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# Shape optimization for surface functionals in Navier-Stokes flow using a phase field approach

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We consider shape and topology optimization of an object in fluid flow governed by the Navier–Stokes equations. Shapes are modelled with the help of a phase field approach and the solid body is relaxed to be a porous medium. The phase field method uses a Ginzburg–Landau functional in order to approximate a perimeter penalization. We focus on surface functionals and carefully introduce a new modelling variant, show existence of minimizers and derive first order necessary conditions. These conditions are related to classical shape derivatives by identifying the sharp interface limit with the help of formally matched asymptotic expansions. Finally, we present numerical computations based on a Cahn–Hilliard type gradient descent which demonstrate that the method can be used to solve shape optimization problems for fluids with the help of the new approach.

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## 1. Introduction

Shape optimization problems are a very challenging field in mathematical analysis and has attracted more and more attention in the last decade. One of the most discussed and oldest problems is certainly the task of finding the shape of a body inside a fluid having the least resistance. This problem dates back at least to Newton, who proposed this topic in a rotationally symmetric setting.

Nowadays, there are a lot of important industrial applications leading to this kind of questions. Among others we mention in particular the problem of optimizing the shape of airplanes, cars and wind turbine blades in order to have the least resistance or biomechanical applications like bypass constructions. The wide fields of applications may be one of the reasons that shape optimization problems in fluids received growing attention recently. Nevertheless, those problems turn out to be very challenging and so far no overall mathematical concept has been successful in a general sense.

One of the main difficulties certainly is that shape optimization problems are often not well-posed, i.e., no minimizer exists, compare for instance [20, 23, 28]. There are some contributions leading to mathematically well-posed problem formulations, see for instance [25], but the geometric restrictions are difficult to handle numerically. The most common approaches used in practice parametrize the boundary of the unknown optimal shape by functions, see for instance [6, 24]. However, those formulations do not inherit a minimizer in general. For numerical simulations typically shape sensitivity analysis is used. Here, one uses local boundary variations in order to find a gradient of the cost function with respect to the design variable, which is in this case the shape of the body. The necessary calculations are carried out without considering the existence or regularity of a minimizer. But in the end one obtains a mathematical structure that can be used for numerical implementations.

In [14], a phase field approach was introduced for minimizing general volume functionals in a Navier–Stokes flow. For this purpose, the porous medium approach proposed by Borrvall and Petersson [4] and a Ginzburg–Landau regularization as in the work of Bourdin and Chambolle [5] were combined. The latter is a diffuse interface approximation of a perimeter regularization. This leads to a model where existence of a minimizer can be guaranteed, and at the same time necessary optimality conditions can be derived and used for numerical simulations, see [15]. In particular, this approach replaces the free boundary $\Gamma$ of the body $B$ by a diffuse interface. Hence, it is a priori not clear how to deal with objective functionals that are defined on the free boundary $\Gamma$.

In this work, we study the following boundary objective functional:

$$\int_\Gamma h(x, \nabla \boldsymbol{u}, p, \boldsymbol{\nu}) \, \mathrm{d}\mathcal{H}^{d-1}, \tag{1.1}$$

where $h$ is a given function, $\boldsymbol{u}$ denotes the velocity field of the fluid, $p$ denotes the pressure, $\boldsymbol{\nu}$ is the *inner* unit normal of the fluid region, i.e., pointing from the body $B$ into the complementary fluid region $E = B^c$. The velocity $\boldsymbol{u}$ and pressure $p$ are assumed to obey the stationary Navier–Stokes equations inside the fluid region $E$, and the no-slip condition on $\Gamma$, namely,

$$-\operatorname{div} \boldsymbol{\sigma} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{f} \text{ in } E, \tag{1.2a}$$

$$\operatorname{div} \boldsymbol{u} = 0 \text{ in } E, \tag{1.2b}$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma, \tag{1.2c}$$

where $\boldsymbol{\sigma} := \mu \left( \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T \right) - p\,\mathrm{I}$ denotes the stress tensor of the velocity field $\boldsymbol{u}$, $\mu > 0$ denotes the viscosity of the fluid, $\boldsymbol{f}$ denotes an external body force, and $\mathrm{I}$ denotes the identity tensor.

An important example of $h$ is the hydrodynamic force component acting on $\Gamma$ with the force direction defined by the unit vector $\boldsymbol{a}$:

$$h(x, \nabla \boldsymbol{u}, p, \boldsymbol{\nu}) = \boldsymbol{a} \cdot (\boldsymbol{\sigma} \boldsymbol{\nu}) = \boldsymbol{a} \cdot \Big( \mu \left( \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T \right) - p\,\mathrm{I} \Big) \boldsymbol{\nu}, \tag{1.3}$$

and so (1.1) becomes

$$\int_\Gamma \boldsymbol{a}\cdot(\boldsymbol{\sigma}\boldsymbol{\nu})\,\mathrm{d}\mathcal{H}^{d-1} = \boldsymbol{a}\cdot\Big(\int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^{d-1}\Big). \tag{1.4}$$

If $\boldsymbol{a}$ is parallel to the direction of the flow, then (1.4) represents the drag of the object $B$. If $\boldsymbol{a}$ is perpendicular to the direction of the flow, then (1.4) represents the lift of the object.

In the work at hand we propose an approach on how to deal with boundary objective functionals in the phase field setting. To be precise, we aim to minimize an appropriate phase field approximation of the functional (1.1), and also the functional (1.4), which can be considered as one of the most important objectives in the field of shape optimization in fluids. The fluid is assumed to be an incompressible, Newtonian fluid described by the stationary Navier–Stokes equations (1.2).

For this purpose, we first discuss how we model the integral over the free boundary $\Gamma$ if it is replaced by a diffuse interface and how the normal $\boldsymbol{\nu}$ can be defined in this setting, see Section 3. Afterwards, we analyze the phase field problem for both (1.1) and (1.4) and discuss the existence of a minimizer and optimality conditions, see Section 4. In Section 5, we focus on the hydrodynamic force functional (1.4) and the corresponding phase field problem is then related to the sharp interface free boundary problem with a perimeter regularization by the method of matched formal asymptotic expansions. We find that the formal sharp interface limit of the optimality system gives the same results as can be found in the shape sensitivity literature.

We then solve the phase field problem numerically, see Section 6. For this purpose, we derive a gradient flow equation for the reduced objective functional and arrive at a Cahn–Hilliard type system. After time discretization, this system is treated in every time step by a Newton method. We numerically solve shape optimization problems involving drag and the lift-to-drag ratio.

## 2. Notation and problem formulation

Let us assume that $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a fixed domain with Lipschitz boundary. Inside this fixed domain $\Omega$ we may have certain parts filled with fluid, denoted by $E$, and the complement $B := \overline{\Omega} \setminus E$ is some non-permeable medium. In the following we will denote by $\boldsymbol{\nu}$ the outer unit normal of $B$, i.e., the inner unit normal of the fluid region. The aim is to minimize the functional, given by (1.1), where $\Gamma := \partial B \cap \Omega$, subject to the Navier–Stokes equations (1.2). We additionally impose a volume constraint on the amount of fluid. For this purpose we choose $\beta \in (-1,1)$ and only use fluid regions $E \subset \Omega$ fulfilling the constraint $|E| = \frac{(\beta+1)}{2}|\Omega|$.

We prescribe some inflow or outflow regions on the boundary of $\Omega$ and choose for this purpose $\boldsymbol{g} \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ such that $\int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{\nu}_{\partial\Omega}\,\mathrm{d}\mathcal{H}^{d-1} = 0$. Additionally, we may have some body force $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ acting on the design domain. Note that throughout this paper we denote $\mathbb{R}^d$-valued functions and spaces consisting of $\mathbb{R}^d$-valued functions in boldface.

As already mentioned in the introduction, problems like this are generally not well-posed in the sense that the existence of a minimizer can not be guaranteed. Hence, we use an additional perimeter regularization. For this purpose, we add a multiple of the perimeter of the obstacle to the cost functional (1.1). In order to properly formulate the resulting problem we introduce a design function $\varphi : \Omega \to \{\pm 1\}$, where $\{\varphi = 1\} = E$ describes the fluid region and $\{\varphi = -1\} = B$ is its complement. The volume constraint reads in this setting as $\int_\Omega \varphi\,\mathrm{d}x = \beta|\Omega|$.

The design functions are chosen to be functions of bounded variation, such that the fluid region has finite perimeter, i.e., $\varphi \in BV(\Omega,\{\pm 1\})$. We shall write $P_\Omega(E)$ for the perimeter of some set of

bounded variation $E \subseteq \Omega$ in $\Omega$. Besides, if $\varphi$ is a function of bounded variation, its distributional derivative $\mathrm{D}\varphi$ is a finite Radon measure and we can define the total variation by $|\mathrm{D}\varphi|(\Omega)$. For $\varphi \in BV(\Omega, \{\pm 1\})$, it holds that

$$|\mathrm{D}\varphi|(\Omega) = 2P_\Omega(\{\varphi = 1\}). \tag{2.1}$$

For a more detailed introduction to the theory of sets of finite perimeter and functions of bounded variation we refer to [11, 17]. We hence arrive in the following space of admissible design functions:

$$\Phi^0_{ad} := \left\{ \varphi \in BV(\Omega, \{\pm 1\}) \mid \int_\Omega \varphi \,\mathrm{dx} = \beta\,|\Omega| \right\}. \tag{2.2}$$

Let $\gamma > 0$ denote the weighting factor for the perimeter regularization. Then, we arrive at the following shape optimization problem for the functional (1.1) with additional perimeter regularization:

$$\min_{(\varphi,\boldsymbol{u},p)} J_0(\varphi,\boldsymbol{u},p) := \int_\Omega \frac{1}{2} h(x, \nabla \boldsymbol{u}, p, \boldsymbol{\nu}_\varphi)\, \mathrm{d}\,|\mathrm{D}\varphi| + \frac{\gamma}{2} |\mathrm{D}\varphi|(\Omega), \tag{2.3}$$

subject to $\varphi \in \Phi^0_{ad}$ and $(\boldsymbol{u}, p) \in \boldsymbol{H}^1(E) \times L^2(E)$ fulfilling

$$-\mu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } E = \{\varphi = 1\}, \tag{2.4a}$$

$$\operatorname{div} \boldsymbol{u} = 0 \text{ in } E, \tag{2.4b}$$

$$\boldsymbol{u} = \boldsymbol{g} \text{ on } \partial\Omega \cap \partial E, \tag{2.4c}$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma = \Omega \cap \partial E. \tag{2.4d}$$

Here, we used the relation (2.1) to replace the perimeter of $E$ with $\frac{1}{2}|\mathrm{D}\varphi|(\Omega)$. Furthermore, by the polar decomposition

$$\mathrm{D}\varphi = \boldsymbol{\nu}_\varphi |\mathrm{D}\varphi| \text{ for } \varphi \in BV\big(\Omega, \{\pm 1\}\big), \tag{2.5}$$

of the Radon measure $\mathrm{D}\varphi$ into a positive measure $|\mathrm{D}\varphi|$ and a $S^{d-1}$-valued function $\boldsymbol{\nu}_\varphi \in L^1(\Omega, |\mathrm{D}\varphi|)^d$, see for instance [1, Corollary 1.29], we replace the product of the normal and the Hausdorff measure in (1.4) by $\frac{1}{2}\boldsymbol{\nu}_\varphi \,\mathrm{d}\,|\mathrm{D}\varphi|$. In particular, $\boldsymbol{\nu}_\varphi$ can be considered as a generalised unit normal on $\partial E$.

We remark that the shape optimization problem (2.3) for the hydrodynamic force component (1.3) have been studied extensively in the literature. In the work of [2], the boundary integral (1.4) is transformed into a volume integral. This is also done in [7, 25], but in the latter, the compressible Navier–Stokes equations are considered. We also mention [21], which utilises the approach of Borrvall and Petersson [4] and the volume integral formulation. The shape derivatives for general volume and boundary objective functionals in Navier–Stokes flow have been derived in [26]. Finally, we mention the work of [3], which bears the most similarity to our set-up. Under the assumption that the set $E = \{\varphi = 1\}$ is $C^2$ and that there is a unique, sufficiently regular solution $\boldsymbol{u}$ to (1.2), the analysis of [3] obtained, via the speed method, that the shape derivative of

$$J(E) = \int_\Gamma \boldsymbol{a} \cdot \Big( \mu\big(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\big) - p\boldsymbol{I} \Big) \boldsymbol{\nu}\, \mathrm{d}\mathcal{H}^{d-1}$$

with respect to vector field $V$ is given by (see [3, Theorem 4, Equation 39])[1]

$$\mathrm{D}J(E)[V] = \int_\Gamma \langle V(0), \boldsymbol{\nu}\rangle (\boldsymbol{f}\cdot\boldsymbol{a} + \mu\partial_{\boldsymbol{\nu}}\boldsymbol{q}\cdot\partial_{\boldsymbol{\nu}}\boldsymbol{u})\,\mathrm{d}\mathcal{H}^{d-1}, \tag{2.6}$$

where $\boldsymbol{q}$ is the solution to the adjoint system (see [3, Equation 33.2]):

$$-\mu\Delta\boldsymbol{q} + (\nabla\boldsymbol{u})^T\boldsymbol{q} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{q} + \nabla\pi = \boldsymbol{0} \text{ in } E, \tag{2.7a}$$
$$\operatorname{div}\boldsymbol{q} = 0 \text{ in } E, \tag{2.7b}$$
$$\boldsymbol{q} = \boldsymbol{a} \text{ on } \Gamma, \tag{2.7c}$$
$$\boldsymbol{q} = \boldsymbol{0} \text{ on } \partial\Omega\cap\partial E. \tag{2.7d}$$

Here, we denote the normal derivative of a scalar $\alpha$ and of a vector $\boldsymbol{\beta}$ as

$$\partial_{\boldsymbol{\nu}}\alpha := \nabla\boldsymbol{\alpha}\cdot\boldsymbol{\nu}, \quad \partial_{\boldsymbol{\nu}}\boldsymbol{\beta} := (\nabla\boldsymbol{\beta})\boldsymbol{\nu}. \tag{2.8}$$

We note that as $\boldsymbol{u}$ satisfies the no-slip boundary condition (2.4d), $\boldsymbol{u}$ has no tangential components on $\Omega\cap\partial E$. Thus, we obtain

$$\nabla\boldsymbol{u} = \partial_{\boldsymbol{\nu}}\boldsymbol{u}\otimes\boldsymbol{\nu} \text{ on } \Gamma = \Omega\cap\partial E. \tag{2.9}$$

Using the divergence free condition (2.4b), and the no-slip condition (2.4d), we obtain on $\Gamma$:

$$0 = \operatorname{div}\boldsymbol{u} = \operatorname{tr}(\nabla\boldsymbol{u}) = \sum_{i=1}^{d}\partial_{\boldsymbol{\nu}}u_i\nu_i = \partial_{\boldsymbol{\nu}}\boldsymbol{u}\cdot\boldsymbol{\nu} \Longrightarrow (\nabla\boldsymbol{u})^T\boldsymbol{\nu} = (\partial_{\boldsymbol{\nu}}\boldsymbol{u}\cdot\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{0}, \tag{2.10}$$

which in turn implies that

$$J(E) = \int_\Gamma \boldsymbol{a}\cdot(\boldsymbol{\sigma}\boldsymbol{\nu})\,\mathrm{d}\mathcal{H}^{d-1} = \int_\Gamma \boldsymbol{a}\cdot(\mu\nabla\boldsymbol{u} - p\mathrm{I})\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^{d-1}. \tag{2.11}$$

This is similar to the setting of [26, Remark 12] and by following the computations in [26] one obtains (2.7) as the adjoint system and the shape derivative of (2.11) for a $C^2$ domain in the direction of $V$ is[2]

$$\begin{aligned}\mathrm{D}J(E)[V] = \int_\Gamma \langle V(0), \boldsymbol{\nu}\rangle\,(-\mu\partial_{\boldsymbol{\nu}}(\partial_{\boldsymbol{\nu}}\boldsymbol{u})\cdot\boldsymbol{a} + \partial_{\boldsymbol{\nu}}p(\boldsymbol{a}\cdot\boldsymbol{\nu}) + \mu\partial_{\boldsymbol{\nu}}\boldsymbol{q}\cdot\partial_{\boldsymbol{\nu}}\boldsymbol{u})\,\mathrm{d}\mathcal{H}^{d-1}\\ - \int_\Gamma \langle V(0), \boldsymbol{\nu}\rangle \operatorname{div}_\Gamma\left(\mu(\nabla\boldsymbol{u})^T\boldsymbol{a} - p\boldsymbol{a}\right)\mathrm{d}\mathcal{H}^{d-1},\end{aligned} \tag{2.12}$$

where $\operatorname{div}_\Gamma$ denotes the surface divergence. We introduce the surface gradient of $f$ on $\Gamma$ by $\nabla_\Gamma f$ with components $(\underline{D}_k f)_{1\le k\le d}$, and with this definition we obtain $\operatorname{div}_\Gamma\boldsymbol{\nu} = \sum_{k=1}^{d}\underline{D}_k\nu_k$ for a vector field $\boldsymbol{\nu}$. Moreover, in components, we have

$$\partial_{\boldsymbol{\nu}}(\partial_{\boldsymbol{\nu}}\boldsymbol{u})\cdot\boldsymbol{a} = \sum_{i,j,k=1}^{d}\nu_i\partial_i(\nu_j\partial_j u_k)a_k.$$

[1] We remark that in [3], the normal $\boldsymbol{n}$ is pointing from the fluid domain to the obstacle, i.e., in comparison with our set-up, $\boldsymbol{n} = -\boldsymbol{\nu}$.

[2] We remark that in [26, Remark 12] the term $\operatorname{div}_\Gamma(\mu(\nabla\boldsymbol{u})\boldsymbol{a})$ appears instead of $\operatorname{div}_\Gamma(\mu(\nabla\boldsymbol{u})^T\boldsymbol{a})$, which we believe is a typo.

REMARK 2.1 In [26, Remark 12], the term $\mu\partial_{\boldsymbol{\nu}}(\partial_{\boldsymbol{\nu}}\boldsymbol{u})\cdot\boldsymbol{a}$ appearing on the right hand side of (2.12) is originally given as $\sum_{i,j,k=1}^{d}\nu_i\frac{\partial^2 u_k}{\partial x_i\partial x_j}\nu_j a_k$. This is related to $\partial_{\boldsymbol{\nu}}(\partial_{\boldsymbol{\nu}}\boldsymbol{u})\cdot\boldsymbol{a}$ by the formula

$$\sum_{i,j,k=1}^{d}\nu_i\frac{\partial^2 u_k}{\partial x_i\partial x_j}\nu_j a_k=\partial_{\boldsymbol{\nu}}(\partial_{\boldsymbol{\nu}}\boldsymbol{u})\cdot\boldsymbol{a}-\sum_{i,j,k=1}^{d}\nu_i\partial_i\tilde{\nu}_j\partial_j u_k a_k, \tag{2.13}$$

where $\tilde{\boldsymbol{\nu}}=(\tilde{\nu}_j)_{1\le j\le d}$ denotes an extension of $\boldsymbol{\nu}$ off the boundary $\Gamma$ to a neighborhood $U\supset\Gamma$ with $|\tilde{\boldsymbol{\nu}}|=1$ near $\Gamma$ and $\tilde{\boldsymbol{\nu}}|_\Gamma=\boldsymbol{\nu}$.

By (2.9), we see that $\partial_j u_k=\partial_{\boldsymbol{\nu}}u_k\nu_j$ on $\Gamma$, and so

$$\sum_{i,j,k=1}^{d}\nu_i\partial_i\tilde{\nu}_j\partial_j u_k a_k=\sum_{i,j,k=1}^{d}\nu_i\partial_i\tilde{\nu}_j\nu_j\partial_{\boldsymbol{\nu}}u_k a_k=\sum_{i,j,k=1}^{d}\tfrac{1}{2}\nu_i\partial_i\big(|\tilde{\nu}_j|^2\big)\partial_{\boldsymbol{\nu}}u_k a_k=0. \tag{2.14}$$

Thus, the last term in (2.13) is zero and we have the relation

$$\sum_{i,j,k=1}^{d}\nu_i\frac{\partial^2 u_k}{\partial x_i\partial x_j}\nu_j a_k=\partial_{\boldsymbol{\nu}}(\partial_{\boldsymbol{\nu}}\boldsymbol{u})\cdot\boldsymbol{a}, \tag{2.15}$$

when $\boldsymbol{u}=\boldsymbol{0}$ on $\Gamma$.

Based on Remark 2.1, if $(\boldsymbol{u},p)$ are sufficiently regular, then a short computation involving (2.15) shows that on $\Gamma$,

$$\begin{aligned}-\mu\,\mathrm{div}_\Gamma\big((\nabla\boldsymbol{u})^T\boldsymbol{a}\big)-\mu\partial_{\boldsymbol{\nu}}(\partial_{\boldsymbol{\nu}}\boldsymbol{u})\cdot\boldsymbol{a}+\partial_{\boldsymbol{\nu}}p(\boldsymbol{a}\cdot\boldsymbol{\nu})+\mathrm{div}_\Gamma(p\boldsymbol{a})&\\ =-\mu\sum_{i=1}^{d}\underline{D}_i(\partial_i u_j)a_j-\mu\sum_{i,j,k=1}^{d}\nu_i\partial_k(\partial_i u_j)\nu_k a_j+\nabla p\cdot\boldsymbol{a}&\\ =-\mu\Delta\boldsymbol{u}\cdot\boldsymbol{a}+\nabla p\cdot\boldsymbol{a}=\boldsymbol{f}\cdot\boldsymbol{a}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{a}=\boldsymbol{f}\cdot\boldsymbol{a},&\end{aligned}$$

where we have used the no-slip condition (2.4d), and hence (2.12) is equivalent to (2.6).

## 3. Derivation of the phase field formulation

The problem derived in the previous section has several drawbacks. First, it is not clear if this is well-posed, i.e., if for every $\varphi\in\Phi^0_{ad}$ there is a solution of the state equations (2.4) and if there exists a minimizer $(\varphi,\boldsymbol{u},p)$ of the overall problem (2.3)–(2.4). Second, optimizing in the space $BV(\Omega)$ is not very practical. Deriving optimality conditions is not easy and it is not clear how to perform numerical simulations on this problem. Hence, we now want to approximate the complex shape optimization problem (2.3)–(2.4) by a problem that can be treated by well-known approaches. To this end we introduce a diffuse interface version of the free boundary problem by using a phase field approach.

### 3.1 *The state equations in the phase field setting*

In this setting, the design variable $\varphi:\Omega\to\mathbb{R}$ is now allowed to have values in $\mathbb{R}$, instead of only the two discrete values $\pm1$, and inherits $H^1(\Omega)$ regularity. In addition to the two phases $\{\varphi=1\}$ (fluid

region $E$) and $\{\varphi = -1\}$ (solid region $B$), we also have an interfacial region $\{-1 < \varphi < 1\}$ which is related to a small parameter $\varepsilon > 0$. By [22], we know that the Ginzburg–Landau energy

$$\mathcal{E}_\varepsilon : H^1(\Omega) \to \mathbb{R}, \quad \mathcal{E}_\varepsilon(\varphi) := \int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\,\mathrm{dx} \tag{3.1}$$

approximates $\varphi \mapsto c_0|\mathrm{D}\varphi|(\Omega) = 2c_0 P_\Omega(\{\varphi = 1\})$ in the sense of $\Gamma$-convergence. Here,

$$c_0 := \frac{1}{2}\int_{-1}^{1}\sqrt{2\psi(s)}\,\mathrm{ds} \tag{3.2}$$

and $\psi : \mathbb{R} \to \mathbb{R}$ is a potential with two equal minima at $\pm 1$, and in this paper we focus on an arbitrary double-well potential satisfying the assumption below:

ASSUMPTION 3.1 Let $\psi \in C^{1,1}(\mathbb{R})$ be a non-negative function such that $\psi(s) = 0$ if and only if $s \in \{\pm 1\}$, and the following growth condition is fulfilled for some constants $c_1, c_2, t_0 > 0$ and $k \geq 2$:

$$c_1 t^k \leq \psi(t) \leq c_2 t^k \quad \forall\, |t| \geq t_0.$$

Additionally, we use the so-called porous medium approach for the state equations, see also [14, 15]. This means that, we relax the non-permeability of the solid region $B$ outside the fluid by placing a porous medium of small permeability $(\overline{\alpha}_\varepsilon)^{-1} \ll 1$ outside the fluid region $E$. In the interfacial region $\{-1 < \varphi < 1\}$ we interpolate between the equations describing the flow through the porous medium and the stationary Navier–Stokes equations by using an interpolation function $\alpha_\varepsilon$ satisfying the following assumption:

ASSUMPTION 3.2 We assume that $\alpha_\varepsilon \in C^{1,1}(\mathbb{R})$ is non-negative, with $\alpha_\varepsilon(1) = 0$, $\alpha_\varepsilon(-1) = \overline{\alpha}_\varepsilon > 0$, and there exist $s_a, s_b \in \mathbb{R}$ with $s_a \leq -1$ and $s_b \geq 1$ such that

$$\begin{aligned} \alpha_\varepsilon(s) &= \alpha_\varepsilon(s_a) \text{ for } s \leq s_a, \\ \alpha_\varepsilon(s) &= \alpha_\varepsilon(s_b) \text{ for } s \geq s_b. \end{aligned} \tag{3.3}$$

Moreover, we assume that the inverse permeability vanishes as $\varepsilon \searrow 0$, i.e., $\lim_{\varepsilon \searrow 0} \overline{\alpha}_\varepsilon = \infty$.

