v_\infty(x_0) = a_{z_0}(z_0) - a_{z_0}(x_0) = \lambda_1|x_0 - z_0|,$$

a contradiction to the fact $\|\nabla u_\infty\|_{L^\infty(\Omega)} \leqslant \lambda_1$. Therefore we conclude that $u_\infty \geqslant v_\infty$ in the set $\{v_\infty > 0\}$ and hence

$$\bigcup_{z\in\partial\Omega,\sigma(z)>0} B_{\sigma(z)/\lambda_1}(z)\cap\Omega = \{v_\infty > 0\} \subseteq \{u_\infty > 0\}.$$

In the following, we characterize the limit function u_∞ through a variational problem.

As before, u_p is a minimizer of the functional J_p. Take any Lipschitz continuous function θ_∞ with Lipschitz constant less than or equal to λ_1, which verifies $\theta_\infty = \sigma$ on $\partial\Omega$. Note that σ is such a function. The function θ_∞ can be taken as a competitor for u_p for the functional J_p, and hence we obtain

$$\frac{1}{p}\int_\Omega |\nabla u_p|^p + \int_\Omega \lambda(u_p) \leqslant \frac{1}{p}\int_\Omega |\nabla\theta_\infty|^p + \int_\Omega \lambda(\theta_\infty).$$

Hence

$$\lambda_1^p\big|\{u_p > 0\}\big| \leqslant \frac{1}{p}\lambda_1^p|\Omega| + \lambda_1^p\big|\{\theta_\infty > 0\}\big| + \lambda_2^p\big|\{\theta_\infty < 0\}\big|.$$

Therefore

$$\big|\{u_p > 0\}\big| \leqslant \frac{1}{p}|\Omega| + \big|\{\theta_\infty > 0\}\big| + |\Omega|\frac{\lambda_2^p}{\lambda_1^p}. \tag{8}$$

Now we observe that

$$\{u_\infty > 0\} = \bigcup_{\eta>0}\{u_\infty > \eta\}.$$

Hence,

$$\big|\{u_\infty > 0\}\big| = \lim_{\eta\to 0}\big|\{u_\infty > \eta\}\big|,$$

and then, given any $\epsilon > 0$, one can find an $\eta > 0$ such that

$$\big|\{u_\infty > 0\}\big| - \big|\{u_\infty > \eta\}\big| \leqslant \epsilon.$$

Now we observe that, from the uniform convergence of u_p to u_∞, one gets

$$\{u_\infty > \eta\} \subset \{u_p > 0\}$$

for every $p \geqslant p_0$, and hence

$$\big|\{u_\infty > 0\}\big| \leqslant \big|\{u_\infty > \eta\}\big| + \epsilon \leqslant |\{u_p > 0\}| + \epsilon.$$

We conclude that, since ϵ is arbitrary,

$$\big|\{u_\infty > 0\}\big| \leqslant \liminf_{p\to\infty} |\{u_p > 0\}|.$$

With this in mind we can take limit in (8) as $p \to \infty$ and we get

$$\big|\{u_\infty > 0\}\big| \leqslant \big|\{\theta_\infty > 0\}\big|,$$

for any Lipschitz continuous function θ_∞ with Lipschitz constant less than or equal to λ_1 that verifies $\theta_\infty = \sigma$ on $\partial\Omega$.

Therefore we have that any uniform limit of u_p is a solution of the minimization problem of

$$\text{minimizing } |\{u > 0\}|, \ \text{ subject to } \ Lip(u) \leqslant \lambda_1,\ u|_{\partial\Omega} = \sigma \tag{9}$$

We observe that v_∞ satisfies the hypothesis imposed on θ_∞. Therefore, we conclude that

$$\big|\{v_\infty > 0\}\big| \geqslant \big|\{u_\infty > 0\}\big|.$$

As a result, both v_∞ and u_∞ are solutions to the minimization problem (9), and

$$\{u_\infty > 0\} = \{v_\infty > 0\} \cup Z$$

for a set Z of measure zero, due to the fact that $\{v_\infty > 0\} \subseteq \{u_\infty > 0\}$.

