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Convergence of a threshold-type algorithm using the signed distance function

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We consider a threshold-type algorithm for curvature-dependent motions of hypersurfaces. This algorithm was numerically studied by [27], [9] and [35], where they used the signed distance function. It is also regarded as a variant of the Bence–Merriman–Osher algorithm for the mean curvature flow ([4]). In this paper we prove the convergence of our algorithm under the nonfattening condition, applying the method of [30] which is based on the notion of the generalized flow due to [3]. Then we derive the rate of convergence of our algorithm to the smooth and compact curvature-dependent motions and show its optimality to the special case of a circle evolving by its curvature. We also give a local estimate on the convergence to a regular portion of the generalized curvature-dependent motion.

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1. Introduction

In this paper we are concerned with the convergence of a threshold-type algorithm for curvature-dependent motions (CDM for short) of hypersurfaces. This was numerically studied by [27], [9] and [35] and is also regarded as a variant of the so-called Bence–Merriman–Osher algorithm to the mean curvature flow (MCF for short, cf. [4]).

Let $\{\Gamma(t)\}_{t\in[0,T)}$ be a family of compact hypersurfaces in $\mathbb{R}^N$. We say this family is a CDM if $\Gamma(t)$ moves by the following equation:

$$V = \kappa + \langle \mathbf{b}, \mathbf{n}\rangle + g \quad \text{on } \Gamma(t),\ t \in (0, T). \tag{1.1}$$

Here $T > 0$, $\mathbf{n} = \mathbf{n}(t, x)$ is the inner unit normal vector field on $\Gamma(t)$, $V = V(t, x)$ is the velocity of $\Gamma(t)$ in the direction of $\mathbf{n}$, $\kappa = \kappa(t, x)(:= -\operatorname{div}\mathbf{n}(t, x))$ is the $((N-1)$-times) mean curvature of $\Gamma(t)$, $\mathbf{b} = \mathbf{b}(t, x) = (b^1(t, x), \cdots, b^N(t, x))$ denotes a given vector field in $\mathbb{R}^N$, $g = g(t, x)$ is a forcing term and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^N$. As well known, the MCF is the case of $\mathbf{b} \equiv \mathbf{0}$ and $g \equiv 0$. The CDM arises in various fields such as two-phase Stefan problems, phase transitions, image processing, two-phase fluid flows and so on.

The main feature of the CDM is the development of singularities in finite time even if $\mathbf{b}$, g and the initial hypersurface are sufficiently smooth. To interpret the evolution past the singularities, the level set approach was introduced for numerical computations by [37] and was rigorously developed by [7] and independently by [11]. In these papers the authors proposed the notion of *generalized motion* by (1.1) and proved the well-posedness of the level set equation to (1.1) in the sense of viscosity solutions and the well-definedness of the generalized motion by (1.1). Since then many people have developed the theory of the generalized motion and its applications. See [38], [24] and [3] and the references therein. The book [15] provides a self-contained introduction to the level set approach for various surface evolution equations.

From the viewpoints of the above applications, many people have studied numerical computations for CDM. Among many numerical methods for CDM, we treat the following algorithm: Let C_0 be a compact set in $\mathbb{R}^N$ and fix a time step $h > 0$. For $k = 0, 1, 2, \ldots$, set $\mathbf{b}_k(t, x) := \mathbf{b}(t + kh, x)$ and $g_k(t, x) := g(t + kh, x)$. Let $w_0 = w_0(t, x)$ be a unique solution of the initial value problem for the linear parabolic equation with $k = 0$:

$$w_t - \Delta w + \langle \mathbf{b}_k, Dw \rangle + g_k = 0 \quad \text{in } (0, h] \times \mathbb{R}^N, \tag{1.2}$$

$$w(0, x) = d(x, C_k) \quad \text{for } x \in \mathbb{R}^N. \tag{1.3}$$

Here $d(x, D)$ is the signed distance function to ∂D defined by

$$d(x, D) := \begin{cases} \text{dist}\,(x, \partial D) & \text{for } x \in D, \\ -\text{dist}\,(x, \partial D) & \text{for } x \notin D, \end{cases} \tag{1.4}$$

for each closed subset $D(\neq \emptyset)$ of $\mathbb{R}^N$. We then set

$$C_1 := \{w_0(h, \cdot) \geqslant 0\}. \tag{1.5}$$

Let w_1 be a unique solution of (1.2) - (1.3) with $k = 1$. Again we define C_2 as the set in (1.5) with w_1 replacing w_0. Repeating this process, we have a sequence $\{C_k\}_{k=0}^{+\infty}$ of compact subsets of $\mathbb{R}^N$. We then set

$$C^h(t) := C_k \quad \text{for } t \in [kh, (k+1)h),\ k = 0, 1, 2, \ldots \tag{1.6}$$

Letting $h \to 0$, we formally obtain a limit flow $\{C(t)\}_{t \geqslant 0}$ of compact sets in $\mathbb{R}^N$ and observe that $\partial C(t)$ moves by (1.1) with the initial data ∂C_0.

The above algorithm was numerically studied by [27] and [9]. In [27] Kimura and Notsu proposed a fully discrete finite element scheme based on the above level set method of the signed distance function. In [27, Section 4] they gave some numerical examples for MCF with a forcing term. In [9] Esedoḡlu, Ruuth and Tsai considered various geometric motions with using the signed distance function, including CDM, MCF with triple junctions and the motion by surface diffusion. The extension of the signed distance approach to vector setting for numerical computation of multiphase problems was addressed in [35]. This algorithm is regarded as a variant of the Bence–Merriman–Osher (BMO for short) algorithm to MCF (cf. [4]), which utilizes the solutions of the usual heat equation, continually reinitialized after short time steps. The BMO algorithm and its generalizations are studied by many people. See [34], [10], [1], [18] and [25] etc. for the convergence of the BMO algorithm and [20], [23], [39], [30] etc. for some generalizations. In particular, in [39] and [30] Vivier and Leoni generalized the BMO algorithm with using the linear/semilinear parabolic equations and proved the convergence of their scheme to the anisotropic CDM's associated with these equations. Our algorithm is quite similar to theirs on the point that we use the linear parabolic equation (1.2) to construct the approximate sequence for CDM. However, the choice of the initial data of each time step is the main difference between the (generalized) BMO algorithm and ours. In the (generalized) BMO algorithm they choose the initial data

$$w(0,x) = \begin{cases} 1 & \text{for } x \in C_k, \\ -1 & \text{for } x \notin C_k, \end{cases} \quad (= \operatorname{sgn}^*(d(x, C_k)))$$

instead of (1.3), where $\operatorname{sgn}^*(r) := 1$ for $r \geqslant 0$, $:= -1$ for $r < 0$. In addition, we refer to [6], where Chambolle and Novaga considered an algorithm to the anisotropic mean curvature flow with using a subdifferential inclusion and proved the convergence to a compact and smooth flow. Their algorithm is also quite similar to ours, but several points are different, e.g., the approximate equation, the choice of the initial data etc.

The purposes of this paper are to derive the convergence of the above algorithm to the generalized CDM under the nonfattening condition, the optimal rate of convergence of this algorithm to the smooth and compact one and a local estimate on the convergence to the generalized CDM. Related to our results, [25] derived the optimal rate of convergence of the BMO algorithm for MCF and [36] gave a local estimate on the convergence of a bilateral obstacle problem to the generalized CDM.

The strategy in proving the convergence of the flow $\{C^h(t)\}_{t\in[0,T),h>0}$ is to apply the method of [30], where she made use of the notion of the generalized flow by [3] and constructed suitable sub- and super-solutions to her approximation scheme. We also use the arguments in [26] to show the convergence of the sequences $\{w^h\}_{h>0}$, $\{d^h\}_{h>0}$ of the functions given by $w^h(t,x) := w_k(t - kh, x)$ and $d^h(t,x) := d(x, C_k)$ for $(t,x) \in [kh, (k+1)h) \times \mathbb{R}^N$ and $k = 0, 1, 2, \ldots, [T/h]$. Hence we are able to prove that

$$\begin{aligned} &\lim_{h\to 0} w^h = \lim_{h\to 0} d^h = d \quad \text{locally uniformly in } [0,T) \times \mathbb{R}^N, \\ &\lim_{h\to 0} d_H(C^h(t), C(t)) = 0 \quad \text{locally uniformly in } [0,T), \\ &C(t) := \{d(t,\cdot) \geqslant 0\}, \\ &d = d(t,x) \ : \ \text{signed distance function to } \partial C(t), \end{aligned}$$

and that $\{\partial C(t)\}_{t\in[0,T)}$ is a generalized CDM. Here d_H is the Hausdorff distance defined at the end of this introduction. In order to derive the rate of convergence to the smooth and compact CDM, we directly calculate the distance between CDM and the approximate motion along the characteristics. For this purpose the estimate of Dw_k from below plays an important role. Consequently, we obtain that for any $\varepsilon > 0$, there are constants $L_1, h_0 > 0$ such that

$$\sup_{t\in[0,T-\varepsilon]} d_H(C^h(t), C(t)) \leqslant L_1 h \quad \text{for all } h \in (0, h_0). \tag{1.7}$$

The optimality of this estimate is obtained by precise calculations in the case of a circle evolving by curvature. The ideas in considering a local estimate on the convergence to the generalized CDM are to introduce the graph-like equation of (1.1) and to get a local regularity of the generalized CDM. They are based on [13, Section 5].

This paper is organized as follows. In Section 2 we state our assumptions and recall the notion and some results of the generalized CDM. As for the latter one, we briefly explain the notions of the generalized CDM in the sense of [7], [11] and that of [3] in Section 2.3. Note that these two notions are equivalent under the nonfattening condition. Section 3 is devoted to some estimates on solutions of (1.2) and $\{C^h(t)\}_{t\in[0,T),h>0}$. In Section 4 we study semicontinuous limits of $\{w^h\}_{h>0}$ and $\{d^h\}_{h>0}$ and characterize those of $\{d^h\}_{h>0}$ by means of the eikonal equations. Section 5 is devoted to the convergence of our algorithm. In Section 6 we obtain (1.7) in the case of the smooth and compact CDM and show its optimality. In Section 7 we treat the rate of convergence to a regular portion of the generalized CDM. Section 8 is the appendix.

We do not precisely explain the definition and the theory of viscosity solutions of the level set equation to (1.1). We refer to [8], [28] and [15] for them. Throughout this paper, we use the following notations: For $m \in \mathbb{N}\cup\{0\}$, $\alpha \in (0,1)$, $Q \subset [0,T)\times\mathbb{R}^N$, $f : Q \longrightarrow \mathbb{R}$ and $\mathbf{f} = (f^1,\cdots,f^N) : Q \longrightarrow \mathbb{R}^N$,

$$
\begin{aligned}
&Df = D_x f := (\partial f/\partial x_1,\cdots,\partial f/\partial x_N),\, D_t f := f_t = \partial f/\partial t,\\
&D_x^l f := \partial^{|l|} f/\partial x_1^{l_1}\cdots\partial x_N^{l_N},\, |l| = l_1+\cdots+l_N \text{ for } l=(l_1,\cdots,l_N)\in\big(\mathbb{N}\cup\{0\}\big)^N,\\
&D^2 f := (\partial^2 f/\partial x_i\partial x_j)_{1\leqslant i,j\leqslant N},\ \mathbf{Df} := (\partial f^i/\partial x_j)_{1\leqslant i,j,\leqslant N},\\
&|f|_{0,Q} := \sup_{(t,x)\in Q}|f(t,x)|,\ \langle f\rangle_{x,\alpha,Q} := \sup_{(t,x),(t,y)\in Q}\frac{|f(t,x)-f(t,y)|}{|x-y|^\alpha},\\
&\langle f\rangle_{t,\alpha/2,Q} := \sup_{(t,x),(s,x)\in Q}\frac{|f(t,x)-f(s,x)|}{|t-s|^{\alpha/2}},\\
&\|f\|_Q^{(m+\alpha)} := \sum_{2k+|l|\leqslant m}|D_t^k D_x^l f|_{0,Q} + \sum_{2k+|l|=m}\langle D_t^k D_x^l f\rangle_{x,\alpha,Q}\\
&\qquad\qquad + \sum_{0<(m+\alpha)-2k-|l|<2}\langle D_t^k D_x^l f\rangle_{t,(m+\alpha-2k-|l|)/2,Q},\\
&\|\mathbf{f}\|_Q^{(m+\alpha)} := \sqrt{\sum_{i=1}^N\left(\|f^i\|_Q^{(m+\alpha)}\right)^2},\ |\mathbf{f}|_{0,Q} := \sqrt{\sum_{i=1}^N|f^i|_{0,Q}^2},\, |D\mathbf{f}|_{0,Q} := \sqrt{\sum_{i,j=1}^N\left|\frac{\partial f^i}{\partial x_j}\right|_{0,Q}^2}.
\end{aligned}
$$

For $m \in \mathbb{N}\cup\{0\}$, $\Omega\subset\mathbb{R}^N$ and $f:\Omega\longrightarrow\mathbb{R}$,

$$\|f\|_\Omega^{(m)} := \sum_{|l|\leqslant m}\sup_{x\in\Omega}|D_x^l f(x)|.$$

For $u:\mathbb{R}^N\longrightarrow\mathbb{R}$, $v:[0,T)\times\mathbb{R}^N\longrightarrow\mathbb{R}$ and $\mu\in\mathbb{R}$,

$$
\begin{aligned}
&\{u\geqslant\mu\} := \{x\in\mathbb{R}^N \mid u(x)\geqslant\mu\},\\
&\{v\geqslant\mu\} := \{(t,x)\in[0,T)\times\mathbb{R}^N \mid v(t,x)\geqslant\mu\},\\
&\{v(t,\cdot)\geqslant\mu\} := \{x\in\mathbb{R}^N \mid v(t,x)\geqslant\mu\}, \text{ etc.}
\end{aligned}
$$

Let U be a metric space and V a dense subset of U.

$$UC(U) = \text{the set of all uniformly continuous functions.}$$

For $u : V\longrightarrow\mathbb{R}$ and $x\in U$,

$$u^*(x) := \limsup_{V\ni y\to x} u(y),\ u_*(x) := \liminf_{V\ni y\to x} u(y).$$

For $Q \subset [0,T) \times \mathbb{R}^N$, $f : Q \longrightarrow \mathbb{R}$ (or $\mathbb{R}^N, \mathbb{S}^N$) and $g, v : Q \longrightarrow \mathbb{R}$,

$$f(t,x) = O\big(g(t,x)\big) \Longleftrightarrow |f(t,x)| \leqslant Kg(t,x) \text{ for some } K > 0 \text{ independent of } (t,x) \in Q,$$
$$\widetilde{f}_v = \widetilde{f}_v(t,x) := f(t, x - v(t,x)Dv(t,x)).$$

For $Q' \subset (0,T) \times \mathbb{R}^{N-1}$ and $v : Q' \longrightarrow \mathbb{R}$,

$$D'v := (\partial v/\partial x_1, \cdots, \partial v/\partial x_{N-1}),\ D'^2 v := (\partial^2 v/\partial x_i \partial x_j)_{1 \leqslant i,j \leqslant N-1},$$
$$\Delta' v := \frac{\partial^2 v}{\partial x_1^2} + \cdots + \frac{\partial^2 v}{\partial x_{N-1}^2}.$$

Besides we use the following symbols.

$\langle p, q\rangle$ = the inner product between $p, q \in \mathbb{R}^N$,
$\mathrm{cl}\, A$ = the closure of A, $\mathrm{int}\, A$ = the interior of A for at set $A \subset \mathbb{R}^N$,
χ_A = the characteristic function for a set $A \subset \mathbb{R}^N$,
$P(x,\delta) := \prod_{i=1}^{N}(x_i - \delta, x_i + \delta)$ for $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$ and $\delta > 0$
= N-dimensional open cube centered at x,
$Q_T := (0,T) \times \mathbb{R}^N$, $Q_h := (0,h) \times \mathbb{R}^N$, $Q((t,x),r) := (t-r, t+r) \times B(x,r)$,
$Q'\big((t,x),r\big) := (t-r,t+r) \times B'(x,r)$, $B'(x,r) := B(x,r) \cap \big(\{x\} + \mathbb{R}^{N-1}\big)$,
$[r]$ = Gauss symbol for $r \in \mathbb{R}$,
$\mathbb{S}^N$ = the set of all $N \times N$-real symmetric matrices,
$\mathrm{tr}\, X$ = the trace of $X \in \mathbb{S}^N$,
$d_H(A,B) := \max\left\{\sup_{x\in A} \mathrm{dist}\,(x,B), \sup_{x \in B} \mathrm{dist}\,(x,A)\right\}$ for $A, B \subset \mathbb{R}^N$
= Hausdorff distance between the sets A and B,
$U(t,x) := (4\pi t)^{-N/2} \exp(-|x|^2/4t)$.

2. Preliminaries

2.1 *Assumptions*

Fix $T, h > 0$. Throughout this paper we assume that the functions $\mathbf{b}$ and g are so smooth that for some $\alpha \in (0,1)$

$$\|f\|^{(3+\alpha)}_{\mathrm{cl}Q_T} < +\infty \quad (f = \mathbf{b}, g). \tag{2.1}$$

Then for each $k = 0, 1, 2, \ldots, [T/h]$, there is a unique classical solution $w_k \in C(\mathrm{cl}\, Q_h) \cap C^{(5+\alpha)/2, 5+\alpha}(Q_h)$ of (1.2) - (1.3) satisfying

$$\sup_{\substack{(t,x)\in \mathrm{cl}Q_h \\ k=0,1,2,\cdots,[T/h]}} \frac{|w_k(t,x)|}{|x|+1} < +\infty, \quad \sup_{\substack{k=0,1,2,\cdots,[T/h] \\ 0<h<1}} \|Dw_k\|_{L^\infty(Q_h)} =: K_1 < +\infty. \tag{2.2}$$

See, e.g., [29]. Note that the former bound may depend on $h > 0$.

For a given compact hypersurface $\Gamma_0 \subset \mathbb{R}^N$, assume that

$$\Gamma_0 \in C^{5+\alpha} \quad \text{for some } \alpha \in (0,1). \tag{2.3}$$

As will be seen in Appendix, there uniquely exists a smooth and compact CDM $\{\Gamma(t)\}_{t\in[0,T_0)}$ with $\Gamma(0) = \Gamma_0$ for some $T_0 > 0$. Define the signed distance function $\rho(t,x)$ to $\Gamma(t)$ by

$$\rho(t,x) := d\big(x, D(t)\big) \tag{2.4}$$

where $D(t)$ denotes the compact set such that $\partial D(t) = \Gamma(t)$ and $d(x, D(t))$ is defined by (1.4) with $D = D(t)$ for each $t \in [0, T_0)$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\rho\|^{(5+\alpha)}_{N_{\varepsilon,10\delta}} < +\infty, \; N_{\varepsilon,10\delta} := \big\{(t,x) \in [0, T_0 - \varepsilon] \times \mathbb{R}^N \mid |\rho(t,x)| \leqslant 10\delta\big\}. \tag{2.5}$$

This property will be used in section 5.

2.2 *Signed distance function and CDM*

Let $\{\Gamma(t)\}_{t\in[0,T)}$ be a smooth and compact CDM and let ρ be defined by (2.4) and satisfy (2.5). As $V = -\rho_t$, $\mathbf{n} = D\rho$ and $\kappa = -\Delta\rho$ on $\Gamma(t)$, (1.1) is equivalent to

$$\rho_t - \Delta\rho + \langle \mathbf{b}, D\rho\rangle + g = 0 \quad \text{on } \Gamma := \bigcup_{t\in[0,T)} \big(\{t\} \times \Gamma(t)\big). \tag{2.6}$$

We show some inequalities and an equation which d satisfies in $N_{\varepsilon,10\delta}$. For any $(t,x) \in N_{\varepsilon,10\delta}$ the point $y = x - \rho(t,x)D\rho(t,x)$ is a unique minimizer of $|y - x|$ among $y \in \Gamma(t)$. The eigenvalues $\{\lambda^i = \lambda^i(D^2\rho(t,x))\}_{i=1}^N$ of the matrix $D^2\rho$ at (t,x) are

$$\lambda^i := -\frac{\widetilde{\kappa}^i_\rho}{1 - \widetilde{\kappa}^i_\rho \rho} \quad \text{for } i = 1, 2, \dots, N-1, \; \lambda^N = 0. \tag{2.7}$$

Here $\{\kappa^i = \kappa^i(t,y)\}_{i=1}^{N-1}$ denotes the principal curvatures of $\Gamma(t)$ at $y \in \Gamma(t)$ with respect to the direction $\mathbf{n}(= D\rho)$ (cf. [17, Section 14.6]). We get from (2.7)

$$\widetilde{\kappa}^i_\rho = \frac{\lambda^i}{\lambda^i\rho - 1} \quad \text{for } i = 1, 2, \dots, N-1. \tag{2.8}$$

Since $\widetilde{(\Delta\rho)}_\rho = -\widetilde{\kappa}_\rho = -\sum_{i=1}^{N-1} \widetilde{\kappa}^i_\rho$, (2.6) is rewritten as

$$\rho_t - \widetilde{(\Delta\rho)}_\rho + \langle \widetilde{\mathbf{b}}_\rho, D\rho\rangle + \widetilde{g}_\rho = 0 \quad \text{in } N_{\varepsilon,10\delta} \tag{2.9}$$

because $\widetilde{(\rho_t)}_\rho = \rho_t$ and $\widetilde{\mathbf{n}}_\rho = \widetilde{(D\rho)}_\rho = D\rho$ on $N_{\varepsilon,10\delta}$. Hence it follows from [17, Section 14.6] that

$$\rho_t - \Delta\rho + \langle \widetilde{\mathbf{b}}_\rho, D\rho\rangle + \widetilde{g}_\rho \geqslant 0 \quad \text{in } N_{\varepsilon,10\delta} \cap \{\rho \geqslant 0\}, \tag{2.10}$$

$$\rho_t - \Delta\rho + \langle \widetilde{\mathbf{b}}_\rho, D\rho\rangle + \widetilde{g}_\rho \leqslant 0 \quad \text{in } N_{\varepsilon,10\delta} \cap \{\rho \leqslant 0\}. \tag{2.11}$$

On the other hand, substituting (2.8) into (2.9) we have

$$\rho_t - F_0(\rho, D^2\rho) + \langle \widetilde{\mathbf{b}}_\rho, D\rho\rangle + \widetilde{g}_\rho = 0 \quad \text{in } N_{\varepsilon,10\delta}, \tag{2.12}$$

$$F_0(r, X) = \sum_{i=1}^{N} \frac{\lambda^i(X)}{1-\lambda^i(X)r} \quad \text{for } (r, X) \in \mathbb{R}\times\mathbb{S}^N. \tag{2.13}$$

Note by [12, p.323] that F_0 is smooth and uniformly elliptic in a neighborhood of $(0, O)$ in $\mathbb{R}\times\mathbb{S}^N$.

2.3 *Level set equation and generalized CDM*

In this subsection we collect the notions of the level-set flow and the generalized flow by (1.1) and known results on them according to [7], [16], [24], [3] and [15].

The level set equation to (1.1) is given by

$$u_t + F(t, x, Du, D^2u) = 0 \quad \text{in } Q_T, \tag{2.14}$$

$$F(t, x, p, X) := -\mathrm{tr}X + \frac{\langle Xp, p\rangle}{|p|^2} + \langle \mathbf{b}(t, x), p\rangle + g(t, x)|p|$$
$$\text{for } \big((t, x), p, X\big) \in Q_T \times \big(\mathbb{R}^N\backslash\{0\}\big)\times\mathbb{S}^N.$$

See [7] and [15] for the derivation of this equation. The F is degenerate elliptic, that is, for each $((t, x), p) \in Q_T \times (\mathbb{R}^N\backslash\{0\})$

$$F(t, x, p, X) \leqslant F(t, x, p, Y) \quad \text{for all } X, Y \in \mathbb{S}^N \text{ satisfying } X \geqslant Y.$$

In addition, it satisfies the property called geometric:

$$F(t, x, \lambda p, \lambda X + \mu p\otimes p) = \lambda F(t, x, p, X)$$

for all $\lambda > 0$, $\mu \in \mathbb{R}$, $(t, x) \in Q_T$, $p \in \mathbb{R}^N\backslash\{0\}$ and $X \in \mathbb{S}^N$. Since (2.14) has a singularity at $p = 0$, we adopt the notion of viscosity solutions to consider weak solutions of (2.14).