In particular, we have that

$$0 \leq \alpha_\varepsilon(s) \leq \sup_{t \in [s_a, s_b]} \alpha_\varepsilon(t) < \infty \quad \forall s \in \mathbb{R},$$

i.e., $\alpha_\varepsilon \in L^\infty(\mathbb{R})$. The resulting state equations for the phase field problem are then given in the strong form by the following system:

$$\alpha_\varepsilon(\varphi)\boldsymbol{u} - \mu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega, \tag{3.4a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega, \tag{3.4b}$$

$$\boldsymbol{u} = \boldsymbol{g} \text{ on } \partial\Omega. \tag{3.4c}$$

Later we add $\int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2\,\mathrm{dx}$ to the objective functional and this ensures that in the limit $\varepsilon \searrow 0$, the velocity $\boldsymbol{u}$ vanishes outside the fluid region, and hence the medium can really be considered as non-permeable again. In the following, we will use the following function spaces:

$$\boldsymbol{H}^1_{0,\sigma}(\Omega) := \left\{\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) \mid \operatorname{div}\boldsymbol{v} = 0\right\}, \quad \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) := \left\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \mid \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{g},\ \operatorname{div}\boldsymbol{v} = 0\right\},$$

and for the pressure we use the space $L^2_0(\Omega) := \left\{p \in L^2(\Omega) \mid \int_\Omega p \,\mathrm{d}x = 0\right\}$. The function space of admissible design functions for the phase field optimization problem will be given correspondingly to (2.2) as

$$\Phi_{ad} := \left\{\varphi \in H^1(\Omega) \mid \int_\Omega \varphi \,\mathrm{d}x = \beta\,|\Omega|\right\}.$$

### 3.2 *The cost functional in the phase field setting*

We are now left to transfer the boundary integral in (2.3) to the diffuse interface setting where the free boundary $\Gamma$ is replaced by an interfacial region. To this end, we apply a result of [22] and approximate the perimeter regularization term with $\frac{1}{2c_0}\mathcal{E}_\varepsilon(\varphi)$. Meanwhile, keeping in mind the polar decomposition (2.5) and the relation (2.1), we consider the vector-valued measure with density $\frac{1}{2}\nabla\varphi$ as an approximation to $\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^{d-1}$. Thus, for functions $h$ that are positively one homogeneous with respect to its last variable, we may approximate (2.3) with

$$\int_\Omega \frac{1}{2} h(x, \nabla \boldsymbol{u}, p, \nabla\varphi)\,\mathrm{d}x + \frac{\gamma}{2c_0}\mathcal{E}_\varepsilon(\varphi).$$

Alternatively, we may appeal to the property of equipartition for the Ginzburg–Landau energy, i.e., it holds asymptotically that (see for instance, (5.28) in Section 5, or [9, Section 5.1]):

$$\int_\Omega \left|\frac{1}{\varepsilon}\psi(\varphi_\varepsilon) - \frac{\varepsilon}{2}|\nabla\varphi_\varepsilon|^2\right| \mathrm{d}x \sim 0 \text{ as } \varepsilon \searrow 0.$$

Hence, together with (2.1), and the fact that $\Gamma$-limit of $\mathcal{E}_\varepsilon(\varphi)$ is the functional $c_0\,|\mathrm{D}\varphi|(\Omega)$, defined for functions with values in $\{\pm 1\}$, and $+\infty$ otherwise, we have loosely speaking

$$2c_0\mathcal{H}^{d-1}\llcorner\Gamma \sim c_0\,|\mathrm{D}\varphi| \sim \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi) \sim \frac{2}{\varepsilon}\psi(\varphi), \tag{3.5}$$

where $\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)$ and $\frac{2}{\varepsilon}\psi(\varphi)$ are interpreted as measures on $\Omega$, by using their values as densities. Here, we have identified $\Gamma = \partial\{\varphi = 1\} \cap \Omega$ with its reduced boundary, then it holds that $\frac{1}{2}|\mathrm{D}\varphi| = \left|\mathrm{D}\chi_{\{\varphi=1\}}\right| = \mathcal{H}^{d-1}\llcorner\Gamma$, see for instance [1, Theorem 3.59].

The generalised unit normal $\boldsymbol{\nu}$ can be approximated by $\frac{\nabla\varphi}{|\nabla\varphi|}$. To rewrite this into a more convenient form, which is in particular differentiable with respect to $\varphi$, we use equipartition of energy and replace $|\nabla\varphi|$ by $\frac{1}{\varepsilon}\sqrt{2\psi(\varphi)}$, and obtain the approximation

$$c_0\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^{d-1} \sim \varepsilon\frac{\nabla\varphi}{\sqrt{2\psi(\varphi)}}\frac{1}{\varepsilon}\psi(\varphi)\,\mathrm{d}x = \sqrt{\frac{\psi(\varphi)}{2}}\nabla\varphi\,\mathrm{d}x. \tag{3.6}$$

Hence, we may also approximate (2.3) with

$$\frac{1}{c_0}\int_\Omega \sqrt{\tfrac{\psi(\varphi)}{2}}\,h(x, \nabla\boldsymbol{u}, p, \nabla\varphi)\,\mathrm{d}x + \frac{\gamma}{2c_0}\mathcal{E}_\varepsilon(\varphi), \tag{3.7}$$

when we have again used that $h$ is positively one homogeneous with respect to its last variable.

We note that in the bulk regions $\{\varphi = \pm 1\}$, we have $\psi(\varphi) = 0$ and hence the functional (3.7) is not differentiable with respect to $\varphi$. Hence, we add a small positive constant $\delta_\varepsilon$ to $\psi$ in order to have

$\psi(s)+\delta_\varepsilon > 0$ for all $s \in \mathbb{R}$. However, we neglect the addition of this constant for the Ginzburg–Landau regularization $\mathcal{E}_\varepsilon(\varphi)$ in the objective functional because adding a constant to the cost functional will not change the optimization problem.

In fact, for the analysis of the phase field problem, it is only important that $\delta_\varepsilon > 0$. In Section 5 where we perform a formal asymptotic analysis, we will require $\lim_{\varepsilon \searrow 0} \delta_\varepsilon = 0$ at a superlinear rate (see Remark 5.1).

### 3.3 *Optimization problem in the phase field setting*

Combining the above ideas, we arrive in the following phase field approximation:

$$\min_{(\varphi,\boldsymbol{u},p)} J_\varepsilon^h(\varphi,\boldsymbol{u},p) := \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2 + \frac{\gamma}{2c_0}\left(\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\right)\mathrm{d}x + \int_\Omega \mathcal{M}(\varphi)h(x,\nabla\boldsymbol{u},p,\nabla\varphi)\,\mathrm{d}x, \quad (3.8)$$

subject to $\varphi \in \Phi_{ad}$ and $(\boldsymbol{u},p) \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) \times L^2_0(\Omega)$ fulfilling

$$\int_\Omega \alpha_\varepsilon(\varphi)\boldsymbol{u}\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v} - p\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega). \quad (3.9)$$

Notice, that (3.9) is a weak formulation of the state equations (3.4). Moreover, based on the discussions in Section 3.2, the function $\mathcal{M}(\varphi)$ can be chosen to be

$$\mathcal{M}(\varphi) = \frac{1}{2} \text{ or } \mathcal{M}(\varphi) = \frac{1}{c_0}\sqrt{\tfrac{\psi(\varphi)+\delta_\varepsilon}{2}}. \quad (3.10)$$

The phase field approximation for the shape optimization problem with the hydrodynamic force (1.3) is obtained from (3.8) by substituting

$$h(x,\nabla\boldsymbol{u},p,\nabla\varphi) = \nabla\varphi\cdot(\mu(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T) - p\mathrm{I})\boldsymbol{a}.$$

That is,

$$\min_{(\varphi,\boldsymbol{u},p)} J_\varepsilon(\varphi,\boldsymbol{u},p) := \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2 + \frac{\gamma}{2c_0}\left(\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\right)\mathrm{d}x + \int_\Omega \mathcal{M}(\varphi)\nabla\varphi\cdot(\mu(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T) - p\mathrm{I})\boldsymbol{a}\,\mathrm{d}x, \quad (3.11)$$

subject to $\varphi \in \Phi_{ad}$ and $(\boldsymbol{u},p) \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) \times L^2_0(\Omega)$ fulfilling (3.9).

Let us point out the main novel contributions of the present paper when compared to earlier works of [14, 15]. We now consider surface functionals which are important for many applications. This was not possible with the earlier set-up in [14, 15], and leads to severely new difficulties as we first of all have to approximate surface integrals with the help of the phase field diffuse interfacial layer. Secondly, we now also need to approximate the surface normal with the help of the gradient of the phase field variable. The additional highly non-linear terms lead to serious new difficulties with respect to the analysis, asymptotics and numerical simulations.

### 3.4 *Possible modifications*

3.4.1 *Double obstacle potential.* We could also use a double obstacle potential $\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ instead of the double-well potential in Assumption 3.1, i.e.,

$$\psi(\varphi) = \begin{cases} \frac{1}{2}(1-\varphi^2) & \text{if } \varphi \in [-1,1], \\ +\infty & \text{if } |\varphi| > 1. \end{cases} \tag{3.12}$$

Then, one has to treat the constraint $|\varphi| \le 1$ a.e. in the necessary optimality system either by writing the gradient equation in form of a variational inequality or by including additional Lagrange parameters. Numerical simulations could be implemented by a Moreau-Yosida relaxation as in [15]. A Moreau-Yosida relaxation also leads to a differentiable double-well potential, and here we restrict ourselves to a differentiable potential where both settings can then be included in the above mentioned way.

3.4.2 *Inequality constraint for fluid volume.* Another possible modification of the problem setting would be to replace the equality constraint $\int_\Omega \varphi \,\mathrm{dx} = \beta\,|\Omega|$ by an inequality constraint $\int_\Omega \varphi \,\mathrm{dx} \le \beta\,|\Omega|$. This would make sense in certain settings, if a maximal amount of fluid that can be used during the optimization process is prescribed and not the exact volume fraction. This would not change anything in the analysis, only that the Lagrange multiplier for this constraint would have a sign and an additional complementarity constraint appears in the optimality system.

3.4.3 *Objective functionals with no dependency on the unit normal.* We may also consider objective functionals with no dependence on the normal, i.e., the boundary objective functional (2.3) takes the form

$$\int_\Gamma k(x, \nabla \boldsymbol{u}, p)\,\mathrm{d}\mathcal{H}^{d-1}. \tag{3.13}$$

An example of (3.13) is the best approximation to a target surface pressure distribution in the sense of least squares:

$$k(x, \nabla \boldsymbol{u}, p) = \frac{1}{2}\,|p - p_d|^2,$$

where $p_d$ denotes the target surface pressure distribution. Then, using (3.5), we deduce that the phase field approximation of (3.13) is given by

$$\frac{1}{c_0} \int_\Omega \frac{1}{\varepsilon} \psi(\varphi) k(x, \nabla \boldsymbol{u}, p)\,\mathrm{dx}.$$

If $k(\cdot,\cdot,\cdot)$ satisfies similar assumptions to Assumptions 4.1 and 4.2 (see below), one can adapt the proofs of Theorems 4.6 and 4.10 to obtain existence of a minimiser and the corresponding first order necessary optimality conditions.

## 4. Analysis of the phase field problem

In this section we want to analyze the phase field problem (3.8)–(3.9) derived in the previous section as a diffuse interface approximation of the shape optimization problem of minimizing (1.1) for

a Navier–Stokes flow. For this purpose, we introduce some notation for the nonlinearity in the stationary Navier–Stokes equations. We define the trilinear form

$$b : \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \to \mathbb{R},$$
$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := \int_\Omega (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{v} \cdot \boldsymbol{w} \, \mathrm{dx} = \sum_{i,j=1}^{d} \int_\Omega u_i \partial_i v_j w_j \, \mathrm{dx}.$$

From this, we directly obtain the following properties, which are also used in [14].

**Lemma 4.1** *The form $b$ is well-defined and continuous in the space $\boldsymbol{H}^1_0(\Omega) \times \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1_0(\Omega)$. Moreover we have:*

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le K_\Omega \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \|\nabla \boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{u}, \boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega), \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \tag{4.1}$$

*with*

$$K_\Omega = \begin{cases} \frac{1}{2} |\Omega|^{1/2} & \text{if } d = 2, \\ \frac{2\sqrt{2}}{3} |\Omega|^{1/6} & \text{if } d = 3. \end{cases} \tag{4.2}$$

*Additionally, the following properties are satisfied:*

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega), \operatorname{div} \boldsymbol{u} = 0, \quad \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega), \tag{4.3}$$
$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) \quad \forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega), \operatorname{div} \boldsymbol{u} = 0, \quad \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega). \tag{4.4}$$

*Proof.* The stated continuity and estimate (4.1) can be found in [13, Lemma IX.1.1] and (4.3)–(4.4) are considered in [13, Lemma IX.2.1]. □

Next, we have the following important continuity property.

**Lemma 4.2** *Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}, (\boldsymbol{v}_n)_{n\in\mathbb{N}}, (\boldsymbol{w}_n)_{n\in\mathbb{N}} \subset \boldsymbol{H}^1(\Omega)$, $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ be such that $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$, $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$ and $\boldsymbol{w}_n \rightharpoonup \boldsymbol{w}$ in $\boldsymbol{H}^1(\Omega)$ where $\boldsymbol{v}_n|_{\partial\Omega} = \boldsymbol{v}|_{\partial\Omega}$ for all $n \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} b(\boldsymbol{u}_n, \boldsymbol{v}_n, \tilde{\boldsymbol{w}}) = b(\boldsymbol{u}, \boldsymbol{v}, \tilde{\boldsymbol{w}}) \quad \forall \tilde{\boldsymbol{w}} \in \boldsymbol{H}^1(\Omega). \tag{4.5}$$

*Moreover, one can show that*

$$\boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \ni (\boldsymbol{u}, \boldsymbol{v}) \mapsto b(\boldsymbol{u}, \cdot, \boldsymbol{v}) \in \boldsymbol{H}^{-1}(\Omega) \tag{4.6}$$

*is strongly continuous, and thus*

$$\lim_{n\to\infty} b(\boldsymbol{u}_n, \boldsymbol{v}_n, \boldsymbol{w}_n) = b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}). \tag{4.7}$$

*Proof.* We apply the idea of [32, Lemma 72.5] and make in particular use of the compact embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$ and the continuous embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$. The strong continuity of (4.6) follows from [32, Lemma 72.5]. In addition, from the boundedness of the sequences

$(\boldsymbol{u}_n)_{n\in\mathbb{N}}, (\boldsymbol{v}_n)_{n\in\mathbb{N}}, (\boldsymbol{w}_n)_{n\in\mathbb{N}}$, and (4.6), we have

$$\begin{aligned}
&|b(\boldsymbol{u}_n,\boldsymbol{v}_n,\boldsymbol{w}_n) - b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \\
&\quad = |b(\boldsymbol{u}_n-\boldsymbol{u},\boldsymbol{v}_n,\boldsymbol{w}_n)| + |b(\boldsymbol{u},\boldsymbol{v}_n,\boldsymbol{w}_n-\boldsymbol{w})| + |b(\boldsymbol{u},\boldsymbol{v}_n-\boldsymbol{v},\boldsymbol{w})| \\
&\quad \le \underbrace{\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}}_{\xrightarrow{n\to\infty}0} \underbrace{\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{w}_n\|_{\boldsymbol{L}^6(\Omega)}}_{\le C} + \underbrace{\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}}_{\le C}\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^2(\Omega)} \underbrace{\|\boldsymbol{w}_n-\boldsymbol{w}\|_{\boldsymbol{L}^3(\Omega)}}_{\xrightarrow{n\to\infty}0} \\
&\qquad\qquad + \underbrace{|b(\boldsymbol{u},\boldsymbol{v}_n-\boldsymbol{v},\boldsymbol{w})|}_{\xrightarrow{n\to\infty}0 \text{ by (4.6)}}.
\end{aligned}$$

□

### 4.1 *Existence results*

In this section, we want to analyze the solvability of the state equations (3.9). Afterwards, we will show existence of a minimizer for the overall optimization problem (3.8)–(3.9).

**Lemma 4.3** *Let Assumption 3.2 hold. Then, for every $\varphi \in L^1(\Omega)$ there exists at least one pair $(\boldsymbol{u},p) \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)\times L^2_0(\Omega)$ such that the state equations (3.4) are fulfilled in the sense of (3.9). This solution $(\boldsymbol{u},p)$ fulfils the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|p\|_{L^2(\Omega)} \le C(\mu,\alpha_\varepsilon,\boldsymbol{f},\boldsymbol{g},\Omega), \tag{4.8}$$

*with a constant $C = C(\mu,\alpha_\varepsilon,\boldsymbol{f},\boldsymbol{g},\Omega)$ independent of $\varphi$.*

*Proof.* We refer to [14, Lemma 4], where the existence of the velocity field $\boldsymbol{u}$ is discussed. We point out, that the restriction to functions $\varphi \in L^1(\Omega)$ with $|\varphi| \le 1$ a.e. in $\Omega$ used in [14] is only necessary because the function $\alpha_\varepsilon$ in [14] is only defined on the interval $[-1,1]$. But of course, the same arguments apply to our case where $\alpha_\varepsilon$ is bounded and $\varphi \in L^1(\Omega)$.

Now for every $\varphi \in L^1(\Omega)$, there exists a $\boldsymbol{u} \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$ fulfilling

$$\int_\Omega \alpha_\varepsilon(\varphi)\boldsymbol{u}\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_{0,\sigma}(\Omega).$$

Since $\boldsymbol{g} \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ with $\int_{\partial\Omega}\boldsymbol{g}\cdot\boldsymbol{\nu}_{\partial\Omega}\,\mathrm{d}\mathcal{H}^{d-1} = 0$, by [13, Lemma IX.4.2] or [14, Lemma 3], for any $\eta > 0$, there exists some $\delta = \delta(\eta,\boldsymbol{g},\boldsymbol{\nu},\Omega) > 0$ and a vector field $\boldsymbol{G} = \boldsymbol{G}(\delta)$ such that $\boldsymbol{G} \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$ and

$$\left|\int_\Omega \boldsymbol{v}\cdot\nabla\boldsymbol{G}\cdot\boldsymbol{v}\,\mathrm{d}x\right| \le \eta\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega). \tag{4.9}$$

Then, $\boldsymbol{w} := \boldsymbol{u} - \boldsymbol{G} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$ satisfies

$$\begin{aligned}
&\int_\Omega \alpha_\varepsilon(\varphi)\boldsymbol{w}\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{w}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x + b(\boldsymbol{w},\boldsymbol{w},\boldsymbol{v}) + b(\boldsymbol{w},\boldsymbol{G},\boldsymbol{v}) + b(\boldsymbol{G},\boldsymbol{w},\boldsymbol{v}) \\
&\qquad = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} - \alpha_\varepsilon(\varphi)\boldsymbol{G}\cdot\boldsymbol{v} - \mu\nabla\boldsymbol{G}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x - b(\boldsymbol{G},\boldsymbol{G},\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_{0,\sigma}(\Omega).
\end{aligned}$$

Testing with $\boldsymbol{v} = \boldsymbol{w}$ and using (4.3) so that $b(\boldsymbol{w},\boldsymbol{w},\boldsymbol{w}) = 0$, $b(\boldsymbol{G},\boldsymbol{w},\boldsymbol{w}) = 0$, and (4.9) with $\eta = \frac{\mu}{8}$, $\boldsymbol{v} = \boldsymbol{w}$ so that $|b(\boldsymbol{w},\boldsymbol{G},\boldsymbol{w})| \leq \frac{\mu}{8}\|\nabla \boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}$, we obtain after applying Hölder's inequality, Poincaré's inequality, and Young's inequality

$$\int_\Omega \frac{\alpha_\varepsilon(\varphi)}{2}|\boldsymbol{w}|^2\,\mathrm{dx} + \frac{7}{8}\mu\|\nabla \boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)} \leq \int_\Omega \frac{\alpha_\varepsilon(\varphi)}{2}|\boldsymbol{G}|^2\,\mathrm{dx} + 2C_p^2\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + 2\mu\|\nabla \boldsymbol{G}\|^2_{\boldsymbol{L}^2(\Omega)} + 2(C_p+1)^2\|\boldsymbol{G}\cdot\nabla \boldsymbol{G}\|^2_{\boldsymbol{H}^{-1}(\Omega)} + \frac{3}{8}\mu\|\nabla \boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}.$$

Rearranging and applying the triangle inequality, we obtain that

$$\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \leq \frac{4}{\mu}\Big(\big(\max_{s\in[s_a,s_b]}\alpha_\varepsilon(s)+2\mu\big)\|\boldsymbol{G}\|^2_{\boldsymbol{H}^1(\Omega)} + 2C_p^2\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}\Big) + \frac{4}{\mu}\Big(2(C_p+1)^2\|\boldsymbol{G}\cdot\nabla \boldsymbol{G}\|^2_{\boldsymbol{H}^{-1}(\Omega)}\Big) + 2\|\nabla \boldsymbol{G}\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{4.10}$$

Furthermore, by [27, Lemma II.2.1.1] we find a unique $p \in L^2_0(\Omega)$ such that (3.9) together with

$$\|p\|_{L^2(\Omega)} \leq C(\Omega)\|\alpha_\varepsilon(\varphi)\boldsymbol{u} - \mu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)}$$

is fulfilled. Combining this with (4.10) and the Poincaré inequality yields the estimate (4.8). □

This motivates the definition of a set-valued solution operator

$$\boldsymbol{S}_\varepsilon(\varphi) := \{(\boldsymbol{u},p) \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)\times L^2_0(\Omega) \mid (\boldsymbol{u},p) \text{ fulfil (3.9)}\} \text{ for } \varphi \in L^1(\Omega). \tag{4.11}$$

REMARK 4.1 If there is some $\boldsymbol{u} \in \boldsymbol{S}_\varepsilon(\varphi)$ with $\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{K_\Omega}$, where $K_\Omega$ is defined in (4.2). Then $\boldsymbol{S}_\varepsilon(\varphi) = \{(\boldsymbol{u},p)\}$. That is, there is exactly one solution of (3.9) corresponding to $\varphi$ (see for instance [18, Lemma 12.2] or [14, Lemma 5]). From (4.10), the condition $\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{K_\Omega}$ can be achieved for sufficiently large viscosity $\mu$ or small data $\boldsymbol{f}$ and $\boldsymbol{g}$.