Now, let us show that $\partial\{u_\infty > 0\} \cap \Omega \subset P \cap \Omega$. In fact, take a point $x_0 \in \partial\{u_\infty > 0\} \cap \Omega$, then we have that $u_\infty(x_0) = 0$. Hence, using again that $\{v_\infty > 0\} \subseteq \{u_\infty > 0\}$, we get $v_\infty(x_0) = 0$. If $x_0 \notin P \cap \Omega$ then, there is a small ball $B_\delta(x_0) \subset \{v_\infty = 0\}$ (note that $P = \{v_\infty > 0\}$). As $x_0 \in \partial\{u_\infty > 0\}$ we have $|\{u_\infty > 0\} \cap B_\delta(x_0)| > 0$, but this leads to a contradiction with the fact that $|\{u_\infty > 0\}| = |\{v_\infty > 0\}|$.

Next, we assume that $\lambda_1 < Lip(\sigma)$. Take any Lipschitz continuous function θ_∞ such that $\theta_\infty = \sigma$ on $\partial\Omega$. Note that σ is such a function, and that $Lip(\theta_\infty) \geqslant Lip(\sigma)$ for any such θ_∞. This function θ_∞ can be viewed as a competitor for u_p in the minimization problem for the functional J_p and hence

$$\begin{aligned}&\left(\frac{1}{p}\int_\Omega |\nabla u_p|^p + \lambda_1^p|\{u_p > 0\}| + \lambda_2^p\big|\{u_p \leqslant 0\}\big|\right)^{\frac{1}{p}}\\ &\qquad\qquad \leqslant \left(\frac{1}{p}\int_\Omega |\nabla \theta_\infty|^p + \lambda_1^p\big|\{\theta_\infty > 0\}\big| + \lambda_2^p\big|\{\theta_\infty \leqslant 0\}\big|\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore,

$$\left(\frac{1}{p}\int_\Omega |\nabla u_p|^p\right)^{\frac{1}{p}} \leqslant \left(\frac{1}{p}\int_\Omega |\nabla \theta_\infty|^p + \lambda_1^p\big|\{\theta_\infty > 0\}\big| + \lambda_2^p\big|\{\theta_\infty \leqslant 0\}\big|\right)^{\frac{1}{p}}.$$

Hence,

$$\left(\frac{1}{p}\int_\Omega |\nabla u_p|^p\right)^{\frac{1}{p}} \leqslant 3^{\frac{1}{p}} \max\left\{\left(\frac{1}{p}\int_\Omega |\nabla \theta_\infty|^p\right)^{\frac{1}{p}}; \lambda_1\big|\{\theta_\infty > 0\}\big|^{\frac{1}{p}}; \lambda_2\big|\{\theta_\infty \leqslant 0\}\big|^{\frac{1}{p}}\right\}.$$

On account of the reason stated in the proof of Lemma 2.10, one may conclude that

$$Lip(u_\infty) \leqslant \liminf_{p\to\infty} \left(\frac{1}{p} \int_\Omega |\nabla u_p|^p \right)^{\frac{1}{p}},$$

In addition, since θ_∞ is Lipschitz, one gets

$$\lim_{p\to\infty} \left(\frac{1}{p} \int_\Omega |\nabla \theta_\infty|^p \right)^{\frac{1}{p}} = Lip(\theta_\infty).$$

Using the above two inequalities and one equation and the fact that

$$Lip(\theta_\infty) \geqslant Lip(\sigma) > \lambda_1 > \lambda_2,$$

one gets

$$Lip(u_\infty) \leqslant Lip(\theta_\infty).$$

Therefore we conclude that u_∞ is a minimizer of the Lipschitz norm $Lip(u)$ over the region Ω in the set of Lipschitz functions that take on the boundary value σ on $\partial\Omega$.

To finish the proof, we show that, when $Lip(\sigma) \leqslant \lambda_1$, there is a boundary condition on the boundary of the set $\{u_\infty > 0\} \cap \Omega$. In fact, we show that the limit u_∞ satisfies $u_\nu^+ = \lambda_1$ on $\partial\{u_\infty > 0\} \cap \Omega$ in the sense that, if $x_0 \in \partial\{u_\infty > 0\} \cap \Omega$ then

$$u_\nu^+(x_0) = \lim_{\epsilon\downarrow 0} \frac{u_\infty(x_0 - \epsilon\nu) - u_\infty(x_0)}{\epsilon} = \lambda_1,$$

where ν is an external normal vector to the set $\{u_\infty > 0\}$ at x_0.

