

Interfaces determined by capillarity and gravity in a two-dimensional porous medium

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We consider a two-dimensional model of a porous medium where circular grains are uniformly distributed in a squared container. We assume that such medium is partially filled with water and that the stationary interface separating the water phase from the air phase is described by the balance of capillarity and gravity. Taking the unity as the average distance between grains, we identify four asymptotic regimes that depend on the Bond number and the size of the container. We analyse, in probabilistic terms, the possible global interfaces that can form in each of these regimes. In summary, we show that in the regimes where gravity dominates the probability of configurations of grains allowing solutions close to the horizontal solution is close to one. Moreover, in such regimes where the size of the container is sufficiently large we can describe deviations from the horizontal in probabilistic terms. On the other hand, when capillarity dominates while the size of the container is sufficiently large, we find that the probability of finding interfaces close to the graph of a given smooth curve without self-intersections is close to one.

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1. Introduction

In this paper, we present a two-dimensional model of a porous medium that consists of a squared container partially filled with a liquid and where the pore throats are formed by fixed circular grains that are uniformly distributed. We analyse in probabilistic terms the structure of the stationary interfaces that separate the liquid from air. We consider several parameter regimes where, essentially, either gravity or capillarity dominates.

In this model the form of the interfaces, both at the pore scale and at the macro scale, is determined by the balance between gravity and capillary forces. For a given volume of liquid, these

interfaces are the union of single interfaces (see [6]) that meet the solid matrix at given contact angles following Young's law (see, e.g., [1]).

The quasi-stationary system seems to evolve by means of a sequence of local minimizers of the energy. The energy for steady states is the sum of surface energy and potential gravitational energy. The contact angle conditions can be encoded in the different interfacial energies. One of the most relevant features in these flows is the existence of a large number of equilibria that are local minimizers of the total energy of the system. In quasi-stationary situations, if the volume of the liquid phase changes slowly, transitions occur between different minimizers. Understanding the evolution of such a system would require the description of the global motion of the interface that explains the jumps between these multiple local energy minimizers. A first step towards this goal consists in understanding the structure of the minimizers.

The aim of this paper is thus to describe the structure of equilibrium interfaces for different parameter regimes in the model described above. There are several characteristic lengths in the problem, namely, the distance between grains, their size, as well as the macroscopic size of the system. On the other hand, there are other length scales (the capillary length and the inverse of the Bond number) that, as we shall see, give the distances at which either surface tension or gravity dominate (typically, surface tension dominates for small distances and gravity for large ones).

This study is motivated by the experiments presented in Furuberg-Maloy-Feder [11] and [12]. The authors reproduced Haines jumps (see [8]) in a slow drainage process in a container consisting of two plates separated by thin cylinders that act as obstacles or 'grains'. We recall that Haines jumps are abrupt changes of the pressure in the liquid phase due to sudden changes in the geometry of the interface. From the mathematical point of view these jumps seem to correspond to a fast transition from one equilibrium state to a different one in the capillarity equations.

We mention that there are several mathematical studies related to the study of Multiphase Flow in Porous Media. However, much of the mathematical analysis has focused on the qualitative study of semi-empirical macroscopic descriptions of these flows (see, e.g., [2]). There are also rigorous results concerning the derivation of such macroscopic laws from the flows taking place at the pore scale. Not surprisingly, such attempts are restricted to single-phase flow (see the chapter of [15], [20]) or simplified models mimicking essential features of multi-phase flow (e.g., [9], [16], [17], [18]). Although our model falls into the second category, we believe that one can analyse simple, but not phenomenological, models in probabilistic terms to get a better understanding of the effect that processes at the pore scale have in the whole system. In this regard, our work is also related to for example [4] (and the references therein), where the propagation of fronts through a random field of obstacles is analysed in relation to hysteresis.

In the next section we set up the mathematical model and discern the parameter regimes under consideration. We outline the contents of the paper at the end of this introduction.

1.1 *The mathematical model and main results*

We consider a two dimensional rectangular container of length L that contains N uniformly distributed circular grains. If the average distance between grains is $d > 0$ (understood as the average of the distances of every center to its nearest neighbour), then $N \propto L^2/d^2$. We then let $\nu > 0$ represent the density of grains, i.e. ν satisfies $N = \nu L^2/d^2$. We assume, for simplicity, that all grains are of equal size and we denote by $R < d/2$ their radius.

The container is partially filled with a liquid so that an interface forms separating the liquid from the air. We assume that such an interface splits the domain into two regions, i.e. it does not intersect

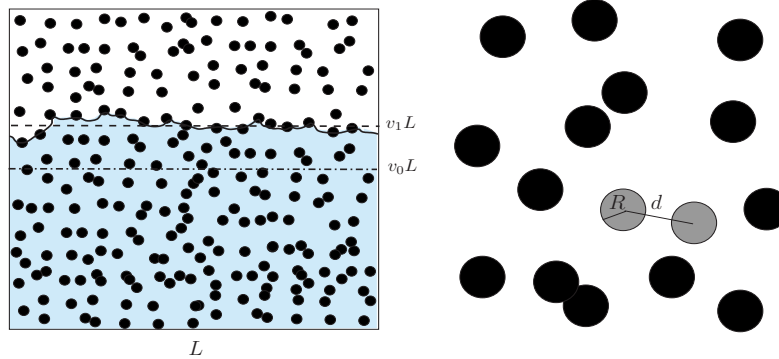


FIG. 1. Schematic picture showing the problem set-up and the three relevant length scales

itself and there are no isolated bubbles of either of the fluid phases. Let us denote by Γ the curve that joins grains and forms the interface, that is the union of capillary(-pressure) curves γ joining every two grains that meet the interface and that we call *elemental components* of Γ . A sketch of this setting is shown in Figure 1.

Let us fix the canonical basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ such that $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$, then \mathbf{e}_2 points in the opposite direction to the gravitational field. We shall denote the coordinates of a point $x \in \mathbb{R}^2$ by $x = (x_1, x_2)$ in this basis. The symbol $\|\cdot\|$ denotes the euclidean norm in \mathbb{R}^2 . The $|\cdot|$ will denote two things; the absolute value when applied to a coordinate or real number, and when applied to a subset of $[0, L]^2$ it will denote its Lebesgue measure.

Every elemental component of Γ , γ , satisfies the capillary equation (see, e.g., [6])

$$\sigma H = \rho g x_2 - p,$$

that must be complemented with Young's condition at the grains and/or the walls of the container. We shall assume that the contact angle is the same at all solid surfaces. Here H denotes the signed curvature of each γ that forms Γ , ρ is the (constant) density of the liquid, g is the gravity constant and σ denotes surface tension. Here p is the pressure associated to the volume fraction of the domain (including grains), $v_1 \in (0, 1)$, occupied by the liquid. Then

$$p = \rho g \lambda \quad \text{with} \quad \lambda = v_1 L.$$

We take as unit of length the average distance between the grains of the system, d . Up-scaling the length variables with d , we obtain that Γ satisfies the non-dimensional equation

$$H = B (x_2 - \lambda), \tag{1.1}$$

subject to a contact angle boundary condition that we specify below. The non-dimensional parameter B is the Bond number, given by

$$B = \frac{\rho g d^2}{\sigma}, \tag{1.2}$$

and measures the balance between gravitational forces and surface tension in this units of length.

For simplicity, we use the same notation for the non-dimensional variables. Also we let L and R be the non-dimensional size of the domain (i.e. L/d) and grain radius (i.e. R/d), respectively. This, in particular, means that, with this notation, $\lambda = v_1 L$ in (1.1). We observe that v_1 depends on the configuration of grains, so if we let $v_0 \in [0, 1]$ be the volume fraction of the domain occupied by the liquid, the volume occupied by the liquid is $v_0 L^2$ and then

$$v_1 L^2 \approx v_0 L^2 + n\pi R^2 \quad (1.3)$$

is the volume below the interface Γ , if n is the number of grains below Γ .

We assume thereafter that v_0 is fixed, and we take the horizontal line $x_2 = v_1 L (= \lambda)$ as a reference height for each configuration of grains. We observe that this is a solution of (1.1) with contact angle $\alpha = 0$ at the walls according to the definition of α given below.

We also observe that with the new notation, where L and R are non-dimensional lengths, then the number of grains in the container is $N = \nu L^2$. Henceforth, with this scaling we can think of the equivalent problem where the liquid and air occupy some space in a square domain $[0, L]^2$ with $L \gg 1$, where νL^2 grains are uniformly distributed with an average distance 1 between them.

We indicate the position of a center by the variable $\xi = (\xi^{(1)}, \xi^{(2)}) \in \mathbb{R}^2$ and write its boundary in the form $\partial B_R(\xi)$. We take \mathbf{n} as the unit normal vector to Γ pointing in the direction of air, and \mathbf{t} the tangent vector to Γ such that $\{\mathbf{t}, \mathbf{n}\}$ has the same orientation as the canonical basis (i.e. that $\det(\mathbf{t}, \mathbf{n}) = 1$).

Let us denote by \mathbf{N} the outer normal vector to $\partial B_R(\xi)$ at a point P where Γ and $\partial B_R(\xi)$ intersect (i.e. P is a contact point). Let us denote by \mathbf{t}_c the value of \mathbf{t} at P . Then, the Young condition is defined as follows: if $\mathbf{N} \cdot \mathbf{t}_c \geq 0$,

$$\mathbf{N} \wedge \mathbf{t}_c = \sin(\alpha) \mathbf{e}_3 \quad (1.4)$$

where $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the contact angle and where $\mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2$. If $\mathbf{N} \cdot \mathbf{t}_c < 0$, then with the same choice of \mathbf{e}_1 , we set $\mathbf{N} \wedge \mathbf{t}_c = \sin(\alpha - \pi) \mathbf{e}_3$.

We shall distinguish different parameter regimes. The following observations are then in order. We recall that B is the Bond number with respect to the scale given by the distance between grains. In the case $B \gg 1$ the gravitational effects are the dominant ones. In this limit the liquid-gas interfaces tend to be horizontal lines, corrected by boundary layers of height $1/\sqrt{B}$. In the limit $B \ll 1$, the interfaces are dominated by surface tension and this allows vertical variations in the connection between grains (of the order of the distance between the grains). Thus, unlike in the case $B \gg 1$, if $B \ll 1$ the connections can take place in any direction, not just horizontally.

The above regimes take into account the size of B only, but we shall relate it to L as well. In particular, the limit $L \rightarrow \infty$ gives an ever increasing number of grains and can in principle give rise to ‘anomalous’ situations (not sketched above) that we analyse in detail in Section 3 and that we summarise below.

We distinguish several cases depending on the relative size of B and L . In the cases where $B \gg 1$ we also assume that $R \ll 1$ (this means that in the dimensional problem, d is much larger than R), and we compare L to $\min\{\sqrt{B}, R^{-1}\}$, which relates the height of the boundary layers to R :

1. *The case $B \gg 1, R \ll 1$:*

- (a) *The regime $1 \lesssim L \ll \min\{\sqrt{B}, R^{-1}\}$ (Section 3.1, Theorem 3.4):* In this case gravity dominates and the size of the container is not sufficiently large to allow significant capillarity effects. We find that as $B \rightarrow \infty$ there are configurations with probability close to one for which the solution interfaces tend to the horizontal solution. One can in fact distinguish the

cases $L = O(1)$ and $L \rightarrow \infty$, with either $R \ll 1/\sqrt{B}$ or $R \gg 1/\sqrt{B}$, with the same result.

- (b) *The regime $L = O(\min\{\sqrt{B}, R^{-1}\})$ (Section 3.2, Theorem 3.7):* In this case we find that the probability of finding solutions close to the horizontal line is close to one. On the other hand we also find that the probability of finding configurations that have a given number of elemental components is close to zero, but bounded from above by a Poisson distribution. Again, the distinction between the cases $1/\sqrt{B} \gg R$ and $1/\sqrt{B} \ll R$ is not significant.
 - (c) *The regime $L \gg \min\{\sqrt{B}, R^{-1}\}$ (Section 3.3, Theorem 3.12).* This is a regime where gravity in principle dominates. We find an interval of L in terms of B where with probability close to one solutions stay in a narrow strip around the horizontal reference line. For L above this range there is no guarantee that solutions are necessarily close to the horizontal reference line. If $1/\sqrt{B} \gg R$, the interval is $\sqrt{B} \ll L \ll \sqrt{B} \log(B)$, and if $1/\sqrt{B} \ll R$ with $R \rightarrow 0$, the result holds replacing $1/\sqrt{B}$ by R .
2. *The case $B = O(1)$.* In this case, capillary and gravity forces are comparable and it is not possible to derive general properties about the structure of the solution interfaces using perturbative methods. Therefore this regime is not included in our analysis.
 3. *The case $B \rightarrow 0$ (Section 3.4, Theorem 3.14).* We will see that in this case the only meaningful limit is that of $L = O(1/B)$. When $B \rightarrow 0$, the maximum fluctuations of height of a solution Γ are of the order of $1/B$. In this regime, we can show that with probability close to one, we can find interfaces that are suitably close to any smooth curve without self-intersections that connect (following its orientation) the vertical boundaries $\{0\} \times [0, L]$ to $\{L\} \times [0, L]$.

In order to understand the probabilistic problem we need to analyse the elemental components that an interface Γ can have. This is done in Section 2 where we analyse the solution curves of (1.1) and characterise them in terms of a free parameter that controls the height reached by the curve and its curvature. In fact, equation (1.1) implies that the curvature of an elemental component increases in absolute value with the distance to the reference height. The main results and their proofs are presented in Section 3, where the regimes outlined above are studied separately. Finally, Appendix A deals with the proof of a technical lemma that is needed in Section 3.3.

2. Preliminary analysis: Gluing of solutions

In this section we analyse the basic features of elemental components of an interface Γ . We first observe that equation (1.1) in the absence of boundary conditions can be solved explicitly in terms of elliptic integrals, see [13]. One can achieve this by first introducing the variable $\bar{x}_2 = x_2 - \lambda$ and using the angle $\beta \in [-\pi, \pi]$ from \mathbf{e}_1 to the tangent vector $\mathbf{t} = (x'_1(s), \bar{x}'_2(s))$ where s is the arc-length. Then multiplying (1.1) by $\bar{x}'_2(s)$ and integrating once we get

$$B \frac{\bar{x}_2^2}{2} + \cos \beta = C, \tag{2.1}$$

where C is a constant of integration that can have either sign or be zero, but clearly $C \geq -1$. Then solving (1.1), with given initial conditions, amounts to solving

$$x'_1(s) = \cos \beta \quad \bar{x}'_2(s) = \sin \beta \tag{2.2}$$

subject to the same initial conditions, together with (2.1) and a compatible constant C .

The are the obvious symmetries of equation (1.1), $(x_1, s) \rightarrow (2a - x_1, -s)$ (for any $a \in \mathbb{R}$) and $\bar{x}_2 \rightarrow -\bar{x}_2$, that will be used to simplify the computations below.

Using (2.2) and the definition of arc-length, one can write (2.1) as

$$\left[1 - \left(C - B \frac{\bar{x}_2^2}{2}\right)^2\right] (dx_1)^2 = \left(C - B \frac{\bar{x}_2^2}{2}\right)^2 (d\bar{x}_2)^2 \quad (2.3)$$

The fact that we are taking squares to go from (1.1) to (2.3) means that the set of solutions of the latter is larger than that of the former. In the construction one has to impose that the resulting curves are indeed solutions of (1.1), as we shall see. We also notice that all solutions must have $|\bar{x}_2| \leq \sqrt{2(C+1)/B}$.

Below, we find the solutions of (1.1) by gluing together pieces of curve solutions that are graphs of a function of the form $x_1(\bar{x}_2)$ and that solve (2.3). We then integrate (2.3) after taking the positive or negative square root of $(dx_1)^2/(d\bar{x}_2)^2$; we use the equations:

$$\frac{dx_1}{d\bar{x}_2} = \pm \frac{C - B \frac{\bar{x}_2^2}{2}}{\sqrt{1 - \left(C - B \frac{\bar{x}_2^2}{2}\right)^2}}. \quad (2.4)$$

Observe that β , for such graphs, is characterised by

$$\frac{dx_1}{d\bar{x}_2} = \frac{1}{\tan \beta}.$$

There are solutions, however, that cannot be constructed the way explained above. These are the solutions with $\bar{x}_2 \equiv 0$, that is the curves $(s, 0)$ (air is on top of the liquid) and $(-s, 0)$ (air is below the liquid). The former corresponds to a solution of (2.1) with $C = 1$, and the latter to a solution of (2.1) with $C = -1$.