**Lemma 4.4** *Under Assumption 3.2, assume that for* $(\varphi_k)_{k\in\mathbb{N}} \subset L^1(\Omega)$, $(\boldsymbol{u}_k)_{k\in\mathbb{N}} \subset \boldsymbol{L}^2(\Omega)$ *and* $\varphi \in L^1(\Omega)$, $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$,

$$\lim_{k\to\infty}\|\varphi_k - \varphi\|_{L^1(\Omega)} = 0, \quad \varphi_k \to \varphi \text{ a.e. and } \lim_{k\to\infty}\|\boldsymbol{u}_k - \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} = 0.$$

*Then it holds that*

$$\lim_{k\to\infty}\int_\Omega \alpha_\varepsilon(\varphi_k)|\boldsymbol{u}_k|^2\,\mathrm{dx} = \int_\Omega \alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2\,\mathrm{dx} \text{ and } \lim_{k\to\infty}\|\alpha_\varepsilon(\varphi_k)\boldsymbol{u}_k - \alpha_\varepsilon(\varphi)\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} = 0.$$

*Proof.* Using the ideas of [18, Theorem 5.1] and [14, Theorem 1] we find that

$$\left|\int_\Omega \alpha_\varepsilon(\varphi_k)|\boldsymbol{u}_k|^2 - \alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2\,\mathrm{dx}\right| = \int_\Omega \alpha_\varepsilon(\varphi_k)\big(|\boldsymbol{u}_k|^2 - |\boldsymbol{u}|^2\big)\,\mathrm{dx} + \int_\Omega \big(\alpha_\varepsilon(\varphi_k) - \alpha_\varepsilon(\varphi)\big)|\boldsymbol{u}|^2\,\mathrm{dx},$$

and from $\alpha_\varepsilon \in L^\infty(\mathbb{R})$ we obtain

$$\int_\Omega \alpha_\varepsilon(\varphi_k)\big(|\boldsymbol{u}_k|^2 - |\boldsymbol{u}|^2\big)\,\mathrm{dx} \leq \|\alpha_\varepsilon\|_{L^\infty(\mathbb{R})}\|\boldsymbol{u}_k + \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}_k - \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \xrightarrow{k\to\infty} 0.$$

Moreover, the uniform bound on $\alpha_\varepsilon$ yields by Lebesgue's dominated convergence theorem

$$\lim_{k\to\infty} \int_\Omega \big(\alpha_\varepsilon(\varphi_k) - \alpha_\varepsilon(\varphi)\big)|\boldsymbol{u}|^2 \,\mathrm{dx} = 0,$$

which combined with the previous step yields the first assertion. Using a similar idea we find

$$\begin{aligned}\|\alpha_\varepsilon(\varphi_k)\boldsymbol{u}_k - \alpha_\varepsilon(\varphi)\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} &\le \|\alpha_\varepsilon(\varphi_k)(\boldsymbol{u}_k - \boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)} + \|\big(\alpha_\varepsilon(\varphi_k) - \alpha_\varepsilon(\varphi)\big)\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \\ &\le \|\alpha_\varepsilon\|_{L^\infty(\mathbb{R})}\|\boldsymbol{u}_k - \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\big(\alpha_\varepsilon(\varphi_k) - \alpha_\varepsilon(\varphi)\big)\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \xrightarrow{k\to\infty} 0,\end{aligned}$$

where we applied Lebesgue's dominated convergence theorem in order to deduce from $\alpha_\varepsilon \in L^\infty(\mathbb{R})$ that $\lim_{k\to\infty} \|(\alpha_\varepsilon(\varphi_k) - \alpha_\varepsilon(\varphi))\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} = 0$. □

With the help of the above lemma, we show a certain continuity property of the solution operator.

**Lemma 4.5** *Under Assumption 3.2, assume $(\varphi_k)_{k\in\mathbb{N}} \subset L^1(\Omega)$ converges strongly to $\varphi \in L^1(\Omega)$ in the $L^1$-norm and $(\boldsymbol{u}_k, p_k)_{k\in\mathbb{N}} \subset \boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ are given such that $(\boldsymbol{u}_k, p_k) \in \boldsymbol{S}_\varepsilon(\varphi_k)$ for all $k \in \mathbb{N}$. Then there is a subsequence, which will be denoted by the same, such that $(\boldsymbol{u}_k, p_k)_{k\in\mathbb{N}}$ converges strongly in $\boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ to some element $(\boldsymbol{u}, p) \in \boldsymbol{S}_\varepsilon(\varphi)$.*

*Proof.* Let $(\varphi_k)_{k\in\mathbb{N}}$ and $(\boldsymbol{u}_k, p_k)_{k\in\mathbb{N}}$ be chosen as in the statement. By passing to another subsequence, denoted the same, we can without loss of generality assume that $\varphi_k \to \varphi$ almost everywhere. Invoking (4.8), we obtain a uniform bound on $(\boldsymbol{u}_k, p_k)$ in $\boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ because $(\boldsymbol{u}_k, p_k) \in \boldsymbol{S}_\varepsilon(\varphi_k)$. And so there is a subsequence, which will be denoted by the same, such that $\boldsymbol{u}_k$ converges weakly in $\boldsymbol{H}^1(\Omega)$ and strongly in $\boldsymbol{L}^2(\Omega)$ to some limit element $\boldsymbol{u} \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$ and $p_k$ converges weakly in $L^2(\Omega)$ to some limit element $p \in L^2_0(\Omega)$. We now aim to show that

$$\begin{aligned} F_k &: \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) \to \mathbb{R}, \\ F_k(\boldsymbol{v}) &:= \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_k)|\boldsymbol{v}|^2 + \frac{\mu}{2}|\nabla\boldsymbol{v}|^2 + (\boldsymbol{u}_k\cdot\nabla)\boldsymbol{u}_k\cdot\boldsymbol{v} - \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{dx},\end{aligned}$$

$\Gamma$-converges in $\boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$ equipped with the weak topology to

$$\begin{aligned} F_\infty &: \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) \to \mathbb{R}, \\ F_\infty(\boldsymbol{v}) &:= \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi)|\boldsymbol{v}|^2 + \frac{\mu}{2}|\nabla\boldsymbol{v}|^2 + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v} - \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{dx},\end{aligned}$$

as $k\to\infty$. To see this we first notice that for any sequence $(\boldsymbol{v}_k)_{k\in\mathbb{N}} \subseteq \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$ converging weakly in $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{v} \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$, by Fatou's lemma it holds that

$$\int_\Omega \alpha_\varepsilon(\varphi)|\boldsymbol{v}|^2\,\mathrm{dx} \le \liminf_{k\to\infty}\int_\Omega \alpha_\varepsilon(\varphi_k)|\boldsymbol{v}_k|^2\,\mathrm{dx}.$$

Applying the boundedness and continuity properties of the trilinear form $b(\cdot,\cdot,\cdot)$, see Lemma 4.1 and 4.2, we can deduce that $\lim_{k\to\infty} b(\boldsymbol{u}_k,\boldsymbol{u}_k,\boldsymbol{v}_k) = b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})$. As the remaining terms of $F_k$ are weakly lower semicontinuous in $\boldsymbol{H}^1(\Omega)$ and independent of $\varphi_k$, we directly obtain

$$F_\infty(\boldsymbol{v}) \le \liminf_{k\to\infty} F_k(\boldsymbol{v}_k).$$

Let $\boldsymbol{v} \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$ be chosen. We will show, that the constant sequence $(\boldsymbol{v})_{k\in\mathbb{N}}$ defines a recovery sequence. For this purpose, we notice that due to the boundedness and continuity of $\alpha_\varepsilon$, we have from Lebesgue's dominated convergence theorem

$$\lim_{k\to\infty} \int_\Omega \alpha_\varepsilon(\varphi_k)\,|\boldsymbol{v}|^2 \,\mathrm{dx} = \int_\Omega \alpha_\varepsilon(\varphi)\,|\boldsymbol{v}|^2 \,\mathrm{dx}. \tag{4.12}$$

Invoking (4.5) in Lemma 4.2, we deduce that

$$\lim_{k\to\infty} b(\boldsymbol{u}_k, \boldsymbol{u}_k, \boldsymbol{v}) = b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}),$$

and thus, we obtain that $\lim_{k\to\infty} F_k(\boldsymbol{v}) = F_\infty(\boldsymbol{v})$. This shows that the $\Gamma$-limit of $(F_k)_{k\in\mathbb{N}}$ in $\boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$ with respect to the weak topology equals $F_\infty$.

Now we notice, that $\boldsymbol{u}_k$ is exactly the unique minimizer of $F_k$ in $\boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$, as it fulfils per definition the necessary and sufficient first order optimality conditions for the convex optimization problem $\min_{\boldsymbol{u}\in\boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)} F_k(\boldsymbol{u})$. Hence, the weak $\boldsymbol{H}^1(\Omega)$ limit of $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$, which is $\boldsymbol{u} \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$, is the unique solution of $\min_{\boldsymbol{u}\in\boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)} F_\infty(\boldsymbol{u})$. Thus it holds that

$$\int_\Omega \alpha_\varepsilon(\varphi)\boldsymbol{u}\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{dx} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{dx} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_{0,\sigma}(\Omega). \tag{4.13}$$

By [27, Lemma II.2.1.1] we can associate to (4.13) a unique $\tilde{p} \in L^2_0(\Omega)$ such that (3.9) is fulfilled, and hence $\tilde{p} = p$. Altogether we have shown $(\boldsymbol{u}, p) \in \boldsymbol{S}_\varepsilon(\varphi)$.

To show the strong convergence in $\boldsymbol{H}^1(\Omega)\times L^2(\Omega)$, we note that from the $\Gamma$-convergence of $(F_k)_{k\in\mathbb{N}}$ to $F_\infty$ we obtain additionally that $\lim_{k\to\infty} F_k(\boldsymbol{u}_k) = F_\infty(\boldsymbol{u})$. Invoking Lemma 4.4 we find

$$\lim_{k\to\infty} \int_\Omega \alpha_\varepsilon(\varphi_k)\,|\boldsymbol{u}_k|^2 \,\mathrm{dx} = \int_\Omega \alpha_\varepsilon(\varphi)\,|\boldsymbol{u}|^2 \,\mathrm{dx}.$$

In addition, by means of (4.7) from Lemma 4.2 we have

$$\lim_{k\to\infty} b(\boldsymbol{u}_k, \boldsymbol{u}_k, \boldsymbol{u}_k) = b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}).$$

These two results allow us to deduce from the convergence of the minimal functional values of $(F_k)_{k\in\mathbb{N}}$ that $\lim_{k\to\infty}\int_\Omega |\nabla\boldsymbol{u}_k|^2\,\mathrm{dx} = \int_\Omega |\nabla\boldsymbol{u}|^2\,\mathrm{dx}$. Then, together with $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ this yields that $\lim_{k\to\infty} \|\boldsymbol{u}_k - \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} = 0$.

Subtracting the state equations (3.9) written for $\varphi$ from the state equations (3.9) written for $\varphi_k$, we find from Lemma 4.4 and (4.5) that

$$\begin{aligned}
\int_\Omega (p_k - p)\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{dx} &= \int_\Omega \big(\alpha_\varepsilon(\varphi_k)\boldsymbol{u}_k - \alpha_\varepsilon(\varphi)\boldsymbol{u}\big)\cdot\boldsymbol{v} + \mu\nabla(\boldsymbol{u}_k - \boldsymbol{u})\cdot\nabla\boldsymbol{v}\,\mathrm{dx} + b(\boldsymbol{u}_k, \boldsymbol{u}_k, \boldsymbol{v}) - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) \\
&\le \underbrace{\|\alpha_\varepsilon(\varphi_k)\boldsymbol{u}_k - \alpha_\varepsilon(\varphi)\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}}_{\xrightarrow{k\to\infty}0} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \mu \underbrace{\|\boldsymbol{u}_k - \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}}_{\xrightarrow{k\to\infty}0} \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \\
&\quad + \underbrace{\|b(\boldsymbol{u}_k, \boldsymbol{u}_k, \cdot) - b(\boldsymbol{u}, \boldsymbol{u}, \cdot)\|_{\boldsymbol{H}^{-1}(\Omega)}}_{\xrightarrow{k\to\infty}0} \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}.
\end{aligned}$$

Thus $\lim_{k\to\infty} \|\nabla(p_k - p)\|_{\boldsymbol{H}^{-1}(\Omega)} = 0$. Using now the pressure estimate, see for instance [27, Lemma II.1.5.4], we find

$$\|p_k - p\|_{L^2(\Omega)} \le c\|\nabla(p_k - p)\|_{\boldsymbol{H}^{-1}(\Omega)} \xrightarrow{k\to\infty} 0.$$

Therefore, we deduce that $(p_k)_{k\in\mathbb{N}}$ converges strongly in $L^2(\Omega)$ to $p$. □

We make the following assumption regarding $h$.

ASSUMPTION 4.1 Let $h : \Omega \times \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function, which fulfils
1. $h(\cdot, \boldsymbol{A}, s, \boldsymbol{w}) : \Omega \to \mathbb{R}$ is measurable for each $\boldsymbol{w} \in \mathbb{R}^d, s \in \mathbb{R}, \boldsymbol{A} \in \mathbb{R}^{d\times d}$, and
2. $h(x, \cdot, \cdot, \cdot) : \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is continuous for almost every $x \in \Omega$.

Moreover, there exist non-negative functions $a \in L^1(\Omega)$, $b_1, b_2, b_3 \in L^\infty(\Omega)$ such that for almost every $x \in \Omega$ it holds

$$|h(x, \boldsymbol{A}, s, \boldsymbol{w})| \le a(x) + b_1(x)|\boldsymbol{A}|^2 + b_2(x)|s|^2 + b_3(x)|\boldsymbol{w}|^2,$$

for all $\boldsymbol{w} \in \mathbb{R}^d, s \in \mathbb{R}, \boldsymbol{A} \in \mathbb{R}^{d\times d}$. Furthermore, the functional $\mathcal{H} : \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined as

$$\mathcal{H}(\boldsymbol{u}, p, \varphi) := \int_\Omega \mathcal{M}(\varphi) h(x, \nabla \boldsymbol{u}, p, \nabla \varphi)\, \mathrm{d}x,$$

satisfy the following properties
(i) $\mathcal{H}|_{\boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)\times L^2_0(\Omega)\times \Phi_{ad}}$ is bounded from below, and
(ii) for all $\varphi_n \rightharpoonup \varphi$ in $H^1(\Omega)$, $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$, $p_n \to p$ in $L^2(\Omega)$, it holds that

$$\mathcal{H}(\boldsymbol{u}, p, \varphi) \le \liminf_{n\to\infty} \mathcal{H}(\boldsymbol{u}_n, p_n, \varphi_n).$$

We then obtain the following existence result for (3.8)–(3.9).

**Theorem 4.6** *Under Assumptions 3.1, 3.2 and 4.1, there exists at least one minimizer of the optimal control problem* (3.8)–(3.9).

*Proof.* We may restrict ourselves to considering $\varphi \in \Phi_{ad}$ with $\varphi \in [s_a, s_b]$ a.e. in $\Omega$. In fact, we define as in [22, Proof of Proposition 1] for arbitrary $\varphi \in \Phi_{ad}$ the truncated functions $\tilde{\varphi} := \max\{s_a, \min\{\varphi, s_b\}\}$ and find $\mathcal{E}_\varepsilon(\tilde{\varphi}) \le \mathcal{E}_\varepsilon(\varphi)$, where $\mathcal{E}_\varepsilon$ is defined in (3.1). Moreover, by (3.3), we have $\alpha_\varepsilon(\varphi) = \alpha_\varepsilon(\tilde{\varphi})$ and hence also $\boldsymbol{S}_\varepsilon(\varphi) = \boldsymbol{S}_\varepsilon(\tilde{\varphi})$. Therefore we obtain

$$J^h_\varepsilon(\tilde{\varphi}, \boldsymbol{u}, p) \le J^h_\varepsilon(\varphi, \boldsymbol{u}, p) \text{ for all } (\boldsymbol{u}, p) \in \boldsymbol{S}_\varepsilon(\varphi) = \boldsymbol{S}_\varepsilon(\tilde{\varphi}).$$

By Assumption 4.1, $\mathcal{H}|_{\boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)\times L^2_0(\Omega)\times\Phi_{ad}}$ is bounded below by a constant $C_0$, and so $J^h_\varepsilon : \Phi_{ad} \times \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) \times L^2_0(\Omega)$ is bounded from below by a constant $C_1$. Thus, we can choose a minimizing sequence $(\varphi_n, \boldsymbol{u}_n, p_n)_{n\in\mathbb{N}} \subset \Phi_{ad} \times \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) \times L^2_0(\Omega)$ with $(\boldsymbol{u}_n, p_n) \in \boldsymbol{S}_\varepsilon(\varphi_n)$ for all $n$ and

$$\lim_{n\to\infty} J^h_\varepsilon(\varphi_n, \boldsymbol{u}_n, p_n) = \inf_{\varphi\in\Phi_{ad}, (\boldsymbol{u},p)\in\boldsymbol{S}_\varepsilon(\varphi)} J^h_\varepsilon(\varphi, \boldsymbol{u}, p) > -\infty.$$

In particular, from the non-negativity of $\psi$ and $\alpha_\varepsilon$, we see that for $\rho > 0$, there exists an $N$ such that $n > N$ implies

$$C_0 + \frac{\gamma\varepsilon}{4c_0}\|\nabla\varphi_n\|^2_{\boldsymbol{L}^2(\Omega)} \le J^h_\varepsilon(\varphi_n, \boldsymbol{u}_n, p_n) \le \inf_{\varphi\in\Phi_{ad}, (\boldsymbol{u},p)\in\boldsymbol{S}_\varepsilon(\varphi)} J^h_\varepsilon(\varphi, \boldsymbol{u}, p) + \rho.$$

Thus, $\{\nabla\varphi_n\}_{n\in\mathbb{N}}$ is bounded uniformly in $\boldsymbol{L}^2(\Omega)$. Moreover, without loss of generality, we may assume that $\varphi_n(x) \in [s_a, s_b]$ for a.e. $x \in \Omega$ and every $n \in \mathbb{N}$. And so, we deduce that $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded uniformly in $H^1(\Omega) \cap L^\infty(\Omega)$, and we may choose a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ that converges strongly in $L^2(\Omega)$ and pointwise almost everywhere in $\Omega$ to some limit element $\varphi \in \Phi_{ad}$.

Using Lemma 4.5 we can deduce that there is a subsequence of $(\boldsymbol{u}_{n_k}, p_{n_k})_{k\in\mathbb{N}}$, denoted by the same index, such that

$$\lim_{k\to\infty} \|\boldsymbol{u}_{n_k} - \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} = 0, \quad \lim_{k\to\infty} \|p_{n_k} - p\|_{L^2(\Omega)} = 0, \tag{4.14}$$

and $(\boldsymbol{u}, p) \in \boldsymbol{S}_\varepsilon(\varphi)$. From Lemma 4.4 we deduce additionally that

$$\lim_{k\to\infty} \int_\Omega \alpha_\varepsilon(\varphi_{n_k}) \left|\boldsymbol{u}_{n_k}\right|^2 \mathrm{dx} = \int_\Omega \alpha_\varepsilon(\varphi) \left|\boldsymbol{u}\right|^2 \mathrm{dx}. \tag{4.15}$$

As $\sup_{k\in\mathbb{N}} \|\psi(\varphi_{n_k})\|_{L^\infty(\Omega)} < \infty$ we can use Lebesgue's dominated convergence theorem to deduce $\lim_{k\to\infty} \int_\Omega \psi(\varphi_{n_k})\,\mathrm{dx} = \int_\Omega \psi(\varphi)\,\mathrm{dx}$. Finally, the weak lower semicontinuity of $H^1(\Omega) \ni \varphi \mapsto \int_\Omega |\nabla\varphi|^2\,\mathrm{dx}$ yields

$$\int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\,\mathrm{dx} \le \liminf_{k\to\infty} \int_\Omega \frac{\varepsilon}{2}\left|\nabla\varphi_{n_k}\right|^2 + \frac{1}{\varepsilon}\psi(\varphi_{n_k})\,\mathrm{dx}. \tag{4.16}$$

Together with the lower semicontinuity assumption on $\mathcal{H}$ from Assumption 4.1, we deduce that

$$J_\varepsilon^h(\varphi, \boldsymbol{u}, p) \le \liminf_{k\to\infty} J_\varepsilon^h(\varphi_{n_k}, \boldsymbol{u}_{n_k}, p_{n_k}) = \inf_{\varphi\in\Phi_{ad}, (\boldsymbol{u},p)\in\boldsymbol{S}_\varepsilon(\varphi)} J_\varepsilon^h(\varphi, \boldsymbol{u}, p),$$

and so $(\varphi, \boldsymbol{u}, p)$ is a minimizer of (3.8)–(3.9). □

By the same arguments, one can show an analogous existence result for the optimal control problem $\{(3.9), (3.11)\}$ involving the hydrodynamic force (1.3).

**Theorem 4.7** *Under Assumptions 3.1 and 3.2, there exists at least one minimizer of the optimization problem* $\{(3.9), (3.11)\}$ *involving the hydrodynamic force* (1.3).

*Proof.* We will prove the assertion for the choice $\mathcal{M}(\varphi) = \sqrt{\frac{\psi(\varphi)+\delta_\varepsilon}{2}}$, and the analogous assertion for the choice $\mathcal{M}(\varphi) = \frac{1}{2}$ follows along the same lines.

We first show that $\{J_\varepsilon(\varphi, \boldsymbol{u}, p) \mid \varphi \in \Phi_{ad}, (\boldsymbol{u}, p) \in \boldsymbol{S}_\varepsilon(\varphi)\}$ is bounded from below. We may restrict ourselves to considering $\varphi \in \Phi_{ad}$ with $\varphi \in [s_a, s_b]$ a.e. in $\Omega$ as in the proof of Theorem 4.6.

