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# Zero width limit of the heat equation on moving thin domains

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We study the behavior of a variational solution to the Neumann type problem of the heat equation on a moving thin domain $\Omega_\varepsilon(t)$ that converges to an evolving surface $\Gamma(t)$ as the width of $\Omega_\varepsilon(t)$ goes to zero. We show that, under suitable assumptions, the average in the normal direction of $\Gamma(t)$ of a variational solution to the heat equation converges weakly in a function space on $\Gamma(t)$ as the width of $\Omega_\varepsilon(t)$ goes to zero, and that the limit is a unique variational solution to a limit equation on $\Gamma(t)$, which is a new type of linear diffusion equation involving the mean curvature and the normal velocity of $\Gamma(t)$. We also estimate the difference between variational solutions to the heat equation on $\Omega_\varepsilon(t)$ and the limit equation on $\Gamma(t)$.

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## 1. Introduction

For $t \in [0, T]$, $T > 0$, let $\Omega_\varepsilon(t)$ be a moving thin domain in $\mathbb{R}^n$, $n \geqslant 2$, with width of order $\varepsilon > 0$ that converges to an evolving closed hypersurface $\Gamma(t)$ as $\varepsilon \to 0$. We consider the Neumann type problem of the heat equation of the form

$$\begin{cases} \partial_t u^\varepsilon - \Delta u^\varepsilon = 0 & \text{in} \quad Q_{\varepsilon,T}, \\ \partial_{\nu_\varepsilon} u^\varepsilon + v_\varepsilon^N u^\varepsilon = 0 & \text{on} \quad \partial_\ell Q_{\varepsilon,T}, \\ u^\varepsilon(0) = u_0^\varepsilon & \text{in} \quad \Omega_\varepsilon(0). \end{cases} \tag{$H_\varepsilon$}$$

Here $Q_{\varepsilon,T} := \bigcup_{t\in(0,T)} \Omega_\varepsilon(t) \times \{t\}$, $\partial_\ell Q_{\varepsilon,T} := \bigcup_{t\in(0,T)} \partial\Omega_\varepsilon(t) \times \{t\}$, and $\nu_\varepsilon$, $v_\varepsilon^N$ are the unit outward normal vector field of $\partial\Omega_\varepsilon(t)$ and the outer normal velocity of $\partial\Omega_\varepsilon(t)$, respectively. The term $v_\varepsilon^N u^\varepsilon$ in the boundary condition is added so that the total amount of heat $\int_{\Omega_\varepsilon(t)} u^\varepsilon \, dx$ is conserved, see the beginning of Section 3. Also, if $u^\varepsilon$ denotes the concentration of some chemicals, the boundary condition says that chemicals near the boundary move along it and do not go into and out of the moving thin domain.

We are interested in the behavior of a solution $u^\varepsilon$ to ($H_\varepsilon$) as $\varepsilon \to 0$. Our goal is to characterize its limit as well as its convergence. Let us explain the simplest case when $\Omega_\varepsilon(t)$ is the set of all points in $\mathbb{R}^n$ with distance less than $\varepsilon$ from $\Gamma(t)$ so that the width of $\Omega_\varepsilon(t)$ is $2\varepsilon$. Let $\nu$ be the unit outward normal vector field of $\Gamma(t)$ and $V_\Gamma = v_\Gamma^N \nu + V_\Gamma^T$ be the total velocity of $\Gamma(t)$, where $v_\Gamma^N$ and $V_\Gamma^T$ are the outer normal velocity of $\Gamma(t)$ and a given tangential velocity field. Then our main result formally implies that, under suitable assumptions on the initial data $u_0^\varepsilon$ of ($H_\varepsilon$), the limit $v$ is a solution to

$$\partial^\circ v - v_\Gamma^N H v - \Delta_{\Gamma(t)} v = 0 \quad \text{on} \quad S_T. \tag{1.1}$$

Here $S_T := \bigcup_{t\in(0,T)} \Gamma(t)\times\{t\}$ and $\partial^\circ v = \partial_t v + v_\Gamma^N \nu\cdot\nabla v$ is the normal time derivative of $v$. (The notation $\partial^\circ$ is used in [2, 5]. We refer to [3] for the normal time derivative.) Also, $H := -\mathrm{div}_{\Gamma(t)}\nu$ and $\Delta_{\Gamma(t)} := \mathrm{div}_{\Gamma(t)}\nabla_{\Gamma(t)}$ are the mean curvature of $\Gamma(t)$ and the Laplace–Beltrami operator on $\Gamma(t)$, where $\mathrm{div}_{\Gamma(t)}$ and $\nabla_{\Gamma(t)}$ are the surface divergence operator and the tangential gradient on $\Gamma(t)$, respectively (see Section 2 for their definitions). We will give a heuristic derivation of the limit equation (1.1) in Appendix A. The equation (1.1) is equivalent to

$$\partial^\bullet v + (\mathrm{div}_{\Gamma(t)} V_\Gamma)v - \Delta_{\Gamma(t)} v - \mathrm{div}_{\Gamma(t)}\big(v V_\Gamma^T\big) = 0 \quad \text{on} \quad S_T, \tag{1.2}$$

which we will actually derive in Section 6. Here $\partial^\bullet v = \partial^\circ v + V_\Gamma^T\cdot\nabla_{\Gamma(t)} v$ denotes the material derivative of $v$ (see Section 4 for its precise definition). Note that the equation (1.1) is independent of the tangential velocity $V_\Gamma^T$. In other words, the evolution of the limit $v$ is not affected by advection along $\Gamma(t)$. Such a phenomenon does not occur in an advection-diffusion equation widely studied in recent years [2, 4–9, 19, 28]:

$$\partial^\bullet v + (\mathrm{div}_{\Gamma(t)} V_\Gamma)v - \Delta_{\Gamma(t)} v = 0 \quad \text{on} \quad S_T. \tag{1.3}$$

This equation is derived from a conservation law such that, for an arbitrary portion $\mathfrak{M}(t)$ of $\Gamma(t)$,

$$\frac{d}{dt}\int_{\mathfrak{M}(t)} v\, d\mathcal{H}^{n-1} = -\int_{\partial\mathfrak{M}(t)} q\cdot\mu\, d\mathcal{H}^{n-2}$$

holds, where $\mathcal{H}^k$ is the $k$-dimensional Hausdorff measure for $k\in\mathbb{N}$, $\mu$ is the co-normal to the boundary $\partial\mathfrak{M}(t)$, and $q$ is the surface flux, see [4, Section 3] and [5, Section 3.1] for details.

