Interfaces and Free Boundaries **18** (2016), 545–578
DOI 10.4171/IFB/373

Reduced models for linearly elastic thin films allowing for fracture, debonding or delamination

JEAN-FRANÇOIS BABADJIAN
Sorbonne Universités, UPMC Univ Paris 06, CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, 75005 Paris, France
E-mail: jean-francois.babadjian@upmc.fr

DUVAN HENAO
Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Macul, Santiago, Chile
E-mail: dhenao@mat.puc.cl

[Received 21 September 2015 and in revised form 17 June 2016]

In this work, we study the emergence of different crack modes in linearly elastic thin films by means of a Γ-convergence analysis as the thickness tends to zero. We first consider a purely elastic body made of a film deposited on an infinitely stiff substrate through a bonding layer. The displacement mismatch between the film and the substrate generates a cohesive type energy depending on the displacement jump. Then, we consider a single linearly elastic brittle thin film. We show that the limit admissible displacements are of Kirchhoff–Love type outside the cracks, which are themselves transverse. Finally, we study the interplay between transverse cracks and debonding. We come back to the first system made of a film, a bonding layer and a substrate, but now allow it to crack. In the simplified anti-plane setting, in addition to transverse cracks, a threshold criterion acting on the displacement activates either a cohesive or a delamination energy. Some partial results in the general vectorial case are discussed.

2010 Mathematics Subject Classification: Primary 49J45, 49Q20, 74R10, 74K35; Secondary 26B30, 35R35.

Keywords: Free discontinuity problems, functions of bounded deformation, Γ-convergence, fracture mechanics, thin films.

1. Introduction

It is experimentally observed that thin films can essentially develop two different crack patterns: either transverse cracks channeling through the thickness of the film, or planar debonding at the interface of two layers. In classical fracture mechanics [28], a threshold criterion on the energy release rate drives the propagation of a crack along a prescribed path. As proved in [13], this so-called Griffith's law turns out to be equivalent to some notion of stationarity of the total energy (the sum of the potential energy and the surface energy) coupled with the irreversibility of the process and an energy balance. Within this framework, [31] described different possibilities of failure modes. In [43], a reduced two-dimensional model of a thin film system on an elastic foundation is proposed, and the propagation of different crack modes is discussed.

On the basis of this work, a phenomenological one-dimensional model accounting for transverse cracks and debonding is adopted in [32] (see also [33] in the two-dimensional case). This problem is set in the framework of the variational approach to fracture [13]: the evolution is governed by a least energy principle coupled with an irreversibility condition and an energy balance, which, as shown in [13], select particular stationary points of the total energy. At the discrete time level, one considers successive minimization problems. At time t_i, one minimizes an energy functional, say $E(\boldsymbol{u}, \Gamma, \Delta)$, among all kinematically admissible displacements $\boldsymbol{u}$ at time t_i, and among all transverse cracks Γ and all delamination sets Δ satisfying the irreversibility constraints $\Gamma \supset \Gamma_{t_{i-1}}$ and $\Delta \supset \Delta_{t_{i-1}}$.

In [33], the energy considered takes the form

$$E(\boldsymbol{u}, \Gamma, \Delta) = \int_{\omega\setminus\Gamma} \left[\frac{\lambda_f \mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\boldsymbol{u}) e_{\beta\beta}(\boldsymbol{u}) + \mu_f e_{\alpha\beta}(\boldsymbol{u}) e_{\alpha\beta}(\boldsymbol{u}) \right] \mathrm{d}x + \frac{\mu_b}{2} \int_{\omega\setminus\Delta} |\boldsymbol{u}|^2 \,\mathrm{d}x + \kappa_f \operatorname{length}(\Gamma) + \kappa_b \operatorname{area}(\Delta).$$

The region $\omega \subset \mathbb{R}^2$ denotes the basis of a thin film $\Omega_f^\varepsilon = \omega \times (0, \varepsilon)$ bonded on a infinitely stiff substrate $\Omega_s^\varepsilon := \omega \times (-\infty, -\varepsilon)$ through a bonding layer $\Omega_b^\varepsilon = \omega \times [-\varepsilon, 0]$. The transverse cracks are of the form $\Gamma \times (0, \varepsilon)$ where $\Gamma \subset \omega$ is a one-dimensional object, while the delamination zone $\Delta \subset \omega$ is two-dimensional.

The in-plane displacement of the film at the interface with the substrate is denoted by $\boldsymbol{u} : \omega \setminus \Gamma \to \mathbb{R}^2$. The fracture toughness κ_f is a material property of the film, while κ_b measures the strength of the bonding between the film and the substrate. The reduced linearly elastic energy is well-known and rigorously derived in the Kirchhoff–Love theory of elastic plates [17].

The substrate is supposed to be infinitely stiff so that the displacement inside that region is given by a prescribed (planar) function which, for simplicity, we assume to be zero. The term $\frac{\mu_b}{2} \int_{\omega\setminus\Delta} |\boldsymbol{u}|^2 \,\mathrm{d}x$ represents the cost for the film to deform differently from the substrate. It only has to be paid in the bonded region $\omega \setminus \Delta$ because in Δ the film is no longer attached to the substrate.

The goal of the present work is to isolate a particular meaningful scaling law of the elasticity and fracture parameters (with respect to the thickness of the film) which enables one to rigorously justify the emergence of these different crack modes in linearly elastic thin plates. A Γ-convergence analysis is carried out, as the thickness of the plate tends to zero, starting from three-dimensional linearized elasticity and the variational approach to fracture. The three-dimensional energy functional is given by

$$(\boldsymbol{v}, \Gamma) \mapsto \frac{1}{2} \int_{\Omega_f^\varepsilon \cup \Omega_b^\varepsilon} \left[\lambda^\varepsilon e_{ii}(\boldsymbol{v}) e_{jj}(\boldsymbol{v}) + 2\mu^\varepsilon e_{ij}(\boldsymbol{v}) e_{ij}(\boldsymbol{v}) \right] \mathrm{d}x + \int_{\Gamma\cap(\Omega_f^\varepsilon \cup \Omega_b^\varepsilon)} \kappa^\varepsilon \,\mathrm{d}\mathcal{H}^2,$$

where any kinematically admissible displacement $\boldsymbol{v} : \Omega_f^\varepsilon \cup \Omega_b^\varepsilon \cup \Omega_s^\varepsilon \to \mathbb{R}^3$ is required to satisfy the condition $\boldsymbol{v} = 0$ in the substrate Ω_s^ε. As is to be expected, the results strongly depend on the scaling of the elasticity and fracture parameter with respect to ε. In this paper, the Lamé coefficients and the toughness are supposed to be piecewise constant functions scaling like

$$(\lambda^\varepsilon, \mu^\varepsilon) = \begin{cases} (\lambda_f, \mu_f) & \text{in } \Omega_f^\varepsilon, \\ \varepsilon^2(\lambda_b, \mu_b) & \text{in } \Omega_b^\varepsilon, \end{cases} \qquad \kappa^\varepsilon = \begin{cases} \kappa_f & \text{in } \Omega_f^\varepsilon, \\ \varepsilon\kappa_b & \text{in } \Omega_b^\varepsilon. \end{cases} \tag{1.1}$$

This choice is *a posteriori* justified by the fact that the models derived below correspond to our initial goal.

First, we examine the case in which cracks are absent. We prove in Theorem 4.1 that the limit energy for the bilayer system is given by

$$\int_\omega \left[\frac{\lambda_f \mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}}) + \mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}}) e_{\alpha\beta}(\bar{\boldsymbol{u}}) \right] \mathrm{d}x + \frac{\mu_b}{2} \int_\omega |\bar{\boldsymbol{u}}|^2 \,\mathrm{d}x, \quad \bar{\boldsymbol{u}} \in H^1(\omega;\mathbb{R}^2)$$

(a result obtained also in [34, Thm. 2.1]; here we present an alternative and simpler proof). The analysis rests on the standard dimension reduction for thin elastic plates in linear elasticity [17], the main new difficulty being the lack of coerciveness in the compliant bonding layer. The limiting displacements are planar (not just of Kirchhoff–Love type), due to the presence of the bonding layer and the homogeneous condition in the substrate. On the other hand, a bonding type energy $\frac{\mu_b}{2} \int_\omega |\bar{\boldsymbol{u}}|^2 \,\mathrm{d}x$ appears in the limit, due to the displacement mismatch between the film and the substrate (which is accommodated by an affine transition in the vertical variable).

Integrals of the form $\frac{\mu_b}{2} \int_\omega |\bar{\boldsymbol{u}}|^2 \,\mathrm{d}x$ have appeared in the literature either as the effective energy of a Winkler [42] elastic foundation (whose reaction force $\mu_b \bar{\boldsymbol{u}}$ is assumed to be linear with respect to the relative displacement of the film; see [34] and the references therein), or in the form of a Barenblatt [9] cohesive-zone surface energy (where the film and the substrate are regarded as a single elastic body). They have been used, in particular, in the existing analytical studies of delamination problems, *e.g.* [11, 26, 27, 37].

The problem of rigorously deriving cohesive energies such as $\frac{\mu_b}{2} \int_\omega |\bar{\boldsymbol{u}}|^2 \,\mathrm{d}x$ (and of more general forms), for domains with a fixed positive thickness (as opposed to in the vanishing thickness limit $\varepsilon \to 0$ considered in our work), has been considered by other authors. In [18, 20, 23, 29] they are obtained in the Γ-limit of a particular type of Ambrosio-Tortorelli functionals. In [3, 4] they are derived by homogenization as the limit of a Neumann sieve, debonding being regarded as the effect of the interaction of two films through a suitably periodically distributed contact zone. Finally, in [15], they appear in the homogenization of brittle composites with soft inclusions.

Next, we consider only a single linearly elastic brittle thin film (not the full bilayer system), and we show the emergence of a limit energy which is finite on Kirchhoff–Love type displacements outside the cracks which are transverse. More precisely, the limit energy in that case is given by

$$\int_{\Omega_f} \left[\frac{\lambda_f \mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\boldsymbol{u}) e_{\beta\beta}(\boldsymbol{u}) + \mu_f e_{\alpha\beta}(\boldsymbol{u}) e_{\alpha\beta}(\boldsymbol{u}) \right] \mathrm{d}x + \kappa_f \mathcal{H}^2(J_{\boldsymbol{u}}), \quad \boldsymbol{u} \in SBD(\Omega_f).$$

The mathematical analysis falls in the framework of free discontinuity problems. A satisfactory mathematical treatment can be done in the space SBD of special functions of bounded deformation where the cracks are assimilated to the jump set $J_{\boldsymbol{u}}$ of the displacement $\boldsymbol{u}$.

In this limit model, the admissible limit displacements are not planar anymore but present a Kirchhoff–Love type structure outside the jump set. It turns out that the function u_3 is independent of x_3 and it belongs to $SBV(\omega)$. In addition, its approximate gradient $\nabla u_3 = (\partial_1 u_3, \partial_2 u_3) \in SBD(\omega)$ and, for $\alpha = 1, 2$, we have $u_\alpha(x_1, x_2, x_3) = \bar{u}_\alpha(x_1, x_2) + \left(\frac{1}{2} - x_3\right) \partial_\alpha u_3(x_1, x_2)$ where $\bar{\boldsymbol{u}} = (\bar{u}_1, \bar{u}_2) \in SBD(\omega)$. For what concerns the jump set itself, one shows that it is transverse and that it essentially coincides with $(J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) \times (0, 1)$. The proof of this structure result relies on tools of geometric measure theory and fine properties of bounded deformation functions. Inserting this special structure of admissible displacements inside the elastic energy gives rise to the usual membrane and flexural energies of plate theory.

Previous studies of this type of problems include [6, 7, 12, 14] in the case of nonlinear elasticity, where the elastic energy is a function of the deformation gradient. In [25] the dimension reduction of linearly elastic brittle thin films is carried out under the additional assumptions that the cracks are closed and vertical. In Theorem 5.3 below, we present the first complete result without prior assumptions on the geometry or the topology of the cracks.

Finally, we study the interplay between transverse cracks and debonding. We come back to our original thin film system made of a film, a bonding layer and a substrate, but now allow it to crack. In a simplified anti-plane like setting, we show the appearance of transverse cracks. In addition, if the displacement is below a critical threshold, debonding is energetically favorable, while if the threshold is overpassed, the body prefers to delaminate, *i.e.*, to create a brittle planar crack. We recover in that case the simplified one-dimensional model introduced in [32] (see Theorem 6.1) where the limit energy is given by

$$\frac{\mu_f}{2}\int_0^L |u'|^2\,\mathrm{d}x_1 + \frac{\mu_b}{2}\int_{(0,L)\setminus\Delta} |u|^2\,\mathrm{d}x_1 + \kappa_f \#(J_u) + \kappa_b \mathfrak{L}^1(\Delta), \quad u \in SBV(0,L),$$

$\Delta := \{|u| > \sqrt{2\kappa_b/\mu_b}\}$ being the delamination set.

The proof follows the lines of [33] where we take advantage of the scalar nature of the displacement. The limit model then couples transverse cracks, cohesive transitions and delamination. In the full vectorial linearly elastic case, we have been unable to prove the convergence to $E(\boldsymbol{u}, \Gamma, \Delta)$. We limit ourselves to present some partial results which, in our opinion, ought to be considered in any attempt to establish the desired Γ-convergence. We prove an energy upper bound by constructing, for every admissible limit displacement, an optimal recovery sequence (Proposition 6.3). What remains open is to establish the optimality of the affine transitions in the vertical variable in order to accommodate the mismatch between the film and the substrate. Indeed, the ability to break gives the bonding layer the opportunity to reduce its elastic energy by performing a periodic sequence of small rotations (Example 6.5). This implies that the delamination zone cannot be identified just by taking the orthogonal projections of the jump set of the displacement, as is done in the Sobolev and scalar cases. As a possible remedy, we consider instead, “almost vertical” projections. We are able to prove a surface energy lower bound (although with a bad multiplicative constant) and to show the validity of the desired bulk energy lower bound under the assumption that the minimizing sequence satisfies better *a priori* estimates than just the energy bound (Lemma 6.9).

In this spirit, let us mention [36] where a Griffith energy for the debonding at the interface is obtained as the limit elastic energy of a thin bonding layer in a problem involving a damage internal variable.

To close this introduction, let us briefly comment our assumptions on the scaling laws (1.1). In the purely elastic case, an exhaustive study of the different scalings has been performed in [34]. Our choice of Lamé coefficients is a particular situation considered in that paper. The choice of the toughness might seem arbitrary. It is actually justified by the nature of the models we derive. We claim neither that these are the only interesting scalings nor that they are the only ones leading to such multifissuration models.

The paper is organized as follows: Section 2 is devoted to introduce various notations used throughout this work. In Section 3, we precisely describe the model and perform a scaling to make the problem more tractable from a mathematical point of view. Section 4 investigates the asymptotic analysis in the absence of cracks, and evidences the appearance of a debonding type limiting energy (Theorem 4.1). In Section 5, we carry out the analysis of a linearly elastic thin film, and show the emergence of transverse cracks (Theorem 5.3). Finally, Section 6 discusses the interplay between transverse cracks, debonding, and delamination.

2. Notation and preliminaries

If a and $b \in \mathbb{R}^n$, we write $a \cdot b = \sum_{i=1}^n a_i b_i$ for the Euclidean scalar product, and we denote the norm by $|a| = \sqrt{a \cdot a}$. The open ball of center x and radius ϱ is denoted by $B_\varrho(x)$. If $x = 0$, we simply write B_ϱ instead of $B_\varrho(0)$.

We denote by $\mathbb{M}^{m\times n}$ the set of real $m \times n$ matrices, and by $\mathbb{M}^{n\times n}_{\rm sym}$ the set of all real symmetric $n \times n$ matrices. Given two matrices A and $B \in \mathbb{M}^{m\times n}$, we let $A : B := \mathrm{tr}(A^T B)$ for the Frobenius scalar product, and $|A| := \sqrt{\mathrm{tr}(A^T A)}$ for the associated norm (A^T is the transpose of A, and $\mathrm{tr}(A)$ is its trace). We recall that for any two vectors $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$, $a \otimes b \in \mathbb{M}^{m\times n}$ stands for the tensor product, *i.e.*, $(a \otimes b)_{ij} = a_i b_j$ for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. If $m = n$, then $a \odot b := \frac{1}{2}(a \otimes b + b \otimes a) \in \mathbb{M}^{n\times n}_{\rm sym}$ denotes the symmetric tensor product.

Given an open subset U of $\mathbb{R}^n$ and a finite dimensional Euclidean space X, we use standard notations for Lebesgue spaces $L^p(U; X)$ and Sobolev spaces $H^1(U; X)$ or $W^{1,p}(U; X)$. We denote by $\mathfrak{M}(U; X)$ the space of all X-valued Radon measures with finite total variation. If the target space $X = \mathbb{R}$, we omit to write it for simplicity. According to the Riesz representation Theorem, it is identified with the topological dual of $\mathcal{C}_0(U; X)$ (the space of all continuous functions $\varphi : U \to X$ such that $\{\varphi \geqslant \varepsilon\}$ is compact for every $\varepsilon > 0$), and a weak* topology is defined according to this duality. The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$, and the k-dimensional Hausdorff measure by $\mathcal{H}^k$. Sometimes, the notation # will be used instead of $\mathcal{H}^0$ for the counting measure, and $|\cdot|$ instead of the Lebesgue measure $\mathcal{L}^n$. In dimension n, equality or inclusion of sets up to a $\mathcal{H}^{n-1}$-negligible set will be respectively denoted by $\cong$ and $\tilde{\subset}$.

Given a function $\boldsymbol{u} \in L^1(U; \mathbb{R}^m)$ with $m \geqslant 1$, we say that $\boldsymbol{u}$ has an approximate limit at $x \in U$ if there exists $\tilde{\boldsymbol{u}}(x) \in \mathbb{R}^m$ such that

$$\lim_{\varrho \to 0} \frac{1}{\varrho^n} \int_{B_\varrho(x)} |\boldsymbol{u}(y) - \tilde{\boldsymbol{u}}(x)| \, \mathrm{d}y = 0.$$

The set $S_{\boldsymbol{u}}$ where this property fails is called the approximate discontinuity set.

We say that $\boldsymbol{u}$ has one-sided Lebesgue limits $\boldsymbol{u}^\pm(x) \in \mathbb{R}^m$ at $x \in U$ with respect to a direction $\nu_{\boldsymbol{u}}(x) \in \mathbb{S}^{n-1} := \{\zeta \in \mathbb{R}^n : |\zeta| = 1\}$ if

$$\lim_{\varrho \to 0} \frac{1}{\varrho^n} \int_{B^\pm_\varrho(x, \nu_{\boldsymbol{u}}(x))} |\boldsymbol{u}(y) - \boldsymbol{u}^\pm(x)| \, \mathrm{d}y = 0,$$

where $B^\pm_\varrho(x, \nu_{\boldsymbol{u}}(x)) := \{y \in B_\varrho(x) : \pm\nu_{\boldsymbol{u}}(x) \cdot (y - x) \geqslant 0\}$. We will denote by $[\boldsymbol{u}](x) := \boldsymbol{u}^+(x) - \boldsymbol{u}^-(x)$ the jump of $\boldsymbol{u}$ at x. The jump set $J_{\boldsymbol{u}}$ of $\boldsymbol{u}$ is defined as the set of points $x \in U$ such that the one-sided Lebesgue limits with respect to a direction $\nu_{\boldsymbol{u}}(x)$ exist, and in addition $\boldsymbol{u}^+(x) \neq \boldsymbol{u}^-(x)$. Clearly we have $J_{\boldsymbol{u}} \subset S_{\boldsymbol{u}}$.

2.1 *Functions of bounded variation*

The space $BV(U; \mathbb{R}^m)$ of functions of bounded variation in U with values in $\mathbb{R}^m$ is made of all functions $\boldsymbol{u} \in L^1(U; \mathbb{R}^m)$ such that the distributional derivative satisfies $D\boldsymbol{u} \in \mathfrak{M}(U; \mathbb{M}^{m\times n})$. The measure $D\boldsymbol{u}$ can be decomposed as

$$D\boldsymbol{u} = \nabla \boldsymbol{u} \mathcal{L}^n + (\boldsymbol{u}^+ - \boldsymbol{u}^-) \otimes \nu_{\boldsymbol{u}} \mathcal{H}^{n-1} \llcorner J_{\boldsymbol{u}} + D^c \boldsymbol{u},$$

where $\nabla \boldsymbol{u}$ is the Radon–Nikodým derivative of $D\boldsymbol{u}$ with respect to the Lebesgue measure $\mathcal{L}^n$, which coincides with the approximate gradient of $\boldsymbol{u}$. For any $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, we denote by $\partial_j u_i := (\nabla \boldsymbol{u})_{ij}$ the entries of $\nabla \boldsymbol{u}$. The measure $D^c\boldsymbol{u}$ is the Cantor part of $D\boldsymbol{u}$ which has the property of vanishing on any σ-finite set with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$. The jump set $J_{\boldsymbol{u}}$ is a countably $\mathcal{H}^{n-1}$-rectifiable Borel set, $\nu_{\boldsymbol{u}}$ is an approximate unit normal to $J_{\boldsymbol{u}}$, and $\boldsymbol{u}^{\pm}(x)$ are the one-sided Lebesgue limits of $\boldsymbol{u}$ at $x \in U$ in the direction $\nu_{\boldsymbol{u}}(x)$. In addition, we have $\mathcal{H}^{n-1}(S_{\boldsymbol{u}} \setminus J_{\boldsymbol{u}}) = 0$.