We have the explicit form for the positive set of the limit

$$\{u_\infty > 0\} \supseteq P = \bigcup_{z\in\partial\Omega,\sigma(z)>0} B_{\sigma(z)/\lambda_1}(z) \cap \Omega = \{v_\infty > 0\}.$$

Hence, given $x_0 \in \partial\{u_\infty > 0\} \cap \Omega \subset P \cap \Omega$, there exists a $z_0 \in \partial\Omega \cap \{z : \sigma(z) > 0\}$ such that

$$0 = u_\infty(x_0) = \max_{z\in\partial\Omega,\sigma(z)>0} \big(\sigma(z) - \lambda_1|x - z|\big)_+ = \sigma(z_0) - \lambda_1|x_0 - z_0|.$$

Take

$$\nu = \frac{x_0 - z_0}{|x_0 - z_0|}.$$

We have that ν is a normal exterior vector to the set $\{u_\infty > 0\}$ (in fact we have that $\{x \in \Omega : \sigma(z_0) - \lambda_1|x - z_0| > 0\} \subset \{u_\infty > 0\}$).

By the same arguments used before we have that for any $\epsilon > 0$ small enough,

$$v_\infty(x_0 - \epsilon\nu) = \sigma(z_0) - \lambda_1|x_0 - z_0 - \epsilon\nu|,$$

then

$$u_\infty(x_0 - \epsilon\nu) \geqslant v_\infty(x_0 - \epsilon\nu) = \sigma(z_0) - \lambda_1|x_0 - z_0 - \epsilon\nu| = \sigma(z_0) - \lambda_1(|x_0 - z_0| - \epsilon)$$

and, from the fact that $Lip(u_\infty) \leqslant \lambda_1$ and the explicit formulas we obtain

$$\lambda_1 \geqslant u_\nu^+(x_0) = \lim_{\epsilon \downarrow 0} \frac{u_\infty(x_0 - \epsilon\nu) - u_\infty(x_0)}{\epsilon} \geqslant \lim_{\epsilon \downarrow 0} \frac{\lambda_1 \epsilon}{\epsilon} = \lambda_1,$$

as we wanted to show. □

REMARK 2.12 The properties of the positive set for the limit given in (4) are given in terms of the set P that is exactly the positive set of the function

$$v_\infty(x) = \max_{z \in \partial\Omega, \sigma(z) > 0} \big(\sigma(z) - \lambda_1 |x - z|\big)_+. \tag{10}$$

Also note that we have that $\{u_\infty > 0\} = \{v_\infty > 0\} \cup Z$ for a set Z of measure zero, and the free boundary of u_∞ is included in the boundary of the positive set of v_∞.

REMARK 2.13 If we consider the same problem with λ_1, λ_2 instead of λ_1^p, λ_2^p in the definition of $\lambda(u)$, our arguments show that u_p converges uniformly to a limit, u_∞, that is a solution of

$$\begin{aligned} &\min_{Lip(u) \leqslant 1, u = \sigma \text{ on } \partial\Omega} \lambda_1 \big|\{u > 0\}\big| + \lambda_2 \big|\{u < 0\}\big|, &&\text{if } Lip(\sigma) \leqslant 1,\\ &\min_{u = \sigma \text{ on } \partial\Omega} Lip(u), &&\text{if } Lip(\sigma) > 1. \end{aligned}$$

REMARK 2.14 Note that if we have that u_∞ is ∞-harmonic in $\Omega \backslash \overline{\{u_\infty > 0\}}$ since it has boundary data σ on $\partial\Omega \cap \partial(\Omega \backslash \overline{\{u_\infty > 0\}})$ and 0 on $\Omega \cap \partial\{u_\infty > 0\}$, we get that the limit is unique.

Also note that up to this point we only had uniform convergence of a subsequence of u_p but if we have uniqueness of the limit (and it holds u_∞ is ∞-harmonic in $\Omega \backslash \{u_\infty > 0\}$), we have convergence of the whole family u_p as $p \to \infty$.