Partial differential equations on thin domains are studied over the years [12–16, 20–24, 26, 27], and many researchers deal with a nonmoving thin domain of the form

$$\Omega_\varepsilon = \big\{(x', x_n)\in\mathbb{R}^{n-1}\times\mathbb{R} \mid x'\in\omega,\ \varepsilon g_0(x') < x_n < \varepsilon g_1(x')\big\}, \quad \varepsilon > 0, \tag{1.4}$$

where $\omega$ is a domain in $\mathbb{R}^{n-1}$ and $g_0$, $g_1$ are functions on $\omega$. In their pioneering works [12, 13], Hale and Raugel compared the dynamics of reaction-diffusion equations and damped wave equations on $\Omega_\varepsilon$ of the form (1.4) (with $g_0 = 0$ and slightly modified $g_1$) and that of corresponding limit equations on $\omega$ by the scaling argument. They transformed the equations on $\Omega_\varepsilon$ into scaled equations on a fixed reference domain $\Omega_0 = \omega\times(0,1)$ by the change of variables, and formally derived the limit equations on $\omega$ by letting $\varepsilon\to 0$ in the scaled equations on $\Omega_0$ and omitting divergent terms. Then they compared the dynamics of the scaled equations on $\Omega_0$ and that of the limit equations on $\omega$ by analyzing weighted bilinear forms that appear in variational formulations of the scaled equations and the limit equations. Their scaling argument is applicable to more general thin domains such as a thin L-shaped domain [14] and a moving thin domain of the form (1.4) where $g_0 = 0$ and $g_1$ depends on time [23]. Prizzi and Rybakowski [21] generalized the scaling argument in [12, 13] to study reaction-diffusion equations on a (nonmoving) thin domain with holes around a lower dimensional domain. The generalized scaling argument in [21] is also valid for a (nonmoving) thin domain with holes around a lower dimensional manifold [20, 22]. We refer to [24] and references therein for other examples of thin domains.

In contrast to the above papers, the limit hypersurface $\Gamma(t)$ of our thin domain $\Omega_\varepsilon(t)$ evolves. Such a situation has been considered only in the paper [8], which deals with a diffuse interface model for the advection-diffusion equation (1.3). See also [9] for numerical computations of the advection-diffusion equation (1.3) based on the diffuse interface model. In [8], however, the limit equation (1.3) on the evolving surface is given and a bulk equation on the moving thin domain involves a weight function that vanishes on the boundary of the domain. Therefore, there is no literature on initial-boundary value problems of partial differential equations on moving thin domains around evolving surfaces whose limit equations are unknown in advance, even in the case of the heat equation.

The difficulty caused by the evolution of the hypersurface $\Gamma(t)$ is in transforming equations on $\Omega_\varepsilon(t)$ and $\Gamma(t)$ into equations on fixed (in time and width) domain and hypersurface. In particular, transformations of differential operators on $\Gamma(t)$ into those on a fixed hypersurface is so complicated that we can hardly find a limit equation on the fixed hypersurface and convert it into an equation on $\Gamma(t)$, see [7] for the actual transformations of differential operators.

To avoid this difficulty, we employ another method that does not require transformations of $\Omega_\varepsilon(t)$ and $\Gamma(t)$. Let us explain our idea of derivation of a limit equation on $\Gamma(t)$. We start from a variational formulation of ($H_\varepsilon$) (see (3.2)) that consists of integrals over the noncylindrical domain $Q_{\varepsilon,T}$ of a variational solution $u^\varepsilon$ to ($H_\varepsilon$) and a test function defined on $Q_{\varepsilon,T}$. In this variational formulation, we take a test function independent of the normal direction of $\Gamma(t)$ and apply the co-area formula (see (5.1)) and a weighted average operator $M_\varepsilon$ (see Definition 5.1) to get a variational formulation (with some residual term) of the average $M_\varepsilon u^\varepsilon$ (see (6.1)) that consists of integrals over the space-time manifold $S_T$ of $M_\varepsilon u^\varepsilon$ and a test function defined on $S_T$. Then we obtain a variational formulation of a limit equation on $\Gamma(t)$ (see (6.13)) by omitting the residual term in the variational formulation of $M_\varepsilon u^\varepsilon$. Moreover, we prove that $M_\varepsilon u^\varepsilon$ converges weakly in a function space on $S_T$ as $\varepsilon \to 0$ and that the limit is a unique variational solution to the limit equation (see Theorem 6.9), and estimate the $L^2(Q_{\varepsilon,T})$-norm of the difference between variational solutions to ($H_\varepsilon$) and the limit equation (see Theorem 6.12). These results indicate that our limit equation on $\Gamma(t)$ derived as above is indeed the "limit" of ($H_\varepsilon$).

In our derivation of a limit equation, Lemma 5.6 and Lemma 5.13 play an important role. In Lemma 5.6 we approximate an $H^1$-bilinear form on $\Omega_\varepsilon(t)$ for each $t \in [0, T]$ by that on $\Gamma(t)$ with the tangential gradient of the average $M_\varepsilon u$ of a function $u$ on $\Omega_\varepsilon(t)$. The proof of Lemma 5.6 is based on simple representations of the gradient in $\mathbb{R}^n$ and the tangential gradient on $\Gamma(t)$ under a special local coordinate system for each fixed point on $\Gamma(t)$. On the other hand, Lemma 5.13 gives an integral formula that formally represents a relation between the weak time derivative of a function $u$ on $Q_{\varepsilon,T}$ and the weak material derivative of its average $M_\varepsilon u$ (in fact, we do not explicitly deal with the time derivative of $u$). Lemma 5.13 essentially follows from Lemma 5.11, which gives a relation between the time derivative and the material derivative of functions defined on $S_T$.