Now let $\varphi \in \Phi_{ad}$ be arbitrarily chosen with $\varphi \in [s_a, s_b]$ for a.e. $x \in \Omega$ and choose $(\boldsymbol{u}, p) \in \boldsymbol{S}_\varepsilon(\varphi)$. From (4.8), we find a constant $C_2 > 0$ independent of $\varphi$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|p\|_{L^2(\Omega)} < C_2.$$

By construction, we have

$$\varphi \in [s_a, s_b] \Longrightarrow \|\psi(\varphi)\|_{L^\infty(\Omega)} < C_3,$$

for some constant $C_3 > 0$ independent of $\varphi$. Then, using the Cauchy–Schwarz inequality, and

Young's inequality we have

$$\begin{aligned}\frac{1}{c_0}\int_\Omega \sqrt{\tfrac{\psi(\varphi)+\delta_\varepsilon}{2}}\nabla\varphi\cdot\left(\mu\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T\right)-p\mathrm{I}\right)\boldsymbol{a}\,\mathrm{dx}\\ \geq -\frac{1}{c_0\sqrt{2}}\|\nabla\varphi\sqrt{\psi(\varphi)+\delta_\varepsilon}\|_{L^2(\Omega)}\|\mu\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T\right)\boldsymbol{a}-p\boldsymbol{a}\|_{\boldsymbol{L}^2(\Omega)}\\ \geq -\frac{1}{c_0}\sqrt{\tfrac{C_3+\delta_\varepsilon}{2}}\|\nabla\varphi\|_{L^2(\Omega)}\left(2\mu C_2+C_2\right)\geq -\frac{\gamma\varepsilon}{8c_0}\|\nabla\varphi\|^2_{L^2(\Omega)}-C_4,\end{aligned}$$

where $C_4=\frac{(2\mu+1)^2C_2^2}{\gamma\varepsilon c_0}(C_3+\delta_\varepsilon)>0$ is independent of $\varphi$. The non-negativity of $\alpha_\varepsilon$ and $\psi$ yield that

$$\begin{aligned}J_\varepsilon(\varphi,\boldsymbol{u},p)&\geq\int_\Omega\frac{1}{c_0}\sqrt{\tfrac{\psi(\varphi)+\delta_\varepsilon}{2}}\nabla\varphi\cdot\left(\mu\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T\right)-p\mathrm{I}\right)\boldsymbol{a}+\frac{\gamma}{2c_0}\frac{\varepsilon}{2}|\nabla\varphi|^2\,\mathrm{dx}\\ &\geq -\frac{\gamma\varepsilon}{8c_0}\|\nabla\varphi\|^2_{L^2(\Omega)}-C_4+\frac{\gamma\varepsilon}{4c_0}\|\nabla\varphi\|^2_{L^2(\Omega)}=\frac{\gamma\varepsilon}{8c_0}\|\nabla\varphi\|^2_{L^2(\Omega)}-C_4\geq -C_4.\end{aligned}\tag{4.17}$$

This shows that $\{J_\varepsilon(\varphi,\boldsymbol{u},p)\mid\varphi\in\Phi_{ad},(\boldsymbol{u},p)\in\boldsymbol{S}_\varepsilon(\varphi)\}$ is bounded from below. Hence we may choose a minimizing sequence $(\varphi_n,\boldsymbol{u}_n,p_n)_{n\in\mathbb{N}}\subset\Phi_{ad}\times\boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)\times L^2_0(\Omega)$ with

$$\lim_{n\to\infty}J_\varepsilon(\varphi_n,\boldsymbol{u}_n,p_n)=\inf_{\varphi\in\Phi_{ad},(\boldsymbol{u},p)\in\boldsymbol{S}_\varepsilon(\varphi)}J_\varepsilon(\varphi,\boldsymbol{u},p)>-\infty.$$

As before, we deduce that $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded uniformly in $H^1(\Omega)\cap L^\infty(\Omega)$, together with Lemma 4.5, we have subsequences $(\varphi_{n_k},\boldsymbol{u}_{n_k},p_{n_k})_{k\in\mathbb{N}}$ that satisfy

$$\lim_{k\to\infty}\|\varphi_{n_k}-\varphi\|_{L^2(\Omega)}=0,\quad\lim_{k\to\infty}\|\boldsymbol{u}_{n_k}-\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}=0,\quad\lim_{k\to\infty}\|p_{n_k}-p\|_{L^2(\Omega)}=0,$$

and $(\boldsymbol{u},p)\in\boldsymbol{S}_\varepsilon(\varphi)$. To deduce that $(\varphi,\boldsymbol{u},p)$ is a minimizer of $\{(3.9),(3.11)\}$, we only need to show that

$$\begin{aligned}\liminf_{k\to\infty}\int_\Omega\sqrt{\psi(\varphi_{n_k})+\delta_\varepsilon}\nabla\varphi_{n_k}\cdot\left(\mu\left(\nabla\boldsymbol{u}_{n_k}+(\nabla\boldsymbol{u}_{n_k})^T\right)-p_{n_k}\mathrm{I}\right)\boldsymbol{a}\,\mathrm{dx}\\ \geq\int_\Omega\sqrt{\psi(\varphi)+\delta_\varepsilon}\nabla\varphi\cdot\left(\mu\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T\right)-p\mathrm{I}\right)\boldsymbol{a}\,\mathrm{dx},\end{aligned}\tag{4.18}$$

as the other integrals in (3.11) are shown to be weakly lower semicontinuous in the proof of Theorem 4.6. We apply now an idea of [22] and define

$$\phi(t):=\int_{s_a}^t\sqrt{\psi(s)+\delta_\varepsilon}\,\mathrm{d}s,\quad w_{n_k}(x):=\phi\left(\varphi_{n_k}(x)\right).$$

Then we see that

$$\mathrm{D}w_{n_k}(x)=\phi'\left(\varphi_{n_k}(x)\right)\mathrm{D}\varphi_{n_k}(x)=\left(\sqrt{\psi\left(\varphi_{n_k}(x)\right)+\delta_\varepsilon}\right)\mathrm{D}\varphi_{n_k}(x).$$

By the uniform boundedness of $(\varphi_{n_k})_{k\in\mathbb{N}}$ in $H^1(\Omega)\cap L^\infty(\Omega)$, we find that $(\psi(\varphi_{n_k}))_{k\in\mathbb{N}}$ is uniformly

bounded in $L^\infty(\Omega)$, and so by the Cauchy–Schwarz inequality,

$$\begin{aligned}\|w_{n_k}\|^2_{L^2(\Omega)} &\le \int_\Omega (\varphi_{n_k} - s_a)\left(\int_{s_a}^{\varphi_{n_k}} (\psi(s)+\delta_\varepsilon)\,\mathrm{d}s\right)\mathrm{d}x\\ &\le \sup_{s\in[s_a,s_b]} (\psi(s)+\delta_\varepsilon)\int_\Omega \left|\varphi_{n_k}-s_a\right|^2\,\mathrm{d}x,\\ \|\mathrm{D}w_{n_k}\|^2_{\boldsymbol{L}^2(\Omega)} &\le \sup_{k\in\mathbb{N}}(\psi(\varphi_{n_k})+\delta_\varepsilon)\|\mathrm{D}\varphi_{n_k}\|^2_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Thus, we deduce that $(w_{n_k})_{k\in\mathbb{N}}$ is bounded uniformly in $H^1(\Omega)$, and hence there is a subsequence, denoted by the same index, that converges weakly in $H^1(\Omega)$ and pointwise almost everywhere in $\Omega$ to some limit element $w\in H^1(\Omega)$. Since $\phi$ is continuous and $\lim_{k\to\infty}\varphi_{n_k}(x)=\varphi(x)$ for almost every $x\in\Omega$, we know that $w=\phi(\varphi)$. In particular, the weak convergence of $\mathrm{D}w_{n_k}$ to $\mathrm{D}w$ implies that

$$\sqrt{\psi(\varphi_{n_k})+\delta_\varepsilon}\nabla\varphi_{n_k} \rightharpoonup \sqrt{\psi(\varphi)+\delta_\varepsilon}\nabla\varphi \quad \text{in } \boldsymbol{L}^2(\Omega). \tag{4.19}$$

Combining (4.14) and (4.19) we obtain from the product of weak-strong convergence:

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega \sqrt{\psi(\varphi_{n_k})+\delta_\varepsilon}\nabla\varphi_{n_k}\cdot\left(\mu\left(\nabla\boldsymbol{u}_{n_k}+(\nabla\boldsymbol{u}_{n_k})^T\right)-p_{n_k}\mathrm{I}\right)\boldsymbol{a}\,\mathrm{d}x&\\ = \int_\Omega \sqrt{\psi(\varphi)+\delta_\varepsilon}\nabla\varphi\cdot\left(\mu\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T\right)-p\mathrm{I}\right)\boldsymbol{a}\,\mathrm{d}x.&\end{aligned} \tag{4.20}$$

Using (4.20), (4.15) and (4.16), we deduce that

$$J_\varepsilon(\varphi,\boldsymbol{u},p)\le\liminf_{k\to\infty}J_\varepsilon(\varphi_{n_k},\boldsymbol{u}_{n_k},p_{n_k})=\inf_{\varphi\in\Phi_{ad},(\boldsymbol{u},p)\in\boldsymbol{S}_\varepsilon(\varphi)}J_\varepsilon(\varphi,\boldsymbol{u},p),$$

and so $(\varphi,\boldsymbol{u},p)$ is a minimizer of $\{(3.9),(3.11)\}$. □

REMARK 4.2 Note that, for the choice $\mathcal{M}(\varphi)=\frac{1}{2}$, the proof of Theorem 4.7 is completed once we showed that $J_\varepsilon$ is bounded from below, which can be shown similarly as in (4.17), and (ii) in Assumption 4.2 has been verified. This follows the product of weak-strong convergence:

$$\lim_{k\to\infty}\int_\Omega \nabla\varphi_{n_k}\cdot\left(\mu\left(\nabla\boldsymbol{u}_{n_k}+(\nabla\boldsymbol{u}_{n_k})^T\right)-p_{n_k}\mathrm{I}\right)\boldsymbol{a}\,\mathrm{d}x=\int_\Omega\nabla\varphi\cdot\left(\mu\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T\right)-p\mathrm{I}\right)\boldsymbol{a}\,\mathrm{d}x. \tag{4.21}$$

## 4.2 *Optimality conditions*

This section is devoted to the derivation of a first order necessary optimality system for the optimal control problem (3.8)–(3.9). For this purpose, we first show Fréchet differentiability of the solution operator. We will only be able to show differentiability at certain points where the solution to the state equations is unique. Otherwise we cannot apply the implicit function theorem in order to deduce the statement. In the following we will use Remark 4.1 and the condition $\|\nabla\boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}<\frac{\mu}{K_\Omega}$ to ensure the uniqueness of solutions to the state equations for a given $\varphi_\varepsilon\in H^1(\Omega)\cap L^\infty(\Omega)$. From (4.10), this is achievable for large viscosities $\mu$ or small data $\boldsymbol{f}$ and $\boldsymbol{g}$. To be precise, we obtain the following result.

**Lemma 4.8** *Under Assumption 3.2, let $\varphi_\varepsilon \in H^1(\Omega)\cap L^\infty(\Omega)$ be given such that there is $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$ with $\|\nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{K_\Omega}$. Then there is a neighborhood $N$ of $\varphi_\varepsilon$ in $H^1(\Omega)\cap L^\infty(\Omega)$ such that for every $\varphi \in N$ the solution operator consists of exactly one pair, and hence we may write $\boldsymbol{S}_\varepsilon : N \subset H^1(\Omega)\cap L^\infty(\Omega) \to \boldsymbol{H}^1(\Omega)\times L^2(\Omega)$. This mapping is then differentiable at $\varphi_\varepsilon$ with $\mathrm{D}\boldsymbol{S}_\varepsilon(\varphi_\varepsilon)(\varphi) =: (\boldsymbol{u}, p) \in \boldsymbol{H}^1_0(\Omega)\times L^2_0(\Omega)$ being the unique solution of the linearized state system*

$$\alpha'_\varepsilon(\varphi_\varepsilon)\varphi \boldsymbol{u}_\varepsilon + \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u} - \mu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_\varepsilon + (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{0} \text{ in } \Omega, \tag{4.22a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega, \tag{4.22b}$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega. \tag{4.22c}$$

*Proof.* As already mentioned, we want to apply the implicit function theorem to prove the statements of the lemma. For this purpose, we first note that, by [13, Lemma IX.4.2], there exists a $\boldsymbol{G} \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$, i.e., $\boldsymbol{G}$ satisfies

$$\operatorname{div}\boldsymbol{G} = 0 \text{ in } \Omega, \quad \boldsymbol{G}|_{\partial\Omega} = \boldsymbol{g}.$$

We define

$$F : \big(H^1(\Omega)\cap L^\infty(\Omega)\big)\times \boldsymbol{H}^1_0(\Omega)\times L^2_0(\Omega) \to \boldsymbol{H}^{-1}(\Omega)\times L^2_0(\Omega), \quad F = (F_1, F_2),$$

by

$$\begin{aligned} F_1(\varphi,\boldsymbol{u},p)\boldsymbol{v} &:= \int_\Omega \alpha_\varepsilon(\varphi)\boldsymbol{u}\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v} - p\operatorname{div}\boldsymbol{v} - \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{dx} \\ &\quad + \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{G}\cdot\boldsymbol{v} + (\boldsymbol{G}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v} + \alpha_\varepsilon(\varphi)\boldsymbol{G}\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{G}\cdot\nabla\boldsymbol{v} + (\boldsymbol{G}\cdot\nabla)\boldsymbol{G}\cdot\boldsymbol{v}\,\mathrm{dx}, \\ F_2(\varphi,\boldsymbol{u},p) &:= \operatorname{div}\boldsymbol{u}, \end{aligned}$$

for all $\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)$. Hence, $F(\varphi, \boldsymbol{u}-\boldsymbol{G}, p) = 0$ if and only if $(\boldsymbol{u}, p) \in \boldsymbol{S}_\varepsilon(\varphi)$. Thus in particular we have $F(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{G}, p_\varepsilon) = 0$. Besides, we directly see that the Fréchet differential $\mathrm{D}_{(\boldsymbol{u},p)}F$ exists and is given at $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{G}, p_\varepsilon)$ as

$$\begin{aligned} \mathrm{D}_{(\boldsymbol{u},p)}F_1(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{G}, p_\varepsilon)(\boldsymbol{u},p)\boldsymbol{v} &= \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v}\,\mathrm{dx} \\ &\quad + \int_\Omega (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v} - p\operatorname{div}\boldsymbol{v}\,\mathrm{dx} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega), \\ \mathrm{D}_{(\boldsymbol{u},p)}F_2(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{G}, p_\varepsilon)(\boldsymbol{u},p) &= \operatorname{div}\boldsymbol{u}. \end{aligned}$$

The assumption $\|\nabla\boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{K_\Omega}$, equations (4.1) and (4.3) ensure that, after taking $\operatorname{div}\boldsymbol{v} = 0$ into account,

$$\boldsymbol{H}^1_{0,\sigma}(\Omega)\times\boldsymbol{H}^1_{0,\sigma}(\Omega) \ni (\boldsymbol{u},\boldsymbol{v}) \mapsto \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v} + (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{dx}$$

defines a coercive, continuous bilinear form. Hence, we may use the Lax–Milgram theorem and standard results for the solvability of the divergence operator, see for instance [27, Lemma II.2.1.1], in order to obtain that $\mathrm{D}_{(\boldsymbol{u},p)}F(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{G}, p_\varepsilon)$ is an isomorphism.

Next, we want to consider the differentiability of $F$ with respect to its first argument. For this purpose, we have to consider $\alpha_\varepsilon : L^6(\Omega) \to L^{\frac{3}{2}}(\Omega)$ as a Nemytskii operator, making in particular

use of the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$. The results in [29, Section 4.3.3] ensure that $\alpha_\varepsilon : L^6(\Omega) \to L^{\frac{3}{2}}(\Omega)$ defines a Fréchet-differentiable Nemytskii operator, which follows from the assumption $\alpha_\varepsilon \in L^\infty(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$. We can then conclude directly that $F$ is Fréchet differentiable with respect to its first argument with

$$\mathrm{D}_\varphi F_1(\varphi, \boldsymbol{u} - \boldsymbol{G}, p)(\tilde{\varphi})\boldsymbol{v} = \int_\Omega \alpha_\varepsilon'(\varphi)\tilde{\varphi}\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{dx}, \quad \mathrm{D}_\varphi F_2(\varphi, \boldsymbol{u} - \boldsymbol{G}, p) = 0.$$

Additionally, we need that $F$ is Fréchet differentiable in a neighborhood of $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon)$. To show this, we will use [31, Proposition 4.14], i.e., we show that the partial derivatives are continuous in order to conclude that $F$ is Fréchet differentiable. Thus, let $(\varphi_k, \boldsymbol{u}_k, p_k)_{k\in\mathbb{N}} \subset (H^1(\Omega) \cap L^\infty(\Omega)) \times \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ be sequences with

$$\lim_{k\to\infty} \|\boldsymbol{u}_k - \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} = 0, \quad \lim_{k\to\infty} \|p_k - p\|_{L^2(\Omega)} = 0, \quad \lim_{k\to\infty} \|\varphi_k - \varphi\|_{H^1(\Omega)\cap L^\infty(\Omega)} = 0.$$

As $\alpha_\varepsilon : L^6(\Omega) \to L^{\frac{3}{2}}(\Omega)$ defines a continuous Nemytskii-operator, making additionally use of the continuity properties of the trilinear form as stated in Lemma 4.2, we can deduce that

$$\lim_{k\to\infty} \|\mathrm{D}_{(\boldsymbol{u},p)} F(\varphi_k, \boldsymbol{u}_k, p_k) - \mathrm{D}_{(\boldsymbol{u},p)} F(\varphi, \boldsymbol{u}, p)\|_{\mathcal{L}(\boldsymbol{H}_0^1(\Omega)\times L_0^2(\Omega), \boldsymbol{H}^{-1}(\Omega)\times L_0^2(\Omega))} = 0.$$

Moreover, from $\alpha_\varepsilon' \in C^{0,1}$ and standard results for Nemytskii operators we find that $L^6(\Omega) \ni \varphi \mapsto \alpha_\varepsilon'(\varphi) \in L^6(\Omega)$ is continuous. And thus we also find by direct calculations that $\lim_{k\to\infty} \|\mathrm{D}_\varphi F(\varphi_k, \boldsymbol{u}_k, p_k) - \mathrm{D}_\varphi F(\varphi, \boldsymbol{u}, p)\|_{\mathcal{L}(H^1(\Omega), \boldsymbol{H}^{-1}(\Omega)\times L_0^2(\Omega))} = 0$. Therefore, we obtain that $F$ is Fréchet differentiable.

Finally, applying the implicit function theorem, we obtain for $\|\varphi - \varphi_\varepsilon\|_{H^1(\Omega)\cap L^\infty(\Omega)} \ll 1$ the existence and uniqueness of a pair $(\boldsymbol{u}, p)$ such that $F(\varphi, \boldsymbol{u} - \boldsymbol{G}, p) = 0$, i.e., $(\boldsymbol{u}, p) \in \boldsymbol{S}_\varepsilon(\varphi)$. This implies the first part of the statement. The second part of the lemma is a consequence of the differentiability statement of the implicit function theorem:

$$\mathrm{D}\boldsymbol{S}_\varepsilon(\varphi_\varepsilon) = -\left(\mathrm{D}_{(\boldsymbol{u},p)} F(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{G}, p_\varepsilon)\right)^{-1} \circ \mathrm{D}_\varphi F(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{G}, p_\varepsilon),$$

which reads in our setting as $\operatorname{div}\boldsymbol{u} = 0$ and

$$\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\varphi\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v} + \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{dx} + \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v} + (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v} - p\operatorname{div}\boldsymbol{v}\,\mathrm{dx} = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{4.23}$$

□

We denote by $\mathrm{D}_i h(x, \boldsymbol{A}, s, \boldsymbol{w})$ for $i \in \{1,2,3,4\}$ as the differential of

$$\Omega \times \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R}^d \ni (x, \boldsymbol{A}, s, \boldsymbol{w}) \mapsto h(x, \boldsymbol{A}, s, \boldsymbol{w})$$

with respect to the $i$-th variable, respectively.

ASSUMPTION 4.2 In addition to Assumption 4.1, assume further that $x \mapsto h(x, \boldsymbol{A}, s, \boldsymbol{w})$ is in $W^{1,1}(\Omega)$ for all $(\boldsymbol{A}, s, \boldsymbol{w}) \in \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R}^d$ and the partial derivatives

$$\mathrm{D}_2 h(x, \cdot, s, \boldsymbol{w}),\ \mathrm{D}_3 h(x, \boldsymbol{A}, \cdot, \boldsymbol{w}),\ \mathrm{D}_4 h(x, \boldsymbol{A}, s, \cdot)$$

exist for all $\boldsymbol{w} \in \mathbb{R}^d$, $s \in \mathbb{R}$, $\boldsymbol{A} \in \mathbb{R}^{d\times d}$, and almost all $x \in \Omega$. Moreover, we assume that

$$|\mathrm{D}_i h(x,\boldsymbol{A},s,\boldsymbol{w})| \le \tilde{a}(x) + \tilde{b}_1(x)\,|\boldsymbol{A}| + \tilde{b}_2(x)\,|s| + \tilde{b}_3(x)\,|\boldsymbol{w}|, \text{ for } i \in \{2,3,4\}, \tag{4.24}$$

for some non-negative $\tilde{a} \in L^2(\Omega)$, $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3 \in L^\infty(\Omega)$.

From Assumption 4.2 we see that

$$\begin{aligned}\left(L^2(\Omega)\right)^{d\times d} \ni \boldsymbol{A} &\mapsto \mathrm{D}_2 h(\cdot,\boldsymbol{A},s,\boldsymbol{w}) \in L^2(\Omega),\\ L^2(\Omega) \ni s &\mapsto \mathrm{D}_3 h(\cdot,\boldsymbol{A},s,\boldsymbol{w}) \in L^2(\Omega),\\ \left(L^2(\Omega)\right)^{d} \ni \boldsymbol{w} &\mapsto \mathrm{D}_4 h(\cdot,\boldsymbol{A},s,\boldsymbol{w}) \in L^2(\Omega),\end{aligned}$$

are well-defined Nemytskii operators for $\boldsymbol{A} \in (L^2(\Omega))^{d\times d}$, $s \in L^2(\Omega)$, and $\boldsymbol{w} \in (L^2(\Omega))^d$ if and only if (4.24) is fulfilled. Moreover, the operator

$$\left(L^2(\Omega)\right)^{d\times d} \times L^2(\Omega) \times \left(L^2(\Omega)\right)^{d} \ni (\boldsymbol{A},s,\boldsymbol{w}) \mapsto h(\cdot,\boldsymbol{A},s,\boldsymbol{w}) \in L^1(\Omega)$$

is continuously Fréchet differentiable.

Next, by Assumption 3.1, $\psi \in C^{1,1}(\mathbb{R})$, we have that $\mathrm{D}_y(\sqrt{\psi(y)+\delta_\varepsilon})$ is locally Lipschitz and thus the Nemytskii operator

$$L^\infty(\Omega) \ni \varphi \mapsto \sqrt{\psi(\varphi)+\delta_\varepsilon} \in L^\infty(\Omega)$$

is continuously Fréchet differentiable. Hence, we find that

$$\mathcal{H} : \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \left(H^1(\Omega) \cap L^\infty(\Omega)\right) \ni (\boldsymbol{u},p,\varphi) \mapsto \int_\Omega \mathcal{M}(\varphi) h(x,\nabla\boldsymbol{u},p,\nabla\varphi)\,\mathrm{d}x$$

is continuously Fréchet differentiable and its distributional derivative is given as

$$\begin{aligned}\mathrm{D}\mathcal{H}(\boldsymbol{u},p,\varphi)(\boldsymbol{v},s,\eta) = \int_\Omega \mathcal{M}(\varphi)(\mathrm{D}_2 h, \mathrm{D}_3 h, \mathrm{D}_4 h)\,|_{(x,\nabla\boldsymbol{u},p,\nabla\varphi)} \cdot (\nabla\boldsymbol{v}, s, \nabla\eta)\,\mathrm{d}x\\ + \int_\Omega h(x,\nabla\boldsymbol{u},p,\nabla\varphi)\mathcal{M}'(\varphi)\eta\,\mathrm{d}x.\end{aligned} \tag{4.25}$$

We note that for the choice $\mathcal{M}(\varphi) = \frac{1}{2}$, the second integral on the right hand side of (4.25) vanishes as the Fréchet derivative of $\frac{1}{2}$ is the zero functional. On the other hand, for the choice $\mathcal{M}(\varphi) = \frac{1}{c_0}\sqrt{\frac{\psi(\varphi)+\delta_\varepsilon}{2}}$, the Fréchet derivative is given as

$$\mathcal{M}'(\varphi) = \frac{1}{c_0}\frac{\psi'(\varphi)}{2\sqrt{2(\psi(\varphi)+\delta_\varepsilon)}}. \tag{4.26}$$

Before formulating the optimality system we want to discuss the adjoint system. The pair of adjoint variables $(\boldsymbol{q}_\varepsilon, \pi_\varepsilon) \in \boldsymbol{H}^1_0(\Omega) \times L^2(\Omega)$ is the weak solution of the adjoint system, which is given as follows: find $(\boldsymbol{q}_\varepsilon, \pi_\varepsilon) \in \boldsymbol{H}^1_0(\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned}\alpha_\varepsilon(\varphi_\varepsilon)(\boldsymbol{q}_\varepsilon - \boldsymbol{u}_\varepsilon) - \mu\,\mathrm{div}\,(\nabla\boldsymbol{q}_\varepsilon + (\nabla\boldsymbol{q}_\varepsilon)^T) + (\nabla\boldsymbol{u}_\varepsilon)^T\boldsymbol{q}_\varepsilon - (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{q}_\varepsilon + \nabla\pi_\varepsilon&\\ = -\mathrm{div}\,(\mathcal{M}(\varphi_\varepsilon)\mathrm{D}_2 h) &\quad \text{in } \Omega, &\quad (4.27a)\\ \mathrm{div}\,\boldsymbol{q}_\varepsilon = -\mathcal{M}(\varphi_\varepsilon)\mathrm{D}_3 h + \vartheta_\varepsilon &\quad \text{in } \Omega, &\quad (4.27b)\\ \boldsymbol{q}_\varepsilon = \boldsymbol{0} &\quad \text{on } \partial\Omega, &\quad (4.27c)\end{aligned}$$

where $\mathrm{D}_2 h, \mathrm{D}_3 h$ are evaluated at $(x, \nabla \boldsymbol{u}_\varepsilon, p_\varepsilon, \nabla \varphi_\varepsilon)$ and

$$\vartheta_\varepsilon := \fint_\Omega \mathcal{M}(\varphi_\varepsilon) \mathrm{D}_3 h(x, \nabla \boldsymbol{u}_\varepsilon, p_\varepsilon, \nabla \varphi_\varepsilon) \, \mathrm{d}x. \tag{4.28}$$

REMARK 4.3 The parameter $\vartheta_\varepsilon \in \mathbb{R}$ can be interpreted as a Lagrange multiplier for the constraint $\int_\Omega p \, \mathrm{d}x = 0$. By carrying out the formal Lagrange method as described for instance in [19, 29] and appending the mean value condition on the pressure $p$ with some Lagrange multiplier $\vartheta_\varepsilon$ to the Lagrangian, one obtains that $\vartheta_\varepsilon$ appears in the adjoint system as in (4.27).

The next lemma shows that the system (4.27) is uniquely solvable.