We say that $\boldsymbol{u}$ is a special function of bounded variation, and we write $\boldsymbol{u} \in SBV(U;\mathbb{R}^m)$, if $D^c\boldsymbol{u} = 0$. If further $\nabla \boldsymbol{u} \in L^p(U;\mathbb{R}^{m\times n})$ for some $p > 1$, and $\mathcal{H}^{n-1}(J_{\boldsymbol{u}}) < \infty$, we write $\boldsymbol{u} \in SBV^p(U;\mathbb{R}^m)$. We refer to [2] for general properties of BV-functions.

2.2 *Functions of bounded deformation*

The space $BD(U)$ of functions of bounded deformation is made of all vector fields $\boldsymbol{u} \in L^1(U;\mathbb{R}^n)$ whose distributional symmetric gradient satisfies

$$E\boldsymbol{u} = \frac{D\boldsymbol{u} + D\boldsymbol{u}^T}{2} \in \mathfrak{M}(U;\mathbb{M}^{n\times n}_{\mathrm{sym}}).$$

This measure can be decomposed as

$$E\boldsymbol{u} = e(\boldsymbol{u})\mathcal{L}^n + (\boldsymbol{u}^+ - \boldsymbol{u}^-) \odot \nu_{\boldsymbol{u}}\mathcal{H}^{n-1} \llcorner J_{\boldsymbol{u}} + E^c\boldsymbol{u}. \tag{2.1}$$

In the previous expression, $e(\boldsymbol{u})$ denotes the absolutely continuous part of $E\boldsymbol{u}$ with respect to $\mathcal{L}^n$. For any $1 \leqslant i, j \leqslant n$, we denote by $e_{ij}(\boldsymbol{u}) = (e(\boldsymbol{u}))_{ij}$ the entries of $e(\boldsymbol{u})$. The measure $E^c\boldsymbol{u}$ is the Cantor part of $E\boldsymbol{u}$ which has the property to vanish on any σ-finite set with respect to $\mathcal{H}^{n-1}$. The jump set $J_{\boldsymbol{u}}$ of $\boldsymbol{u}$ is a countably $\mathcal{H}^{n-1}$-rectifiable Borel set, $\nu_{\boldsymbol{u}}$ is an approximate unit normal to $J_{\boldsymbol{u}}$, and $\boldsymbol{u}^{\pm}(x)$ are the one-sided Lebesgue limits of $\boldsymbol{u}$ at $x \in U$ in the direction $\nu_{\boldsymbol{u}}(x)$. If $E^c\boldsymbol{u} = 0$, we say that $\boldsymbol{u}$ is a special function of bounded deformation and we write $\boldsymbol{u} \in SBD(U)$. We refer to [1, 5, 8, 10, 16, 19, 35, 38, 40, 41] for general properties of BD-functions.

2.3 *General conventions*

In the sequel we will always work in dimensions 1, 2 or 3. Latin indices $i, j, k, l, \dots$ (except f and b) take their values in the set $\{1, 2, 3\}$ unless otherwise indicated. Greek indices $\alpha, \beta, \gamma, \dots$ (except ε) take their values in the set $\{1, 2\}$. The repeated index summation convention is systematically used.

3. Description of the problem

3.1 *In the original configuration*

Let ω be a bounded and connected open subset of $\mathbb{R}^2$ with Lipschitz boundary which denotes the basis of a thin domain occupying the open set $\Omega^\varepsilon := \omega \times (-2\varepsilon, \varepsilon)$ in its reference configuration. We assume that this domain is made of the union of a film $\Omega^\varepsilon_f := \omega \times (0, \varepsilon)$, a bonding layer $\Omega^\varepsilon_b := \omega \times [-\varepsilon, 0]$, and a substrate $\Omega^\varepsilon_s := \omega \times (-2\varepsilon, -\varepsilon)$. Let us underline that the set Ω^ε_b is not open. Any kinematically admissible displacement $\boldsymbol{v} : \Omega^\varepsilon \to \mathbb{R}^3$ is required to satisfy the condition $\boldsymbol{v} = 0$ in Ω^ε_s. In the sequel we shall denote by $x' := (x_1, x_2)$ the in-plane variable.

The background behavior of this medium in that of an isotropic linearly elastic material whose Lamé coefficients are given by

$$(\lambda^\varepsilon, \mu^\varepsilon) = \begin{cases} (\lambda_f, \mu_f) & \text{in } \Omega_f^\varepsilon, \\ \varepsilon^2(\lambda_b, \mu_b) & \text{in } \Omega_b^\varepsilon. \end{cases}$$

The elastic energy associated to a displacement $\boldsymbol{v} \in H^1(\Omega^\varepsilon; \mathbb{R}^3)$ satisfying $\boldsymbol{v} = 0$ $\mathcal{L}^3$-a.e. in Ω_s^ε is given by

$$\frac{1}{2}\int_{\Omega^\varepsilon} \Big[\lambda^\varepsilon e_{ii}(\boldsymbol{v}) e_{jj}(\boldsymbol{v}) + 2\mu^\varepsilon e_{ij}(\boldsymbol{v}) e_{ij}(\boldsymbol{v})\Big] \, \mathrm{d}x. \tag{3.1}$$

If the body undergoes cracks, according to the variational approach to fracture (see [13, 24]), the presence of cracks is penalized by means of a surface energy of Griffith type where the toughness is given by

$$\kappa^\varepsilon = \begin{cases} \kappa_f & \text{in } \Omega_f^\varepsilon, \\ \varepsilon\kappa_b & \text{in } \Omega_b^\varepsilon. \end{cases}$$

In this case, Sobolev spaces cannot describe admissible displacements since they may jump across the cracks. The natural framework is to consider displacements which are special functions of bounded deformation. Identifying the cracks with the jump set of the displacement, denoted by $J_{\boldsymbol{v}}$, the surface energy is given by

$$\int_{J_{\boldsymbol{v}} \cap \Omega^\varepsilon} \kappa^\varepsilon \, \mathrm{d}\mathcal{H}^2. \tag{3.2}$$

The total energy is then given by the sum of the bulk energy, given by (3.1), where $e(\boldsymbol{v})$ is intended as the absolutely continuous part of the strain with respect to the Lebesgue measure, and the surface energy, given by (3.2). It is well defined for any displacement $\boldsymbol{v} \in SBD(\Omega^\varepsilon)$ satisfying $\boldsymbol{v} = 0$ $\mathcal{L}^3$-a.e. in the substrate Ω_s^ε.

3.2 *In the rescaled configuration*

As usual in dimension reduction, we rescale the problem on a fixed domain of unit thickness (see [17]). We denote by $\Omega := \Omega^1$, $\Omega_f := \Omega_f^1$, $\Omega_b := \Omega_b^1$, and $\Omega_s := \Omega_s^1$. For every original displacement $\boldsymbol{v} \in H^1(\Omega^\varepsilon; \mathbb{R}^3)$ (resp. $\boldsymbol{v} \in SBD(\Omega^\varepsilon)$) such that $\boldsymbol{v} = 0$ $\mathcal{L}^3$-a.e. in Ω_s^ε, we define the rescaled displacement $\boldsymbol{u}$ in the rescaled configuration by

$$\begin{cases} u_\alpha(x', x_3) = v_\alpha(x', \varepsilon x_3), \\ u_3(x', x_3) = \varepsilon v_3(x', \varepsilon x_3), \end{cases} \quad \text{for all } x = (x', x_3) \in \Omega.$$

Replacing $\boldsymbol{v}$ by this expression in the energy (3.1), and dividing the resulting expression by ε yields the following rescaled elastic energy (see [17])

$$J_\varepsilon(\boldsymbol{u}) = J_\varepsilon(\boldsymbol{u}, \Omega_f) + J_\varepsilon(\boldsymbol{u}, \Omega_b),$$

where

$$J_\varepsilon(\boldsymbol{u}, \Omega_f) := \frac{1}{2}\int_{\Omega_f}\Big[\lambda_f e_{\alpha\alpha}(\boldsymbol{u})e_{\beta\beta}(\boldsymbol{u}) + 2\mu_f e_{\alpha\beta}(\boldsymbol{u})e_{\alpha\beta}(\boldsymbol{u})\Big]\,\mathrm{d}x + \frac{1}{2\varepsilon^2}\int_{\Omega_f}\Big[2\lambda_f e_{\alpha\alpha}(\boldsymbol{u})e_{33}(\boldsymbol{u}) + 4\mu_f e_{\alpha 3}(\boldsymbol{u})e_{\alpha 3}(\boldsymbol{u})\Big]\,\mathrm{d}x + \frac{1}{2\varepsilon^4}\int_{\Omega_f}(\lambda_f + 2\mu_f)e_{33}(\boldsymbol{u})e_{33}(\boldsymbol{u})\,\mathrm{d}x, \tag{3.3}$$

and

$$J_\varepsilon(\boldsymbol{u}, \Omega_b) := \frac{\varepsilon^2}{2}\int_{\Omega_b}\Big[\lambda_b e_{\alpha\alpha}(\boldsymbol{u})e_{\beta\beta}(\boldsymbol{u}) + 2\mu_b e_{\alpha\beta}(\boldsymbol{u})e_{\alpha\beta}(\boldsymbol{u})\Big]\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_b}\Big[2\lambda_b e_{\alpha\alpha}(\boldsymbol{u})e_{33}(\boldsymbol{u}) + 4\mu_b e_{\alpha 3}(\boldsymbol{u})e_{\alpha 3}(\boldsymbol{u})\Big]\,\mathrm{d}x + \frac{1}{2\varepsilon^2}\int_{\Omega_b}(\lambda_b + 2\mu_b)e_{33}(\boldsymbol{u})e_{33}(\boldsymbol{u})\,\mathrm{d}x. \tag{3.4}$$

In the case of cracks, the total energy is obtained by adding the surface energy. In the rescaled configuration, it is given by (see [6, 7, 12, 14])

$$E_\varepsilon(\boldsymbol{u}) = E_\varepsilon(\boldsymbol{u}, \Omega_f) + E_\varepsilon(\boldsymbol{u}, \Omega_b),$$

where

$$E_\varepsilon(\boldsymbol{u}, \Omega_f) = J_\varepsilon(\boldsymbol{u}, \Omega_f) + \kappa_f \int_{J_{\boldsymbol{u}}\cap\Omega_f}\left|\left((\nu_{\boldsymbol{u}})', \frac{1}{\varepsilon}(\nu_{\boldsymbol{u}})_3\right)\right|\,\mathrm{d}\mathcal{H}^2,$$

and

$$E_\varepsilon(\boldsymbol{u}, \Omega_b) = J_\varepsilon(\boldsymbol{u}, \Omega_b) + \kappa_b \int_{J_{\boldsymbol{u}}\cap\Omega_b}\left|\left(\varepsilon(\nu_{\boldsymbol{u}})', (\nu_{\boldsymbol{u}})_3\right)\right|\,\mathrm{d}\mathcal{H}^2.$$

4. Debonding of thin films

In this section, we assume that the body is purely elastic, *i.e.*, no cracks are allowed. Through an asymptotic analysis as the thickness ε tends to zero, we rigorously recover a reduced two-dimensional model of a thin film system as an elastic membrane on an in-plane elastic foundation. A similar model has been derived in [34, Theorem 2.1] by means of a different method. The original three-dimensional energy $J_\varepsilon : L^2(\Omega;\mathbb{R}^3) \to [0, +\infty]$ is defined by

$$J_\varepsilon(\boldsymbol{u}) := \begin{cases} J_\varepsilon(\boldsymbol{u}, \Omega_f) + J_\varepsilon(\boldsymbol{u}, \Omega_b) & \text{if } \boldsymbol{u} \in H^1(\Omega;\mathbb{R}^3) \text{ and } \boldsymbol{u} = 0\ \mathcal{L}^3\text{-a.e. in } \Omega_s, \\ +\infty & \text{otherwise,} \end{cases}$$

while the reduced two dimensional energy $J_0 : L^2(\Omega;\mathbb{R}^3) \to [0, +\infty]$ is given by

$$J_0(\boldsymbol{u}) := \begin{cases} \displaystyle\int_\omega \left[\frac{\lambda_f\mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}})e_{\beta\beta}(\bar{\boldsymbol{u}}) + \mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}})e_{\alpha\beta}(\bar{\boldsymbol{u}})\right]\mathrm{d}x' + \frac{\mu_b}{2}\int_\omega |\bar{\boldsymbol{u}}|^2\,\mathrm{d}x' & \text{if } \begin{cases} \boldsymbol{u} = (\bar{\boldsymbol{u}}, 0), \\ \bar{\boldsymbol{u}} \in H^1(\omega;\mathbb{R}^2), \end{cases} \\ +\infty & \text{otherwise.} \end{cases}$$

Our first main result in the following Γ-convergence.

Theorem 4.1 *Let $\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^3)$, then*
- *for any sequence $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega;\mathbb{R}^3)$ with $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$, then*

$$J_0(\boldsymbol{u}) \leqslant \liminf_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon);$$

- *there is a recovery sequence $(\boldsymbol{u}^*_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega;\mathbb{R}^3)$ such that $\boldsymbol{u}^*_\varepsilon \to \boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$, and*

$$J_0(\boldsymbol{u}) \geqslant \limsup_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}^*_\varepsilon).$$

Proof. Although some parts of the proof are already well known (see [17, Theorem 1.11.2]), it will be convenient for us to reproduce the entire argument.

Step 1. Compactness. Let $(\boldsymbol{u}_\varepsilon) \subset L^2(\Omega;\mathbb{R}^3)$ be such that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$. If $\liminf_\varepsilon J_\varepsilon(\boldsymbol{u}_\varepsilon) = +\infty$, there is nothing to prove. We therefore assume that $\liminf_\varepsilon J_\varepsilon(\boldsymbol{u}_\varepsilon) < \infty$. Up to a subsequence, there is no loss of generality to suppose that

$$J_\varepsilon(\boldsymbol{u}_\varepsilon) = J_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_f) + J_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_b) \leqslant C,$$

for some constant $C > 0$ independent of ε. The expression (3.3) of the energy in the film Ω_f combined with Korn's inequality implies that $(\boldsymbol{u}_\varepsilon)$ is actually bounded in $H^1(\Omega_f;\mathbb{R}^3)$, and that $\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u}$ weakly in $H^1(\Omega_f;\mathbb{R}^3)$ with $\boldsymbol{u} \in H^1(\Omega_f;\mathbb{R}^3)$. Contrary to the case of a standard linearly elastic plate model (see [17]), we will show that, thanks to the Dirichlet condition in the substrate, the limit displacement $\boldsymbol{u}$ is planar instead of just Kirchhoff–Love type. Indeed, using also the expression of the energy (3.3)–(3.4), the fact that $\boldsymbol{u}_\varepsilon = 0$ $\mathfrak{L}^3$-a.e. in Ω_s, and Poincaré's inequality, we get that

$$\int_{\Omega_f} |(u_\varepsilon)_3|^2 \,\mathrm{d}x \leqslant \int_{\Omega_f\cup\Omega_b} |e_{33}(\boldsymbol{u}_\varepsilon)|^2 \,\mathrm{d}x \leqslant C\varepsilon^2 \to 0,$$

so that $u_3 = 0$. Thanks again to the bound of the energy in the film (3.3), we have

$$\|e_{\alpha 3}(\boldsymbol{u}_\varepsilon)\|_{L^2(\Omega_f)} \leqslant C\varepsilon \to 0,$$

which shows that $e_{\alpha 3}(\boldsymbol{u}) = 0$. It thus follows that $\partial_3 u_\alpha = -\partial_\alpha u_3 = 0$ which implies that $u_\alpha(x', x_3) = \bar{u}_\alpha(x')$ for $\mathfrak{L}^3$-a.e. $x \in \Omega_f$, for some $\bar{\boldsymbol{u}} \in H^1(\omega;\mathbb{R}^2)$. We have thus identified the right limit space.

Step 2. Lower bound. We next derive the lower bound. Up to a further subsequence, we may assume that

$$\begin{cases} \varepsilon^{-2} e_{33}(\boldsymbol{u}_\varepsilon) \rightharpoonup \zeta_3 \\ \varepsilon^{-1} e_{\alpha 3}(\boldsymbol{u}_\varepsilon) \rightharpoonup \zeta_\alpha \end{cases} \quad \text{weakly in } L^2(\Omega_f),$$

for some functions ζ_1, ζ_2 and $\zeta_3 \in L^2(\Omega_f)$. Then, by lower semicontinuity of the norm with respect to weak convergence, we get that

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_f) \geqslant \frac{1}{2}\int_{\Omega_f} \left[\lambda_f (e_{\alpha\alpha}(\bar{\boldsymbol{u}}) + \zeta_3)^2 + 2\mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}}) e_{\alpha\beta}(\bar{\boldsymbol{u}}) + 4\mu_f \zeta_\alpha\zeta_\alpha + 2\mu_f \zeta_3\zeta_3\right] \mathrm{d}x.$$

Minimizing with respect to $(\zeta_1, \zeta_2, \zeta_3)$, we find that the minimal value is attained when $\zeta_\alpha = 0$ and $\zeta_3 = -\frac{\lambda_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}})$, and thus

$$\liminf_{\varepsilon \to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_f) \geqslant \int_\omega \left[\frac{\lambda_f \mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}}) + \mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}}) e_{\alpha\beta}(\bar{\boldsymbol{u}}) \right] \mathrm{d}x.$$

We now examine the contribution of the bonding layer. To this aim, according to (3.4), isolating the only term of order 1 leads to

$$\begin{aligned} J_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_b) &\geqslant 2\mu_b \int_{\Omega_b} e_{\alpha 3}(\boldsymbol{u}_\varepsilon) e_{\alpha 3}(\boldsymbol{u}_\varepsilon)\,\mathrm{d}x \\ &\geqslant \frac{\mu_b}{2} \int_\omega \left| \int_{-1}^0 [\partial_3 (u_\varepsilon)_1 + \partial_1 (u_\varepsilon)_3]\,\mathrm{d}x_3 \right|^2 \mathrm{d}x' + \frac{\mu_b}{2} \int_\omega \left| \int_{-1}^0 [\partial_3 (u_\varepsilon)_2 + \partial_2 (u_\varepsilon)_3]\,\mathrm{d}x_3 \right|^2 \mathrm{d}x', \end{aligned}$$

thanks to the Cauchy-Schwarz inequality with respect to the x_3 variable. Since $\boldsymbol{u}_\varepsilon = 0$ $\mathcal{L}^3$-a.e. in Ω_s, then

$$\int_{-1}^0 \partial_3 \boldsymbol{u}_\varepsilon(x', x_3)\,\mathrm{d}x_3 = \boldsymbol{u}_\varepsilon(x', 0) \quad \text{ for } \mathcal{L}^2\text{-a.e. } x' \in \omega,$$

where $\boldsymbol{u}_\varepsilon(\cdot, 0)$ denotes the trace of $\boldsymbol{u}_\varepsilon$ on $\{x_3 = 0\}$. On the other hand, setting

$$\bar{u}_3^\varepsilon = \int_{-1}^0 (u_\varepsilon)_3(\cdot, x_3)\,\mathrm{d}x_3 \in H^1(\omega),$$

we have

$$\int_{-1}^0 \partial_\alpha (u_\varepsilon)_3(x', x_3)\,\mathrm{d}x_3 = \partial_\alpha \bar{u}_3^\varepsilon(x') \quad \text{ for } \mathcal{L}^2\text{-a.e. } x' \in \omega.$$

Gathering everything, we infer that

$$J_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_b) \geqslant \frac{\mu_b}{2} \int_\omega \left| (u_\varepsilon)_1(x', 0) + \partial_1 \bar{u}_3^\varepsilon(x') \right|^2 \mathrm{d}x' + \frac{\mu_b}{2} \int_\omega \left| (u_\varepsilon)_2(x', 0) + \partial_2 \bar{u}_3^\varepsilon(x') \right|^2 \mathrm{d}x'. \quad (4.1)$$

According to the trace theorem, and since $\bar{u}_\alpha$ is independent of x_3, we have $(u_\varepsilon)_\alpha(\cdot, 0) \to \bar{u}_\alpha$ strongly in $L^2(\omega)$. On the other hand, the energy in the bonding layer (3.4) together with the Cauchy-Schwarz and Poincaré inequalities yield

$$\int_\omega |\bar{u}_3^\varepsilon|^2\,\mathrm{d}x' \leqslant \int_{\Omega_b} |e_{33}(\boldsymbol{u}_\varepsilon)|^2\,\mathrm{d}x \leqslant C\varepsilon^2 \to 0,$$

while (4.1) shows that the sequence $(\nabla \bar{u}_3^\varepsilon)$ is bounded in $L^2(\omega; \mathbb{R}^2)$. Consequently, $\nabla \bar{u}_3^\varepsilon \rightharpoonup 0$ weakly in $L^2(\omega; \mathbb{R}^2)$, and combining all the convergences established so far, we deduce that

$$\liminf_{\varepsilon \to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_b) \geqslant \frac{\mu_b}{2} \int_\omega |\bar{\boldsymbol{u}}|^2\,\mathrm{d}x',$$

which completes the proof of the lower bound.