REMARK 2.15 An argument as in [3] shows that u_p is p-subharmonic in Ω. If we call z_p the p-harmonic function, $-\Delta_p z_p = 0$, with boundary conditions $z_p = \sigma$ then we have that

$$u_p \leqslant z_p$$

and passing to the limit we conclude that

$$u_\infty \leqslant z_\infty$$

where z_∞ is the AMLE of $\sigma \mid_{\partial\Omega}$. This implies that

$$\{u_\infty > 0\} \subset \{z_\infty > 0\}.$$

And in fact, when $\lambda_1 \geqslant Lip(\sigma)$ we have obtained this property in the previous proof, but this inclusion holds also for the case $\lambda_1 < Lip(\sigma)$.

The explicit formula that we have for the limit in the positive set in the case $Lip(\sigma) \leqslant \lambda_1$ is monotone decreasing with λ_1. Therefore the positive set of the limit decreases as λ_1 increases in this case.

In general we do not have a two-sided free boundary condition as the following example shows (in fact in this simple $1 - d$ example one can see all the features described in the general case in Theorem 1.1).

EXAMPLE 2.16 The $1-d$ example. Let us solve the problem in $\Omega = (0,1)$ with boundary conditions $u_p(0) = \sigma_0 > 0$ and $u_p(1) = \sigma_1 < 0$.

Recall that the functional that we want to minimize is given by

$$J_p(u) = \frac{1}{p}\int_0^1 |u'|^p + \lambda_1^p\big|\{u > 0\}\big| + \lambda_2^p\big|\{u < 0\}\big|.$$

First, let us tackle the case in which we have a flat zero region. That is, there are two points

$$0 < x_p^+ < x_p^- < 1$$

such that

$$u_p \equiv 0, \qquad \text{in } (x_p^+, x_p^-).$$

In this case the energy is minimized by a function of the form

$$u_p(x) = \begin{cases} -\dfrac{\sigma_0}{x_p^+}(x - x_p^+), & x \in (0, x_p^+), \\ 0, & x \in [x_p^+, x_p^-], \\ \dfrac{\sigma_1}{1 - x_p^-}(x - x_p^-), & x \in (x_p^-, 1), \end{cases}$$

and is given by

$$J_p(u_p) = \frac{1}{p}\sigma_0^p (x_p^+)^{1-p} + \frac{1}{p}|\sigma_1|^p(1 - x_p^-)^{1-p} + \lambda_1^p x_p^+ + \lambda_2^p(1 - x_p^-).$$

Since J_p attains its minimum at u_p just minimizing the previous expression with respect to x_p^+ and x_p^- we get that

$$x_p^+ = \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \frac{\sigma_0}{\lambda_1} \qquad \text{and} \qquad 1 - x_p^- = \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \frac{|\sigma_1|}{\lambda_2}.$$

As we have assumed that $0 < x_p^+ < x_p^- < 1$ we conclude that a solution with a zero region exists if and only if

$$\frac{\sigma_0}{\lambda_1} - \frac{\sigma_1}{\lambda_2} < 1.$$

In this case the limit as $p \to \infty$ of x_p^+ and x_p^- are given by

$$x_\infty^+ = \frac{\sigma_0}{\lambda_1} \qquad \text{and} \qquad x_\infty^- = \frac{|\sigma_1|}{\lambda_2}$$

and hence the limit of u_p is

$$u_\infty(x) = \begin{cases} -\lambda_1(x - x_\infty^+), & x \in (0, x_\infty^+), \\ 0, & x \in [x_\infty^+, x_\infty^-], \\ -\lambda_2(x - x_\infty^-), & x \in (x_\infty^-, 1), \end{cases}$$

Now, assume that there is no flat zero region, that is, $x_p^+ = x_p^-$. We have that u_p vanishes at only one point, that we call $x_p \in (0,1)$, and that must verify

$$\left|\frac{\sigma_0}{x_p}\right|^p - \left|\frac{\sigma_1}{1-x_p}\right|^p = \frac{p}{p-1}\left(\lambda_1^p - \lambda_2^p\right). \tag{11}$$

Once this point is fixed then u_p is given by

$$u_p(x) = \begin{cases} \sigma_0 - \dfrac{\sigma_0}{x_p}x, & x \in (0, x_p), \\ \sigma_1 - \dfrac{\sigma_1}{1-x_p}(1-x), & x \in (x_p, 1). \end{cases}$$