**Lemma 4.9** *Let Assumptions 3.1, 3.2, and 4.2 hold, and let $\varphi_\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega)$ and $\boldsymbol{u}_\varepsilon \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$ such that $\|\nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{K_\Omega}$ be given. Then there exists a unique solution pair $(\boldsymbol{q}_\varepsilon, \pi_\varepsilon) \in \boldsymbol{H}^1_0(\Omega) \times L^2(\Omega)$ of the adjoint system (4.27).*

*Proof.* First, we notice that by definition of $\vartheta_\varepsilon$ (4.28), it holds that

$$\int_\Omega \mathcal{M}(\varphi_\varepsilon) \mathrm{D}_3 h(x, \nabla \boldsymbol{u}_\varepsilon, p_\varepsilon, \nabla \varphi_\varepsilon) - \vartheta_\varepsilon \, \mathrm{d}x = 0.$$

As $\varphi_\varepsilon \in L^\infty(\Omega)$, we have $\mathcal{M}(\varphi_\varepsilon) \in L^\infty(\Omega)$ for either choices. Thus, by Assumption 4.2, we obtain that $\mathcal{M}(\varphi_\varepsilon)\mathrm{D}_3 h \in L^2(\Omega)$. So, from standard results, see for instance [27, Lemma II.2.1.1], we deduce the existence of some $\boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega)$ such that

$$\operatorname{div} \boldsymbol{w} = -\mathcal{M}(\varphi_\varepsilon)\mathrm{D}_3 h + \vartheta_\varepsilon.$$

Note that, by the density of $\boldsymbol{C}^\infty_{0,\sigma}(\Omega) := \{\boldsymbol{v} \in (C^\infty_0(\Omega))^d \,|\, \operatorname{div} \boldsymbol{v} = 0\}$ in $\boldsymbol{H}^1_{0,\sigma}(\Omega)$ (see [27, Lemma II.2.2.3]), for any $\boldsymbol{v} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$, there exists a sequence $\{\boldsymbol{v}^n\}_{n\in\mathbb{N}} \subset \boldsymbol{C}^\infty_{0,\sigma}(\Omega)$ such that

$$\|\boldsymbol{v}^n - \boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \to 0 \text{ as } n \to \infty.$$

Thus, for any $\boldsymbol{y} \in \boldsymbol{H}^1_0(\Omega), \boldsymbol{v} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$, we find that by the commutativity of second derivatives,

$$\begin{aligned}
\int_\Omega \nabla \boldsymbol{y} \cdot (\nabla \boldsymbol{v})^T \, \mathrm{d}x &= \lim_{n\to\infty} \int_\Omega \nabla \boldsymbol{y} \cdot (\nabla \boldsymbol{v}^n)^T \, \mathrm{d}x \\
&= \lim_{n\to\infty} \sum_{i,j=1}^d \int_\Omega \partial_i y_j \partial_j v_i^n \, \mathrm{d}x \\
&= \lim_{n\to\infty} \sum_{i,j=1}^d \left( \int_{\partial\Omega} y_j \partial_j v_i^n \nu_{\partial\Omega,i} \, \mathrm{d}\mathcal{H}^{d-1} - \int_\Omega y_j \partial_j \partial_i v_i^n \, \mathrm{d}x \right) \\
&= \lim_{n\to\infty} \int_{\partial\Omega} (\boldsymbol{y} \cdot \nabla) \boldsymbol{v}^n \cdot \boldsymbol{\nu}_{\partial\Omega} \, \mathrm{d}\mathcal{H}^{d-1} - \int_\Omega \boldsymbol{y} \cdot \nabla (\operatorname{div} \boldsymbol{v}^n) \, \mathrm{d}x = 0.
\end{aligned} \tag{4.29}$$

We define the bilinear form $a : \boldsymbol{H}^1_{0,\sigma}(\Omega) \times \boldsymbol{H}^1_{0,\sigma}(\Omega) \to \left(\boldsymbol{H}^1_{0,\sigma}(\Omega)\right)'$ by

$$\begin{aligned}
a(\boldsymbol{u}, \boldsymbol{v}) &:= \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) \boldsymbol{u} \cdot \boldsymbol{v} + \mu \nabla \boldsymbol{u} \cdot (\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T) + (\nabla \boldsymbol{u}_\varepsilon)^T \boldsymbol{u} \cdot \boldsymbol{v} - (\boldsymbol{u}_\varepsilon \cdot \nabla) \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}x \\
&= \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) \boldsymbol{u} \cdot \boldsymbol{v} + \mu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} + (\nabla \boldsymbol{u}_\varepsilon)^T \boldsymbol{u} \cdot \boldsymbol{v} - (\boldsymbol{u}_\varepsilon \cdot \nabla) \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}x,
\end{aligned} \tag{4.30}$$

where we have used (4.29) for $\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$. Making use of $\|\nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{K_\Omega}$, (4.1), (4.3), and the Poincaré inequality, we can establish that $a(\cdot,\cdot)$ is a coercive bilinear form, i.e., there exists a constant $c(\mu,|\Omega|) > 0$ such that,

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{u}) &= \int_\Omega \underbrace{\alpha_\varepsilon(\varphi_\varepsilon)}_{\geq 0} |\boldsymbol{u}|^2 + \mu\,|\nabla \boldsymbol{u}|^2 \,\mathrm{dx} + b(\boldsymbol{u},\boldsymbol{u}_\varepsilon,\boldsymbol{u}) - \underbrace{b(\boldsymbol{u}_\varepsilon,\boldsymbol{u},\boldsymbol{u})}_{=0 \text{ by } (4.3)} \\ &\geq \mu \|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} - K_\Omega \|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \|\nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} \geq c(\mu,|\Omega|) \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1_0(\Omega)}. \end{aligned}$$

Meanwhile, the boundedness of the bilinear form $a(\cdot,\cdot)$ in $\boldsymbol{H}^1_{0,\sigma}(\Omega) \times \boldsymbol{H}^1_{0,\sigma}(\Omega)$ can be shown using (4.1), the boundedness of $\alpha_\varepsilon$, Hölder's inequality and the assumption $\|\nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{K_\Omega}$. Thus, by the Lax–Milgram theorem, we obtain a unique $\hat{\boldsymbol{q}} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$ such that

$$a(\hat{\boldsymbol{q}},\boldsymbol{v}) = \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} + \mathcal{M}(\varphi_\varepsilon)(\mathrm{D}_2 h \cdot \nabla \boldsymbol{v})\,\mathrm{dx} - a(\boldsymbol{w},\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_{0,\sigma}(\Omega). \tag{4.31}$$

We note that the integral terms are well-defined due to Assumption 4.2 and the boundedness of $\alpha_\varepsilon$. We set $\boldsymbol{q}_\varepsilon := \hat{\boldsymbol{q}} + \boldsymbol{w}$. The existence of $\pi_\varepsilon \in L^2(\Omega)$ follows from standard results, see for instance [27, Lemma II.2.2.1]. Thus, $(\boldsymbol{q}_\varepsilon,\pi_\varepsilon)$ is the unique weak solution of the adjoint system (4.27). □

Now we can formulate necessary optimality conditions for our optimal control problem.

**Theorem 4.10** *Let $(\varphi_\varepsilon,\boldsymbol{u}_\varepsilon,p_\varepsilon) \in (\Phi_{ad} \cap L^\infty(\Omega)) \times \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) \times L^2_0(\Omega)$ be a minimizer of $J^h_\varepsilon$ such that $\|\nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{K_\Omega}$. Then, there exists a Lagrange multiplier $\lambda_\varepsilon \in \mathbb{R}$ for the integral constraint such that*

$$\begin{aligned} &\left(\alpha'_\varepsilon(\varphi_\varepsilon)\left(\frac{1}{2}|\boldsymbol{u}_\varepsilon|^2 - \boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon\right) + \frac{\gamma}{2c_0\varepsilon}\psi'(\varphi_\varepsilon) + \lambda_\varepsilon + \mathcal{M}'(\varphi_\varepsilon)h(x,\nabla \boldsymbol{u}_\varepsilon,p_\varepsilon,\nabla\varphi_\varepsilon),\zeta\right)_{L^2(\Omega)} \\ &\qquad + \left(\mathcal{M}(\varphi_\varepsilon)\mathrm{D}_4 h(x,\nabla \boldsymbol{u}_\varepsilon,p_\varepsilon,\nabla\varphi_\varepsilon) + \frac{\gamma\varepsilon}{2c_0}\nabla\varphi_\varepsilon,\nabla\zeta\right)_{\boldsymbol{L}^2(\Omega)} \\ &\qquad\qquad = 0 \quad \forall \zeta \in H^1(\Omega) \cap L^\infty(\Omega). \end{aligned} \tag{4.32}$$

*Here, $(\boldsymbol{q}_\varepsilon,\pi_\varepsilon) \in \boldsymbol{H}^1_0(\Omega) \times L^2(\Omega)$ is the unique weak solution of the adjoint system (4.27).*

*Proof.* We rewrite the problem (3.8)–(3.9) as a minimizing problem for a reduced objective functional defined on an open set in $H^1(\Omega) \cap L^\infty(\Omega)$ by making use of Lemma 4.8. In particular, at least in a neighborhood $N \subset H^1(\Omega) \cap L^\infty(\Omega)$ of $\varphi_\varepsilon$, the solution operator $\boldsymbol{S}_\varepsilon$ is not set-valued, but for every $\varphi \in N$ we have $\boldsymbol{S}_\varepsilon(\varphi) = \{(\boldsymbol{u},p)\}$. Thus we may define the reduced functional $j^h_\varepsilon : N \to \mathbb{R}$ by

$$j^h_\varepsilon(\varphi) := J^h_\varepsilon\big(\varphi, \boldsymbol{S}_\varepsilon(\varphi)\big).$$

Then, $\varphi_\varepsilon$ is also a local minimizer of $j^h_\varepsilon$. Hence, the gradient equation

$$\mathrm{D} j^h_\varepsilon(\varphi_\varepsilon)(\zeta) = 0 \quad \forall \zeta \in H^1(\Omega) \cap L^\infty(\Omega),\ \int_\Omega \zeta\,\mathrm{dx} = 0, \tag{4.33}$$

would be fulfilled if $j^h_\varepsilon$ would be differentiable.

We will show in the next step that $j_\varepsilon^h$ is differentiable at $\varphi_\varepsilon$ as a mapping from $H^1(\Omega)\cap L^\infty(\Omega)$ to $\mathbb{R}$. Lemma 4.8 already ensures that the solution operator $\boldsymbol{S}_\varepsilon$ is differentiable from $H^1(\Omega)\cap L^\infty(\Omega)$ to $\boldsymbol{H}^1(\Omega)\times L^2(\Omega)$. Thus we now look at dependence of $J_\varepsilon^h$ on the first variable. For this purpose we find first as in the proof of Lemma 4.8 that $\alpha_\varepsilon : L^6(\Omega)\to L^{\frac{3}{2}}(\Omega)$ is a Fréchet differentiable Nemytskii operator, and hence

$$H^1(\Omega)\ni\varphi\mapsto\int_\Omega \alpha_\varepsilon(\varphi)\,|\boldsymbol{u}|^2\,\mathrm{dx}$$

is Fréchet differentiable for any $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)$. With similar results, i.e., by making use of [29, Section 4.3.3], we also find that

$$\begin{aligned} &L^\infty(\Omega)\ni\varphi\mapsto\psi(\varphi)\in L^\infty(\Omega), \quad && L^\infty(\Omega)\ni\varphi\mapsto\int_\Omega\psi(\varphi)\,\mathrm{dx},\\ &H^1(\Omega)\ni\varphi\mapsto\nabla\varphi\in\boldsymbol{L}^2(\Omega), && H^1(\Omega)\ni\varphi\mapsto\int_\Omega|\nabla\varphi|^2\,\mathrm{dx} \end{aligned}$$

are differentiable. Combining these results and the Fréchet differentiability of $\mathcal{H}$, we find that $j_\varepsilon^h : N\to\mathbb{R}$ is differentiable. Hence we may conclude by the minimizing property of $\varphi_\varepsilon$ that the gradient equation (4.33) is fulfilled. Then, from (4.33) we find that

$$0=\mathrm{D}j_\varepsilon^h(\varphi_\varepsilon)\Big(\zeta-\fint_\Omega\zeta\,\mathrm{dx}\Big)=\mathrm{D}j_\varepsilon^h(\varphi_\varepsilon)(\zeta)+\lambda_\varepsilon\int_\Omega\zeta\,\mathrm{dx}\quad\forall\zeta\in H^1(\Omega)\cap L^\infty(\Omega),\tag{4.34}$$

where we defined

$$\lambda_\varepsilon:=-|\Omega|^{-1}\,\mathrm{D}j_\varepsilon(\varphi_\varepsilon)\in\mathbb{R}.\tag{4.35}$$

In particular, we interpret $\lambda_\varepsilon\in\mathbb{R}$ as a Lagrange multiplier for the integral constraint $\int_\Omega\varphi\,\mathrm{dx}=\beta\,|\Omega|$.

We now want to rewrite (4.34) into a more convenient form by using the adjoint variable $\boldsymbol{q}_\varepsilon$, which is defined as the solution of (4.27). For this purpose we start calculating the derivative of $j_\varepsilon^h$. We find for every $\zeta\in H^1(\Omega)\cap L^\infty(\Omega)$ the following formula:

$$\begin{aligned}\mathrm{D}j_\varepsilon^h(\varphi_\varepsilon)(\zeta)=&\int_\Omega\frac{1}{2}\alpha_\varepsilon'(\varphi_\varepsilon)\zeta\,|\boldsymbol{u}_\varepsilon|^2+\alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{u}\,\mathrm{dx}\\&+\frac{\gamma}{2c_0}\int_\Omega\varepsilon\nabla\varphi_\varepsilon\cdot\nabla\zeta+\frac{1}{\varepsilon}\psi'(\varphi_\varepsilon)\zeta\,\mathrm{dx}\\&+\int_\Omega\mathcal{M}(\varphi_\varepsilon)(\mathrm{D}_2h,\mathrm{D}_3h,\mathrm{D}_4h)\,|_{(x,\nabla\boldsymbol{u}_\varepsilon,p_\varepsilon,\nabla\varphi_\varepsilon)}\cdot(\nabla\boldsymbol{u},p,\nabla\zeta)\,\mathrm{dx}\\&+\int_\Omega h(x,\nabla\boldsymbol{u}_\varepsilon,p_\varepsilon,\nabla\varphi_\varepsilon)\mathcal{M}'(\varphi_\varepsilon)\zeta\,\mathrm{dx}.\end{aligned}\tag{4.36}$$

where $\boldsymbol{S}_\varepsilon(\varphi_\varepsilon)=\{(\boldsymbol{u}_\varepsilon,p_\varepsilon)\}$ and $(\boldsymbol{u},p):=\mathrm{D}\boldsymbol{S}_\varepsilon(\varphi_\varepsilon)(\zeta)$ is the solution of the linearized state equation (4.22). Now we use the adjoint state $\boldsymbol{q}_\varepsilon$ as a test function in the linearized state equation (4.22) and find that

$$\begin{aligned}&\int_\Omega\alpha_\varepsilon'(\varphi_\varepsilon)\varphi\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon+\alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}\cdot\boldsymbol{q}_\varepsilon+\mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{q}_\varepsilon\,\mathrm{dx}\\&\qquad+\int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon+(\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{q}_\varepsilon+p\big(\mathcal{M}(\varphi_\varepsilon)\mathrm{D}_3h-\vartheta_\varepsilon\big)\,\mathrm{dx}=0,\end{aligned}\tag{4.37}$$

where $\mathrm{D}_3 h$ is evaluated at $(x, \nabla \boldsymbol{u}_\varepsilon, p_\varepsilon, \nabla \varphi_\varepsilon)$. Then we use the linearized state $\boldsymbol{u} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$ as a test function in (4.31) and obtain

$$\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{q}_\varepsilon \cdot \boldsymbol{u} + \mu \nabla \boldsymbol{q}_\varepsilon \cdot \nabla \boldsymbol{u} + (\nabla \boldsymbol{u}_\varepsilon)^T \boldsymbol{q}_\varepsilon \cdot \boldsymbol{u} - (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{q}_\varepsilon \cdot \boldsymbol{u} \,\mathrm{dx} = \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon \cdot \boldsymbol{u} + \mathcal{M}(\varphi_\varepsilon)\left(\mathrm{D}_2 h \cdot \nabla \boldsymbol{u}\right) \mathrm{dx}, \quad (4.38)$$

where $\mathrm{D}_2 h$ is evaluated at $(x, \nabla \boldsymbol{u}_\varepsilon, p_\varepsilon, \nabla \varphi_\varepsilon)$. Comparing (4.37) and (4.38) yields the following identity

$$\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\varphi \boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon + \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon \cdot \boldsymbol{u} + \mathcal{M}(\varphi_\varepsilon)\left(\mathrm{D}_2 h \cdot \nabla \boldsymbol{u} + p \mathrm{D}_3 h\right) \mathrm{dx} = 0, \quad (4.39)$$

where we have used that $p \in L^2_0(\Omega)$, $\operatorname{div} \boldsymbol{u}_\varepsilon = 0$ in $\Omega$, $\boldsymbol{u} = \boldsymbol{q}_\varepsilon = \boldsymbol{0}$ on $\partial\Omega$, and thus

$$\int_\Omega p \vartheta_\varepsilon \,\mathrm{dx} = \vartheta_\varepsilon \int_\Omega p \,\mathrm{dx} = 0,$$
$$\int_\Omega (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{q}_\varepsilon \cdot \boldsymbol{u} + (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{q}_\varepsilon \,\mathrm{dx} = \int_\Omega \boldsymbol{u}_\varepsilon \cdot \nabla(\boldsymbol{q}_\varepsilon \cdot \boldsymbol{u}) \,\mathrm{dx} = 0.$$

Hence, by using (4.39), we can rewrite (4.36) as follows:

$$\begin{aligned} \mathrm{D} j_\varepsilon(\varphi_\varepsilon)(\zeta) = \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\zeta\left(\frac{1}{2}|\boldsymbol{u}_\varepsilon|^2 - \boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon\right) + \frac{\gamma\varepsilon}{2c_0}\nabla\varphi_\varepsilon \cdot \nabla\zeta + \frac{\gamma}{2c_0\varepsilon}\psi'(\varphi_\varepsilon)\zeta \,\mathrm{dx} \\ + \int_\Omega \mathcal{M}(\varphi_\varepsilon)\mathrm{D}_4 h(x, \nabla \boldsymbol{u}_\varepsilon, p_\varepsilon, \nabla\varphi_\varepsilon) \cdot \nabla\zeta + h(x, \nabla \boldsymbol{u}_\varepsilon, p_\varepsilon, \nabla\varphi_\varepsilon)\mathcal{M}'(\varphi_\varepsilon)\zeta \,\mathrm{dx}. \quad (4.40) \end{aligned}$$

Together with (4.34), this yields the statement of the theorem. □

The analogous optimality condition for the optimization problem $\{(3.9), (3.11)\}$ involving the hydrodynamic force (1.3) is given as follows.

**Theorem 4.11** *Let $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon) \in (\Phi_{ad} \cap L^\infty(\Omega)) \times \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) \times L^2_0(\Omega)$ be a minimizer of optimization problem $\{(3.9), (3.11)\}$ involving the hydrodynamic force (1.3) with $\|\nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{K_\Omega}$, thus in particular, $\boldsymbol{S}_\varepsilon(\varphi_\varepsilon) = \{(\boldsymbol{u}_\varepsilon, p_\varepsilon)\}$. Then, there exists a Lagrange multiplier $\lambda_\varepsilon \in \mathbb{R}$ for the integral constraint such that*

$$\begin{aligned} \left(\alpha_\varepsilon'(\varphi_\varepsilon)\left(\frac{1}{2}|\boldsymbol{u}_\varepsilon|^2 - \boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon\right) + \frac{\gamma}{2c_0\varepsilon}\psi'(\varphi_\varepsilon) + \lambda_\varepsilon + \mathcal{M}'(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot (\boldsymbol{\sigma}_\varepsilon \boldsymbol{a}), \zeta\right)_{L^2(\Omega)} \\ + \left(\mathcal{M}(\varphi_\varepsilon)\boldsymbol{\sigma}_\varepsilon \boldsymbol{a} + \frac{\gamma\varepsilon}{2c_0}\nabla\varphi_\varepsilon, \nabla\zeta\right)_{\boldsymbol{L}^2(\Omega)} \\ = 0 \quad \forall \zeta \in H^1(\Omega) \cap L^\infty(\Omega), \quad (4.41) \end{aligned}$$

*where $\boldsymbol{\sigma}_\varepsilon := \mu\left(\nabla \boldsymbol{u}_\varepsilon + (\nabla \boldsymbol{u}_\varepsilon)^T\right) - p_\varepsilon \mathrm{I}$, and $(\boldsymbol{q}_\varepsilon, \pi_\varepsilon) \in \boldsymbol{H}^1_0(\Omega) \times L^2(\Omega)$ is the unique weak solution of the adjoint system*

$$\begin{aligned} \alpha_\varepsilon(\varphi_\varepsilon)(\boldsymbol{q}_\varepsilon - \boldsymbol{u}_\varepsilon) - \mu\nabla \cdot (\nabla \boldsymbol{q}_\varepsilon + (\nabla \boldsymbol{q}_\varepsilon)^T) + (\nabla \boldsymbol{u}_\varepsilon)^T \boldsymbol{q}_\varepsilon - (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{q}_\varepsilon + \nabla\pi_\varepsilon \\ = -\mu\left(\operatorname{div}\left(\mathcal{M}(\varphi_\varepsilon)\nabla\varphi_\varepsilon\right)\boldsymbol{a} + \nabla\left(\mathcal{M}(\varphi_\varepsilon)\nabla\varphi_\varepsilon\right)\boldsymbol{a}\right) \quad \text{in } \Omega, \quad (4.42a) \\ \operatorname{div} \boldsymbol{q}_\varepsilon = \mathcal{M}(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot \boldsymbol{a} - \fint_\Omega \mathcal{M}(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot \boldsymbol{a} \,\mathrm{dx} \quad \text{in } \Omega, \quad (4.42b) \\ \boldsymbol{q}_\varepsilon = \boldsymbol{0} \quad \text{on } \partial\Omega. \quad (4.42c) \end{aligned}$$

*Proof.* Note that for the hydrodynamic force (1.3), we have

$$h(x, \nabla \boldsymbol{u}_\varepsilon, p_\varepsilon, \nabla \varphi_\varepsilon) = \nabla \varphi_\varepsilon \cdot \Big( \mu \big( \nabla \boldsymbol{u}_\varepsilon + (\nabla \boldsymbol{u}_\varepsilon)^T \big) - p_\varepsilon \, \mathrm{I} \Big) \boldsymbol{a},$$

and so we compute that

$$\begin{aligned} \mathrm{D}_2 h &= \mu \left( \nabla \varphi_\varepsilon \otimes \boldsymbol{a} + \boldsymbol{a} \otimes \nabla \varphi_\varepsilon \right), \quad \mathrm{D}_3 h = -\boldsymbol{a} \cdot \nabla \varphi_\varepsilon, \\ \mathrm{D}_4 h &= \Big( \mu \big( \nabla \boldsymbol{u}_\varepsilon + (\nabla \boldsymbol{u}_\varepsilon)^T \big) - p_\varepsilon \, \mathrm{I} \Big) \boldsymbol{a}. \end{aligned}$$

As $\boldsymbol{a}$ is a constant vector, (4.24) in Assumption 4.2 is satisfied and the statements follow from the application of Theorem 4.10. □

REMARK 4.4 After using integration by parts, we find that we can rewrite the gradient equation (4.41) for the hydrodynamic force formally in the strong form as

$$-\frac{\gamma}{2c_0} \Big( \varepsilon \Delta \varphi_\varepsilon - \frac{1}{\varepsilon} \psi'(\varphi_\varepsilon) \Big) + \lambda_\varepsilon + \alpha_\varepsilon'(\varphi_\varepsilon) \Big( \frac{1}{2} |\boldsymbol{u}_\varepsilon|^2 - \boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon \Big) - \mathcal{M}(\varphi_\varepsilon) \operatorname{div} (\boldsymbol{\sigma}_\varepsilon \boldsymbol{a}) = 0 \text{ in } \Omega, \tag{4.43}$$

with the boundary condition

$$\frac{\gamma \varepsilon}{2c_0} \nabla \varphi_\varepsilon \cdot \boldsymbol{\nu}_{\partial\Omega} + \mathcal{M}(\varphi_\varepsilon) \boldsymbol{\nu}_{\partial\Omega} \cdot (\boldsymbol{\sigma}_\varepsilon \boldsymbol{a}) = 0 \text{ on } \partial\Omega. \tag{4.44}$$

Moreover, with sufficiently smooth solutions, we can make use of the state equation (3.4a) to rewrite (4.43) as:

$$\begin{aligned} -\frac{\gamma}{2c_0} \Big( \varepsilon \Delta \varphi_\varepsilon - \frac{1}{\varepsilon} \psi'(\varphi_\varepsilon) \Big) + \lambda_\varepsilon + \alpha_\varepsilon'(\varphi_\varepsilon) \Big( \frac{1}{2} |\boldsymbol{u}_\varepsilon|^2 - \boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon \Big) \\ + \mathcal{M}(\varphi_\varepsilon) \left( \boldsymbol{f} - \alpha_\varepsilon(\varphi_\varepsilon) \boldsymbol{u}_\varepsilon - (\boldsymbol{u}_\varepsilon \cdot \nabla) \boldsymbol{u}_\varepsilon \right) \cdot \boldsymbol{a} = 0. \end{aligned} \tag{4.45}$$

REMARK 4.5 We note that the above analysis of (3.8)–(3.9) can be modified to include a Dirichlet condition for the design function $\varphi_\varepsilon$ on $\partial\Omega$, for instance $\varphi_\varepsilon = 1$ on $\partial\Omega$. This amounts to changing the space of admissible design functions to

$$\Phi_{ad} = \Big\{ \varphi \in H^1(\Omega) \mid \int_\Omega \varphi \, \mathrm{d}x = \beta \, |\Omega| \text{ and } \varphi = 1 \text{ on } \partial\Omega \Big\}.$$

Then, in the optimality conditions (4.32) and (4.41), and also in (4.33) and (4.34), we use test functions $\zeta \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Moreover, from Remark 4.4, the strong form of the resulting gradient equation (4.41) remains as (4.43) (or (4.45)), but now with the boundary condition

$$\varphi_\varepsilon = 1 \text{ on } \partial\Omega.$$

## 5. Sharp interface asymptotics for the hydrodynamic force

In Section 3, we introduced the diffuse interface problem (3.8)–(3.9) as an approximation of the shape optimization problem (2.3)–(2.4) for a general functional $h$. In Section 4, the existence of a minimizer $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon)$ to (3.8)–(3.9) for every fixed $\varepsilon > 0$ is guaranteed by Theorem 4.6, and the

first order necessary optimality condition is given in Theorem 4.10. The analogous results for the hydrodynamic force problem $\{(3.9),(3.11)\}$ are also presented in Theorem 4.7 and Theorem 4.11.