Step 3. Upper bound. We assume without loss of generality that $\boldsymbol{u} = (\bar{\boldsymbol{u}}, 0)$ for some $\bar{\boldsymbol{u}} \in H^1(\omega;\mathbb{R}^2)$, otherwise the limit energy is infinite. We now define a recovery sequence $(\boldsymbol{u}_\varepsilon^*)_{\varepsilon>0}$. For all $\varepsilon > 0$, let

$$\boldsymbol{u}_\varepsilon^*(x', x_3) = \begin{cases} \big(\bar{\boldsymbol{u}}(x'), \varepsilon^2 x_3 h_\varepsilon(x')\big) & \text{if } x \in \Omega_f, \\ (x_3+1)(\bar{\boldsymbol{u}}(x'), 0) & \text{if } x \in \Omega_b, \\ 0 & \text{if } x \in \Omega_s, \end{cases}$$

where $(h_\varepsilon)_{\varepsilon>0}$ is a sequence in $\mathcal{C}_c^\infty(\omega)$ such that

$$h_\varepsilon \to -\frac{\lambda_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}}) \text{ in } L^2(\omega), \quad \lim_{\varepsilon\to 0} \varepsilon \|\nabla h_\varepsilon\|_{L^2(\omega;\mathbb{R}^2)} = 0. \tag{4.2}$$

Clearly, $\boldsymbol{u}_\varepsilon^* \in H^1(\Omega;\mathbb{R}^3)$ and $\boldsymbol{u}_\varepsilon^* = 0$ $\mathcal{L}^3$-a.e. in Ω_s. Using (3.3) we have that

$$\begin{aligned} J_\varepsilon(\boldsymbol{u}_\varepsilon^*, \Omega_f) = \frac{1}{2}\int_{\Omega_f} &\big[\lambda_f e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}}) + 2\mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}}) e_{\alpha\beta}(\bar{\boldsymbol{u}})\big]\,\mathrm{d}x \\ &+ \frac{1}{2\varepsilon^2}\int_{\Omega_f} \big[2\lambda_f e_{\alpha\alpha}(\bar{\boldsymbol{u}})\varepsilon^2 h_\varepsilon + \mu_f \varepsilon^4 x_3^2 |\nabla h_\varepsilon|^2\big]\,\mathrm{d}x \\ &+ \frac{1}{2\varepsilon^4}\int_{\Omega_f} (\lambda_f + 2\mu_f)\varepsilon^4 |h_\varepsilon|^2\,\mathrm{d}x, \end{aligned}$$

and according to the convergence properties (4.2), we get that

$$\begin{aligned} \lim_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon^*, \Omega_f) &= \frac{1}{2}\int_\omega \big[\lambda_f e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}}) + 2\mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}}) e_{\alpha\beta}(\bar{\boldsymbol{u}})\big]\,\mathrm{d}x' \\ &\quad - \frac{1}{2}\int_\omega \frac{2\lambda_f^2}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}})\,\mathrm{d}x' + \frac{1}{2}\int_\omega \frac{\lambda_f^2}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}})\,\mathrm{d}x' \\ &= \frac{1}{2}\int_\omega \left[\frac{2\lambda_f\mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}}) + 2\mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}}) e_{\alpha\beta}(\bar{\boldsymbol{u}})\right]\mathrm{d}x'. \end{aligned}$$

On the other hand, (3.4) yields

$$J_\varepsilon(\boldsymbol{u}_\varepsilon^*, \Omega_b) = \frac{\varepsilon^2}{2}\int_{\Omega_b} (x_3+1)^2\big[\lambda_b e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}}) + 2\mu_b e_{\alpha\beta}(\bar{\boldsymbol{u}}) e_{\alpha\beta}(\bar{\boldsymbol{u}})\big]\,\mathrm{d}x + \frac{\mu_b}{2}\int_\omega \bar{u}_\alpha \bar{u}_\alpha\,\mathrm{d}x',$$

and thus

$$\lim_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon^*, \Omega_f) = \frac{\mu_b}{2}\int_\omega |\bar{\boldsymbol{u}}|^2\,\mathrm{d}x',$$

which completes the proof of the upper bound. □

5. Transverse cracks in thin films

In this section, we assume that the body can fracture. We first only address the analysis of the film Ω_f in order to highlight the appearance of transverse cracks in the reduced model. This property is already known in the framework of nonlinear elasticity where energies depend on the deformation gradient [6, 7, 12, 14]. The difficulty here is to consider a linearly elastic material outside the crack so that the energy depends on the elastic strain.

5.1 *Compactness*

From a mathematical point of view, the natural functional setting is to consider displacement fields $\boldsymbol{u} \in SBD(\Omega_f)$. For technical reasons, we also assume that all the deformations take place in a fixed container K which is a compact subset of $\mathbb{R}^3$. Therefore, we assume that any displacement is uniformly bounded by some fixed positive constant $M > 0$.

Throughout this section, we assume that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0} \subset SBD(\Omega_f)$ is a sequence of displacements in the film such that $\|\boldsymbol{u}_\varepsilon\|_{L^\infty(\Omega_f;\mathbb{R}^3)} \leqslant M$, and

$$\sup_{\varepsilon>0} E_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_f) < \infty.$$

We establish that any admissible sequence of displacements with uniformly bounded energy converges to some limit displacement having a Kirchhoff–Love type structure.

Proposition 5.1 *Up to a subsequence, there exists* $\boldsymbol{u} \in SBD(\Omega_f) \cap L^\infty(\Omega_f;\mathbb{R}^3)$ *such that*

(i) $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ *strongly in* $L^2(\Omega_f;\mathbb{R}^3)$ *and* $\boldsymbol{u}_\varepsilon \overset{*}{\rightharpoonup} \boldsymbol{u}$ *weakly* in* $L^\infty(\Omega_f;\mathbb{R}^3)$*;*
(ii) $e(\boldsymbol{u}_\varepsilon) \rightharpoonup e(\boldsymbol{u})$ *weakly in* $L^2(\Omega_f;\mathbb{M}^{3\times3}_{\text{sym}})$*;*
(iii) $e_{\alpha 3}(\boldsymbol{u}) = e_{33}(\boldsymbol{u}) = 0$ $\mathcal{L}^3$*-a.e. in* Ω_f *and* $(\nu_{\boldsymbol{u}})_3 = 0$ $\mathcal{H}^2$*-a.e. on* $J_{\boldsymbol{u}} \cap \Omega_f$.

Proof. From the hypotheses and the definition of $E_\varepsilon(\cdot, \Omega_f)$, we have that

$$\|\boldsymbol{u}_\varepsilon\|_{L^\infty(\Omega_f;\mathbb{R}^3)} + \|e(\boldsymbol{u}_\varepsilon)\|_{L^2(\Omega_f;\mathbb{M}^{3\times3}_{\text{sym}})} + \mathcal{H}^2(J_{\boldsymbol{u}_\varepsilon} \cap \Omega_f) \leqslant C, \tag{5.1}$$

for some constant $C > 0$ independent of ε. According to the compactness theorem in SBD [10, Theorem 1.1], we deduce the existence of a subsequence (not relabeled) and a function $\boldsymbol{u} \in SBD(\Omega_f)$ such that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$, $\boldsymbol{u}_\varepsilon \overset{*}{\rightharpoonup} \boldsymbol{u}$ weakly* in $L^\infty(\Omega_f;\mathbb{R}^3)$, $e(\boldsymbol{u}_\varepsilon) \rightharpoonup e(\boldsymbol{u})$ weakly in $L^2(\Omega_f;\mathbb{M}^{3\times3}_{\text{sym}})$, and

$$\mathcal{H}^2(J_{\boldsymbol{u}} \cap \Omega_f) \leqslant \liminf_{\varepsilon\to0} \mathcal{H}^2(J_{\boldsymbol{u}_\varepsilon} \cap \Omega_f) \leqslant \liminf_{\varepsilon\to0} \int_{\Omega_f \cap J_{\boldsymbol{u}_\varepsilon}} \left| \left((\nu_{\boldsymbol{u}_\varepsilon})', \frac{1}{\varepsilon}(\nu_{\boldsymbol{u}_\varepsilon})_3 \right) \right| \, \mathrm{d}\mathcal{H}^2. \tag{5.2}$$

Using the expression of the energy in the film, we deduce that

$$\|e_{\alpha3}(\boldsymbol{u}_\varepsilon)\|_{L^2(\Omega_f)} + \int_{\Omega_f \cap J_{\boldsymbol{u}_\varepsilon}} |(\nu_{\boldsymbol{u}_\varepsilon})_3| \, \mathrm{d}\mathcal{H}^2 \leqslant C\varepsilon, \quad \|e_{33}(\boldsymbol{u}_\varepsilon)\|_{L^2(\Omega_f)} \leqslant C\varepsilon^2 \tag{5.3}$$

for some C independent of ε.

According to the lower semicontinuity of the L^2-norm with respect to weak convergence, (5.3) implies that $e_{\alpha3}(\boldsymbol{u}) = e_{33}(\boldsymbol{u}) = 0$ $\mathcal{L}^3$-a.e. in Ω_f.

Let us now study the surface integral in (5.3). For fixed $k \in \mathbb{N}$ and $\mathcal{L}^3$-a.e. $x = (x', x_3) \in \omega \times (0, 1/k) = \Omega_f^{1/k}$, we define

$$\begin{cases} (u_\varepsilon^k)_\alpha(x', x_3) = (u_\varepsilon)_\alpha(x', kx_3), \\ (u_\varepsilon^k)_3(x', x_3) = k(u_\varepsilon)_3(x', kx_3), \end{cases} \qquad \begin{cases} (u^k)_\alpha(x', x_3) = u_\alpha(x', kx_3), \\ (u^k)_3(x', x_3) = ku_3(x', kx_3), \end{cases}$$

Clearly, $\boldsymbol{u}^k$ and $\boldsymbol{u}_\varepsilon^k \in SBD(\Omega_f^{1/k})$, $\boldsymbol{u}_\varepsilon^k \to \boldsymbol{u}^k$ strongly in $L^2(\Omega_f^{1/k};\mathbb{R}^3)$, $\boldsymbol{u}_\varepsilon^k \overset{*}{\rightharpoonup} \boldsymbol{u}^k$ weakly* in $L^\infty(\Omega_f^{1/k};\mathbb{R}^3)$, $e(\boldsymbol{u}_\varepsilon^k) \rightharpoonup e(\boldsymbol{u}^k)$ weakly in $L^2(\Omega_f^{1/k};\mathbb{M}^{3\times3}_{\text{sym}})$ and

$$\mathcal{H}^2\big(J_{\boldsymbol{u}^k} \cap \Omega_f^{1/k}\big) \leqslant \liminf_{\varepsilon\to0} \mathcal{H}^2\big(J_{\boldsymbol{u}_\varepsilon^k} \cap \Omega_f^{1/k}\big).$$

Changing variable in the previous inequality yields

$$\int_{\Omega_f \cap J_{\boldsymbol{u}}} \left| \left(\frac{1}{k}(\nu_{\boldsymbol{u}})', (\nu_{\boldsymbol{u}})_3 \right) \right| \mathrm{d}\mathcal{H}^2 \leqslant \liminf_{\varepsilon \to 0} \int_{\Omega_f \cap J_{\boldsymbol{u}_\varepsilon}} \left| \left(\frac{1}{k}(\nu_{\boldsymbol{u}_\varepsilon})', (\nu_{\boldsymbol{u}_\varepsilon})_3 \right) \right| \mathrm{d}\mathcal{H}^2$$

or still

$$\int_{\Omega_f \cap J_{\boldsymbol{u}}} |(\nu_{\boldsymbol{u}})_3| \, \mathrm{d}\mathcal{H}^2 \leqslant \liminf_{\varepsilon \to 0} \int_{\Omega_f \cap J_{\boldsymbol{u}_\varepsilon}} |(\nu_{\boldsymbol{u}_\varepsilon})_3| \, \mathrm{d}\mathcal{H}^2 + \frac{\sqrt{2}}{k} \mathcal{H}^2(\Omega_f \cap J_{\boldsymbol{u}_\varepsilon}).$$

Thanks to the last bound in (5.1), and letting $k \to +\infty$, we get

$$\int_{\Omega_f \cap J_{\boldsymbol{u}}} |(\nu_{\boldsymbol{u}})_3| \, \mathrm{d}\mathcal{H}^2 \leqslant \liminf_{\varepsilon \to 0} \int_{\Omega_f \cap J_{\boldsymbol{u}_\varepsilon}} |(\nu_{\boldsymbol{u}_\varepsilon})_3| \, \mathrm{d}\mathcal{H}^2,$$

and (5.3) yields $(\nu_{\boldsymbol{u}})_3 = 0$ $\mathcal{H}^2$-a.e. on $J_{\boldsymbol{u}} \cap \Omega_f$. □

In the sequel, $\boldsymbol{u}$ denotes a displacement as in the conclusion of Proposition 5.1. Our next goal is to get a more precise structure of such displacements. Contrary to the case of linear elasticity (see [17]) or linearly elastic-perfectly plastic plates (see [21]), they in general are not of Kirchhoff–Love type (*i.e.* such that $E_{i3}\boldsymbol{u} = 0$) since we do not control the full distributional strain $E\boldsymbol{u}$. In particular, the singular part of the shearing strain $E_{\alpha 3}\boldsymbol{u}$ is given by $\frac{[\boldsymbol{u}]_3 \nu_\alpha}{2} \mathcal{H}^2 \llcorner J_{\boldsymbol{u}}$ which might not vanish. However, we shall prove below that they have the same structure in the sense that the transverse displacement u_3 only depends on the planar variable x', while the in-plane displacement (u_1, u_2) is affine with respect to the transverse variable x_3.

Proposition 5.2 *Let $\boldsymbol{u} \in SBD(\Omega_f) \cap L^\infty(\Omega_f; \mathbb{R}^3)$ be such that $e_{i3}(\boldsymbol{u}) = 0$ $\mathcal{L}^3$-a.e. in Ω_f, and $(\nu_{\boldsymbol{u}})_3 = 0$ $\mathcal{H}^2$-a.e. on $J_{\boldsymbol{u}} \cap \Omega_f$. Then the following properties hold:*

- *the function u_3 is independent of x_3 and it (is identified with a function which) belongs to $SBV(\omega) \cap L^\infty(\omega)$. In addition, its approximate gradient $\nabla u_3 = (\partial_1 u_3, \partial_2 u_3) \in SBD(\omega) \cap L^\infty(\omega; \mathbb{R}^2)$;*
- *for $\mathcal{L}^3$-a.e. $(x', x_3) \in \Omega_f$,*

$$u_\alpha(x', x_3) = \bar{u}_\alpha(x') + \left(\frac{1}{2} - x_3 \right) \partial_\alpha u_3(x'), \tag{5.4}$$

  where $\bar{u}_\alpha := \int_0^1 u_\alpha(\cdot, x_3) \, \mathrm{d}x_3$, and $\bar{\boldsymbol{u}} := (\bar{u}_1, \bar{u}_2) \in SBD(\omega) \cap L^\infty(\omega; \mathbb{R}^2)$;
- *$J_{\boldsymbol{u}} \cong (J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) \times (0, 1)$;*

Proof. Step 1. First of all, by virtue of (2.1), the distributional derivative of u_3 with respect to x_3 satisfies

$$D_3 u_3 = E_{33}\boldsymbol{u} = e_{33}(\boldsymbol{u})\mathcal{L}^3 + [\boldsymbol{u}]_3 (\nu_{\boldsymbol{u}})_3 \mathcal{H}^2 \llcorner J_{\boldsymbol{u}} = 0.$$

This implies that u_3 is independent of x_3, and that it can be identified with a function defined on ω.

Step 2. We next show that $u_3 \in SBV(\omega)$ and that formula (5.4) holds. This will be obtained thanks to a suitable mollification of $\boldsymbol{u}$. We first extend $\boldsymbol{u}$ to the whole space in the following way: since the trace of an $SBD(\Omega_f)$ function belongs to $L^1(\partial\Omega_f; \mathbb{R}^3)$ (see [8, Theorem 3.2]), according to Gagliardo's Theorem, $\boldsymbol{u}$ may be extended to $\mathbb{R}^3$ by a function, still denoted by $\boldsymbol{u}$, that is compactly supported in $\mathbb{R}^3$ and such that $\boldsymbol{u} \in W^{1,1}(\mathbb{R}^3 \setminus \Omega_f; \mathbb{R}^3)$ with $|E\boldsymbol{u}|(\partial\Omega_f) = 0$.

Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ be an even and non negative function such that $\int_{\mathbb{R}} \chi(t)\,\mathrm{d}t = 1$ and $\mathrm{Supp}\chi \subset (-1,1)$. For all $x = (x_1, x_2, x_3) = (x', x_3) \in \mathbb{R}^3$, we define $\bar\rho(x') := \chi(x_1)\chi(x_2)$ and $\rho(x) := \chi(x_1)\chi(x_2)\chi(x_3)$. We then denote by $\bar\rho_\delta(x') = \delta^{-2}\bar\rho(x'/\delta)$ a sequence of two-dimensional mollifiers, and by $\rho_\delta(x) = \delta^{-3}\rho(x/\delta)$ a sequence of three-dimensional mollifiers. Since $\boldsymbol{u}_\delta := \boldsymbol{u} * \rho_\delta \in \mathcal{C}^1(\mathbb{R}^3;\mathbb{R}^3)$ and

$$\partial_3(\boldsymbol{u}_\delta)_\alpha = 2e_{\alpha 3}(\boldsymbol{u}_\delta) - \partial_\alpha(\boldsymbol{u}_\delta)_3,$$

it follows from the fundamental theorem of calculus that for each $(x', x_3) \in \Omega_f$,

$$(\boldsymbol{u}_\delta)_\alpha(x', x_3) = (\boldsymbol{u}_\delta)_\alpha(x', 0) + 2\int_0^{x_3} e_{\alpha 3}(\boldsymbol{u}_\delta)(x', s)\,\mathrm{d}s - \int_0^{x_3} \partial_\alpha(\boldsymbol{u}_\delta)_3(x', s)\,\mathrm{d}s. \tag{5.5}$$

Let us study each of the above terms separately. The term in the left hand side of (5.5) clearly satisfies $(\boldsymbol{u}_\delta)_\alpha \to u_\alpha$ strongly in $L^2(\Omega_f)$, and thus (for a suitable subsequence)

$$(\boldsymbol{u}_\delta)_\alpha \to u_\alpha \quad \mathcal{L}^3\text{-a.e. in } \Omega_f. \tag{5.6}$$

Concerning the first term on the right-hand side of (5.5), standard properties of convolution of measures ensure that $E\boldsymbol{u}_\delta \overset{*}{\rightharpoonup} E\boldsymbol{u}$ weakly* in $\mathfrak{M}(\mathbb{R}^3;\mathbb{M}^{3\times 3}_{\mathrm{sym}})$ and $|E\boldsymbol{u}_\delta|(\mathbb{R}^3) \to |E\boldsymbol{u}|(\mathbb{R}^3)$. Therefore, since $|E\boldsymbol{u}|(\partial\Omega_f) = 0$, we deduce that $|E\boldsymbol{u}_\delta|(\Omega_f) \to |E\boldsymbol{u}|(\Omega_f)$ which implies, by the continuity property of the trace (see [8, Proposition 3.4]) that $(\boldsymbol{u}_\delta)_\alpha \to u_\alpha$ strongly in $L^1(\partial\Omega_f)$. Thus, denoting by $u_\alpha^+(\cdot, 0)$ the upper trace of u_α on $\omega \times \{0\}$, there is a subsequence such that

$$(\boldsymbol{u}_\delta)_\alpha(\cdot, 0) \to u_\alpha^+(\cdot, 0) \quad \mathcal{L}^2\text{-a.e. in } \omega. \tag{5.7}$$