Next, we identify the points of the solution curves where either $x_1'(s) = 0$ or $\bar{x}_2'(s) = 0$, i.e. the values of \bar{x}_2 where either the left or the right-hand side of (2.3) vanishes. For simplicity, we use the following notation:

$$x_2^{\min}(C; B) := \begin{cases} \sqrt{\frac{2(C-1)}{B}} & \text{if } C > 1, \\ 0 & \text{otherwise} \end{cases} \quad (2.5)$$

and

$$x_2^{\text{med}}(C; B) := \sqrt{\frac{2C}{B}}, \quad x_2^{\max}(C; B) := \sqrt{\frac{2(C+1)}{B}} \quad (2.6)$$

Lemma 2.1 *The points at which either $x_1'(s) = 0$ or $\bar{x}_2'(s) = 0$ are as follows:*

- (i) *If $C \geq 0$, when $x_1'(s) = 0$ then $\bar{x}_2 = \pm x_2^{\text{med}}(C; B)$, and when $x_2'(s) = 0$ then either $\bar{x}_2 = \pm x_2^{\min}(C; B)$ or $\bar{x}_2 = \pm x_2^{\max}(C; B)$, except if $0 \leq C < 1$, that $x_2'(s) = 0$ only occurs at $\bar{x}_2 = \pm x_2^{\max}(C; B)$.*
- (ii) *If $C < 0$, then there is no value of s such that $x_1'(s) = 0$, and as before, $x_2'(s) = 0$ occurs at $\bar{x}_2 = \pm x_2^{\max}(C; B)$.*

The next lemma describes the way solutions of (1.1) can be obtained by gluing solutions of (the branch equations) (2.4) at points with either vertical or horizontal tangent vector.

Lemma 2.2 (Gluing rule) *For $C > -1$, given the initial condition $(x_1(0), \bar{x}_2(0)) = (a, x_2^{\max}(C; B))$ with $a \in \mathbb{R}$ and $(x'_1(0), \bar{x}'_2(0)) = (-1, 0)$ for (1.1), this specifies the branch equation of (2.4) that the corresponding solution (1.1) satisfies. If this solution reaches a point with $x'_1(s) = 0$ then it can only be continued further by solving the same branch of (2.4). On the other hand, if the curve reaches a point with $\bar{x}'_2(s) = 0$ then it can only be continued further by solving the branch equation with opposite sign.*

Proof. A solution stops being a graph of a function of the form $x_1(\bar{x}_2)$ when, seen as a curve, $\bar{x}'_2(s) = 0$. It is easy to see that, this never occurs at a point with $\bar{x}_2 = 0$. At those points we can further continue the solution curve by gluing smoothly with another solution curve that solves (2.4) with the opposite sign (notice that these curves, as graphs, have the opposite orientation, and hence the change of sign in (2.4) implies that they have the same curvature at that point).

On the other hand, at points with $x'_1(s) = 0$, the solution is still a graph, and in principle, one could glue it to a solution that solves (2.4) with the opposite sign. However, if this happens at a point with $\bar{x}_2 \neq 0$, then H would change sign, which contradicts (1.1). If, this happens at a point with $\bar{x}_2 = 0$, then H keeps its sign, but this also contradicts (1.1). \square

We now construct pieces of curve solutions from which one can recover complete solutions of (1.1) (i.e. defined for all $s \in \mathbb{R}$) by the symmetries $(x_1, s) \rightarrow (2a - x_1, -s)$, for any $a \in \mathbb{R}$, and $\bar{x}_2 \rightarrow -\bar{x}_2$. The next lemma gives these pieces for every $C > -1$ by distinguishing the cases $C > 1$, $1 > C > 0$ and $0 \geq C > -1$.

Lemma 2.3 (The pieces) *Given the initial condition $(x_1(0), \bar{x}_2(0)) = (a, x_2^{\max}(C; B))$, with $C > -1$ and $a \in \mathbb{R}$, and $(x'_1(0), \bar{x}'_2(0)) = (-1, 0)$ for (1.1), the elemental component of the solution curve is obtained by integrating (2.4), with the minus sign, up to a $\bar{x}_2 = 0$ or up to $\bar{x}_2 = x_2^{\min}(C; B)$. Namely,*

- (i) *For $C > 1$, the curve starts with $dx_1/d\bar{x}_2 \geq 0$ and reaches $\bar{x}_2 = x_2^{\text{med}}(C; B)$ for some value of s , for which $x'_1(s) = 0$ and $\bar{x}'_2(s) = -1$. The curve continues with $dx_1/d\bar{x}_2 \leq 0$ up to a point where $\bar{x}_2 = x_2^{\min}(C; B) > 0$ for some other value of s , for which $x'_1(s) = 1$ and $\bar{x}'_2(s) = 0$.*
- (ii) *For $C = 1$, the curve starts with $dx_1/d\bar{x}_2 \geq 0$ and reaches $\bar{x}_2 = \sqrt{\frac{2}{B}}$, for which $x'_1(s) = 0$ and $\bar{x}'_2(s) = -1$. The curve continues with $dx_1/d\bar{x}_2 \leq 0$ and has an asymptote at $\bar{x}_2 = 0$, that is with $x_1(s) \rightarrow \infty$ and $\bar{x}_2(s) \rightarrow 0^+$ as $s \rightarrow \infty$.*
- (iii) *For $0 < C < 1$, the curve starts with $dx_1/d\bar{x}_2 \geq 0$ and reaches $\bar{x}_2 = x_2^{\text{med}}(C; B)$ at some value of s , for which $x'_1(s) = 0$ and $\bar{x}'_2(s) = -1$. The curve continues with $dx_1/d\bar{x}_2 \leq 0$ up to a point where $\bar{x}_2 = 0$ at some other value of s , for which $x'_1(s) > 0$ and $\bar{x}'_2(s) < 0$.*
- (iv) *For $-1 < C \leq 0$, the curve starts with $dx_1/d\bar{x}_2 \geq 0$ and reaches $\bar{x}_2 = 0$ at some value of s , for which $x'_1(s) = 0$ and $\bar{x}'_2(s) = -1$, if $C = 0$, and $x'_1(s) < 0$ and $\bar{x}'_2(s) < 0$, otherwise.*

Proof. We only prove (i), thus, let us assume that $C > 1$. We first observe that due to the fact that the maximum value that \bar{x}_2 can take is the initial one, the direction of the initial tangent vector implies that away from, but close to, the initial point $dx_1/d\bar{x}_2 > 0$. Then the curve solution can be seen as a graph of a function of the form $x_1(\bar{x}_2)$ that solves (2.4) with the minus sign. Then the solution $x_1(\bar{x}_2)$ is convex as long as $\bar{x}_2 \in (x_2^{\min}(C; B), x_2^{\max}(C; B))$ (see (1.1), that implies $\text{sign}(d^2x_1/d\bar{x}_2^2) = \text{sign}(\bar{x}_2)$). In particular, this curve reaches a point with $\bar{x}_2 = x_2^{\text{med}}(C; B)$ (i.e. with $(x'_1(s), \bar{x}'_2(s)) = (0, -1)$ for some value of s). In order to prove that the curve also reaches a point with $\bar{x}_2 = x_2^{\min}(C; B)$ (i.e. with $(x'_1(s), \bar{x}'_2(s)) = (1, 0)$ for some value of s), we construct

a curve ending up at such a point, and by a similar argument this curve also passes through a point with $\bar{x}_2 = x_2^{med}(C; B)$. Then, adjusting the initial value of x_1 so that both curves intersect at such a point (which they do tangentially), and using the uniqueness provided by Lemma 2.2 imply that they are the same solution curve.

All the other cases are straight forward by convexity of the solution curve. □

We show in Figures 2(a) (for $C > 1$), 2(b) (for $0 < C < 1$), 3 (for $C = 1$), 2(c) (for $C = 0$) and 2(d) (for $-1 < C < 0$) the pieces described in Lemma 2.3. The next lemma describes how complete curves are obtained from these pieces.

Lemma 2.4 (Complete solution curves) *For each $C > -1$, the pieces of solution curves obtained in Lemma 2.3 give rise to complete curves, as follows:*

- (i) *There are three additional pieces of curve solutions obtained by the symmetries $x_1 \rightarrow -x_1$ and $\bar{x}_2 \rightarrow -\bar{x}_2$.*

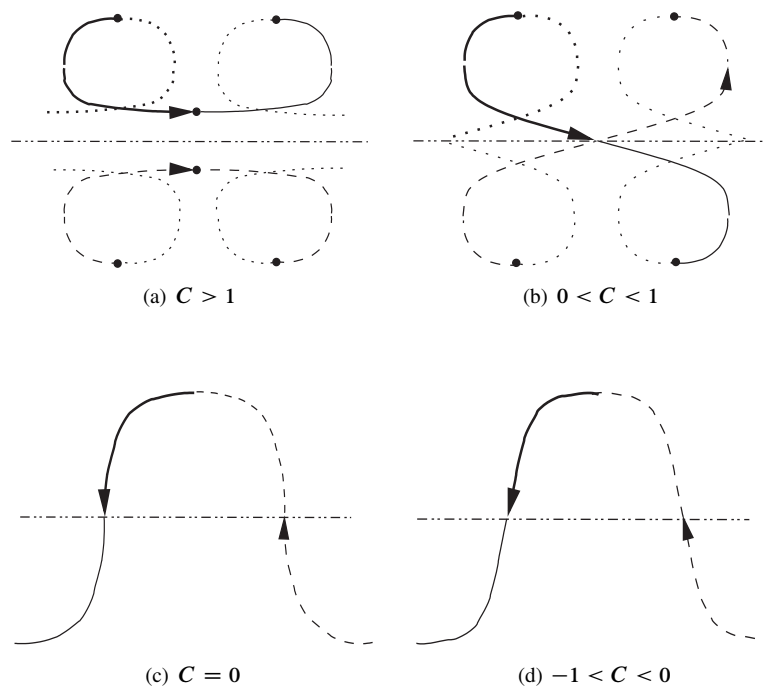


FIG. 2. This picture shows solution curves of (2.1) with $C > -1$ and $C \neq 1$. The dashed-dotted line represents the horizontal reference line $\bar{x}_2 = 0$. The thick solid lines represent the pieces of curves as specified in Lemma 2.3. The remaining solid and dashed lines give solutions obtained by the symmetries $\bar{x}_2 \rightarrow -\bar{x}_2$ and $x_1 \rightarrow -x_1$, see Lemma 2.4. Any other solution is a translation of the ones shown. Complete solutions curves to (1.1) are achieved by the smooth gluing at $\bar{x}'_2(s) = 0$ (dots), as illustrated by the dotted lines. (a) shows the case $C > 1$ and (b) shows the case in which $0 < C < 1$. According to the orientation shown in these cases the air lies 'on top' of the (slid lined) graphs. (c) shows the case $C = 0$ and (d) shows the case in which $-1 < C < 0$. In these two later cases, air lies below the graphs.

- (ii) *In all cases above the curves obtained can be glued together smoothly, translating as necessary in x_1 , at the points where $\bar{x}'_2(s) = 0$, and at points with $\bar{x}_2 = 0$ (choosing pieces with the same tangent vector at the point).*

REMARK 2.5 (i) The complete curves that we obtain are open and periodic except for a particular value of C for which the complete curves are closed: by continuity, there exists a value of $C \in (0, 1)$, say C_0 (actually $C_0 \approx 0.6522$, see [13]), such that the corresponding elemental component satisfies that $x_1(0) = x_1(x_2^{max}(C; B))$. In particular, if $C > C_0$, the curve obtained in Lemma 2.4 has, globally, increasing x_1 as $s \rightarrow \infty$, whereas the corresponding curve if $C < C_0$ has, globally, decreasing x_1 as $s \rightarrow \infty$.

Physically, one can interpret the solution curves with $C < C_0$ as describing a situation in which air lies below the liquid, and *vice versa* for the ones with $C > C_0$.

- (ii) It is also worth mentioning that the complete curve obtained for $C \in (-1, 0]$ that oscillates around $\bar{x}_2 = 0$ does not have self-intersections and can in principle connect two grains very far apart (in the horizontal direction).

The next lemma gives the range of β that complete curve have depending on C :

Lemma 2.6 (Range of β) *For the complete curves obtained in Lemma 2.4 the values of the angle β range in different intervals depending on C :*

- (i) *If $C > 1$, β takes all values in $[-\pi, \pi]$.*
- (ii) *If $C = 1$, β takes all values in $[-\pi, 0) \cup (0, \pi]$, unless the solution is the horizontal line, for which $\beta \equiv 0$.*
- (iii) *If $-1 \leq C < 1$, β takes all values in $[-\pi, -\eta] \cup [\eta, \pi]$ for some $\eta \in (0, \pi]$, and η increases when C decreases (in fact, $\eta = \pi$ if and only if $C = -1$, i.e. $\bar{x}_2 \equiv 0$ with \mathbf{n} pointing in the direction of gravity, and $\eta = \pi/2$ if and only if $C = 0$).*

REMARK 2.7 (i) We observe that by the determination of β that we have chosen, its value jumps from π to $-\pi$ at gluing points with $\bar{x}_2 = x_2^{max}(C; B)$, and, similarly, its value jumps from $-\pi$ to π at gluing points with $\bar{x}_2 = -x_2^{max}(C; B)$.

- (ii) We observe that given the contact angle α and the radius of grains R , the contact point at a grain is given by the angle ρ_c measured from the horizontal placed at the center of the grain to the contact point, and satisfies

$$\rho_c = \alpha - \beta_c \tag{2.7}$$

where β_c is β at the contact point.

Then Lemma 2.6 gives the possible range of ρ_c for a given α . In particular, if $C \geq 1$, ρ_c covers one period and a unique solution exists between two given grains that are sufficiently close by adjusting C . If $-1 \leq C < 1$, then $\rho_c \in [\alpha - \pi, \alpha - \eta] \cup [\alpha + \eta, \alpha + \pi]$. This means that there are some restrictions on the available contact surface $\partial B_R(\xi)$ for grains that lie near the horizontal if air is situated below the interface.

We give the solutions of (2.1) with $C = 1$ explicitly in the next lemma, see also Figure 3.

Lemma 2.8 *The solutions of (2.1) with $C = 1$ are explicitly given by the horizontal line $\bar{x}_2 \equiv 0$ oriented with $\mathbf{n} = \mathbf{e}_2$, and by two one-parameter families of solution branches:*

$$x_1^+(\bar{x}_2; a) = -\frac{1}{\sqrt{B}}F(\sqrt{B}\bar{x}_2) + a, \quad \bar{x}_2 \in \left(-\frac{2}{\sqrt{B}}, 0\right) \cup \left(0, \frac{2}{\sqrt{B}}\right),$$

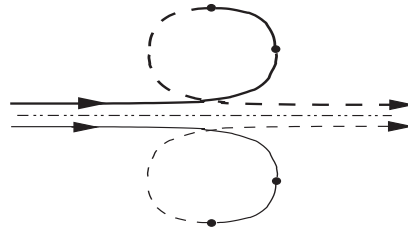


FIG. 3. This picture shows two solution branches of (2.1) (solid lines) with $C = 1$ and the solution $\bar{x}_2 \equiv 0$ (dashed-dotted line). The complete solution curves to (1.1) are achieved by smoothly gluing at $\bar{x}_2 = x_2^{med}(C; B)$ or at $\bar{x}_2 = -x_2^{med}(C; B)$ (dots), as illustrated by the dashed lines.

$$x_1^-(\bar{x}_2; a) = \frac{1}{\sqrt{B}}F(\sqrt{B}\bar{x}_2) + a, \quad \bar{x}_2 \in \left(-\frac{2}{\sqrt{B}}, 0\right) \cup \left(0, \frac{2}{\sqrt{B}}\right),$$

where the parameter a is a real number and F is given by

$$F(Z) = \frac{1}{2} \log \left(\frac{2 - \sqrt{4 - Z^2}}{2 + \sqrt{4 - Z^2}} \right) + \sqrt{4 - Z^2}, \quad Z \in (-2, 0) \cup (0, 2). \tag{2.8}$$

Moreover, F satisfies the following properties: $\lim_{Z \rightarrow -2^+} F(Z) = \lim_{Z \rightarrow -2^-} F(Z) = 0$ and $\lim_{Z \rightarrow 0^\pm} F(Z) = -\infty$. Observe also that $F'(Z) = 0$ at $Z = \pm\sqrt{2}$ and that $\lim_{Z \rightarrow -2^+} F'(Z) = \infty$, $\lim_{Z \rightarrow -2^-} F'(Z) = -\infty$ and $\lim_{Z \rightarrow 0^\pm} F'(Z) = \pm\infty$. Regarding convexity; $F''(Z) < 0$ with $Z \in (-2, 0) \cup (0, 2)$.