Average operators in the thin direction were originally introduced by Hale and Raugel [12, 13], but they took the average of functions on the scaled domain $\Omega_0 = \omega \times (0, 1)$. Average operators on actual thin domains $\Omega_\varepsilon$ appears in the study of the Navier-Stokes equations on three-dimensional thin domains [15, 16, 26, 27]. Temam and Ziane [26, 27] first employed them to study the global existence of strong solutions to the Navier-Stokes equations for large initial data and external forces and the behavior of solutions as $\varepsilon \to 0$ when $\Omega_\varepsilon$ is a three-dimensional thin product domain $\Omega_\varepsilon = \omega \times (0, \varepsilon)$ with a bounded domain $\omega$ in $\mathbb{R}^2$ and a thin spherical domain

$\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid a < |x| < (1+\varepsilon)a\}$ with a constant $a > 0$. In [15, 16], average operators were employed to study the dynamics of the Navier-Stokes equations on $\Omega_\varepsilon$ of the form (1.4). In particular, the authors of [16] compared the dynamics of the Navier-Stokes equations with that of limit equations by estimating the difference of the average of solutions to the Navier-Stokes equations and solutions to the limit equations.

We point out that our weighted average operator given in Definition 5.1 is a generalization of average operators given in [15, 16, 26] and that its weight function is different from that of an average operator given in [27]. In fact, the weight function of our average operator is a Jacobian that appears when we change variables of integrals over a tubular neighborhood of $\Gamma(t)$ in terms of the normal coordinate system around $\Gamma(t)$. Our choice of the weight function enables us to avoid including the material derivative of a test function in the estimate for the residual term in the variational formulation of the average of a variational solution to ($H_\varepsilon$), which is essential for derivation of its energy estimate, see Lemma 5.13 and Remark 5.14. We also note that, contrary to our case, Kublik, Tanushev, and Tsai [17] employed the same Jacobian and co-area formula to transform integrals over boundaries of domains into those over their tubular neighborhoods. Based on this transformation, they proposed a new approach to numerical computations of boundary integrals without explicit parametrizations of boundaries and a simple formulation for constructing boundary integral methods to solve Poisson's equation. Their method of the numerical computations of boundary integrals is also applicable to integrals over nonclosed manifolds of higher codimension, such as curves in $\mathbb{R}^3$ with different endpoints, see [18] for details.

Finally we mention variational formulations of partial differential equations on evolving surfaces. There are several kinds of variational frameworks for equations on evolving surfaces, mainly the advection-diffusion equation (1.3), see [4, 19, 28] for example. In addition, Alphonse, Elliott, and Stinner [1, 2] proposed an abstract variational setting with evolving Hilbert spaces and applied it to some equations on moving domains and evolving surfaces. Among these variational frameworks, we adopt the one introduced by Olshanskii, Reusken, and Xu [19]. Their variational formulation is imposed on function spaces on $S_T$, which is suitable for our calculation of bilinear forms on function spaces on $S_T$ and $Q_{\varepsilon,T}$ performed in Section 5 and Section 6.

This paper is organized as follows. In Section 2 we introduce notations related to the evolving surface $\Gamma(t)$ and define the moving thin domain $\Omega_\varepsilon(t)$. In Section 3 we define a variational solution to ($H_\varepsilon$) and prove its existence and uniqueness. We also derive an energy estimate of a variational solution to ($H_\varepsilon$) with a constant independent of $\varepsilon$. In Section 4 we define function spaces on $S_T$ introduced in [19] and give their properties. In Section 5 we define the weighted average operator $M_\varepsilon$ and establish estimates and formulas related to $M_\varepsilon$. In Section 6, we derive a limit equation on $\Gamma(t)$ of the form (1.2) via its variational formulation and prove our main theorems (Theorem 6.9 and Theorem 6.12). In Appendix A, we give a heuristic derivation of the limit equation (1.1) when $\Omega_\varepsilon(t)$ is the set of all points in $\mathbb{R}^n$ with distance less than $\varepsilon$ from $\Gamma(t)$. In Appendix B, we give complete proofs of some results in Section 4 related to integrals over $\Gamma(t)$. In Appendix C, we show detailed calculations in proofs of some lemmas in Section 5 involving the differential geometry of tubular neighborhoods of $\Gamma(t)$.

## 2. Evolving surfaces and moving thin domains

For each $t \in [0,T]$, let $\Gamma(t)$ be a closed (that is, compact and without boundary), connected and oriented smooth hypersurface in $\mathbb{R}^n$. We set $\Gamma_0 := \Gamma(0)$ and define a space-time manifold $S_T \subset \mathbb{R}^{n+1}$ as $S_T := \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\}$. We assume that each point $y$ on $\Gamma(t)$ evolves with velocity $V_\Gamma(y,t)$, which is not necessarily normal to $\Gamma(t)$, and the velocity field $V_\Gamma\colon \overline{S_T} \to \mathbb{R}^n$ is smooth. Let $\Phi(\cdot,t)\colon \Gamma_0 \to \Gamma(t)$ be the flow map of $V_\Gamma$, that is, $\Phi(\cdot,t)$ is a diffeomorphism from $\Gamma_0$ onto $\Gamma(t)$ with its inverse $\Phi^{-1}(\cdot,t)$ for each $t \in [0,T]$ and satisfies

$$\Phi(Y,0) = Y, \quad \frac{\partial \Phi}{\partial t}(Y,t) = V_\Gamma\big(\Phi(Y,t),t\big) \quad \text{for all} \quad Y \in \Gamma_0,\ t \in [0,T].$$

We assume that $\Phi$ and $\Phi^{-1}$ are smooth on $\Gamma_0 \times [0,T]$ and $\overline{S_T}$, respectively. Due to this assumption, $\overline{S_T}$ is a compact smooth manifold in $\mathbb{R}^{n+1}$.