Regarding the second term on the right-hand side of (5.5), we have $e_{\alpha 3}(\boldsymbol{u}_\delta) = (E_{\alpha 3}\boldsymbol{u}) * \rho_\delta$ with $E_{\alpha 3}\boldsymbol{u} = \frac{[\boldsymbol{u}]_3(\nu_{\boldsymbol{u}})_\alpha}{2}\mathcal{H}^2 \llcorner J_{\boldsymbol{u}}$, and thus

$$\begin{aligned}\mathcal{E}(x', x_3) &:= \int_0^{x_3} e_{\alpha 3}(\boldsymbol{u}_\delta)(x', s)\,\mathrm{d}s \\ &= \frac{1}{2}\int_0^{x_3}\int_{J_{\boldsymbol{u}}} \rho_\delta(x' - y', s - y_3)[\boldsymbol{u}]_3(y)(\nu_{\boldsymbol{u}})_\alpha(y)\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}s.\end{aligned}$$

Since $\boldsymbol{u} \in L^\infty(\Omega_f;\mathbb{R}^3)$ with $\|\boldsymbol{u}\|_{L^\infty(\Omega_f;\mathbb{R}^3)} \leqslant M$, then $|[\boldsymbol{u}]| \leqslant 2M$ which leads to

$$\begin{aligned}|\mathcal{E}(x', x_3)| &\leqslant M\int_0^1\int_{J_{\boldsymbol{u}}} \rho_\delta(x' - y', s - y_3)\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}s \\ &= M\int_{J_{\boldsymbol{u}}}\int_0^1 \rho_\delta(x' - y', s - y_3)\,\mathrm{d}s\,\mathrm{d}\mathcal{H}^2(y),\end{aligned}$$

where we used Fubini's Theorem in the last equality. We next denote by $Q'(x', \delta) := x' + (-\delta, \delta)^2$ the open square of $\mathbb{R}^2$ (parallel to the coordinate axis) centered at x' and of edge length 2δ. Observing that $\rho_\delta(x' - y', s - y_3) = 0$ if $y' \notin Q'(x', \delta)$ and that $\rho_\delta(x' - y', s - y_3) = \bar\rho_\delta(x' - y')\delta^{-1}\chi((s - y_3)/\delta)$ with $\int_{\mathbb{R}} \chi(t)\,\mathrm{d}t = 1$, we get that

$$\begin{aligned}|\mathcal{E}(x', x_3)| &\leqslant M\int_{J_{\boldsymbol{u}} \cap [Q'(x',\delta)\times(0,1)]} \bar\rho_\delta(x' - y')\left(\int_{\mathbb{R}} \delta^{-1}\chi((s - y_3)/\delta)\,\mathrm{d}s\right)\mathrm{d}\mathcal{H}^2(y) \\ &= M\int_{J_{\boldsymbol{u}} \cap [Q'(x',\delta)\times(0,1)]} \bar\rho_\delta(x' - y')\,\mathrm{d}\mathcal{H}^2(y).\end{aligned}$$

For any Borel set $B \subset \omega$, let us define the measure $\mu(B) := \mathcal{H}^2(J_{\boldsymbol{u}} \cap (B \times (0,1)))$ which is nothing but the push-forward of $\mathcal{H}^2 \llcorner J_{\boldsymbol{u}}$ by the orthogonal projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2 \times \{0\}$. Note that μ is concentrated on $\pi(J_{\boldsymbol{u}})$ since $\mu(\omega \setminus \pi(J_{\boldsymbol{u}})) = \mathcal{H}^2(J_{\boldsymbol{u}} \cap [(\omega \setminus \pi(J_{\boldsymbol{u}})) \times (0,1)]) = 0$. On the other hand, the generalized coarea formula (see [2, Theorem 293]) yields

$$\mathcal{L}^2(\pi(J_{\boldsymbol{u}})) \leqslant \int_{\pi(J_{\boldsymbol{u}})} \mathcal{H}^0(J_{\boldsymbol{u}} \cap \pi^{-1}(x'))\, dx' = \int_{J_{\boldsymbol{u}}} |(\nu_{\boldsymbol{u}})_3|\, \mathrm{d}\mathcal{H}^2 = 0.$$

Therefore, μ and $\mathcal{L}^2$ are mutually singular which ensures that the Radon-Nikodým derivative satisfies $\frac{d\mu}{d\mathcal{L}^2}(x') = 0$ at $\mathcal{L}^2$-a.e. $x' \in \omega$. It follows that for $\mathcal{L}^2$-a.e. $x' \in \omega$,

$$\sup_{x_3 \in (0,1)} |\mathcal{E}(x', x_3)| \leqslant M \|\chi\|^2_{L^\infty(\mathbb{R})} \frac{\mu(Q'(x',\delta))}{\delta^2} \to 0,$$

and thus, in particular,

$$\int_0^{x_3} e_{\alpha 3}(\boldsymbol{u}_\delta)(x', s)\, \mathrm{d}s \to 0 \text{ for } \mathcal{L}^3\text{-a.e. } (x', x_3) \in \Omega_f. \tag{5.8}$$

For what concerns the last term on the right-hand side of (5.5), since u_3 is independent of x_3, we infer that $(\boldsymbol{u}_\delta)_3$ is independent of x_3 as well since $(\boldsymbol{u}_\delta)_3(x) = u_3 * \bar\rho_\delta(x')$ for all $x \in \mathbb{R}^3$. Therefore,

$$\int_0^{x_3} \partial_\alpha(\boldsymbol{u}_\delta)_3(x', s)\, \mathrm{d}s = x_3 \partial_\alpha(u_3 * \bar\rho_\delta)(x'),$$

and (5.5)–(5.8) thus imply that

$$\partial_\alpha(u_3 * \bar\rho_\delta)(x') \to \frac{u_\alpha^+(x',0) - u_\alpha(x',x_3)}{x_3} := \psi_\alpha(x') \quad \text{for } \mathcal{L}^3\text{-a.e. } (x', x_3) \in \Omega_f.$$

That ψ_α only depends on x' is due to the fact that the left-hand side only depends on x'. Moreover, since $u_\alpha^+(\cdot, 0) \in L^1(\omega)$ and $u_\alpha(\cdot, x_3) \in L^2(\omega)$ for a.e. $x_3 \in (0,1)$, we deduce that $\psi_\alpha \in L^1(\omega)$. From the last formula we get that

$$u_\alpha(x', x_3) = u_\alpha^+(x', 0) - x_3 \psi_\alpha(x'), \tag{5.9}$$

which in particular implies that $D_3 u_\alpha = -\psi_\alpha \mathcal{L}^3$, and

$$D_\alpha u_3 = -D_3 u_\alpha + 2E_{\alpha 3}\boldsymbol{u} = \psi_\alpha \mathcal{L}^3 + [\boldsymbol{u}]_3 (\nu_{\boldsymbol{u}})_\alpha \mathcal{H}^2 \llcorner J_{\boldsymbol{u}}.$$

As a consequence, the distributional derivative in Ω_f of u_3 is a bounded Radon measure in Ω_f, and therefore $u_3 \in BV(\Omega_f)$. Since the singular part of the above measure is concentrated on $J_{\boldsymbol{u}}$ which is σ-finite with respect to $\mathcal{H}^2$, we deduce thanks to [2, Proposition 3.92] that $u_3 \in SBV(\Omega_f)$. Finally, since u_3 is independent of x_3, we actually infer that $u_3 \in SBV(\omega)$. In addition, by uniqueness of the Lebesgue decomposition, it follows that

$$\psi_\alpha = \partial_\alpha u_3, \quad [\boldsymbol{u}]_3 (\nu_{\boldsymbol{u}})_\alpha \mathcal{H}^2 \llcorner J_{\boldsymbol{u}} = [u_3](\nu_{u_3})_\alpha \mathcal{H}^2 \llcorner [J_{u_3} \times (0,1)]$$

so that

$$J_{u_3} \times (0,1) \stackrel{\sim}{\subset} J_{\boldsymbol{u}}. \tag{5.10}$$

Integrating relation (5.9) with respect to x_3 yields

$$\bar{u}_\alpha(x') := \int_0^1 u_\alpha(x', x_3)\,dx_3 = u_\alpha^+(x', 0) - \frac{1}{2}\partial_\alpha u_3(x') \quad \text{for } \mathcal{L}^2\text{-a.e. } x' \in \omega,$$

from where (5.4) follows.

Step 3. Let us prove that the approximate gradient of u_3, denoted by $\nabla u_3 := (\partial_1 u_3, \partial_2 u_3)$, and the averaged planar displacement $\bar{\boldsymbol{u}} := (\bar{u}_1, \bar{u}_2)$ belong to $BD(\omega)$. For any $\varphi \in \mathcal{C}_c^\infty(\omega; \mathbb{M}^{2\times 2}_{\rm sym})$, according to the integration by parts formula in BD (see [8, Theorem 3.2]), we infer that

$$-\int_\omega \partial_\beta \varphi_{\alpha\beta} \bar{u}_\alpha\,dx' = -\int_{\Omega_f} \partial_\beta \varphi_{\alpha\beta} u_\alpha\,dx = \int_{\Omega_f} \varphi_{\alpha\beta}\,dE_{\alpha\beta}\boldsymbol{u} - \int_{\partial\Omega_f} \varphi_{\alpha\beta} u_\alpha \nu_\beta\,d\mathcal{H}^2.$$

Since $\varphi = 0$ in a neighborhood of $\partial\omega \times (0,1)$ and $\nu = \pm e_3$ on $\omega \times \{0,1\}$, we get that the boundary term in the previous expression is zero. Therefore

$$-\int_\omega \partial_\beta \varphi_{\alpha\beta} \bar{u}_\alpha\,dx' = \int_{\Omega_f} \varphi_{\alpha\beta} e_{\alpha\beta}(\boldsymbol{u})\,dx + \int_{J_{\boldsymbol{u}}} \varphi_{\alpha\beta}([\boldsymbol{u}] \odot \nu_{\boldsymbol{u}})_{\alpha\beta}\,d\mathcal{H}^2 \tag{5.11}$$

which shows that $\bar{\boldsymbol{u}} \in BD(\omega)$. According to slicing properties of BD functions (see [1, Proposition 3.4]), for $\mathcal{L}^1$-a.e. $x_3 \in (0,1)$, the function $(u_1(\cdot, x_3), u_2(\cdot, x_3)) \in BD(\omega)$ so that relation (5.4) yields in turn that $\nabla u_3 \in BD(\omega)$.

Step 4. We next establish that $J_{\boldsymbol{u}} \cong (J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) \times (0,1)$. To this aim, let us define the functions $\boldsymbol{v} := (\bar{u}_1, \bar{u}_2, u_3)$ and $\boldsymbol{g} := (\partial_1 u_3, \partial_2 u_3, 0)$. Since $u_3 \in SBV(\omega)$, $\bar{\boldsymbol{u}} \in BD(\omega)$ and $\nabla u_3 \in BD(\omega)$, then clearly both $\boldsymbol{v}$, $\boldsymbol{g} \in BD(\Omega_f)$, and

$$J_{\boldsymbol{g}} = J_{\nabla u_3} \times (0,1). \tag{5.12}$$

Moreover [2, Proposition 3.92 (b)] and [1, Proposition 3.5] imply that

$$J_{\bar{\boldsymbol{u}}} \cong \left\{ x' \in \omega : \limsup_{\varrho\to 0} \frac{|E\bar{\boldsymbol{u}}|\big(B'_\varrho(x')\big)}{\varrho} > 0 \right\},\ J_{u_3} \cong \left\{ x' \in \omega : \limsup_{\varrho\to 0} \frac{|Du_3|\big(B'_\varrho(x')\big)}{\varrho} > 0 \right\},$$

and

$$J_{\boldsymbol{v}} \cong \Theta_{\boldsymbol{v}} := \left\{ x \in \Omega_f : \limsup_{\varrho\to 0} \frac{|E\boldsymbol{v}|\big(B_\varrho(x)\big)}{\varrho^2} > 0 \right\},$$

where $B'_\varrho(x')$ stands for the two-dimensional open ball of center x' and radius ϱ, while $B_\varrho(x)$ stands for the three-dimensional open ball of center x and radius ϱ. Since $\boldsymbol{v}$ is independent of x_3, then

$$J_{\boldsymbol{v}} \cong (J_{\bar{\boldsymbol{u}}} \cup J_{u_3}) \times (0,1). \tag{5.13}$$

According to (5.12) and (5.13), it is thus enough to show that $J_{\boldsymbol{u}} \cong J_{\boldsymbol{v}} \cup J_{\boldsymbol{g}}$.

Let us also define the sets

$$\Theta_{\boldsymbol{u}} := \left\{ x \in \Omega_f : \limsup_{\varrho\to 0} \frac{|E\boldsymbol{u}|\big(B_\varrho(x)\big)}{\varrho^2} > 0 \right\},$$

$$\Theta_{\boldsymbol{g}} := \left\{ x \in \Omega_f : \limsup_{\varrho\to 0} \frac{|E\boldsymbol{g}|\big(B_\varrho(x)\big)}{\varrho^2} > 0 \right\},$$

and recall that, according again to [1, Proposition 3.5], $\Theta_{\boldsymbol{u}} \cong J_{\boldsymbol{u}}$ and $\Theta_{\boldsymbol{g}} \cong J_{\boldsymbol{g}}$. Using the expression of the displacement (5.4), we have $\boldsymbol{u} = \boldsymbol{v} + (\frac{1}{2} - x_3)\boldsymbol{g}$. Since $\boldsymbol{u} \in L^\infty(\Omega;\mathbb{R}^3)$, then $\boldsymbol{v} \in L^\infty(\omega;\mathbb{R}^3)$ as well, and the previous relation yields $\boldsymbol{g} \in L^\infty(\omega;\mathbb{R}^3)$ with

$$\lim_{\varrho\to 0} \frac{1}{\varrho^2} \int_{B_\varrho(x)} |\boldsymbol{g}|\,\mathrm{d}y = 0 \quad \text{for all } x \in \Omega_f.$$

Consequently since $E\boldsymbol{u} = E\boldsymbol{v} + (\frac{1}{2} - x_3)E\boldsymbol{g} - e_3 \odot \boldsymbol{g}$, we deduce that $\Omega_f \setminus (\Theta_{\boldsymbol{v}} \cup \Theta_{\boldsymbol{g}}) \subset \Omega_f \setminus \Theta_{\boldsymbol{u}}$ *i.e.* $\Theta_{\boldsymbol{u}} \subset \Theta_{\boldsymbol{v}} \cup \Theta_{\boldsymbol{g}}$ and

$$J_{\boldsymbol{u}} \,\tilde{\subset}\, J_{\boldsymbol{v}} \cup J_{\boldsymbol{g}}. \tag{5.14}$$

We now prove the converse inclusion. From the relations $\boldsymbol{v} = \boldsymbol{u} + (x_3 - \frac{1}{2})\boldsymbol{g}$ and $(\frac{1}{2} - x_3)\boldsymbol{g} = \boldsymbol{u} - \boldsymbol{v}$, and the fact that $\boldsymbol{g}$ is independent of x_3, we similarly obtain that $\Theta_{\boldsymbol{v}} \subset \Theta_{\boldsymbol{u}} \cup \Theta_{\boldsymbol{g}}$ and $\Theta_{\boldsymbol{g}} \subset \Theta_{\boldsymbol{u}} \cup \Theta_{\boldsymbol{v}}$ which imply that

$$J_{\boldsymbol{v}} \setminus J_{\boldsymbol{g}} \,\tilde{\subset}\, J_{\boldsymbol{u}}, \quad J_{\boldsymbol{g}} \setminus J_{\boldsymbol{v}} \,\tilde{\subset}\, J_{\boldsymbol{u}}. \tag{5.15}$$

It thus remains to prove that

$$J_{\boldsymbol{v}} \cap J_{\boldsymbol{g}} \,\tilde{\subset}\, J_{\boldsymbol{u}}. \tag{5.16}$$

According to (5.10), (5.12) and (5.13), we get

$$(J_{\boldsymbol{v}} \cap J_{\boldsymbol{g}}) \setminus J_{\boldsymbol{u}} \cong ([(J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}) \times (0,1)] \setminus J_{\boldsymbol{u}} \,\tilde{\subset}\, [(J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}) \setminus S_{u_3}] \times (0,1),$$

where we used that, since $u_3 \in SBV(\omega)$, then $J_{u_3} \cong S_{u_3}$. Assume by contradiction that

$$\mathcal{H}^2((J_{\boldsymbol{v}} \cap J_{\boldsymbol{g}}) \setminus J_{\boldsymbol{u}}) > 0, \tag{5.17}$$

then there is some $x = (x', x_3) \in (J_{\boldsymbol{v}} \cap J_{\boldsymbol{g}}) \setminus J_{\boldsymbol{u}}$ with $x' \in (J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}) \setminus S_{u_3}$ such that $\nu_{\bar{\boldsymbol{u}}}(x') = \pm\nu_{\nabla u_3}(x')$. Let us assume without loss of generality that $\nu_{\bar{\boldsymbol{u}}}(x') = \nu_{\nabla u_3}(x') =: \nu(x')$, the other case can be dealt with similarly. Since x' is a Lebesgue point of u_3, then the one-sided Lebesgue limits of u_3 at x' in the direction $\nu(x')$ are equal and coincide with its approximate limit. On the other hand, since $x' \in J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}$, then the functions $\bar{\boldsymbol{u}}$ and ∇u_3 admit one-sided Lebesgue limits at x' in the direction $\nu(x')$. Next, from the expression (5.4) of the displacement, we deduce that for all $\alpha \in \{1, 2\}$, the functions u_α admit as well one-sided Lebesgue limits at x in the direction $(\nu(x'), 0)$. Gathering all previous informations, we get that the full displacement $\boldsymbol{u}$ admits one-sided Lebesgue limits at x in the direction $(\nu(x'), 0)$. Using the fact that $x \notin J_{\boldsymbol{u}}$, we infer that necessarily $[\boldsymbol{u}](x) = 0$, and thus, using again (5.4) yields

$$[\bar{u}_\alpha](x') + \Big(\frac{1}{2} - x_3\Big)[\partial_\alpha u_3](x') = 0 \quad \text{for all } \alpha \in \{1, 2\}. \tag{5.18}$$

We observe that, by (5.12) and (5.13), the sets $J_{\boldsymbol{v}}$ and $J_{\boldsymbol{g}}$ are invariant in the transverse direction, and consequently $(x', y_3) \in J_{\boldsymbol{v}} \cap J_{\boldsymbol{g}}$ for any $y_3 \in (0, 1)$. Therefore if $(x', y_3) \notin J_{\boldsymbol{u}}$ for some $y_3 \neq x_3$, then reproducing the same argument as above implies that

$$[\bar{u}_\alpha](x') + \Big(\frac{1}{2} - y_3\Big)[\partial_\alpha u_3](x') = 0 \quad \text{for all } \alpha \in \{1, 2\}.$$

Subtracting the previous relation to (5.18) yields $[\bar{u}_\alpha](x') = [\partial_\alpha u_3](x') = 0$ for all $\alpha \in \{1, 2\}$, which is against the fact that $x' \in J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}$. As a consequence, $(x', y_3) \in J_{\boldsymbol{u}}$ for all $y_3 \in (0, 1)$

with $y_3 \neq x_3$. In addition, since $x' \in J_{\nabla u_3}$, there is some $\alpha \in \{1, 2\}$ such that $[\partial_\alpha u_3](x') \neq 0$, and x_3 is therefore given by

$$x_3 = \frac{1}{2} + \frac{[\bar{u}_\alpha](x')}{[\partial_\alpha u_3](x')}.$$

Consequently, we have proved that

$$(J_{\boldsymbol{v}} \cap J_{\boldsymbol{g}}) \setminus J_{\boldsymbol{u}} \tilde{\subset} \bigcup_{\alpha=1}^{2} \left\{(x', x_3) : x' \in J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3},\ [\partial_\alpha u_3](x') \neq 0,\ x_3 = \frac{1}{2} + \frac{[\bar{u}_\alpha](x')}{[\partial_\alpha u_3](x')}\right\} =: A.$$

The set A is Borel measurable, and, for each $x' \in J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}$, its transverse section passing through x', denoted by $A^{x'} := \{x_3 \in (0, 1) : (x', x_3) \in A\}$ is reduced to at most two points. Since the set $J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}$ is countably $\mathcal{H}^1$-rectifiable, [22, Theorem 3.2.23] ensures that $\mathcal{H}^2 \llcorner ((J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}) \times (0, 1)) = (\mathcal{H}^1 \llcorner (J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3})) \otimes (\mathcal{L}^1 \llcorner (0, 1))$, and Fubini's Theorem yields

$$\mathcal{H}^2(A) = \int_{J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}} \mathcal{L}^1(A^{x'})\, d\mathcal{H}^1(x') = 0,$$

which is against (5.17), and therefore completes the proof of (5.16). Gathering (5.14) – (5.16) leads to $J_{\boldsymbol{u}} \cong J_{\boldsymbol{v}} \cup J_{\boldsymbol{g}}$, and thus $J_{\boldsymbol{u}} \cong (J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) \times (0, 1)$.