DEFINITION 2.1 Let u be a locally bounded function in Q_T. We say that u is a viscosity subsolution (resp., supersolution) of (2.14) provided that for any $\varphi \in C^\infty(Q_T)$, if $u^* - \varphi$ (resp., $u_* - \varphi$) takes a local maximum (resp., minimum) at (t_0, x_0), then

$$\varphi_t(t_0, x_0) + F_*\big(t_0, x_0, D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)\big) \leqslant 0$$
$$\big(\text{resp., } \varphi_t(t_0, x_0) + F^*\big(t_0, x_0, D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)\big) \geqslant 0\big).$$

We say that u is a viscosity solution of (2.14) if u is a viscosity subsolution and a viscosity supersolution of (2.14).

DEFINITION 2.2 Let $u \in C([0, T)\times\mathbb{R}^N)$ be a viscosity solution of (2.14). Set

$$\Gamma_L(t) := \{u(t,\cdot) = 0\},\ \Omega_L^+(t) := \{u(t,\cdot) > 0\},\ \Omega_L^-(t) := \{u(t,\cdot) < 0\} \tag{2.15}$$

for each $t \in [0, T)$. We call the family $(\Gamma_L(t), \Omega_L^+(t), \Omega_L^-(t))_{t\in[0,T)}$ a level-set flow by (1.1).

We recall the comparison principle and existence of viscosity solutions of (2.14) and the well-definedness of the level-set flow by (1.1), according to [16], [24] and [15].

Theorem 2.3 *Assume* (2.1). *Let u and v be, respectively, a viscosity subsolution and a viscosity supersolution of* (2.14). *If $u^*(0,\cdot) \leqslant v_*(0,\cdot)$ in $\mathbb{R}^N$, $u^*(t,x) \leqslant C(1+|x|)$, $v_*(t,x) \geqslant -C(1+|x|)$ for all $(t,x) \in [0,T)\times\mathbb{R}^N$ and some $C>0$ and either $u^*(0,\cdot)$ or $v_*(0,\cdot) \in UC(\mathbb{R}^N)$, then $u^* \leqslant v_*$ in $[0,T)\times\mathbb{R}^N$. Moreover, for any $u_0 \in UC(\mathbb{R}^N)$ there is a unique viscosity solution $u \in UC([0,T)\times\mathbb{R}^N)$ of* (2.14).

Theorem 2.4 *Assume* (2.1). *Let $(\varGamma_L(t), \varOmega_L^+(t), \varOmega_L^-(t))_{t\in[0,T)}$ be defined by (2.15). Here $u \in UC([0,T)\times\mathbb{R}^N)$ is a unique viscosity solution of* (2.14) *with the initial data $u_0 \in UC(\mathbb{R}^N)$. Then this family is determined independently of the choice of $u_0 \in UC(\mathbb{R}^N)$ satisfying $\varGamma_L(0) = \{u_0 = 0\}$, $\varOmega_L^+(0) = \{u_0 > 0\}$ and $\varOmega_L^-(0) = \{u_0 < 0\}$.*

REMARK 2.5 In the following of this paper, based on Theorems 2.3 and 2.4 we shall always consider that the level-set flow $(\varGamma_L(t), \varOmega_L^+(t), \varOmega_L^-(t))_{t\in[0,T)}$ by (1.1) is given in the same way as (2.15) by using a viscosity solution $u \in UC([0,T)\times\mathbb{R}^N)$ satisfying $u(0,\cdot) \in UC(\mathbb{R}^N)$ and $\varGamma_L(0) = \{u(0,\cdot) = 0\}$, $\varOmega_L^+(0) = \{u(0,\cdot) > 0\}$, $\varOmega_L^-(0) = \{u(0,\cdot) < 0\}$.

We assume that the level-set flow $(\varGamma_L(t), \varOmega_L^+(t), \varOmega_L^-(t))_{t\in[0,T)}$ by (1.1) satisfies the nonfattening condition:

$$\varGamma_L(t) = \partial\varOmega_L^+(t) = \partial\varOmega_L^-(t) \quad \text{for all } t \in [0,T). \tag{2.16}$$

Then we have the uniqueness of viscosity solution of (2.14) with a discontinuous initial data.

Theorem 2.6 (cf. [2, Theorem 2.1], [3, Proposition 2.1]) *Assume (2.1). Let $(\varGamma_L(t), \varOmega_L^+(t), \varOmega_L^-(t))_{t\in[0,T)}$ be a level-set flow by (1.1).*

(1) *The condition* (2.16) *holds if and only if the initial value problem* (2.14) *with $u(0,x) = \chi_{\varOmega_L^+(0)} - \chi_{\varOmega_L^-(0)}$ has a unique discontinuous viscosity solution.*
(2) *Assume that* (2.16) *fails. Then for any upper semicontinuous viscosity subsolution $w = w(t,x)$ of* (2.14) *with $w(0,\cdot) \leqslant \chi_{\varOmega_L^+(0)\cup\varGamma_L(0)} - \chi_{\varOmega_L^-(0)}$ in $\mathbb{R}^N$, we have $w \leqslant \chi_{\varOmega_L^+(t)\cup\varGamma_L(t)} - \chi_{\varOmega_L^-(t)}$ in $[0,T)\times\mathbb{R}^N$. Similarly, for any lower semicontinuous viscosity supersolution $w = w(t,x)$ of* (1.1) *with $w(0,\cdot) \geqslant \chi_{\varOmega_L^+(0)} - \chi_{\varOmega_L^-(0)\cup\varGamma_L(0)}$ in $\mathbb{R}^N$, we have $w \geqslant \chi_{\varOmega_L^+(t)} - \chi_{\varOmega_L^-(t)\cup\varGamma_L(t)}$ in $[0,T)\times\mathbb{R}^N$.*

In Section 4 we apply the notion of the generalized flow by (1.1). This notion is proposed in [3] and is equivalent to the level-set flow under (2.16).

DEFINITION 2.7 Let $\{\varOmega_G(t)\}_{t\in[0,T)}$ be a family of open subsets of $\mathbb{R}^N$. We say $\{\varOmega_G(t)\}_{t\in[0,T)}$ is a generalized superflow (resp., subflow) by (1.1) provided for any $t>0$, $x_0 \in \mathbb{R}^N$, $r>0$ and $\alpha>0$ and for any $\phi \in C^\infty(\mathbb{R}^N)$ such that $\{\phi \geqslant 0\} \subset \varOmega_G(t) \cap B(x_0,r)$ (resp., $\{\phi \leqslant 0\} \subset (\mathbb{R}^N\backslash \mathrm{cl}\,\varOmega(t)) \cap B(x_0,r)$) with $|D\phi| \neq 0$ on $\{\phi = 0\}$, there exists $s_0>0$ depending only on α, t and $\|\phi\|^{(3)}_{B(x_0,r)}$ such that for all $s \in (0,s_0)$,

$$\begin{aligned}
&\big\{\phi - s[F(t,\cdot,D\phi,D^2\phi) + \alpha] > 0\big\} \cap \mathrm{cl}\,B(x_0,r) \subset \varOmega_G(t+s)\\
&(resp., \big\{\phi - s[F(t,\cdot,D\phi,D^2\phi) - \alpha] < 0\big\} \cap \mathrm{cl}\,B(x_0,r) \subset \mathbb{R}^N\backslash \mathrm{cl}\,\varOmega_G(t+s).
\end{aligned}$$

We say $\{\varOmega_G(t)\}_{t\in[0,T)}$ is a generalized flow by (1.1) if it is both a generalized superflow and a generalized subflow by (1.1).

This definition is based on the avoidance/inclusion property of the evolution by (2.14), which is a direct consequence of the comparison principle for (2.14). Let $(\Gamma_L(t), \Omega_L^+(t), \Omega_L^-(t))_{t\in[0,T)}$ be the level set flow by (1.1). Assume that $\phi \in C^\infty(\mathbb{R}^N)$ and that $E(0) := \{\phi \geqslant 0\} \subset \Omega_L^+(t)$ for some fixed $t \in [0, T)$. Let $\varphi = \varphi(s, x)$ be a unique viscosity solution of

$$\begin{cases} \varphi_s + F(t+s, x, D\varphi, D^2\varphi) = -\alpha|D\varphi| & \text{in } (0, T-t) \times \mathbb{R}^N, \\ \varphi(0, x) = \phi(x) & \text{for } x \in \mathbb{R}^N, \end{cases}$$

with an arbitrary $\alpha > 0$. For $s \geqslant 0$ set $E^+(s) := \{\varphi(s, \cdot) \geqslant 0\}$. Then $E^+(s) \subset \Omega_L^+(t+s)$ for all $s \geqslant 0$. Similarly, if $E^-(0) := \{\phi \leqslant 0\} \subset \Omega_L^-(t)$ for some $t \in [0, T)$ (or equivalently, $E^+(0) \supset \Omega_L^+(t)$), we define the evolution $E^-(s)$ by use of a unique viscosity solution of

$$\begin{cases} \varphi_s + F(t+s, x, D\varphi, D^2\varphi) = \alpha|D\varphi| & \text{in } (0, T-t) \times \mathbb{R}^N, \\ \varphi(0, x) = \phi(x) & \text{for } x \in \mathbb{R}^N, \end{cases}$$

with an arbitrary $\alpha > 0$. Then $E^-(s) \subset \Omega_L^-(t+s)$ for all $s \geqslant 0$. This property characterizes the family $(\Gamma_L(t), \Omega_L^+(t), \Omega_L^-(t))_{t\in[0,T)}$.

From the above observations we expect that the surface $\partial\{\phi - h[F(t, \cdot, D\phi, D^2\phi) + \alpha] > 0\}$ evolves in a weak sense with a normal velocity smaller than $-F$. This idea is justified by the following lemma, which says that such sets are contained in some smooth and open sets evolving with a normal velocity smaller than $-F$.

Lemma 2.8 (cf. [3, Lemma 2.2]) *Let $t \geqslant 0$, $x_0 \in \mathbb{R}^N$, $r > 0$ and $\alpha > 0$. Let $\phi \in C^\infty(\mathbb{R}^N)$ satisfy $\{\phi \geqslant 0\} \subset B(x_0, r)$ and $|D\phi| \neq 0$ on $\{\phi = 0\}$. Then there are a constant $s_0 > 0$ depending on t, α and $\|\phi\|^{(3)}_{B(x_0,r)}$, a smooth function $\varphi_\alpha = \varphi_\alpha(s, x)$ in $[0, s_0) \times \mathbb{R}^N$ with $\varphi_\alpha(0, \cdot) = \phi$ and a neighborhood $V \subset B(x_0, r)$ of the set $\{\phi = 0\}$ such that for all $s \in [0, s_0)$,*

$$\partial\Omega_1(t+s) \subset V, \ \Omega_1(t+s) := \{\varphi_\alpha(s, \cdot) > 0\}, \tag{2.17}$$

$$\{\phi - h[F(t, \cdot, D\phi, D^2\phi) + \alpha] > 0\} \cap \operatorname{cl} B(x_0, r) \subset \Omega_1(t+s), \tag{2.18}$$

$$|D\varphi_\alpha| \neq 0, \ \varphi_s + F(t+s, x, D\varphi, D^2\varphi) \leqslant 0 \quad \text{in } (0, s_0) \times V. \tag{2.19}$$

Similarly, let $t \geqslant 0$, $x_0 \in \mathbb{R}^N$, $r > 0$ and $\alpha > 0$ and let $\phi \in C^\infty(\mathbb{R}^N)$ satisfy $\{\phi \leqslant 0\} \subset B(x_0, r)$ and $|D\phi| \neq 0$ on $\{\phi = 0\}$. Then there are a constant $s_0 > 0$ depending on t, α and $\|\phi\|^{(3)}_{B(x_0,r)}$, a smooth function $\varphi_\alpha = \varphi_\alpha(s, x)$ in $[0, s_0) \times \mathbb{R}^N$ with $\varphi_\alpha(0, \cdot) = \phi$ and a neighborhood $V \subset B(x_0, r)$ of the set $\{\phi = 0\}$ such that for all $s \in [0, s_0)$,

$$\partial\Omega_2(t+s) \subset V, \ \Omega_2(t+s) := \{\varphi_\alpha(s, \cdot) < 0\},$$

$$\{\phi - h[F(t, \cdot, D\phi, D^2\phi) + \alpha] < 0\} \cap \operatorname{cl} B(x_0, r) \subset \Omega_2(t+s),$$

$$|D\varphi_\alpha| \neq 0, \ \varphi_s + F(t+s, x, D\varphi, D^2\varphi) \geqslant 0 \quad \text{in } (0, s_0) \times V.$$

The following proposition shows the relation between the generalized flow and the level-set flow by (1.1).

Proposition 2.9 (cf. [3, Theorem 2.4]) *Assume (2.1). A family $\{\Omega_G(t)\}_{t\in[0,T)}$ of open subsets of $\mathbb{R}^N$ is a generalized superflow (resp., subflow) by (1.1) if and only if the function $\chi_{\Omega_G(t)} - \chi_{\mathbb{R}^N \setminus \Omega_G(t)}$ is a viscosity supersolution (resp., subsolution) of (2.14).*

3. Some estimates

Let $\{w_k\}_{k=0}^{[T/h]}$ be the sequence of classical solutions of (1.2)–(1.3) and let $C^h(t)$ be given by (1.6). Define $\rho = \rho(t,x)$ as (2.4) to the smooth and compact CDM $\{\Gamma(t)\}_{t\in[0,T)}$. For each $h > 0$ and $k = 0, 1, 2, \ldots$, set W_k as the solution of (1.2) satisfying $W_k(0,\cdot) = \rho(kh,\cdot)$. In this section we derive some estimates for $\{w_k\}_{k=0}^{[T/h]}$, $\{C^h(t)\}_{t\in[0,T),h>0}$ and $\{W_k\}_{k=0}^{[T/h]}$. We assume (2.1) throughout this section.

3.1 *Estimates on* $\{w_k\}_{k=0}^{[T/h]}$ *and* $\{C^h(t)\}_{t\in[0,T),h>0}$

First, we show the uniform boundedness of $\{C^h(t)\}_{t\in[0,T),h>0}$.

Proposition 3.1 *Let $C_0 \subset \mathbb{R}^N$ be compact and take $R_0 > 0$ so that $C_0 \subset B(0,R_0)$. Then $C^h(t) \subset \operatorname{cl} B(0, R_0 + K_2 t)$ for all $t \in [0,T)$ and $h > 0$. Here $K_2 := |\mathbf{b}|_{0,Q_T} + |g|_{0,Q_T}$.*

Proof. For any $x_0 \in \partial B(0,R_0)$ and $k = 0, 1, \ldots, [T/h]$, set $D_k(x_0) := \{x \in \mathbb{R}^N \mid \langle x - x_0, x_0/R_0\rangle \leqslant K_2 kh\}$. We remark that for each $k = 0, 1, \ldots, [T/h]$, $\partial D_k(x_0)$ is a hyperplane and $D_k(x_0) = \{d(\cdot, D_{k-1}(x_0)) + K_2 h \geqslant 0\}$, where $d(\cdot, D_{k-1}(x_0))$ be the signed distance function given by (1.4) with $D = D_{k-1}(x_0)$.

Set $\overline{w}_0 = \overline{w}_0(t,x) := d(x, D(x_0)) + K_2 t$. Noting that $\Delta \overline{w}_0 = \Delta d(\cdot, D_0(x_0)) = 0$ in $\mathbb{R}^N$, we easily see that $\overline{w}_0$ is a classical supersolution of (1.2) satisfying $\overline{w}_0(0,\cdot) \geqslant d(\cdot, C_0)$ in $\mathbb{R}^N$. Hence we use the maximum principle to have $w_0(t,x) \leqslant \overline{w}_0(t,x)$ for $(t,x) \in [0,h] \times \mathbb{R}^N$. Thus $C_1 \subset D_1(x_0)$.

Let $d(\cdot, D_1(x_0))$ be the signed distance function given by (1.4) with $D = D_1(x_0)$ and $\overline{w}_1 = \overline{w}_1(t,x) := d(x, D_1(x_0)) + K_2 t$. The same argument as above yields that $w_1(t,x) \leqslant d(x, D_1(x_0)) + K_2 t$ and hence $C_2 \subset D_2(x_0)$. Repeating these arguments, we get $C_k \subset D_k(x_0)$ for $k = 1, 2, \ldots, [T/h]$.

Since $d(\cdot, D_k(x_0)) = d(\cdot, D_0(x_0)) + K_2 kh$, we obtain

$$C^h(t) = C_{[t/h]} \subset D_{[t/h]}(x_0) \subset \{d(\cdot, D_0(x_0)) + K_2 t \geqslant 0\}$$

for all $t \in [0,T)$ and $h > 0$. As $x_0 \in \partial B(0,R_0)$ is arbitrary, we conclude that

$$C^h(t) \subset \bigcap_{x_0 \in \partial B(0,R_0)} \{d(\cdot, D_0(x_0)) + K_2 t \geqslant 0\} = \operatorname{cl} B(0, R_0 + K_2 t)$$

for all $t \in [0,T)$ and $h > 0$. □

We improve the estimates of (2.2).

Proposition 3.2 *For all $h > 0$, $k = 0, 1, 2, \ldots, [T/h]$ and $(t,x) \in \operatorname{cl} Q_h$, we get*

$$-\sqrt{|x|^2 + 2Nt} - R_0 - K_2(kh + t) \leqslant w_k(t,x) \leqslant -|x| + R_0 + K_2(kh + t), \qquad (3.1)$$

where R_0 and K_2 are given in Proposition 3.1.

The (3.1) implies that the first bound of (2.2) is independent of $h > 0$.

Proof of Proposition 3.2. Fix $h > 0$ and $k = 0, 1, 2, \ldots, [T/h]$. As for the upper estimate, we see from the proof of Proposition 3.1 that for all $(t,x) \in \operatorname{cl} Q_h$ and $x_0 \in \partial B(0,R_0)$,

$$w_k(t,x) \leqslant d(x, D_k(x_0)) + K_2 t \leqslant d(x, D_0(x_0)) + K_2(kh + t).$$

Taking the infimum with respect to $x_0 \in \partial B(0, R_0)$, we have the upper estimate of (3.1) for all $(t, x) \in \mathrm{cl}\, Q_h$

Next we show the lower estimate of (3.1). Set $\underline{w}_0 = \underline{w}_0(t, x) := -\sqrt{|x|^2 + 2Nt} - R_0 - K_2 t$. Noting that

$$\underline{w}_{0,t} = \frac{-N}{\sqrt{|x|^2 + 2Nt}} - K_2,\ D\underline{w}_0 = \frac{-x}{\sqrt{|x|^2 + 2Nt}},\ \Delta \underline{w}_0 = \frac{-(N-1)|x|^2 + 2N^2 t}{\{|x|^2 + 2Nt\}^{3/2}}, \quad (3.2)$$

we easily observe that $\underline{w}_0$ is a classical subsolution of (1.2) with $k = 0$. Moreover, we see that if $d(x, C_0) < 0$, then

$$d(x, C_0) = -|x - y| \geqslant -|x| - |y| \quad \text{for some } y \in \partial C_0.$$

Since $C_0 \subset \mathrm{cl}\, B(0, R_0)$, we get $d(x, C_0) \geqslant -|x| - R_0$. Hence $\underline{w}(0, \cdot) \leqslant d(\cdot, C_0)$ in $\mathbb{R}^N$. We thus obtain the lower estimate of (3.1) by the maximum principle. In the case of $k \geqslant 1$, it follows from Proposition 3.1 and similar arguments to the above that $-|\cdot| - R_0 - K_2 kh \leqslant d(\cdot, C_k)$ in $\mathbb{R}^N$ and that $\underline{w}_k(t, x) := -\sqrt{|x|^2 + 2Nt} - R_0 - K_2(kh + t)$ is a classical subsolution of (1.2) satisfying $\underline{w}_k(0, \cdot) \leqslant d(\cdot, C_k)$ in $\mathbb{R}^N$. Therefore we get the result. □

Proposition 3.3 *We have* $|Dw_k(t, x)| \leqslant 1 + K_3 t$ *for all* $(t, x) \in \mathrm{cl}\, Q_h$, $k = 0, 1, 2, \ldots, [T/h]$, $h > 0$ *and some* $K_3 > 0$.

Proof. Fix $h > 0$, $k = 0, 1, 2, \ldots, [T/h]$ and any $\xi \in \mathbb{R}^N$, $|\xi| = 1$. Taking the derivative in the direction of ξ to (1.2) and denoting by ∂_ξ its symbol, we see that $v := \partial_\xi w_k$ is a classical solution of

$$\begin{aligned} &v_t - \Delta v + \langle \mathbf{b}_k, Dv \rangle + \partial_\xi g_k + \langle \partial_\xi \mathbf{b}_k, Dw_k \rangle = 0 \quad \text{in } Q_h, \\ &v(0, x) = \partial_\xi d(x, C_k) \leqslant 1 \quad \text{for almost all } x \in \mathbb{R}^N. \end{aligned} \quad (3.3)$$

Define $\overline{w} := 1 + K_3 t$ and $K_3 := K_1 |D\mathbf{b}|_{0,Q_T} + |Dg|_{0,Q_T}$, where K_1 is the same constant as that of (2.2). Then $\overline{w}$ is a classical supersolution of (3.3) satisfying $\overline{w}(0, \cdot) = 1$ in $\mathbb{R}^N$. Hence we obtain $v \leqslant \overline{w}$ in $\mathrm{cl}\, Q_h$ by the maximum principle . As ξ is arbitrary, we have the desired estimate because $|p| = \sup_{q \in \mathbb{R}^N, |q| \leqslant 1} \langle p, q \rangle$ for $p \in \mathbb{R}^N$. □

3.2 *Local estimates for* $\{W_k\}_{k=0}^{[T/h]}$

This subsection is devoted to some local estimates for $\{W_k\}_{k=0}^{[T/h]}$ under (2.1) and (2.5).

Applying the regularity theory for parabolic equations, we get the following estimate.

$$\sup_{\substack{k=0,1,2,\ldots,[T/h] \\ h>0}} \|W_k\|^{(5+\alpha)}_{[0,h]\times\{|\rho(kh,\cdot)| \leqslant 5\delta\}} =: K_4 < +\infty. \quad (3.4)$$

We need an estimate from below for $\{DW_k\}_{k=0}^{[T/h]}$ to obtain the rate of convergence of our algorithm to a smooth and compact CDM.

Proposition 3.4 *There are constants* $K_5 > 0$ *and* $t_1 > 0$ *such that*

$$\langle DW_k, D\rho(kh, \cdot) \rangle \geqslant 1 - K_5 t (> 0) \quad \text{on } [0, h] \times \big\{ |\rho(kh, \cdot)| \leqslant 5\delta \big\} \quad (3.5)$$

for all $k = 0, 1, 2, \ldots, [T/h]$ *and* $h \in (0, t_1)$.