- REMARK 2.9 (i) We notice that the functions x_1^+ and x_1^- can be extended to $\bar{x}_2 = \pm 2/\sqrt{B}$, with $x_1^+(\pm 2/\sqrt{B}; a) = a$ and $x_1^-(\pm 2/\sqrt{B}; a) = a$, but the derivative at those points is not finite. We also observe that gluing together the branch x_1^+ and x_1^- with $\bar{x}_2 \in [-2/\sqrt{B}, 0)$, and the same value of a we obtain a complete planar curve. Similarly, x_1^+ and x_1^- with $\bar{x}_2 \in (0, 2/\sqrt{B}]$ for the same a give another complete curve, that can be simply recovered from the previous one by the transformation $\bar{x}_2 \rightarrow -\bar{x}_2$.
- (ii) We observe that the limit $C \rightarrow 1$ can be studied independently of the value of B . One can easily show that solutions with positive C , and either $C > 1$ or $C < 1$ converge to the solutions found in Lemma 2.8 on compact sets of x_2 .

The rest of the section is devoted to deriving estimates on the maximal horizontal length of a solution curve depending on the value of C . First, we derive the horizontal length at points where the height is minimum or 0 to points where the height is x_2^{med} or x_2^{max} .

Lemma 2.10 *Given a elemental component solution as in Lemma 2.3:*

- (i) *If $-1 < C \leq 0$, then*

$$\log \left(\frac{2 + \sqrt{2}}{1 + \sqrt{3}} \right) \frac{1}{\sqrt{B}} < x_1(x_2^{max}(C; B)) - x_1(0) < \frac{\pi}{\sqrt{2B}}.$$

- (ii) *If $0 < C < 1$, then*

$$\frac{1}{\sqrt{B}}h_1(C) < x_1(0) - x_1(x_2^{med}(C; B)) < \frac{1}{\sqrt{B}}h_2(C).$$

where $h_i(C) \rightarrow 0$ as $C \rightarrow 0$ and $h_i(C) = O(-\log(\sqrt{1-C}))$ as $C \rightarrow 1^-$, for both $i = 1, 2$.
 (iii) If $C > 1$, then there exist non-negative functions $h_3(C)$ and $h_4(C)$, continuous in C , such that,

$$\frac{1}{\sqrt{B}}h_3(C) < x_1(x_2^{min}(C; B)) - x_1(x_2^{med}(C; B)) < \frac{1}{\sqrt{B}}h_4(C),$$

where $h_i(C) = O(-\log \sqrt{C-1})$ as $C \rightarrow 1^+$, for both $i = 3, 4$ and $h_i(C) = O(1/\sqrt{C})$ as $C \rightarrow \infty$, for both $i = 3, 4$.

Proof. In order to prove the estimates we use Lemma 2.3 and integrate (2.4) over \bar{x}_2 in the appropriate interval where $x_1(\bar{x}_2)$ is monotone. Then we estimate the integrand (that is the right-hand side of (2.4)) with the minus sign by using straightforward bounds on the numerator and on the denominator. Namely, we estimate the parabola $C - B\frac{Z^2}{2}$ from above and below. We also write

$$\left(1 - \left(C - B\frac{Z^2}{2}\right)^2\right)^{\frac{1}{2}} = \left(1 - \left(C - B\frac{Z^2}{2}\right)\right)^{\frac{1}{2}} \left(1 + \left(C - B\frac{Z^2}{2}\right)\right)^{\frac{1}{2}}.$$

Then we either estimate the first or the second factor using the bounds on the parabola. The resulting integrals are elementary. We outline the proof of (i), the rest is left to the reader.

(i) Let us assume first that $-1 < C \leq 0$. Then

$$0 < x_1\left(\sqrt{\frac{2(C+1)}{B}}\right) - x_1(0) = - \int_0^{\sqrt{\frac{2(C+1)}{B}}} \frac{C - B\frac{Z^2}{2}}{\left(1 - \left(B\frac{Z^2}{2} - C\right)^2\right)^{\frac{1}{2}}} dZ,$$

therefore

$$x_1\left(\sqrt{\frac{2(C+1)}{B}}\right) - x_1(0) \leq \frac{1}{\sqrt{1-C}} \int_0^{\sqrt{\frac{2(C+1)}{B}}} \frac{dZ}{\left(1 + C - B\frac{Z^2}{2}\right)^{\frac{1}{2}}} = \frac{\pi}{\sqrt{2B(1-C)}},$$

and this shows the estimate on the right-hand side.

On the other hand,

$$\begin{aligned} x_1\left(\sqrt{\frac{2(C+1)}{B}}\right) - x_1(0) &\geq x_1\left(\sqrt{\frac{2(C+1)}{B}}\right) - x_1\left(\sqrt{\frac{C+1}{B}}\right) \\ &\geq \frac{1-C}{\sqrt{2(C+1)}} \int_{\sqrt{\frac{C+1}{B}}}^{\sqrt{\frac{2(C+1)}{B}}} \frac{dZ}{\left(1 - C + B\frac{Z^2}{2}\right)^{\frac{1}{2}}} \\ &= \frac{1-C}{\sqrt{B(C+1)}} \log\left(\frac{\sqrt{2(C+1)} + 2}{\sqrt{C+1} + \sqrt{3-C}}\right) > 0 \end{aligned}$$

One can check that the right-hand tends to $(\sqrt{2}-1)/\sqrt{B}$ as $C \rightarrow -1$, this implies continuity of this term for $C \in [-1, 0]$. Its minimum there is achieved at $C = 0$, computing its value gives the result.

□

We now give the estimate on the horizontal length from points at height x_2^{med} to points at height x_2^{max} .

Lemma 2.11 *For any $C > 0$ the following holds*

$$\frac{1}{\sqrt{B}}h_5(C) \leq x_1(x_2^{max}(C; B)) - x_1(x_2^{med}(C; B)) \leq \frac{1}{\sqrt{B}}h_6(C)$$

where $h_5(C)$ and $h_6(C)$ decrease for $C \in [1, +\infty)$, $h_5(C), h_6(C) \rightarrow 0$ as $C \rightarrow +\infty$ and $h_5(C)$ tends to a positive constant as $C \rightarrow 0$.

We now adapt these estimates to actual curves that solve (1.1) subject to a Young condition on the grains connected. Before that we distinguish two types of solution curves.

DEFINITION 2.12 Let p_c and $q_c \in \mathbb{R}^2$ be the first and second, following orientation, contact points of a solution curve γ . We say that γ goes from left to right if $p_c^{(1)} \leq q_c^{(1)}$ and write γ_{lr} . Similarly, we say that γ goes from right to left if $p_c^{(1)} \geq q_c^{(1)}$, and we write γ_{rl} .

We also denote by $\beta(p_c)$ the angle that \mathbf{t} forms with the horizontal at the contact point $p_c \in \mathbb{R}^2$. With this we define the parameter

$$d(v_1, R, C, p_c, q_c) := 2R + \frac{|\cos \beta(p_c) - \cos \beta(q_c)|}{(2B(C + 1))^{\frac{1}{2}}} > 0. \tag{2.9}$$

Moreover, we shall indicate with a super-index + that γ lies above $x_2 = v_1L$ (since it then has positive curvature) and with a super-index - that γ lies below $x_2 = v_1L$ (negative curvature). When γ crosses $x_2 = v_1L$ (its curvature changes sign across $x_2 = v_1L$) we indicate it with the super-index c .

Lemma 2.13 *Let ξ and ζ be centers of two grains connected by a solution γ of (1.1) with a Young condition at contact points p_c and q_c (following orientation) and that solves (2.1) with $C > 1$, then:*

(i) *If $\gamma = \gamma_{lr}$,*

$$d(v_1, R, C, p_c, q_c) < \|\xi - \zeta\| < 2R + 2 \left(\frac{5}{B(C - 1)} \right)^{\frac{1}{2}} \tag{2.10}$$

(ii) *If $\gamma = \gamma_{rl}$,*

$$d(v_1, R, C, p_c, q_c) < \|\xi - \zeta\| < 2R + \left(\frac{\pi^2 + 2}{B(C + 1)} \right)^{\frac{1}{2}}. \tag{2.11}$$

Proof. We write $D := \|\xi - \zeta\|$ (euclidean norm), then clearly $D = 2R + \|p_c - q_c\|$ and we can write:

$$\|p_c - q_c\|^2 = |q_c^{(1)} - p_c^{(1)}|^2 + |p_c^{(2)} - q_c^{(2)}|^2$$

and

$$\|p_c - q_c\| \geq |p_c^{(2)} - q_c^{(2)}|.$$

Using (2.1) then

$$B \frac{(p_c^{(2)})^2}{2} - \frac{(q_c^{(2)})^2}{2} = \cos \beta(q_c) - \cos \beta(p_c)$$

which gives the right-hand side of (2.10) and (2.11).

On the other hand if $\gamma = \gamma_{lr}$ then

$$0 \leq |p_c^{(1)} - q_c^{(1)}| \leq 2|x_1(x_2^{med}(C; B)) - x_1(x_2^{min}(C; B))|$$

and

$$|p_c^{(2)} - q_c^{(2)}| \leq x_2^{max}(C; B) - x_2^{min}(C; B) = \frac{\sqrt{2}}{\sqrt{B}} \frac{2}{\sqrt{C+1} + \sqrt{C-1}}$$

If $\gamma = \gamma_{rl}$ then

$$0 \leq |p_c^{(1)} - q_c^{(1)}| \leq 2|x_1(x_2^{max}(C; B)) - x_1(x_2^{med}(C; B))|$$

and

$$|p_c^{(2)} - q_c^{(2)}| \leq \frac{\sqrt{2}}{\sqrt{B}} \frac{2}{\sqrt{C+1} + \sqrt{C-1}}$$

The rest of the proof follows from lemmas 2.10 and 2.11 and the fact that $-\log|1 - C| \ll 1/\sqrt{|1 - C|}$ as $C \rightarrow 1$. □

The following lemmas are proved in a similar way as Lemma 2.13.

Lemma 2.14 *Let ξ and ζ be centers of two grains connected by a solution γ of (1.1) with a Young condition at contact points p_c and q_c (following orientation) and that solves (2.1) $0 < C < 1$:*

(i) *If $\gamma = \gamma_{lr}$, then*

$$d(v_1, R, C, p_c, q_c) < \|\xi - \zeta\| < 2R + \frac{6}{(B(1 - C))^{\frac{1}{2}}}. \tag{2.12}$$

(ii) *If $\gamma = \gamma_{rl}$, then*

$$d(v_1, R, C, p_c, q_c) < \|\xi - \zeta\| < 2R + \left(\frac{\pi^2 + 2}{B(C + 1)}\right)^{\frac{1}{2}}. \tag{2.13}$$

We recall that for $-1 < C \leq 0$ the complete curve does not intersect, so the estimates in Lemma 2.4 have to be summed up depending on the number of pieces of the curve (see Remark 2.5(ii)):

Lemma 2.15 *Let ξ and ζ be centers of two grains connected by a solution γ of (1.1) with a Young condition and that solves (2.1) with $-1 < C \leq 0$, then it is necessarily of the form γ_{rl} and satisfies*

$$d(v_1, R, C, p_c, q_c) < \|\xi - \zeta\| < 2R + n \left(\frac{2(\pi^2 + 4)}{B}\right)^{\frac{1}{2}}, \tag{2.14}$$

where n is the number of elemental components of γ_{rl} .

3. The probabilistic analysis

In this section we shall give the main results and prove some of them. Before we do this, we complete the problem setting by describing its probabilistic features in detail.

We observe first that the probability of finding at least one grain center in $V \subset [0, L]^2$, $P(V)$, is computed by

$$P(V) = \frac{1}{L^2} \int_V dx. \tag{3.1}$$

We shall denote by $\Omega_\nu(L)$ the set of all possible configurations of νL^2 centers in $[0, L]^2$ that are distributed uniformly and independently. In particular, there are νL^2 centers in $[0, L]^2$, and we can make the identification $\Omega_\nu(L) = ([0, L]^2)^{\nu L^2}$. We assume that the grains have the same size, where their radius $R < 1/2$. Notice that we allow situations of overlapping grains. Then, we assign the following probability measure to $\Omega_\nu(L)$:

$$\mu_\nu(d\xi) = \frac{1}{L^{2\nu L^2}} \prod_{k=1}^{\nu L^2} d\xi_k, \quad \text{with} \quad d\xi = \prod_{k=1}^{\nu L^2} d\xi_k. \tag{3.2}$$

The probability space is $(\Omega_\nu(L), A_\nu, \mu_\nu)$, where A_ν is the σ -algebra of Borel sets of $\Omega_\nu(L)$.

For simplicity of notation, we introduce the parameter

$$\theta = L \max \left\{ \frac{1}{\sqrt{B}}, R \right\} \tag{3.3}$$

that relates the three (non-dimensionalised) lengths. As mentioned in the introduction, θ is relevant in the cases where $B \gg 1$. In the other cases we use it to simplify notation; it will indicate that the solutions depend on B , R and L .

For every $\omega \in \Omega_\nu(L)$ we denote by $\Gamma(\omega, \lambda, \theta)$ an interface solution of (1.1) subject to the Young condition. We recall that such an interface connects the boundary $x_1 = 0$ to the boundary $x_1 = L$ and that $\Gamma(\omega, \lambda, \theta)$ is the union of non-intersecting elemental components that are solutions of (1.1) and the Young condition between a pair of grains.

In particular, each component of $\Gamma(\omega, \lambda, \theta)$ solves (2.1) with a particular value of C , and we denote by $C(\Gamma)$ the maximum C for each such Γ .

We index the elemental components of any $\Gamma(\omega, \lambda, \theta)$ following its orientation and let $J = \{1, \dots, m + 1\}$ be this set of indexes if the interface connects m grains. We shall write that a center $\xi_i \in \Gamma(\omega, \lambda, \theta)$ with $i = 1, \dots, m$ if it is connected (by the Young conditions) to the elemental components γ_{i-1} and γ_i of $\Gamma(\omega, \lambda, \theta)$. We also let D_i denote $\|\xi_i - \xi_{i+1}\|$ for every $i \in \{1, \dots, m\}$.

We shall need a number of results. The first one is a consequence of the Lemmas 2.13, 2.14 and 2.15, and will be used in Sections 3.1 and 3.2:

Corollary 3.1 *For every L , $\omega \in \Omega_\nu(L)$ and ν_0 , there exists a constant D_0 , such that if a $\Gamma(\omega, \lambda, \theta)$ satisfies that*

$$\Gamma(\omega, \lambda, \theta) \cap \left\{ |x_2 - \nu_1 L| \geq \frac{D}{\sqrt{B}} \right\} \neq \emptyset \text{ for } D \geq D_0,$$

then there exists a constant $K > 0$ such that every pair of centers of grains joined by Γ , ξ_i and ξ_l , and contained in $\left\{ |x_2 - \nu_1 L| \geq \frac{D}{\sqrt{B}} \right\}$, satisfy $\|\xi_i - \xi_l\| \leq \frac{K}{D\sqrt{B}} + 2R$. In fact, $D_0 \propto \sqrt{C(\Gamma)}$.

In many of the probability results we will use Stirling estimates (see, e.g., [5]). Namely,

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{r_n} \quad \text{with} \quad \frac{1}{12(n+1)} < r_n < \frac{1}{12n}. \quad (3.4)$$

Thus we can write for any $n, m \in \mathbb{N}$ with $1 < m < n$:

$$\binom{n}{m} = \frac{e^{r_n - r_m - r_{n-m}}}{\sqrt{2\pi}} \left(\frac{n}{(n-m)m}\right)^{\frac{1}{2}} (n)^n (m)^{-m} (n-m)^{-n+m}. \quad (3.5)$$

We have the following lemma:

Lemma 3.2 *Given $n_0, n, m \in \mathbb{N}$, with $1 < n_0 \leq m \leq n - 1$ then*

$$\binom{n}{m} \leq C(n)e^{W(n,m)} \quad \text{with} \quad W(n,m) = n\Psi\left(\frac{m}{n}\right) - \frac{1}{2}\varphi\left(\frac{m}{n}\right) - \frac{1}{2}\log(n)$$

with

$$\Psi(x) = -x \log(x) - (1-x) \log(1-x), \quad \varphi(x) = \log(x) + \log(1-x),$$

and $C(n) \leq e^{-c/n}/\sqrt{2\pi}$ for some $c > 0$. Moreover, at $m = n/2$, W reaches its maximum $W(n, n/2) = -(n+1) \log(1/2) - (1/2) \log(n) > 0$, attaining its minimum at either $m = n - 1$ or $m = n_0$ (depending on the value of n_0 and n) for n large enough.