Step 5. We complete the proof of the proposition by establishing that $\bar{\boldsymbol{u}}$ and ∇u_3 are actually $SBD(\omega)$ functions. Indeed, since we know that $J_{\boldsymbol{u}} \cong \Gamma \times (0, 1)$ for some countably $\mathcal{H}^1$-rectifiable set $\Gamma \subset \omega$, equation (5.11) reads

$$-\int_\omega \partial_\beta \varphi_{\alpha\beta} \bar{u}_\alpha\, dx' = \int_\omega \varphi_{\alpha\beta} \left(\int_0^1 e_{\alpha\beta}(\boldsymbol{u})\, dx_3\right) dx' + \int_\Gamma \varphi_{\alpha\beta} \left(\int_0^1 ([\boldsymbol{u}] \odot \nu_\Gamma)_{\alpha\beta}\, dx_3\right) d\mathcal{H}^1,$$

which implies that $e_{\alpha\beta}(\bar{\boldsymbol{u}}) = \int_0^1 e_{\alpha\beta}(\boldsymbol{u})(\cdot, x_3)\, dx_3$ by uniqueness of the Lebesgue decomposition, and that the singular part of $E\bar{\boldsymbol{u}}$ is concentrated on a countably $\mathcal{H}^1$-rectifiable set. It follows from [1, Proposition 4.7] that $\bar{\boldsymbol{u}} \in SBD(\omega)$ and the same can be said, therefore, first for ∇u_3 and then for $(u_1^+(\cdot, 0), u_2^+(\cdot, 0))$. □

Propositions 5.1 and 5.2 suggest one to define the limiting space of all kinematically admissible displacements by

$$\begin{aligned}
\mathcal{A}_{KL} := \Big\{ & \boldsymbol{u} \in SBD(\Omega_f) : \|\boldsymbol{u}\|_{L^\infty(\Omega_f;\mathbb{R}^3)} \leqslant M,\ u_3 \in SBV(\omega) \cap L^\infty(\omega) \\
& \text{with } \nabla u_3 \in SBD(\omega) \cap L^\infty(\omega; \mathbb{R}^2), \\
& u_\alpha(x', x_3) = \bar{u}_\alpha(x') + \Big(\frac{1}{2} - x_3\Big)\partial_\alpha u_3(x') \text{ for } \mathcal{L}^3\text{-a.e. } x = (x', x_3) \in \Omega_f, \qquad (5.19) \\
& \text{where } \bar{\boldsymbol{u}} := (\bar{u}_1, \bar{u}_2) \in SBD(\omega) \cap L^\infty(\omega; \mathbb{R}^2), \\
& \text{and } J_{\boldsymbol{u}} \cong (J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) \times (0, 1) \Big\}. \qquad (5.20)
\end{aligned}$$

5.2 *Γ-limit in the film*

For each $\varepsilon > 0$, let us define the functionals $\mathcal{E}^f_\varepsilon$ and $\mathcal{E}^f_0 : L^2(\Omega_f;\mathbb{R}^3) \to [0,+\infty]$ by

$$\mathcal{E}^f_\varepsilon(\boldsymbol{u}) := \begin{cases} E_\varepsilon(\boldsymbol{u},\Omega_f) & \text{if } \boldsymbol{u} \in SBD(\Omega_f) \text{ and } \|\boldsymbol{u}\|_{L^\infty(\Omega_f;\mathbb{R}^3)} \leqslant M, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\mathcal{E}^f_0(\boldsymbol{u}) := \begin{cases} \displaystyle\int_\omega \left[\frac{\lambda_f\mu_f}{\lambda_f+2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}})e_{\beta\beta}(\bar{\boldsymbol{u}}) + \mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}})e_{\alpha\beta}(\bar{\boldsymbol{u}})\right] \mathrm{d}x' \\ \displaystyle\quad + \frac{1}{12}\int_\omega \left[\frac{\lambda_f\mu_f}{\lambda_f+2\mu_f} e_{\alpha\alpha}(\nabla u_3)e_{\beta\beta}(\nabla u_3) + \mu_f e_{\alpha\beta}(\nabla u_3)e_{\alpha\beta}(\nabla u_3)\right] \mathrm{d}x' \\ \quad + \kappa_f \mathcal{H}^1(J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) & \text{if } \boldsymbol{u} \in \mathcal{A}_{KL}, \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 5.3 *The sequence of functionals $(\mathcal{E}^f_\varepsilon)_{\varepsilon>0}$ Γ-converges to $\mathcal{E}^f_0$ with respect to the strong $L^2(\Omega_f;\mathbb{R}^3)$-topology.*

Proof. Step 1. We start by deriving a lower bound inequality, *i.e.*, for any $\boldsymbol{u} \in L^2(\Omega_f;\mathbb{R}^3)$ and any sequence $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega_f;\mathbb{R}^3)$ such that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$, then

$$\liminf_{\varepsilon\to 0} \mathcal{E}^f_\varepsilon(\boldsymbol{u}_\varepsilon) \geqslant \mathcal{E}^f_0(\boldsymbol{u}).$$

If $\liminf_\varepsilon \mathcal{E}^f_\varepsilon(\boldsymbol{u}_\varepsilon) = +\infty$, the result is obvious. Otherwise, up to a subsequence, we can assume that

$$\lim_{\varepsilon\to 0} \mathcal{E}^f_\varepsilon(\boldsymbol{u}_\varepsilon) = \liminf_{\varepsilon\to 0} \mathcal{E}^f_\varepsilon(\boldsymbol{u}_\varepsilon) < \infty.$$

By virtue of the above energy bound, we can assume without loss of generality that the conclusions of Propositions 5.1 and 5.2 hold so that $\boldsymbol{u} \in \mathcal{A}_{KL}$. Using a very similar argument as that used in the proof of the lower bound in Theorem 4.1, combined with the lower semicontinuity of the surface energy established in (5.2), we obtain that

$$\liminf_{\varepsilon\to 0} E_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_f) \geqslant \int_{\Omega_f} \left[\frac{\lambda_f\mu_f}{\lambda_f+2\mu_f} e_{\alpha\alpha}(\boldsymbol{u})e_{\beta\beta}(\boldsymbol{u}) + \mu_f e_{\alpha\beta}(\boldsymbol{u})e_{\alpha\beta}(\boldsymbol{u})\right] \mathrm{d}x + \kappa_f \mathcal{H}^2(J_{\boldsymbol{u}} \cap \Omega_f).$$

According to (5.19), we get that

$$\begin{aligned} \int_{\Omega_f} e_{\alpha\beta}(\boldsymbol{u})e_{\alpha\beta}(\boldsymbol{u})\,\mathrm{d}x &= \int_{\Omega_f} \Big[e_{\alpha\beta}(\bar{\boldsymbol{u}})e_{\alpha\beta}(\bar{\boldsymbol{u}}) + 2\left(\frac{1}{2}-x_3\right) e_{\alpha\beta}(\bar{\boldsymbol{u}})e_{\alpha\beta}(\nabla u_3) \\ &\qquad + \left(\frac{1}{2}-x_3\right)^2 e_{\alpha\beta}(\nabla u_3)e_{\alpha\beta}(\nabla u_3)\Big]\,\mathrm{d}x \\ &= \int_\omega e_{\alpha\beta}(\bar{\boldsymbol{u}})e_{\alpha\beta}(\bar{\boldsymbol{u}})\,\mathrm{d}x' + \frac{1}{12}\int_\omega e_{\alpha\beta}(\nabla u_3)e_{\alpha\beta}(\nabla u_3)\,\mathrm{d}x', \end{aligned}$$

and similarly for the other term

$$\int_{\Omega_f} e_{\alpha\alpha}(\boldsymbol{u})e_{\beta\beta}(\boldsymbol{u})\,\mathrm{d}x = \int_\omega e_{\alpha\alpha}(\bar{\boldsymbol{u}})e_{\beta\beta}(\bar{\boldsymbol{u}})\,\mathrm{d}x' + \frac{1}{12}\int_\omega e_{\alpha\alpha}(\nabla u_3)e_{\beta\beta}(\nabla u_3)\,\mathrm{d}x'.$$

Therefore (5.20) yields the announced energy lower bound.

Step 2. We next derive an upper bound through the construction of a recovery sequence, *i.e.*, for every $\boldsymbol{u} \in L^2(\Omega_f; \mathbb{R}^3)$, there exists a recovery sequence $(\boldsymbol{u}_\varepsilon^*)_{\varepsilon>0} \subset L^2(\Omega_f; \mathbb{R}^3)$ such that $\boldsymbol{u}_\varepsilon^* \to \boldsymbol{u}$ strongly in $L^2(\Omega_f; \mathbb{R}^3)$, and

$$\limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon^f(\boldsymbol{u}_\varepsilon^*) \leqslant \mathcal{E}_0^f(\boldsymbol{u}).$$

If $\boldsymbol{u} \notin \mathcal{A}_{KL}$, then $\mathcal{E}_0^f(\boldsymbol{u}) = +\infty$ and the result is obvious. It therefore suffices to assume that $\boldsymbol{u} \in \mathcal{A}_{KL}$. We now define a recovery sequence $(\boldsymbol{u}_\varepsilon^*)_{\varepsilon>0}$. For $\mathcal{L}^3$-a.e. $x = (x', x_3) \in \Omega_f$ and all $\varepsilon > 0$, let

$$\boldsymbol{u}_\varepsilon^*(x', x_3) = c_\varepsilon\big(\boldsymbol{u}(x) + (0, 0, \varepsilon^2 x_3 h_\varepsilon(x'))\big),$$

where $(h_\varepsilon)_{\varepsilon>0}$ is a sequence in $\mathcal{C}_c^\infty(\omega)$ such that

$$h_\varepsilon \to -\frac{\lambda_f}{\lambda_f + 2\mu_f} \boldsymbol{e}_{\alpha\alpha}(\boldsymbol{u}) \text{ in } L^2(\omega), \quad \lim_{\varepsilon \to 0} \varepsilon \|\nabla h_\varepsilon\|_{L^2(\omega;\mathbb{R}^2)} = \lim_{\varepsilon \to 0} \varepsilon \|h_\varepsilon\|_{L^\infty(\omega)} = 0, \tag{5.21}$$

and

$$c_\varepsilon := \frac{M}{M + \varepsilon^2 \|h_\varepsilon\|_{L^\infty(\omega)}}.$$

Clearly, $\boldsymbol{u}_\varepsilon^* \in SBD(\Omega_f)$ and $\|\boldsymbol{u}_\varepsilon^*\|_{L^\infty(\Omega_f;\mathbb{R}^3)} \leqslant M$. Using (3.3) we get that

$$\begin{aligned} J_\varepsilon(\boldsymbol{u}_\varepsilon^*, \Omega_f) = {} & \frac{c_\varepsilon^2}{2} \int_{\Omega_f} \big[\lambda_f \boldsymbol{e}_{\alpha\alpha}(\boldsymbol{u}) \boldsymbol{e}_{\beta\beta}(\boldsymbol{u}) + 2\mu_f \boldsymbol{e}_{\alpha\beta}(\boldsymbol{u}) \boldsymbol{e}_{\alpha\beta}(\boldsymbol{u})\big]\, \mathrm{d}x \\ & + \frac{c_\varepsilon^2}{2\varepsilon^2} \int_{\Omega_f} \big[2\lambda_f \boldsymbol{e}_{\alpha\alpha}(\boldsymbol{u}) \varepsilon^2 h_\varepsilon + \mu_f \varepsilon^4 x_3^2 |\nabla h_\varepsilon|^2\big]\, \mathrm{d}x \\ & + \frac{c_\varepsilon^2}{2\varepsilon^4} \int_{\Omega_f} (\lambda_f + 2\mu_f) \varepsilon^4 |h_\varepsilon|^2 \, \mathrm{d}x. \end{aligned}$$

Thus, since $c_\varepsilon \to 1$ and according to the convergence properties (5.21), we get that

$$\lim_{\varepsilon \to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon^*, \Omega_f) = \frac{1}{2} \int_{\Omega_f} \left[\frac{2\lambda_f \mu_f}{\lambda_f + 2\mu_f} \boldsymbol{e}_{\alpha\alpha}(\boldsymbol{u}) \boldsymbol{e}_{\beta\beta}(\boldsymbol{u}) + 2\mu_f \boldsymbol{e}_{\alpha\beta}(\boldsymbol{u}) \boldsymbol{e}_{\alpha\beta}(\boldsymbol{u})\right] \mathrm{d}x.$$

Concerning the surface energy, since $J_{\boldsymbol{u}_\varepsilon^*} = J_{\boldsymbol{u}} \cong (J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) \times (0, 1)$ it follows that

$$\int_{\Omega_f \cap J_{\boldsymbol{u}_\varepsilon^*}} \left| \left((\boldsymbol{\nu}_{\boldsymbol{u}_\varepsilon^*})', \frac{1}{\varepsilon} (\boldsymbol{\nu}_{\boldsymbol{u}_\varepsilon^*})_3 \right) \right| \mathrm{d}\mathcal{H}^2 = \mathcal{H}^1(J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}),$$

which completes the proof of the upper bound. □

6. Multifissuration: debonding and delamination vs. transverse cracks

In this section, we consider the full model of a film Ω_f deposited on a substrate Ω_s through a bonding layer Ω_b, and we assume that both Ω_f and Ω_b can crack.

6.1 *The anti-plane case*

Following [32], it is assumed that the geometry is invariant in the direction $\boldsymbol{e}_2$, *i.e.*, $\omega = I \times \mathbb{R}$, where I is a bounded open interval, and that the admissible displacements take the form

$$\boldsymbol{u}(x) = u(x_1, x_3)\boldsymbol{e}_2.$$

In this case the elastic energy reduces to

$$\tilde{J}_\varepsilon(u) = \frac{\mu_f}{2} \int_{I\times(0,1)} (|\partial_1 u|^2 + \varepsilon^{-2}|\partial_3 u|^2)\,\mathrm{d}x_1\,\mathrm{d}x_3 + \frac{\mu_b}{2} \int_{I\times(-1,0)} (\varepsilon^2|\partial_1 u|^2 + |\partial_3 u|^2)\,\mathrm{d}x_1\,\mathrm{d}x_3,$$

and the total energy is given by

$$\tilde{E}_\varepsilon(u) := \tilde{J}_\varepsilon(u) + \kappa_f \int_{J_u\cap[I\times(0,1)]} \left|\left((\nu_u)_1, \varepsilon^{-1}(\nu_u)_3\right)\right|\,\mathrm{d}\mathcal{H}^1 + \kappa_b \int_{J_u\cap[I\times[-1,0]]} |(\varepsilon(\nu_u)_1, (\nu_u)_3)|\,\mathrm{d}\mathcal{H}^1.$$

The natural functional setting is to consider (scalar) displacements in the class

$$\tilde{\mathfrak{A}} := \{u \in SBV(I\times(-2,1)) : u = 0\ \mathfrak{L}^2\text{-a.e. in } I\times(-2,-1) \text{ and } \|u\|_{L^\infty(I\times(0,1))} \leqslant M\},$$

where $M > 0$ is an arbitrary fixed constant.

In [32], the following one-dimensional energy, defined for all $u \in SBV(I)$, was proposed as an approximation of the previous two-dimensional energy

$$\tilde{E}_0(u) := \frac{\mu_f}{2} \int_I |u'|^2\,\mathrm{d}x_1 + \frac{\mu_b}{2} \int_{I\setminus\Delta_u} |u|^2\,\mathrm{d}x_1 + \kappa_f \#(J_u) + \kappa_b \mathfrak{L}^1(\Delta_u),$$

where $\Delta_u := \{|u| > \sqrt{2\kappa_b/\mu_b}\}$ is the delamination set. An easy adaptation of the proof of [33, Theorem A.1] justifies rigorously this conjecture through the following Γ-convergence type result.

Theorem 6.1 *Let $u \in SBV(I)$, then*

- *for any sequence $(u_\varepsilon)_{\varepsilon>0} \subset \tilde{\mathfrak{A}}$ satisfying $u_\varepsilon \to u$ strongly in $L^2(I\times(0,1))$, then*

$$\tilde{E}_0(u) \leqslant \liminf_{\varepsilon\to 0} \tilde{E}_\varepsilon(u_\varepsilon);$$

- *there exists a recovery sequence $(u^*_\varepsilon)_{\varepsilon>0} \subset \tilde{\mathfrak{A}}$ such that $u^*_\varepsilon \to u$ strongly in $L^2(I\times(0,1))$, and*

$$\tilde{E}_0(u) \geqslant \liminf_{\varepsilon\to 0} \tilde{E}_\varepsilon(u^*_\varepsilon).$$

Let us observe that if u_ε is a sequence of minimizers of $\tilde{E}_\varepsilon$ (under suitable loadings), the (characteristic function of the) delamination set Δ_u is constructed as the L^1-limit of the orthogonal projection of the jump sets J_{u_ε} onto the mid-surface $\{x_3 = 0\}$. In particular, the vertical cracks in the bonding layer do not contribute to delamination.

6.2 *The general case*

We conjecture that Theorem 6.1 can be extended to the general three-dimensional vectorial case. In this situation, the space of kinematically admissible displacements is given by

$$\mathcal{A} := \left\{ \boldsymbol{u} \in SBD(\Omega) : \boldsymbol{u} = 0 \ \mathcal{L}^3\text{-a.e. on } \Omega_s, \text{ and } \|\boldsymbol{u}\|_{L^\infty(\Omega_f;\mathbb{R}^3)} \leqslant M \right\}.$$

Let us define the energy functionals $\mathcal{E}_\varepsilon$ and $\mathcal{E}_0 : L^2(\Omega;\mathbb{R}^3) \to [0, +\infty]$ by

$$\mathcal{E}_\varepsilon(\boldsymbol{u}) := \begin{cases} E_\varepsilon(\boldsymbol{u}) & \text{if } \boldsymbol{u} \in \mathcal{A}, \\ +\infty & \text{otherwise,} \end{cases} \tag{6.1}$$

and

$$\mathcal{E}_0(\boldsymbol{u}) := \begin{cases} \displaystyle\int_\omega \left[\frac{\lambda_f \mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}}) + \mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}}) e_{\alpha\beta}(\bar{\boldsymbol{u}}) \right] \mathrm{d}x' \\ \quad + \displaystyle\frac{1}{12} \int_\omega \left[\frac{\lambda_f \mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\nabla u_3) e_{\beta\beta}(\nabla u_3) + \mu_f e_{\alpha\beta}(\nabla u_3) e_{\alpha\beta}(\nabla u_3) \right] \mathrm{d}x' \\ \quad + \displaystyle\frac{\mu_b}{2} \int_{\omega \setminus \Delta} |\bar{\boldsymbol{u}}|^2 \, \mathrm{d}x' + \kappa_f \mathcal{H}^1(J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) + \kappa_b \mathcal{L}^2(\Delta) \text{ if } \boldsymbol{u} \in \mathcal{A}_{KL}, \\ +\infty \qquad \text{otherwise,} \end{cases}$$

where the delamination set is defined by

$$\Delta := \left\{ x' \in \omega : |\bar{\boldsymbol{u}}(x')| > \sqrt{\frac{2\kappa_b}{\mu_b}} \right\} \cup \{x' \in \omega : u_3 \neq 0\}. \tag{6.2}$$

We expect $\mathcal{E}_0$ to be the Γ-limit of $\mathcal{E}_\varepsilon$ as $\varepsilon \to 0$, but have been unable to prove the corresponding lower bound inequality:

Conjecture 6.2 *If $\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^3)$ and $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega;\mathbb{R}^3)$ is any sequence converging strongly to $\boldsymbol{u}$ in $L^2(\Omega_f;\mathbb{R}^3)$, then*

$$\mathcal{E}_0(\boldsymbol{u}) \leqslant \liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(\boldsymbol{u}_\varepsilon).$$

Our aim here is only to prove the Γ-lim sup inequality and to present some partial results and techniques which could be relevant in future investigations of this problem.

Proposition 6.3 *For every $\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^3)$, there exists a sequence $(\boldsymbol{u}^*_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega;\mathbb{R}^3)$ such that $\boldsymbol{u}^*_\varepsilon \to \boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$, and*

$$\mathcal{E}_0(\boldsymbol{u}) \geqslant \limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon(\boldsymbol{u}^*_\varepsilon).$$

Proof. If $\boldsymbol{u} \notin \mathcal{A}_{KL}$, then $\mathcal{E}_0(\boldsymbol{u}) = +\infty$ and there is nothing to prove. Therefore, we assume from now on that $\boldsymbol{u} \in \mathcal{A}_{KL}$.