Proof. We consider only the case $k = 0$ since the other ones are similarly proved. Set $f(t,x) := \langle \mathbf{b}_0(t,x), DW_0(t,x)\rangle + g_0(t,x)$. Recall that $\rho(0,\cdot) \in C^{5+\alpha}(\{|\rho(0,\cdot)| \leqslant 10\delta\})$ by (2.5). The solution W_0 of (1.2) satisfying $W_0(0,\cdot) = \rho(0,\cdot)$ is represented as follows:

$$W_0(t,x) = \int_{\mathbb{R}^N} U(t,x-y)\rho(0,y)dy + \int_0^t \int_{\mathbb{R}^N} U(t-s,x-y)f(s,y)dyds.$$

Thus

$$\begin{aligned} W_{0,x_i}(t,x) &= \int_{\mathbb{R}^N} U_{x_i}(t,x-y)\rho(0,y)dy + \int_0^t \int_{\mathbb{R}^N} U_{x_i}(t-s,x-y)f(s,y)dyds \\ &=: I_1 + I_2 \end{aligned}$$

Set $\delta' := 5\delta/\sqrt{N}$ so that $P(x,\delta') \subset \{|\rho(0,\cdot)| \leqslant 10\delta\}$ for all $x \in \{|\rho(0,\cdot)| \leqslant 5\delta\}$. Then it is observed by Green's formula, (2.1) and Proposition 3.3 that

$$\begin{aligned} I_1 &= \int_{P(x,\delta')} U(t,y-x)\rho_{x_i}(0,y)dy + O(e^{-(\delta')^2/8t}) =: I_{1,1} + O(e^{-(\delta')^2/8t}), \\ I_2 &= \int_0^t \int_{P(x,\delta')} U(t-s,y-x)f_{x_i}(s,y)dyds + O(te^{-(\delta')^2/8t}) =: I_{2,1} + O(te^{-(\delta')^2/8t}). \end{aligned}$$

Step 1. We estimate $I_{1,1}$.

We observe by the change of variables $y - x \mapsto y$ and Taylor's theorem that for some $\theta \in (0,1)$ and small $t > 0$,

$$\begin{aligned} I_{1,1} = \int_{P(0,\delta')} U(t,y) \Bigg\{ & \rho_{x_i}(0,x) + \langle D\rho_{x_i}(0,x), y\rangle + \frac{1}{2}\langle D^2\rho_{x_i}(0,x)y, y\rangle \\ & + \frac{1}{3!}\left(\sum_{i=1}^N y_i \frac{\partial}{\partial x_i}\right)^3 \rho_{x_i}(0,x+\theta y)\Bigg\} dy. \end{aligned}$$

By virtue of

$$\int_{P(0,\delta')} U(t,y)y_i dy = \int_{P(0,\delta')} U(t,y)y_i y_j dy = 0, \quad \int_{P(0,\delta')} U(t,y)y_i^2 dy = 2t + O(e^{-(\delta')^2/8t})$$

for all $i,j = 1,2,\ldots,N$ $(i \neq j)$, we get

$$\left|I_{1,1} - \{\rho_{x_i}(0,x) + t\Delta\rho_{x_i}(0,x)\}\right| \leqslant K_{5,1}t^{3/2}.$$

for all $(t,x) \in [0,t_{1,1}] \times \{|\rho(0,\cdot)| \leqslant 5\delta\}$ and some $K_{5,1}, t_{1,1} > 0$.

Step 2. We estimate $I_{2,1}$.

We calculate similarly to the previous step with using (2.1) and (3.4) to yield that

$$\left|I_{2,1} - f_{x_i}(0,x)t\right| \leqslant K_{5,2}t^{3/2}$$

for all $(t,x) \in [0,t_{1,2}] \times \{|\rho(0,\cdot)| \leqslant 5\delta\}$ and some $K_{5,2}, t_{1,2} > 0$. Choosing $K_5 \geqslant K_{5,1} + K_{5,2}$ and $t_1 \leqslant \min\{t_{1,1}, t_{1,2}\}$, we obtain the desired result. □

REMARK 3.5 (1) Proposition 3.4 implies that

$$|DW_k(t,x)| \geqslant 1 - K_5 t$$

for all $(t,x) \in [0,h] \times \{|\rho(kh,\cdot)| \leqslant 5\delta\}$, $k = 0,1,2,\dots,[T/h]$ and $h \in (0,t_1)$. Proposition 6.5 below shows that we cannot improve this estimate in the following sense: There is a nonnegative and continuous function $\omega(t)$ satisfying $\omega(0) = 0$ for which

$$|DW_0(t,x)| \geqslant 1 - t\omega(t) \quad \text{for small } t > 0 \text{ and } x \in \big\{|\rho(0,\cdot)| \leqslant 5\delta\big\}.$$

(2) In the case where $\mathbf{b} = 0$ and $g = 0$, we are able to get the following estimate by calculating $I_{1,1}$ more precisely: For some $K_6, t_2 > 0$,

$$\big|DW_k(t,x) - \big(D\rho(kh+t,x) + tD\Delta\rho(kh+t,x)\big)\big| \leqslant K_6 t^2$$
$$\text{for } (t,x) \in [0,h] \times \big\{|\rho(kh,\cdot)| \leqslant 5\delta\big\}, k = 0,1,2\dots,[T/h] \text{ and } h \in (0,t_2).$$

4. Semicontinuous limits of $\{w_k\}_{k=0}^{[T/h]}$ and $\{d(\cdot,C_k)\}_{k=0}^{[T/h]}$

In this section we assume (2.1) and consider the semicontinuous limits of $\{w_k\}_{k=0}^{[T/h]}$ and $\{d(\cdot,C_k)\}_{k=0}^{[T/h]}$ as $h \to 0$. These are based on [2, Section 5] and [26, Section 4].

For any compact set $C_0 \subset \mathbb{R}^N$ let $\{C^h(t)\}_{t\in[0,T),h>0}$ be defined by (1.6). Set

$$d^h(t,x) := d\big(x,C^h(t)\big)\big(= d(x,C_k)\big),\ w^h(t,x) := w_k(t-kh,x) \tag{4.1}$$

for $(t,x) \in [kh,(k+1)h) \times \mathbb{R}^N$, $k = 0,1,2,\dots,[T/h]$ and $h > 0$. Define

$$\overline{f}(t,x) := \limsup_{(h,s,y)\to(0,t,x)} f^h(s,y), \tag{4.2}$$
$$\underline{f}(t,x) := \liminf_{(h,s,y)\to(0,t,x)} f^h(s,y) \quad (f^h = d^h,\ w^h),$$
$$\Omega^+(t) := \big\{\underline{d}(t,\cdot) > 0\big\},\ \Omega^-(t) := \big\{\overline{d}(t,\cdot) < 0\big\} \tag{4.3}$$
$$\Gamma(t) := \mathbb{R}^N \backslash \big(\Omega^+(t) \cup \Omega^-(t)\big) \quad \big(= \big\{\underline{d}(t,\cdot) \leqslant 0 \leqslant \overline{d}(t,\cdot)\big\}\big). \tag{4.4}$$

REMARK 4.1 It follows from the fact $\|\nabla d^h\|_{L^\infty([0,T)\times\mathbb{R}^N)} = 1$ and Proposition 3.3 that

$$|f(t,x) - f(t,y)| \leqslant |x-y| \quad \text{for all } t \in [0,T) \text{ and } x,y \in \mathbb{R}^N\ (f = \overline{d}, \underline{d}, \overline{w}, \underline{w}).$$

Besides (4.2) turns to

$$\overline{f}(t,x) := \limsup_{(h,s)\to(0,t)} f^h(s,x),\ \underline{f}(t,x) := \liminf_{(h,s)\to(0,t)} f^h(s,x) \quad (f^h = d^h, w^h). \tag{4.5}$$

We explain some properties of $\overline{d}$, $\underline{d}$, $\overline{w}$ and $\underline{w}$.

Proposition 4.2 *$\overline{d} = \overline{w}$ and $\underline{d} = \underline{w}$ in $[0,T) \times \mathbb{R}^N$.*

Proof. The solution w_k of (1.2)–(1.3) is given by

$$w_k(t,x) = \int_{\mathbb{R}^N} U(t,x-y)d(y,C_k)dy + \int_0^t \int_{\mathbb{R}^N} U(t-s,x-y)f_k(s,y)dyds,$$

where $f_k(t,x) := \langle \mathbf{b}_k(t,x), Dw_k(t,x)\rangle + g_k(t,x)$. It follows from (2.1), the Lipschitz continuity of $d(\cdot,C_k)$, Proposition 3.3 and (4.1) that for any small $h>0$

$$\sup_{(t,x)\in[0,T)\times\mathbb{R}^N} |w^h(t,x) - d^h(t,x)| = \sup_{\substack{(t,x)\in \mathrm{cl}\, Q_h \\ k=0,1,\ldots,[T/h]}} |w_k(t,x) - d(x,C_k)| \leqslant K_7\sqrt{h}.$$

Here $K_7 > 0$ is independent of $h>0$. Taking such limits as (4.2), we have the result. □

Proposition 4.3 *We have*

$$\overline{w} = \overline{d} = 0 \quad on \bigcup_{0\leqslant t<T} [\{t\}\times\partial\Omega^-(t)], \ \underline{w} = \underline{d} = 0 \quad on \bigcup_{0\leqslant t<T} [\{t\}\times\partial\Omega^+(t)].$$

Proof. We show only $\overline{d} = 0$ on $\cup_{0\leqslant t<T}[\{t\}\times\partial\Omega^-(t)]$ since the others can be similarly proved.

Note that $\overline{d} \geqslant 0$ in $\cup_{0\leqslant t<T}[\{t\}\times\partial\Omega^-(t)]$, for each $t\in[0,T)$. Suppose $\overline{d}(t_0,x_0) > 0$ for some $(t_0,x_0) \in \cup_{0\leqslant t<T}[\{t\}\times\partial\Omega^-(t)]$. By (4.5) there are sequences $\{h_n\}_{n=1}^{+\infty}$, $\{t_n\}_{n=1}^{+\infty}$ such that $(h_n, t_n, d^{h_n}(t_n,x_0)) \longrightarrow (0,t_0,\overline{d}(t_0,x_0))$ as $n\to+\infty$. Hence we may consider $d^{h_n}(t_n,x_0) > \overline{d}(t_0,x_0)/2$ for all $n\in\mathbb{N}$. Since $\{d^h(t,\cdot)\}_{t\in[0,T),h>0}$ is equi-Lipschitz continuous, we observe that there exists $r_0>0$ such that $d^{h_n}(t_n,x) \geqslant \overline{d}(t_0,x_0)/4 > 0$ for all $x\in B(x_0,r_0)$ and $n\in\mathbb{N}$. Take $x\in B(x_0,r_0)\cap\Omega^-(t)$. Letting $n\to+\infty$, we get $\overline{d}(t_0,x) > \overline{d}(t_0,x_0)/4 > 0$. This contradicts to $\overline{d}(t_0,x) < 0$. Therefore we have $\overline{d} = 0$ in $\cup_{0\leqslant t<T}[\{t\}\times\partial\Omega^-(t)]$. □

Proposition 4.4 $\overline{w}(0,\cdot) = \underline{w}(0,\cdot) = \overline{d}(0,\cdot) = \underline{d}(0,\cdot) = d(\cdot,C_0)$ *in* $\mathbb{R}^N$.

Proof. We have only to prove the desired equalities in $\mathrm{int}\,C_0$ since we have already shown in Proposition 4.3 that $\overline{w}(0,\cdot) = \underline{w}(0,\cdot) = \overline{d}(0,\cdot) = \underline{d}(0,\cdot) = 0 = d(\cdot,C_0)$ in ∂C_0 and we are able to show by a similar way that $\overline{w}(0,\cdot) = \underline{w}(0,\cdot) = \overline{d}(0,\cdot) = \underline{d}(0,\cdot) = d(\cdot,C_0)$ in $\mathbb{R}^N\backslash C_0$.

Fix $x_0\in \mathrm{int}\,C_0$ and set $r_0 := d(x_0,C_0)(>0)$. Define

$$\overline{v}_0 = \overline{v}_0(t,x) := r_0 + \sqrt{|x-x_0|^2 + 2Nt} + K_2 t \quad \text{for } (t,x)\in \mathrm{cl}\, Q_h.$$

Here $K_2 := |\mathbf{b}|_{0,Q_T} + |g|_{0,Q_T}$. Note that $\{\overline{v}_0(0,\cdot) \leqslant 2r_0\} = B(x_0,r_0) \subset C_0$. We observe by similar calculations to (3.2) that $\overline{v}_0$ is a classical supersolution of (1.2) with $k=0$ and that $\overline{v}_0(0,\cdot) \geqslant d(\cdot,C_0)$ in $\mathbb{R}^N$. Thus the inequality $w^h \leqslant \overline{v}_0$ in $[0,h]\times\mathbb{R}^N$ follows from the maximum principle. Some calculations yield that

$$\{\overline{v}_0(h,\cdot)\leqslant 2r_0\} \subset \mathrm{cl}\, B(x_0.r_1) \subset C_1,\ r_1 := \sqrt{(r_0-K_2h)^2 - 2Nh}.$$

By use of this inclusion we get

$$w^h(h,\cdot) = d(\cdot,C_1) \leqslant 2r_0 - r_1 + |\cdot - x_0| \quad \text{in } \mathbb{R}^N.$$

Next we define

$$\overline{v}_1 = \overline{v}_1(t,x) := 2r_0 - r_1 + \sqrt{|x-x_0|^2 + 2Nt} + K_2 t \quad \text{for } (t,x)\in \mathrm{cl}\, Q_h.$$

By a similar argument to the above we have $w^h(h+\cdot,\cdot) \leqslant \overline{v}_1$ on $\mathrm{cl}\, Q_h$ and

$$\{\overline{v}_1(h,\cdot) \leqslant 2r_0\} \subset \mathrm{cl}\, B(x_0.r_2) \subset C_2,\ r_2 := \sqrt{(r_1 - K_2 h)^2 - 2Nh}.$$

Thus we inductively obtain

$$\begin{aligned}
&w^h(kh+\cdot,\cdot) \leqslant \overline{v}_k \quad \text{on } \mathrm{cl}\, Q_h,\\
&\overline{v}_k = \overline{v}_k(t,x) := 2r_0 - r_k + \sqrt{|x-x_0|^2 + 2Nt} + K_2 t \quad \text{for } (t,x) \in \mathrm{cl}\, Q_h,\\
&\{\overline{v}_k(h,\cdot) \leqslant 2r_0\} \subset \mathrm{cl}\, B(x_0.r_{k+1}) \subset C_{k+1},\ r_{k+1} := \sqrt{(r_k - K_2 h)^2 - 2Nh},\\
&w^h((k+1)h,\cdot) = d(\cdot, C_{k+1}) \leqslant 2r_0 - r_{k+1} + |\cdot - x_0| \quad \text{in } \mathbb{R}^N.
\end{aligned} \tag{4.6}$$

We estimate $\{r_k\}_{k=0}^{[T/h]}$. Since $\sqrt{1-s} \geqslant 1-s$ for all $s \in [0,1/4]$, it is easy to see that there exists $t_3 \in (0, r_0/(8N+K_2))$ such that $r_1 \geqslant r_0 - (K_2+2N)h$ for any $h \in (0,t_3)$. Moreover, we can show by induction that

$$r_k \geqslant r_0 - (K_2+2N)kh \quad \text{for all } k = 0,1,\ldots,[t_3/h].$$

Combining (4.6) with the above estimate, we obtain

$$d^h(t,x_0) = d(kh,x_0) = d(x_0,C_k) \leqslant r_0 + (K_2+2N)kh$$

for $t \in [kh,(k+1)h)$ and $k = 0,1,2\ldots,[t_3/h]$. Letting $h \to 0$, $kh \to 0$, we get $\overline{d}(0,x_0) \leqslant d(x_0,C_0)$ in $\mathrm{int}\, C_0$. Since the inequality $d(x_0,C_0) \leqslant \underline{d}(0,x_0)$ in $\mathrm{int}\, C_0$ is proved by the same way, we have the result from $\overline{d}(0,\cdot) = \underline{d}(0,\cdot) = d(\cdot,C_0)$ in $\mathbb{R}^N$ and Proposition 4.4. ☐

For fixed $h > 0$ and $t \in [0,T)$, the function $d^h(t,\cdot)$ is a viscosity solution of

$$|Dd| - 1 = 0 \quad \text{in } \mathrm{int}\, C^h(t),\ -|Dd| + 1 = 0 \quad \text{in } \mathbb{R}^N \backslash C^h(t).$$

The semicontinuous envelopes $d^{h*}(t,x)$ and $d^h_*(t,x)$ are given by, respectively,

$$d^{h*}(t,x) := \limsup_{s\to t} d^h(s,x),\ d^h_*(t,x) := \liminf_{s\to t} d^h(s,x),$$

since $\{d^h(t,\cdot)\}_{t\in[0,T),h>0}$ is equi-Lipschitz continuous in $\mathbb{R}^N$. Therefore, we get

$$\begin{aligned}
d^{h*}(t,x) &= \begin{cases} \max\{d^h((k-1)h,x), d^h(kh,x)\} & \text{for } t = kh,\\ d^h(kh,x) & \text{for } kh < t < (k+1)h,\end{cases}\\
d^h_*(kh,x) &= \begin{cases} \min\{d^h((k-1)h,x), d^h(kh,x)\} & \text{for } t = kh,\\ d^h(kh,x) & \text{for } kh < t < (k+1)h,\end{cases}
\end{aligned} \tag{4.7}$$

for all $h > 0$ and $k \in \mathbb{N}$. Applying the stability for viscosity solutions, we are able to characterize $\overline{d}$ and $\underline{d}$ as follows.

Theorem 4.5 *Assume (2.1). Let $\overline{d}$ and $\underline{d}$ be defined by (4.2). Then $\overline{d}$ and $\underline{d}$ satisfy, respectively,*

$$|D\overline{d}| - 1 \leqslant 0 \quad \text{in } \{\underline{d} > 0\},\ -|D\overline{d}| + 1 \leqslant 0 \quad \text{in } \{\overline{d} < 0\}, \tag{4.8}$$

$$|D\underline{d}| - 1 \geqslant 0 \quad \text{in } \{\underline{d} > 0\},\ -|D\underline{d}| + 1 \geqslant 0 \quad \text{in } \{\overline{d} < 0\}, \tag{4.9}$$

in the viscosity sense.

Proof. We show only $|D\overline{d}| - 1 \leqslant 0$ in $\{\underline{d} > 0\}$ since the remaining inequalities are similarly proved. Recall that $C^h(t) = C_k$ for $t \in [kh, (k+1)h)$ and $k = 0, 1, 2, \ldots, [T/h]$.

For any $\varphi \in C^\infty((0,T) \times \mathbb{R}^N)$ we assume that $\overline{d} - \varphi$ takes its strict maximum in $\{\underline{d} > 0\}$ at $(t_0, x_0) \in \{\underline{d} > 0\}$. Since $\{\underline{d} > 0\}$ is open in $(0,T) \times \mathbb{R}^N$, we are able to find $r_0 > 0$ such that $Q((t_0, x_0), r_0) \subset \{\underline{d} > 0\}$. In view of $\underline{d}(t_0, x_0) > 0$, we may assume that $d^h \geqslant \underline{d}(t_0, x_0)/2$ in $\mathrm{cl}\, Q((t_0, x_0), r_0)$ for any small $h > 0$, replacing r_0 with a smaller one if necessary. This implies that

$$\mathrm{cl}\, Q\big((t_0, x_0), r_0\big) \subset \mathrm{int} \left(\bigcup_{k=0}^{[T/h]} [kh, (k+1)h) \times C_k \right) \quad \text{for any small } h > 0.$$

Besides there are sequences $\{h_n\}_{n=1}^{+\infty}$ and $\{(t_n, x_n)\}_{n=1}^{+\infty} \subset \{\underline{d} > 0\} (\subset \{\overline{d} > 0\})$ satisfying

$$\begin{aligned} &\big(h_n, t_n, x_n, d^{h_n *}(t_n, x_n)\big) \longrightarrow (0, t_0, x_0, \overline{d}(t_0, x_0)) \quad \text{as } n \to +\infty, \\ &d^{h_n *}(t_n, x_n) - \varphi(t_n, x_n) = \max_{Q((t_0, x_0), r_0)} (d^{h_n *} - \varphi) \quad \text{for each } n \in \mathbb{N}. \end{aligned}$$

We take $k_n \in \mathbb{N} \cup \{0\}$ such that $k_n h_n \leqslant t_n < (k_n + 1) h_n$.

Case 1. $t_n \neq k_n h_n$.

Since $d^{h_n *}(t_n, \cdot) - \varphi(t_n, \cdot)$ takes a maximum in $B(x_0, r_0) (\subset \mathrm{int}\, C^{h_n}(t_n))$ at x_n and $d^{h_n *}(t_n, \cdot) = d(\cdot, C_{k_n})$, we have $|D\varphi(t_n, x_n)| - 1 \leqslant 0$. Letting $n \to \infty$, we obtain $|D\varphi(t_0, x_0)| - 1 \leqslant 0$.

Case 2. $t_h = k_n h_n$.

Note that $B(x_0, r_0) \subset C^{h_n}((k_n - 1)h_n) \cap C^{h_n}(k_n h_n)$. It follows from (4.7) and the stability of viscosity solutions that $d^{h_n *}(t_n, \cdot)$ is a viscosity subsolution of $|Dd| - 1 = 0$ in $B(x_0, r_0)$. Hence we get $|D\varphi(t_0, x_0)| - 1 \leqslant 0$ in the same way as in Case 1. □

5. Convergence

In this section we present the convergence of $\{C^h(t)\}_{t \in [0,T), h > 0}$ to the level-set flow by (1.1) under (2.1) and (2.16). The following arguments are based on [30, Section 3], constructions of suitable sub- and super-solution of (1.2) by applying the theory of viscosity solutions. At first we show the following theorem.

Theorem 5.1 *Assume* (2.1). *Let $t \geqslant 0$, $x_0 \in \mathbb{R}^N$, $r > 0$, $\alpha > 0$ and $\phi \in C^\infty(\mathbb{R}^N)$ be such that $\{\phi \geqslant 0\} \subset B(x_0, r)$ (resp., $\{\phi \leqslant 0\} \subset B(x_0, r)$) with $|D\phi| \neq 0$ on $\{\phi = 0\}$. For $h > 0$ and $k \in \mathbb{N} \cup \{0\}$ let $v_k = v_k(t, x)$ be a classical solution of* (1.2) *satisfying $v_k(0, \cdot) = d(\cdot, E_k)$, where $E_0 := \{\phi \geqslant 0\}$ and $E_k := \{v_{k-1}(h, \cdot) \geqslant 0\}$ ($k \in \mathbb{N}$). Then there exists $s_1 > 0$ depending only on t, α and $\|\phi\|^{(3)}_{B(x_0, r)}$ such that for all $s \in (0, s_1)$*

$$\liminf_{(h, kh, y) \to (0, s, x)} d(x, E_k) > 0 \quad \left(\textit{resp.,} \limsup_{(h, kh, y) \to (0, s, x)} d(x, E_k) < 0 \right), \tag{5.1}$$

provided

$$\begin{aligned} &x \in \mathrm{cl}\, B(x_0, r) \cap \big\{\phi - h[F^*(t, \cdot, D\phi, D^2\phi) + \alpha] > 0\big\}, \\ &(\textit{resp.,}\ x \in \mathrm{cl}\, B(x_0, r) \cap \{\phi - h[F_*(t, \cdot, D\phi, D^2\phi) - \alpha] < 0\}). \end{aligned}$$

Here $d(x, E_k)$ is the signed distance function defined by (1.4) *with $D = E_k$.*

We prepare two lemmas to prove this theorem. Let V and φ_α be given in Lemma 2.8 and for small $r \geqslant 0$ let $D_{\alpha,r}(s) := \{\varphi_\alpha(s,\cdot) \geqslant r\}$. Note that $D_\alpha(s) := D_{\alpha,0}(s) = \mathrm{cl}\,\Omega(t+s)$. Define $\rho_{\alpha,r} = \rho_{\alpha,r}(s,x) = d(x, D_{\alpha,r}(s))$ as in (2.4) with $D = D_{\alpha,r}(s)$. We then observe by (2.17) and (2.19) that

$$\rho_{\alpha,r} \text{ is smooth in } \mathcal{M} := \bigcup_{s\in[0,s_0)} \{s\} \times \big\{|\rho_{\alpha,0}(s,\cdot)| \leqslant 3r_0\big\}\big(\subset [0,s_0)\times V\big),$$
$$|D\varphi_\alpha| \neq 0,\ \varphi_s + F(t+s, x, D\varphi, D^2\varphi) \leqslant 0 \quad \text{in } \mathcal{M}, \tag{5.2}$$

for all $r \in [0, r_0)$, taking $r_0 > 0$ small and replacing s_0 in Lemma 2.8 with a smaller one if necessary. Furthermore, slightly modifying φ_α if necessary, we may consider that

$$\varphi_\alpha > 0 \quad \text{in } [0,s_0)\times\big(\{\phi > 0\}\backslash V\big).$$

Let $\mu_{\alpha,r} = \mu_{\alpha,r}(s,x)$ be the sum of the squares of all the principal curvatures of $\partial D_{\alpha,r}(s)$. We assume (2.1) in the following lemmas.