Let $\nu : \overline{S_T} \to \mathbb{R}^n$ be the unit outward normal vector field of $\Gamma(t)$. The velocity $V_\Gamma$ is decomposed into $V_\Gamma = v_\Gamma^N \nu + V_\Gamma^T$, where $v_\Gamma^N\colon \overline{S_T} \to \mathbb{R}$ is the outer normal velocity and $V_\Gamma^T\colon \overline{S_T} \to \mathbb{R}^n$ is a tangential velocity field. Note that to describe the geometric motion of $\Gamma(t)$ it is sufficient to prescribe the normal velocity. However, to describe a limit equation on $\Gamma(t)$ we will derive in Section 6, we also need to consider a tangential velocity, which represents advection along $\Gamma(t)$.

For each $t \in [0,T]$, let $d(\cdot,t)$ be the signed distance function from $\Gamma(t)$ that increases in the direction of $\nu(\cdot,t)$. By the smoothness (in space and time) and compactness of $\Gamma(t)$, there is an open set $N(t)$ in $\mathbb{R}^n$ of the form $N(t) = \{x \in \mathbb{R}^n \mid -\delta < d(x,t) < \delta\}$ for each $t \in [0,T]$, where $\delta > 0$ is a constant independent of $t$, that satisfies the following conditions:

- The signed distance function $d$ is smooth on $\overline{N_T}$, where $N_T := \bigcup_{t\in(0,T)} N(t)\times\{t\} \subset \mathbb{R}^{n+1}$.
- For each $(x,t) \in \overline{N_T}$, there is a unique point $p(x,t) \in \Gamma(t)$ such that

$$x = p(x,t) + d(x,t)\nu(p(x,t),t), \quad \nabla d(x,t) = \nu(p(x,t),t).$$

The set $N(t)$ is called a tubular neighborhood of $\Gamma(t)$. Based on the above equality, we extend the outward normal $\nu$ to $\overline{N_T}$ by setting $\nu(x,t) := \nabla d(x,t)$ for $(x,t) \in \overline{N_T}$. Then, by the smoothness of $d$, the extended outward normal $\nu$ and the projection mapping $p$ are smooth on $\overline{N_T}$. Also, the normal velocity $v_\Gamma^N$ of $\Gamma(t)$ is given by $v_\Gamma^N = -\partial_t d$ on $\overline{S_T}$.

Next, we give definitions of differential operators on evolving surfaces. For a function $v$ and a vector field $F$ on $S_T$, we define the tangential gradient of $v$ and the surface divergence of $F$ as

$$\nabla_{\Gamma(t)} v(y,t) := \big[I_n - \nu(y,t)\otimes\nu(y,t)\big]\nabla\overline{v}(y,t),$$
$$\operatorname{div}_{\Gamma(t)} F(y,t) := \operatorname{trace}\big[\{I_n - \nu(y,t)\otimes\nu(y,t)\}\nabla\overline{F}(y,t)\big]$$

for $(y,t) \in S_T$. Here $I_n$ is the identity matrix of size $n$ and $\nu\otimes\nu := (\nu_i\nu_j)_{i,j}$ is the tensor product of $\nu$. Also, $\overline{v}$ and $\overline{F}$ are the constant extensions of $v$ and $F$ in the normal direction of $\Gamma(t)$ given by

$$\overline{v}(x,t) := v\big(p(x,t),t\big), \quad \overline{F}(x,t) := F\big(p(x,t),t\big), \quad (x,t) \in N_T.$$

By definition, $\nu \cdot \nabla_{\Gamma(t)} v = 0$ holds. Hereafter we use the same notations for functions and vector fields on $\Gamma(t)$ with each fixed $t \in [0,T]$.

Finally, we define a moving thin domain. Let $g_0$ and $g_1$ be smooth functions on $\overline{S_T}$. We assume that there is a constant $c > 0$ such that

$$g(y,t) := g_1(y,t) - g_0(y,t) \geqslant c \quad \text{for all} \quad (y,t) \in \overline{S_T}. \tag{2.1}$$

Then we define a moving thin domain $\Omega_\varepsilon(t) \subset \mathbb{R}^n$ as

$$\Omega_\varepsilon(t) := \big\{y + \rho\nu(y,t) \mid y \in \Gamma(t),\ \varepsilon g_0(y,t) < \rho < \varepsilon g_1(y,t)\big\}, \quad t \in [0,T],\ \varepsilon > 0$$

and a space-time noncylindrical domain $Q_{\varepsilon,T} \subset \mathbb{R}^{n+1}$ as $Q_{\varepsilon,T} := \bigcup_{t\in(0,T)} \Omega_\varepsilon(t) \times \{t\}$. Note that $\Omega_\varepsilon(t)$ does not necessarily include $\Gamma(t)$, since we do not assume that $g_0$ is negative and $g_1$ is positive. Since $g_0$ and $g_1$ are smooth and thus bounded on the compact manifold $\overline{S_T}$, there is a positive number $\varepsilon_0$ such that $\overline{\Omega_\varepsilon(t)} \subset N(t)$ for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T]$. Hereafter we assume that $\varepsilon \in (0, \varepsilon_0)$.