In this section, we focus only the hydrodynamic force problem $\{(3.9),(3.11)\}$ and carry out a sharp interface limit of the system $\{(3.4),(4.42),(4.45)\}$ by the method of formally matched asymptotic expansions. We hereby recover the optimality conditions expected from the classical shape sensitivity analysis presented in Section 2 in the limit $\varepsilon \searrow 0$. For an introduction and more detailed discussion of the techniques and basic assumptions used in the method of formally matched asymptotic analysis we refer for instance to [12, 16].

In the asymptotic analysis, we assume there are sufficient smooth solutions to the system $\{(3.4),(4.42),(4.43)\}$, and hence we consider (4.45) instead of (4.43) in the sequel as the analysis is comparatively easier.

ASSUMPTION 5.1 We assume that for small $\varepsilon$, the domain $\Omega$ can be divided into two open subdomains $\Omega^{\pm}(\varepsilon)$, separated by an interface $\Gamma(\varepsilon)$. Furthermore, we assume that there is a family $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon, \boldsymbol{q}_\varepsilon, \pi_\varepsilon, \lambda_\varepsilon, \vartheta_\varepsilon)_{\varepsilon>0}$ of solutions to $\{(3.4),(4.42),(4.45)\}$, which are sufficiently smooth and have an asymptotic expansion in $\varepsilon$ in the bulk regions away from $\Gamma(\varepsilon)$ (the outer expansion, see Section 5.1), and another expansion in the interfacial region (inner expansions, see Section 5.2), see also [12, 16] for a detailed formulation.

The general idea is to show that the leading order expansions $(\varphi_0, \boldsymbol{u}_0, p_0, q_0, \pi_0, \lambda_0, \vartheta_0)$ satisfy (2.4), (2.7) and the strong form of (2.6) with additional perimeter regularization and volume constraint. For the remainder of this section, we will make use of the following assumptions extensively.

ASSUMPTION 5.2 The correction constant $\delta_\varepsilon$ and the interpolation function $\alpha_\varepsilon$ fulfill

$$\delta_\varepsilon = \varepsilon^k, k > 1, \quad \alpha_\varepsilon(t) = \frac{1}{\varepsilon}\hat{\alpha}(t),$$

where $\hat{\alpha} \in C^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfies

$$\hat{\alpha}(-1) > 0, \quad \hat{\alpha}(1) = \hat{\alpha}'(1) = 0, \quad \hat{\alpha}(t) \neq 0 \text{ for } t \neq 1. \tag{5.1}$$

Moreover, we assume that the potential $\psi \in C^2(\mathbb{R})$ satisfies

$$\psi(\pm 1) = \psi'(\pm 1) = 0. \tag{5.2}$$

For the terms involving the square root, we make use of the following expansion for $a = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots$, which holds due to Taylor's theorem,

$$\begin{aligned}\sqrt{a+\delta_\varepsilon} = \sqrt{a_0 + \varepsilon a_1 + \ldots + \varepsilon^k(a_k+1) + \ldots} = \sqrt{a_0} + \frac{1}{2\sqrt{a_0}}\left[\varepsilon a_1 + \ldots + \varepsilon^k(a_k+1) + \ldots\right] \\ - \frac{1}{4\sqrt{a_0^3}}\left[\varepsilon a_1 + \ldots + \varepsilon^k(a_k+1) + \ldots\right]^2 + \ldots.\end{aligned} \tag{5.3}$$

## 5.1 *Outer expansions*

We assume that for $v_\varepsilon \in \{\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon, \lambda_\varepsilon, \vartheta_\varepsilon, \boldsymbol{q}_\varepsilon, \pi_\varepsilon\}$, the following outer expansions hold:

$$v_\varepsilon = v_0 + \varepsilon v_1 + \ldots.$$

Applying Taylor's theorem and (5.3), for the choice $\mathcal{M}(\varphi_\varepsilon) = \frac{1}{\sqrt{2}c_0}\sqrt{\psi(\varphi_\varepsilon)+\delta_\varepsilon}$, we obtain following outer expansion

$$\begin{aligned}\mathcal{M}(\varphi_\varepsilon) &= \mathcal{M}(\varphi_0+\varepsilon\varphi_1+\dots)\\ &= \frac{1}{\sqrt{2}c_0}\left(\sqrt{\psi(\varphi_0)+\psi'(\varphi_0)(\varepsilon\varphi_1+\dots)+\dots+\psi^{(k)}(\varphi_0)(\varepsilon\varphi_1+\dots)^k+\dots}\right)\\ &= \frac{1}{\sqrt{2}c_0}\left(\sqrt{\psi(\varphi_0)}+\frac{\varepsilon\psi'(\varphi_0)\varphi_1}{2\sqrt{\psi(\varphi_0)}}+\mathcal{O}(\varepsilon^2)\right) =: \mathcal{M}_0(\varphi_0)+\varepsilon\mathcal{M}_1(\varphi_0)\varphi_1+\text{ h.o.t.}.\end{aligned} \tag{5.4}$$

We remark that, for the classical smooth double-well potential $\psi(\varphi) = \frac{1}{4}(1-\varphi^2)^2$, one can compute that

$$\lim_{s\searrow -1}\frac{\psi'(s)}{\sqrt{\psi(s)}} = 2, \quad \lim_{s\nearrow 1}\frac{\psi'(s)}{\sqrt{\psi(s)}} = -2,$$

and so $\mathcal{M}_1(\pm 1)$ is well-defined for the smooth double-well potential. We denote $(\cdot)_O^\beta$ to be the order $\beta$ outer expansions of equation $(\cdot)$. To leading order $(3.4a)_O^{-1}$ gives

$$\hat{\alpha}(\varphi_0)\boldsymbol{u}_0 = \boldsymbol{0}. \tag{5.5}$$

By (5.1), if $\varphi_0 \neq 1$, we then obtain $\boldsymbol{u}_0 = \boldsymbol{0}$. Similarly, to leading order $(4.42a)_O^{-1}$ gives

$$\hat{\alpha}(\varphi_0)\boldsymbol{q}_0 = \hat{\alpha}(\varphi_0)\boldsymbol{u}_0. \tag{5.6}$$

Thus, if $\varphi_0 \neq 1$, then $\boldsymbol{q}_0 = \boldsymbol{u}_0 = \boldsymbol{0}$. Meanwhile, $(3.4b)_O^0$, $(3.4c)_O^0$, and $(4.42c)_O^0$ give

$$\begin{aligned}\operatorname{div}\boldsymbol{u}_0 &= 0 \text{ in } \Omega,\\ \boldsymbol{u}_0 = \boldsymbol{g}, \quad \boldsymbol{q}_0 &= \boldsymbol{0} \text{ on } \partial\Omega.\end{aligned}$$

To order $-1$, $(4.45)_O^{-1}$ gives

$$\hat{\alpha}'(\varphi_0)\left(\frac{1}{2}|\boldsymbol{u}_0|^2 - \boldsymbol{u}_0\cdot\boldsymbol{q}_0\right) = -\frac{\gamma}{2c_0}\psi'(\varphi_0). \tag{5.7}$$

If $\varphi_0 \neq 1$, then from (5.5), (5.6), and (5.1), we have that

$$-\psi'(\varphi_0) = 0. \tag{5.8}$$

Hence, $\varphi_0$ must be a piecewise constant function that takes values equal to the roots of $\psi'(\cdot)$. The stable solutions to (5.8) are $\varphi_0 = \pm 1$. In particular, we can define the fluid region and the solid region by

$$E := \{x\in\Omega \mid \varphi_0(x) = 1\}, \quad B := \{x\in\Omega \mid \varphi_0(x) = -1\},$$

respectively. Moreover, from (5.5) and (5.6) we have

$$\boldsymbol{u}_0 = \boldsymbol{q}_0 = \boldsymbol{0} \text{ in } B. \tag{5.9}$$

Furthermore, as $\varphi_0 = \pm 1$, we have $\nabla\varphi_0 = \boldsymbol{0}$ in $E$ and $B$, and so, $\vartheta_0 = 0$ by the definition (4.28). From $(4.42b)_O^0$ we have

$$\operatorname{div}\boldsymbol{q}_0 = 0 \text{ in } E\cup B. \tag{5.10}$$

The next order $(3.4a)_O^0$ gives

$$\hat{\alpha}'(\varphi_0)\varphi_1 \boldsymbol{u}_0 + \hat{\alpha}(\varphi_0)\boldsymbol{u}_1 - \mu\Delta\boldsymbol{u}_0 + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}_0 + \nabla p_0 = \boldsymbol{f}. \tag{5.11}$$

By (5.1), for $\varphi_0 = 1$, we obtain

$$-\mu\Delta\boldsymbol{u}_0 + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}_0 + \nabla p_0 = \boldsymbol{f} \text{ in } E. \tag{5.12}$$

Similarly, $(4.42a)_O^0$ gives

$$\begin{aligned}\hat{\alpha}'(\varphi_0)\varphi_1(\boldsymbol{q}_0 - \boldsymbol{u}_0) + \hat{\alpha}(\varphi_0)(\boldsymbol{q}_1 - \boldsymbol{u}_1) - \mu\,\mathrm{div}\,(\nabla\boldsymbol{q}_0 + (\nabla\boldsymbol{q}_0)^T) \\ + (\nabla\boldsymbol{u}_0)^T\boldsymbol{q}_0 - (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{q}_0 + \nabla\pi_0 = \boldsymbol{0}.\end{aligned} \tag{5.13}$$

For $\varphi_0 = 1$, we obtain

$$-\mu\Delta\boldsymbol{q}_0 + (\nabla\boldsymbol{u}_0)^T\boldsymbol{q}_0 - (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{q}_0 + \nabla\pi_0 = \boldsymbol{0} \text{ in } E,$$

where we have used (5.10) to simplify the divergence term. Thus, we have recovered (2.4a), (2.4b), (2.4c), (2.7a), (2.7b), and (2.7d).

### 5.2 *Inner expansions and matching conditions*

To recover the boundary conditions (2.4d), (2.7c) and the strong form of the shape derivative (2.6) with additional perimeter regularization and volume constraint, we now consider the interfacial region, i.e., near some free boundary $\Gamma = \partial E \cap \partial B$ which is assumed to be the limiting hypersurface of the zero level sets of $\varphi_\varepsilon$. For studying the limiting behaviour in these parts of $\Omega$ we introduce new coordinates with the help of the signed distance function $d(x)$ to $\Gamma$ and set $z = \frac{d}{\varepsilon}$ as the rescaled distance variable. Here we use the sign convention $d(x) > 0$ if $x \in E$ and $d(x) < 0$ if $x \in B$.

Let $\gamma(s)$ denote a parametrization of $\Gamma$ by arc-length $s$, and let $\boldsymbol{\nu}$ denote the outward unit normal of $\Gamma$. Then, in a tubular neighborhood of $\Gamma$, for sufficiently smooth function $v(x)$, we have

$$v(x) = v\Big(\gamma(s) + \varepsilon z\boldsymbol{\nu}\big(\gamma(s)\big)\Big) =: V(s,z).$$

In this new $(s,z)$-coordinate system, the following change of variables apply, see [16]:

$$\nabla_x v = \frac{1}{\varepsilon}\partial_z V\boldsymbol{\nu} + \nabla_\Gamma V + \text{ h.o.t.},$$

where $\nabla_\Gamma f$ denotes the surface gradient of $f$ on $\Gamma$ with components $(\underline{D}_k f)_{1\le k\le d}$ and h.o.t. denotes higher order terms with respect to $\varepsilon$. Moreover, if $\boldsymbol{\nu}$ is a vector-valued function, then we obtain

$$\mathrm{div}_x\boldsymbol{\nu} = \frac{1}{\varepsilon}\partial_z\boldsymbol{V}\cdot\boldsymbol{\nu} + \mathrm{div}_\Gamma\boldsymbol{V} + \text{ h.o.t.},$$

where $\mathrm{div}_\Gamma$ denotes the surface divergence. In particular, using the fact that the normal $\boldsymbol{\nu}$ is independent of $z$, we have

$$\Delta v = \mathrm{div}_x(\nabla_x v) = \frac{1}{\varepsilon^2}\partial_{zz}V + \frac{1}{\varepsilon}\underbrace{\mathrm{div}_\Gamma(\partial_z V\boldsymbol{\nu})}_{=-\kappa\partial_z V} + \text{h.o.t.},$$

where $\kappa = -\operatorname{div}_\Gamma \boldsymbol{\nu}$ is the mean curvature.

We denote the variables $\varphi_\varepsilon$, $\boldsymbol{u}_\varepsilon$, $p_\varepsilon$, $\boldsymbol{q}_\varepsilon$, $\pi_\varepsilon$ in the new coordinate system by $\Phi_\varepsilon$, $\boldsymbol{U}_\varepsilon$, $P_\varepsilon$, $\boldsymbol{Q}_\varepsilon$, $\Pi_\varepsilon$. We further assume that they have the following inner expansions:

$$V_\varepsilon(s,z) = V_0(s,z) + \varepsilon V_1(s,z) + \dots,$$

for $V_\varepsilon \in \{\Phi_\varepsilon, \boldsymbol{U}_\varepsilon, P_\varepsilon, \boldsymbol{Q}_\varepsilon, \Pi_\varepsilon\}$. We then obtain,

$$\mathcal{M}(\Phi_\varepsilon) = \mathcal{M}_0(\Phi_0) + \varepsilon \mathcal{M}_1(\Phi_0)\Phi_1 + \text{ h.o.t.},$$

where $\mathcal{M}_0$, $\mathcal{M}_1$ are as defined in (5.4) if we consider $\mathcal{M}(\varphi) = \frac{1}{c_0}\sqrt{\frac{\psi(\varphi)+\delta_\varepsilon}{2}}$. We remark that, for a sufficiently smooth function $\boldsymbol{f}$ independent of $\varepsilon$,

$$\begin{aligned}\boldsymbol{f}(x) &= \boldsymbol{f}\big(\gamma(s) + \varepsilon z \boldsymbol{\nu}(s)\big) = \boldsymbol{f}\big(\gamma(s)\big) + \varepsilon z \nabla \boldsymbol{f}\big(\gamma(s)\big)\cdot \boldsymbol{\nu} + \text{ h.o.t.} \\ &=: \boldsymbol{F}_0(s) + \varepsilon \boldsymbol{F}_1(s,z) + \text{ h.o.t.},\end{aligned}$$

for $x$ in a neighborhood of $\Gamma$. As a consequence, we see that

$$\partial_z \boldsymbol{F}_0 = \boldsymbol{0}. \tag{5.14}$$

As the Lagrange multipliers $\lambda_\varepsilon$ and $\vartheta_\varepsilon$ are constants by definition, we assume that their inner expansions are the same as their outer expansions. In particular, the leading order inner expansions of the Lagrange multipliers do not depend on $z$, as in (5.14). Furthermore, the assumption that the zero level set of $\varphi_\varepsilon$ converge to $\Gamma$ implies that

$$\Phi_0(0) = 0. \tag{5.15}$$

In a tubular neighborhood of $\Gamma$ the solution $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon, q_\varepsilon, \pi_\varepsilon, \lambda_\varepsilon, \vartheta_\varepsilon)$ is represented by an outer and an inner expansion. Thus, we require that these two expansions match. In order to match the inner expansions valid in the interfacial region to the outer expansions of Section 5.1 we employ the following matching conditions (for the derivation we refer to [16, Appendix D]),

$$\lim_{z\to\pm\infty} V_0(s,z) = v_0^\pm, \tag{5.16}$$

$$\lim_{z\to\pm\infty} \partial_z V_0(s,z) = 0, \tag{5.17}$$

$$\lim_{z\to\pm\infty} \partial_z V_1(s,z) = \nabla v_0^\pm \cdot \boldsymbol{\nu}, \tag{5.18}$$

where $v_0^\pm := \lim_{\delta \searrow 0} v_0(p \pm \delta \boldsymbol{\nu})$ for $p \in \Gamma$. For vector-valued functions (5.18) read as

$$\lim_{z\to\pm\infty} \partial_z \boldsymbol{V}_1(s,z) = \partial_{\boldsymbol{\nu}} \boldsymbol{v}_0^\pm.$$

As $\operatorname{div} \boldsymbol{u}_\varepsilon = 0$, we can rewrite

$$\Delta \boldsymbol{u}_\varepsilon = \operatorname{div}\big(\nabla \boldsymbol{u}_\varepsilon + (\nabla \boldsymbol{u}_\varepsilon)^T\big).$$

For a tensor $\boldsymbol{A}$, let $\mathcal{E}(\boldsymbol{A}) = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T)$. Then we can compute

$$\begin{aligned}\Delta \boldsymbol{u}_\varepsilon &= \frac{2}{\varepsilon^2}\partial_z\big(\mathcal{E}(\partial_z \boldsymbol{U}_\varepsilon \otimes \boldsymbol{\nu})\boldsymbol{\nu}\big) + \frac{2}{\varepsilon}\partial_z\big(\mathcal{E}(\nabla_\Gamma \boldsymbol{U}_\varepsilon)\boldsymbol{\nu}\big) + \frac{2}{\varepsilon}\operatorname{div}_\Gamma\big(\mathcal{E}(\partial_z \boldsymbol{U}_\varepsilon \otimes \boldsymbol{\nu})\big) + \dots \\ &= \frac{1}{\varepsilon^2}\partial_{zz}\boldsymbol{U}_\varepsilon + \frac{1}{\varepsilon^2}\partial_z(\partial_z \boldsymbol{U}_\varepsilon \cdot \boldsymbol{\nu})\boldsymbol{\nu} + \frac{2}{\varepsilon}\partial_z\big(\mathcal{E}(\nabla_\Gamma \boldsymbol{U}_\varepsilon)\boldsymbol{\nu}\big) + \frac{2}{\varepsilon}\operatorname{div}_\Gamma\big(\mathcal{E}\big(\partial_z \boldsymbol{U}_\varepsilon \otimes \boldsymbol{\nu}\big)\big) + \dots.\end{aligned}$$

We note that the same expansion holds for the divergence term in (4.42a). The goal in this section is to deduce properties satisfied by the inner expansions and to relate these properties to boundary conditions on $\Gamma$ for the outer expansions on $\Gamma$ by matching. As in Section 5.1, we will denote $(\cdot)_I^\beta$ to be the order $\beta$ inner expansions of equation $(\cdot)$.

5.2.1 *Inner expansions of the state equations.* To order $-1$, $(3.4b)_I^{-1}$ gives

$$\partial_z \boldsymbol{U}_0 \cdot \boldsymbol{\nu} = \partial_z (\boldsymbol{U}_0 \cdot \boldsymbol{\nu}) = 0, \tag{5.19}$$

while to leading order $(3.4a)_I^{-2}$ gives

$$-\mu \partial_z \big( \partial_z \boldsymbol{U}_0 + (\partial_z \boldsymbol{U}_0 \cdot \boldsymbol{\nu}) \boldsymbol{\nu} \big) = -\mu \partial_{zz} \boldsymbol{U}_0 = \boldsymbol{0}, \tag{5.20}$$

where we have used (5.19). Integrating with respect to $z$ and applying the matching condition (5.17) leads to

$$\partial_z \boldsymbol{U}_0(s,z) = \boldsymbol{0}, \tag{5.21}$$

and so $\boldsymbol{U}_0$ is independent of $z$. Integrating once more with respect to $z$ and by the matching condition (5.16), we hence find that

$$\boldsymbol{U}_0(s,z) = \boldsymbol{u}_0^- = \boldsymbol{0}, \tag{5.22}$$

where we made use of (5.9) for the second equality. This implies that

$$\boldsymbol{u}_0^+ = \boldsymbol{u}_0^- = \boldsymbol{0}, \tag{5.23}$$

which is the no-slip condition (2.4d). Since $\boldsymbol{U}_0 = \boldsymbol{0}$, to first order $(3.4b)_I^0$ gives

$$\partial_z \boldsymbol{U}_1 \cdot \boldsymbol{\nu} + \operatorname{div}_\Gamma \boldsymbol{U}_0 = \partial_z \boldsymbol{U}_1 \cdot \boldsymbol{\nu} = 0. \tag{5.24}$$

Using (5.22) and (5.24), to first order $(3.4a)_I^{-1}$ gives

$$-\mu \partial_{zz} \boldsymbol{U}_1 + \partial_z P_0 \boldsymbol{\nu} = \boldsymbol{0}. \tag{5.25}$$

5.2.2 *Phase field equation to leading order.* To leading order $(4.45)_I^{-1}$ gives

$$-\frac{\gamma}{2c_0}\big(\partial_{zz}\Phi_0 - \psi'(\Phi_0)\big) + \hat{\alpha}'(\Phi_0)\big(\tfrac{1}{2}|\boldsymbol{U}_0|^2 - \boldsymbol{U}_0 \cdot \boldsymbol{Q}_0\big) - \mathcal{M}(\Phi_0)\alpha(\Phi_0)\boldsymbol{U}_0 \cdot \boldsymbol{a} = 0. \tag{5.26}$$

Using (5.22), the above simplifies to

$$\partial_{zz}\Phi_0 - \psi'(\Phi_0) = 0. \tag{5.27}$$

Along with the following matching conditions (5.16) for $\Phi_0$,

$$\Phi_0(s, z = \pm\infty) = \pm 1,$$

we can choose $\Phi_0$ to be a function independent of $s$ and as the unique monotone solution to (5.27) satisfying $\Phi_0(z=0) = 0$ (recall (5.15)). Moreover, taking the product of (5.27) with $\Phi_0'(z)$, integrating with respect to $z$ and applying matching (5.16) and (5.17) leads to the so-called equipartition of energy

$$\frac{1}{2}\left|\Phi_0'(z)\right|^2 = \psi(\Phi_0(z)) \text{ for } |z| < \infty. \tag{5.28}$$