Since x_p is bounded we can extract a converging subsequence $x_p \to x_\infty$. Now, we just take the limit in (11),

$$\left|\frac{\sigma_0}{x_p}\right|^p \left(1 - \left|\frac{\sigma_1 x_p}{\sigma_0(1-x_p)}\right|^p\right) = \frac{p}{p-1}\left(\lambda_1^p - \lambda_2^p\right) \sim \lambda_1^p$$

to obtain

$$\frac{\sigma_0}{x_\infty} = \lambda_1,$$

this can be done provided that

$$\frac{-\sigma_1 x_\infty}{\sigma_0(1-x_\infty)} < 1,$$

that is,

$$\frac{-\sigma_1}{\lambda_1(1-\frac{\sigma_0}{\lambda_1})} < 1.$$

It holds if and only if

$$\frac{-\sigma_1}{\lambda_1 - \sigma_0} < 1,$$

that is,

$$\sigma_0 - \sigma_1 < \lambda_1,$$

and hence u_∞ (the uniform limit of the u_p) is uniquely determined and is given by

$$u_\infty(x) = \begin{cases} \sigma_0 - \dfrac{\sigma_0}{x_\infty}x, & x \in (0, x_\infty), \\ \sigma_1 - \dfrac{\sigma_1}{1-x_\infty}(1-x), & x \in (x_\infty, 1). \end{cases}$$

In the case $\sigma_0 - \sigma_1 \geqslant \lambda_1$ we get from our previous results that u_∞ is a Lipschitz function with boundary values σ_0 and σ_1 and Lipschitz constants less or equal to $\sigma_0 - \sigma_1$ so the only possibility is the strait line,

$$u_\infty(x) = \sigma_0 + (\sigma_1 - \sigma_0)x.$$

Note that in this case we lost the free boundary condition since the limit does not depends on λ_1 and λ_2.

Summarizing, we have:

- If
$$\frac{\sigma_0}{\lambda_1} - \frac{\sigma_1}{\lambda_2} < 1$$
then there is a zero flat region for large p (and also for $p = \infty$).
- If
$$\frac{\sigma_0}{\lambda_1} - \frac{\sigma_1}{\lambda_2} \geqslant 1 \qquad \text{and} \qquad \sigma_0 - \sigma_1 < \lambda_1$$
there is no flat region for p large and the limit problem shows a free boundary condition governed by λ_1.
- If
$$\sigma_0 - \sigma_1 \geqslant \lambda_1$$
there is no flat region for large p and in the limit the free boundary condition is lost (the limit is just the AMLE (in this simple 1−d case the strait line)).

2.3 *The limit as $p \to \infty$ for $Q \neq 1$*

Proof of Theorem 1.2. First, we obtain the analogous to Lemma 2.9. We observe that using σ as a competitor for u_p we get $J_p(u_p) \leqslant J_p(\sigma)$ and hence

$$\int_\Omega |\nabla u_p|^p \leqslant \int_\Omega |\nabla\sigma|^p + p\int_\Omega Q^p\lambda(\sigma)$$
$$\leqslant \big(Lip(\sigma)\big)^p|\Omega| + p\lambda_1^p\|Q\|^p_{L^\infty(\{\sigma>0\})}\big|\{\sigma>0\}\big| + p\lambda_2^p\|Q\|^p_{L^\infty(\{\sigma\leqslant 0\})}\big|\{\sigma\leqslant 0\}\big|.$$

Then
$$\left(\int_\Omega |\nabla u_p|^p\right)^{\frac{1}{p}} \leqslant C(p,\sigma),$$
where
$$\lim_{p\to\infty} C(p,\sigma) = \max\big\{Lip(\sigma); \lambda_1\|Q\|_{L^\infty(\{\sigma>0\})}; \lambda_2\|Q\|_{L^\infty(\{\sigma\leqslant 0\})}\big\}.$$
From this fact we can (arguing as in Lemma 2.10) obtain that there is a uniform limit, u_∞, of a subsequence of $\{u_p\}_p$, as $p\to\infty$. Moreover, the limit u_∞ satisfies

$$u_\infty = \sigma \text{ on } \partial\Omega,$$

and $u_\infty \in W^{1,\infty}(\Omega)$ with

$$\|\nabla u_\infty\|_{L^\infty(\Omega)} \leqslant \max\big\{Lip(\sigma); \lambda_1\|Q\|_{L^\infty(\{\sigma>0\})}; \lambda_2\|Q\|_{L^\infty(\{\sigma\leqslant 0\})}\big\}.$$