## 3. Heat equation on moving thin domains

In this section, we consider the initial-boundary problem ($H_\varepsilon$) of the heat equation on the moving thin domain $\Omega_\varepsilon(t)$. First we show that the boundary condition of ($H_\varepsilon$) yields the conservation of the total amount of heat. Suppose that $u^\varepsilon$ satisfies the heat equation in $Q_{\varepsilon,T}$. By the Reynolds transport theorem and Green's formula (see [10, Appendix C]) we have

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_\varepsilon(t)} u^\varepsilon\,dx &= \int_{\Omega_\varepsilon(t)} \partial_t u^\varepsilon\,dx + \int_{\partial\Omega_\varepsilon(t)} v_\varepsilon^N u^\varepsilon\,d\mathcal{H}^{n-1} \\ &= \int_{\Omega_\varepsilon(t)} \Delta u^\varepsilon\,dx + \int_{\partial\Omega_\varepsilon(t)} v_\varepsilon^N u^\varepsilon\,d\mathcal{H}^{n-1} \\ &= \int_{\partial\Omega_\varepsilon(t)} \left(\partial_{\nu_\varepsilon} u^\varepsilon + v_\varepsilon^N u^\varepsilon\right)\,d\mathcal{H}^{n-1}.\end{aligned}$$

Hence if $u^\varepsilon$ additionally satisfies the boundary condition of ($H_\varepsilon$), then $\frac{d}{dt}\int_{\Omega_\varepsilon(t)} u^\varepsilon\,dx = 0$ for all $t \in (0, T)$, that is, the total amount of heat $\int_{\Omega_\varepsilon(t)} u^\varepsilon\,dx$ is conserved.

Next, we give a definition of a variational solution to ($H_\varepsilon$). For each $\varepsilon > 0$, we define a function space $L^2_{H^1(\varepsilon)}$ on $Q_{\varepsilon,T}$ and an inner product on $L^2_{H^1(\varepsilon)}$ as

$$\begin{aligned} L^2_{H^1(\varepsilon)} &:= \left\{u \in L^2(Q_{\varepsilon,T}) \mid \nabla u \in L^2(Q_{\varepsilon,T})\right\}, \\ (u_1, u_2)_{L^2_{H^1(\varepsilon)}} &:= \int_0^T \int_{\Omega_\varepsilon(t)} (u_1u_2 + \nabla u_1 \cdot \nabla u_2)\,dx\,dt. \end{aligned} \tag{3.1}$$

The space $L^2_{H^1(\varepsilon)}$ is a Hilbert space endowed with the above inner product. Let $\|\cdot\|_{L^2_{H^1(\varepsilon)}}$ denote the norm of $L^2_{H^1(\varepsilon)}$ induced by the inner product $(\cdot,\cdot)_{L^2_{H^1(\varepsilon)}}$.

DEFINITION 3.1 Let $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$. A function $u^\varepsilon \in L^2_{H^1(\varepsilon)}$ is said to be a *variational solution* to the initial-boundary value problem ($H_\varepsilon$) if it satisfies

$$\int_0^T \int_{\Omega_\varepsilon(t)} (-u^\varepsilon \partial_t w + \nabla u^\varepsilon \cdot \nabla w)\,dx\,dt - \int_{\Omega_\varepsilon(0)} u_0^\varepsilon w(0)\,dx = 0 \tag{3.2}$$

for all $w \in C^1(\overline{Q_{\varepsilon,T}})$ with $w(T) = 0$ in $\Omega_\varepsilon(T)$.

The variational formulation (3.2) is derived as follows. Suppose that $u^\varepsilon$ is a classical solution to ($H_\varepsilon$). We multiply both sides of the heat equation in $Q_{\varepsilon,T}$ by an arbitrary function $w \in C^1(\overline{Q_{\varepsilon,T}})$ with $w(T) = 0$ in $\Omega_\varepsilon(T)$ and integrate them over $Q_{\varepsilon,T}$ to get

$$\int_0^T \int_{\Omega_\varepsilon(t)} (\partial_t u^\varepsilon - \Delta u^\varepsilon) w\,dx\,dt = 0.$$

We calculate the left-hand side of the above equality. By the Reynolds transport theorem and the conditions $u^\varepsilon(0) = u_0^\varepsilon$ in $\Omega_\varepsilon(0)$ and $w(T) = 0$ in $\Omega_\varepsilon(T)$, we have

$$\int_0^T \int_{\Omega_\varepsilon(t)} (\partial_t u^\varepsilon) w \, dx \, dt = \\ -\int_0^T \int_{\Omega_\varepsilon(t)} u^\varepsilon \partial_t w \, dx \, dt - \int_0^T \int_{\partial\Omega_\varepsilon(t)} v_\varepsilon^N u^\varepsilon w \, d\mathcal{H}^{n-1} \, dt - \int_{\Omega_\varepsilon(0)} u_0^\varepsilon w(0) \, dx.$$

On the other hand, by integration by parts,

$$-\int_{\Omega_\varepsilon(t)} (\Delta u^\varepsilon) w \, dx \, dt = \int_{\Omega_\varepsilon(t)} \nabla u^\varepsilon \cdot \nabla w \, dx - \int_{\partial\Omega_\varepsilon(t)} (\partial_{\nu_\varepsilon} u^\varepsilon) w \, d\mathcal{H}^{n-1}.$$

Hence it follows that

$$\int_0^T \int_{\Omega_\varepsilon(t)} (-u^\varepsilon \partial_t w + \nabla u^\varepsilon \cdot \nabla w) \, dx \, dt - \int_0^T \int_{\partial\Omega_\varepsilon(t)} (\partial_{\nu_\varepsilon} u^\varepsilon + v_\varepsilon^N u^\varepsilon) w \, d\mathcal{H}^{n-1} \, dt \\ - \int_{\Omega_\varepsilon(0)} u_0^\varepsilon w(0) \, dx = 0$$

and we obtain (3.2) by applying the boundary condition of ($H_\varepsilon$) to the second term of the left-hand side in the above equality.

Our goal in this section is to obtain a unique variational solution to ($H_\varepsilon$) that satisfies an energy estimate with a constant independent of $\varepsilon$. To this end, we transform (3.2) into a variational formulation of some equation on the fixed (in time) domain $\Omega_\varepsilon(0)$ with the aid of a suitable diffeomorphism between $\Omega_\varepsilon(0)$ and $\Omega_\varepsilon(t)$.