Lemma 5.2 *For fixed $\alpha \geqslant 0$ and $r \in [0, r_0)$, set $\rho = \rho_{\alpha,r}$, $\mu = \mu_{\alpha,r}$ to simplify the notations. Define*

$$\underline{w} = \underline{w}(s,x) := \big(1 - s\omega(s,x)\big)\rho(s,x) - M_1\big(\rho(s,x)\big)^2 s - M_2 s^2,$$
$$\overline{w} = \overline{w}(s,x) := \big(1 - s\omega(s,x)\big)\rho(s,x) + M_1\big(\rho(s,x)\big)^2 s + M_2 s^2,$$
$$\omega = \omega(s,x) := \widetilde{\mu}_\rho(s,x) + \big\langle D\mathbf{b}(s,x)D\rho(s,x), D\rho(s,x)\big\rangle + \big\langle Dg(s,x), D\rho(s,x)\big\rangle,$$

For large M_1, $M_2 > 0$ depending on (2.1) and $\|\varphi\|^{(4)}_{\mathcal{M}}$, the function $\underline{w}(= \underline{w}_{\alpha,r})$ (resp., $\overline{w}(= \overline{w}_{\alpha,r})$) is a classical subsolution (resp., supersolution) of (1.2) in $\mathcal{M}$.

Proof. We treat only the subsolution case since we are able to similarly handle the other case. We assume $t = 0$ in Lemma 2.8 because if otherwise, we have only to replace $\mathbf{b}(s,x)$ and $g(s,x)$ with $\mathbf{b}(t+s,x)$ and $g(t+s,x)$, respectively.

Step 1. We derive from (5.2)

$$\rho_s - \Delta\rho - \omega\rho + \langle \mathbf{b}, D\rho\rangle + g \leqslant M_1\rho^2 \quad \text{in } \mathcal{M} \quad \text{for some } M_1 > 0. \tag{5.3}$$

It follows from [17, Lemma 14.17] that

$$|(\widetilde{\Delta\rho})_\rho - (\Delta\rho + \widetilde{\mu}_\rho\rho)| \leqslant M_{1,1}\rho^2 \quad \text{on } \mathcal{M} \quad \text{for some } M_{1,1} > 0. \tag{5.4}$$

Here and in the sequel $M_{1,j}$'s ($j \in \mathbb{N}$) are constants depending on (2.1) and $\|\varphi\|^{(4)}_{\mathcal{M}}$. Besides we easily get in $\mathcal{M}$

$$\rho_s = (\widetilde{\rho_s})_\rho = \frac{(\widetilde{\varphi_{\alpha,s}})_\rho}{|(\widetilde{D\varphi_\alpha})_\rho|},\ D\rho = (\widetilde{D\rho})_\rho = \frac{(\widetilde{D\varphi_\alpha})_\rho}{|(\widetilde{D\varphi_\alpha})_\rho|}, \tag{5.5}$$

$$(\widetilde{\Delta\rho})_\rho = \frac{1}{|(\widetilde{D\varphi_\alpha})_\rho|}\mathrm{tr}\left\{\left(I - \frac{(\widetilde{D\varphi_\alpha})_\rho \otimes (\widetilde{D\varphi_\alpha})_\rho}{|(\widetilde{D\varphi_\alpha})_\rho|^2}\right)(\widetilde{D^2\varphi_\alpha})_\rho\right\}. \tag{5.6}$$

We see from (5.2), (5.5) and (5.6) that

$$\rho_s - (\widetilde{\Delta\rho})_\rho + \langle \widetilde{\mathbf{b}}_\rho, D\rho\rangle + \widetilde{g}_\rho \leqslant 0 \quad \text{in } \mathcal{M}. \tag{5.7}$$

Applying Taylor's theorem to the term $\langle \widetilde{\mathbf{b}}_\rho, D\rho\rangle + \widetilde{g}_\rho$, we get

$$\left|\langle \widetilde{\mathbf{b}}_\rho, D\rho\rangle + \widetilde{g}_\rho - \left\{\langle \mathbf{b}, D\rho\rangle + g - \rho(\langle D\mathbf{b}D\rho, D\rho\rangle + \langle Dg, D\rho\rangle\right\}\right| \leqslant M_{1,2}\rho^2.$$

Hence combining (5.4), (5.7) with this, we have (5.3) with $M_1 := M_{1,1} + M_{1,2}$.

Step 2. We show that $\underline{w}$ is a classical subsolution of (1.2) in $\mathcal{M}$.

We calculate that

$$\begin{aligned}
\underline{w}_s &= \rho_s - \omega\rho - M_1\rho^2 - s\{(\omega + 2M_1 s\rho)\rho_s + \rho\omega_s + 2M_2 s\},\\
D\underline{w} &= D\rho - s\{(\omega + 2M_1\rho)D\rho + \rho D\omega\},\\
\Delta\underline{w} &= \Delta\rho - s\{(\omega + 2M_1\rho)\Delta\rho + \rho\Delta\omega + 2\langle D\omega, D\rho\rangle + 2M_1\}.
\end{aligned}$$

We then observe by (5.3) that

$$\begin{aligned}
\underline{w}_s - \Delta\underline{w} + \langle \mathbf{b}, D\underline{w}\rangle + g &= \rho_s - \Delta\rho - \omega\rho + \langle \mathbf{b}, D\rho\rangle + g - M_1\rho^2\\
&\quad - s\Big\{(\omega + 2M_1\rho)\big(\rho_s - \Delta\rho + \langle \mathbf{b}, D\rho\rangle\big) + \rho\big(\omega_s - \Delta\omega + \langle \mathbf{b}, D\omega\rangle\big)\\
&\qquad + 2\langle D\omega, D\rho\rangle - 2M_1 + 2M_2\Big\}\\
&\leqslant -s\big\{2M_2 - (2M_1 + 1)M_{1,3} - 2M_1\big\}.
\end{aligned}$$

Hence choosing $M_2 := (M_1 + 1)M_{1,3} + M_1$, we get the result. ☐

Based on Lemma 5.2, we construct suitable subsolutions and supersolutions of (1.2). Hence we obtain the following inclusions for $\{E_k\}_{k=0}^{[T/h]}$ in Theorem 5.1.

Lemma 5.3 *Take $r_1 \in (0, r_0/10)$. For any $h > 0$ and $k = 0, 1, 2, \ldots$ let E_k and v_k be defined in Theorem 5.1. Set $\widehat{E}_k := \{\varphi_\alpha(kh, \cdot) \geqslant \mu_k\}$ for some small $\mu_k \geqslant 0$. There exist $M_3 > 0$ and $t_4 \in (0, s_0)$ depending on (2.1), r_1 and $\|\varphi_\alpha\|_{\mathcal{M}}^{(4)}$ such that for each $h \in (0, t_4)$ and $k = 0, 1, 2, \ldots$ if $\widehat{E}_k \subset E_k$, then*

$$\widehat{E}_{k+1} \subset E_{k+1},\ \mu_{k+1} := \mu_k + \frac{M_3 h^{3/2}}{1 - M_3 h}. \tag{5.8}$$

Similarly, set $\widehat{E}_k := \{\varphi_\alpha(kh, \cdot) \geqslant -\mu_k\}$ for some small $\mu_k \geqslant 0$. For each $h \in (0, t_4)$ and $k = 0, 1, 2, \ldots$, if $E_k \subset \widehat{E}_k$, then

$$E_{k+1} \subset \widehat{E}_{k+1},\ \mu_{k+1} := \mu_k + \frac{M_3 h^{3/2}}{1 - M_3 h}. \tag{5.9}$$

Proof. We prove only (5.8) since (5.9) is proved by a similar way. Recall that $\rho(s, x) = \rho_{\alpha,r}(s, x) = d(x, D_{\alpha,r}(s))$ and $D_{\alpha,r}(s) = \{\varphi_\alpha(s, \cdot) \geqslant r\}$ for $s \in [0, s_1)$ and $r \in [0, r_0)$. For simplicity we assume $s_0, r_0 \leqslant 1$. Let $h \in (0, s_1)$ and $k = 0, 1, 2, \ldots$ satisfy $0 \leqslant kh < s_1$.

Step 1. We modify $\underline{w}$ in Lemma 5.2 to obtain a classical subsolution of (1.2) in $(0, h] \times \{|\rho(kh, \cdot)| \leqslant 6r_1\}$.

We use all the notations in the proof of Lemma 5.2. Let $\eta_{k,1} \in C^\infty(\mathbb{R}^N)$ be a cut-off function by

$$\eta_{k,1}(x) := \begin{cases} 0 & \text{if } |\rho(kh,x)| \leqslant 2r_1, \\ 1 & \text{if } |\rho(kh,x)| \geqslant 3r_1, \end{cases} \quad 0 \leqslant \eta_{k,1} \leqslant 1 \quad \text{in } \mathbb{R}^N,$$
$$r_1 \|D\eta_{k,1}\|_{L^\infty(\mathbb{R}^N)} + r_1^2 \|D^2\eta_{k,1}\|_{L^\infty(\mathbb{R}^N)} \leqslant M_{3,1}.$$

Define $\underline{v}_k^1 = \underline{v}_k^1(s,x)$ by

$$\underline{v}_k^1(s,x) = \underline{w}(s,x) - M_{3,2}\eta_{k,1}(x)\sqrt{s} - M_{3,3}s^{3/2}, \; M_{3,2} := 2N + 4r_1|\omega|_{0,\mathcal{M}}.$$

Applying Lemma 5.2, we observe that

$$\underline{v}_{k,s}^1 - \Delta \underline{v}_k^1 + \langle \mathbf{b}, D\underline{v}_k^1 \rangle + g \leqslant \left(M_{3,1}M_{3,2}r_1^{-2}(1 + |\mathbf{b}|_{0,Q}) - \frac{3}{2}M_{3,3} \right) \sqrt{s}.$$

Taking $M_{3,3} \geqslant M_{3,1}M_{3,2}r_1^{-2}(1 + |\mathbf{b}|_{0,Q})$, we see that $\underline{v}_k^1$ is a classical subsolution of (1.2) in $(0,h] \times \{|\rho(kh,\cdot)| \leqslant 6r_1\}$.

Step 2. We construct a classical subsolution of (1.2) in $(0,h] \times \{2r_1 \leqslant \rho(kh,\cdot) \leqslant 6r_1\}$ and that in $(0,h] \times \{\rho(kh,\cdot) \leqslant -2r_1\}$.

Fix any $y \in \{\rho(kh,\cdot) = 4r_1\}$. Choose a smooth cut-off function $\eta_{k,2} = \eta_{k,2}(x;y)$ satisfying

$$\eta_{k,2}(x;y) := \begin{cases} 0 & \text{for } x \in \operatorname{cl} B(y,r_1), \\ 1 & \text{for } x \in \{2r_1 \leqslant \rho(kh,\cdot) \leqslant 6r_1\} \backslash B(y,2r_1), \end{cases} \quad 0 \leqslant \eta_{k,2} \leqslant 1 \quad \text{in } \mathbb{R}^N,$$
$$r_1 \|D\eta_{k,2}\|_{L^\infty(\mathbb{R}^N)} + r_1^2 \|D^2\eta_{k,2}\|_{L^\infty(\mathbb{R}^N)} \leqslant M_{3,4}.$$

Define $\underline{v}_k^2 = \underline{v}_k^2(s,x;y)$ by

$$\underline{v}_k^2(s,x;y) := 4r_1 - \sqrt{|x-y|^2 + 2Ns} - M_{3,5}s - \eta_{k,2}(x;y)(M_2 + M_{3,3})\sqrt{s}, \tag{5.10}$$

where M_2 is the constant given in Lemma 5.2. Similar calculations to the proof of Proposition 3.2 and the above ones yield that

$$\underline{v}_{k,s}^2 - \Delta \underline{v}_k^2 + \langle \mathbf{b}, D\underline{v}_k^2 \rangle + g \leqslant -M_{3,5} + |\mathbf{b}|_{0,Q} + |g|_{0,Q} + M_{3,1}r_1^{-2}(1 + |\mathbf{b}|_{0,Q})(M_2 + M_{3,3})\sqrt{s}.$$

Taking

$$M_{3,5} \geqslant |\mathbf{b}|_{0,Q} + |g|_{0,Q} + M_{3,1}r_1^{-2}(1 + |\mathbf{b}|_{0,Q})(M_2 + M_{3,3}) + 2r_1|\omega|_{0,\mathcal{M}} + 16M_1r_1^2,$$

we see that $\underline{v}_k^2$ is a classical subsolution of (1.2) in $(0,h] \times \{2r_1 \leqslant \rho(kh,\cdot) \leqslant 6r_1\}$.

Fix any $y \in \{\rho(kh,\cdot) = -4r_1\}$. Choose a smooth cut-off function $\eta_{k,3} = \eta_{k,3}(x;y)$ satisfying

$$\eta_{k,3}(x;y) := \begin{cases} 0 & \text{for } x \in \operatorname{cl} B(y,r_1), \\ 1 & \text{for } x \in \{\rho(kh,\cdot) \leqslant -2r_1\} \backslash B(y,2r_1), \end{cases} \quad 0 \leqslant \eta_{k,3} \leqslant 1 \quad \text{in } \mathbb{R}^N,$$
$$r_1 \|D\eta_{k,3}\|_{L^\infty(\mathbb{R}^N)} + r_1^2 \|D^2\eta_{k,3}\|_{L^\infty(\mathbb{R}^N)} \leqslant M_{3,4}.$$

Define $\underline{v}_k^3 = \underline{v}_k^3(s,x;y)$ by

$$\underline{v}_k^3(s,x;y) := -4r_1 - \sqrt{|x-y|^2 + 2Ns} - M_{3,5}s - \eta_{k,3}(x;y)(M_2 + M_{3,3})s^{1/3}.$$

Here M_2, $M_{3,3}$ and $M_{3,5}$ are the same constants as above. We observe by similar calculations to the above that $\underline{v}_k^3$ is a classical subsolution of (1.2) in $(0,h] \times \{\rho(0,\cdot) \leqslant -2r_1\}$.

Step 3. We obtain a viscosity subsolution of (1.2) in $(0,h) \times \mathbb{R}^N$ for all $h \in (0, t_{3,1})$ and some $t_{4,1} \in (0, s_1)$.

Define $\underline{V}_k^2 = \underline{V}_k^2(s,x)$ and $\underline{V}_k^3 = \underline{V}_k^3(s,x)$ by

$$\underline{V}_k^2(s,x) := \sup_{y \in \{\rho(0,\cdot)=4r_1\}} \underline{v}_k^2(s,x;y), \quad \underline{V}_k^3(s,x) := \sup_{y \in \{\rho(0,\cdot)=-4r_1\}} \underline{v}_k^3(s,x;y)$$

Then $\underline{V}_k^2$ (resp., $\underline{V}_k^3$) is a viscosity subsolution of (1.2) in $(0,h) \times \{2r_1 \leqslant \rho(kh,\cdot) \leqslant 6r_1\}$ (resp., in $(0,h) \times \{\rho(kh,\cdot) \leqslant -2r_1\}$).

We verify that

$$\begin{cases} \underline{v}_k^1 > \underline{V}_k^2 & \text{for } s \in (0,h) \text{ and } x \in \{\rho(kh,\cdot) = 2r_1\}, \\ \underline{v}_k^1 < \underline{V}_k^2 & \text{for } s \in (0,h) \text{ and } x \in \{\rho(kh,\cdot) = 4r_1\}, \end{cases} \tag{5.11}$$

$$\begin{cases} \underline{v}_k^1 > \underline{V}_k^3 & \text{for } s \in (0,h) \text{ and } x \in \{\rho(kh,\cdot) = -4r_1\}, \\ \underline{v}_k^1 < \underline{V}_k^3 & \text{for } s \in (0,h) \text{ and } x \in \{\rho(kh,\cdot) = -6r_1\}. \end{cases} \tag{5.12}$$

For any $x \in \{\rho(kh,\cdot) = 2r_1\}$, there exists $y_x \in \{\rho(kh,\cdot) = 4r_1\}$ such that $|x - y_x| = 2r_1$. Then it is observed by the choice of $M_{3,5}$ in Step 2 that

$$\begin{aligned} \underline{v}_k^1(s,x) &= 2r_1\big(1 - s\omega(s,x)\big) - 4M_1 s r_1^2 - M_2 s^2 - M_{3,3}s^{3/2} \\ &\geqslant 2r_1 - s\big\{|\omega|_{0,\mathcal{M}} + 4M_1 r_1^2 + M_2 + M_{3,3}\sqrt{s}\big\} \\ &> 2r_1 - M_{3,5}s = \underline{v}_k^2(s,x;y_x) = \underline{V}_k^2(s,x). \end{aligned}$$

On the other hand, we observe that there exists a $t_{4,2} \in (0, s_1)$ such that for any small $s \in (0, t_{4,2})$ and $x \in \{\rho(kh,\cdot) = 4r_1\}$

$$\begin{aligned} \underline{v}_k^1(s,x) &\leqslant 4r_1(1 + s|\omega|_{0,\mathcal{M}}) - M_{3,2}\sqrt{s} < 4r_1 - \sqrt{2Ns} - M_{3,5}s \\ &= \underline{v}_k^2(s,x;x) = \underline{V}_k^2(s,x). \end{aligned}$$

Thus (5.11) is obtained for all $h \in (0, t_{4,2})$. We omit the proof of (5.12) because it is quite similar.

Consequently, set $t_4 := \min\{t_{4,1}, t_{4,2}\}$ and $\underline{V}_k = \underline{V}_k(s,x)$ as

$$\underline{V}_k(s,x) := \begin{cases} \max\big\{\underline{v}_k^1(s,x), \underline{V}_k^2(s,x)\big\} & \text{for } (s,x) \in [0,h] \times \big\{2r_1 \leqslant \rho(kh,\cdot) \leqslant 6r_1\big\}, \\ \underline{v}_k^1(s,x) & \text{for } (s,x) \in [0,h] \times \big\{|\rho(kh,\cdot)| \leqslant 2r_1\big\}, \\ \max\big\{\underline{v}_k^1(s,x), \underline{V}_k^3(s,x)\big\} & \text{for } (s,x) \in [0,h] \times \big\{-6r_1 \leqslant \rho(kh,\cdot) \leqslant -2r_1\big\}, \\ \underline{V}_k^3(s,x) & \text{for } (s,x) \in [0,h] \times \big\{\rho(kh,\cdot) \leqslant -6r_1\big\}. \end{cases}$$

Then for all $h \in (0, t_4)$, $\underline{V}_k$ is a viscosity subsolution of (1.2) in $(0,h) \times \{\rho(kh,\cdot) \leqslant 6r_1\}$ satisfying $\underline{V}_k(0,\cdot) = \rho(0,\cdot)$ on $\{\rho(0,\cdot) \leqslant 6r_1\}$. In addition $\underline{V}_k(s,x) = 4r_1 - M_{3,5}s - (M_2 + M_{3,2})\sqrt{s}$ for $s \in [0,h]$ and $x \in \{\rho(kh,\cdot) = 6r_1\}$.

Finally we extend $\underline{V}_k$ to the set $[0, h) \times \{\rho(kh, \cdot) \geqslant 6r_1\}$. For $y \in \{\rho(kh, \cdot) > 4r_1\}$, let $\eta_{k,4} = \eta_{k,4}(x; y)$ be a smooth cut-off function such that

$$\eta_{k,4}(x; y) := \begin{cases} 0 & \text{for } x \in \operatorname{cl} B(y, r_1), \\ 1 & \text{for } x \in \mathbb{R}^N \setminus B(y, 2r_1), \end{cases} \quad 0 \leqslant \eta_{k,4} \leqslant 1 \quad \text{in } \mathbb{R}^N,$$
$$r_1 \|D\eta_{k,4}\|_{L^\infty(\mathbb{R}^N)} + r_1^2 \|D^2\eta_{k,4}\|_{L^\infty(\mathbb{R}^N)} \leqslant M_{3,4}.$$

Define

$$\underline{v}_k^4 = \underline{v}_k^4(s, x; y) := 4r_1 - \sqrt{|x-y|^2 + 2Ns} - M_{3,5}s - \eta_{k,4}(x; y)(M_2 + M_{3,3})\sqrt{s}.$$

Then $\underline{v}_k^4$ is a classical subsolution of (1.2). Set $\underline{v}_k = \underline{v}_k(s, x)$ as

$$\underline{v}_k(s, x) = \begin{cases} \sup\limits_{y \in \{\rho(kh,\cdot) > 4r_1\}} \underline{v}_k^4(s, x; y) & \text{for } (s, x) \in [0, h) \times \{\rho(kh, \cdot) \geqslant 6r_1\}, \\ \max\left\{ \underline{V}_k(s, x), \sup\limits_{y \in \{\rho(kh,\cdot) > 4r_1\}} \underline{v}_k^4(s, x; y) \right\} & \\ & \text{for } (s, x) \in [0, h) \times \{4r_1 < \rho(kh, \cdot) < 6r_1\}, \\ \underline{V}_k(s, x) & \text{for } (s, x) \in [0, h) \times \{\rho(kh, \cdot) \leqslant 4r_1\}. \end{cases}$$

Then $\underline{v}_k$ is a viscosity subsolution of (1.2) in $(0, h] \times \mathbb{R}^N$ for all $h \in (0, t_3)$. Note that $\underline{v}_k^4(s, x; y) = \underline{v}_k^2(s, x; y)$ for $s \in [0, h]$, $x \in \{\rho(kh, \cdot) = 6r_1\}$, $y \in \{\rho(kh, \cdot) = 4r_1\}$ and $h \in (0, t_3)$.

Step 4. We derive (5.8).

Fix $h \in (0, t_4)$. For $k = 0, 1, 2, \ldots$ set $\rho_k = \rho_{\alpha,\mu_k}$. Since we see by $\widehat{E}_k \subset E_k$ that $\underline{v}_k(0, \cdot) \leqslant \rho_k(0, \cdot) = v_k(0, \cdot)$ in $\mathbb{R}^N$, we get $\underline{v}_k \leqslant v_k$ in $[0, h] \times \mathbb{R}^N$ by the comparison principle for viscosity solutions. Thus $\{\underline{v}_k(s, \cdot) \geqslant 0\} \subset \{v_k(s, \cdot) \geqslant 0\}$ for all $s \in [0, h]$.