Now let us look for a variational problem verified by u_∞. To this end, let us consider

$$A = \big\{u \,:\, Lip(u) \leqslant \max\{Lip(\sigma); \lambda_1\|Q\|_{L^\infty(\sigma>0)}; \lambda_2\|Q\|_{L^\infty(\sigma\leqslant 0)}\}\big\}$$

We have that u_p is a minimizer of the functional J_p. Take any $\theta_\infty \in A$ such that $\theta_\infty = \sigma$ on $\partial\Omega$ (note that σ verifies this, so the set of such functions is not empty). This function θ_∞ can be viewed as a competitor for u_p and we obtain

$$\frac{1}{p}\int_\Omega |\nabla u_p|^p + \int_\Omega Q^p\lambda(u_p) \leqslant \frac{1}{p}\int_\Omega |\nabla\theta_\infty|^p + \int_\Omega Q^p\lambda(\theta_\infty).$$

Hence

$$\left(\frac{1}{p}\int_{\Omega}|\nabla u_p|^p+\lambda_1^p\int_{\{u_p>0\}}Q^p+\lambda_2^p\int_{\{u_p\leqslant 0\}}Q^p\right)^{\frac{1}{p}}$$
$$\leqslant\left(\frac{1}{p}\int_{\Omega}|\nabla \theta_\infty|^p+\lambda_1^p\int_{\{\theta_\infty>0\}}Q^p+\lambda_2^p\int_{\{\theta_\infty\leqslant 0\}}Q^p\right)^{\frac{1}{p}} \tag{12}$$

Since

$$\limsup_{p\to\infty}(a_p+b_p+c_p)^{\frac{1}{p}}\leqslant \max\left\{\limsup_{p\to\infty}(a_p)^{\frac{1}{p}};\limsup_{p\to\infty}(b_p)^{\frac{1}{p}};\limsup_{p\to\infty}(c_p)^{\frac{1}{p}}\right\}$$

we have that the limsup of the right hand side in (12) is bounded by

$$\max\left\{Lip(\theta_\infty);\lambda_1\|Q\|_{L^\infty(\theta_\infty>0)};\lambda_2\|Q\|_{L^\infty(\theta_\infty\leqslant 0)}\right\}.$$

Therefore, from (12), we obtain

$$\max\left\{\liminf_{p\to\infty}\left(\frac{1}{p}\int_{\Omega}|\nabla u_p|^p\right)^{\frac{1}{p}};\liminf_{p\to\infty}\left(\lambda_1^p\int_{\{u_p>0\}}Q^p\right)^{\frac{1}{p}}\right\}$$
$$\leqslant \max\left\{Lip(\theta_\infty);\lambda_1\|Q\|_{L^\infty(\theta_\infty>0)};\lambda_2\|Q\|_{L^\infty(\theta_\infty\leqslant 0)}\right\}. \tag{13}$$

From our previous discussion we have that

$$Lip(u_\infty)\leqslant\liminf_{p\to\infty}\left(\frac{1}{p}\int_{\Omega}|\nabla u_p|^p\right)^{\frac{1}{p}}$$

and hence we get

$$Lip(u_\infty)\leqslant\max\left\{Lip(\theta_\infty);\lambda_1\|Q\|_{L^\infty(\theta_\infty>0)};\lambda_2\|Q\|_{L^\infty(\theta_\infty\leqslant 0)}\right\}.$$

Now, using that Q is continuous, given $\epsilon>0$, one fixes $\eta>0$ such that

$$\left|\|Q\|_{L^\infty(\{u_\infty>0\})}-\|Q\|_{L^\infty(\{u_\infty>\eta\})}\right|\leqslant\epsilon.$$

We observe that, from the uniform convergence of u_p to u_∞, one gets

$$\{u_\infty>\eta\}\subset\{u_p>0\}$$

for every $p\geqslant p_0$, and hence

$$\begin{aligned}\|Q\|_{L^\infty(\{u_\infty>0\})}&\leqslant\|Q\|_{L^\infty(\{u_\infty>\eta\})}+\epsilon\\&\leqslant\lim_{p\to\infty}\left(\int_{\{u_\infty>\eta\}}Q^p\right)^{\frac{1}{p}}+\epsilon\\&\leqslant\liminf_{p\to\infty}\left(\int_{\{u_p>0\}}Q^p\right)^{\frac{1}{p}}+\epsilon.\end{aligned}$$