Proof. The estimates follow from (3.5). First, we observe that $\exp(r_n - r_m - r_{n-m}) \leq C(n)$ with

$$\begin{aligned} C(n) &= \exp\left(\frac{1}{12n} - \frac{1}{12m+1} - \frac{1}{12(n-m)+1}\right) \\ &\leq \exp\left(\frac{1}{12n} - \frac{1}{12n+1} - 1\right) < \exp\left(-\frac{11}{12n}\right) \end{aligned}$$

where we use the estimates on r_n (3.4). The estimate on the other factor of (3.5) follows by a calculus exercise. First, we observe that $W_n(x) = n\Psi(x) - \varphi(x)/2 - \log(n)/2$ has a maximum at $x = 1/2$ if $n > 1$ with $W_n(1/2) > 0$. Also since $\lim_{x \rightarrow 0} W_n(x) = \lim_{x \rightarrow 1} W_n(x) = \infty$ for all n , there must be at least two relative minima, one in the interval $(0, 1/2)$ and one in $(1/2, 1)$. Indeed, it is easy to check that $W''(x) > 0$ in $(0, x_l(n))$ and $W''(x) < 0$ in $(x_r(n), 1)$, where $x_l(n) = (1 - \sqrt{(n-1)/(n+1)})/2$ and $x_r(n) = (1 + \sqrt{(n-1)/(n+1)})/2$ are the inflexion points. Since $x_r(n) > (n-1)/n$ for all $n \geq 1$ and $x_l(n) < n_0/n$ with $1 \leq n_0 \leq n$, the statement follows. \square

Finally, it is key to the estimations of the probabilities in each regime to identify events in terms of grains contained in certain region of the domain. Such events follow a Binomial distribution as follows:

Lemma 3.3 *Let $k \in \mathbb{N} \cup \{0\}$ and $V \subset [0, L]^2$. Define $\Omega_{k,V} = \{\omega \in \Omega_\nu(L) : \text{there are } k \text{ grain centers in } V\}$, then*

$$\mu_\nu(\Omega_{k,V}) = \binom{\nu L^2}{k} \left(\frac{|V|}{L^2}\right)^k \left(1 - \frac{|V|}{L^2}\right)^{\nu L^2 - k}.$$

In particular, the expected number of grains centers in V is $\nu|V|$.

Proof. According to our definition

$$\mu_\nu(\Omega_{k,V}) = \frac{1}{L^{2N}} \sum_{\substack{\{l_1, \dots, l_k\} \subset \{1, \dots, N\} \\ l_i \neq l_j}} \left(\prod_{i=1}^k \int_{\xi_{l_i} \in V} d\xi_{l_i} \right) \left(\prod_{\substack{m=1 \\ m \neq l_i}}^N \int_{\xi_m \notin V} d\xi_m \right).$$

Where the integrals are equal to $|V|$ and $L^2 - |V|$ respectively, then counting the number of terms in the sum gives the result.

This means that the probability of the events $\Omega_{k,V}$ are given by the binomial distribution with νL^2 being the maximum number of trials and $|V|/L^2$ the probability of success. Therefore the expected number of grains in V is $\nu L^2 |V|/L^2$, and the second statement follows. \square

3.1 *The regime* $1 \lesssim L \ll \min\{\sqrt{B}, R^{-1}\}$

The subcases of this regime can be treated similarly. Namely, the following theorem holds.

Theorem 3.4 *For all $\varepsilon > 0$ there exists θ_ε such that for all $\theta \leq \theta_\varepsilon$ there exists $\Omega_\varepsilon \subset \Omega_\nu(L)$ with $\mu_\nu(\Omega_\varepsilon) \geq 1 - \varepsilon$ and such that for any $\omega \in \Omega_\varepsilon$, the solutions $\Gamma(\omega, \lambda, \theta)$ satisfy $\Gamma(\omega, \lambda, \theta) \equiv \{x_2 - v_1 L = h(\omega)\}$ with*

$$|h(\omega)| \leq \max \left\{ \frac{2D_0}{\sqrt{B}}, 2R \right\} + 2Kv_1LR^2,$$

where the positive constant K is as in Corollary 3.1.

Proof. For simplicity of notation we introduce the parameter

$$\delta(\theta) = \max \left\{ \frac{2D_0}{\sqrt{B}}, 2R \right\} + 2Kv_1LR^2,$$

that satisfies $\delta(\theta) \rightarrow 0$ as $\theta \rightarrow 0$.

We first compute the probability of having a configuration ω without centers in the strip $|x_2 - v_1 L| < \delta(\theta)$. We call the set of such configurations Ω_0 , then, clearly,

$$\mu_\nu(\Omega_0) = \left(1 - \frac{2\delta(\theta)}{L} \right)^{\nu L^2} \rightarrow 1 \quad \text{as } \theta \rightarrow 0. \tag{3.6}$$

On the other hand, if there exists $\omega \in \Omega_0$ and a $\Gamma(\omega, \lambda, \theta)$ such that all its points satisfy $|x_2 - v_0 L| \geq \delta$ then, by Corollary 3.1, every two consecutive grains, ξ_i and ξ_l , joined by Γ , satisfy $\|\xi_i - \xi_l\| \leq M(\theta)$ where $M(\theta) = K/D_0\sqrt{B} + 2R$ (observe that $M(\theta) \rightarrow 0$ as $\theta \rightarrow 0$). The number of grains contained in Γ is at least $N_0 = L/M(\theta) \gg \nu L^2$, then this implies that the probability of such an event tends to 0 as $\theta \rightarrow 0$. \square

REMARK 3.5 We remark that the computation of (3.6) shows that one must indeed distinguish three regimes if $B \gg 1$: the current one that makes $\mu_\nu(\Omega_0) \rightarrow 1$, a second regime with $L = O(\min\{\sqrt{B}, R^{-1}\})$ and a third with $L \gg \min\{\sqrt{B}, R^{-1}\}$. In the former $0 < \mu_\nu(\Omega_0) < 1$ as $B \rightarrow \infty$ and this allows solutions in the strip that join several grains. In the latter, then $\mu_\nu(\Omega_0) \rightarrow 0$ as $B \rightarrow \infty$, and this implies that all solutions in the strip do join several grains. In these two cases one needs to take into account for scenarios that allow interfaces outside the strip.

3.2 The regime $L = O(\min\{\sqrt{B}, R^{-1}\})$ and $L \rightarrow \infty$

We first observe that if we denote by U_0 the set of configurations for which the ‘horizontal’ solutions exist (no grains, therefore, are connected), then $\mu_\nu(U_0) = 1 - \mu_\nu(\Omega_0)$ where Ω_0 is the set of configurations with at least one grain in the strip around the horizontal reference height of width $2R$, thus $\mu_\nu(\Omega_0) = 2R/L$ and $\mu_\nu(U_0) = 1 - 2R/L$. In the current limit this means that $\mu_\nu(U_0) \rightarrow 1$ as $L \rightarrow \infty$.

If we now let U_1 be the set of configurations that have at least a solution Γ with exactly one grain connected to the walls, then $\mu_\nu(U_1) \leq \mu_\nu(\Omega_1)$ where Ω_1 is the set of configurations that have at least one center at a distance smaller than or equal than $2/\sqrt{B} + R$ to the horizontal reference line (recall that the supremum of the distance to the horizontal that can be reached is precisely $2/\sqrt{B}$ for the solution with $C = 1$ different from the horizontal line). Then $\mu_\nu(\Omega_1) = 2(2/\sqrt{B} + R)/L \rightarrow 0$ as $L \rightarrow \infty$ and thus also $\mu_\nu(U_1) \rightarrow 0$ as $L \rightarrow \infty$.

In the next theorem we obtain that in general $\mu_\nu(U_k) \rightarrow 0$ as $L \rightarrow \infty$ for all $k > 0$ and that an upper bound for such behaviour is given by a Poisson distribution.

First let us make precise the definition of the sets of configurations U_k :

DEFINITION 3.6 Let $U_k \subset \Omega_\nu(L)$ be the set of configurations ω for which there is at least one solution $\Gamma(\omega, \lambda, \theta)$ such that $k = \#\{\xi_j \in \omega : \Gamma(\omega, \lambda, \theta) \cap B_R(\xi_j) \neq \emptyset\}$.

Theorem 3.7 There exists $N_0 \leq 2D_0\theta\nu < \nu L^2$ and L sufficiently large such that, if $1 \leq k \leq N_0$,

$$\mu_\nu(U_k) \lesssim e^{-2D_0\theta\nu} \sum_{i=k}^{N_0} \frac{1}{i!} (2D_0\theta\nu)^i.$$

Moreover, for all $\varepsilon > 0$, there exists a $L_\varepsilon > 0$ such that for all $L \geq L_\varepsilon$ there exists $\Omega_\varepsilon \subset \Omega_\nu(L)$ such that $\mu_\nu(\Omega_\varepsilon) \geq 1 - \varepsilon$, and that for all $\omega \in \Omega_\varepsilon$ the solutions $\Gamma(\omega, \lambda, \theta)$ satisfy $|C(\Gamma) - 1| \leq \varepsilon$.

Proof. First, we observe that the probability of having at least k connections is the sum of the probabilities of having exactly i connections with $i = k \dots \nu L^2$. Then we distinguish between the configurations that have connections outside the strip around the horizontal reference line of width $2D_0R$ and those that do not. We first show that the probability of having connections outside the given strip tends to 0. On the other hand, the probability of having an interface connecting exactly k grains that lies inside a strip of width $2D_0R$ is less than that of having k grains in the strip of that same width. Then we compute this probability. We recall the notation $N = \nu L^2$ that we use throughout this proof.

First, we treat the case $R \gg 1/\sqrt{B}$. Let $\Omega_s \subset \Omega_\nu(L)$ be the set of configurations that have a solution $\Gamma(\omega, \lambda, \theta)$, with $\omega \in \Omega_s$, that intersects grains ξ with $|\xi^{(2)} - v_1 L| > D_0R$. In that case, there are at least three centers $\xi_i, \xi_j, \xi_l \in \omega \subset \Omega_s$ with $\|\xi_i - \xi_j\| < (2 + \tilde{K})R$ and, by Corollary 3.1, $\|\xi_j - \xi_l\| < (2 + \tilde{K})R$ for some \tilde{K} of order one. The probability of such events is:

$$\begin{aligned} \mu_\nu(\Omega_s) &\leq \binom{N}{3} \frac{(L^2)^{N-2}}{(L^2)^N} \left(\int_{\|\xi_1 - \xi_2\| < (2 + \tilde{K})R} d\xi_2 \right) \left(\int_{\|\xi_2 - \xi_3\| < (2 + \tilde{K})R} d\xi_3 \right) \\ &= \frac{N(N-1)(N-2)}{6L^4} \pi^2 (2 + \tilde{K})^4 R^4 \lesssim \tilde{K} \frac{1}{L^2} \quad \text{as } L \rightarrow \infty, \end{aligned}$$

since $L = O(1/R)$ and $N = \nu L^2$.

Let Ω_s^c be the complementary event of Ω_s . Then, $\mu_\nu(\Omega_s^c) \rightarrow 1$ as $L \rightarrow \infty$, and this together with Remark 2.9 (ii) shows the second statement of the Theorem.

We let Ω_k the set of configurations that have exactly k grains in a strip of width $2D_0R$. Then, the probability measure of Ω_k is, by Lemma 3.3,

$$\mu_\nu(\Omega_k) = \binom{\nu L^2}{k} \left(\frac{2D_0\theta}{L^2}\right)^k \left(1 - \frac{2D_0\theta}{L^2}\right)^{\nu L^2 - k}.$$

Where we have used that, in the current case, $\theta = RL$.

In a strip of width $2D_0R$ the maximum number of centers expected is proportional to $2\nu LD_0R = 2\nu D_0\theta =: N_0 = O(1)$. To see this, we let $U_{P \geq N_0}$ be the set of configurations for which the number of particles in the strip of width $2D_0R$ is larger or equal to N_0 . Then

$$\mu_\nu(U_{P \geq N_0}) = \sum_{j=N_0}^{\nu L^2} \binom{\nu L^2}{j} \left(\frac{2D_0\theta}{L^2}\right)^j \left(1 - \frac{2D_0\theta}{L^2}\right)^{\nu L^2 - j}$$

Using Lemma 3.2 we can estimate each term (except the last one which is $(2D_0\theta/L^2)^{\nu L^2}$) as follows:

$$\binom{\nu L^2}{j} \left(\frac{2D_0\theta}{L^2}\right)^j \left(1 - \frac{2D_0\theta}{L^2}\right)^{\nu L^2 - j} \leq C(\nu L^2) \exp\left(\bar{W}\left(\nu L^2, \frac{j}{\nu L^2}\right)\right)$$

where $C(\nu L^2)$ is as in Lemma 3.2, and

$$\begin{aligned} \bar{W}(\nu L^2, x) = & -\left(\nu L^2 + \frac{1}{2}\right) (x \log(x) + (1-x) \log(1-x)) - \frac{1}{2} \log(\nu L^2) \\ & + \nu L^2 (x \log(2D_0\theta/L^2) + (1-x) \log(1 - 2D_0\theta/L^2)) \quad \text{with } x \in \left[\frac{2D_0\theta}{L^2}, 1\right]. \end{aligned}$$

Then, $\bar{W}(\nu L^2, x) \rightarrow -\infty$ as $L \rightarrow \infty$, in particular, $\bar{W}(\nu L^2, x) \sim \nu L^2 x \log(2D_0\theta/L^2)$ as $L \rightarrow \infty$ if $x = O(1)$ and $\bar{W}(\nu L^2, x) \sim -\frac{1}{2} \log(\nu L^2)$ as $L \rightarrow \infty$ if $x \sim \frac{2D_0\theta}{L^2}$. Summing up all the terms implies that $\mu_\nu(U_{P \geq N_0}) \rightarrow 0$ as $L \rightarrow \infty$ at an exponential rate.

Finally, we write

$$U_k = (U_k \cap \Omega_s) \cup (U_k \cap \Omega_s^c) = (U_k \cap \Omega_s) \cup (U_k \cap \Omega_s^c \cap U_{P \geq N_0}) \cup (U_k \cap \Omega_s^c \cap U_{P < N_0})$$

where $U_{P < N_0}$ denotes the set of configurations for which the number of particles in the strip is less than N_0 . Then,

$$\mu_\nu(U_k) \leq \mu_\nu(\Omega_s) + \mu_\nu(U_k \cap \Omega_s^c \cap U_{P \geq N_0}) + \sum_{i=k}^{N_0} \binom{\nu L^2}{i} \left(\frac{2D_0\theta}{L^2}\right)^i \left(1 - \frac{2D_0\theta}{L^2}\right)^{\nu L^2 - i}.$$

As we have seen the first two terms tend to 0 as $L \rightarrow \infty$, and the third tends to $e^{-2D_0\theta\nu} \sum_{i=k}^{N_0} \frac{1}{i!} (2D_0\theta\nu)^i$ as $L \rightarrow \infty$.

The case $R \ll 1/\sqrt{B}$ is analogous but working on a strip of width $2D_0/\sqrt{B}$. □

3.3 The regime $L \gg \sqrt{B}$ with $B \rightarrow \infty$ and $R \rightarrow 0$

In this regime we shall prove that the probability of having configurations with solutions that separate from the horizontal reference line a distance of order one tends to zero in a regime where L is not too large compare to some increasing function of B .

We prove the result in the case $1/\sqrt{B} \ll R$, the other case can be shown similarly, as we remark at the end of this section.

Let us give two auxiliary lemmas that will be needed in the proof of the main result of this section. The first one is a technical lemma:

Lemma 3.8 *Let $b > 0$ and a sequence $\{a_k\}$ such that*

$$a_k + \frac{1}{2} \frac{b}{1+a_k} < a_{k+1} < a_k + 2 \frac{b}{1+a_k},$$

and $a_0 > \delta > 0$. Then, there exist positive constants C_1 and C_2 independent of k , such that

$$C_2 \sqrt{k} < a_k < C_1 \sqrt{k}.$$

The proof follows by induction, multiplying the inequalities by a_k and a_{k+1} and summing up the results. The constants C_1 and C_2 satisfy $C_1 > 4b(1 + b/C_2)$ and $C_2 > \max_x (b\sqrt{x} + \sqrt{b^2x + b})/2(1 + C_1\sqrt{x})$.

The next lemma is a geometric result, which we prove in Appendix A.

Lemma 3.9 *If $\Gamma(\omega, \lambda, \theta)$ is a solution with m components joining $(m - 1)$ grains, ξ_1, \dots, ξ_{m-1} , ordered following the orientation of $\Gamma(\omega, \lambda, \theta)$, then*

$$\sum_{i=0}^m \|\xi_i - \xi_{i+1}\| < 4m \left(2R + \frac{K_1}{\sqrt{B}} \right) + K_2L, \tag{3.7}$$

for some positive constants K_1 and K_2 independent of L , B and R .