We estimate $\rho_k(s, x)$ for $x \in \{\underline{v}_k(s, \cdot) = 0\}$. Note that $\{\underline{v}_k(s, \cdot) = 0\}$ and $\{\varphi_\alpha(kh + s, \cdot) = r\}$ ($r \in [0, r_0)$) are smooth surfaces since $|D\underline{v}_k(s, \cdot)| \geqslant 1/2$ on $\{\underline{v}_k(s, \cdot) = 0\}$ and $|D\varphi_\alpha(kh+s, \cdot)| \neq 0$ on $\{\varphi_\alpha(kh + s, \cdot) = r\}$ for all $s \in [0, h]$. From the fact $\underline{v}_k(s, x) = 0$ we easily get

$$\rho_k(s, x) \leqslant \frac{M_{3,6}s^{3/2}}{1 - s\omega(s, x)} \quad \text{for some } M_{3,6} \geqslant M_1 + M_{3,2}.$$

Hence recalling that $\rho_k(s, \cdot) = \rho_{\alpha,\mu_k}(s, \cdot)$ is the signed distance function to $\{\varphi_\alpha(kh + s, \cdot) = \mu_k\}$ we observe that for some $\theta \in (0, 1)$,

$$\begin{aligned} \varphi_\alpha\big((k+1)h, x\big) &= \varphi_\alpha\big((k+1)h, x - \rho_k(h, x)D\rho_k(h, x)\big) \\ &\quad + \rho_k(h, x)\big\langle D\varphi_\alpha\big(h, x - \theta\rho_k(h, x)D\rho_k(h, x)\big), D\rho_k(h, x)\big\rangle \\ &\leqslant \frac{M_3 h^{3/2}}{1 - M_3 h}, \quad M_3 := |D\varphi_\alpha|_{0,\mathcal{M}} M_{3,6} + |\omega|_{0,\mathcal{M}}, \end{aligned}$$

since $x - \rho_k(h, x)D\rho_k(h, x) \in \{\varphi_\alpha((k+1)h, \cdot) = \mu_k\}$. As a result, we have

$$\{\underline{v}_k(h, \cdot) = 0\} \subset \left\{\varphi_\alpha\big((k+1)h, \cdot\big) \leqslant \frac{M_3 h^{3/2}}{1 - M_3 h}\right\}.$$

Since $\{\underline{v}_k(s,\cdot)=0\}$ and $\{\varphi_\alpha(kh+s,\cdot)=r\}$ $(r\in[0,r_0))$ are smooth surfaces as mentioned above, we obtain the desired inclusion. □

Proof of Theorem 5.1. We prove only the case $\{\phi\geqslant 0\}$ since the other case is completely analogous. The strategy of the proof is similar to [30].

Let s_0, φ_α and V be given in Lemma 2.8. We give the proof under the situation mentioned before the proof of Lemma 5.3. By (2.18) it suffices to show (5.1) for all $s\in(0,s_1)$, $x\in\{\varphi_\alpha(x,s)>0\}$ and $s_1=\min\{s_0,t_4\}$.

Fix $s\in(0,s_1)$ and $x\in\{\varphi_\alpha(s,\cdot)>0\}$. Then $B(x,2r_3)\subset\{\varphi_\alpha(s+s',\cdot)>0\}$ for all $s'\in(-2r_3,2r_3)$ and some $r_3>0$.

For any small $h>0$ set $k_0:=[s/h]$ and $l_0:=[(s-r_3)/h]$ so that $s-r_3<l_0h<s-r_3/2$. Then using Lemma 5.3 with $k=l_0$ and $\mu_{l_0}=0$, we have

$$\widehat{E}_{l_0+1}\subset E_{l_0+1},\ \mu_{l_0+1}:=\frac{M_3h^{3/2}}{1-M_3h}.$$

Inductively, we apply Lemma 5.3 with $k=l_0+l$ and small $\mu_{l_0+l}>0$ to obtain

$$\widehat{E}_{l_0+l+1}\subset E_{l_0+l+1},\ \mu_{l_0+l+1}:=\mu_{l_0+l}+\frac{M_3h^{3/2}}{1-M_3h}.$$

Setting $l_1:=[(s+r_3)/h]$ we get

$$\{\varphi_\alpha(lh,\cdot)\geqslant\mu_h\}\subset E_l,\ \mu_h:=\frac{M_3(2r_3+h)\sqrt{h}}{1-M_3h}$$

for all $l=l_0,l_0+1,\cdots,l_1$.

Consequently, taking $h>0$ small enough, we have $B(x,r_3)\subset\{\varphi_\alpha(s+lh,\cdot)\geqslant\mu_h\}\subset E_{[s/h]+l}$ for $l=0,\pm1\cdots,\pm l_0$. This implies that

$$d(y,E_{[s/h]+l})\geqslant\rho(s,y)\geqslant\frac{r_3}{2}\quad\text{for all }(s,y)\in\text{ and small }h>0.$$

Letting $(h,lh,y)\to(0,s',x)$, we obtain (5.1). □

Theorem 5.4 *Assume* (2.1). *Let* $(\varGamma_L(t),\varOmega_L^+(t),\varOmega_L^-(t))_{t\in[0,t)}$ *be a level-set flow by* (1.1) *with the initial data* $(\partial C_0,\operatorname{int}C_0,\mathbb{R}^N\backslash C_0)$. *Let* $(\varGamma(t),\varOmega^+(t),\varOmega^-(t))_{t\in[0,T)}$ *be defined by* (4.3) *and* (4.4). *Then for each* $t\in[0,T)$,

$$\varOmega_L^+(t)\subset\varOmega^+(t)\subset\varOmega_L^+(t)\cup\varGamma_L(t),\ \varOmega_L^-(t)\subset\varOmega^-(t)\subset\varOmega_L^-(t)\cup\varGamma_L(t)$$

Proof. Step 1. From Theorem 2.6 and Proposition 2.9 we have only to show that $\{\varOmega^+(t)\}_{t\in[0,T)}$ and $\{\mathbb{R}^N\backslash\operatorname{cl}\varOmega^-(t)\}_{t\in[0,T)}$ are, respectively, a generalized superflow and a generalized subflow by (1.1).

Indeed, if we do so, then it follows from Proposition 2.9 that $\overline{u}=\overline{u}(t,x):=\chi_{\varOmega^+(t)}(x)-\chi_{\varOmega^-(t)\cup\varGamma(t)}(x)$ and $\underline{u}=\underline{u}(t,x):=\chi_{\mathbb{R}^N\backslash\operatorname{cl}\varOmega^-(t)}(x)-\chi_{\operatorname{cl}\varOmega^-(t)}(x)$ are, respectively, a viscosity supersolution and a viscosity subsolution of (2.14). Since

$$\chi_{\varOmega^+(0)}=\chi_{\mathbb{R}^N\backslash\operatorname{cl}\varOmega^-(t)}=\chi_{\operatorname{int}C_0},\ \chi_{\mathbb{R}^N\backslash\varOmega^+(0)}=\chi_{\varOmega^-(0)}=\chi_{\mathbb{R}^N\backslash C_0}\quad\text{in }\mathbb{R}^N$$

by Proposition 4.4, we have from Theorem 2.6

$$\underline{u} \leqslant \chi_{\Omega_L^+(t)\cup\Gamma_L(t)} - \chi_{\Omega_L^-(t)}, \ \overline{u} \geqslant \chi_{\Omega_L^+(t)} - \chi_{\Omega_L^-(t)\cup\Gamma_L(t)} \quad \text{in } [0,T)\times\mathbb{R}^N.$$

Therefore we obtain the desired inclusions.

Step 2. We show that $\{\Omega^+(t)\}_{t\in[0,T)}$ and $\{\mathbb{R}^N\backslash \mathrm{cl}\,\Omega^-(t)\}_{t\in[0,T)}$ are, respectively, a generalized superflow and a generalized subflow by (1.1).

We prove only the superflow case since the other one is shown by the same way. Let $t > 0$, $x_0 \in \mathbb{R}^N$, $r > 0$, $\alpha > 0$ and $\phi \in C^\infty(\mathbb{R}^N)$ be such that $\{\phi \geqslant 0\} \subset \mathrm{int}\,\Omega^+(t) \cap B(x_0, r)$ with $|D\phi| \neq 0$ on $\{\phi = 0\}$. It follows from (4.3) that there exists $t_5 > 0$ such that for any $h \in (0, t_5)$, $l \in \mathbb{N}$ satisfying $|lh - t| < t_5$, we get

$$w^h(lh,\cdot) = d(\cdot, C_l) \geqslant d(\cdot, E_0) \quad \text{in } \mathbb{R}^N \ \ E_0 := \{\phi \geqslant 0\}. \tag{5.13}$$

We now apply our algorithm with the initial set E_0 and construct a sequence $\{v_k\}_{k\geqslant 0}$ of solutions of (1.2) with $v_k(0,\cdot) = d(\cdot, E_k)$. We repeatedly use the maximum principle for (1.2) to have $E_k \subset C_{l+k}$ and hence

$$w^h\big((l+k)h,\cdot\big) = d(\cdot, C_{l+k}) \geqslant d(\cdot, E_k) \quad \text{in } \mathbb{R}^N \quad \text{for all } k, l \in \mathbb{N} \text{ such that } |lh - t| < t_5.$$

We see by Proposition 4.2, Lemma 5.3 and this inequality that there exists $s_2 > 0$ such that for any $s \in (0, s_2)$ and $x \in \mathrm{cl}\,B(x_0, r) \cap \{\phi - s[F(t,\cdot, D\phi, D^2\phi)] > 0\}$

$$\underline{d}(t+s, x) = \liminf_{(h,lh,kh,y)\to(0,t,s,x)} w^h\big((l+k)h, y\big) \geqslant \liminf_{(h,kh,y)\to(0,s,x)} d(y, E_k) > 0.$$

Hence $\mathrm{cl}\,B(x_0,r) \cap \{\phi - s[F(t,\cdot,D\phi,D^2\phi)] > 0\} \subset \Omega^+(t+s)$ for all $s \in (0, s_1)$. Thus $\{\Omega^+(t)\}_{t\in(0,T)}$ is a generalized superflow by (1.1). ☐

Let $d = d(t,x)$ be the signed distance function to $\Gamma_L(t)$ defined by (1.4) with $D = \mathrm{cl}\,\Omega_L^+(t)$. Then we have the convergences of $\{d^h\}_{h>0}$ and $\{w^h\}_{h>0}$ to d as $h \to 0$.

Theorem 5.5 *Assume* (2.1) *and let* $(\Gamma_L(t), \Omega_L^+(t), \Omega_L^-(t))_{t\in[0,T)}$ *be a level-set flow by* (1.1) *with the initial data* $(\partial C_0, \mathrm{int}\,C_0, \mathbb{R}^N\backslash C_0)$ *satisfying* (2.16). *The sequences* $\{d^h\}_{h>0}$ *and* $\{w^h\}_{h>0}$ *converge to* d_L *locally uniformly in* $[0,T)\times\mathbb{R}^N$ *as* $h \to 0$.

Proof. It is observed by (2.16), (4.3), (4.4), Proposition 4.3 and Theorem 5.4 that

$$\Gamma_L(t) = \Gamma(t) = \big\{\overline{d}(t,\cdot) = \underline{d}(t,\cdot) = 0\big\}, \tag{5.14}$$

$$\Omega_L^+(t) = \Omega^+(t) = \big\{\overline{d}(t,\cdot) > 0\big\} = \big\{\underline{d}(t,\cdot) > 0\big\}, \tag{5.15}$$

$$\Omega_L^-(t) = \Omega^-(t) = \big\{\overline{d}(t,\cdot) < 0\big\} = \big\{\underline{d}(t,\cdot) < 0\big\}, \tag{5.16}$$

for all $t \in [0,T)$. Note that the map $t \mapsto \mathrm{cl}\,\Omega_L^+(t)$ is continuous in the sense that

$$\lim_{s\to t} d_H\big(\mathrm{cl}\,\Omega_L^+(t), \mathrm{cl}\,\Omega_L^+(s)\big) = 0. \tag{5.17}$$

Indeed, we choose a unique viscosity solution $u \in UC([0,T)\times\mathbb{R}^N)$ of (2.14) satisfying $u(0,\cdot) = d(\cdot, C_0)$ in $\mathbb{R}^N$ and thus (2.15) holds (cf. Remark 2.5). Using the continuity of u and (2.16), we get (5.17). Moreover we observe that d satisfies (4.8) and (4.9) in the viscosity sense and by Propositions 4.2–4.4 that $\overline{d} = \overline{w} = \underline{d} = \underline{w} = d$ on $\{\cup_{t\in[0,T)}(\{t\}\times\varGamma_L(t))\}\cup(\{0\}\times\mathbb{R}^N)$. Thus we use Theorem 4.5 and the comparison principle for eikonal equations (cf. [19], [21]) to have $\overline{w} = \overline{d} = \underline{w} = \underline{d} = d$ in Q_T. By [8, Remark 6.4], we have the desired result. □

Theorems 5.4 and 5.5 lead to the convergence of $\{C^h(t)\}_{t\in[0,T),h>0}$ to $\{\mathrm{cl}\,\varOmega_L^+(t)\}_{t\in[0,T)}$.

Theorem 5.6 *Assume* (2.1). *Let* $(\varGamma_L(t),\varOmega_L^+(t),\varOmega_L^-(t))_{t\in[0,T)}$ *be the level-set flow by* (1.1) *with the initial data* $(\partial C_0,\mathrm{int}\,C_0,\mathbb{R}^N\backslash C_0)$ *satisfying* (2.16). *For any* $\varepsilon\in(0,T)$

$$\lim_{h\to 0}\sup_{t\in[0,T-\varepsilon]} d_H(C^h(t),\mathrm{cl}\,\varOmega^+(t)) = \lim_{h\to 0}\sup_{t\in[0,T-\varepsilon]} d_H(C^h(t),\mathrm{cl}\,\varOmega_L^+(t)) = 0, \tag{5.18}$$

Proof. By the continuity of d, (5.14), (5.15) and (5.16) turn to

$$\varGamma_L(t)=\{d(t,\cdot)=0\},\ \varOmega_L^+(t)=\{d(t,\cdot)>0\},\ \varOmega_L^-(t)=\{d(t,\cdot)<0\}.$$

Thus we get (5.18) by applying Theorem 5.5 and [15, Lemma 4.6.5]. □

REMARK 5.7 (1) Since $\{C^h(t)\}_{t\in[0,T),h>0}$ and $\{\mathrm{cl}\,\varOmega_L(t)\}_{t\in[0,T)}$ are bounded (cf. Proposition 3.1), from [5, Appendix A], one can prove the uniform convergences of $\{d^h\}_{h>0}$ and $\{w^h\}_{h>0}$ to d on $[0,T-\varepsilon]\times\mathbb{R}^N$ for each $\varepsilon>0$.

(2) We see from Theorem 4.5, [19] and [21] that the limit d of $\{d^h\}_{h>0}$ and $\{w^h\}_{h>0}$ is a unique viscosity solution of (4.8) and (4.9). In addition, d satisfies a weak form of (2.10) and (2.11): $\min\{d,0\}$ (resp., $\max\{d,0\}$) is a viscosity subsolution (resp., supersolution) of

$$u_t + F(t,x-uDu,Du,D^2u)=0 \quad \text{in } Q_T.$$

See [38, Theorem 11.1], [2, Theorem 3.1, Lemma 5.3] for the proof.

(3) Vivier [39] and Leoni [30] considered the approximation schemes for the anisotropic CDM related to ours. The choice of the initial data is the main different point from our scheme. They choose the initial data $w(0,x)=\chi_{C_k}(x)-\chi_{\mathbb{R}^N\backslash C_k}(x)$ for $x\in\mathbb{R}^N$ instead of (1.3). In addition, Chambolle and Novaga also treated in [6] an algorithm to the anisotropic MCF. The differences between [6] and our algorithm are the approximate equation and the choice of the initial data. Theorem 5.6 is a similar result to those in [39], [30] and [6].

(4) From the viewpoint of numerical analysis we are able to replace the initial data (1.3) with

$$w_k(0,x)=\eta(d(x,C_k)) \quad \text{for } x\in\mathbb{R}^N, \tag{5.19}$$

in constructing $\{C_k\}_{k=0}^{[T/h]}$. Here η is a Lipschitz continuous function defined by

$$\eta(r) := \begin{cases} -3\delta & \text{if } r\leqslant -4\delta,\\ r & \text{if } |r|\leqslant 2\delta, \quad \eta'(r)\geqslant 0 \quad \text{for a.e. } r\in\mathbb{R}\\ 3\delta & \text{if } r\geqslant 4\delta,\end{cases}$$

For $h>0$ and $k=0,1,\ldots,[T/h]$, let $\widehat{w}_k$ be a solution of (1.2) - (5.19). Then we observe by lengthy calculations that

$$\sup_{k=0,1,\ldots,[T/h]}\|w_k-\widehat{w}_k\|_{L^\infty([kh,(k+1)h)\times\{|d(\cdot,C_k)|\leqslant\delta\})} = O(e^{-\delta^2/8h})$$

for any small $h>0$. Thus applying the results of this section, we have the convergence of $\{\widehat{w}_k\}_{k=0}^{[T/h]}$ to d as $h\to 0$ uniformly in $\cup_{t\in[0,T-\varepsilon]}[\{t\}\times\{|d(t,\cdot)|\leqslant\delta\}]$ for each $\varepsilon>0$.

6. Rate of convergence to smooth and compact CDM's

This section is devoted to the rate of convergence to the smooth and compact CDM and to its optimality in the case of a circle evolving by (1.1) with $\mathbf{b} = 0$ and $g = 0$.

6.1 *Rate of convergence*

In order to derive the rate of convergence we reformulate our algorithm in the following way: Let C_0 be a compact subset of $\mathbb{R}^N$ whose boundary is of class $C^{5+\alpha}$. For each $h > 0$ let $\{w_k\}_{k=0}^{[T_0/h]}$ be a sequence of solutions of (1.2) - (1.3) with setting $C_k := \{w_{k-1}(h, \cdot) \geqslant 0\}$ $(k = 1, 2, \ldots, [T_0/h])$. Define $\{w^h\}_{h>0}$ as in (4.1) and $C^h(t)$ as

$$C^h(t) := \{w^h(t, \cdot) \geqslant 0\} \quad \text{for } t \in [0, T_0) \text{ and } h > 0 \tag{6.1}$$

instead of (1.6). Notice that $C^h(kh) = C_k$ for $k = 0, 1, 2, \ldots, [T_0/h]$ and $h > 0$. We then obtain the following theorem.

Theorem 6.1 *Assume* (2.1) *and* (2.3). *Let* $\{\Gamma(t)\}_{t\in[0,T_0)}$ *be a smooth and compact CDM with* $\Gamma(0) = \partial C_0$ *and let* $\rho = \rho(t, x)$ *be defined by* (2.4). *Set* $C^h(t)$ *as* (6.1) *and* $C(t) := \{\rho(t, \cdot) \geqslant 0\}$ *for each* $t \in [0, T_0)$ *and* $h > 0$. *For any* $\varepsilon > 0$, *there exist* L_1 *and* $h_0 > 0$ *depending on* (2.1) *and* (2.5) *such that*

$$\sup_{t\in[0,T_0-\varepsilon]} d_H\big(C^h(t), C(t)\big) \leqslant L_1 h \quad \textit{for all } h \in (0, h_0).$$

See Appendix for the existence and uniqueness of $\{\Gamma(t)\}_{t\in[0,T_0)}$. Since $\Gamma(t)$ is a hypersurface for every $t \in [0, T_0)$, Theorem 5.6 yields that for any $\varepsilon, \eta_0 > 0$, there exists $h_{0,1} > 0$ such that

$$\sup_{t\in[0,T_0-\varepsilon]} d_H\big(C^h(t), C(t)\big) \leqslant \eta_0 \quad \text{for all } h \in (0, h_{0,1}). \tag{6.2}$$

Hence the above theorem is deduced from the following lemma.

Lemma 6.2 *Under the conditions in Theorem* 6.1, *if* $d_H(C^h(kh), C(kh)) \leqslant \eta$ *for small* $\eta \in [0, \eta_0)$, *then for some* $K_8 > 0$, $h_{0,2} \in (0, h_{0,1})$ *depending on* (2.1) *and* (2.5),

$$d_H\big(C^h(kh+\bar{t}), C(kh+\bar{t})\big) \leqslant \frac{\eta + K_8\bar{t}^2}{1 - K_8\bar{t}} \quad \textit{for all } \bar{t} \in [0, h] \textit{ and } h \in (0, h_{0,2}).$$

Proof. Assume that $(0 \leqslant) d_H(C^h(kh), C(kh)) \leqslant \eta$. Let W_k be a solution of (1.2) satisfying $W_k(0, \cdot) = d(\cdot, C(kh))$ in $\mathbb{R}^N$ and set $D^{\pm}_{\eta}(kh+\bar{t}) := \{W_k(\bar{t}, \cdot) \geqslant \pm\eta\}$ and $\Omega^{\pm}_{\eta}(kh+\bar{t}) := \{\rho(kh+\bar{t}, \cdot) \geqslant \pm\eta\}$.

We easily get $W_k - \eta \leqslant w_k \leqslant W_k + \eta$ on $[0, h] \times \mathbb{R}^N$ from the maximum principle since $W_k(0, \cdot) - \eta \leqslant w_k(0, \cdot) \leqslant W_k(0, \cdot) + \eta$ in $\mathbb{R}^N$. Hence we have $D^+_{\eta}(kh+\bar{t}) \subset C^h(kh+\bar{t}) \subset D^-_{\eta}(kh+\bar{t})$ for all $\bar{t} \in [0, h]$. Since $\Omega^+_{\eta}(kh+\bar{t}) \subset C(kh+\bar{t}) \subset \Omega^+_{\eta}(kh+\bar{t})$, we obtain for all $\bar{t} \in [0, h]$

$$\Omega^+_{\eta}(kh+\bar{t}) \cap D^+_{\eta}(kh+\bar{t}) \subset C(kh+\bar{t}), C^h(kh+\bar{t}) \subset \Omega^-_{\eta}(kh+\bar{t}) \cup D^-_{\eta}(kh+\bar{t}).$$

Therefore we observe that for all $\bar{t} \in [0, h]$,

$$\begin{aligned} d_H\big(C^h(kh+\bar{t}), C^h(kh+\bar{t})\big) \leqslant \max\big\{& d_H\big(\Omega^+_{\eta}(kh+\bar{t}) \cap D^+_{\eta}(kh+\bar{t}), C^h(kh+\bar{t})\big), \\ & d_H\big(\Omega^-_{\eta}(kh+\bar{t}) \cup D^-_{\eta}(kh+\bar{t}), C^h(kh+\bar{t})\big)\big\}. \end{aligned} \tag{6.3}$$

We estimate the right-hand side of (6.3). It is easily seen that

$$\begin{aligned} &d_H\big(\Omega_\eta^+(kh+\bar t)\cap D_\eta^+(kh+\bar t), C^h(kh+\bar t)\big)\\ &\qquad\leqslant d_H\big(D_\eta^+(kh+\bar t), C^h(kh+\bar t)\big)+d_H\big(\Omega_\eta^+(kh+\bar t), D_\eta^+(kh+\bar t)\big),\\ &d_H\big(\Omega_\eta^-(kh+\bar t)\cup D_\eta^-(kh+\bar t), C^h(kh+\bar t)\big)\\ &\qquad\leqslant d_H(D_\eta^-(kh+\bar t), C^h(kh+\bar t))+d_H\big(\Omega_\eta^-(kh+\bar t), D_\eta^-(kh+\bar t)\big). \end{aligned}$$

As W_k satisfies Proposition 3.4, we get from some calculations

$$d_H\big(D_\eta^\pm(kh+\bar t), C^h(kh+\bar t)\big)\leqslant \frac{\eta}{1-K_5\bar t}\quad \text{for all } \bar t\in[0,h] \text{ and } h>0.$$

Step 1. We derive an estimate for $\sup_{x\in D_\eta^+(kh+\bar t)}\operatorname{dist}(x,\Omega_\eta^+(kh+\bar t))$.