We conclude that, since ϵ is arbitrary,

$$\lambda_1 \|Q\|_{L^\infty(\{u_\infty>0\})} \leqslant \liminf_{p\to\infty} \left(\lambda_1^p \int_{\{u_p>0\}} Q^p\right)^{\frac{1}{p}},$$

and hence from (13) we get

$$\lambda_1 \|Q\|_{L^\infty(\{u_\infty>0\})} \leqslant \max\left\{Lip(\theta_\infty); \lambda_1 \|Q\|_{L^\infty(\theta_\infty>0)}; \lambda_2 \|Q\|_{L^\infty(\theta_\infty\leqslant 0)}\right\}.$$

To finish the proof we need a bound for

$$\lambda_2 \|Q\|_{L^\infty(\{u_\infty\leqslant 0\})}.$$

This task is different from the previous one since we can not assert that the sets $\{u_\infty \leqslant 0\}$ and $\{u_p \leqslant 0\}$ are similar from the uniform convergence.

From (12) we get

$$\begin{aligned}
&\left(\lambda_1^p \int_{\{u_p>0\}} Q^p + \lambda_2^p \int_{\{u_p\leqslant 0\}} Q^p\right)^{\frac{1}{p}} \\
&\qquad \leqslant \left(\frac{1}{p}\int_\Omega |\nabla\theta_\infty|^p + \lambda_1^p \int_{\{\theta_\infty>0\}} Q^p + \lambda_2^p \int_{\{\theta_\infty\leqslant 0\}} Q^p\right)^{\frac{1}{p}}. \qquad (14)
\end{aligned}$$

Using that $\lambda_1 < \lambda_2$ and that $\Omega = \{u_p > 0\} \cap \{u_p \leqslant 0\}$ we get

$$\left(\lambda_2^p \int_{\{u_\infty\leqslant 0\}} Q^p\right)^{\frac{1}{p}} \leqslant \left(\lambda_1^p \int_{\{u_p>0\}} Q^p + \lambda_2^p \int_{\{u_p\leqslant 0\}} Q^p\right)^{\frac{1}{p}}.$$

Taking $p \to \infty$, using (14) and our previous argument, we obtain

$$\begin{aligned}
\lambda_2 \|Q\|_{L^\infty(\{u_\infty\leqslant 0\})} &\leqslant \lim_{p\to\infty} \left(\lambda_2^p \int_{\{u_\infty\leqslant 0\}} Q^p\right)^{\frac{1}{p}} \\
&\leqslant \limsup_{p\to\infty} \left(\frac{1}{p}\int_\Omega |\nabla\theta_\infty|^p + \lambda_1^p \int_{\{\theta_\infty>0\}} Q^p + \lambda_2^p \int_{\{\theta_\infty\leqslant 0\}} Q^p\right)^{\frac{1}{p}} \\
&\leqslant \max\left\{Lip(\theta_\infty); \lambda_1 \|Q\|_{L^\infty(\theta_\infty>0)}; \lambda_2 \|Q\|_{L^\infty(\theta_\infty\leqslant 0)}\right\}.
\end{aligned}$$

Therefore, collecting all these bounds, we have obtained that any uniform limit of u_p is a solution of the minimization problem

$$\min_{u\in A,\, u|_{\partial\Omega}=\sigma} \max\left\{Lip(u); \lambda_1 \|Q\|_{L^\infty(u>0)}; \lambda_2 \|Q\|_{L^\infty(u\leqslant 0)}\right\}. \qquad (15)$$

□

REMARK 2.17 Remark that the limit problem be scaled as follows: if u is a solution to the limit problem with constants λ_1, λ_2 and boundary datum σ, then $u_k(x) = ku(x)$, for $k > 0$, is a also a solution with constants λ_1/k, λ_2/k and boundary datum $\sigma_k(x) = \sigma(x)/k$. Moreover if we let $u_k(x) = u(x/k)$ then we obtain a solution in the domain $\Omega_k = k\Omega$ with constants λ_1/k, λ_2/k and boundary datum $\sigma_k(x) = \sigma(x/k)$. Note that the Lipschitz constant of σ_k is the Lipschitz constant of σ over k. These facts are easy consequences of Remark 2.8 or can be obtained directly by scaling the limit minimization problem (15) as described above.

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