Step 1. In order to define the recovery sequence, we start by approximating the delamination set defined in (6.2) by a sequence of sets of finite perimeter. Let $(\rho_m)_{m\in\mathbb{N}}$ be a standard sequence of mollifiers in $\mathbb{R}^2$, and set $\chi_m := \rho_m * \chi_\Delta$. We know that $\chi_m \to \chi_\Delta$ strongly in $L^1(\omega)$. Set

$$\delta_m := \sqrt{\|\chi_m - \chi_\Delta\|_{L^1(\omega)}} \to 0.$$

By the coarea formula [2, Theorem 3.40], for every $m \in \mathbb{N}$ large enough, there exists $\frac{1}{2} \leqslant t_m \leqslant 1 - \delta_m$ such that

$$\Delta_m := \{x' \in \omega : \chi_m(x') > t_m\}$$

has finite perimeter. We claim that

$$\chi_{\Delta_m} \to \chi_\Delta \text{ in } L^1(\omega). \tag{6.3}$$

Indeed,

$$\mathcal{L}^2(\Delta_m \setminus \Delta) \leqslant \frac{1}{t_m} \int_{\Delta_m \setminus \Delta} \chi_m \,\mathrm{d}x' \leqslant \frac{1}{t_m} \int_{\Delta_m \setminus \Delta} |\chi_m - \chi_\Delta| \,\mathrm{d}x' \to 0,$$

and

$$\begin{aligned}\mathcal{L}^2(\Delta \setminus \Delta_m) &\leqslant \mathcal{L}^2(\{x' \in \Delta : \chi_\Delta(x') = 1 \text{ and } \chi_m(x') \leqslant 1 - \delta_m\}) \\ &\leqslant \frac{1}{\delta_m} \int_\Delta |\chi_\Delta - \chi_m| \,\mathrm{d}x' \leqslant \delta_m \to 0,\end{aligned}$$

hence $\|\chi_\Delta - \chi_{\Delta_m}\|_{L^1(\omega)} = \mathcal{L}^2(\Delta_m \setminus \Delta) + \mathcal{L}^2(\Delta \setminus \Delta_m) \to 0$. In addition, it is possible to find a sequence $\varepsilon_m \overset{m\to\infty}{\longrightarrow} 0$ such that $\varepsilon_m \mathcal{H}^1(\partial^* \Delta_m) \overset{m\to\infty}{\longrightarrow} 0$. With a slight abuse of notation, we refer to the sequences (ε_m) and (Δ_m) simply as (ε) and (Δ_ε) and henceforth assume that

$$\lim_{\varepsilon \to 0} \varepsilon \mathcal{H}^1(\partial^* \Delta_\varepsilon) = 0. \tag{6.4}$$

As in the proof of Theorem 5.3, we consider a sequence $(h_\varepsilon)_{\varepsilon>0} \subset \mathcal{C}_c^\infty(\omega)$ satisfying (5.21).

We now define the recovery sequence by setting, for all $\varepsilon > 0$ and for $\mathcal{L}^3$-a.e. $x = (x', x_3) \in \Omega$,

$$\boldsymbol{u}_\varepsilon^*(x', x_3) = \begin{cases} c_\varepsilon\big(\boldsymbol{u}(x) + \big(0, 0, \varepsilon^2 x_3 h_\varepsilon(x')\big)\big) & \text{if } (x', x_3) \in \Omega_f, \\ c_\varepsilon(x_3 + 1)\big(\bar{\boldsymbol{u}}(x'), 0\big) & \text{if } (x', x_3) \in (\omega \setminus \Delta_\varepsilon) \times [-1, 0], \\ 0 & \text{if } (x', x_3) \in (\Delta_\varepsilon \times [-1, 0]) \cup \Omega_s, \end{cases}$$

where

$$c_\varepsilon = \frac{M}{M + \varepsilon^2 \|h_\varepsilon\|_{L^\infty(\omega)}} \to 1.$$

Since the set Δ_ε has finite perimeter in ω and $u_3 \in SBV(\omega; \mathbb{R}^2) \cap L^\infty(\omega; \mathbb{R}^2)$, then according to [1, Theorem 3.84], we have $u_3 \chi_{\omega \setminus \Delta_\varepsilon} \in SBV(\omega; \mathbb{R}^2)$. Similarly, since $\bar{\boldsymbol{u}} \in SBD(\omega) \cap L^\infty(\omega; \mathbb{R}^2)$ then $\bar{\boldsymbol{u}} \chi_{\omega \setminus \Delta_\varepsilon} \in SBD(\omega)$. Indeed, according to [16, Theorem 3] (see also [30, Theorem 3]), there exists a sequence $(\bar{\boldsymbol{u}}_k)_{k\in\mathbb{N}}$ in $SBV(\omega; \mathbb{R}^2)$ such that $\bar{\boldsymbol{u}}_k \to \bar{\boldsymbol{u}}$ strongly in $L^2(\omega; \mathbb{R}^2)$, $e(\bar{\boldsymbol{u}}_k) \to e(\bar{\boldsymbol{u}})$ strongly in $L^2(\omega; \mathbb{M}^{2\times2}_{\mathrm{sym}})$, $\mathcal{H}^1(J_{\bar{\boldsymbol{u}}_k} \setminus J_{\bar{\boldsymbol{u}}}) + \mathcal{H}^1(J_{\bar{\boldsymbol{u}}} \setminus J_{\bar{\boldsymbol{u}}_k}) \to 0$, and $\|\bar{\boldsymbol{u}}_k\|_{L^\infty(\omega;\mathbb{R}^2)} \leqslant \|\bar{\boldsymbol{u}}\|_{L^\infty(\omega;\mathbb{R}^2)}$. Using again [1, Theorem 3.84], we get that $\bar{\boldsymbol{u}}_k \chi_{\omega\setminus\Delta_\varepsilon} \in SBV(\omega; \mathbb{R}^2)$. Since this sequence satisfies the assumptions of the compactness theorem of [10] in SBD and $\bar{\boldsymbol{u}}_k \chi_{\omega\setminus\Delta_\varepsilon} \to \bar{\boldsymbol{u}} \chi_{\omega\setminus\Delta_\varepsilon}$ strongly in $L^2(\omega; \mathbb{R}^2)$, we deduce that $\bar{\boldsymbol{u}} \chi_{\omega\setminus\Delta_\varepsilon} \in SBD(\omega)$ as required.

As a consequence of the previous discussion, $\boldsymbol{u}_\varepsilon^* \in SBD(\Omega)$, $\boldsymbol{u}_\varepsilon^* = 0$ $\mathcal{L}^3$-a.e. in Ω_s and $\|\boldsymbol{u}_\varepsilon^*\|_{L^\infty(\Omega_f;\mathbb{R}^3)} \leqslant M$ which ensures that $\boldsymbol{u}_\varepsilon^* \in \mathcal{A}$. The sequence $(\boldsymbol{u}_\varepsilon^*)_{\varepsilon>0}$ is thus admissible, and clearly $\boldsymbol{u}_\varepsilon^* \to \boldsymbol{u}$ strongly in $L^2(\Omega_f; \mathbb{R}^3)$.

Step 2. Arguing as in the proof of Theorem 5.3, we get that

$$\limsup_{\varepsilon\to 0} E_\varepsilon(\boldsymbol{u}_\varepsilon^*, \Omega_f) \leqslant \mathcal{E}_0^f(\boldsymbol{u}).$$

It thus remains to compute the energy associated to this sequence in the bonding layer. First, the bulk energy in the bonding layer gives

$$J_\varepsilon(\boldsymbol{u}_\varepsilon^*, \Omega_b) = \frac{c_\varepsilon^2\varepsilon^2}{2}\int_{(\omega\setminus\Delta_\varepsilon)\times(-1,0)} (x_3+1)^2\Big[\lambda_b e_{\alpha\alpha}(\bar{\boldsymbol{u}})e_{\beta\beta}(\bar{\boldsymbol{u}}) + 2\mu_b e_{\alpha\beta}(\bar{\boldsymbol{u}})e_{\alpha\beta}(\bar{\boldsymbol{u}})\Big]\,\mathrm{d}x + \frac{c_\varepsilon^2\mu_b}{2}\int_{\omega\setminus\Delta_\varepsilon}|\bar{\boldsymbol{u}}|^2\,\mathrm{d}x',$$

and thus

$$\limsup_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon^*, \Omega_b) \leqslant \frac{\mu_b}{2}\int_{\omega\setminus\Delta}|\bar{\boldsymbol{u}}|^2\,\mathrm{d}x'.$$

Concerning the surface energy in the bonding layer, we first observe that for each $\varepsilon > 0$,

$$J_{\boldsymbol{u}_\varepsilon^*}\cap\Omega_b \subset \Big[J_{\bar{\boldsymbol{u}}}\times[-1,0]\Big]\cup\Big[\Delta_\varepsilon\times\{0\}\Big]\cup\Big[\big(\big\{(u_3,\nabla u_3)\neq 0\big\}\setminus\Delta_\varepsilon\big)\times\{0\}\Big]\cup\Big[\partial^*\Delta_\varepsilon\times[-1,0]\Big],$$

where $\partial^*\Delta_\varepsilon$ stands for the reduced boundary of Δ_ε [2, Definition 3.54]. Let us observe that $\omega\setminus\Delta\subset\{u_3=0\}\,\tilde{\subset}\,\{(u_3,\nabla u_3)=0\}$ since, by locality of the approximate gradient, $\nabla u_3 = 0$ $\mathcal{L}^2$-a.e. in $\{u_3 = 0\}$ (see [2, Proposition 3.73 (c)]). Then

$$\limsup_{\varepsilon\to 0}\int_{J_{\boldsymbol{u}_\varepsilon^*}\cap\Omega_b}\Big|\big(\varepsilon(\boldsymbol{\nu}_{\boldsymbol{u}_\varepsilon^*})', (\boldsymbol{\nu}_{\boldsymbol{u}_\varepsilon^*})_3\big)\Big|\,\mathrm{d}\mathcal{H}^2$$
$$\leqslant \limsup_{\varepsilon\to 0}\Big[\varepsilon\mathcal{H}^1(J_{\bar{\boldsymbol{u}}}) + \mathcal{L}^2(\Delta_\varepsilon) + \mathcal{L}^2\big(\big\{(u_3,\nabla u_3)\neq 0\big\}\setminus\Delta_\varepsilon\big) + \varepsilon\mathcal{H}^1(\partial^*\Delta_\varepsilon)\Big] = \mathcal{L}^2(\Delta),$$

thanks to (6.2), (6.3) and (6.4). □

6.2.1 *Partial results for the lower bound.* Let $\boldsymbol{u}\in L^2(\Omega;\mathbb{R}^3)$, and $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}\subset L^2(\Omega;\mathbb{R}^3)$ be a sequence such that $\boldsymbol{u}_\varepsilon\to\boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$. If $\liminf_\varepsilon \mathcal{E}_\varepsilon(\boldsymbol{u}_\varepsilon) = +\infty$ there is nothing to prove. Otherwise by (6.1), up to a subsequence, we can assume without loss of generality that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}\subset\mathcal{A}$, and that

$$\sup_{\varepsilon>0} E_\varepsilon(\boldsymbol{u}_\varepsilon) < +\infty. \tag{6.5}$$

As a consequence, all the compactness results in the film Ω_f established in Section 5.1 hold. In particular, Propositions 5.1 and 5.2 show that $\boldsymbol{u}\in\mathcal{A}_{KL}$, and the lower bound established in Theorem 5.3 yields the terms in $\mathcal{E}_0(\boldsymbol{u})$ corresponding to the energy in Ω_f. The main problem is to deal with the bonding layer. Following the scalar case treated in [33], it is enough to show that the energy in Ω_b is bounded from below by some functional where the delamination set is replaced by a function $\theta\in L^\infty(\omega;[0,1])$, which can be interpreted as a delamination volume fraction density. On $\{\theta = 1\}$, the film is entirely debonded from the substrate, while on $\{\theta = 0\}$ it continuously accommodates the prescribed zero displacement on the substrate exactly as in the Sobolev case (Theorem 4.1). All intermediate states are contained in the set $\{0 < \theta < 1\}$.

Proposition 6.4 *Assume there exists $\theta \in L^\infty(\omega;[0,1])$ such that $(1-\theta)u_3 = 0$ $\mathcal{L}^2$-a.e. in ω, and*

$$\frac{\mu_b}{2}\int_\omega (1-\theta)|\bar{\boldsymbol{u}}|^2\,\mathrm{d}x' + \kappa_b \int_\omega \theta\,\mathrm{d}x' \leqslant \liminf_{\varepsilon\to 0} E_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_b). \tag{6.6}$$

Then

$$\frac{\mu_b}{2}\int_{\omega\setminus\Delta} |\bar{\boldsymbol{u}}|^2\,\mathrm{d}x' + \kappa_b \mathcal{L}^2(\Delta) \leqslant \liminf_{\varepsilon\to 0} E_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_b),$$

where Δ is the delamination set defined in (6.2)*.*

Proof. By assumption, we have that

$$\int_\omega \min_{\{\eta\in[0,1]:(1-\eta)u_3(x')=0\}} \left(\frac{\mu_b}{2}(1-\eta)|\bar{\boldsymbol{u}}(x')|^2 + \kappa_b\eta\right)\mathrm{d}x' \leqslant \liminf_{\varepsilon\to 0} E_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_b).$$

The result follows by solving the pointwise minimization problem explicitly. □

The main point is to construct such a function θ. As in the scalar case [33], θ is supposed to be obtained as the $L^\infty(\omega)$-weak* limit of a sequence $(\chi_{\Delta_\varepsilon})_{\varepsilon>0}$ of suitable measurable sets $\Delta_\varepsilon \subset \omega$. However, it is unclear what is the right notion of an ε-delamination set Δ_ε in the vectorial case. In particular, the following example shows that vertical cracks in the bonding layer cannot be neglected, so it is not enough to define Δ_ε as the orthogonal projection of $J_{\boldsymbol{u}_\varepsilon}$ onto the mid-plane $\omega\times\{0\}$, as in the anti-plane and in the Sobolev case (Thm. 6.1, [33, Prop. B.2], and Thm. 4.1).

EXAMPLE 6.5 (Microstructure example) Suppose that $\omega = (0,1)^2$ and $\varepsilon = \frac{1}{2N}$ for some $N \in \mathbb{N}$. In the film, set

$$\boldsymbol{u}_\varepsilon(x) = \boldsymbol{u}(x) = (0,\ell,0) \quad \text{for all } x \in \Omega_f.$$

In Ω_b set, for each $i = 0,\dots,N-1$ and all $2i\varepsilon \leqslant x_2 \leqslant (2i+2)\varepsilon, -1 \leqslant x_3 \leqslant 0 < x_1 < 1$,

$$u_\varepsilon(x_1,x_2,x_3) = \left(0, \ell(1+x_3), \ell\varepsilon v\left(\frac{x_2-2i\varepsilon}{\varepsilon}, 1+x_3\right)\right),$$

where $v \in H^1\big((0,2)\times(0,1)\big)$ is any function such that $v(s,0) = v(s,1) = 0$ $\forall s \in [0,1]$ and

$$q := \fint_{s=0}^{2}\int_{t=0}^{1} \left((1+\partial_s v)^2 + 2\partial_t v^2\right)\mathrm{d}s\,\mathrm{d}t < 1.$$

If Δ_ε is defined as $\pi(J_{\boldsymbol{u}_\varepsilon}\cap\Omega)$, then

$$\int_{\Omega_b} \left(2\mu_b e_{\alpha 3}(\boldsymbol{u}_\varepsilon)e_{\alpha 3}(\boldsymbol{u}_\varepsilon) + \varepsilon^{-2}\mu_b e_{33}(\boldsymbol{u}_\varepsilon)e_{33}(\boldsymbol{u}_\varepsilon)\right)\mathrm{d}x + \kappa_b\mathcal{L}^2(\Delta_\varepsilon) = \frac{q\mu_b\ell^2}{2}. \tag{6.7}$$

On the other hand, if $\Delta = \{|\bar{\boldsymbol{u}}| > \sqrt{2\kappa_b/\mu_b}\}$ is the expected limit delamination set, then

$$\int_{\omega\setminus\Delta} \frac{\mu_b}{2} u_\alpha u_\alpha\,\mathrm{d}x' + \kappa_b\mathcal{L}^2(\Delta) = \begin{cases} \frac{\mu_b\ell^2}{2} & \text{if} \quad \ell \leqslant \sqrt{\frac{2\kappa_b}{\mu_b}}, \\ \kappa_b & \text{if} \quad \ell > \sqrt{\frac{2\kappa_b}{\mu_b}}. \end{cases} \tag{6.8}$$

Choosing $\ell \in \left(\sqrt{\frac{2\kappa_b}{\mu_b}}, \sqrt{\frac{2\kappa_b}{q\mu_b}}\right)$ shows that (6.8) is not always a lower bound for (6.7). □

Regardless of the notion of an ε-delamination set Δ_ε one tries to define, it is convenient to impose that it should contain the set

$$P_\varepsilon := \pi(J_{\boldsymbol{u}_\varepsilon} \cap \Omega_f),$$

where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, $\pi(x) := x'$, is the orthogonal projection onto $\mathbb{R}^2 \times \{0\}$. On the one hand, there is no loss of generality in doing this, since it converges to a Lebesgue negligible set. Indeed, according to the coarea formula (see [2, Theorem 2.93]) and the surface energy bound (6.5) in the film, we have

$$\mathcal{L}^2(P_\varepsilon) \leqslant \int_{\mathbb{R}^2} \mathcal{H}^0(J_{\boldsymbol{u}_\varepsilon} \cap \Omega_f \cap \pi^{-1}(x'))\,\mathrm{d}x' = \int_{J_{\boldsymbol{u}_\varepsilon} \cap \Omega_f} |(\nu_{\boldsymbol{u}_\varepsilon})_3|\,\mathrm{d}\mathcal{H}^2 \leqslant C\varepsilon \to 0.$$

On the other hand, excluding P_ε enables one to slightly improve the convergences in the film, as in the following lemma which proves the convergence of the planar gradient of the anti-plane displacement.

It will be assumed henceforth that $u_\varepsilon \in SBV^2(\Omega;\mathbb{R}^3)$ and that $J_{\boldsymbol{u}_\varepsilon}$ is closed in Ω and contained in a finite union of closed connected pieces of $\mathcal{C}^1$ hypersurfaces. In doing this no generality is lost, thanks to the density result in SBD of [16, Thm. 1]. In particular, we have that $\boldsymbol{u}_\varepsilon \in H^1((\omega \setminus P_\varepsilon) \times (0,1);\mathbb{R}^3)$.

Lemma 6.6 *Let $(\Delta_\varepsilon)_{\varepsilon>0}$ be a sequence of closed sets be such that $P_\varepsilon \subset \Delta_\varepsilon$ for each $\varepsilon > 0$. Assume that there exists a function $\theta \in L^\infty(\omega;[0,1])$ such that $\chi_{\Delta_\varepsilon} \overset{*}{\rightharpoonup} \theta$ weakly* in $L^\infty(\omega)$, and $(1-\theta)u_3 = 0$ $\mathcal{L}^2$-a.e. in ω. Then*

$$\chi_{\omega\setminus\Delta_\varepsilon}\partial_\alpha(u_\varepsilon)_3 \overset{*}{\rightharpoonup} 0 \quad \textit{weakly* in } L^2(\omega; H^{-1}(0,1)).$$

Proof. First note that for $\mathcal{L}^2$-a.e. $x' \notin P_\varepsilon$ and $\mathcal{L}^1$-a.e. $x_3 \in (0,1)$

$$\begin{aligned}\zeta_\alpha^\varepsilon(x) &:= \int_0^{x_3} \partial_\alpha(u_\varepsilon)_3(x',s)\,\mathrm{d}s + (u_\varepsilon)_\alpha(x) - (u_\varepsilon)_\alpha^+(x',0)\\ &= \int_0^{x_3} \left[\partial_\alpha(u_\varepsilon)_3(x',s) + \partial_3(u_\varepsilon)_\alpha(x',s)\right]\mathrm{d}s = 2\int_0^{x_3} e_{\alpha 3}(\boldsymbol{u}_\varepsilon)(x',s)\,\mathrm{d}s.\end{aligned}$$

Thanks to the bulk energy bound (6.5) in the film (see also (5.3)), we have that

$$\|\zeta_\alpha^\varepsilon\|_{L^2((\omega\setminus\Delta_\varepsilon)\times(0,1))} \leqslant 2\|e_{\alpha 3}(\boldsymbol{u}_\varepsilon)\|_{L^2(\Omega_f)} \leqslant C\varepsilon \to 0. \tag{6.9}$$

Integrating (6.9) we obtain that also $\|\bar{\zeta}_\alpha^\varepsilon\|_{L^2(\omega\setminus\Delta_\varepsilon)} \to 0$, where

$$\bar{\zeta}_\alpha^\varepsilon(x') := \int_0^1 \zeta_\alpha^\varepsilon(x',x_3)\,\mathrm{d}x_3 = \int_0^1\int_0^{x_3} \partial_\alpha(u_\varepsilon)_3(x',s)\,\mathrm{d}s\,\mathrm{d}x_3 + (\bar{u}_\varepsilon)_\alpha(x') - (u_\varepsilon)_\alpha^+(x',0)$$

and $(\bar{u}_\varepsilon)_\alpha(x') := \int_0^1 (u_\varepsilon)_\alpha(x',x_3)\,\mathrm{d}x_3$. As a consequence,

$$\begin{aligned}(u_\varepsilon)_\alpha(x) &= (u_\varepsilon)_\alpha^+(x',0) - \int_0^{x_3} \partial_\alpha(u_\varepsilon)_3(x',s)\,\mathrm{d}s + \zeta_\alpha^\varepsilon(x)\\ &= (\bar{u}_\varepsilon)_\alpha(x') + \int_0^1\int_0^{x_3} \partial_\alpha(u_\varepsilon)_3(x',s)\,\mathrm{d}s\,\mathrm{d}x_3 - \int_0^{x_3} \partial_\alpha(u_\varepsilon)_3(x',s)\,\mathrm{d}s + \eta_\alpha^\varepsilon(x),\end{aligned} \tag{6.10}$$

where $\|\eta_\alpha^\varepsilon\|_{L^2((\omega\setminus\Delta_\varepsilon)\times(0,1))} \to 0$.