Moreover, a short calculation using (5.28), the monotonicity of $\Phi_0$, and a change of variables $s \mapsto \Phi_0(z)$ shows that

$$c_0 = \frac{1}{2}\int_{-1}^{1} \sqrt{2\psi(s)}\,\mathrm{d}s = \frac{1}{2}\int_{\mathbb{R}} \sqrt{2\psi(\Phi_0(z))}\Phi_0'(z)\,\mathrm{d}z = \frac{1}{2}\int_{\mathbb{R}} \left|\Phi_0'(z)\right|^2 \mathrm{d}z\,. \tag{5.29}$$

5.2.3 *Inner expansions of the adjoint equation.* Before we analyze the adjoint equation, we first compute the following expansions

$$\operatorname{div}\left(\mathcal{M}(\varphi_\varepsilon)\nabla\varphi_\varepsilon\right) = \frac{1}{\varepsilon^2}\partial_z\left(\mathcal{M}(\Phi_\varepsilon)\partial_z\Phi_\varepsilon\right) + \operatorname{div}_\Gamma\left(\mathcal{M}(\Phi_\varepsilon)\left(\frac{1}{\varepsilon}\partial_z\Phi_\varepsilon\boldsymbol{\nu} + \nabla_\Gamma\Phi_\varepsilon\right)\right) + \text{ h.o.t.}, \tag{5.30}$$

and for any $1 \le j \le d$,

$$\begin{aligned}\left(\nabla\left(\mathcal{M}(\varphi_\varepsilon)\nabla\varphi_\varepsilon\right)\boldsymbol{a}\right)_j &= \sum_{i=1}^d \partial_i\left(\mathcal{M}(\varphi_\varepsilon)\partial_j\varphi_\varepsilon\right)a_i \\ &= \sum_{i=1}^d \frac{1}{\varepsilon}\nu_i\partial_z\left(\mathcal{M}(\Phi_\varepsilon)\left(\frac{1}{\varepsilon}\partial_z\Phi_\varepsilon\nu_j + \underline{D}_j\Phi_\varepsilon\right)\right)a_i \\ &\quad + \underline{D}_i\left(\mathcal{M}(\Phi_\varepsilon)\left(\frac{1}{\varepsilon}\partial_z\Phi_\varepsilon\nu_j + \underline{D}_j\Phi_\varepsilon\right)\right)a_i + \text{ h.o.t.},\end{aligned}$$

so that

$$\begin{aligned}\nabla\left(\mathcal{M}(\varphi_\varepsilon)\nabla\varphi_\varepsilon\right)\boldsymbol{a} &= \frac{1}{\varepsilon^2}(\boldsymbol{\nu}\cdot\boldsymbol{a})\boldsymbol{\nu}\partial_z\left(\mathcal{M}(\Phi_\varepsilon)\partial_z\Phi_\varepsilon\right) \\ &\quad + \frac{1}{\varepsilon}\left((\boldsymbol{\nu}\cdot\boldsymbol{a})\partial_z\left(\mathcal{M}(\Phi_\varepsilon)\nabla_\Gamma\Phi_\varepsilon\right) + \nabla_\Gamma\left(\mathcal{M}(\Phi_\varepsilon)\partial_z\Phi_\varepsilon\boldsymbol{\nu}\right)\boldsymbol{a}\right) \\ &\quad + \nabla_\Gamma(\nabla_\Gamma\Phi_\varepsilon)\boldsymbol{a} + \text{ h.o.t..}\end{aligned} \tag{5.31}$$

To leading order $(4.42b)_I^{-1}$ gives

$$\partial_z\boldsymbol{Q}_0\cdot\boldsymbol{\nu} = \mathcal{M}_0(\Phi_0)\Phi_0'(\boldsymbol{\nu}\cdot\boldsymbol{a}), \tag{5.32}$$

while to leading order $(4.42a)_I^{-2}$ gives

$$-\mu\partial_{zz}\boldsymbol{Q}_0 - \mu\partial_z(\partial_z\boldsymbol{Q}_0\cdot\boldsymbol{\nu})\boldsymbol{\nu} = -\mu\partial_z\left(\mathcal{M}_0(\Phi_0)\Phi_0'\right)\left((\boldsymbol{\nu}\cdot\boldsymbol{a})\boldsymbol{\nu} + \boldsymbol{a}\right), \tag{5.33}$$

where we have used (5.30), (5.31) and that $\boldsymbol{\nu}$ is independent of $z$. Integrating (5.33) with respect to $z$ and using the matching condition (5.17) leads to

$$\partial_z\boldsymbol{Q}_0 + (\partial_z\boldsymbol{Q}_0\cdot\boldsymbol{\nu})\boldsymbol{\nu} = \mathcal{M}_0(\Phi_0)\Phi_0'\left((\boldsymbol{\nu}\cdot\boldsymbol{a})\boldsymbol{\nu} + \boldsymbol{a}\right),$$

and upon adding the product of (5.32) with $\boldsymbol{\nu}$ leads to

$$\partial_z\boldsymbol{Q}_0(s,z) = \mathcal{M}_0(\Phi_0)\Phi_0'\boldsymbol{a}. \tag{5.34}$$

Then, integrating (5.34) with respect to $z$ and using the matching condition (5.16) and $\boldsymbol{q}_0^- = \boldsymbol{0}$ (see (5.9)) leads to

$$\boldsymbol{Q}_0(s,y) = \left(\int_{-\infty}^y \mathcal{M}_0\left(\Phi_0(z)\right)\Phi_0'(z)\,\mathrm{d}z\right)\boldsymbol{a} \quad \forall\, |y| < \infty. \tag{5.35}$$

In particular, the right hand side is independent of $s$, and so we can deduce that $\boldsymbol{Q}_0$ is also independent of $s$. Using the matching condition (5.16), we hence have

$$\boldsymbol{q}_0^+ = \left(\int_{\mathbb{R}} \mathcal{M}_0(\Phi_0(z))\Phi_0'(z)\,\mathrm{d}z\right)\boldsymbol{a}. \tag{5.36}$$

For the choice $\mathcal{M}(\varphi) = \frac{1}{2}$, we see that

$$\int_{\mathbb{R}} \mathcal{M}_0\big(\Phi_0(z)\big)\Phi_0'(z)\,\mathrm{d}z = \frac{1}{2}\int_{\mathbb{R}} \Phi_0'(z)\,\mathrm{d}z = 1, \tag{5.37}$$

while for the choice $\mathcal{M}(\varphi) = \frac{1}{c_0}\sqrt{\frac{\psi(\varphi)+\delta_\varepsilon}{2}}$, we see that by (5.4), (5.28), and (5.29),

$$\int_{\mathbb{R}} \mathcal{M}_0\big(\Phi_0(z)\big)\Phi_0'(z)\,\mathrm{d}z = \frac{1}{c_0}\int_{\mathbb{R}} \frac{1}{\sqrt{2}}\sqrt{\psi\big(\Phi_0(z)\big)}\Phi_0'(z)\,\mathrm{d}z = \frac{1}{c_0}\int_{\mathbb{R}} \frac{1}{2}\big|\Phi_0'(z)\big|^2\,\mathrm{d}z = 1.$$

Thus, in both cases, we obtain

$$\boldsymbol{q}_0^+ = \boldsymbol{a}, \tag{5.38}$$

which is the boundary condition (2.7c). To the next order, we obtain from $(4.42b)_I^0$

$$\partial_z \boldsymbol{Q}_1 \cdot \boldsymbol{\nu} = \mathcal{M}_0(\Phi_0)\partial_z\Phi_1(\boldsymbol{\nu}\cdot\boldsymbol{a}) + \mathcal{M}_1(\Phi_0)\Phi_1\Phi_0'(\boldsymbol{\nu}\cdot\boldsymbol{a}), \tag{5.39}$$

where we used that $\boldsymbol{Q}_0$ and $\Phi_0$ are functions of $z$ only, and $\vartheta_0 = 0$ from the outer expansions. Meanwhile, from (5.30) and (5.31), and the fact that $\boldsymbol{U}_0 = \boldsymbol{0}$, we obtain from $(4.42a)_I^{-1}$

$$\begin{aligned}\hat{\alpha}(\Phi_0)\boldsymbol{Q}_0 - \mu\partial_{zz}\boldsymbol{Q}_1 - \mu\partial_z(\partial_z\boldsymbol{Q}_1\cdot\boldsymbol{\nu})\boldsymbol{\nu} - 2\mu\,\mathrm{div}_\Gamma\big(\mathcal{E}(\boldsymbol{Q}_0'\otimes\boldsymbol{\nu})\big) + \partial_z\Pi_0\nu \\ = -\mu\big(\boldsymbol{a} + (\boldsymbol{\nu}\cdot\boldsymbol{a})\boldsymbol{\nu}\big)\partial_z\big(\mathcal{M}_0(\Phi_0)\partial_z\Phi_1 + \mathcal{M}_1(\Phi_0)\Phi_1\Phi_0'\big) \\ - \mu\,\mathrm{div}_\Gamma(\mathcal{M}_0(\Phi_0)\Phi_0'\boldsymbol{\nu})\boldsymbol{a} - \mu\nabla_\Gamma\big(\mathcal{M}_0(\Phi_0)\Phi_0'\boldsymbol{\nu}\big)\boldsymbol{a}.\end{aligned} \tag{5.40}$$

Moreover, applying the product rule, we have

$$\begin{aligned}2\,\mathrm{div}_\Gamma\big(\mathcal{E}(\boldsymbol{Q}_0'\otimes\boldsymbol{\nu})\big) &= \nabla_\Gamma(\boldsymbol{Q}_0')\boldsymbol{\nu} + (\,\mathrm{div}_\Gamma\boldsymbol{\nu})\boldsymbol{Q}_0' + (\nabla_\Gamma\boldsymbol{\nu})\boldsymbol{Q}_0' + (\,\mathrm{div}_\Gamma\boldsymbol{Q}_0')\boldsymbol{\nu} \\ &= -\kappa\boldsymbol{Q}_0' + (\nabla_\Gamma\boldsymbol{\nu})\boldsymbol{Q}_0', \\ \mathrm{div}_\Gamma(\mathcal{M}_0(\Phi_0)\Phi_0'\boldsymbol{\nu}) &= -\mathcal{M}_0(\Phi_0)\Phi_0'\kappa, \\ \nabla_\Gamma(\mathcal{M}_0(\Phi_0)\Phi_0'\boldsymbol{\nu}) &= \mathcal{M}_0(\Phi_0)\Phi_0'\nabla_\Gamma\boldsymbol{\nu}.\end{aligned}$$

Then, using the relation (5.34) for $\boldsymbol{Q}_0'$, we obtain from (5.40)

$$\begin{aligned}\hat{\alpha}(\Phi_0)\boldsymbol{Q}_0 - \mu\partial_{zz}\boldsymbol{Q}_1 - \mu\partial_z(\partial_z\boldsymbol{Q}_1\cdot\boldsymbol{\nu})\boldsymbol{\nu} + \mu\kappa\boldsymbol{Q}_0' - \mu(\nabla_\Gamma\boldsymbol{\nu})\boldsymbol{Q}_0' + \partial_z\Pi_0\nu \\ = -\mu(\boldsymbol{a} + (\boldsymbol{\nu}\cdot\boldsymbol{a})\boldsymbol{\nu})\partial_z(\mathcal{M}_0(\Phi_0)\partial_z\Phi_1 + \mathcal{M}_1(\Phi_0)\Phi_1\Phi_0') + \mu\boldsymbol{Q}_0'\kappa - \mu(\nabla_\Gamma\boldsymbol{\nu})\boldsymbol{Q}_0',\end{aligned} \tag{5.41}$$

and thus, upon cancelling the common terms, we have

$$\begin{aligned}\hat{\alpha}(\Phi_0)\boldsymbol{Q}_0 - \mu\partial_{zz}\boldsymbol{Q}_1 - \mu\partial_z(\partial_z\boldsymbol{Q}_1\cdot\boldsymbol{\nu})\boldsymbol{\nu} + \partial_z\Pi_0\nu \\ = -\mu\big(\boldsymbol{a} + (\boldsymbol{\nu}\cdot\boldsymbol{a})\boldsymbol{\nu}\big)\partial_z\big(\mathcal{M}_0(\Phi_0)\partial_z\Phi_1 + \mathcal{M}_1(\Phi_0)\Phi_1\Phi_0'\big).\end{aligned} \tag{5.42}$$

5.2.4 *Phase field equation to first order.* Using (5.22), we obtain from $(4.45)_I^0$ to first order:

$$\frac{\gamma}{2c_0}\big(-\partial_{zz}\Phi_1 + \kappa\Phi_0' + \psi''(\Phi_0)\Phi_1\big) + \lambda_0 - \hat{\alpha}'(\Phi_0)\boldsymbol{U}_1\cdot\boldsymbol{Q}_0 + \mathcal{M}_0(\Phi_0)\big(\boldsymbol{F}_0 - \hat{\alpha}(\Phi_0)\boldsymbol{U}_1\big)\cdot\boldsymbol{a} = 0. \tag{5.43}$$

Making use of (5.34), after taking the product of (5.43) with $\Phi_0'$ we have

$$\frac{\gamma}{2c_0}\Big(-\partial_{zz}\Phi_1\Phi_0'+\Phi_1\big(\psi'(\Phi_0)\big)'+\kappa\big|\Phi_0'\big|^2\Big)+\lambda_0\Phi_0'-\big(\hat\alpha(\Phi_0)\big)'\boldsymbol{U}_1\cdot\boldsymbol{Q}_0+\boldsymbol{Q}_0'\cdot\big(\boldsymbol{F}_0-\hat\alpha(\Phi_0)\boldsymbol{U}_1\big)=0. \tag{5.44}$$

Integrating by parts leads to

$$-\int_{\mathbb{R}}\big(\hat\alpha(\Phi_0)\big)'\boldsymbol{U}_1\cdot\boldsymbol{Q}_0\,\mathrm{dz}=\int_{\mathbb{R}}\hat\alpha(\Phi_0)(\boldsymbol{U}_1\cdot\boldsymbol{Q}_0'+\partial_z\boldsymbol{U}_1\cdot\boldsymbol{Q}_0)\,\mathrm{dz}-\big[\hat\alpha(\Phi_0)\boldsymbol{U}_1\cdot\boldsymbol{Q}_0\big]_{z=-\infty}^{z=+\infty}.$$

We use that $\hat\alpha(1)=0$, $\boldsymbol{Q}_0(z=-\infty)=\boldsymbol{q}_0^-=\boldsymbol{0}$ to deduce that the jump term is zero. Hence,

$$-\int_{\mathbb{R}}\big(\hat\alpha(\Phi_0)\big)'\boldsymbol{U}_1\cdot\boldsymbol{Q}_0\,\mathrm{dz}=\int_{\mathbb{R}}\hat\alpha(\Phi_0)(\boldsymbol{U}_1\cdot\boldsymbol{Q}_0'+\partial_z\boldsymbol{U}_1\cdot\boldsymbol{Q}_0)\,\mathrm{dz}. \tag{5.45}$$

So, from integrating (5.44) over $\mathbb{R}$ and using (5.45) we obtain

$$\int_{\mathbb{R}}\frac{\gamma}{2c_0}\Big(-\partial_{zz}\Phi_1\Phi_0'+\Phi_1(\psi'(\Phi_0))'+\kappa\big|\Phi_0'\big|^2\Big)+\lambda_0\Phi_0'\,\mathrm{dz}+\int_{\mathbb{R}}\hat\alpha(\Phi_0)\partial_z\boldsymbol{U}_1\cdot\boldsymbol{Q}_0+\boldsymbol{Q}_0'\cdot\boldsymbol{F}_0\,\mathrm{dz}=0. \tag{5.46}$$

We find that, after integrating by parts and applying matching conditions (5.16) and (5.17) for $\Phi_0$,

$$\begin{aligned}\int_{\mathbb{R}}\frac{\gamma}{2c_0}\Big(-\partial_{zz}\Phi_1\Phi_0'+\Phi_1\big(\psi'(\Phi_0)\big)'+\kappa\big|\Phi_0'\big|^2\Big)+\lambda_0\Phi_0'\,\mathrm{dz}&\\ =\frac{\gamma}{2c_0}\int_{\mathbb{R}}\partial_z\Phi_1\big(\Phi_0''-\psi'(\Phi_0)\big)+\frac{\gamma}{2c_0}[\psi'(\Phi_0)\Phi_1-\Phi_0'\partial_z\Phi_1]_{z=-\infty}^{z=+\infty}&\\ +\kappa\frac{\gamma}{2c_0}\underbrace{\int_{\mathbb{R}}\big|\Phi_0'\big|^2\,\mathrm{dz}}_{=2c_0}+\lambda_0\underbrace{\int_{\mathbb{R}}\Phi_0'\,\mathrm{dz}}_{=2}=\kappa\gamma+2\lambda_0,&\end{aligned} \tag{5.47}$$

where we made use of (5.27), the relation (5.29), and that $\kappa$ is independent of $z$. Thus it remains to identify

$$\int_{\mathbb{R}}\boldsymbol{F}_0\cdot\partial_z\boldsymbol{Q}_0+\hat\alpha(\Phi_0)\partial_z\boldsymbol{U}_1\cdot\boldsymbol{Q}_0\,\mathrm{dz}. \tag{5.48}$$

To this end, we take the scalar product of (5.42) with $\partial_z\boldsymbol{U}_1$ and use (5.24) to obtain

$$\hat\alpha(\Phi_0)\boldsymbol{Q}_0\cdot\partial_z\boldsymbol{U}_1-\mu\partial_{zz}\boldsymbol{Q}_1\cdot\partial_z\boldsymbol{U}_1=-\mu(\partial_z\boldsymbol{U}_1\cdot\boldsymbol{a})\partial_z\big(\mathcal{M}_0(\Phi_0)\partial_z\Phi_1+\mathcal{M}_1(\Phi_0)\Phi_1\Phi_0'\big). \tag{5.49}$$

Integrating (5.49) over $\mathbb{R}$ with respect to $z$, and applying integration by parts leads to

$$\begin{aligned}\int_{\mathbb{R}}\hat\alpha(\Phi_0)\boldsymbol{Q}_0\cdot\partial_z\boldsymbol{U}_1\,\mathrm{dz}&=\mu\int_{\mathbb{R}}\partial_{zz}\boldsymbol{Q}_1\cdot\partial_z\boldsymbol{U}_1-(\partial_z\boldsymbol{U}_1\cdot\boldsymbol{a})\partial_z\big(\mathcal{M}_0(\Phi_0)\partial_z\Phi_1+\mathcal{M}_1(\Phi_0)\Phi_1\Phi_0'\big)\,\mathrm{dz}\\ &=\mu\Big[\partial_z\boldsymbol{Q}_1\cdot\partial_z\boldsymbol{U}_1-(\partial_z\boldsymbol{U}_1\cdot\boldsymbol{a})\big(\mathcal{M}_0(\Phi_0)\partial_z\Phi_1+\mathcal{M}_1(\Phi_0)\Phi_1\Phi_0'\big)\Big]_{z=-\infty}^{z=+\infty}\\ &\quad-\mu\int_{\mathbb{R}}\partial_{zz}\boldsymbol{U}_1\cdot\big(\partial_z\boldsymbol{Q}_1-\boldsymbol{a}(\mathcal{M}_0(\Phi_0)\partial_z\Phi_1+\mathcal{M}_1(\Phi_0)\Phi_1\Phi_0')\big)\,\mathrm{dz}.\end{aligned} \tag{5.50}$$

Using (5.2), the matching conditions (5.16), (5.17), and (5.18) for $\Phi_0$, and (5.18) for $\boldsymbol{Q}_1$ and $\boldsymbol{U}_1$, we see that the jump term is

$$\Big[\partial_z\boldsymbol{Q}_1\cdot\partial_z\boldsymbol{U}_1-(\partial_z\boldsymbol{U}_1\cdot\boldsymbol{a})\big(\mathcal{M}_0(\Phi_0)\partial_z\Phi_1+\mathcal{M}_1(\Phi_0)\Phi_1\Phi_0'\big)\Big]_{z=-\infty}^{z=+\infty}=[\partial_{\boldsymbol{\nu}}\boldsymbol{q}_0\cdot\partial_{\boldsymbol{\nu}}\boldsymbol{u}_0]_-^+, \tag{5.51}$$

since, in the case $\mathcal{M}(\varphi) = \frac{1}{2}$, we have $\mathcal{M}_0(\Phi_0) = \frac{1}{2}$, $\mathcal{M}_1(\Phi_0)\Phi_1 = 0$, and so $\left[(\partial_z \boldsymbol{U}_1 \cdot \boldsymbol{a})\partial_z \Phi_1\right]_{z=-\infty}^{z=+\infty} = 0$. While for the case $\mathcal{M}(\varphi) = \frac{1}{c_0}\sqrt{\frac{\psi(\varphi)+\delta_\varepsilon}{2}}$, using (5.4), (5.28) and the matching conditions, we have

$$\begin{aligned}&\Big[(\partial_z \boldsymbol{U}_1 \cdot \boldsymbol{a})\big(\mathcal{M}_0(\Phi_0)\partial_z\Phi_1 + \mathcal{M}_1(\Phi_0)\Phi_1\Phi_0'\big)\Big]_{z=-\infty}^{z=+\infty}\\ &\qquad = \frac{1}{\sqrt{2}c_0}\left[(\partial_z \boldsymbol{U}_1 \cdot \boldsymbol{a})\left(\sqrt{\psi(\Phi_0)}\partial_z\Phi_1 + \frac{\psi'(\Phi_0)\Phi_1}{\sqrt{2}}\frac{\Phi_0'}{\sqrt{2\psi(\Phi_0)}}\right)\right]_{z=-\infty}^{z=+\infty} = 0.\end{aligned}$$

Meanwhile, using (5.39) and (5.25), the integral term is

$$\begin{aligned}&\int_{\mathbb{R}} \mu\partial_{zz}\boldsymbol{U}_1 \cdot \big(\partial_z \boldsymbol{Q}_1 - \boldsymbol{a}(\mathcal{M}_0(\Phi_0)\partial_z\Phi_1 + \mathcal{M}_1(\Phi_0)\Phi_1\Phi_0')\big)\,\mathrm{d}z\\ &\qquad = \int_{\mathbb{R}} -\partial_z P_0 \boldsymbol{\nu} \cdot \big(\partial_z \boldsymbol{Q}_1 - \boldsymbol{a}(\mathcal{M}_0(\Phi_0)\partial_z\Phi_1 + \mathcal{M}_1(\Phi_0)\Phi_1\Phi_0')\big)\,\mathrm{d}z = 0. \quad (5.52)\end{aligned}$$

Together with (5.14), i.e., $\boldsymbol{F}_0$ is independent of $z$, we obtain from (5.51), (5.52) that (5.48) can be expressed as

$$\begin{aligned}\int_{\mathbb{R}} \boldsymbol{F}_0 \cdot \partial_z \boldsymbol{Q}_0 + \hat{\alpha}(\Phi_0)\partial_z \boldsymbol{U}_1 \cdot \boldsymbol{Q}_0\,\mathrm{d}z &= \boldsymbol{f}_0 \cdot [\boldsymbol{q}_0]_-^+ + \mu[\partial_{\boldsymbol{\nu}}\boldsymbol{q}_0 \cdot \partial_{\boldsymbol{\nu}}\boldsymbol{u}_0]_-^+\\ &= \boldsymbol{f}_0 \cdot \boldsymbol{a} + \mu\partial_{\boldsymbol{\nu}}\boldsymbol{q}_0^+ \cdot \partial_{\boldsymbol{\nu}}\boldsymbol{u}_0^+,\end{aligned} \quad (5.53)$$

where we have used $\boldsymbol{q}_0^- = \boldsymbol{u}_0^- = \boldsymbol{0}$, and $\boldsymbol{q}_0^+ = \boldsymbol{a}$ from (5.38). Thus, we obtain from (5.46)

$$2\lambda_0 + \kappa\gamma + \boldsymbol{f}_0 \cdot \boldsymbol{a} + \mu\partial_{\boldsymbol{\nu}}\boldsymbol{q}_0^+ \cdot \partial_{\boldsymbol{\nu}}\boldsymbol{u}_0^+ = \boldsymbol{0} \text{ on } \Gamma, \quad (5.54)$$

which is the strong form of (2.6) taking into account the volume constraint and the additional perimeter regularization.