Next, we define the configurations that may allow solution interfaces that separate a distance of order one from the reference horizontal line. The centers of such configurations that might be joined by a solution interface must satisfy certain conditions, according to Corollary 3.1. We gather these in the following definition.

DEFINITION 3.10 We let the union of the horizontal reference line to the left boundary and to right boundary be denoted by

$$Z_L^l = \{x_2 = v_1L\} \cup \{x_1 = 0\}, \quad Z_L^r = \{x_2 = v_1L\} \cup \{x_1 = L\}.$$

DEFINITION 3.11 For given constants $K > 0$ and $h > 0$ of order one, and a given θ in the current regime, we let $U_h(K, D_0, \theta, L) \subset \Omega_\nu(L)$ be the set of configurations of grains such that for every $\omega \in U(K, D_0, \theta, L)$ there exists $N_0 \leq \nu L^2$ and an injective map $g : \{1, \dots, N_0\} \rightarrow \{1, \dots, \nu L^2\}$ so that the centers $\{\xi_{g(n)}\}_{n=1, \dots, N_0}$ satisfy the following conditions:

- (i) $\max_n \{|\xi_{g(n)}^{(2)} - v_1L|\} \geq h$.
- (ii) $|\xi_{g(n+1)}^{(2)} - \xi_{g(n)}^{(2)}| \leq \frac{1}{\sqrt{B}} \frac{K}{(1 + \sqrt{B}|\xi_{g(n)}^{(2)} - v_1L)}.$

- (iii) If $\max\{|\xi_{g(n)}^{(2)} - v_1 L|, |\xi_{g(n+1)}^{(2)} - v_1 L|\} \geq D_0/\sqrt{B}$ then $\|\xi_{g(n+1)} - \xi_{g(n)}\| \leq \frac{1}{B} \frac{K}{|\xi_{g(n)}^{(2)} - v_1 L|}$.
- (iv) $dist(\xi_{g(1)}, Z_L^l) \leq D_0 \min\left\{\frac{1}{\sqrt{B}}, \frac{1}{B|\xi_{g(1)}^{(2)} - v_1 L|}\right\}$,
 $dist(\xi_{g(N_0)}, Z_L^r) \leq D_0 \min\left\{\frac{1}{\sqrt{B}}, \frac{1}{B|\xi_{g(N_0)}^{(2)} - v_1 L|}\right\}$.
- (v) Moreover, they satisfy (3.7).

We observe that condition (ii) is suggested by the statement of Corollary 3.1 and the values of x_2^{max}, x_2^{med} in (2.5) and (2.6). Condition (iii) is also suggested by Corollary 3.1. Finally, condition (iv) implies that the sequence starts either near the horizontal or near Z_L^l and ends either near the horizontal or near Z_L^r .

We can now prove the following theorem:

Theorem 3.12 *Let $\Omega_h \subset \Omega_v(L)$ denote the set of all configurations ω such that there exists a $\Gamma(\omega, \lambda, \theta)$ satisfying $\max |\xi_j^{(2)} - v_1 L| \geq h$ for some $h > 0$ of order one. Then $\mu_v(\Omega_h) \rightarrow 0$ as $B \rightarrow \infty$ if $\sqrt{B} \ll L \ll \sqrt{B} \log B$.*

Proof. Definition 3.11 implies that $\mu_v(\Omega_h) \leq \mu_v(U_h(K, D_0, \theta, L))$. Let us then get an estimate on $\mu_v(U_h(K, D_0, \theta, L))$.

We denote by $\Sigma_m := \{g : \{1, \dots, m\} \rightarrow \{1, \dots, N\} : \text{injective}\}$ and for every $m \leq N$ and every $g \in \Sigma_m$ we let $U^g \subset U_h(K, D_0, \theta, L)$ be the set configurations for which there exists a family of centers $\{\xi_{g(n)}\}_{n=1, \dots, m}$ satisfying the conditions of Definition 3.11. Observe that these sets of configurations are not, in general, disjoint. Then

$$U_h(K, D_0, \theta, L) = \bigcup_{m=1}^N \bigcup_{g \in \Sigma_m} U^g,$$

and we estimate its probability as follows

$$\begin{aligned} \mu_v(U_h(K, D_0, \theta, L)) &= \frac{1}{L^{2N}} \left(\prod_{n=1}^N \int_{U_h(K, D_0, \theta, L)} d\xi_n \right) \\ &\leq \frac{1}{L^{2N}} \sum_{m=1}^N \sum_{g \in \Sigma_m} \left(\prod_{n=1}^N \int_{U^g} d\xi_n \right) \\ &= \frac{1}{L^{2N}} \sum_{m=1}^N N(N-1) \dots (N-m+1) \left(\prod_{n=1}^N \int_{U^{\bar{g}}} d\xi_n \right) \\ &\leq \sum_{m=1}^N \frac{N(N-1) \dots (N-m+1)}{L^{2m}} \left(\prod_{n=1}^m \int_{U^{\bar{g}}} d\xi_n \right), \end{aligned}$$

where we have reordered the first indexes by setting $\bar{g}(j) = j$ with $j \in \{1, \dots, m\}$ for every m , and in the last inequality we have simply counted the number of elements in each Σ_m and also estimated the measure of the elements ξ_n with $n > m$ by the total measure L^2 .

Now, we consider two types of configurations in each $U^{\bar{g}}$, the ones for which all points ξ_n satisfy that

$$\left| \xi_n^{(2)} - v_1 L \right| \geq \frac{D_0}{\sqrt{B}} \quad \text{for all } n \in \{1, \dots, m\},$$

and the ones for which there exists at least one j with $\left| \xi_j^{(2)} - v_1 L \right| < \frac{D_0}{\sqrt{B}}$. In the first case, we have, due to the properties in Definition 3.11, that

$$\|\xi_n - \xi_{n+1}\| \leq \frac{K}{D_0 \sqrt{B}}, \quad \text{for } n \in \{0, \dots, m\},$$

where we abuse notation by referring to the vertical lines $x_1 = 0$ and $x_1 = L$ by ξ_0 and ξ_{m+1} , respectively, and denoting the distance of points to these lines with the norm symbol. Observe that then $m \geq LD_0 \sqrt{B}/K$.

Let us denote by

$$U_1^{\bar{g}} = U^{\bar{g}} \cap \{\omega \in \Omega_v(L) : |\xi_n^{(2)} - v_1 L| \geq \frac{D_0}{\sqrt{B}} \quad \forall n \in \{1, \dots, m\}\}$$

and

$$U_2^{\bar{g}} = U^{\bar{g}} \setminus \{\omega \in \Omega_v(L) : |\xi_n^{(2)} - v_1 L| \geq \frac{D_0}{\sqrt{B}} \quad \forall n \in \{1, \dots, m\}\},$$

then

$$\mu_v(U_h(K, D_0, \theta, L)) \leq I_1 + I_2 \tag{3.8}$$

with

$$I_j = \sum_{m=M_j}^N \frac{N(N-1)\dots(N-m+1)}{L^{2m}} \prod_{n=1}^m \int_{U_j^{\bar{g}}} d\xi_n, \quad j = 1, 2, \tag{3.9}$$

where $M_1 \geq LD_0 \sqrt{B}/K$ and M_2 will be determined bellow.

We first estimate I_1 . For B large enough so that $v\pi K^2 / ((D_0)^2 B) < 1$, we have that

$$\begin{aligned} I_1 &\leq \sum_{m \geq LD_0 \sqrt{B}/K}^N N(N-1)\dots(N-m+1) \frac{L^2}{L^{2m}} \pi^{m-1} \left(\frac{K}{D_0 \sqrt{B}} \right)^{2m-2} \\ &\leq vL^2 \sum_{m \geq LD_0 \sqrt{B}/K}^N \left(\frac{v\pi K^2}{(D_0)^2 B} \right)^{m-1} \leq vL^2 \left(\frac{v\pi K^2}{(D_0)^2 B} \right)^{\frac{LD_0 \sqrt{B}}{K}}, \end{aligned}$$

and this tends to zero provided that

$$(B)^{L\sqrt{B}} \gg L^2. \tag{3.10}$$

In the current limit, $L^2 \gg B$ and $B \rightarrow \infty$, then for large enough θ it is clear that $(B)^{L\sqrt{B}} > (B)^L$ and so (3.10) holds and $I_1 \rightarrow 0$ as $L \rightarrow \infty$.

In order to estimate I_2 , for a $\omega \in U_2^{\bar{g}}$ we define a subset of the points in ω as follows. First, we observe that, by the definition of $U_2^{\bar{g}}$, there is a $j_0 \in \{1, \dots, m\}$, such that $|\xi_{j_0}^{(2)} - v_1 L| < \frac{D_0}{\sqrt{B}}$, on

the other hand there exist a $j_1 \in \{1, \dots, m\}$ that $|\xi_{j_1}^{(2)} - v_1 L| > 2D_0/\sqrt{B}$. Then we can define the first point of the subset by taking $\xi_{l_1} = \xi_{j_0}$ and $\xi_{l_{M_2}} = \xi_{j_1}$ (the last point of the subset). For each $k \in \{1, \dots, M_2\}$, we choose the points as follows

$$l_{k+1} = \min \left\{ j > l_k : |\xi_{l_{k+1}}^{(2)} - v_1 L| > |\xi_{l_k}^{(2)} - v_1 L| + \frac{K}{2\sqrt{B}} \frac{1}{(1 + \sqrt{B}|\xi_{l_k}^{(2)} - v_1 L|)} \right\},$$

then by the properties of Definition 3.11, and this definition, the subset satisfies:

$$\begin{aligned} |\xi_{l_{k+1}}^{(2)} - v_1 L| &> |\xi_{l_k}^{(2)} - v_1 L| + \frac{K}{2\sqrt{B}} \frac{1}{(1 + \sqrt{B}|\xi_{l_k}^{(2)} - v_1 L|)}, \\ |\xi_{l_{k+1}}^{(2)} - v_1 L| &< |\xi_{l_k}^{(2)} - v_1 L| + 2\frac{K}{\sqrt{B}} \frac{1}{(1 + \sqrt{B}|\xi_{l_k}^{(2)} - v_1 L|)}, \end{aligned}$$

and also there exists a k_0 such that $|\xi_{l_{k_0}}^{(2)} - v_1 L| > D_0/(2\sqrt{B})$.

We can apply Lemma 3.8 with $a_k = \sqrt{B}|\xi_{l_k}^{(2)} - v_1 L|$ to obtain an estimate on M_2 , namely,

$$h < C_1 \frac{\sqrt{M_2}}{\sqrt{B}},$$

so

$$M_2 > M_0 := \left(\frac{h}{C_1} \right)^2 B.$$

We then estimate the second term in (3.8) as follows,

$$\begin{aligned} I_2 &\leq \sum_{m \geq M_0}^N \frac{N(N-1)\dots(N-m+1)}{L^{2m}} \left(\prod_{n=1}^{M_0} \int_{U_2^{\bar{g}} \cap \{\xi_{l_k}\}_k} d\xi_n \right) \left(\prod_{n=1}^{m-M_0} \int_{U_2^{\bar{g}} \setminus \{\xi_{l_k}\}_k} d\xi_n \right) \\ &\leq \sum_{m \geq M_0}^N \frac{N(N-1)\dots(N-m+1)}{L^{2m}} \left(\prod_{n=1}^{M_0} \int_{U_2^{\bar{g}} \cap \{\xi_{l_k}\}_k} d\xi_{l_n}^{(2)} \right) \\ &\quad \left(\prod_{n=1}^{m-M_0} \int_{U_2^{\bar{g}} \setminus \{\xi_{l_k}\}_k} d\xi_{m_n}^{(2)} \right) \left(\prod_{n=1}^m \int_{U_2^{\bar{g}}} d\xi_n^{(1)} \right) \\ &\leq \nu L^3 \sum_{m \geq M_0}^N \nu^{m-1} \left(\prod_{n=2}^{M_0} \frac{2K}{C_2 \sqrt{B} \sqrt{n}} \right) \left(\prod_{n=2}^m D_n^2 \right) \\ &= \nu L^3 \frac{1}{\sqrt{M_0!}} \left(\frac{2K}{C_2 \sqrt{B}} \right)^{M_0-1} \sum_{m \geq M_0}^N \left(\prod_{n=2}^m \sqrt{\nu} D_n \right)^2. \end{aligned}$$

Here we have indexed the elements of $U_2^{\bar{g}} \setminus \{\xi_{l_k}\}_k$ by m_n with $n \in \{1, \dots, m - M_0\}$. Also, in the last inequality, we have used that the elements of $\{\xi_{l_k}\}_k$ satisfy $|\xi_{l_{k+1}}^{(2)} - \xi_{l_k}^{(2)}| \leq 2K|\xi_{l_k}^{(2)} - v_1 L|/B$ and Lemma 3.8. We recall that $D_n = \|\xi_n - \xi_{n+1}\|$.

Then, using the inequality of arithmetic and geometric means and Lemma 3.9, we obtain

$$\begin{aligned} I_2 &\leq v \frac{L^3}{\sqrt{M_0!}} \left(\frac{2K}{C_2 \sqrt{B}} \right)^{M_0-1} \sum_{m \geq M_0}^N \left(\frac{1}{m-1} \sum_{n=2}^m \sqrt{v} D_n \right)^{2(m-1)} \\ &\leq \frac{L^3}{\sqrt{M_0!}} \left(\frac{2K}{C_2 \sqrt{B}} \right)^{M_0-1} \sum_{m \geq M_0}^N \left(\frac{v}{m-1} \left(m \frac{\bar{K}_1}{\sqrt{B}} + K_2 L \right) \right)^{2(m-1)}. \end{aligned}$$

Recalling that $M_0 \propto B$ and rewriting

$$\left(\frac{v}{m-1} \left(m \frac{\bar{K}_1}{\sqrt{B}} + K_2 L \right) \right)^{2(m-1)} = \left(\frac{v \bar{K}_1}{\sqrt{B}} \right)^{2(m-1)} \left(\frac{m}{m-1} + \frac{K_2 L \sqrt{B}}{\bar{K}_1 m-1} \right)^{2(m-1)},$$

we obtain for B large enough

$$I_2 \lesssim \frac{L^3}{\sqrt{M_0!}} \left(\frac{2K}{C_2 \sqrt{B}} \right)^{M_0-1} \left(\frac{v \bar{K}_1}{\sqrt{B}} \right)^{2(M_0-1)} e^{2 \left(\frac{K_2}{\bar{K}_1} L \sqrt{B} + 1 \right)}.$$

We recall that $M_0 \propto B$ and using (3.4) we can conclude that I_2 tends to zero as $L \rightarrow \infty$ provided that the largest exponential growing factor is controlled by the fastest exponentially decreasing one, that is if $L \sqrt{B} \ll B \log B$ as $L \rightarrow \infty$, and this implies the result. \square

Theorem 3.12 can be formulated in the case $1/\sqrt{B} \lesssim R$ by replacing \sqrt{B} by R^{-1} when $R \rightarrow 0$. The proof is analogous, except for the proof of Lemma 3.9. In this case, the conclusion is simpler to achieve, by realising that the longest part of an interface joining the least number of grains is one crossing $x_2 = v_1 L$, all other parts are away from this one at a distance of order $1/\sqrt{B}$, applying Corollary 3.1, hence the lemma follows.

3.4 The regime $L = O(1/B)$ as $B \rightarrow 0$

We only consider here the case $L = O(1/B)$, and to simplify we assume that $L = a/B$ for some $a > 0$ of order one. The reason for this is that we expect the maximum difference in height in a full solution Γ to be of order $1/B$. Indeed, if Γ reaches distances from $x_2 = \lambda$ much larger than $1/B$, the elemental components of Γ on that region would have a very small radius of curvature, and could not join grains at an average distance of order one.

In this section we further restrict the height of the possible Γ 's to the interval $[\lambda - \varepsilon_0 L, \lambda + \varepsilon_0 L]$ where, essentially,

$$0 < \varepsilon_0 < \min \left\{ v_1, 1 - v_1, \frac{1}{2a} \right\}$$

and (for reasons that become apparent below) $\varepsilon_0 \in \mathbb{Q}$. This ensures that the curvature of Γ in this region is small enough and will allow to approximate the elemental components of Γ by straight lines, since for x_2 in this interval $|H| \leq a \varepsilon_0 < 1/2$. We recall, however, that this restriction depends on the configuration of grains, since $\lambda = v_1 L$ depends on Γ . We shall see that we can fix a volume V that is close to v_1 for the configurations of interest.

We show that, in this regime, given *any* regular curve without self-intersections contained in $[0, L] \times [\lambda - \varepsilon_0 L, \lambda + \varepsilon_0 L]$ there are configurations $\omega \in \Omega_v(L)$ with probability close to one, such that an interface solution $\Gamma(\omega, \lambda, \theta)$ is close to such curve in some suitable sense.