On the other hand, for $\mathfrak{L}^3$-a.e. $x \in \Omega_f$, let us define the sequences

$$g_\alpha^\varepsilon(x', x_3) := \chi_{\omega\setminus\Delta_\varepsilon}(x') \int_0^{x_3} \partial_\alpha (u_\varepsilon)_3(x', s)\,\mathrm{d}s,$$

$$\bar{g}_\alpha^\varepsilon(x') := \chi_{\omega\setminus\Delta_\varepsilon}(x') \int_0^1 \int_0^{x_3} \partial_\alpha (u_\varepsilon)_3(x', s)\,\mathrm{d}s\,\mathrm{d}x_3.$$

From (6.9) and the *a priori* bound $\|\boldsymbol{u}_\varepsilon\|_{L^\infty(\Omega_f;\mathbb{R}^3)} \leqslant M$, we get $\|g_\alpha^\varepsilon\|_{L^2(\Omega_f)} \leqslant C$ for some constant $C > 0$ independent of ε. Therefore, up to a subsequence, $g_\alpha^\varepsilon \rightharpoonup g_\alpha$ weakly in $L^2(\Omega_f)$ for some $g_\alpha \in L^2(\Omega_f)$. In addition, $\bar{g}_\alpha^\varepsilon \rightharpoonup \bar{g}_\alpha$ weakly in $L^2(\omega)$, where $\bar{g}_\alpha(x') := \int_0^1 g_\alpha(x', x_3)\,\mathrm{d}x_3$.

Multiplying (6.10) by $\chi_{\omega\setminus\Delta_\varepsilon}$ leads to

$$(u_\varepsilon)_\alpha(x)\chi_{\omega\setminus\Delta_\varepsilon}(x') = (\bar{u}_\varepsilon)_\alpha(x')\chi_{\omega\setminus\Delta_\varepsilon}(x') + \bar{g}_\alpha^\varepsilon(x') - g_\alpha^\varepsilon(x) + \tilde{\eta}_\alpha^\varepsilon(x),$$

where $\|\tilde{\eta}_\alpha^\varepsilon\|_{L^2(\Omega_f)} \to 0$. Passing to the limit as $\varepsilon \to 0$ finally yields

$$\big(1 - \theta(x')\big)\big(u_\alpha(x) - \bar{u}_\alpha(x')\big) = \bar{g}_\alpha(x') - g_\alpha(x),$$

and according to the structure (5.19) of planar displacements, we deduce that

$$\left(\frac{1}{2} - x_3\right)\big(1 - \theta(x')\big)\partial_\alpha u_3(x') = \bar{g}_\alpha(x') - g_\alpha(x).$$

Since by assumption $u_3 = 0$ $\mathfrak{L}^2$-a.e. in $\{\theta < 1\}$, we get by locality of approximate gradients of SBV functions (see [2, Proposition 3.73 (c)]), that $\nabla u_3 = 0$ $\mathfrak{L}^2$-a.e. in $\{\theta < 1\}$, hence $g_\alpha(x) = \bar{g}_\alpha(x')$. As a consequence, $\chi_{\omega\setminus\Delta_\varepsilon}\partial_\alpha(u_\varepsilon)_3 = D_3 g_\alpha^\varepsilon \overset{*}{\rightharpoonup} D_3 g_\alpha = 0$ weakly* in $L^2(\omega; H^{-1}(0,1))$. □

An alternative to the definition of Δ_ε as the orthogonal projection of $J_{\boldsymbol{u}_\varepsilon}$ onto $\omega \times \{0\}$ is to consider its projection along certain almost-vertical oblique directions. Define the unit vectors

$$\boldsymbol{\xi}^\pm = \frac{1}{\sqrt{2}}(\pm 1, 0, 1), \quad \boldsymbol{\eta}^\pm = \frac{1}{\sqrt{2}}(0, \pm 1, 1),$$

and their rescaled versions

$$\boldsymbol{\xi}_\varepsilon^\pm := \frac{1}{\sqrt{2}}\left(\pm 1, 0, \varepsilon^{-1}\right), \quad \boldsymbol{\eta}_\varepsilon^\pm := \frac{1}{\sqrt{2}}\left(0, \pm 1, \varepsilon^{-1}\right).$$

Denote by $\pi_{\boldsymbol{\xi}_\varepsilon^\pm}$ (resp. $\pi_{\boldsymbol{\eta}_\varepsilon^\pm}$) : $\mathbb{R}^3 \to \mathbb{R}^2$ the projection onto $\{x_3 = 0\}$ parallel to the vector $\boldsymbol{\xi}_\varepsilon^\pm$ (resp. $\boldsymbol{\eta}_\varepsilon^\pm$), *i.e.*, for $x := (x', 0) + t\boldsymbol{\xi}_\varepsilon^\pm$ (resp. $x := (x', 0) + t\boldsymbol{\eta}_\varepsilon^\pm$), then $\pi_{\boldsymbol{\xi}_\varepsilon^\pm}(x) := x'$ (resp. $\pi_{\boldsymbol{\eta}_\varepsilon^\pm}(x) := x'$). Finally, consider the set

$$\Delta_\varepsilon := \pi_{\boldsymbol{\xi}_\varepsilon^+}\Big(J_{\boldsymbol{u}_\varepsilon} \cap \big(\omega_\varepsilon \times (-2, 1)\big)\Big) \cup \pi_{\boldsymbol{\eta}_\varepsilon^+}\Big(J_{\boldsymbol{u}_\varepsilon} \cap \big(\omega_\varepsilon \times (-2, 1)\big)\Big)$$
$$\cup\ \pi_{\boldsymbol{\xi}_\varepsilon^-}\Big(J_{\boldsymbol{u}_\varepsilon} \cap \big(\omega_\varepsilon \times (-2, 1)\big)\Big) \cup \pi_{\boldsymbol{\eta}_\varepsilon^-}\Big(J_{\boldsymbol{u}_\varepsilon} \cap \big(\omega_\varepsilon \times (-2, 1)\big)\Big) \cup P_\varepsilon.$$

where $\omega_\varepsilon := \{x' \in \omega : \mathrm{dist}(x', \partial\omega) > 2\varepsilon\}$. Up to a subsequence, it can be assumed that

$$\chi_{\Delta_\varepsilon} \overset{*}{\rightharpoonup} \theta \quad \text{weakly* in } L^\infty(\omega) \quad \text{for some } \theta \in L^\infty(\omega; [0,1]).$$

Using the decomposition

$$\begin{aligned}\left|\left(\varepsilon(\nu_{\boldsymbol{u}_\varepsilon})', (\nu_{\boldsymbol{u}_\varepsilon})_3\right)\right|^2 &= \frac{1}{2}|\varepsilon(\nu_{\boldsymbol{u}_\varepsilon})_1 + (\nu_{\boldsymbol{u}_\varepsilon})_3|^2 + \frac{1}{2}|\varepsilon(\nu_{\boldsymbol{u}_\varepsilon})_1 - (\nu_{\boldsymbol{u}_\varepsilon})_3|^2 + \varepsilon^2|(\nu_{\boldsymbol{u}_\varepsilon})_2|^2 \\ &= \frac{1}{2}|\varepsilon(\nu_{\boldsymbol{u}_\varepsilon})_2 + (\nu_{\boldsymbol{u}_\varepsilon})_3|^2 + \frac{1}{2}|\varepsilon(\nu_{\boldsymbol{u}_\varepsilon})_2 - (\nu_{\boldsymbol{u}_\varepsilon})_3|^2 + \varepsilon^2|(\nu_{\boldsymbol{u}_\varepsilon})_1|^2,\end{aligned}$$

it is possible to prove that

$$\liminf_{\varepsilon\to 0} \int_{J_{\boldsymbol{u}_\varepsilon}\cap\Omega_b} \left|\left(\varepsilon(\nu_{\boldsymbol{u}_\varepsilon})', (\nu_{\boldsymbol{u}_\varepsilon})_3\right)\right| \,\mathrm{d}\mathcal{H}^2 \geqslant \frac{1}{8}\int_\omega \theta \,\mathrm{d}x',$$

which shows that $\|\theta\|_{L^1(\omega)}$ is controlled (up to a multiplicative constant) by the fracture energy in the bonding layer. The constant $1/8$, however, is not optimal, since in order to obtain (6.6) that prefactor should not be present. In most situations (*e.g.*, if the sets $\pi(J_{\boldsymbol{u}_\varepsilon} \cap \Omega_b)$ have uniformly bounded perimeters) it should be possible to obtain the optimal lower bound, but there are pathological cases (such as the microstructure Example 6.5) where $\int_\omega \theta \,\mathrm{d}x'$ is larger than the fracture energy on the left-hand side (because each vertical crack is counted twice in Δ_ε, which is defined as the union of all the oblique projections).

Be it as it may, by including in Δ_ε the oblique projections of the cracks inside the bonding layer, one is able to obtain an optimal estimate for the elastic energy required by the body to accommodate the strain mismatch between the deformations in the film and in the rigid substrate. Before proving this final estimate, we need two preliminary technical results concerning sections of BD-functions along the oblique directions defined above. For $\mathcal{L}^2$-a.e. $x' \in \omega_\varepsilon$ and $\mathcal{L}^1$-a.e. $t \in (-2\sqrt{2}\varepsilon, \sqrt{2}\varepsilon)$, define the functions

$$(\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\xi}^\pm_\varepsilon}(t) := \boldsymbol{u}_\varepsilon\Big((x',0) + t\boldsymbol{\xi}^\pm_\varepsilon\Big)\cdot\boldsymbol{\xi}^\pm_\varepsilon, \quad (\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\eta}^\pm_\varepsilon}(t) := \boldsymbol{u}_\varepsilon\Big((x',0) + t\boldsymbol{\eta}^\pm_\varepsilon\Big)\cdot\boldsymbol{\eta}^\pm_\varepsilon.$$

Lemma 6.7 *For $\mathcal{L}^2$-a.e. $x' \in \omega_\varepsilon \setminus \Delta_\varepsilon$, we have*

$$(\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\xi}^\pm_\varepsilon} \in H^1(-\sqrt{2}\varepsilon, \sqrt{2}\varepsilon) \text{ and } (\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\eta}^\pm_\varepsilon} \in H^1(-\sqrt{2}\varepsilon, \sqrt{2}\varepsilon),$$

with $(\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\xi}^\pm_\varepsilon}(-\sqrt{2}\varepsilon) = (\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\eta}^\pm_\varepsilon}(-\sqrt{2}\varepsilon) = 0$, and

$$x_3 \mapsto (u_\varepsilon)_3(x', x_3) \in H^1(0,1).$$

Proof. Let us denote by

$$\Pi_{\boldsymbol{\xi}^\pm_\varepsilon} := \{\zeta \in \mathbb{R}^3 : \ \zeta\cdot\boldsymbol{\xi}^\pm_\varepsilon = 0\}$$

the plane orthogonal to $\boldsymbol{\xi}^\pm_\varepsilon$ passing through the origin, and, for $y \in \Pi_{\boldsymbol{\xi}^\pm_\varepsilon}$, we define

$$\Omega^y_{\boldsymbol{\xi}^\pm_\varepsilon} := \{t \in \mathbb{R} : y + t\boldsymbol{\xi}^\pm_\varepsilon \in \Omega_f\}.$$

According to slicing properties of functions of bounded deformations (see [1, Theorem 4.5]), we know that for $\mathcal{H}^2$-a.e. $y \in \Pi_{\boldsymbol{\xi}_\varepsilon^\pm}$, the function

$$t \mapsto \boldsymbol{u}_\varepsilon\big(y + t\boldsymbol{\xi}_\varepsilon^\pm\big) \cdot \boldsymbol{\xi}_\varepsilon^\pm \text{ belongs to } SBV^2\big(\Omega^y_{\boldsymbol{\xi}_\varepsilon^\pm}\big),$$

and its jump set is contained in

$$\big\{t \in \Omega^y_{\boldsymbol{\xi}_\varepsilon^\pm} : y + t\boldsymbol{\xi}_\varepsilon^\pm \in J_{\boldsymbol{u}_\varepsilon}\big\}.$$

Let us denote by $N_{\boldsymbol{\xi}_\varepsilon^\pm} \subset \Pi_{\boldsymbol{\xi}_\varepsilon^\pm}$ the exceptional set of zero $\mathcal{H}^2$ measure on which the previous properties fail. Since $\pi_{\boldsymbol{\xi}_\varepsilon^\pm}$ are Lipschitz functions, it follows that the sets $Z_{\boldsymbol{\xi}_\varepsilon^\pm} := \pi_{\boldsymbol{\xi}_\varepsilon^\pm}(N_{\boldsymbol{\xi}_\varepsilon^\pm}) \subset \omega$ are $\mathcal{L}^2$-negligible as well. Consequently, for all $x' \in \omega_\varepsilon \setminus Z_{\boldsymbol{\xi}_\varepsilon^\pm}$ (and thus for $\mathcal{L}^2$-a.e. $x' \in \omega_\varepsilon$), we have that

$$(\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\xi}_\varepsilon^\pm} \in SBV^2(-2\sqrt{2}\varepsilon, \sqrt{2}\varepsilon),$$

and its jump set is contained in

$$\big\{t \in (-2\sqrt{2}\varepsilon, \sqrt{2}\varepsilon) : (x', 0) + t\boldsymbol{\xi}_\varepsilon^\pm \in J_{\boldsymbol{u}_\varepsilon}\big\}.$$

By definition of the set Δ_ε, if $x' \in \omega_\varepsilon \setminus \Delta_\varepsilon$ then $(x', 0) + t\boldsymbol{\xi}_\varepsilon^\pm \notin J_{\boldsymbol{u}_\varepsilon} \cap [\omega_\varepsilon \times (-2, 1)]$ for all $t \in (-2\sqrt{2}\varepsilon, \sqrt{2}\varepsilon)$, and therefore $(\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\xi}_\varepsilon^\pm} \in H^1(-2\sqrt{2}\varepsilon, \sqrt{2}\varepsilon)$ for $\mathcal{L}^2$-a.e. $x' \in \omega_\varepsilon \setminus \Delta_\varepsilon$. In addition since $(\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\xi}_\varepsilon^\pm} = 0$ $\mathcal{L}^1$-a.e. in $(-2\sqrt{2}\varepsilon, -\sqrt{2}\varepsilon)$, it follows that $(\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\xi}_\varepsilon^\pm}(-\sqrt{2}\varepsilon) = 0$. The statement concerning the vectors $\boldsymbol{\eta}_\varepsilon^\pm$ can be proved in an analogous way.

According again to slicing properties of functions of bounded deformations, we have that for $\mathcal{L}^2$-a.e. $x' \in \omega$, the function $x_3 \mapsto (u_\varepsilon)_3(x', x_3)$ belongs to $SBV^2(0, 1)$, and its jump set is contained in $\{x_3 \in (0, 1) : (x', x_3) \in J_{\boldsymbol{u}_\varepsilon}\}$. As a consequence, for $\mathcal{L}^2$-a.e. $x \in \omega \setminus \Delta_\varepsilon$, the function $x_3 \mapsto (u_\varepsilon)_3(x', x_3)$ belongs to $H^1(0, 1)$. □

The following technical result will be useful in the argument leading to a partial bulk energy lower bound.

Lemma 6.8 *Let $\vartheta \in [0, 2\pi)$, $p := \cos\vartheta$, $q := \sin\vartheta$, and define the unit vectors*

$$\boldsymbol{\xi}^\pm := \frac{1}{\sqrt{2}}(\pm p, \pm q, 1), \quad \boldsymbol{\eta}^\pm := \frac{1}{\sqrt{2}}(\mp q, \pm p, 1).$$

For any matrix $\boldsymbol{A} = (a_{ij})_{1\leqslant i,j\leqslant 3} \in \mathbb{M}^{3\times 3}_{sym}$, we have the decomposition

$$\begin{aligned}|\boldsymbol{A}|^2 = {} & |\boldsymbol{A}\boldsymbol{\xi}^+ \cdot \boldsymbol{\xi}^+|^2 + |\boldsymbol{A}\boldsymbol{\xi}^- \cdot \boldsymbol{\xi}^-|^2 + |\boldsymbol{A}\boldsymbol{\eta}^+ \cdot \boldsymbol{\eta}^+|^2 + |\boldsymbol{A}\boldsymbol{\eta}^- \cdot \boldsymbol{\eta}^-|^2 - \frac{1}{2}(\mathrm{tr}\boldsymbol{A})^2 \\ & + \frac{1}{2}\big|q^2 a_{11} + p^2 a_{22} - 2pq a_{12}\big|^2 + \frac{1}{2}\big|p^2 a_{11} + q^2 a_{22} + 2pq a_{12}\big|^2 \\ & + 2\big|(p^2 - q^2)a_{12} + pq(a_{22} - a_{11})\big|^2 + \frac{1}{2}\big(a_{33}^2 + (a_{11} + a_{22})^2\big).\end{aligned}$$

Proof. Let us define $\boldsymbol{\xi}_0 := \boldsymbol{\xi}^+ \wedge \boldsymbol{\xi}^- = (q, -p, 0)$ so that $\{\boldsymbol{\xi}^+, \boldsymbol{\xi}^-, \boldsymbol{\xi}_0\}$ is an orthonormal basis of $\mathbb{R}^3$. Then the family

$$\{\boldsymbol{\xi}^+ \otimes \boldsymbol{\xi}^+, \boldsymbol{\xi}^- \otimes \boldsymbol{\xi}^-, \boldsymbol{\xi}_0 \otimes \boldsymbol{\xi}_0, \sqrt{2}(\boldsymbol{\xi}^+ \odot \boldsymbol{\xi}_0), \sqrt{2}(\boldsymbol{\xi}^- \odot \boldsymbol{\xi}_0), \sqrt{2}(\boldsymbol{\xi}^+ \odot \boldsymbol{\xi}^-)\}$$

defines an orthonormal basis of the set $\mathbb{M}^{3\times 3}_{sym}$, and Pythagoras Theorem ensures that

$$\begin{aligned}|\boldsymbol{A}|^2 = {} & |\boldsymbol{A} : (\boldsymbol{\xi}^+ \otimes \boldsymbol{\xi}^+)|^2 + |\boldsymbol{A} : (\boldsymbol{\xi}^- \otimes \boldsymbol{\xi}^-)|^2 + |\boldsymbol{A} : (\boldsymbol{\xi}_0 \otimes \boldsymbol{\xi}_0)|^2 \\ & + 2|\boldsymbol{A} : (\boldsymbol{\xi}^+ \odot \boldsymbol{\xi}_0)|^2 + 2|\boldsymbol{A} : (\boldsymbol{\xi}^- \odot \boldsymbol{\xi}_0)|^2 + 2|\boldsymbol{A} : (\boldsymbol{\xi}^+ \odot \boldsymbol{\xi}^-)|^2 \\ = {} & |\boldsymbol{A}\boldsymbol{\xi}^+ \cdot \boldsymbol{\xi}^+|^2 + |\boldsymbol{A}\boldsymbol{\xi}^- \cdot \boldsymbol{\xi}^-|^2 + |\boldsymbol{A}\boldsymbol{\xi}_0 \cdot \boldsymbol{\xi}_0|^2 \\ & + 2|\boldsymbol{A}\boldsymbol{\xi}^+ \cdot \boldsymbol{\xi}_0|^2 + 2|\boldsymbol{A}\boldsymbol{\xi}^- \cdot \boldsymbol{\xi}_0|^2 + 2|\boldsymbol{A}\boldsymbol{\xi}^+ \cdot \boldsymbol{\xi}^-|^2.\end{aligned}$$

The conclusion follows from a straightforward computation of each term. □

We now prove a partial bulk energy lower bound.