**Lemma 3.2** *For each $t \in [0, T]$, there exists a diffeomorphism $\Psi_\varepsilon(\cdot, t)\colon \Omega_\varepsilon(0) \to \Omega_\varepsilon(t)$ with its inverse $\Psi_\varepsilon^{-1}(\cdot, t)\colon \Omega_\varepsilon(t) \to \Omega_\varepsilon(0)$ such that $\Psi_\varepsilon$ and $\Psi_\varepsilon^{-1}$ are smooth on $\overline{\Omega_\varepsilon(0)} \times [0, T]$ and $\overline{Q_{\varepsilon,T}}$, respectively, and $\Psi_\varepsilon(\cdot, 0)$ is the identity mapping on $\Omega_\varepsilon(0)$. Moreover, there exists a constant $c > 0$ independent of $\varepsilon$ such that*

$$\left|\partial_X^\alpha \partial_t^k \Psi_\varepsilon(X, t)\right| \leqslant c, \qquad \left|\partial_x^\alpha \partial_t^k \Psi_\varepsilon^{-1}(x, t)\right| \leqslant c \tag{3.3}$$

*for all $(X, t) \in \Omega_\varepsilon(0) \times (0, T)$, $(x, t) \in Q_{\varepsilon,T}$, and $|\alpha| + k \leqslant 2$, $k = 0, 1, 2$.*

*Proof.* We observe that for each $X \in \Omega_\varepsilon(0)$ there is a unique $\theta \in (0, 1)$ such that

$$X = p(X, 0) + \varepsilon\big\{(1-\theta) g_0\big(p(X, 0), 0\big) + \theta g_1\big(p(X, 0), 0\big)\big\} \nu(p(X, 0), 0), \tag{3.4}$$

that is, $X$ divides the line segment $A_0 A_1$ internally in the ratio $\theta\colon 1-\theta$, where

$$A_i := p(X, 0) + \varepsilon g_i\big(p(X, 0), 0\big) \nu\big(p(X, 0), 0\big), \quad i = 0, 1.$$

Based on this observation we define $\Psi_\varepsilon(X, t) \in \Omega_\varepsilon(t)$ as

$$\Psi_\varepsilon(X, t) := \Phi\big(p(X, 0), t\big) \\ + \varepsilon\Big\{(1-\theta) g_0\Big(\Phi\big(p(X, 0), t\big), t\Big) + \theta g_1\Big(\Phi\big(p(X, 0), t\big), t\Big)\Big\} \nu\Big(\Phi\big(p(X, 0), t\big), t\Big), \tag{3.5}$$

that is, $\Psi_\varepsilon(X,t)$ divides the line segment $B_0B_1$ internally in the ratio $\theta\colon 1-\theta$, where

$$B_i := \Phi\big(p(X,0),t\big) + \varepsilon g_i\Big(\Phi\big(p(X,0),t\big),t\Big)\nu\Big(\Phi\big(p(X,0),t\big),t\Big), \quad i=0,1.$$

To eliminate $\theta$ in (3.5), we take the inner product of both sides of (3.4) and $\nu(p(X,0),0)$. Then

$$\big\{X - p(X,0)\big\}\cdot\nu\big(p(X,0),0\big) = \varepsilon\big\{(1-\theta)g_0\big(p(X,0),0\big) + \theta g_1\big(p(X,0),0\big)\big\}.$$

Since $\{X-p(X,0)\}\cdot\nu(p(X,0),0) = d(X,0)$ and $g_1 - g_0 = g > 0$, it follows that

$$\theta = \frac{d(X,0) - \varepsilon g_0\big(p(X,0),0\big)}{\varepsilon g\big(p(X,0),0\big)}.$$

Hence, by substituting this for $\theta$ in (3.5), we obtain

$$\Psi_\varepsilon(X,t) = \Phi\big(p(X,0),t\big) + \big\{d(X,0)\phi_1(X,t) + \varepsilon\phi_2(X,t)\big\}\nu\Big(\Phi\big(p(X,0),t\big),t\Big) \tag{3.6}$$

for $X \in \Omega_\varepsilon(0)$ and $t \in [0,T]$, where

$$\phi_1(X,t) := \frac{g\Big(\Phi\big(p(X,0),t\big),t\Big)}{g\big(p(X,0),0\big)}, \quad \phi_2(X,t) := g_0\Big(\Phi\big(p(X,0),t\big),t\Big) - \phi_1(X,t)g_0\big(p(X,0),0\big).$$

Similarly we define a mapping $\Psi_\varepsilon^{-1}$ as

$$\Psi_\varepsilon^{-1}(x,t) := \Phi^{-1}\big(p(x,t),t\big) + \big\{d(x,t)\phi_3(x,t) + \varepsilon\phi_4(x,t)\big\}\nu\Big(\Phi^{-1}\big(p(x,t),t\big),0\Big) \tag{3.7}$$

for $(x,t) \in Q_{\varepsilon,T}$, where

$$\phi_3(x,t) := \frac{g\Big(\Phi^{-1}\big(p(x,t),t\big),0\Big)}{g\big(p(x,t),t\big)},$$
$$\phi_4(x,t) := g_0\Big(\Phi^{-1}\big(p(x,t),t\big),0\Big) - \phi_3(x,t)g_0\big(p(x,t),t\big).$$