Fix $\bar t\in[0,h]$ and $x\in D_\eta^+(kh+\bar t)$. We may assume that $x\in\partial D_\eta^+(kh+\bar t)\backslash\Omega_\eta^+(kh+\bar t)$. Set $\widetilde\rho(\bar t,x):=\rho(kh+\bar t,x)$. Notice that for $s\in[0,h]$ the point $z(s,x):=x-\widetilde\rho(s,x)D\widetilde\rho(s,x)\in\partial\Omega_\eta^+(kh+s)$ satisfies $|x-z(s,x)|=|\widetilde\rho(s,x)|=\operatorname{dist}(x,\partial\Omega_\eta^+(kh+s))$. Set $\xi(s):=W_k(s,z(s,x))$. By the facts $W\in C^{(5+\alpha)/2,5+\alpha}([0,h]\times\{|\rho(kh,\cdot)|\leqslant 5\delta\})$, $W_t(0,z(0,x))=\widetilde\rho_t(0,x)$, $DW_k(0,z(0,x))=D\widetilde\rho(0,x)$ and $\langle D\widetilde\rho, D\widetilde\rho_t\rangle=0$, we see that

$$\begin{aligned} \xi'(0)&=\widetilde\rho_t(0,x)-\widetilde\rho_t(0,x)\big\langle D\widetilde\rho(0,x), D\widetilde\rho(0,x)\big\rangle-\widetilde\rho(0,x)\big\langle D\widetilde\rho(0,x), D\widetilde\rho_t(0,x)\big\rangle=0\\ \xi''(s)&=W_{k,tt}\big(s,z(s,x)\big)+2\big\langle DW_{k,t}\big(s,z(s,x)\big), z_s(s,x)\big\rangle\\ &\quad+\big\langle D^2W_k\big(s,z(s,x)\big)z_s(s,x), z_s(s,x)\big\rangle+\big\langle DW_k\big(s,z(s,x)\big), z_{ss}(s,x)\big\rangle,\\ z_s(s,x)&=-\big\{\widetilde\rho_t(s,x)D\widetilde\rho(s,x)+\widetilde\rho(s,x)D\widetilde\rho_t(s,x)\big\},\\ z_{ss}(s,x)&=-\big\{\widetilde\rho_{tt}(s,x)D\widetilde\rho(s,x)+2\widetilde\rho_t(s,x)D\widetilde\rho_t(s,x)+\widetilde\rho(s,x)D\widetilde\rho_{tt}(s,x)\big\}. \end{aligned}$$

Here we have used (2.5). Hence we have from $\xi(0)=W_k(0,z(0,x))=\widetilde\rho(0,x)=d(x,C_k)=\eta$, Taylor's theorem and these formulae

$$W_k\big(s,z(s,x)\big)=\xi(s)=\eta+\frac12\xi''(\theta s)s^2\quad\text{for some }\theta\in(0,1).$$

Thus it follows from (3.4) and (2.5) that

$$\sup_{\substack{s\in[0,h],x\in\partial D(kh+s)\\ k=0,1,\dots,[T/h],h>0}}\left|\frac12\xi''(s)\right|\leqslant K_{8,1}. \tag{6.4}$$

Here and in the sequel $K_{8,j}>0$ ($j\in\mathbb N$) is a constant depending on (2.5) and (3.4).

Furthermore we observe that for some $\theta\in(0,1)$

$$\begin{aligned} \eta=W_k(\bar t,x)&=W_k(\bar t,z(\bar t,x))+\widetilde\rho(\bar t,x)\big\langle DW_k\big(\bar t,z^\theta(\bar t,x)\big), D\widetilde\rho(\bar t,x)\big\rangle\\ &=\eta+\frac12\xi''(\theta\bar t)\bar t^2+\widetilde\rho(\bar t,x)\big\langle DW_k\big(\bar t,z^\theta(\bar t,x)\big), D\widetilde\rho(\bar t,x)\big\rangle \end{aligned}$$

where $z^\theta(s,x):=x-\theta\widetilde\rho(s,x)D\widetilde\rho(s,x)$. Setting $s=\bar t$ and combining (6.4) and Remark 3.5 with this formula, we get

$$|\widetilde\rho(\bar t,x)|\leqslant\left|\frac{-K_{8,1}\bar t^2}{2\langle DW_k(\bar t,z^\theta(\bar t,x)), D\widetilde\rho(\bar t,z^\theta(\bar t,x))\rangle}\right|\leqslant\frac{K_{8,1}t^2}{1-K_5t}.$$

Recalling $\overline{\rho}(\overline{t},x) = d(kh+\overline{t},x)$, we obtain

$$\sup_{x\in D_\eta^+(kh+\overline{t})} \operatorname{dist}\big(x, \Omega_\eta^+(kh+\overline{t})\big) \leqslant \frac{K_{8,1}\overline{t}^2}{1-K_5\overline{t}}.$$

Step 2. We estimate $\sup_{x\in\Omega_\eta^+(kh+\overline{t})} \operatorname{dist}(x, D_\eta^+(kh+\overline{t}))$.

The argument is similar to that in Step 1. Fix $\overline{t}\in[0,h]$ and $x\in\Omega_\eta^+(kh+\overline{t})$. We may assume that $x\in\partial\Omega_\eta^+(kh+\overline{t})\backslash D_\eta^+(kh+\overline{t})$. Let $\widehat{\rho}(\overline{t},x)$ be the signed distance function given by (2.4) with $\Gamma(kh+\overline{t}) = \partial D_\eta^+(kh+\overline{t})$. Then

$$\begin{aligned}&\widehat{\rho}(0,\cdot) = \rho(kh,\cdot)-\eta \quad \text{on } \{|\rho(kh,\cdot)|\leqslant 5\delta\},\\ &\widehat{\rho}_t = \frac{W_{k,t}}{|DW_k|},\ D\widehat{\rho} = \frac{DW_k}{|DW_k|},\ D\widehat{\rho}_t = \frac{\partial}{\partial t}\left(\frac{DW_k}{|DW_k|}\right) \quad \text{on } [0,h]\times\{|\rho(kh,\cdot)|\leqslant 5\delta\}.\end{aligned}$$

Here we have used (3.5).

For $s\in[0,h]$, the point $\widehat{z}(s,x) := x-\widehat{\rho}(s,x)D\widehat{\rho}(s,x)\in\partial D_\eta^+(s)$ satisfies $|x-\widehat{z}(s,x)| = |\rho(s,x)| = \operatorname{dist}(x,\partial D_\eta^+(s))$. Similar calculations to those in the previous step yield that

$$\sup_{\substack{\overline{t}\in[0,h],x\in\partial\widehat{C}(kh+s)\\ k=0,1,\dots,[T/h],h>0}} \big|\rho\big(kh+\overline{t},\widehat{z}(\overline{t},x)\big)-\eta\big| \leqslant K_{8,2}\overline{t}^2,$$

$$\begin{aligned}\eta &= \rho(kh+\overline{t},x) = \rho\big(kh+\overline{t},\widehat{z}(\overline{t},x)\big)+\widehat{\rho}(\overline{t},x)\big\langle D\rho\big(kh+\overline{t},x-\theta\widehat{\rho}(\overline{t},x)D\widehat{\rho}(\overline{t},x)\big),D\widehat{\rho}(\overline{t},x)\big\rangle,\\ &= \rho\big(kh+\overline{t},\widehat{z}(\overline{t},x)\big)+\widehat{\rho}(\overline{t},x)\left\langle D\rho(kh+\overline{t},x),\frac{DW_k}{|DW_k|}\right\rangle.\end{aligned}$$

Therefore we have by using Propositions 3.3 and 3.4

$$\big|\widehat{\rho}(\overline{t},x)\big| \leqslant \frac{\big|DW_k(\overline{t},x)\big|\,\big|\rho\big(kh+\overline{t},\widehat{z}(\overline{t},x)\big)-\eta\big|}{|\langle D\rho(kh+\overline{t},x),DW_k\rangle|.} \leqslant \frac{(1+K_2h)K_{8,2}\overline{t}^2}{1-K_5\overline{t}}$$

and consequently

$$\sup_{x\in\Omega_\eta^+(kh+\overline{t})} \operatorname{dist}\big(x, D_\eta^+(kh+\overline{t})\big) \leqslant \frac{K_{8,3}\overline{t}^2}{1-K_5\overline{t}}.$$

Combining the estimates in Step 1, 2 and setting $K_8 := \max\{K_5,K_{8,1},K_{8,3}\}$, we obtain $d_H(\Omega_\eta^+(kh+\overline{t}),D_\eta^+(kh+\overline{t}))\leqslant K_8\overline{t}^2/(1-K_5\overline{t})$. The remaining part can be estimated. □

Proof of Theorem 6.1. In the case $k=0$, we apply Lemma 6.2 with $\eta:=0$ to have

$$\sup_{\overline{t}\in[0,h]} d_H\big(C^h(\overline{t}),C(\overline{t})\big) \leqslant \frac{K_8h^2}{1-K_8h}.$$

In the case $k=1$, it follows from Lemma 6.2 with $\eta := K_8h^2/(1-K_8h)$ to obtain

$$\sup_{\overline{t}\in[0,h]} d_H\big(C^h(h+\overline{t}),C(h+\overline{t})\big) \leqslant \frac{K_8h^2}{(1-K_8h)^2}+\frac{K_8h^2}{1-K_8h}.$$

Repeating this process, we see that for $k = 2, 3, \dots, [T_0/h]$,

$$\begin{aligned}\sup_{\bar{t}\in[0,h]} d_H\big(C^h(kh+\bar{t}), C(kh+\bar{t})\big) &\leqslant \sum_{l=1}^{k+1} \frac{K_8 h^2}{(1-K_8h)^l} \\ &\leqslant \frac{K_8h^2\big\{1-(1-K_8h)^{-[T_0/h]}\big\}}{(1-K_8h)\big\{1-(1-K_8h)^{-1}\big\}} \\ &\leqslant \frac{K_8h^2(e^{K_8T_0}-1)}{K_8h} \\ &= (e^{K_8T_0}-1)h.\end{aligned}$$

Letting $L_1 := (e^{K_8T_0}-1)$ and $h_0 = h_{0,2}$, we get the desired result. □

REMARK 6.3 Assume (2.1) and (2.5). From Theorem 6.1 we can replace the initial data (1.3) with

$$w_k(0,x) = \eta\big(d(x, C_k)\big) \quad \text{for } x \in \mathbb{R}^N, \tag{6.5}$$

in constructing $\{C_k\}_{k=0}^{[T/h]}$. Here η is a Lipschitz continuous function defined by

$$\eta(r) := \begin{cases} -3h^{1/2-\beta} & \text{if } r \leqslant -4h^{1/2-\beta}, \\ r & \text{if } |r| \leqslant 2h^{1/2-\beta}, \\ 3h^{1/2-\beta} & \text{if } r \geqslant 4h^{1/2-\beta}, \end{cases} \quad \eta'(r) \geqslant 0 \quad \text{for a.e. } r \in \mathbb{R},\ \beta \in (0, 1/2).$$

For $h > 0$ and $k = 0, 1, \dots, [T/h]$, let $\widehat{w}_k$ be a solution of (1.2) - (6.5). Then we observe by lengthy calculations that

$$\sup_{k=0,1,\dots,[T/h]} \|w_k - \widehat{w}_k\|_{L^\infty([kh,(k+1)h)\times\{|d(C_k)|\leqslant\sqrt{h}\})} = O(e^{-h^{-\beta/2}})$$

for any small $h > 0$. Combining (3.5) with this estimate, we can get the same estimate as in Theorem 6.1 with $C^h(t)$ replacing $\{\widehat{w}_k(0,\cdot) \geqslant 0\}$.

6.2 *Optimality*

This subsection is devoted to the optimality of the estimate in Theorem 6.1. For simplicity, we set $N = 2$, $\mathbf{b} = 0$, $g = 0$, $R(t) := \sqrt{1-2t}$, $T_0 := 1/2$ and $C(t) := \{x \in \mathbb{R}^2 \mid |x| \leqslant R(t)\}$. Since it suffices to consider the radial solution, the initial value problem (1.2) - (1.3) and the definition of $\{C_k\}_{k=0}^{[T/h]}$ turn to

$$w_{k,t} = w_{k,rr} + \frac{w_{k,r}}{r},\ w_k = w_k(t,r) \quad \text{in } (0,+\infty)\times(0,+\infty), \tag{6.6}$$

$$w_{k,r}(t,0) = 0 \quad \text{for } t > 0, \tag{6.7}$$

$$w_k(0,r) = R_k - r \quad \text{for } r \in [0,+\infty), \tag{6.8}$$

$$C_k := \big\{x \in \mathbb{R}^2 \mid w_k(h,|x|) \geqslant 0\big\},\ C_0 := \mathrm{cl}\, B(0,1),$$

$$R_k := \text{radius of } C_k,\ R_0 := 1.$$

For $t \in [kh, (k+1)h)$, $k = 0, 1, 2, \ldots, [T/h]$ and $h > 0$, set

$$C^h(t) := \{x \in \mathbb{R}^2 \mid w_k(t - kh, |x|) \geqslant 0\}, \ R^h(t) := \text{radius of } C^h(t).$$

The following proposition says that for each $h > 0$, $C^h(t)$ evolves faster than $C(t)$.

Proposition 6.4 *$C^h(t) \subset C(t)$ for all $t \in [0, T_0)$ and $h > 0$.*

Proof. Let $V_0 = V_0(t, r) := 1 - \sqrt{r^2 + 2t}$. Then $C(t) = \{V_0(t, |\cdot|) \geqslant 0\}$ for $t \in [0, h]$ and V_0 is a classical supersolution of (6.6) satisfying (6.7) and (6.8). Hence it follows from the maximum principle that $w_0 \leqslant V_0$ on $[0, h] \times [0, +\infty)$. This inequality yields that $C^h(t) \subset C(t)$ for all $t \in [0, h]$. Set $V_1 = V_1(t, r) := R(h) - \sqrt{r^2 + 2t}$. Then $C(t + h) = \{V_1(t, |\cdot|) \geqslant 0\}$ for $t \in [0, h]$ and V_1 is a classical supersolution of (6.6) satisfying (6.7) and $w_1(0, \cdot) \leqslant V_1(0, \cdot)$ on $[0, +\infty)$. Thus we get $w_1 \leqslant V_1$ on $[0, h] \times [0, +\infty)$ by the maximum principle. Therefore $C^h(h + t) \subset C(h + t)$ for all $t \in [0, h]$. Similarly, setting $V_2(t, r) := R(2h) - \sqrt{r^2 + 2t}$, we are able to show that $C^h(2h + t) \subset C(2h + t)$ for all $t \in [0, h]$. Thus we have the result by induction. □

Proposition 6.5 *For any $\delta \in (0, 1/8)$, there are constants $K_9 > 0$ and $t_6 > 0$ depending on δ such that*

$$\left| w_{k,r}(\bar{t}, r) - \left(-1 + \frac{\bar{t}}{r^2} \right) \right| \leqslant K_9 \bar{t}^2 \quad \textit{for all } \bar{t} \in [0, h],\ r \in [\delta, +\infty) \textit{ and } h \in (0, t_6). \tag{6.9}$$

Proof. Since $|\cdot| \in C^\infty(\mathbb{R}^2 \backslash \{0\})$, Remark 3.5 (2) yields that

$$\left| D w_k(\bar{t}, |x|) - \left(-\frac{x}{|x|} + \bar{t} \frac{x}{|x|^3} \right) \right| \leqslant K_9 \bar{t}^2 \quad (i = 1, 2)$$

for small $\bar{t} > 0$, $x \in \mathbb{R}^N \backslash B(0, \delta)$ and some $K_9 > 0$. Noting the formula $w_{k,r} = \langle D w_k, x/|x| \rangle$, we get the desired result. □

Since we see by Proposition 6.4 and Theorem 6.1 that for any $\varepsilon \in (0, 1/4)$

$$d_H\big(C^h(t), C(t)\big) = R(t) - R^h(t) \leqslant L_1 h,\ R^h(t) \geqslant \sqrt{\varepsilon} \tag{6.10}$$

for all $t \in [0, 1/2 - \varepsilon]$, $h \in (0, t_7)$ and some $t_7 \in (0, h_0)$, we consider the lower bound of $R(t) - R^h(t)$ for small $h > 0$ to prove the optimality of Theorem 6.1. Hence our optimality result is stated as follows.

Theorem 6.6 *Set $C(t) := \{|x| \leqslant R(t)\}$ ($R(t) = \sqrt{1 - 2t}$) and $C^h(t) = \{w_k(t - kh, |x|) \geqslant 0\}$. Let $R^h(t)$ be the radius of $C^h(t)$. Then for any $\varepsilon \in (0, 1/4)$ there exists $h_1 > 0$ such that for all $h \in (0, h_1)$*

$$R(t) - R^h(t) \geqslant \begin{cases} \dfrac{1}{2} t^2 & \textit{for } t \in [0, h], \\ \dfrac{1}{4} th & \textit{for } t \in [h, 1/2 - \varepsilon]. \end{cases}$$

The strategy of the proof of Theorem 6.6 is similar to that of Theorem 6.1.

Lemma 6.7 *Fix $\varepsilon \in (0, 1/4)$. If $R(kh) - R^h(kh) \geqslant \eta$ for small $\eta \geqslant 0$, then there exist $K_{10} > 0$ and $t_8 > 0$ such that*

$$R(kh + \bar{t}) - R^h(kh + \bar{t}) \geqslant \left(\eta + \frac{\bar{t}^2}{(R(kh))^3} - K_{10}(\bar{t} + h)\bar{t}^2\right)\left(1 + \frac{\bar{t}}{(R(kh))^2} - K_{10}\bar{t}^2\right)$$

for all $\bar{t} \in [0, h]$ and $h \in (0, t_8)$.

Proof. Assume that $R(kh) - R^h(kh) \geqslant \eta$ for small $\eta > 0$. Let w_k be a solution of (6.6) - (6.7) - (6.8). Set $\xi(\bar{t}) := w_k(\bar{t}, R(kh + \bar{t}))$ for $\bar{t} \in [0, h]$. The argument is quite similar to that in the proof of Theorem 6.1.

Taylor's theorem yields that

$$\left|\xi(\bar{t}) - \left(\xi(0) + \xi'(0)\bar{t} + \frac{1}{2}\xi''(0)\bar{t}^2\right)\right| \leqslant K_{10,1}\bar{t}^3.$$

Here and hereafter $K_{10,j}$'s ($j \in \mathbb{N}$) are positive constants depending on ε. Then we observe by (6.6) and the regularity of w_k near $r = R(kh)$ that

$$\xi(0) = R^h(kh) - R(kh) \leqslant -\eta, \quad \xi'(0) = 0, \quad \xi''(0) = -\frac{1}{(R(kh))^3}.$$

Hence we have for all $\bar{t} \in [0, h]$ and small $h > 0$

$$w_k\big(\bar{t}, R(kh + \bar{t})\big) \leqslant -\eta - \frac{3\bar{t}^2}{2(R(kh))^3} + K_{10,1}\bar{t}^3. \tag{6.11}$$

On the other hand, we see by the mean value theorem that

$$\begin{aligned} w_k(\bar{t}, R^h(kh + \bar{t})) &= w_k\big(\bar{t}, R(kh + \bar{t})\big) + w_{k,r}\big(\bar{t}, R(kh + \bar{t}) + \widetilde{\theta}\big)\big(R(kh + \bar{t}) - R^h(kh + \bar{t})\big) \\ &= w_{k,r}\big(\bar{t}, R(kh + \bar{t}) + \widetilde{\theta}\big)\big(R^h(kh + \bar{t}) - R(kh + \bar{t})\big), \end{aligned}$$

where $\widetilde{\theta} := \theta(R^h(kh + \bar{t}) - R(kh + \bar{t}))(< 0)$ and $\theta \in (0, 1)$. Hence we obtain

$$R(kh + \bar{t}) - R^h(kh + \bar{t}) = \frac{w_k(\bar{t}, R(kh + \bar{t}))}{w_{k,r}(\bar{t}, R^h(kh + \bar{t}) + \widetilde{\theta})} \tag{6.12}$$

It follows from (6.9) that

$$\left|w_{k,r}\big(\bar{t}, R(kh + \bar{t}) + \widetilde{\theta}\big) - \Big(-1 + \frac{\bar{t}}{(R(kh) + \widetilde{\theta})^2}\Big)\right| \leqslant K_{10,2}\bar{t}^2.$$

Using $(R(kh) - L_1 h)^2 \leqslant (R(kh) + \widetilde{\theta})^2 \leqslant (R(kh))^2$, we have

$$\left|w_{k,r}\big(\bar{t}, R(kh + \bar{t}) + \widetilde{\theta}\big) - \Big(-1 + \frac{\bar{t}}{(R(kh))^2}\Big)\right| \leqslant K_{10,3}(\bar{t} + h)\bar{t}.$$

We obtain the desired result by applying (6.11), (6.12) and this estimate and taking $t_8 > 0$ sufficiently small. □

Proof of Theorem 6.6. In the case $k = 0$, as $R(0) = R^h(0) = 1$, we apply Lemma 6.7 with $\eta = 0$ to have

$$R(\bar{t}) - R^h(\bar{t}) \geqslant \Big(\frac{\bar{t}^2}{(R(0))^3} - K_{10}(\bar{t}+h)\bar{t}^2\Big)\Big(1 + \frac{\bar{t}}{(R(0))^2} - K_{10}\bar{t}^2\Big) \geqslant \frac{\bar{t}^2}{(R(0))^3} - K_{11}(\bar{t}+h)\bar{t}^2$$

for $\bar{t} \in [0, h]$ and some $K_{11} = K_{11}(\varepsilon) > 0$. In the case $k = 1$, we use Lemma 6.7 with $\eta = h^2/(R(0))^3 - K_{10,1}h^3$ to obtain

$$\begin{aligned} R(h+\bar{t}) - R^h(h+\bar{t}) &\geqslant \Big(\eta_0 + \frac{\bar{t}^2}{(R(h))^3} - K_{10}(\bar{t}+h)\bar{t}^2\Big)\Big(1 + \frac{\bar{t}}{(R(h))^3} - K_{10}\bar{t}^2\Big) \\ &\geqslant \Big\{\frac{h^2}{(R(0))^3} - K_{11}h^3\Big\} + \frac{\bar{t}^2}{(R(h))^3} - K_{11}(\bar{t}+h)\bar{t}^2 \end{aligned}$$

for all $\bar{t} \in [0, h]$. Hence we are able to prove by induction that

$$R(kh+\bar{t}) - R^h(kh+\bar{t}) \geqslant h^2 \sum_{l=0}^{k} \Big\{\frac{1}{(R(lh))^3} - K_{11}h\Big\} + \frac{\bar{t}^2}{(R(kh))^2} - K_{11}(\bar{t}+h)\bar{t}^2$$

for all $\bar{t} \in [0, h]$, $k = 0, 1, 2, \ldots, [T/h]$ and $h > 0$.

For any $\varepsilon \in (0, T)$, choosing a small $h_1 > 0$ we get

$$R(kh+\bar{t}) - R^h(kh+\bar{t}) \geqslant \frac{1}{2}\Big\{\sum_{l=0}^{k} \frac{h^2}{(R(lh))^3} + \frac{\bar{t}^2}{(R(kh))^2}\Big\} \geqslant \frac{kh^2 + \bar{t}^2}{2} \geqslant \frac{(kh+\bar{t})h}{4}$$

for all $\bar{t} \in [0, h]$, $k = 1, 2, \ldots, [T/h]$ and $h \in (0, h_1)$. Hence the proof is completed. □

7. Rate of convergence to a generalized CDM

In [36] Nochetto and Verdi gave a local estimate for the convergence of a bilateral obstacle problem to a regular portion of a generalized CDM. In this section we derive a similar estimate in the case of our algorithm.

Let $(\Gamma_L(t), \Omega_L^+(t), \Omega_L^-(t))_{t\in[0,T)}$ be a level-set flow by (1.1) satisfying (2.16). Then the convergence of $\{C^h(t)\}_{t\in[0,T),h>0}$ to $\{\mathrm{cl}\,\Omega_L^+(t)\}_{t\in[0,T)}$ follows from Theorem 5.6. Besides, we have the following local estimate.