Before we make precise this theorem, let us give the main idea of the proof and establish some settings and definitions. We make a partition of the domain into small squares Q . For any given smooth curve Λ without self-intersections and compatible with the prescribed volume condition, we show that the probability that a solution interface Γ satisfies $\Gamma \cap Q \neq \emptyset$ for every Q that has $Q \cap \Lambda \neq \emptyset$ tends to one as $L \rightarrow \infty$. This means that all curves compatible with the fixed volume condition can be approximated by an interface Γ . The proof of the fact that the probability of $\Gamma \cap Q \neq \emptyset$ if $Q \cap \Lambda \neq \emptyset$ tends to zero, will be formulated as a percolation problem, see below for the details.

We can now state the main theorem of this section, but first we define the notion of compatible curves.

DEFINITION 3.13 (Compatible curves) Let $V \in (0, 1)$ and $\varepsilon_0 \in (0, 1) \cap \mathbb{Q}$ with

$$\varepsilon_0 < \min \left\{ V, 1 - V, \frac{1}{2a} \right\}.$$

We say that a C^1 curve without self-intersections $\Lambda : [0, 1] \rightarrow [0, 1]^2$ is compatible with (V, ε_0) if $\Lambda([0, 1]) \subset [0, 1] \times [V - \varepsilon_0, V + \varepsilon_0]$, $\Lambda(0) \in \{0\} \times [V - \varepsilon_0, V + \varepsilon_0]$, $\Lambda(1) \in \{1\} \times [V - \varepsilon_0, V + \varepsilon_0]$ and the area under Λ is equal to V .

We remark that we compare real interface solutions Γ to compatible curves in the rescaled domain $[0, 1] \times [V - \varepsilon_0, V + \varepsilon_0]$. We shall denote by $\tilde{\Gamma}$ the result of expressing Γ in the variables $x' = x/L$. Then:

Theorem 3.14 *Assume that $B = a/L$ for some $a > 0$ of order one. Given $V \in (0, 1)$ let ε_0 as in Definition 3.13 and let $\Lambda \in C^1([0, 1])$ be a curve compatible with (V, ε_0) . Then, there exist $v_0 > 0$ and $L_0 > 0$ such that for all $v \geq v_0$ and $L \geq L_0$ there exists $\alpha_{\max}(R) \geq 0$, such that $\alpha_{\max}(R) \rightarrow \pi/2$ as $R \rightarrow 0$, and $U \in \Omega_v(L)$ and a $\varepsilon(L) > 0$ with $L^{-1} \ll \varepsilon(L)$, $\varepsilon(L) \rightarrow 0$ as $L \rightarrow \infty$ such that*

$$\mu_v(U) \geq \delta_\varepsilon(L) \quad \text{with} \quad 0 < \delta_\varepsilon(L) \rightarrow 1^- \quad \text{as} \quad L \rightarrow \infty,$$

and that for any $\omega \in U$, there exists a solution interface $\Gamma(\omega, \lambda, \theta)$ with contact angle $\alpha \in [-\pi/2, \alpha_{\max}(R)]$ that satisfies

$$\tilde{\Gamma}(\omega, \lambda, \theta) \subseteq T_{\sqrt{2\varepsilon(L)}}(\Lambda), \quad \Lambda \subseteq T_{\sqrt{2\varepsilon(L)}}(\tilde{\Gamma}(\omega, \lambda, \theta)),$$

with $v_1 \rightarrow V$ as $L \rightarrow \infty$.

Here $T_{\sqrt{2\varepsilon}}$ stands for the tube of radius $\sqrt{2\varepsilon}$ around a curve. That is it contains the points of the plane that are at a distance smaller than or equal to $\sqrt{2\varepsilon}$ of the given curve.

REMARK 3.15 We remark, that if $v_0 \in (0, 1)$ is a given volume fraction of liquid, then one could tune the theorem a bit to obtain solutions for this prescribed volume with probability close to one. This can be done by taking $V = v_0/(1 - v_0\pi R^2)$ and R sufficiently small, since the expected number of grains under Λ is vL^2V and v_0 is related to v_1 by (1.3).

Next, we define partitions of $[0, L] \times [(V - \varepsilon_0)L, (V + \varepsilon_0)L]$ into small squares of equal size. In each of such squares that intersect Λ we formulate a percolation-like problem.

DEFINITION 3.16 Let

$$Q := [0, L] \times [(V - \varepsilon_0)L, (V + \varepsilon_0)L]. \quad (3.11)$$

Let $\varepsilon > 0$, depending on L , such that $B \ll \varepsilon \ll 1$ and such that $1/\varepsilon, 2\varepsilon_0/\varepsilon \in \mathbb{N}$.

We define $\{Q\}_{Q \in I_\varepsilon}$ to be a partition of \tilde{Q} into squares of size $\varepsilon L \times \varepsilon L$ such that if $Q, \tilde{Q} \in I_\varepsilon$ then $Q \cap \tilde{Q} = \emptyset$ if $Q \neq \tilde{Q}$, unless Q and \tilde{Q} have one edge or one vertex in common.

We let $M := \varepsilon(L) L$ for simplicity.

Clearly, $\#I_\varepsilon = 2\varepsilon_0/\varepsilon^2 < \infty$. We also observe that this definition requires $\varepsilon_0 \in \mathbb{Q}$.

We now consider a subdivision of each Q in smaller sub-squares of size of order one. We assume that the size is exactly one, for that we either need to assume that $L \in \mathbb{N}$ or to rescale the domain. We henceforth adopt the first choice without loss of generality.

DEFINITION 3.17 Assume that $L \in \mathbb{N}$. Let $\varepsilon(L)$ and $Q \in I_\varepsilon$ as in Definition 3.16, then we let $Q = \cup_{\kappa \in J} S_\kappa$ where $\{S_\kappa\}_{\kappa \in J}$ is a partition of Q into closed squares of size one that cover Q and such that their interior sets are disjoint.

Moreover, for a $s \in (0, 1)$ and for each $\kappa \in J$, we let $R_\kappa \subset S_\kappa$ be a closed squared with $|R_\kappa| = s$ centered in S_κ .

Observe that $\#J$ increases as $L \rightarrow \infty$, but $|S_\kappa| = 1$, and this value equals the average distance between centers (and is of the same order of the minimal radius of curvature in this domain). Then we can guarantee that if there are grains in neighbouring S_κ 's, then there is a connection by a solution γ of (1.1). In this way the proof reduces to estimating the probability of finding grains in the S_κ 's that can form a connection between the given portions of ∂Q that intersect Λ . This problem is reminiscent of a site percolation problem in two dimensions (see, e.g., [7, 19]), although the probabilities assigned to the sites, as we explain below, are weakly correlated.

In fact, we shall restrict the connections to grains that fall in adjacent R_κ 's. This is because we have to guarantee that elemental components do not intersect each other, this is possible by reducing $|R_\kappa| = s$, as we shall see. But first, we have the following result that guarantees the existence of γ 's joining grains in adjacent sites. This result is independent of ε .

Lemma 3.18 (i) *Let S_κ and $S_{\kappa'}$ such that $S_\kappa \cap S_{\kappa'} \neq \emptyset$ with one edge in common. Then if there exist $\xi \in R_\kappa$ and $\tilde{\xi} \in R_{\kappa'}$ then, there exists a $L_0 > 0$ such that for all $L \geq L_0$ there exists a solution of (1.1) that joins $\partial_R B(\xi)$ and $\partial_R B(\tilde{\xi})$.*

(ii) *There exists $s_0 \in (0, 1)$ and $\alpha_{max}(R)$ with $\alpha_{max}(R) \rightarrow \pi/2$ as $R \rightarrow 0$, such that if κ, κ' and $\tilde{\kappa} \in \{1, \dots, M\}^2$ have a vertex in common the arc that joins a grain with center in $\xi \in R_\kappa$ to a grain with center in $\xi' \in R_{\kappa'}$ do not intersect the arc that joins the grain $B_R(\xi')$ to a grain with center $\tilde{\xi} \in R_{\tilde{\kappa}}$.*

Proof of (i): Observe that the centers ξ and $\tilde{\xi}$ satisfy that $\|\xi - \tilde{\xi}\| \leq \sqrt{s + 2s^{\frac{1}{2}} + 1} < 2$. On the other hand, $|H| < a\varepsilon_0$, in particular the radius of curvature of a solution joining these grains, say r , has $|r|_\infty > 1/a\varepsilon_0 > 2$.

If there is a solution γ that joins $\partial B(\xi, R)$ and $\partial B(\tilde{\xi}, R)$, then the variation of the curvature of γ , $\Delta H < a|\xi_2 - \tilde{\xi}_2|/L < 2a/L$ (where $\Delta H = |H_{max}(\gamma) - H_{min}(\gamma)|$) is the difference of the maximum curvature and the minimum curvature of γ). This means that such a γ is very close to a circle with large radius of curvature.

On the other hand, there are straight lines that join $\partial B(\xi, R)$ and $\partial B(\tilde{\xi}, R)$ with any prescribed contact angle. Indeed, there are two parallel lines that are tangential to $\partial B(\xi, R)$ and $\partial B(\tilde{\xi}, R)$. One has contact angle $\pi/2$ and the other has contact angle $-\pi/2$. Then for any given contact angle, there is a segment parallel to these ones that joins $\partial B(\xi, R)$ and $\partial B(\tilde{\xi}, R)$ with that contact angle. This is also true for circumferences of large enough radius, being the smallest such circumference the one that has the two grains inscribed at the largest possible distance; i.e. one with diameter $\|\xi - \tilde{\xi}\| + 2R$.

We recall that $H(s) = \beta'(s)$ where $\beta \in [-\pi, \pi]$ is the angle from \mathbf{e}_1 to the tangent vector \mathbf{t} , thus $|\beta'(s)| < a\varepsilon < 1/2$ and $|\beta''(s)| < a/L$. This implies that for L sufficiently large we can find a solution γ close to a large circumference that joins $\partial B(\xi, R)$ and $\partial B(\tilde{\xi}, R)$ with a prescribed contact angle by adjusting the value of C and the height x_2 (see (2.1)).

Proof of (ii): Let us see that reducing if necessary the size of each R_κ the solutions have no self-intersections. We concentrate in three adjacent sites, say $(1, 1)$, $(1, 2)$ and $(2, 2)$ for definiteness and assume that there is exactly one grain in each of $R_{(1,1)}$, $R_{(1,2)}$ and $R_{(2,2)}$, with centers $\xi_{(1,1)}$, $\xi_{(1,2)}$ and $\xi_{(2,2)}$, respectively. Other situations can be recovered by translation and by a $\pi/2$, π or $3\pi/2$ rotation.

Let $b_s = (1 - \sqrt{s})/2$ and let ϕ_s be the angle that the segments that join the grain in $R_{(1,2)}$ to the other two grains form. The length of such segments is $\sqrt{s+1} \in (1, \sqrt{2})$. Self-intersections may occur in several ways, we aim to avoid them in the situations where the angle between joining elemental components is narrowest. The minimum possible ϕ_s , if s is fixed, is achieved when the centers of the grains are $(1 - b_s, 1 - b_s)$ for the one in $R_{(1,1)}$, $(b_s, 2 - b_s)$ for the one in $R_{(1,2)}$ and $(1 + b_s, 1 + b_s)$ for the one in $R_{(2,2)}$ (see Figure 4). This gives:

$$\cos \phi_s = \frac{2(1 - 2b_s)}{(1 - 2b_s)^2 + 1} = \frac{2\sqrt{s}}{s + 1}. \tag{3.12}$$

This function is an increasing concave function for $s \in (0, 1)$, this implies that $\phi_s \in (0, \pi/2)$ is a decreasing function with $\phi_s = \pi/2$ if $s = 0$ and $\phi_s = 0$ if $s = 1$.

First, we assume that $R = 0$, in that case we do not have to worry about contact angles. Suppose now that the grains are joined by elemental components that are circular arcs tangent at $(b_s, 2 - b_s)$, i.e. at the grain in $R_{(1,2)}$. This gives the minimum radius of curvature

$$r_{min}(s) = \frac{\sqrt{s+1}}{2 \sin(\phi_s/2)}. \tag{3.13}$$

In view of (3.12), we obtain that $r_{min}(s)$ is an increasing function with $r_{min}(0) = 1/\sqrt{2}$ and $\lim_{s \rightarrow 1} r_{min}(s) = \infty$. This implies that if the radius of the circular arcs is smaller than r_{min} they intersect, i.e. this implies that for each $s \in (0, 1)$, if

$$|H| < \min \{2 \sin(\phi_s/2)/\sqrt{s+1}, \varepsilon_0 a\} \tag{3.14}$$

the arcs do not intersect. In particular, if we take $0 < s < s_0$ where s_0 is such that $r_{min}(s_0) = (a\varepsilon_0)^{-1}$, this ensures that $|H| < \varepsilon_0 a$ and non-self intersecting arcs.

Suppose now that $R > 0$, with $R < 1/2$. We can assume that H is nearly constant for fixed $0 < s < 1$, so that (3.14) is satisfied. Let r denote the radius of the corresponding circumference and (c_1, c_2) the coordinates of its center.

The situation is symmetric with respect to the diagonal with negative slope of $S_{(1,2)}$. The arc that joins $B(\xi_{(1,2)})$ to $B(\xi_{(2,2)})$ (following orientation) has its center on the line $x_2 = \frac{1}{\sqrt{s}}(x_1 - 1) + 2$.

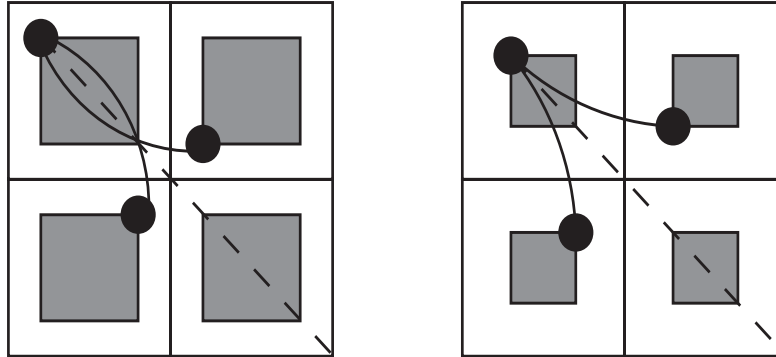


FIG. 4. Schematic picture showing how decreasing s may avoid self intersecting arcs. In this case the centers of the grains are placed at the corners of R_κ which makes the narrowest angle between arcs.

Let us assume, for definiteness, that $H > 0$ at the contact point; in that case, $c_1 > b_s + 1/2$. We recall that if $\rho_c < -\pi/4$, then, by symmetry, the arc that joins $B(\xi_{(1,2)})$ to $B(\xi_{(2,2)})$ intersects the arc that joins $B(\xi_{(1,2)})$ to $B(\xi_{(1,1)})$. Thus we have to find a sufficient condition on s that guarantees that $\rho_c > -\pi/4$, this will also imply a restriction on α .

We can derive the radius of curvature r_{min} for which the contact point has $\rho_c = -\pi/4$ and $\alpha = 0$. This gives

$$r_{min}(s) = \frac{\sqrt{(1+s)^2s + (3-s)^2} - 4\sqrt{s}R}{4\sqrt{s} \tan(\phi_s/2)}$$

which is an increasing function of s . The value $\alpha = 0$ gives the maximum of the contact angle for this radius of curvature. Then, if $s \in (0, s_0)$ with s_0 such that $r_{min}(s_0) = (a\varepsilon_0)^{-1}$, we obtain that $r > r_{min}(s)$, since $|H| < a\varepsilon_0$. Moreover, for such r given, there is a restriction on α , namely $\alpha \leq \alpha_{max}(s, r, R) < 0$ where $\alpha_{max}(s, r, R)$ is the contact angle formed by the arc that intersects $B(\xi_{(1,2)})$ at the point with $\rho_c = -\pi/4$. \square

In order to formulate the problem we identify each Q with a set of coordinates in \mathbb{N}^2 , namely:

DEFINITION 3.19 Let ε and $M = \varepsilon L$ as in Definition 3.16. Adopting percolation terminology, we refer to each sub-square S_κ and, without loss of generality, to R_κ of the partition defined in Definition 3.17 as a **site**.

We identify every $Q \in I_\varepsilon$, via a suitable homeomorphism, with the square $[0, M]^2 \subset \mathbb{R}^2$. We also denote by ∂Q the union of sites that intersect the boundary of Q .

Finally, the sites S_κ of $Q = [0, M]^2$ are identified with the coordinate of their top-right corner; if we identify S_κ with $\kappa = (i, j)$, $i, j \in \{1, 2, 3, \dots, M\}$, this denotes a site with top-right corner in (i, j) . Without loss of generality, $\kappa = (i, j)$ will be identified with R_κ .