By definition, $\Psi_\varepsilon(\cdot,t)\colon \Omega_\varepsilon(0) \to \Omega_\varepsilon(t)$ is a bijection with its inverse $\Psi_\varepsilon^{-1}(\cdot,t)\colon \Omega_\varepsilon(t) \to \Omega_\varepsilon(0)$ for each $t \in [0,T]$. Also, since $\Phi(\cdot,0)$ is the identity mapping on $\Gamma_0$, we have $\phi_1(X,0) = 1$, $\phi_2(X,0) = 0$ and thus

$$\Psi_\varepsilon(X,0) = p(X,0) + d(X,0)\nu\big(p(X,0),0\big) = X \quad \text{for all} \quad X \in \Omega_\varepsilon(0),$$

that is, $\Psi_\varepsilon(\cdot,0)$ is the identity mapping on $\Omega_\varepsilon(0)$. Due to the smoothness of $\Phi$, $\Phi^{-1}$, $d$, $p$, $g_0$, and $g_1$, the right-hand sides of (3.6) and (3.7) are smooth on the compact sets $\overline{N(0)} \times [0,T]$ and $\overline{N_T}$, respectively, and thus bounded independently of $\varepsilon$ along with their derivatives. From this fact and the inclusion $\overline{\Omega_\varepsilon(t)} \subset N(t)$ for each $t \in [0,T]$, it follows that $\Psi_\varepsilon$ and $\Psi_\varepsilon^{-1}$ are smooth on $\overline{\Omega_\varepsilon(0)} \times [0,T]$ and $\overline{Q_{\varepsilon,T}}$, respectively, and that the inequalities (3.3) hold with a constant $c > 0$ independent of $\varepsilon$. In particular, $\Psi_\varepsilon(\cdot,t)\colon \Omega_\varepsilon(0) \to \Omega_\varepsilon(t)$ is a diffeomorphism for each $t \in [0,T]$. □

Let $\Psi_\varepsilon$ and $\Psi_\varepsilon^{-1}$ be the mappings given by Lemma 3.2. In (3.2), we set

$$U^\varepsilon(X,t) := u^\varepsilon\big(\Psi_\varepsilon(X,t),t\big), \quad W(X,t) := w\big(\Psi_\varepsilon(X,t),t\big), \quad (X,t) \in \Omega_\varepsilon(0)\times(0,T).$$

Then, by the change of variables $x = \Psi_\varepsilon(X,t)$, we transform (3.2) into

$$\int_0^T \big\{-\big(U^\varepsilon(t), J^\varepsilon(t)\partial_t W(t)\big)_{L^2} + \big(A^\varepsilon(t)\nabla U^\varepsilon(t) - U^\varepsilon(t)B^\varepsilon(t), \nabla W(t)\big)_{L^2}\big\}\,dt - (u_0^\varepsilon, W(0))_{L^2} = 0. \quad (3.8)$$

Here $(\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(\Omega_\varepsilon(0))$ and

$$\begin{aligned} J^\varepsilon(X,t) &:= \big|\det\nabla\Psi_\varepsilon(X,t)\big| \in \mathbb{R},\\ A^\varepsilon(X,t) &:= J^\varepsilon(X,t)\nabla\Psi_\varepsilon^{-1}\big(\Psi_\varepsilon(X,t),t\big)\big[\nabla\Psi_\varepsilon^{-1}\big(\Psi_\varepsilon(X,t),t\big)\big]^T \in \mathbb{R}^{n\times n},\\ B^\varepsilon(X,t) &:= J^\varepsilon(X,t)\partial_t\Psi_\varepsilon^{-1}\big(\Psi_\varepsilon(X,t),t\big) \in \mathbb{R}^n \end{aligned}$$

for $(X,t) \in \Omega_\varepsilon(0)\times(0,T)$, where

$$\nabla\Psi_\varepsilon^{-1} := \begin{pmatrix} \partial_1(\Psi_\varepsilon^{-1})_1 & \dots & \partial_n(\Psi_\varepsilon^{-1})_1 \\ \vdots & \ddots & \vdots \\ \partial_1(\Psi_\varepsilon^{-1})_n & \dots & \partial_n(\Psi_\varepsilon^{-1})_n \end{pmatrix} \quad \text{for} \quad \Psi_\varepsilon^{-1} = \begin{pmatrix} (\Psi_\varepsilon^{-1})_1 \\ \vdots \\ (\Psi_\varepsilon^{-1})_n \end{pmatrix}$$

and $[\nabla\Psi_\varepsilon^{-1}]^T$ denotes the transposed matrix of $\nabla\Psi_\varepsilon^{-1}$. Note that the vector field $B^\varepsilon$ comes from the differentiation of $w(x,t) = W(\Psi_\varepsilon^{-1}(x,t),t)$ with respect to $t$ holding $x \in \Omega_\varepsilon(t)$ fixed:

$$\partial_t w(x,t) = \partial_t W\big(\Psi_\varepsilon^{-1}(x,t),t\big) + \partial_t\Psi_\varepsilon^{-1}(x,t)\cdot\nabla W\big(\Psi_\varepsilon^{-1}(x,t),t\big).$$

Since $w(T) = 0$ in $\Omega_\varepsilon(T)$ and $\Psi_\varepsilon(\cdot,0)$ is the identity mapping on $\Omega_\varepsilon(0)$, we have $W(T) = 0$ and $J^\varepsilon(0) = 1$ in $\Omega_\varepsilon(0)$. Thus, by integration by parts with respect to $t$, we further transform (3.8) into

$$\int_0^T \{{}_{(H^1)'}\langle\partial_t U^\varepsilon(t), J^\varepsilon(t)W(t)\rangle_{H^1} + \big(U^\varepsilon(t), W(t)\partial_t J^\varepsilon(t)\big)_{L^2} + (A^\varepsilon(t)\nabla U^\varepsilon(t) - U^\varepsilon(t)B^\varepsilon(t), \nabla W(t))_{L^2}\}\,dt = 0. \quad (3.9)$$

Here ${}_{(H^1)'}\langle\cdot,\cdot\rangle_{H^1}$ is the duality product between $H^1(\Omega_\varepsilon(0))$ and its dual space $(H^1(\Omega_\varepsilon(0)))'$.