5.2.5 *Sharp interface limit.* In summary, we obtain the following sharp interface limit,

$$-\mu\Delta\boldsymbol{u}_0 + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_0 + \nabla p_0 = \boldsymbol{f} \text{ in } E, \quad (5.55a)$$

$$-\mu\Delta\boldsymbol{q}_0 + (\nabla\boldsymbol{u}_0)^T\boldsymbol{q}_0 - (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{q}_0 + \nabla\pi_0 = \boldsymbol{0} \text{ in } E, \quad (5.55b)$$

$$\operatorname{div}\boldsymbol{u}_0 = 0, \quad \operatorname{div}\boldsymbol{q}_0 = 0 \text{ in } E, \quad (5.55c)$$

$$\boldsymbol{u}_0 = \boldsymbol{q}_0 = \boldsymbol{0} \text{ in } B, \quad (5.55d)$$

$$\boldsymbol{u}_0 = \boldsymbol{g}, \quad \boldsymbol{q}_0 = \boldsymbol{0} \text{ on } \partial\Omega \cap E, \quad (5.55e)$$

$$\boldsymbol{u}_0 = \boldsymbol{0}, \quad \boldsymbol{q}_0 = \boldsymbol{a} \text{ on } \Gamma, \quad (5.55f)$$

together with the gradient equation

$$\kappa\gamma + 2\lambda_0 + \mu\partial_{\boldsymbol{\nu}}\boldsymbol{q}_0 \cdot \partial_{\boldsymbol{\nu}}\boldsymbol{u}_0 + \boldsymbol{f} \cdot \boldsymbol{a} = 0 \text{ on } \Gamma. \quad (5.56)$$

REMARK 5.1 (Linear scaling for the correction constant $\delta_\varepsilon$) Suppose $\delta_\varepsilon = \varepsilon$, then we observe from (5.3) that

$$\sqrt{\psi(\Phi) + \varepsilon} = \sqrt{\psi(\Phi_0)} + \varepsilon\frac{\psi'(\Phi_0)\Phi_1 + 1}{2\sqrt{\psi(\Phi_0)}} + \text{h.o.t.}.$$

That is,

$$\mathcal{M}_0(\Phi_0) = \frac{1}{\sqrt{2}c_0}\sqrt{\psi(\Phi_0)}, \quad \mathcal{M}_1(\Phi_0)\Phi_1 = \frac{1}{\sqrt{2}c_0}\frac{\psi'(\Phi_0)\Phi_1 + 1}{2\sqrt{\psi(\Phi_0)}}.$$

The presence of this extra factor of $\frac{1}{2\sqrt{\psi(\Phi_0)}}$ in $\mathcal{M}_1(\Phi_0)\Phi_1$ alters the jump term of (5.50) to

$$\begin{aligned}\Big[\partial_z \boldsymbol{Q}_1 \cdot \partial_z \boldsymbol{U}_1 - \partial_z \boldsymbol{U}_1 \cdot \boldsymbol{a}\big(\mathcal{M}_0(\Phi_0)\partial_z\Phi_1 + \mathcal{M}_1(\Phi_0)\Phi_1\Phi_0'\big)\Big]_{z=-\infty}^{z=+\infty}\\ = \big[\partial_{\boldsymbol{\nu}}\boldsymbol{q}_0 \cdot \partial_{\boldsymbol{\nu}}\boldsymbol{u}_0\big]_-^+ - \frac{\boldsymbol{a}}{2c_0}\cdot\left[\frac{\Phi_0'}{\sqrt{2\psi(\Phi_0)}}\partial_z\boldsymbol{U}_1\right]_{z=-\infty}^{z=+\infty} = \partial_{\boldsymbol{\nu}}\boldsymbol{q}_0\cdot\partial_{\boldsymbol{\nu}}\boldsymbol{u}_0 - \frac{\boldsymbol{a}}{2c_0}\cdot\partial_{\boldsymbol{\nu}}\boldsymbol{u}_0,\end{aligned}$$

where we have used (5.28). Thus, instead of (5.54), we obtain

$$\kappa\gamma + 2\lambda_0 + \mu\,\partial_{\boldsymbol{\nu}}\boldsymbol{q}_0\cdot\partial_{\boldsymbol{\nu}}\boldsymbol{u}_0 + \frac{\mu}{2c_0}\partial_{\boldsymbol{\nu}}\boldsymbol{u}_0\cdot\boldsymbol{a} + \boldsymbol{f}\cdot\boldsymbol{a} = 0 \text{ on } \Gamma.$$

Thus, to formally recover the strong form of (2.6), we need to choose $\delta_\varepsilon \sim \mathcal{O}(\varepsilon^k)$ for $k > 1$.

## 6. Numerical computations

In this section we investigate the phase field approach numerically. We minimize the drag and maximize the lift-to-drag ratio of an obstacle in outer flow and apply both phase field approximations of the corresponding surface functionals. Concerning numerical results in the literature we refer to the minimization of the drag functional in [6, 26], where a sharp interface approach is used. In [21] the porous medium approach is used, where the authors argue, that the term $\alpha_\varepsilon \boldsymbol{u}_\varepsilon$ is a valid approximation for the hydrodynamic force.

Let us start with defining the free energy $\psi$. Here we use

$$\begin{aligned}\tilde{\psi}(y) &= \frac{s}{2}\big(\max^2(0, y-1) + \min^2(0, y+1)\big) + \frac{1}{2}(1-y^2),\\ \psi(y) &= \tilde{\psi}\left(\frac{s}{s-1}y\right) + \frac{1}{2(s-1)}.\end{aligned} \tag{6.1}$$

Note that $\tilde{\psi}$ can be obtained by using a Moreau–Yosida relaxation of the double obstacle free energy (3.12) with the relaxation (or penalization) parameter $s \gg 1$, and the scaling of the argument and the shifting are chosen such that $\psi$ has its minima at $y = \pm 1$ with $\psi(\pm 1) = 0$. We further introduce the convex–concave splitting

$$\begin{aligned}\psi &= \psi_+ + \psi_-,\\ \psi_+(y) &= \frac{s}{2}\left(\max^2\Big(0, \frac{s}{s-1}y - 1\Big) + \min^2\Big(0, \frac{s}{s-1}y + 1\Big)\right),\\ \psi_-(y) &= \frac{1}{2}\left(1 - \Big(\frac{s}{s-1}y\Big)^2\right) + \frac{1}{2(s-1)},\end{aligned}$$

where $\psi_+$ is the convex part of $\psi$ and $\psi_-$ is its concave part. Next we define the interpolation function $\alpha_\varepsilon$ as

$$\alpha_\varepsilon(y) = \frac{\overline{\alpha}}{\varepsilon}\begin{cases} 0 & \text{if } y \geq 1, \\ \frac{1}{(1-\theta)(3+\theta)}(y-1)^2 & \text{if } 1 > y \geq \theta, \\ \min\left(1+\frac{2}{3+\theta}, 1-\frac{2}{3+\theta}(y+1)\right) & \text{if } \theta > y, \end{cases} \tag{6.2}$$

where $\overline{\alpha}$ is a given constant, and we choose $\theta = 0.99$. This function $\alpha_\varepsilon(y)$ describes a linear function between $y = -2$ and $y = \theta$ and has a quadratic extension between $y = \theta$ and $y = 1$. We fulfil Assumption 3.2 with $s_a = -2$ and $s_b = 1$. Note that we do not fulfil the regularity $\alpha_\varepsilon \in C^{1,1}(\mathbb{R})$ at $s_a$. But this is not a severe violation since in practice it holds that $-2 < \varphi_\varepsilon$ and we can control the violation of the bound $-1 \leq \varphi_\varepsilon$ by choosing an appropriate relaxation parameter $s$.

For solving the optimization problem (3.8) one can use a gradient flow approach. Introducing an artificial time variable $t$, the gradient flow of $j_\varepsilon^h(\varphi) = \mathcal{J}_\varepsilon^h(\varphi, \boldsymbol{S}_\varepsilon(\varphi))$ is given by

$$\langle \partial_t \varphi_\varepsilon, \zeta \rangle_{\mathcal{X}} = -\mathrm{D} j_\varepsilon^h(\varphi_\varepsilon)(\zeta)$$

for all $\zeta \in H^1(\Omega) \cap L^\infty(\Omega)$ with $\int_\Omega \zeta \,\mathrm{dx} = 0$ with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{X}}$. Due to the volume constraint, we choose the mass conserving inner product

$$\langle \varphi, \zeta \rangle_{\mathcal{X}} = \int_\Omega \left((-\Delta)^{-1}\varphi\right)\eta \,\mathrm{dx},$$

which leads to the well-known $H^{-1}$-gradient flow approach. In turn, we obtain the following system from (4.32) (see also [15] for more details),

$$\begin{aligned} \partial_t \varphi_\varepsilon &= \Delta w_\varepsilon, \\ w_\varepsilon &= -\gamma\varepsilon\Delta\varphi_\varepsilon + \frac{\gamma}{\varepsilon}\psi'(\varphi_\varepsilon) + \alpha_\varepsilon'(\varphi_\varepsilon)\left(\frac{1}{2}|\boldsymbol{u}_\varepsilon|^2 - \boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon\right) + J_\varphi, \\ J_\varphi &= \mathcal{M}'(\varphi_\varepsilon)h(x, \nabla \boldsymbol{u}_\varepsilon, p_\varepsilon, \nabla\varphi_\varepsilon) - \operatorname{div}\left(\mathcal{M}(\varphi_\varepsilon)\mathrm{D}_4 h(x, \nabla \boldsymbol{u}_\varepsilon, p_\varepsilon, \nabla\varphi_\varepsilon)\right), \end{aligned} \tag{6.3}$$

where $\boldsymbol{u}_\varepsilon$ is obtained from (3.4), $\boldsymbol{q}_\varepsilon$ is obtained from (4.27) and $J_\varphi$ abbreviates the terms arising from the differentiation of the functional $h$, as shown in Theorem 4.10. Note that we include the factor $\frac{1}{2c_0}$ into the parameter $\gamma$. The gradient flow approach allows us to use nonlinear parts of the gradient, for example the derivative of $\psi_+$, implicitly in time in a time stepping scheme, which for the chosen free energy is favorable in view of stability reasons.

After time discretization with variable time step size $\tau^{k+1}$ we at each time instance solve the following problem. Given $\varphi_\varepsilon^k$, find $\varphi_\varepsilon^{k+1}$, $w_\varepsilon^{k+1}$, $\boldsymbol{u}_\varepsilon$, $p_\varepsilon$, $\boldsymbol{q}_\varepsilon$, and $\pi_\varepsilon$ fulfilling the primal system

$$\begin{aligned} \alpha_\varepsilon(\varphi_\varepsilon^k)\boldsymbol{u}_\varepsilon - \mu\Delta\boldsymbol{u}_\varepsilon + (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon &= \boldsymbol{f} \text{ in } \Omega, \\ \operatorname{div} \boldsymbol{u}_\varepsilon &= 0 \text{ in } \Omega, \\ \boldsymbol{u}_\varepsilon &= \boldsymbol{g} \text{ on } \partial\Omega, \end{aligned} \tag{6.4}$$

the adjoint system

$$\begin{aligned} \alpha_\varepsilon(\varphi_\varepsilon^k)\boldsymbol{q}_\varepsilon - \mu \operatorname{div}\left(\nabla \boldsymbol{q}_\varepsilon + (\nabla \boldsymbol{q}_\varepsilon)^T\right) + (\nabla \boldsymbol{u}_\varepsilon)^T \boldsymbol{q}_\varepsilon - (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{q}_\varepsilon + \nabla\pi_\varepsilon & \\ = \alpha_\varepsilon(\varphi_\varepsilon^k)\boldsymbol{u}_\varepsilon - \operatorname{div}\left(\mathcal{M}(\varphi_\varepsilon^k)\mathrm{D}_2 h(x, \nabla \boldsymbol{u}_\varepsilon, p_\varepsilon, \nabla\varphi_\varepsilon^{k+1})\right) & \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{q}_\varepsilon = -\mathcal{M}(\varphi_\varepsilon^k)\mathrm{D}_3 h(x, \nabla \boldsymbol{u}_\varepsilon, p_\varepsilon, \nabla\varphi_\varepsilon^{k+1}) + \vartheta_\varepsilon & \quad \text{in } \Omega, \\ \boldsymbol{q}_\varepsilon = \boldsymbol{0} & \quad \text{on } \partial\Omega, \end{aligned} \tag{6.5}$$

and the Cahn–Hilliard system

$$
\begin{aligned}
\varphi_\varepsilon^{k+1} &= \tau^{k+1}\Delta w_\varepsilon^{k+1} + \varphi_\varepsilon^k && \text{in } \Omega,\\
w_\varepsilon^{k+1} &= -\gamma\varepsilon\Delta\varphi_\varepsilon^{k+1} + \frac{\gamma}{\varepsilon}\left(\psi_+'(\varphi_\varepsilon^{k+1}) + \psi_-'(\varphi_\varepsilon^k)\right)\\
&\qquad + \frac{1}{2}\alpha_\varepsilon'(\varphi_\varepsilon^{k+1})\,|\boldsymbol{u}_\varepsilon|^2 - \alpha_\varepsilon'(\varphi_\varepsilon^k)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon + J_\varphi && \text{in } \Omega,\\
J_\varphi &= \mathcal{M}'(\varphi_\varepsilon^k)h(x,\nabla\boldsymbol{u}_\varepsilon,p_\varepsilon,\nabla\varphi_\varepsilon^{k+1})\\
&\qquad - \operatorname{div}\left(\mathcal{M}(\varphi_\varepsilon^k)\mathrm{D}_4h(x,\nabla\boldsymbol{u}_\varepsilon,p_\varepsilon,\nabla\varphi_\varepsilon^{k+1})\right),\\
0 &= \gamma\varepsilon\nabla\varphi_\varepsilon^{k+1}\cdot\boldsymbol{\nu}_{\partial\Omega} + \mathcal{M}(\varphi_\varepsilon^k)\boldsymbol{\nu}_{\partial\Omega}\cdot D_4h(x,\nabla\boldsymbol{u}_\varepsilon,p_\varepsilon,\nabla\varphi_\varepsilon^{k+1}) && \text{on } \partial\Omega,\\
0 &= \nabla w_\varepsilon^{k+1}\cdot\boldsymbol{\nu}_{\partial\Omega} && \text{on } \partial\Omega.
\end{aligned}
\tag{6.6}
$$

As noted above, we evaluate $\psi_+'$ at the new time instance for stability reasons.

For the spatial discretization piecewise linear and globally continuous finite elements are used for the variables $\varphi_\varepsilon^{k+1}$, $w_\varepsilon^{k+1}$, $p_\varepsilon$, and $\pi_\varepsilon$, while piecewise quadratic and globally continuous elements are used for $\boldsymbol{u}_\varepsilon$ and $\boldsymbol{q}_\varepsilon$. The meshes are adapted using the jumps of the normal derivative of $\varphi_\varepsilon^{k+1}$ and $w_\varepsilon^{k+1}$ over edges of the underlying discretization mesh, see [8, 30], together with a Dörfler marking [10].

### 6.1 *Minimization of the hydrodynamic force of an obstacle*

We investigate the minimization of the drag of an obstacle of fixed area in a channel flow with block inflow profile.

The computational domain is $\Omega = (0,1.7)\times(0,0.4)$. The initial phase field $\varphi^0$ is defined as a circle of radius $r = 0.05$ with center at $M = (0.5,0.2)$. The boundary velocity is set to $\boldsymbol{g}(x,y) = (1,0)^T$. We fix $\delta_\varepsilon = 0$, $s = 1\times10^6$, $\boldsymbol{f}\equiv\boldsymbol{0}$ and set

$$
\tau^{k+1} := \xi\min_T(h_T\|\nabla w_\varepsilon^k\|_{L^2(T)}^{-1}),
$$

where the minimization is carried out over all triangles $T$. Here, the diameter of triangle $T$ is denoted by $h_T$, and $\xi$ is a positive scaling parameter typically set to $\xi = 5$. This CFL-like condition prevents the interfacial region from moving too fast for the adaptation process.

We restate the definition of the phase field approximation of the hydrodynamic force in a direction $\boldsymbol{a}$ as

$$
F^{\boldsymbol{a}} := \int_\Omega \mathcal{M}(\varphi_\varepsilon)\nabla\varphi_\varepsilon\cdot\left(\mu\left(\nabla\boldsymbol{u}_\varepsilon + (\nabla\boldsymbol{u}_\varepsilon)^T\right) - p_\varepsilon\,\mathrm{I}\right)\boldsymbol{a}\,\mathrm{d}x\,. \tag{6.7}
$$

When $\boldsymbol{a}$ is equal to the direction of the flow, i.e., $\boldsymbol{a} = (1,0)^T$, we denote the resulting approximation as $F^D$, which corresponds to the drag of the obstacle. Meanwhile, if $\boldsymbol{a}$ is perpendicular to the direction of the flow, i.e., $\boldsymbol{a} = (0,1)^T$, then we denote the resulting approximation as $F^L$, which corresponds to the lift of the obstacle.

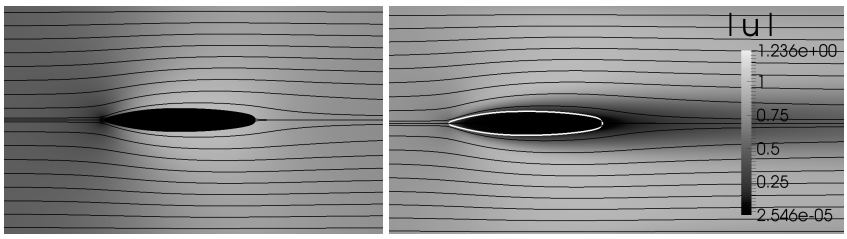


FIG. 1. Result for minimizing the drag using $\mathcal{M}(\varphi_\varepsilon) = \frac{1}{c_0}\sqrt{\frac{\psi(\varphi_\varepsilon)}{2}}$. In the left plot we show the obstacle (i.e., $\varphi_\varepsilon \le 0$) and streamlines of $\boldsymbol{u}_\varepsilon$ in black, and the pressure outside of the obstacle in gray. Darker gray means higher pressure. On the right we show $|\boldsymbol{u}_\varepsilon|$ in gray, where darker gray means lower velocity. The isoline $\varphi_\varepsilon \equiv 0$ is shown in white and again streamlines are displayed in black. The results for $\mathcal{M}(\varphi_\varepsilon) = \frac{1}{2}$ are visually indistinguishable from these results. Note that we only show the computational domain in the neighborhood of the obstacle.

From (4.42) and (4.45), the terms arising from the derivatives of $h$ in systems (6.5) and (6.6) in the present setting are given as

$$
\begin{aligned}
(-\operatorname{div}\left(\mathcal{M}(\varphi_\varepsilon^k)\mathrm{D}_2 h\right), \boldsymbol{v}) &= \mu \int_\Omega \mathcal{M}(\varphi_\varepsilon^k)\nabla\varphi_\varepsilon^k \cdot \left(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\right)\boldsymbol{a}\,\mathrm{dx} && \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),\\
(-\mathcal{M}(\varphi_\varepsilon^k)\mathrm{D}_3 h + \vartheta_\varepsilon, \eta) &= \int_\Omega \left(\mathcal{M}(\varphi_\varepsilon^k)\nabla\varphi_\varepsilon^k \cdot \boldsymbol{a} - \fint_\Omega \mathcal{M}(\varphi_\varepsilon^k)\nabla\varphi_\varepsilon^k \cdot \boldsymbol{a}\,\mathrm{dx}\right)\eta\,\mathrm{dx} && \forall \eta \in L_0^2(\Omega),\\
(J_\varphi, \zeta) &= \int_\Omega \mathcal{M}(\varphi_\varepsilon^k)\left(-\alpha_\varepsilon(\varphi_\varepsilon^k)\boldsymbol{u}_\varepsilon - (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon\right)\cdot \boldsymbol{a}\zeta\,\mathrm{dx} && \forall \zeta \in H^1(\Omega).
\end{aligned}
$$

Next, we report on the numerical results for the case of minimizing $F^D$. The parameters are chosen as $\varepsilon = 0.00025$, $\overline{\alpha} = 0.03$, $\mu = 0.001$, and $\gamma = 0.01$. We note that we use path-following with respect to the value of $\mu$, starting from $\mu = 0.01$, and also for the value of $\gamma$, starting from $\gamma = 0.1$. In Figure 1 we show the results obtained with our approach.

The drag for $\mathcal{M}(\varphi_\varepsilon) = \frac{1}{c_0}\sqrt{\frac{\psi(\varphi_\varepsilon)}{2}}$ is given by $F^D = 3.9454 \times 10^{-2}$ $(3.9492 \times 10^{-2})$, and for $\mathcal{M}(\varphi_\varepsilon) = \frac{1}{2}$ we have $F^D = 3.9117 \times 10^{-2}$ $(3.9499 \times 10^{-2})$. In brackets we give the drag obtained by evaluating the surface formulation (1.3) over the isoline $\varphi_\varepsilon \equiv 0$. We see that both formulations give very similar results.

### 6.2 *Maximization of the lift-drag ratio of an obstacle*

Based on the results of the previous section we now investigate the maximization of the lift-to-drag ratio given by

$$R := F^L / F^D,$$

To this end, we consider replacing $\int_\Omega \mathcal{M}(\varphi_\varepsilon) h(x, \nabla \boldsymbol{u}_\varepsilon, p_\varepsilon, \nabla\varphi_\varepsilon)\,\mathrm{dx}$ in (3.8) with

$$-\frac{\int_\Omega \mathcal{M}(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot (\mu(\nabla \boldsymbol{u}_\varepsilon + (\nabla \boldsymbol{u}_\varepsilon)^T) - p_\varepsilon \mathrm{I})\boldsymbol{a}^\perp\,\mathrm{dx}}{\int_\Omega \mathcal{M}(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot (\mu(\nabla \boldsymbol{u}_\varepsilon + (\nabla \boldsymbol{u}_\varepsilon)^T) - p_\varepsilon \mathrm{I})\boldsymbol{a}\,\mathrm{dx}},$$

where $\boldsymbol{a} = (1,0)^T$ and $\boldsymbol{a}^\perp = (0,1)^T$.

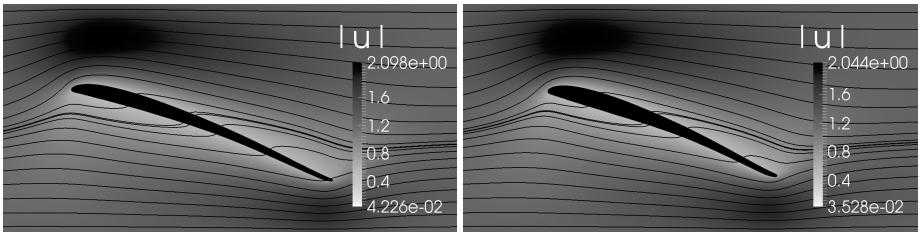


FIG. 2. Result for maximizing the lift-to-drag ratio using $\mathcal{M}(\varphi_\varepsilon) = \frac{1}{c_0}\sqrt{\frac{\psi(\varphi_\varepsilon)}{2}}$ (left) and $\mathcal{M}(\varphi_\varepsilon) = \frac{1}{2}$ (right). The obstacle (i.e., $\varphi_\varepsilon \le 0$) and streamlines are shown in black and the velocity magnitude in gray. Darker gray means larger velocity. Note that we only show the computational domain in the neighborhood of the obstacle.

The numerical set-up is the same as in the previous section and the parameters are chosen as $\varepsilon = 0.0005$, $\overline{\alpha} = 4$, $\mu = 1/15$, and $\gamma = 0.3$. In this example we fix the y-coordinate of the center of mass of the obstacle by a Lagrange multiplier approach in order to keep it fixed at the initial position. We define the center of mass of the obstacle as

$$\mathrm{com} = \frac{\int_\Omega \frac{1-\varphi_\varepsilon}{2} x \,\mathrm{dx}}{\int_\Omega \frac{1-\varphi_\varepsilon}{2} \,\mathrm{dx}}.$$

In Figure 2 we show results for this parameter set. We observe the expected optimal shape for both formulations, but for $\mathcal{M}(\varphi_\varepsilon) = \frac{1}{c_0}\sqrt{\frac{\psi(\varphi_\varepsilon)}{2}}$ we obtain a longer and thinner obstacle. The lift-to-drag ratio for $\mathcal{M}(\varphi_\varepsilon) = \frac{1}{c_0}\sqrt{\frac{\psi(\varphi_\varepsilon)}{2}}$ is $R = 1.1104$, and for $\mathcal{M}(\varphi_\varepsilon) = \frac{1}{2}$ it is $R = 0.9885$. We stress that, here we calculate with a rather small value of viscosity $\mu = 1/15$ and that the minimal magnitude of velocity inside the obstacle is $4 \times 10^{-2}$, which is rather large. However, we think that the results serve as a promising starting point for further investigations.

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