Theorem 7.1 *Assume* (2.1). *Let* $\{C^h(t)\}_{t\in[0,T),h>0}$ *be defined by* (6.1) *and* $(\Gamma_L(t), \Omega_L^+(t), \Omega_L^-(t))_{t\in[0,T)}$ *by the level-set flow by* (1.1). *Assume that for* $t_0 \in (0, T)$ *and* $x_0 \in \Gamma_L(t_0)$, $Du(t_0, x_0) (\neq 0)$ *exists and* $Du \in C(\mathrm{cl}\, Q((t_0, x_0), r_0))$ *for some small* $r_0 > 0$. *Then there exist* L_2, $h_2 > 0$ *such that*

$$\begin{aligned} \sup_{t\in[t_0-r_0,t_0+r_0]} & d_H\big(C^h(t) \cap B(x_0, r_0), \mathrm{cl}\,\Omega_L^+(t) \cap B(x_0, r_0)\big) \\ & \leqslant L_2\Big\{d_H\big(C^h(k_h h) \cap B(x_0, r_0), \mathrm{cl}\,\Omega_L^+(k_h h) \cap B(x_0, r_0)\big) + h\Big\} \quad (7.1) \end{aligned}$$

for all $h \in (0, h_2)$. *Here* $k_h := [(t_0 - r_0)/h]$.

Before proving this theorem we apply the arguments in [13, Section 5] to obtain the local regularity of the level-set flow by (1.1). At first, by $Du \in C(Q((t_0, x_0), r_0))$ and $|Du(t_0, x_0)| \neq 0$ we may assume that

$$\frac{1}{2}|Du(t_0, x_0)| \leqslant |Du(t, x)| \leqslant 2|Du(t_0, x_0)| \quad \text{for all } (t, x) \in Q\big((t_0, x_0), r_0\big). \tag{7.2}$$

Then it follows from the implicit function theorem that for each $t \in (t_0 - r_0, t_0 + t_0)$ the set $\{u(t, \cdot) = 0\} \cap B(x_0, r_0)$ is a portion of a C^1 hypersurface and there is a function $v = v(t, x')$ ($(t, x') \in Q'((t_0, x_0), r_0)$) such that

$$\big\{u(t, \cdot) = 0\big\} \cap B(x_0, r_0) = \big\{x_N = v(t, x')\big\} \cap B(x_0, r_0), \quad D'v(t, x') \in C\Big(Q'\big((t_0, x_0), r_0\big)\Big).$$

In addition,

$$\|v\|^{(0)}_{\mathrm{cl}\, Q'((t_0,x_0),r_0)} + \|D'v\|^{(0)}_{\mathrm{cl}\, Q'((t_0,x_0),r_0)} < +\infty. \tag{7.3}$$

We choose an orthonormal basis $\{e_i\}_{i=1}^N$ for $\mathbb{R}^N$ such that $e_N := -Du(t_0, x_0)/|Du(t_0, x_0)|$.

Lemma 7.2 *Under the conditions of Theorem 7.1, v is a viscosity solution of*

$$v_t - \Delta' v + \frac{\langle D'^2 v D'v, D'v\rangle}{|D'v|^2 + 1} + \langle \mathbf{b}', D'v\rangle - b^{N\prime} + g'\sqrt{|D'v|^2 + 1} = 0 \quad \textit{in } Q'\big((t_0, x_0), r_0\big),$$

where $\mathbf{b}' = \mathbf{b}'(t, x') := (b^1(t, x', v(t, x')), \cdots, b^{N-1}(t, x', v(t, x')))$, $b^{N\prime} = b^{N\prime}(t, x') := b^N(t, x', v(t, x'))$ *and* $g' = g'(t, x') := g(t, x', v(t, x'))$.

Proof. We prove only the subsolution case because the supersolution case is similarly proved. The proof follows from [13, Theorem 5.1].

For any $\psi \in C^\infty(Q'((t_0, x_0), r_0))$, assume that $v - \psi$ takes its maximum at (t_1, x_1'). We may consider that

$$v(t_1, x_1') = \psi(t_1, x_1') = 0. \tag{7.4}$$

Hence $v \leqslant \psi$ in $Q'((t_0, x_0), r_0)$. Note that

$$\begin{aligned} &\{u < 0\} \cap Q\big((t_0, x_0), r_0\big) \supset \big\{(t, x) \in Q\big((t_0, x_0), r_0\big) \mid x_N > \psi(t, x')\big\}, \\ &\big\{(t, x) \in Q\big((t_0, x_0), r_0\big) \mid x_N < \psi(t, x')\big\} \subset \{u > 0\} \cap Q\big((t_0, x_0), r_0\big). \end{aligned} \tag{7.5}$$

Let $\phi(t, x) := \psi(t, x') - x_N$ and set

$$\begin{aligned} E_0 &:= \big\{(t, x') \in Q'\big((t_0, x_0), r_0\big) \mid 1 \leqslant x_N - \psi(t, x')\big\}, \\ E_k &:= \big\{(t, x') \in Q'\big((t_0, x_0), r_0\big) \mid 2^{-k} \leqslant x_N - \psi(t, x') \leqslant 2^{-k+1}\big\}, \\ F_k &:= \big\{(t, x') \in Q'\big((t_0, x_0), r_0\big) \mid -2^{-k+1} \leqslant x_N - \psi(t, x') \leqslant -2^{-k}\big\}, \\ F_0 &:= \big\{(t, x') \in Q'\big((t_0, x_0), r_0\big) \mid x_N - \psi(t, x') \leqslant -1\big\} \end{aligned}$$

for $k = 1, 2, \ldots$ Define

$$\alpha_k := \sup\Big\{u(t, x) \,\Big|\, (t, x) \in \bigcup_{0 \leqslant j \leqslant k} E_j \cap \{u < 0\}\Big\}, \beta_k := \sup\Big\{u(t, x) \,\Big|\, (t, x) \in \bigcup_{0 \leqslant j \leqslant k} F_j\Big\}.$$

Then we easily see by (7.4) that

$$\alpha_k < 0,\ \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_k \leqslant \cdots,\ \lim_{k\to+\infty} \alpha_k = 0,$$
$$\beta_k > 0,\ \beta_0 \geqslant \beta_1 \geqslant \cdots \geqslant \beta_k \geqslant \cdots,\ \lim_{k\to+\infty} \beta_k = 0.$$

We may assume that $\{\alpha_k\}_{k=0}^{+\infty}$ is strictly increasing and that $\{\beta_k\}_{k=0}^{+\infty}$ is strictly decreasing, by reindexing if necessary.

We define a nondecreasing function $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\begin{aligned}&\Phi(\alpha_k) := -2^{-k+1},\ \Phi(\beta_k) := 2^{-k},\ \Phi(0) := 0\\ &\Phi \text{ linear on } [\alpha_k, \alpha_{k+1}] \text{ and } [\beta_{k+1}, \beta_k],\\ &\Phi(r) := -2 \text{ for } r \leqslant \alpha_0,\ = 2 \text{ for } r \geqslant \beta_0\end{aligned}$$

Then if $(t,x) \in E_k \cap \{u < 0\}$, then $u(t,x) \leqslant \alpha_k$ and hence

$$\Phi\big(u(t,x)\big) \leqslant \Phi(\alpha_k) = -2^{-k+1} \leqslant \psi(t,x') - x_N = \phi(t,x).$$

Thus

$$\begin{aligned}\Phi\big(u(t,x)\big) \leqslant \phi(t,x) \quad \text{on} \bigcup_{k=0}^{+\infty} E_k \cap \{u<0\}\\ = \Big\{(t,x) \in Q\big((t_0,x_0),r_0\big) \mid x_N > \psi(t,x')\Big\} \cap \{u<0\}.\end{aligned}$$

Similarly, if $(t,x) \in F_k$, then $u(t,x) \leqslant \beta_k$ and hence

$$\Phi\big(u(t,x)\big) \leqslant \Phi(\beta_k) = 2^{-k} \leqslant \psi(t,x') - x_N = \phi(t,x).$$

Thus

$$\Phi\big(u(t,x)\big) \leqslant \phi(t,x) \quad \text{on} \bigcup_{k=0}^{+\infty} F_k = \Big\{(t,x) \in Q\big((t_0,x_0),r_0\big) \mid x_N < \psi(t,x')\Big\}.$$

In $\{u < 0\} \cap \{x_N < \psi(t,x')\}$ we get

$$\Phi\big(u(t,x)\big) \leqslant 0 \leqslant \psi(t,x') - x_N = \phi(t,x).$$

Therefore we observe from (7.4), (7.5) and this inequality that $\Phi(u) - \phi$ takes a maximum in $Q((t_0,x_0),r_0)$ at $(t_1,(x_1', v(t_1,x_1')))$.

Since $\Phi(u)$ is a viscosity subsolution of (2.14) by the relabeling property (cf. [15, Theorem 4.2.1]), we obtain

$$\phi_t - \Delta\phi + \frac{\langle D^2\phi D\phi, D\phi\rangle}{|D\phi|^2} + \langle \mathbf{b}, D\phi\rangle + g|D\phi| \leqslant 0 \quad \text{at } \Big(t_1, \big(x_1', v(t_1,x_1')\big)\Big).$$

It directly follows from the definition of ϕ that at $(t_1,(x_1', v(t_1,x_1')))$

$$\begin{aligned}&\phi_t = \psi_t,\ \phi_{x_i} = \psi_{x_i} \text{ for } 1 \leqslant i \leqslant N-1,\ \phi_{x_N} = -1,\\ &\phi_{x_i x_j} = \psi_{x_i x_j} \text{ for } 1 \leqslant i,j \leqslant N-1,\ = 0 \text{ otherwise.}\end{aligned}$$

Therefore we have the result. □

To consider the regularity of v, we treat the following initial-boundary value problem:

$$w_t - \Delta w + \frac{\langle D^2 w Dv, Dv\rangle}{|Dv|^2+1} + \langle \mathbf{b}, Dw\rangle - b^N + g\sqrt{|Dv|^2+1} = 0 \quad \text{in } Q((t_0,x_0),r_0), \tag{7.6}$$

$$w(t_0-r_0,x) = v(t_0-r_0,x) \quad \text{for } x \in B(x_0,r_0), \tag{7.7}$$

$$w(t,x) = v(t,x) \quad \text{for } \partial_p Q\big((t_0,x_0),r_0\big). \tag{7.8}$$

For the moment we drop the superscript $'$ for notational simplicity.

It follows from Lemma 7.2 that v is a viscosity solution of (7.6) satisfying (7.7) and (7.8). Note that the principal part of (7.6) is uniformly elliptic because of (7.3)

We show the existence of a solution $w \in W^{1,2,p}_{loc}(Q((t_0,x_0),r_0))$ of (7.6) - (7.7) - (7.8). To do so we approximate $\mathbf{b}$, b^N and g: For $f = \mathbf{b}$, b^N and g, choose a sequence $\{f_\varepsilon\}_{\varepsilon>0} \subset C^\infty(\text{cl}\, Q((t_0,x_0),r_0))$ satisfying

$$\begin{aligned} &(f_\varepsilon, Df_\varepsilon) \longrightarrow (f, Df) \quad \text{uniformly on cl}\, Q\big((t_0,x_0),r_0\big) \quad \text{as } \varepsilon \to 0,\\ &\big\{(f_\varepsilon, Df_\varepsilon)\big\}_{\varepsilon>0} \ : \ \text{equi-continuous on cl}\, Q\big((t_0,x_0),r_0\big). \end{aligned} \tag{7.9}$$

Then for each $\varepsilon > 0$ there is a unique classical solution of w_ε of (7.6) - (7.7) - (7.8) with $\mathbf{b} = \mathbf{b}_\varepsilon$, $b^N = b^N_\varepsilon$ and $g = g_\varepsilon$. We derive some uniform estimates for $\{w_\varepsilon\}_{\varepsilon>0}$. In the following part of this section we always assume (2.1) and (7.3).

First the L^∞-bound readily follows from the maximum principle:

$$\sup_{\varepsilon>0} \|w_\varepsilon\|^{(0)}_{\text{cl}\, Q((t_0,x_0),r_0)} < +\infty. \tag{7.10}$$

Since $\{v_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded on cl $Q((t_0,x_0),r_0)$ and satisfies (7.9) with $f_\varepsilon = v_\varepsilon$, we apply the techniques of [17, Section 14.5] and [31, Section 2] to have the following.

Proposition 7.3 *$\{w_\varepsilon\}_{\varepsilon>0}$ is equi-continuous on $\partial_p Q\big((t_0,x_0),r_0\big)$.*

Furthermore, applying (7.9) and [32, Theorem 7.13], we obtain a uniform $W^{1,2,p}_{loc}$-estimate.

Proposition 7.4 $\sup_{\varepsilon>0}\|w_\varepsilon\|_{W^{1,2,p}(Q)} < +\infty$ *for each compact subset Q of $Q\big((t_0,x_0),r_0\big)$ and $p > N+2$.*

Therefore we have the following result.

Theorem 7.5 *Assume (2.1) and (7.3). There is a solution $w \in W^{1,2,p}_{loc}(Q((t_0,x_0),r_0)) \cap C(\text{cl}\, Q((t_0,x_0),r_0))$ of (7.6) in the almost everywhere sense satisfying (7.7)–(7.8). In addition, $v = w$ on cl $Q((t_0,x_0),r_0)$.*

Proof. From Proposition 7.3 we can use the Sobolev embedding to obtain

$$\sup_{\varepsilon>0}\big(\langle w_\varepsilon\rangle_{\gamma,Q} + \langle Dw_\varepsilon\rangle_{\gamma,Q}\big) < +\infty$$

for each compact set $Q \subset Q((t_0,x_0),r_0)$ and some $\gamma \in (0,1)$. Thus we see by (7.10), Proposition 7.3 and this estimate that $\{w_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded and equi-continuous on cl $Q((t_0,x_0),r_0)$. Therefore we can extract a subsequence $\{\varepsilon_n\}_{n=0}^{+\infty}$, $\varepsilon_n \searrow 0$ such that as $n \to +\infty$

$$w_{\varepsilon_n} \longrightarrow w \quad \text{uniformly on cl } Q\big((t_0,x_0),r_0\big),$$
$$Dw_{\varepsilon_n} \longrightarrow Dw \quad \text{locally uniformly in } Q\big((t_0,x_0),r_0\big),$$
$$(w_{\varepsilon_n,t}, D^2 w_{\varepsilon_n}) \longrightarrow (w_t, D^2 w) \quad \text{weakly in } \Big\{L^p_{loc}\Big(Q\big((t_0,x_0),r_0\big)\Big)\Big\}^{N^2+1} \quad \text{for } p > N+2,$$
$$w \text{ satisfies (7.6) almost everywhere in } Q\big((t_0,x_0),r_0\big).$$

Also w satisfies (7.7)–(7.8). The equality $v = w$ follows from the uniqueness of solutions of (7.6)–(7.7)–(7.8) due to [33, Remark I.16] and [22, Theorem III. 1]. □

For each compact set $Q' \subset Q'((t_0,x_0),r_0)$, $\mathbf{b}'$, $b^{N\prime}$, $g' \in C^{\gamma/2,1+\gamma}(Q')$ since v, $D'v \in C^{\gamma/2,\gamma}(Q')$ and (2.1). Hence $v \in C^{(2+\gamma)/2,2+\gamma}(Q'((t_0,x_0),r_0))$. We use the regularity theory for parabolic equations (cf. [29], [14] etc.) to obtain $v \in C^{(5+\gamma)/2,5+\gamma}(Q((t_0,x_0),r_0))$. Therefore the signed distance function d to $\Gamma_L(t) = \{u(t,\cdot) = 0\}$ satisfies

$$d \in C^{(5+\gamma)/2,5+\gamma}(N), \ N := \Big\{(t,x) \in Q\big((t_0,x_0),r_0\big) \mid |d(t,x)| \leqslant \delta_0\Big\} \tag{7.11}$$

for some $\delta_0 > 0$.

Proof of Theorem 7.1. Fix any $\delta \in (0,\delta_0)$. Applying Theorem 5.6, we get

$$\sup_{t\in[0,t_0-r_0/2]} d_H\big(C^h(t), \mathrm{cl}\,\Omega_L^+(t)\big) \leqslant \delta \quad \text{for all } h \in (0,h_0) \text{ and some } h_0 > 0.$$

Set $k_h := [(t_0 - r_0)/h]$ and $\alpha_h := d_H(C^h(k_h h) \cap B(x_0,r_0), \mathrm{cl}\,\Omega_L^+(k_h h) \cap B(x_0,r_0))$ for all $h \in (0,h_0)$. Let $C^\pm(k_h h)$ be compact sets in $\mathbb{R}^N$ satisfying

$$\begin{aligned}
&\big\{d(k_h h,\cdot) \geqslant \alpha_h\big\} \subset C^+(k_h h) \subset \big\{d(k_h h,\cdot) \geqslant 2\delta\big\},\\
&\big\{d(k_h h,\cdot) \leqslant -2\delta\big\} \subset C^-(k_h h) \subset \big\{d(k_h h,\cdot) \leqslant -\alpha_h\big\},\\
&C^+(k_h h) \cap B(x_0,r_0) = \big\{d(k_h h,\cdot) \geqslant \alpha_h\big\} \cap B(x_0,r_0),\\
&C^+(k_h h)\backslash B(x_0,2r_0) = \big\{d(k_h h,\cdot) \geqslant 2\delta\big\}\backslash B(x_0,r_0),\\
&C^-(k_h h) \cap B(x_0,r_0) = \big\{d(k_h h,\cdot) \leqslant -\alpha_h\big\} \cap B(x_0,r_0),\\
&C^-(k_h h)\backslash B(x_0,2r_0) = \big\{d(k_h h,\cdot) \leqslant -2\delta\big\}\backslash B(x_0,r_0).
\end{aligned}$$

Let $d^\pm(0,\cdot)$ be the signed distance function to $\partial C^\pm(k_h h)$ and $w^\pm$ be a solution of (1.2) - (1.3) with $C^\pm(k_h h)$ replacing C_k. Set $C^\pm(k_h h + \bar{t}) := \{w^\pm(\bar{t},\cdot) \geqslant 0\}$ for $\bar{t} \in [0,h]$. The same arguments as those in subsection 3.2 yields that

$$\|w^\pm\|^{(5+\gamma)}_{[0,h]\times \mathrm{cl}\, B(x_0,r_0)} \leqslant K_{12}, \ |Dw^\pm| \geqslant 1 - K_{12}\bar{t} \quad \text{on } [0,h] \times B(x_0,r_0) \tag{7.12}$$

for some $K_{12} > 0$. Hence applying (7.11), (7.12) and the proof of Lemma 6.2, we get

$$\begin{aligned}
d_H\big(C^h&(k_h h + \bar{t}) \cap B(x_0,r_0), C(k_h h + \bar{t}) \cap B(x_0,r_0)\big)\\
&\leqslant d_H\big(C^\pm(k_h h + \bar{t}) \cap B(x_0,r_0), C(k_h h + \bar{t}) \cap B(x_0,r_0)\big)\\
&\leqslant \frac{\alpha_h}{1 - K_{13}h} + \frac{K_{13}h^2}{2(1-K_{13}h)} \quad \text{for all } \bar{t} \in [0,h] \text{ and some } K_{13} > 0.
\end{aligned}$$

The induction, similar to the proof of Theorem 6.1, gives

$$\begin{aligned} d_H\Big(C^h\big((k_h+l)h+\bar{t}\big)\cap B(x_0,r_0), C\big((k_h+l)h+\bar{t}\big)\cap B(x_0,r_0)\Big) \\ \leqslant d_H\Big(C^{\pm}\big((k_h+l)h+\bar{t}\big)\cap B(x_0,r_0), C\big((k_h+l)h+\bar{t}\big)\cap B(x_0,r_0)\Big) \\ \leqslant \frac{\alpha_h}{(1-K_{13}h)^{l+1}}+\sum_{m=1}^{l+1}\frac{K_{13}h^2}{2(1-K_{13}h)^m} \end{aligned}$$

for all $\bar{t}\in[0,h]$, $l=0,1,\dots,[2r_0/h]$ and $h\in(0,t_9)$. Here $K_{13}, t_9>0$ are constants. Hence we get

$$d_H\big(C^h(t)\cap B(x_0,r_0), C(t)\cap B(x_0,r_0)\big)\leqslant \frac{e^{2K_{14}r_0}\alpha_h}{1-K_{13}h}+\frac{(e^{K_{13}r_0}-1)h}{2}$$

for all $t\in[t_0-r_0,t_0+r_0]$, $h\in(0,t_9)$ and some $K_{14}>0$. Therefore we have (7.1) by choosing L_2 sufficiently large and $h_2=t_9$. □

If $\Gamma_L(t)$ smoothly evolves by (1.1) in $[0,t_0)$, then the following estimate holds.

Corollary 7.6 *Assume the same conditions in Theorem 7.1. If $\Gamma_L(t)$ smoothly evolves by (1.1) in $[0,t_0)$ such that (2.5) is satisfied. Then there exist $L_3>0$ and $h_3>0$ such that*

$$\sup_{t\in[t_0-r_0,t_0+r_0]} d_H\big(C^h(t)\cap B(x_0,r_0), \operatorname{cl}\Omega^+(t)\cap B(x_0,r_0)\big)\leqslant L_3h \quad \textit{for all } h\in(0,h_3).$$

This corollary is a consequence from Theorem 6.1 and 7.1, so we omit the detail.

8. Appendix

In this section we establish the existence and uniqueness for a smooth and compact CDM locally in time.

Theorem 8.1 *Assume (2.1). Let Γ_0 be a compact hypersurface satisfying (2.3). Then for some $T_0>0$, there uniquely exists a smooth and compact CDM $\{\Gamma(t)\}_{t\in[0,T_0)}$ with $\Gamma(0)=\Gamma_0$ satisfying (2.5).*

We give only the outline of the proof of this theorem because it is similar to that in [12].

8.1 *An initial-boundary value problem for (2.12)*

Suppose that $\Gamma(0)\subset\mathbb{R}^N$ is a given compact hypersurface, which is the boundary of a compact set $D(0)$ and that there is a smooth and compact CDM $\{\Gamma(t)\}_{t\in[0,T)}$ starting from $\Gamma(0)$. Let $\rho=\rho(t,x)$ be the signed distance function to $\Gamma(t)$ defined by (2.4) with $D=D(t)$. Here $D(t)$ is the compact set enclosed by $\Gamma(t)$. As seen in Section 2.2, there exists a $\delta_1>0$ for which ρ satisfies (2.12) and $|D\rho|^2=1$ on N_{ε,δ_1}. Therefore, to obtain a smooth and compact CDM $\{\Gamma(t)\}_{t\in[0,T_1]}$, we solve the following initial-boundary value problem: Let ρ_0 be the signed distance function to $\Gamma(0)$ defined by (2.4) with $t=0$. Fix $\delta_2>0$ so that ρ_0 is smooth in $V:=\{|\rho(0,\cdot)|<\delta_2\}$. Then we construct a classical solution of

$$\begin{cases} v_t-F_0(v,D^2v)+\langle\widetilde{\mathbf{b}}_v,Dv\rangle+\widetilde{g}_v=0 & \text{in } Q:=(0,T_1)\times V,\\ |Dv|^2=1 & \text{on } \partial_xQ:=[0.T_1)\times\partial V,\\ v(0,x)=\rho_0(x) & \text{for } x\in\operatorname{cl}V. \end{cases} \tag{8.1}$$

8.2 *Linearized problem for* (8.1)

To construct a classical solution of (8.1) for small $\delta_2, T_1 > 0$, we linearize (8.1). In the following part of this section we use the usual summation convention on repeated indices.

First, we take $\delta_3 > 0$ so small that

$$M_4\delta_3 \leqslant \frac{1}{4},\ M_4 := \sup_{x\in V} \|D^2\rho_0\|. \tag{8.2}$$

Set $G := \{(r, X) \in \mathbb{R} \times \mathbb{S} \mid |r| < \delta_3,\ \|X\| < 2M_4\}$. Then

$$|\lambda_i(X)r| \leqslant \|X\||r| \leqslant \frac{1}{2} \quad \text{for } i = 1, 2, \ldots, N.$$

Hence the function F_0 by (2.13) is smooth on G. We extend F_0 on $\mathbb{R} \times \mathbb{S}^N$ to be smooth, with $|F_0|$, $|DF_0|$, $|D^2F_0|$ bounded. By [12, Lemma 2.1] F_0 satisfies

$$\frac{\partial F_0}{\partial X_{ij}}(r, X)\xi_i\xi_j \geqslant \theta|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N \tag{8.3}$$

for each $(r, X) \in G$ and some $\theta > 0$.