We also identify the sides of ∂Q as follows:

$$\begin{aligned} \partial_1 &= \{1, 2, 3, \dots, M\} \times \{1\}, \quad \partial_2 = \{1\} \times \{1, 2, 3, \dots, M\}, \\ \partial_3 &= \{1, 2, 3, \dots, M\} \times \{M\}, \quad \partial_4 = \{M\} \times \{1, 2, 3, \dots, M\}. \end{aligned}$$

In the following we define what is meant by Q is ‘connected’ or ‘not connected’. This depends on whether the sites contain grains or not, and whether these are connected throughout Q .

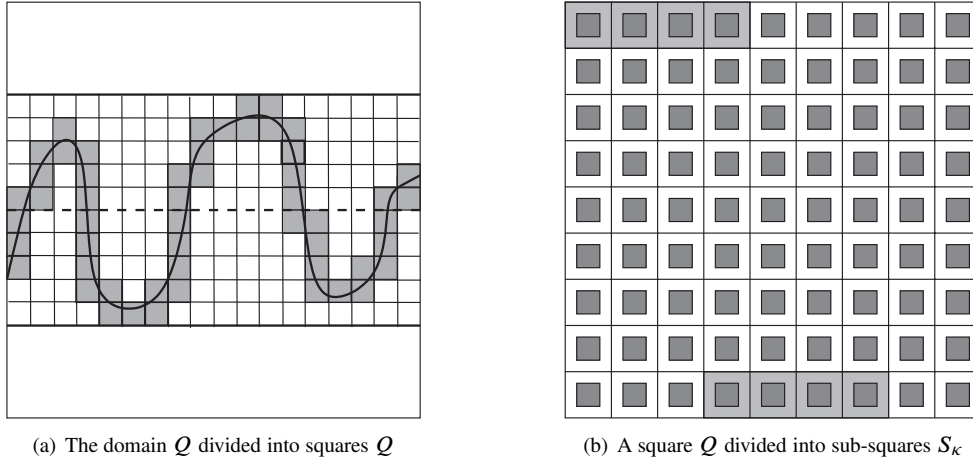


FIG. 5. This figure shows a schematic picture of the division of the domain Q that we use in the proof of Theorem 3.14. In (a) we show a curve Λ contained in Q (thick solid line). The dashed line represents $x_2 = \lambda$ and Q appears divided into squares Q of size $(\varepsilon L)^2$ with $B \ll \varepsilon \ll 1$. The shaded Q 's correspond the ones that intersect Λ . In (b) we show an arbitrary Q subdivided into squares S_κ of size one. Each S_κ contains the shaded region R_κ with $|R_\kappa| = s \in (0, 1)$. Here a portion of ∂Q that intersects Γ is highlighted.

DEFINITION 3.20 Let $\omega \in \Omega_\nu(L)$, where we identify $\omega = \{\xi_j\}_{j=1}^{\nu L^2}$, then:

- (i) We say that a site $\kappa \in \{1, 2, 3, \dots, M\}^2$ is *open* if there exists at least one $j \in \{1, \dots, \nu L^2\}$ such that $\xi_j \in R_\kappa$.
- (ii) Similarly, we say that a site $\kappa \in \{1, 2, 3, \dots, M\}^2$ is *closed* if $\omega \cap R_\kappa = \emptyset$.
- (iii) The following random variable determines whether a site is open or closed:

$$\tau(\kappa; \omega) = \begin{cases} 1, & \text{if } \sum_{j=1}^{\nu L^2} \chi_{R_\kappa}(\xi_j) > 0 \\ 0, & \text{if } \sum_{j=1}^{\nu L^2} \chi_{R_\kappa}(\xi_j) = 0, \end{cases} \quad (3.15)$$

here χ_A denotes the characteristic function for some $A \subset [0, M]^2$.

The following definition sets when open and closed sites are connected, see, e.g., [14] for similar definitions in two-dimensional site percolation.

DEFINITION 3.21 Let $\omega \in \Omega_\nu(L)$, we say that two sites $\kappa_0, \kappa \in \{1, 2, 3, \dots, M\}^2$ are

- (i) *adjacent*, and we write $\kappa_0 \mathcal{R}(\omega) \kappa$, if $\text{dist}(\kappa_0, \kappa) = 1$ and $\tau(\kappa_0, \omega) = \tau(\kappa, \omega) = 1$.
- (ii) *connected* if there exists a finite set of distinct sites, $\kappa_1, \dots, \kappa_j \in \{1, 2, 3, \dots, M\}^2$, such that every two such consecutive sites are adjacent, and $\kappa_0 \mathcal{R}(\omega) \kappa_1$ and $\kappa_j \mathcal{R}(\omega) \kappa$. A set $\{\kappa_0, \kappa_1, \dots, \kappa_j, \kappa\}$ with this property is called a *chain*.
- (iii) **-adjacent*, and we write $\kappa_0 \mathcal{R}^*(\omega) \kappa$, if $\text{dist}(\kappa_0, \kappa) \leq \sqrt{2}$ and $\tau(\kappa_0, \omega) = \tau(\kappa, \omega) = 1$.
- (iv) **-connected* if there exists a finite set of distinct sites, $\kappa_1, \dots, \kappa_j \in \{1, 2, 3, \dots, M\}^2$, such that every two such consecutive sites are *-adjacent, and $\kappa_0 \mathcal{R}^*(\omega) \kappa_1$ and $\kappa_j \mathcal{R}^*(\omega) \kappa$. A set $\{\kappa_0, \kappa_1, \dots, \kappa_j, \kappa\}$ with this property is called a **-chain*.

The next definition gives the concept of regions being *connected* and **-connected*.

DEFINITION 3.22 Let $A_1 \subset \partial_k$ and $A_2 \subset \partial_l$ for $k \neq l$ and $k, l = 1, 2, 3, 4$ be topologically connected regions of ∂Q . Then, for a given $\omega \in \Omega_\nu(L)$, we say that

- (i) A_1 is *connected* to A_2 if there exists a chain that intersects them.
- (ii) A_1 is **-connected* to A_2 if there exists a *-chain that intersects them.

We give a probability space to the configurations of sites induced by μ_ν :

DEFINITION 3.23 We endow $\Omega_M := \{0, 1\}^Q$ where $Q \equiv \{1, 2, 3, \dots, M\}^2$ with the structure of a probability space. We define the probability $\tilde{\mu}_M$ as follows, for any $A \in \Omega_M$ we set:

$$\tilde{\mu}_M(A) = \mu_\nu(\{\omega \in \Omega_\nu(L) : \tau(Q; \omega) \in A\}). \tag{3.16}$$

For example, let $C_\kappa = \{\kappa \in \{1, \dots, M\}^2 \text{ is closed}\}$ then

$$\tilde{\mu}_M(C_\kappa) = \mu_\nu(\{\omega \in \Omega_\nu(L) : \tau(\kappa; \omega) = 0\}) = \left(1 - \frac{|R_\kappa|}{L^2}\right)^{\nu L^2} \tag{3.17}$$

and let $O_\kappa = \{\kappa \in \{1, \dots, M\}^2 \text{ is open}\}$, then

$$\tilde{\mu}_M(O_\kappa) = \mu_\nu(\{\omega \in \Omega_\nu(L) : \tau(\kappa; \omega) = 1\}) = 1 - \tilde{\mu}_M(C_\kappa).$$

We observe, that in contrast to the classical site percolation scenario, the probabilities of two different sites of being open are not mutually independent, because for finite L the number of grains is finite, and that influences the presence of more or less grains in different sites.

On the other hand, since $|R_\kappa| = s$ for all $\kappa \in \{1, \dots, M\}^2$ then $\tilde{\mu}_M(C_\kappa) = q(L) \in [0, 1]$ and $\tilde{\mu}_M(O_\kappa) = 1 - q(L)$ are independent of $\kappa \in \{1, \dots, M\}^2$. Also, from (3.17) we get

$$q(L) \longrightarrow e^{-\nu s} \quad \text{as } L \rightarrow \infty. \tag{3.18}$$

Thus, in the limit $L \rightarrow \infty$, we find a classical site percolation scenario; in this limit the number of grains in the domain (as well as the domain size) goes to infinity, making independent of each other the events $\{\omega \in \Omega_\nu(L) : \tau(\kappa; \omega) = 1\}$. This means that there is a threshold value of ν , say ν_0 sufficiently large, such that the probability of finding an infinite cluster of closed sites is zero, i.e. that the dual event, finding an infinite cluster of open sites, has probability one, see, e.g., [14], [10], [21].

In our setting, we have the following percolation result:

Theorem 3.24 (Percolation result) *Let $A_1 \subset \partial_k$ and $A_2 \subset \partial_l$ for $k \neq l$ and $k, l = 1, 2, 3, 4$ be disjoint and topologically connected regions of Q of size N_1 and N_2 , respectively, with $N_1, N_2 \in (M/4, M/2)$. Let $C(A_1, A_2) \subset \Omega_M$ be the set of configurations of sites for which A_1 is connected to A_2 . Then, there exist $\nu_0 > 0$ and $L_0 > 0$ such that for all $\nu \geq \nu_0$ and $L \geq L_0$ for $\varepsilon(L)$ as in Definition 3.19 such that*

$$\tilde{\mu}_M(C(A_2, A_2)) \geq 1 - e^{-K \varepsilon(L) L}$$

for some $K > 0$ independent of L .

In order to prove this result we show that the probability of the dual or complementary events tends to zero. We identify the complementary events with the aid of the following topological lemma:

Lemma 3.25 *Let $A_1 \subset \partial_k$ and $A_2 \subset \partial_l$ for $k \neq l$ and $k, l = 1, 2, 3, 4$ be disjoint and topologically connected regions of ∂Q . Let A_1^c and A_2^c be the topologically connected components of $\partial Q \setminus (A_1 \cup A_2)$, then A_1 and A_2 are not connected if and only if A_1^c and A_2^c are $*$ -connected.*

Results analogous to Lemma 3.25 are classical in percolation theory and are related to the Jordan curve theorem (in this case the exterior of $A_1 \cup A_2$ in Q is $*$ -connected if and only if A_1 and A_2 are not connected in Q). Notice that the lemma is independent of any probability choice. The proof can be adapted from the literature, see, e.g., Bollobas [3].

We can now show Theorem 3.24.

Proof of Theorem 3.24. In view of Lemma 3.25 and taking into account that A_1 and A_2 have length in $(M/4, M/2)$, the crucial percolation result that we have to prove is that the probability of a $*$ -chain to reach a distance of order M tends to zero.

We first observe that given $\kappa \in \{1, 2, \dots, M\}^2$, let $\text{Chain}(\kappa, j) \subset \Omega_M$ with $j \in \mathbb{N}$ and $j \leq M^2$, be the set of site configurations for which there exist a $*$ -chain of the form $\{\kappa, \kappa_1, \dots, \kappa_{j-1}\}$, then

$$\tilde{\mu}_M(\text{Chain}(\kappa, j)) \leq 9^j \left(1 - \frac{js}{L^2}\right)^{\nu L^2}. \tag{3.19}$$

This is because the probability that a given set of j sites are closed is

$$\left(1 - \frac{js}{L^2}\right)^{\nu L^2},$$

by Lemma 3.3, and there are at most 9^j $*$ -chains with that length (since there are at most 9 sites $*$ -adjacent to any given closed site).

Let $J_h \in \{1, \dots, M\}$ the maximum horizontal distance from A_1 to A_2 and let $J_v \in \{1, \dots, M\}$ the maximal vertical distance from A_1 to A_2 . Then $J^* = \max\{J_h, J_v\}$ satisfies $J^* \in (M/4, M]$.

Let $C^*(A_1^c, A_2^c) \subset \Omega_M$ be the set of site configurations such that A_1^c and A_2^c are $*$ -connected, then for L sufficiently large

$$\tilde{\mu}_M(C^*(A_1^c, A_2^c)) \leq M \sum_{J^* \leq j \leq J^*M} 9^j \left(1 - \frac{js}{L^2}\right)^{\nu L^2} < M \sum_{J^* \leq j \leq J^*M} 9^j e^{-\nu js}. \tag{3.20}$$

Now, taking ν large enough we obtain that $\log(9) - \nu s < -1/2$, thus from (3.20) we get

$$\tilde{\mu}_M(C^*(A_1^c, A_2^c)) \leq M J^*(M - 1) e^{-\frac{J^*}{2}}. \tag{3.21}$$

We write $J^* = lM$ for certain value $l \in (1/4, 1]$, then

$$M^2 J^* e^{-\frac{J^*}{2}} = l M^3 e^{-\frac{1}{2}M}.$$

This and the definition of M (Definition 3.17) concludes the proof. □

We can now prove Theorem 3.14.

Proof of Theorem 3.14. Lemma 3.18 implies that we can determine the probability of a Γ intersecting Q through portions of ∂Q that intersect Λ by computing the probability of finding grain centers in the sub-boxes R_κ . This Lemma also give the restriction in the contact angle.

We let v_0 and L_0 and $\varepsilon(L)$ be as in Theorem 3.24 and assume that $v \geq v_0$ and $L \geq L_0$. Let $J_L(\Lambda) \subset \mathbb{N}$ such that if $i \in J_L(\Lambda)$ then $Q_i \subset I_{\varepsilon(L)}$ satisfies that $Q_i \cap \Lambda \neq \emptyset$. For every $i \in J_L(\Lambda)$, let also $S_1(Q_i), S_2(Q_i) \subset \partial Q_i$ such that $S_j(Q_i) \cap \Lambda \neq \emptyset$ for $j = 1, 2$ and with length in $(M/4, M/2)$ as in Theorem 3.24.

Then, by Theorem 3.24,

$$\tilde{\mu}_M\left(C(S_1(Q_i), S_2(Q_i))\right) \geq 1 - e^{-K\varepsilon(L)L},$$

i.e. for all $\mathcal{U}_i \subset \Omega_v(L)$ compatible with $C(S_1(Q_i), S_2(Q_i))$, $\mu_v(\mathcal{U}_i) \geq 1 - e^{-K\varepsilon(L)L}$. On the other hand

$$\mu_v\left(\bigcap_{i \in J_L(\Lambda)} \mathcal{U}_i\right) \geq 1 - \sum_{i \in J_L(\Lambda)} \mu_v(\mathcal{U}_i^c) \geq 1 - \#J_L(\Lambda)e^{-K\varepsilon(L)L}$$

Observe that $1/\varepsilon(L) < \#J_L(\Lambda) < 1/\varepsilon^2(L)$, thus taking $\varepsilon(L) = L^{-p}$ with $0 < p < 1$ we obtain the result with

$$\delta_\varepsilon(L) = 1 - L^{2p}e^{-KL^{1-p}}.$$

□

Appendix

A. Proof of Lemma 3.9

We divide the proof in several steps. First, we recall that we are assuming

$$L \gg \min\{\sqrt{B}, R^{-1}\} \quad \text{with} \quad B \rightarrow \infty \quad \text{and} \quad R \rightarrow 0. \tag{A1}$$

A rough estimate on the total length of a solution Γ that joins m centers is achieved by thinking of an interface having the longest possible length. We show that this worst-case scenario is given by an interface Γ having as many as possible folds elongating nearly horizontally from one vertical boundary to the other. It is clear that the longest elemental components of such interface will occur near $x_2 = v_1L$ and that their union connects very few grains, since for such pieces C is close to one (see Lemma 2.10).

Taking into account that the volume is fixed and that the grains are uniformly distributed (and do not see gravity) there must be folds on both sides of $x_2 = v_1L$. We can distinguish two types of interfaces, those that zigzag starting above $x_2 = v_1L$ and those starting below it.

Let us make the structure of such hypothetical interface solutions more precise.

As we did in Definition 2.12 we also distinguish different types of elemental components and parts of the interface. First, we distinguish those parts above $x_2 = v_1L$, that we indicate with a subindex $+$ (they have positive curvature), those below $x_2 = v_1L$, that we indicate with a superindex $-$ since (negative curvature), and those that cross $x_2 = v_1L$, that have curvature changing sign across $x_2 = v_1L$, and that we indicate with the superindex c . Then, we can define,

skipping the dependency on $(\omega, \lambda, \theta)$ for simplicity,

$$\Gamma = \Gamma^+ \cup \Gamma^- \cup \Gamma^c \quad \text{with} \quad \Gamma^+ = \bigcup_{i \in J^+} \gamma^{+,i}, \quad \Gamma^- = \bigcup_{j \in J^-} \gamma^{-,j}, \quad \text{and} \quad \Gamma^c = \bigcup_{k \in J^c} \gamma^{c,k}, \tag{A2}$$

and the set of indexes J^+, J^- and J^c are defined in the obvious way, they are disjoint and $J^+ \cup J^- \cup J^c = J$.