Lemma 6.9 *Assume that* $\lambda_b \geqslant \mu_b$. *Then* $(1-\theta)u_3 = 0$ $\mathcal{L}^2$*-a.e. in* ω, *and*

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_b) \geqslant \frac{\mu_b}{2} \liminf_{\varepsilon\to 0} \int_{(\omega\setminus\Delta_\varepsilon)\times(0,1)} \left| u_\alpha(x) + \int_0^{x_3} \partial_\alpha(u_\varepsilon)_3(x', s)\, ds \right|^2 dx. \tag{6.11}$$

If in addition the sequences $(\partial_\alpha(u_\varepsilon)_3)_{\varepsilon>0}$ *are bounded in* $L^2(\Omega_f)$, *then*

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_b) \geqslant \frac{\mu_b}{2} \int_\omega (1-\theta)|\bar{\boldsymbol{u}}|^2\, dx'.$$

Proof. Let us denote by

$$\boldsymbol{A}_\varepsilon := \begin{pmatrix} \varepsilon e_{11}(\boldsymbol{u}_\varepsilon) & \varepsilon e_{12}(\boldsymbol{u}_\varepsilon) & e_{13}(\boldsymbol{u}_\varepsilon) \\ \varepsilon e_{12}(\boldsymbol{u}_\varepsilon) & \varepsilon e_{22}(\boldsymbol{u}_\varepsilon) & e_{23}(\boldsymbol{u}_\varepsilon) \\ e_{13}(\boldsymbol{u}_\varepsilon) & e_{23}(\boldsymbol{u}_\varepsilon) & \varepsilon^{-1} e_{33}(\boldsymbol{u}_\varepsilon) \end{pmatrix}.$$

the scaled strain so that

$$J_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_b) = \frac{\lambda_b}{2} \int_{\Omega_b} \mathrm{tr}(\boldsymbol{A}_\varepsilon)^2\, dx + \mu_b \int_{\Omega_b} |\boldsymbol{A}_\varepsilon|^2\, dx.$$

According to Lemma 6.8 with the angle $\vartheta = 0$, we get that

$$\begin{aligned}J_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_b) \geqslant {} & \frac{\lambda_b - \mu_b}{2} \int_{\Omega_b} \mathrm{tr}(\boldsymbol{A}_\varepsilon)^2\, dx \\ & + \mu_b \int_{\Omega_b} \left[|\boldsymbol{A}_\varepsilon \xi^+ \cdot \xi^+|^2 + |\boldsymbol{A}_\varepsilon \xi^- \cdot \xi^-|^2 + |\boldsymbol{A}_\varepsilon \eta^+ \cdot \eta^+|^2 + |\boldsymbol{A}_\varepsilon \eta^- \cdot \eta^-|^2 \right] dx \\ \geqslant {} & \mu_b \int_{\Omega_b} \left[|\boldsymbol{A}_\varepsilon \xi^+ \cdot \xi^+|^2 + |\boldsymbol{A}_\varepsilon \xi^- \cdot \xi^-|^2 + |\boldsymbol{A}_\varepsilon \eta^+ \cdot \eta^+|^2 + |\boldsymbol{A}_\varepsilon \eta^- \cdot \eta^-|^2 \right] dx,\end{aligned} \tag{6.12}$$

since $\lambda_b \geqslant \mu_b$. It remains to compute each of the four terms in the right hand side of the previous expression. Let us start with the first term. Changing variable $x = (y', 0) + s\boldsymbol{\xi}^+_\varepsilon$ (with $\mathrm{d}x = (\sqrt{2}\varepsilon)^{-1}\,\mathrm{d}y'\,\mathrm{d}s$), and using Fubini's Theorem, we get that

$$\int_{\Omega_b} |\boldsymbol{A}_\varepsilon \boldsymbol{\xi}^+ \cdot \boldsymbol{\xi}^+|^2\,\mathrm{d}x \geqslant \varepsilon^2 \int_{(\omega_\varepsilon\setminus\Delta_\varepsilon)\times(-1,0)} |\nabla \boldsymbol{u}_\varepsilon \boldsymbol{\xi}^+_\varepsilon \cdot \boldsymbol{\xi}^+_\varepsilon|^2\,\mathrm{d}x$$
$$\geqslant \varepsilon^2 \int_{\omega_\varepsilon\setminus\Delta_\varepsilon} \fint_{-\sqrt{2}\varepsilon}^{0} \left|\nabla \boldsymbol{u}_\varepsilon\big((y',0)+s\boldsymbol{\xi}^+_\varepsilon\big)\boldsymbol{\xi}^+_\varepsilon \cdot \boldsymbol{\xi}^+_\varepsilon\right|^2 \mathrm{d}s\,\mathrm{d}y'.$$

According to Lemma 6.7, since $(\boldsymbol{u}_\varepsilon)^{y'}_{\boldsymbol{\xi}^+_\varepsilon} \in H^1(-\sqrt{2}\varepsilon, \sqrt{2}\varepsilon)$ and $(\boldsymbol{u}_\varepsilon)^{y'}_{\boldsymbol{\xi}^\pm_\varepsilon}(-\sqrt{2}\varepsilon) = 0$ for $\mathcal{L}^2$-a.e. $y' \in \omega \setminus \Delta_\varepsilon$, we get that

$$\int_{\Omega_b} |\boldsymbol{A}_\varepsilon \boldsymbol{\xi}^+ \cdot \boldsymbol{\xi}^+|^2\,\mathrm{d}x \geqslant \varepsilon^2 \int_{\omega_\varepsilon\setminus\Delta_\varepsilon} \fint_{-\sqrt{2}\varepsilon}^{0} \left|\frac{d}{ds}[\boldsymbol{u}_\varepsilon((y',0)+s\boldsymbol{\xi}^+_\varepsilon)\cdot\boldsymbol{\xi}^+_\varepsilon]\right|^2 \mathrm{d}s\,\mathrm{d}y'$$
$$\geqslant \varepsilon^2 \int_{\omega_\varepsilon\setminus\Delta_\varepsilon} \left|\fint_{-\sqrt{2}\varepsilon}^{0} \frac{d}{ds}[\boldsymbol{u}_\varepsilon((y',0)+s\boldsymbol{\xi}^+_\varepsilon)\cdot\boldsymbol{\xi}^+_\varepsilon]\,\mathrm{d}s\right|^2 \mathrm{d}y'$$
$$= \frac{1}{4}\int_{\omega_\varepsilon\setminus\Delta_\varepsilon} \left|(u_\varepsilon)^-_1(y',0) + \frac{1}{\varepsilon}(u_\varepsilon)^-_3(y',0)\right|^2 \mathrm{d}y',$$

where $\boldsymbol{u}^-_\varepsilon(\cdot,0)$ denotes the lower trace of $\boldsymbol{u}_\varepsilon$ on $\omega\times\{0\}$. Using again Lemma 6.7, the function $(\boldsymbol{u}_\varepsilon)^{y'}_{\boldsymbol{\xi}^+\varepsilon} \in H^1(-\sqrt{2}\varepsilon, \sqrt{2}\varepsilon)$ does not jump at $t = 0$. Thus according to [1, Theorem 4.5 (iv)], it follows that

$$(u_\varepsilon)^-_1 + \varepsilon^{-1}(u_\varepsilon)^-_3 = (u_\varepsilon)^+_1 + \varepsilon^{-1}(u_\varepsilon)^+_3 \quad \mathcal{H}^2\text{-a.e. on } \omega\times\{0\},$$

and therefore,

$$\int_{\Omega_b} |\boldsymbol{A}\boldsymbol{\xi}^+ \cdot \boldsymbol{\xi}^+|^2\,\mathrm{d}x \geqslant \frac{1}{4}\int_{\omega_\varepsilon\setminus\Delta_\varepsilon} \left|(u_\varepsilon)^+_1(y',0) - \frac{1}{\varepsilon}(u_\varepsilon)^+_3(y',0)\right|^2 \mathrm{d}y'. \tag{6.13}$$

Analogously, we can show that

$$\int_{\Omega_b} |\boldsymbol{A}_\varepsilon \boldsymbol{\xi}^- \cdot \boldsymbol{\xi}^-|^2\,\mathrm{d}x \geqslant \frac{1}{4}\int_{\omega_\varepsilon\setminus\Delta_\varepsilon} \left|(u_\varepsilon)^+_1(y',0) - \frac{1}{\varepsilon}(u_\varepsilon)^+_3(y',0)\right|^2 \mathrm{d}y', \tag{6.14}$$

$$\int_{\Omega_b} |\boldsymbol{A}_\varepsilon \boldsymbol{\eta}^+ \cdot \boldsymbol{\eta}^+|^2\,\mathrm{d}x \geqslant \frac{1}{4}\int_{\omega_\varepsilon\setminus\Delta_\varepsilon} \left|(u_\varepsilon)^+_2(y',0) + \frac{1}{\varepsilon}(u_\varepsilon)^+_3(y',0)\right|^2 \mathrm{d}y', \tag{6.15}$$

$$\int_{\Omega_b} |\boldsymbol{A}_\varepsilon \boldsymbol{\eta}^- \cdot \boldsymbol{\eta}^-|^2\,\mathrm{d}x \geqslant \frac{1}{4}\int_{\omega_\varepsilon\setminus\Delta_\varepsilon} \left|(u_\varepsilon)^+_2(y',0) - \frac{1}{\varepsilon}(u_\varepsilon)^+_3(y',0)\right|^2 \mathrm{d}y'. \tag{6.16}$$

Summing up (6.13), (6.14), (6.15), (6.16) and using (6.12) leads to

$$J_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_b) \geqslant \frac{\mu_b}{2}\int_{\omega_\varepsilon\setminus\Delta_\varepsilon} (u_\varepsilon)^+_\alpha(y',0)(u_\varepsilon)^+_\alpha(y',0)\,\mathrm{d}y' + \frac{\mu_b}{\varepsilon^2}\int_{\omega_\varepsilon\setminus\Delta_\varepsilon} |(u_\varepsilon)^+_3(y',0)|^2\,\mathrm{d}y'. \tag{6.17}$$

Since $P_\varepsilon \subset \Delta_\varepsilon$, Lemma 6.7 together with the fundamental Theorem of calculus yields,

$$\int_{(\omega\setminus\Delta_\varepsilon)\times(0,1)} \left|(u_\varepsilon)_3(x',x_3) - (u_\varepsilon)^+_3(x',0)\right|^2\,\mathrm{d}x \leqslant 4\int_{\Omega_f} |e_{33}(\boldsymbol{u}_\varepsilon)|^2\,\mathrm{d}x \leqslant C\varepsilon^4. \tag{6.18}$$

In particular, (6.17), (6.18) and the energy bound (6.5) ensure that

$$\int_{(\omega_\varepsilon\setminus\Delta_\varepsilon)\times(0,1)} |(u_\varepsilon)_3|^2\,\mathrm{d}x \leqslant C\varepsilon^2,$$

which implies, letting $\varepsilon\to 0$, that $(1-\theta)u_3 = 0$ $\mathcal{L}^2$-a.e. in ω. Therefore Lemma 6.6 shows that $\chi_{\omega\setminus\Delta_\varepsilon}\partial_\alpha(u_\varepsilon)_3 \overset{*}{\rightharpoonup} 0$ weakly* in $L^2(\omega; H^{-1}(0,1))$. In addition, since $P_\varepsilon\subset\Delta_\varepsilon$, we can use (6.9) and the fact that $(u_\varepsilon)_\alpha \to u_\alpha$ strongly in $L^2(\Omega_f)$, to obtain (6.11).

Assume now that the sequences $(\partial_\alpha(u_\varepsilon)_3)_{\varepsilon>0}$ are bounded in $L^2(\Omega_f)$. Then the convergence of the planar gradient improves to $\chi_{\omega\setminus\Delta_\varepsilon}\partial_\alpha(u_\varepsilon)_3 \rightharpoonup 0$ weakly in $L^2(\Omega_f)$, and thus (6.11) gives

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_b) \geqslant \frac{\mu_b}{2}\int_{\Omega_f}(1-\theta)|\boldsymbol{u}|^2\,\mathrm{d}x = \frac{\mu_b}{2}\int_\omega (1-\theta)|\bar{\boldsymbol{u}}|^2\,\mathrm{d}x',$$

since θ is independent of x_3. □

Acknowledgment. D. H. has been supported by the FONDECYT project 1150038 of the Chilean Ministry of Education and by the Millennium Nucleus Center for Analysis of PDE NC130017 of the Chilean Ministry of Economy.

References

1. Ambrosio, L., Coscia, A. & Dal Maso, G., Fine properties of functions with bounded deformation. *Arch. Rational Mech. Anal.* **139** (1997), 201–238. Zbl0890.49019 MR1480240
2. Ambrosio, L., Fusco, N. & Pallara, D., *Functions of bounded variation and free discontinuity problems*, The Clarendon Press, Oxford University Press, New York (2000). Zbl0957.49001 MR1857292
3. Ansini, N., The nonlinear sieve problem and applications to thin films. *Asympt. Anal.* **39** (2004), 113–145. Zbl1210.35117 MR2093896
4. Ansini, N., Babadjian, J.-F. & Zeppieri, C. I., The Neumann sieve problem and dimensional reduction: a multiscale approach. *M3AS* **17** (2007), 681–735. Zbl1120.74040 MR2325836
5. Anzellotti, G. & Giaquinta, M., Existence of the displacement field for an elasto-plastic body subject to Hencky's law and Von Mises' yield criterion. *Manuscripta Math.* **32** (1980), 101–131. Zbl0465.73022 MR0592713
6. Babadjian, J.-F., Quasi-static evolution of a brittle thin film. *Calc. Var. Partial Differential Equations* **26** (2006), 69–118. Zbl1089.74037 MR2217484
7. Babadjian, J.-F., Lower semicontinuity of quasiconvex bulk energies in SBV and integral representation in dimension reduction. *SIAM J. Math. Anal.* **39** (2008), 1921–1950. Zbl1151.74025 MR2390319
8. Babadjian, J.-F., Traces of functions of bounded deformation. *Indiana Univ. Math. J.* **64** (2015), 1271–1290. Zbl1339.26030 MR3385790
9. Barenblatt, G., The mathematical theory of equilibrium cracks in brittle fracture. *Advances in Applied Mechanics* **7** (1962), 55–129. MR0149728
10. Bellettini, G., Coscia, A. & Dal Maso, G., Compactness and lower semicontinuity properties in $SBD(\Omega)$. *Math. Z.* **228** (1998), 337–351. Zbl0914.46007 MR1630504
11. Bhattacharya, K., Fonseca, I. & Francfort, G. A., An asymptotic study of the debonding of thin films. *Arch. Rational Mech. Anal.* **161** (2002), 205–229. Zbl0999.74079 MR1894591
12. Bouchitté, G., Fonseca, I., Leoni, G. & Mascarenhas, L., A global method for relaxation in $W^{1,p}$ and in SBV_p. *Arch. Rational Mech. Anal.* **165** (2002), 187–242. Zbl1028.49009 MR1941478
13. Bourdin, B., Francfort, G. A. & Marigo, J.-J., The variational approach to fracture. *J. Elasticity* **91** (2008), 5–148.

14. Braides, A., & Fonseca, I., Brittle thin films. *Appl. Math. Optim.* **44** (2001), 299–323. Zbl1176.74018
15. Barchiesi, M., Lazzaroni, G. & Zeppieri, C. I., A bridging mechanism in the homogenisation of brittle composites with soft inclusions. *SIAM J. Math. Anal.* **48** (2016), 1178–1209. Zbl1337.49015 MR3477758
16. Chambolle, A., An approximation result for special functions with bounded deformation. *J. Math. Pures Appl.* **83** (2004), 929–954. Addendum in *J. Math. Pures Appl.* **84**, (2005)137–145. Zbl1084.49038 MR2074682
17. Ciarlet, P. G., *Mathematical Elasticity*. Vol. II. **27**, Studies in Mathematics and its Applications. North-Holland Publishing Co., Amsterdam (1997). Zbl0888.73001 MR1477663
18. Conti, S., Focardi, M. & Iurlano, F., Phase field approximation of cohesive fracture models. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **33** (2016), 1033–1067. Zbl1345.49012 MR3519531
19. Dal Maso, G., Generalised functions of bounded deformation. *J. Eur. Math. Soc. (JEMS)* **15** (2013), 1943–1997. Zbl1271.49029 MR3082250
20. Dal Maso, G. & Iurlano, F., Fracture models as Γ-limits of damage models. *Comm. Pure Appl. Anal.* **12** (2013), 1657–1686. Zbl1345.74097 MR2997535
21. Davoli, E. & Mora, M. G., A quasistatic evolution model for perfectly plastic plates derived by Gamma-convergence. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **30** (2013), 615–660. Zbl06295435 MR3082478
22. Federer, F. *Geometric measure theory*, Springer-Verlag, Berlin-Heidelberg-New York (1969). Zbl0176.00801 MR0257325
23. Focardi, M. & Iurlano, F., Asymptotic analysis of Ambrosio-Tortorelli energies in linearized elasticity. *SIAM J. Math. Anal.* **46** (2014), 2936–2955. Zbl1301.49030 MR3249854
24. Francfort, G. A. & Marigo, J.-J., Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids* **46** (1998), 1319–1342. Zbl0966.74060 MR1633984
25. Freddi, L., Paroni, R., & Zanini, C. Dimension reduction of a crack evolution problem in a linearly elastic plate. *Asympt. Anal.* **70** (2010) 101–123. Zbl1219.35306 MR2731652
26. Freddi, L., Paroni, R., Roubicek, T., & Zanini, C. Quasistatic delamination models for Kirchhoff-Love plates. *Z. Angew. Math. Mech.* **91** (2012) 845–865. Zbl1280.74032 MR2847147
27. Freddi, L., Roubicek, T. & Zanini, C., Quasistatic delamination of sandwitch-like Kirchhoff–Love plates. *J. Elasticity* **113** (2013), 219–250. Zbl1275.49019 MR3102596
28. Griffith, A. A., The phenomena of rupture and flow in solids. *Philos. Trans. R. Soc. Lond.* **221A** (1920), 163–198.
29. Iurlano, F., Fracture and plastic models as Γ-limits of damage models under different regimes. *Adv. Calc. Var.* **6** (2013), 165–189. Zbl06162477 MR3043575
30. Iurlano, F., A density result for $GSBD$ and its application to the approximation of brittle fracture energies. *Calc. Var. Partial Differential Equations* **51** (2014), 315–342. Zbl1297.49080 MR3247391
31. Hutchinson, J. W., & Suo, Z., Mixed mode cracking in layered materials. *Adv. Appl. Mech.* **29** (1991), 63–191. Zbl0790.73056
32. León Baldelli, A. A, Bourdin, B., Marigo, J.-J., & Maurini, C. Fracture and debonding of a thin film on a stiff substrate: analytical and numerical solutions of a one-dimensional variational model. *Cont. Mech. Thermodyn.* **25** (2013) 243–268. Zbl1343.74032 MR3035809
33. León Baldelli, A. A, Babadjian, J.-F., Bourdin, B., Henao, D. & Maurini, C., A variational model for fracture and debonding of thin films under in-plane loadings. *J. Mech. Phys. Solids* **70** (2014), 320–348. Zbl1328.74059 MR3245750
34. Leòn Baldelli, A. A. & Bourdin, B., On the asymptotic derivation of Winkler-type energies from 3D elasticity. *J. Elasticity* **121** (2015), 275–301. Zbl1330.35439 MR3415074
35. Matthies, H., Strang, G. & Christiansen, E., The saddle point of a differential program. *Energy methods in finite element analysis.* Ed. Rodin, Zienkiewicz, Glowinski, Wiley, New York (1979).
36. Mielke, A., Roubicek, T. & Thomas, M., From damage to delamination in nonlinearly elastic materials at small strains. *J. Elast.* **109** (2012), 235–273.

37. Roubicek, T., Scardia, L. & Zanini, C., Quasistatic delamination problem. *Cont. Mech. Thermodyn.* **21** (2009), 223–235. Zbl1179.74130 MR2529453
38. Strang, G. & Temam, R., Functions of bounded deformations. *Arch. Rational Mech. Anal.* **75** (1980), 7–21. Zbl0472.73031 MR0592100
39. Souganidis, P. E., Max-min representations and product formulas for the viscosity solutions of Hamilton–Jacobi equations with applications to differential games. *Nonlinear Anal.* **9** (1985), 217–257. Zbl0526.35018 MR0803085
40. Suquet, P., Un espace fonctionnel pour les équations de la plasticité. *Ann. Fac. Sc. Toulouse* **1** (1979), 77–87. Zbl0405.46027 MR0533600
41. Temam, R., *Problèmes mathématiques en plasticité.* Gauthier-Villars, Paris (1983). Zbl0547.73026 MR0711964
42. Winkler, E., *Die Lehre Von der Elasticität und Festigkeit* (1867). Zbl01.0350.01
43. Xia, Z. C. & Hutchinson, J. W., Crack patterns in thin films. *J. Mech. Phys. Solids* **48** (2000), 1107–1131. Zbl0966.74063