We look for the solution v of (8.1) in the form

$$v = \rho_0 + th + w,\ h := F_0(\rho_0, D^2\rho_0) - \langle \widetilde{\mathbf{b}}_{\rho_0}(0, x), D\rho_0\rangle - \widetilde{g}_{\rho_0}(0, x). \tag{8.4}$$

We substitute (8.4) into (8.1) to derive the equation for w. Then using Taylor's theorem, we compute that

$$\begin{aligned}
&F_0(\rho_0 + th + r, D^2\rho + rD^2h + X)\\
&\quad = F_0(\rho_0, D^2\rho_0) + \frac{\partial F_0}{\partial r}(\rho_0, D^2\rho_0)r + \frac{\partial F_0}{\partial X_{ij}}(\rho_0, D^2\rho_0)X_{ij} + A_1(t, x, r, X),\\
&A_1(t, x, r, X) := t\left\{\frac{\partial F_0}{\partial r}(\rho_0, D^2\rho_0)h + \frac{\partial F_0}{\partial X_{ij}}(\rho_0, D^2\rho_0)X_{ij}\right\}\\
&\quad + \int_0^1 (1-s)\frac{\partial^2 F_0}{\partial r^2}(\rho_0 + sth + sr, D^2\rho_0 + stD^2h + sX)ds(th + r)^2\\
&\quad +2\int_0^1 (1-s)\frac{\partial^2 F_0}{\partial r\partial X_{ij}}(\rho_0 + sth + sr, D^2\rho_0 + stD^2h + sX)ds\\
&\qquad\qquad \cdot(th + r)(th_{x_ix_j} + X_{ij})\\
&\quad + \int_0^1 (1-s)\frac{\partial^2 F_0}{\partial X_{ij}\partial X_{kl}}(\rho_0 + sth + sr, D^2\rho_0 + stD^2h + sX)ds\\
&\qquad\qquad \cdot(th_{x_ix_j} + X_{ij})(th_{x_kx_l} + X_{kl}).
\end{aligned} \tag{8.5}$$

Set

$$\begin{aligned}
&\xi := (\rho_0 + th + r)(D\rho_0 + tDh + p) - \rho_0 D\rho_0 = \rho_0 p + r\xi_1 + t\xi_2 + t^2\xi_3,\\
&\xi_1 := D\rho_0 + p,\ \xi_2 := h(D\rho_0 + p) + (\rho_0 + r)Dh,\ \xi_3 := hDh.
\end{aligned}$$

Taylor's expansion around $x - \rho_0 D\rho_0$ yields that

$$
\begin{aligned}
&\big\langle \widetilde{\mathbf{b}}_{\rho_0}(t, x-\xi), D\rho_0 + tDh + p\big\rangle = \big\langle \widetilde{\mathbf{b}}_{\rho_0}(0,x), D\rho_0\big\rangle + \big\langle \widetilde{\mathbf{b}}_{\rho_0}(0,x), p\big\rangle \\
&\qquad -\rho_0\big\langle \widetilde{(D\mathbf{b})}_{\rho_0}(0,x)p, D\rho_0\big\rangle - \big\langle \widetilde{(D\mathbf{b})}_{\rho_0}(0,x)D\rho_0, D\rho_0\big\rangle r + A_2(t,x,r,p),
\end{aligned}
$$

$$
\begin{aligned}
A_2(t,x,r,p) &:= -\rho_0\big\langle \widetilde{(D\mathbf{b})}_{\rho_0}(0,x)p, p\big\rangle - r\big\{\big\langle \widetilde{(D\mathbf{b})}_{\rho_0}(0,x)D\rho_0, p\big\rangle \\
&\quad + \big\langle \widetilde{(D\mathbf{b})}_{\rho_0}(0,x)p, D\rho_0\big\rangle\big\} - r\big\langle \widetilde{(D\mathbf{b})}_{\rho_0}(0,x)p, p\big\rangle \\
&\quad + t\Big\{\big\langle \widetilde{\mathbf{b}}_{\rho_0}(0,x), Dh\big\rangle + \big\langle \widetilde{(\mathbf{b}_t)}_{\rho_0}(0,x), \xi_1\big\rangle - \big\langle \widetilde{(D\mathbf{b})}_{\rho_0}(0,x)(r\xi_1 + \rho_0 p), Dh\big\rangle \\
&\qquad - \big\langle \widetilde{(D\mathbf{b})}_{\rho_0}(0,x)\xi_2, \xi_1\big\rangle \\
&\qquad - 2\int_0^1 (1-s)\big\langle \widetilde{(D\mathbf{b}_t)}_{\rho_0}(st, x-s\xi)\xi, D\rho_0 + tDh + p\big\rangle ds\Big\} \\
&\quad + t^2\Big\{\big\langle \widetilde{(\mathbf{b}_t)}_{\rho_0}(0,x), Dh\big\rangle + \int_0^1 (1-s)\big\langle \widetilde{(\mathbf{b}_{tt})}_{\rho_0}(st, x-s\xi)), D\rho_0 + tDh + p\big\rangle ds \\
&\qquad - \big\langle \widetilde{(D\mathbf{b})}_{\rho_0}(0,x)\xi_2, Dh\big\rangle - \big\langle \widetilde{(D\mathbf{b})}_{\rho_0}(0,x)\xi_3, \xi_1\big\rangle\Big\} \\
&\quad + \Big\langle \int_0^1 (1-s)\widetilde{(D^2\mathbf{b})}_{\rho_0}(st, x-s\xi)(\xi,\xi), D\rho_0 + tDh + p\Big\rangle \\
&\quad - t^3\big\langle \widetilde{(D\mathbf{b})}_{\rho_0}(0,x)\xi_3, Dh\big\rangle,
\end{aligned} \tag{8.6}
$$

$$
\widetilde{g}_{\rho_0}(t, x-\xi) = \widetilde{g}_{\rho_0}(0,x) - \rho_0\big\langle \widetilde{(Dg)}_{\rho_0}(0,x), p\big\rangle - \big\langle \widetilde{(Dg)}_{\rho_0}(0,x), D\rho_0\big\rangle r + A_3(t,x,r,p),
$$

$$
\begin{aligned}
A_3(t,x,r,p) &:= -\big\langle \widetilde{(Dg)}_{\rho_0}(0,x), p\big\rangle r + t\Big\{\widetilde{(g_t)}_{\rho_0}(0,x) - \big\langle \widetilde{(Dg)}_{\rho_0}(0,x), \xi_2\big\rangle\Big\} \\
&\quad - t^2\Big\{\langle \widetilde{(Dg)}_{\rho_0}(0,x), \xi_2\rangle + \int_0^1 (1-s)\widetilde{(g_{tt})}_{\rho_0}(st, x-s\xi)ds\Big\} \\
&\quad - 2t\int_0^1 (1-s)\big\langle \widetilde{(Dg_t)}_{\rho_0}(st, x-s\xi), \xi\big\rangle ds + \int_0^1 \big\langle \widetilde{(D^2 g)}_{\rho_0}(st, x-s\xi)\xi, \xi\big\rangle ds,
\end{aligned} \tag{8.7}
$$

where

$$
\widetilde{(D^2\mathbf{b})}_{\rho_0}(t,y)(q,q) := \Big(\big\langle \widetilde{(D^2 b^i)}_{\rho_0}(t,y)q, q\big\rangle\Big)_{1\leqslant i\leqslant N}.
$$

Thus setting

$$
\begin{aligned}
a_{ij} &:= \frac{\partial F_0}{\partial X_{ij}}(\rho_0, D^2\rho_0),\ \mathbf{B} := \widetilde{\mathbf{b}}_{\rho_0}(0,\cdot) - \rho_0\big({}^T\widetilde{(D\mathbf{b})}_{\rho_0}(0,\cdot)D\rho_0 + Dg\big), \\
c &:= \frac{\partial F_0}{\partial r}(\rho_0, D^2\rho_0) - \big\langle \widetilde{(D\mathbf{b})}_{\rho_0}(0,\cdot)D\rho_0, D\rho_0\big\rangle - \big\langle \widetilde{(Dg)}_{\rho_0}(0,\cdot), D\rho_0\big\rangle, \\
A &:= A_1 + A_2 + A_3,
\end{aligned} \tag{8.8}
$$

we have the equation for w:

$$
w_t - a_{ij}w_{x_i x_j} + \langle \mathbf{B}, Dw\rangle + cw = A(t,x,w,Dw,D^2w) \quad \text{in } Q.
$$

We next derive the boundary condition for w. Let $\nu = \nu(x)$ be the outer unit normal of ∂V. Recalling that

$$|D\rho_0 + tDh + Dw|^2 = |Dv|^2 = 1 \quad \text{on } \partial_x Q, \ D\rho_0 = \begin{cases} \nu & \text{on } \{\rho_0 = \delta_2\}, \\ -\nu & \text{on } \{\rho_0 = -\delta_2\}, \end{cases}$$

we have the boundary condition for w:

$$\frac{\partial w}{\partial \nu} = a(t, x, Dw) \quad \text{on } \partial_x Q,$$
$$a(t, x, p) := \begin{cases} -\frac{1}{2}|tDh + p|^2 - t\frac{\partial h}{\partial \nu} & \text{on } \{\rho_0 = \delta_2\}, \\ \frac{1}{2}|tDh + p|^2 - t\frac{\partial h}{\partial \nu} & \text{on } \{\rho_0 = -\delta_2\}. \end{cases} \tag{8.9}$$

Therefore what we solve is the following initial-boundary value problem:

$$\begin{cases} w_t - a_{ij} w_{x_i x_j} + \langle \mathbf{B}, Dw \rangle + cw = A(t, x, w, Dw, D^2 w) & \text{in } Q, \\ \frac{\partial w}{\partial \nu} = a(t, x, Dw) & \text{on } \partial_x Q, \\ w(0, x) = 0 & \text{for } x \in \text{cl } V. \end{cases} \tag{8.10}$$

8.3 *Solvability of* (8.10)

Similar to [12], we solve (8.10) by the fixed point theorem of the mapping T defined by inserting a given function into the nonlinear terms A and a and solving the resulting linear equation. In addition to the notations at the end of section 1, we define

$$\|f\|^{(1+\alpha)}_{\partial_x Q} := \inf\left\{ \|\overline{f}\|^{(1+\alpha)}_{\text{cl } Q} \;\middle|\; \overline{f} \in C^{(1+\alpha)/2, 1+\alpha}(\text{cl } Q), \ \overline{f} = f \text{ on } \partial_x Q \right\}.$$

We consider the linear and uniformly parabolic equation:

$$\begin{cases} w_t - a_{ij} w_{x_i x_j} + \langle \mathbf{B}, Dw \rangle + cw = B & \text{in } Q, \\ \frac{\partial w}{\partial \nu} = b & \text{on } \partial_x Q, \\ w(0, x) = 0 & \text{for } x \in \text{cl } V. \end{cases} \tag{8.11}$$

Here $B \in C^{\alpha/2, \alpha}(\text{cl } Q)$ and $b \in C^{(1+\alpha)/2, 1+\alpha}(\partial_x Q)$. Suppose that $b = 0$ on $\{0\} \times \partial V$ for the compatibility condition. Then we are able to verify that the result of [12, Lemma 2.2] holds for (8.11). That is, there is a unique classical solution $w \in C^{(2+\alpha)/2, 2+\alpha}(\text{cl } Q)$ of (8.11) and it satisfies the estimate

$$\|w\|^{(2+\alpha)}_{\text{cl } Q} \leqslant C\left(\|B\|^{(\alpha)}_{\text{cl } Q} + \|b\|^{(1+\alpha)}_{\partial_x Q}\right). \tag{8.12}$$

Here and hereafter C denotes various constants depending only on known ones.

We introduce a Banach space:

$$X := \left\{ w \in C^{(2+\alpha)/2, 2+\alpha}(\text{cl } Q) \mid w(0, \cdot) = 0 \text{ on cl } V \right\}.$$

For a given $\widehat{w} \in X$, we set

$$B = B(t,x) := B_1(t,x) + B_2(t,x) + B_3(t,x),\ b(t,x) := a\big(t,x,D\widehat{w}(t,x)\big) \tag{8.13}$$
$$B_1 = B_1(t,x) := A\big(t,x,\widehat{w}(t,x), D^2\widehat{w}(t,x)\big),$$
$$B_i = B_i(t,x) := A_i\big(t,x,\widehat{w}(t,x), D\widehat{w}(t,x)\big)\ (i = 2,3)$$

for A, a defined by (8.8), (8.9). Then we define $T(\widehat{w}) = w$ as a solution of (8.11) with B, b as above. We look for a fixed point of the mapping $T : X \longrightarrow X$. Given $r_0 > 0$, we set

$$Y := \big\{w \in X \mid \|w\|^{(2+\alpha)}_{\mathrm{cl}\,Q)} \leqslant r_0\big\}.$$

Lemma 8.2 *For sufficiently small r_0, $T_1 > 0$, T maps from Y into Y.*

Outline of the proof. Fix any $\widehat{w} \in Y$. Define B and b by (8.13) and let w be a classical solution of (8.11). We prove that the $\|\widehat{w}\|^{(2+\alpha)}_{\mathrm{cl}\,Q} \leqslant r_0$ implies $\|w\|^{(2+\alpha)}_{\mathrm{cl}\,Q} \leqslant r_0$ for sufficiently small r_0, $T_1 > 0$.

As for A_1 given by (8.5), we see by [12, Lemma 2.3] that $\|B_1\|^{(\alpha)}_{\mathrm{cl}\,Q} \leqslant C(r_0^2 + T_1^{1-\alpha/2})$. Lengthy calculations with using (8.2) yield that $\|B_2\|^{(\alpha)}_{\mathrm{cl}\,Q} + \|B_3\|^{(\alpha)}_{\mathrm{cl}\,Q} \leqslant C(r_0^2 + T_1^{1-\alpha/2})$. Hence we get

$$\|B\|^{(\alpha)}_{\mathrm{cl}\,Q} \leqslant C\big(r_0^2 + T_1^{1-\alpha/2}\big).$$

We have the estimate for b in the proof of [12, Lemma 2.3]:

$$\|b\|^{(1+\alpha)}_{\partial_x Q} \leqslant C\big(r_0^2 + T_1^{(1-\alpha)/2}\big).$$

Combining these estimates and (8.12) , we obtain

$$\|w\|^{(2+\alpha)}_{\mathrm{cl}\,Q} \leqslant C\big(r_0^2 + T_1^{(1-\alpha)/2}\big).$$

Thus taking a sufficiently small $r_0 > 0$ and then $T_1 > 0$, we get the desired result. □

Lemma 8.3 *For sufficiently small r_0, $T_1 > 0$, we have*

$$\|T(\widehat{w}_1) - T(\widehat{w}_2)\|^{(2+\alpha)}_{\mathrm{cl}\,Q} \leqslant \frac{1}{2}\|\widehat{w}_1 - \widehat{w}_2\|^{(2+\alpha)}_{\mathrm{cl}\,Q} \quad \textit{for all } \widehat{w}_1, \widehat{w}_2 \in Y.$$

Outline of the proof. Fix any $\widehat{w}_1, \widehat{w}_2 \in Y$. For $i = 1, 2$ set

$$w_i := T(\widehat{w}_i),\ B_i(t,x) := A(t,x,\widehat{w}_i, D\widehat{w}_i, D^2\widehat{w}_i),\ b_i(t,x) := a(t,x,D\widehat{w}_i).$$

Then it readily follows from (8.12) that

$$\|w_1 - w_2\|^{(2+\alpha)}_{\mathrm{cl}\,Q} \leqslant C\big(\|B_1 - B_2\|^{(\alpha)}_{\mathrm{cl}\,Q} + \|b_1 - b_2\|^{(\alpha)}_{\partial_x Q}\big)$$

First we estimate $\|B_1 - B_2\|^{(\alpha)}_{\mathrm{cl}\,Q}$. Note that $|D^2F_0|$ is bounded and A_1, A_2 and A_3 are polynomials of degree not less than two with respect to (t, r, p, X) (cf. (8.5), (8.6) and (8.7)). We then observe by tedious calculations that

$$\|B_1 - B_2\|^{(\alpha)}_{\mathrm{cl}\,Q} \leqslant C\big(r_0 + T_1^{1-\alpha/2}\big)\|\widehat{w}_1 - \widehat{w}_2\|^{(2+\alpha)}_{\mathrm{cl}\,Q}.$$

We have the following estimate from the proof of [12, Lemma 2.4]:

$$\|b_1 - b_2\|^{(1+\alpha)}_{\partial_x Q} \leqslant C\big(r_0 + T_1^{(1-\alpha)/2}\big)\|\widehat{w}_1 - \widehat{w}_2\|^{(2+\alpha)}_{\mathrm{cl}\,Q}.$$

Thus we have from the above estimates

$$\|w_1 - w_2\|^{(2+\alpha)}_{\mathrm{cl}\,Q} \leqslant C\big(r_0 + T_1^{(1-\alpha)/2}\big)\|\widehat{w}_1 - \widehat{w}_2\|^{(2+\alpha)}_{\mathrm{cl}\,Q}.$$

Choosing $r_0, T_1 > 0$ sufficiently small, we obtain the desired result. □

We now establish the existence and uniqueness of solutions of (8.11) by applying Banach's fixed point theorem. Moreover, applying the regularity theory for linear parabolic equations, we obtain the following result.

Theorem 8.4 *For r_0, $T_1 > 0$ sufficiently small, there is a unique classical solution $w \in C^{(2+\alpha)/2,2+\alpha}(\mathrm{cl}\,Q) \cap C^{(3+\alpha)/2,3+\alpha}(Q)$ of (8.10).*

Hence we have a classical solution $v(= \rho_0 + th + w) \in C^{(2+\alpha)/2,2+\alpha}(\mathrm{cl}\,Q) \cap C^{(3+\alpha)/2,3+\alpha}(Q)$ of (8.1) and $(v, D^2v) \in G$. The regularity results (cf. [32]) yields the following theorem.

Theorem 8.5 *For r_0, $T_1 > 0$ sufficiently small, there is a unique classical solution $v \in C^{(5+\alpha)/2,5+\alpha}(\mathrm{cl}\,Q)$ of (8.1) satisfying $(v, D^2v) \in G$.*

8.4 *Motion of the zero level set of* (8.1)

In this section we show that the level set $\Gamma(t) := \{v(t,\cdot) = 0\}$ for $t \in [0, T_1]$ moves by (1.1).

Theorem 8.6 *We obtain $|Dv|^2 = 1$ on $\mathrm{cl}\,Q$, where $Q = (0, T_1) \times \{|\rho(0,\cdot)| < \delta_2\}$.*

Proof. Set $w := |Dv|^2 - 1 \in C^{(4+\alpha)/2,4+\alpha}(Q)$. Then

$$w = 0 \quad \text{on } \partial_x Q,\ w(0,\cdot) = 0 \quad \text{on cl}\, V. \tag{8.14}$$

We derive the equation for w. Differentiate (8.1) with respect to x_k and multiply v_{x_k} ($k = 1, 2, \ldots, N$). Then we get

$$v_{x_k}v_{x_k,t} - \frac{\partial F_0}{\partial X_{ij}}(v, D^2v)v_{x_ix_jx_k}v_{x_k} - \frac{\partial F_0}{\partial r}(v, D^2v)v^2_{x_k} - (\widetilde{b}^i_v)_{x_j}(v_{x_k}v_{x_j} + vv_{x_jx_k})v_{x_i}v_{x_k}$$
$$+ (\widetilde{b}^i_v)_{x_k}v^2_{x_k} - (\widetilde{g}_v)_{x_i}(v_{x_k}v_{x_j} + vv_{x_jx_k})v_{x_k} + (\widetilde{g}_v)_{x_k}v_{x_k} = 0.$$

The same computations as in the proof of [12, Theorem 3.1], we have

$$-\frac{\partial F_0}{\partial X_{ij}}(v, D^2v)v_{x_ix_jx_k}v_{x_k} - \frac{\partial F_0}{\partial r}(v, D^2v)v^2_{x_k} = -\frac{1}{2}\frac{\partial F_0}{\partial X_{ij}}(v, D^2v)w_{x_ix_j} - \frac{\partial F_0}{\partial r}(v, D^2v)w.$$

Besides we observe that

$$-(\widetilde{b^i}_v)_{x_j}(v_{x_k}v_{x_j} + vv_{x_jx_k})v_{x_i}v_{x_k} + (\widetilde{b^i}_v)_{x_k}v^2_{x_k}$$
$$= -\langle\widetilde{(D\mathbf{b})_v}Dv, Dv\rangle w - \frac{1}{2}v\langle\widetilde{(D\mathbf{b})_v}Dv, Dw\rangle,$$

$$-(\widetilde{g}_v)_{x_i}(v_{x_k}v_{x_j} + vv_{x_jx_k})v_{x_k} + (\widetilde{g}_v)_{x_k}v_{x_k} = -\langle\widetilde{(Dg)_v}, Dv\rangle w - \frac{1}{2}v\langle\widetilde{(Dg)_v}, Dw\rangle.$$

Hence we obtain

$$w_t - \frac{\partial F_0}{\partial X_{ij}}(v, D^2 v) w_{x_i x_j} - v\langle \widetilde{(D\mathbf{b})}_v Dv + \widetilde{(Dg)}_v, Dw\rangle \\ - 2\left\{\frac{\partial F_0}{\partial r}(v, D^2 v) + \langle \widetilde{(D\mathbf{b})}_v Dv + \widetilde{(Dg)}_v, Dv\rangle\right\} w = 0 \quad \text{in } Q. \tag{8.15}$$

Combining (8.3), (8.14) and this equation, we obtain $w = 0$ on cl Q by the maximum principle. □

Proof of Theorem 8.1. The vector $\mathbf{n} = \mathbf{n}(t, x) := Dv(t, x)$ is the inner unit normal to $\Gamma(t)$ for each $t \in [0, T_1]$ and $x \in \Gamma(t)$ by Theorem 8.6. Since $\kappa_i = \lambda_i$ on $\Gamma(t)$, we get $F_0(v, D^2 v) = \sum_{i=1}^{N-1} \kappa_i = \kappa$ on $\Gamma(t)$. Hence v satisfies

$$v_t - \widetilde{\kappa}_v + \langle \widetilde{\mathbf{b}}_v, \mathbf{n})\rangle + \widetilde{g}_v = v_t - \kappa + \langle \mathbf{b}, \mathbf{n}\rangle + g = 0 \quad \text{on } \Gamma(t).\ t \in (0, T_1].$$

Fix $t \in [0, T_1)$ and $x_0 \in \Gamma(t)$. Let $x(s) : (t, T_1) \longrightarrow \mathbb{R}^N$ be a solution of

$$\begin{cases} \dot{x}(s) = \{-\widetilde{\kappa}_v(s, x(s)) + \langle \widetilde{\mathbf{b}}_v(s, x(s)), \mathbf{n}(s, x(s))\rangle + \widetilde{g}_v(s, x(s))\}\mathbf{n}(s, x(s)) & \text{for } s > t, \\ x(t) = x_0 \end{cases}$$

Then we observe that

$$\begin{aligned} \frac{d}{ds} v(s, x(s)) &= v_t(s, x(s)) + \langle Dv(s, x(s)), \dot{x}(s)\rangle \\ &= v_t(s, x(s)) - \widetilde{\kappa}_v(s, x(s)) + \langle \widetilde{\mathbf{b}}_v(s, x(s)), \mathbf{n}(s, x(s))\rangle + \widetilde{g}_v(s, x(s)) \\ &= 0. \end{aligned}$$

Thus $v(s, x(s)) = 0$ for $s > t$. This implies that $\widetilde{f}_v = f$ for $f = \kappa, \mathbf{b}, f$. Therefore $\{\Gamma(t)\}_{t\in[0,T_1]}$ is a smooth and compact CDM. Hence we take $T_0 > 0$ as the maximal existence time of $\Gamma(t)$. The uniqueness of smooth and compact CDM's follows from [15, Theorem 4.2.8]. □

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