We distinguish, also in Definition 2.12, two types of elemental components, those that, following its orientation, go from left to right (with subscript lr), and those that go from right to left (subscript rl).

We further decompose Γ into horizontal levels:

DEFINITION A.1 (Horizontal level) Given an interface Γ , we say that a portion $\tilde{\Gamma} \subset \Gamma$ is a horizontal level of Γ if

- (i)
$$\min_{\xi_j \in \tilde{\Gamma}} \text{dist}(\xi_j, x_1 = 0) < 2R + 1 \quad \text{and} \quad \min_{\xi_j \in \tilde{\Gamma}} \text{dist}(\xi_j, x_1 = L) < 2R + 1,$$
- (ii) And, if we let $d_{\min}(\tilde{\Gamma}) = \min_{x \in \tilde{\Gamma}} \text{dist}(x, x_2 = v_1 L)$ and $d_{\max}(\tilde{\Gamma}) = \max_{x \in \tilde{\Gamma}} \text{dist}(x, x_2 = v_1 L)$, then

$$d_{\max}(\tilde{\Gamma}) - d_{\min}(\tilde{\Gamma}) \leq 2R + \frac{4}{\sqrt{B}} \frac{1}{1 + \sqrt{B} d_{\min}(\tilde{\Gamma})} \tag{A3}$$

We now give a definition of a curve that zigzags horizontally:

DEFINITION A.2 (Zigzag interface) A solution Γ of (1.1) with Young conditions is a zigzag interface if:

- (i) There exists a set of consecutive levels Γ_0 that contains Γ^c . One of these levels, that we denote by Γ_0^0 , must ‘cross’ $x_2 = v_1 L$, in the sense that if ξ_c is the first and ζ_c the last contact point connected by Γ_0^0 , then ξ_c and ζ_c lie on opposite sides of $x_2 = v_1 L$.
- (ii) There exists N^+ and $N^- \in \mathbb{N}$ with $N^-, N^+ \geq 0$ such that

$$\Gamma = \left(\bigcup_{i=1 \dots N^+} \Gamma_i^+ \right) \cup \left(\bigcup_{i=1 \dots N^-} \Gamma_i^- \right) \cup \Gamma_0$$

where $\Gamma_i^+ \in \Gamma^+$ with $i \in \{1, \dots, N^+\}$ and $\Gamma_j^- \in \Gamma^-$ with $j \in \{1, \dots, N^-\}$ are horizontal levels of Γ , indexed in decreasing order with proximity to $x_2 = v_1 L$.

REMARK A.3 We distinguish four type of interfaces, namely:

- (i) Type I: the level Γ_0^0 has elemental components of the form γ_{lr} and crosses from positive to negative curvature.
- (ii) Type II: the level Γ_0^0 has elemental components of the form γ_{rl} and crosses from positive to negative curvature.
- (iii) Type III: the level Γ_0^0 has elemental components of the form γ_{lr} and crosses from negative to positive curvature.
- (iv) Type IV: the level Γ_0^0 has elemental components of the form γ_{rl} and crosses from negative to positive curvature.

This definition does not exclude the possibility that other levels rather than the ones contained in Γ_0 have elemental components of the form γ_{lr} with $C < 1$.

We shall write

$$\Gamma_0 = \left(\bigcup_{j \in J_0^+} \gamma^{+,j} \right) \cup \left(\bigcup_{j \in J_0^-} \gamma^{-,j} \right) \cup \left(\bigcup_{j \in J_0^c} \gamma^{c,j} \right)$$

where, depending on the type of interface, J_0^+ and J_0^- might be empty.

Observe that there exists at most three consecutive levels in Γ_0 : Γ_0^0 , one above it Γ_0^1 , and one below it, Γ_0^{-1} . The levels Γ_0^1 and Γ_0^{-1} do not cross $x_2 = v_1 L$ in the sense that Γ_0^0 does, but individual elemental components of them can. This is clear from the orientation of the elemental components and Definition A.2 (specify in at least one case of the types above).

We might write:

$$\Gamma_0^k = \left(\bigcup_{j \in J_0^{k,+}} \gamma^{+,j} \right) \cup \left(\bigcup_{j \in J_0^{k,-}} \gamma^{-,j} \right) \cup \left(\bigcup_{j \in J_0^{k,c}} \gamma^{c,j} \right) \quad \text{for } k = -1, 0, 1.$$

We shall first derive Lemma 3.9 for zigzag interfaces in which Γ_0 has three levels. Then we prove that given the set of interfaces connecting m grains, then these interfaces maximise the length.

Observe that such Γ_0 's connect at least 4 centers, a pathological case given by a Γ_0 composed of just three elemental components, each being a level of Γ_0 : $\Gamma_0^0 = \{\gamma_{rl}^c\}$, oscillating around $x_2 = v_1 L$, $\Gamma_0^1 = \{\gamma_{lr}^+\}$, just above $x_2 = v_1 L$, and $\Gamma_0^{-1} = \{\gamma_{lr}^-\}$.

Let us first derive some elementary properties of zigzag interfaces:

Lemma A.4 *If an interface Γ is a zigzag interface of type I or III and such that its Γ_0 has exactly three levels, then:*

- (i) N^+ and N^- are odd numbers.
- (ii)

$$\Gamma_{2i-1}^\pm = \bigcup_{j \in J_{2i-1}^\pm} \gamma_{lr}^{\pm,j} \quad \text{and} \quad \Gamma_{2i}^\pm = \bigcup_{j \in J_{2i}^\pm} \gamma_{rl}^{\pm,j} \quad \text{with } i = 1, \dots, \frac{N^\pm + 1}{2}.$$

- (iii) *The index sets $\{J_i^+\}$ are disjoint with $\cup_{i=1 \dots N} J_i^+ \subset J^+$ and also $\{J_i^-\}$ are disjoint with $\cup_{i=1 \dots N} J_i^- \subset J^-$. These sets might be empty.*
- (iv) *If Γ connects m grains then $\gamma_1 \in \Gamma_{N^+}^+$ and $\gamma_{m+1} \in \Gamma_{N^-}^-$ if Γ is of type I and $\gamma_1 \in \Gamma_{N^+}^+$ and $\gamma_{m+1} \in \Gamma_{N^+}^+$ if Γ is of type III.*

We leave the analogous lemma for type II and IV to the reader. Figure A shows a sketch of the four types of zigzag interfaces with three levels in Γ_0 .

Proof of Lemma 3.9. Assume there exists an $\omega \in \Omega_v(L)$ and a $\Gamma(\omega, \lambda, \theta)$ that is a zigzag interface and connects m grains where $m \geq 3$ and has three levels in Γ_0 . It is clear any other interface that connects m grains is necessarily of the same length or shorter, the length of the interface in a strip of the size of Γ_0 being at most of the order of $3L$, outside the strip the length between connections of grains is of the same order as for a zigzag interface, according to Definition A.1(ii) and Corollary 3.1.

We can now prove the result for zigzag interfaces that join grains near $x_2 = v_1 L$ with the maximal possible length. Let us assume that Γ is of type I. Then, it has three levels in Γ_0 with three elemental components of the form γ^c whose length is of the order of L . Let

$$n_0 = \#J_0, \quad n_i^+ = \#J_i^+ \quad \text{and} \quad n_i^- = \#J_i^-,$$

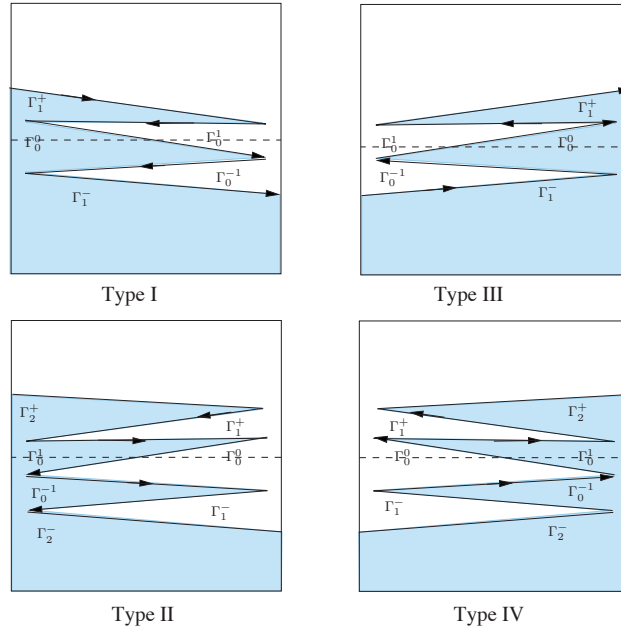


FIG. A. This picture shows a sketch of the four types of zigzag solutions of Definition A.2. The dashed line represents $x_2 = v_1 L$, the solid lines represent the interface. In the examples shown, the central horizontal level, Γ_0 , has three parts (Γ_0^0 , that crosses $x_2 = v_1 L$, Γ_0^1 , above $x_2 = v_1 L$, and Γ_0^{-1} , below $x_2 = v_1 L$) each represented by a straight line. In the examples of type I and III there is just one horizontal level above and below Γ_0 , and in the examples of type II and IV there are two above and two below.

where $J_0 = J_0^+ \cap J_0^- \cap J_0^c$, and J_i^\pm are as in Lemma A.4. Let us make abuse of notation by saying that $\#J_0^c = 3$, so that we index in this group only the pieces γ^c with length of order L , and the other pieces γ^c of smaller order of length we index them in J_0^+ if they are above the of the former, and in J_0^- if they are below. We then let $n_0^\pm = \#J_0^\pm$.

We recall that lemmas 2.10 and 2.11 imply that the other elemental components γ^c or γ^{lr} and γ^{rl} in Γ_0 have horizontal length comparable to the horizontal of elemental components in the even horizontal layers.

Thus we write

$$\sum_{i=0}^m D_i = \sum_{\substack{i=0 \dots m \\ i \in J_0^c}} D_i + \sum_{\substack{i=0 \dots m \\ i \in J_0^+}} D_i + \sum_{\substack{i=0 \dots m \\ i \in J_0^-}} D_i + \sum_{\substack{i=0 \dots m \\ i \notin J_0}} D_i. \tag{A4}$$

Using Definition A.1 (A3), we can estimate the first term in (A4) by

$$\sum_{\substack{i=0 \dots m \\ i \in J_0^c}} D_i \leq 3 \left(L^2 + \left(2R + \frac{4}{\sqrt{B}} \right)^2 \right)^{\frac{1}{2}} \leq 3 \left(L + 2R + \frac{4}{\sqrt{B}} \right), \tag{A5}$$

because $d_{min}(\Gamma_0^0) = d_{min}(\Gamma_0^{\pm 1}) = 0$.

We write $D_i = ((D_i^h)^2 + (D_i^v)^2)^{\frac{1}{2}} \leq D_i^h + D_i^v$ where $D_i^h = |\xi_i^{(1)} - \xi_i^{(1)}|$ and $D_i^v = |\xi_i^{(2)} - \xi_i^{(2)}|$. In particular,

$$\sum_{\substack{i=0\dots m \\ i \in J_0^+}} D_i^v + \sum_{\substack{i=0\dots m \\ i \in J_0^-}} D_i^v + \sum_{\substack{i=0\dots m \\ i \notin J_0}} D_i^v \leq (N^+ + N^- + 2) \left(2R + \frac{4}{\sqrt{B}} \right). \quad (\text{A6})$$

Where we still need to control $N^+ + N^-$ in terms of m .

We let $n_{\min}^{\pm, \text{even}} = \min_{j=0, \dots, (N^\pm - 1)/2} \{n_{2j}^\pm\}$ and $n_{\min}^{\pm, \text{odd}} = \min_{j=1, \dots, (N^\pm + 1)/2} \{n_{2j-1}^\pm\}$, and define $n_{\min}^{\text{even}} = \min\{n_{\min}^{\text{even}, +}, n_{\min}^{\text{even}, -}\}$ and $n_{\min}^{\text{odd}} = \min\{n_{\min}^{\text{odd}, +}, n_{\min}^{\text{odd}, -}\}$. It is clear that $n_{\min}^{\text{odd}} \geq 2$ and that $n_0 \geq 4$, then

$$\begin{aligned} m + 1 &= \sum_{i=1}^{N^+} n_i^+ + \sum_{i=1}^{N^-} n_i^- + n_0 = \sum_{i=0}^{N^+} n_i^+ + \sum_{i=0}^{N^-} n_i^- + 3 \\ &\geq (N^+ + 1) + (N^- + 1) + \sum_{j=0}^{(N^+ - 1)/2} n_{2j}^+ + \sum_{j=0}^{(N^- - 1)/2} n_{2j}^- + 3 \end{aligned}$$

and therefore,

$$m + 1 \geq N^+ + N^- + 5 + n_{\min}^{\text{even}} \frac{N^+ + N^- + 2}{2}. \quad (\text{A7})$$

Which in particular gives that $m \geq N^+ + N^- + 2$.

Let us get an estimate on the horizontal distances. Observe that

$$\sum_{\substack{j=0, \dots, (N^\pm - 1)/2 \\ i \in J_{2j}^\pm}} D_i^h \leq n_{\min}^{\text{even}} \frac{N^\pm + 1}{2} \max_{\substack{j=1, \dots, (N^\pm - 1)/2 \\ i \in J_{2j}}} D_i^h. \quad (\text{A8})$$

On the other hand, since the absolute value of the curvature of the pieces increases with the distance to the horizontal, for each $j \in \{1, \dots, (N^\pm - 1)/2\}$ we have $\sum_{i \in J_{2j-1}^\pm} D_i^h \leq 2 \sum_{i \in J_{2j}^\pm} D_i^h$. Then,

$$\sum_{\substack{j=0, \dots, \frac{N^\pm - 1}{2} \\ i \in J_{2j}^\pm \cup J_{2j+1}^\pm}} D_i^h \leq 3 \sum_{\substack{j=0, \dots, \frac{N^\pm - 1}{2} \\ i \in J_{2j}}} D_i^h. \quad (\text{A9})$$

We can now estimate the horizontal levels using (A9) and (A8), this gives

$$\sum_{\substack{i=0\dots m \\ i \in J_0^+}} D_i^h + \sum_{\substack{i=0\dots m \\ i \in J_0^-}} D_i^h + \sum_{\substack{i=0\dots m \\ i \notin J_0}} D_i^h \leq 3n_{\min}^{\text{even}} \frac{N^+ + N^- + 2}{2} \max_{\substack{j=0, \dots, (N^\pm - 1)/2 \\ i \in J_{2j}^+ \cup J_{2j}^-}} D_i^h + 2L, \quad (\text{A10})$$

where the horizontal size of Γ_{N^+} and of Γ_{N^-} is estimated by L .

Using the estimates (A5), (A6), (A7) and (A10) in (A4) we get

$$\sum_{i=1}^m D_i \leq (m - 2) \left(2R + \frac{4}{\sqrt{B}} \right) + 3(m - 4) \max_{i \in J_{2j}^+ \cup J_{2j}^-} D_i^h + 5L. \quad (\text{A11})$$

It is clear that the maximal distance between connected centers in the levels Γ_{2i}^\pm is achieved at an elemental component γ with $C = C^*$ in an elemental component of the form $\gamma_{r_l}^\pm$. Although we can only guarantee that $C^* > -1$, we observe that the horizontal distances between centers are uniformly bounded. Namely, if $C^* > 1$, Lemma 2.13 (ii) implies

$$\max_{i \in J_{2j}^+ \cup J_{2j}^-} D_i \leq 2R + \left(\frac{\pi^2 + 2}{B(C^* + 1)} \right)^{\frac{1}{2}} \leq 2R + \left(\frac{\pi^2 + 2}{2B} \right)^{\frac{1}{2}}.$$

If $0 < C^* < 1$ then Lemma 2.14 ((ii)) implies

$$\max_{i \in J_{2j}^+ \cup J_{2j}^-} D_i \leq 2R + \left(\frac{\pi^2 + 2}{B(C^* + 1)} \right)^{\frac{1}{2}} \leq 2R + \left(\frac{\pi^2 + 2}{B} \right)^{\frac{1}{2}},$$

and if $-1 < C^* \leq 0$, then Lemma 2.15 implies

$$\max_{i \in J_{2j}^+ \cup J_{2j}^-} D_i \leq 2R + \left(\frac{2(\pi^2 + 4)}{B} \right)^{\frac{1}{2}}.$$

Thus, taking the maximum of these upper bounds we obtain that

$$\sum_{i=1}^m D_i \leq (4m - 14) \left(2R + \frac{K}{\sqrt{B}} \right) + 5L$$

where $K = \sqrt{2(\pi^2 + 4)} \approx 5.3$ and this gives (3.7).

The proof for the other types of zigzag interfaces is similar and is left to the reader. \square

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