**Theorem 3.3** *For every $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$, there exists a unique function*

$$U^\varepsilon \in L^\infty\Big(0,T;L^2\big(\Omega_\varepsilon(0)\big)\Big) \cap L^2\Big(0,T;H^1\big(\Omega_\varepsilon(0)\big)\Big) \quad \textit{with} \quad \partial_t U^\varepsilon \in L^2\bigg(0,T;\Big(H^1\big(\Omega_\varepsilon(0)\big)\Big)'\bigg)$$

*that satisfies (3.9) for all $W \in L^2(0,T;H^1(\Omega_\varepsilon(0)))$ and $U^\varepsilon(0) = u_0^\varepsilon$ in $L^2(\Omega_\varepsilon(0))$. Moreover, there exists a constant $c > 0$ independent of $u_0^\varepsilon$, $U^\varepsilon$, and $\varepsilon$ such that*

$$\sup_{t\in(0,T)}\|U^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(0))} + \int_0^T \|\nabla U^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(0))}\,dt \leq c\|u_0^\varepsilon\|^2_{L^2(\Omega_\varepsilon(0))}. \quad (3.10)$$

*Proof.* For $i, j = 1, \dots, n$, let $A^\varepsilon_{ij}$ be the $(i, j)$-entry of $A^\varepsilon$ and $B^\varepsilon_i$ be the $i$-th component of $B^\varepsilon$. Suppose that there is a positive constant $C$ independent of $\varepsilon$ such that

$$C^{-1} \leqslant J^\varepsilon(X, t) \leqslant C, \tag{3.11}$$

$$|\nabla J^\varepsilon(X, t)| \leqslant C, \quad |\partial_t J^\varepsilon(X, t)| \leqslant C, \quad |A^\varepsilon_{ij}(X, t)| \leqslant C, \quad |B^\varepsilon_i(X, t)| \leqslant C, \tag{3.12}$$

$$A^\varepsilon(X, t)\zeta \cdot \zeta \geqslant C|\zeta|^2 \tag{3.13}$$

for all $(X, t) \in \Omega_\varepsilon(0) \times (0, T)$, $\zeta \in \mathbb{R}^n$, and $i, j = 1, \dots, n$. Then the theorem is proved by a standard Galerkin method and Gronwall argument, see [10, Section 7.1] for details. In particular, the constant $c$ in (3.10) depends only on the above $C$ and thus it is independent of $\varepsilon$.

Let us prove (3.11), (3.12), and (3.13). The inequalities (3.12) and the right-hand inequality of (3.11) immediately follow from (3.3). Since $\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X, t), t)\nabla\Psi_\varepsilon(X, t) = I_n$, it follows that

$$\left|\det \nabla\Psi_\varepsilon^{-1}\big(\Psi_\varepsilon(X, t), t\big)\right| J^\varepsilon(X, t) = 1, \quad \big[\nabla\Psi_\varepsilon(X, t)\big]^T \big[\nabla\Psi_\varepsilon^{-1}\big(\Psi_\varepsilon(X, t), t\big)\big]^T = I_n$$

for all $(X, t) \in \Omega_\varepsilon(0) \times (0, T)$. The first equality yields the left-hand inequality of (3.11) because $|\det \nabla\Psi_\varepsilon^{-1}|$ is bounded on $Q_{\varepsilon,T}$ independently of $\varepsilon$ by (3.3). Moreover, the above equality and (3.3) imply that, for all $(X, t) \in \Omega_\varepsilon(0) \times (0, T)$ and $\zeta \in \mathbb{R}^n$,

$$\begin{aligned}
|\zeta|^2 &= \left|\big[\nabla\Psi_\varepsilon(X, t)\big]^T \big[\nabla\Psi_\varepsilon^{-1}\big(\Psi_\varepsilon(X, t), t\big)\big]^T \zeta\right|^2 \\
&\leqslant c\left|\big[\nabla\Psi_\varepsilon^{-1}\big(\Psi_\varepsilon(X, t), t\big)\big]^T \zeta\right|^2 \\
&= c\left\{\nabla\Psi_\varepsilon^{-1}\big(\Psi_\varepsilon(X, t), t\big)\big[\nabla\Psi_\varepsilon^{-1}\big(\Psi_\varepsilon(X, t), t\big)\big]^T \zeta\right\} \cdot \zeta \\
&= c\left|\det \nabla\Psi_\varepsilon^{-1}\big(\Psi_\varepsilon(X, t), t\big)\right| A^\varepsilon(X, t)\zeta \cdot \zeta \leqslant cA^\varepsilon(X, t)\zeta \cdot \zeta
\end{aligned}$$

with a constant $c > 0$ independent of $\varepsilon$. Thus (3.13) follows. □

Now we can show the existence and uniqueness of a variational solution to ($H_\varepsilon$) and its energy estimate with a constant independent of $\varepsilon$.

**Theorem 3.4** *For every $u^\varepsilon_0 \in L^2(\Omega_\varepsilon(0))$, there exists a unique variational solution $u^\varepsilon$ to ($H_\varepsilon$). Moreover, $u^\varepsilon$ satisfies that $u^\varepsilon(0) = u^\varepsilon_0$ in $L^2(\Omega_\varepsilon(0))$ and*

$$\sup_{t\in(0,T)} \|u^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(t))} + \int_0^T \|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(t))}\, dt \leqslant c\|u^\varepsilon_0\|^2_{L^2(\Omega_\varepsilon(0))} \tag{3.14}$$

*with a constant $c > 0$ independent of $u^\varepsilon_0$, $u^\varepsilon$, and $\varepsilon$.*

*Proof.* For each $u^\varepsilon_0 \in L^2(\Omega_\varepsilon(0))$, let $U^\varepsilon$ be the unique function given by Theorem 3.3 and we set

$$u^\varepsilon(x, t) := U^\varepsilon\big(\Psi_\varepsilon^{-1}(x, t), t\big), \quad (x, t) \in Q_{\varepsilon,T}.$$

Since $\Psi_\varepsilon(\cdot, 0)$ is the identity mapping on $\Omega_\varepsilon(0)$ by Lemma 3.2 and $U^\varepsilon(0) = u^\varepsilon_0$ in $L^2(\Omega_\varepsilon(0))$ by Theorem 3.3, we have $u^\varepsilon(0) = u^\varepsilon_0$ in $L^2(\Omega_\varepsilon(0))$. Let us show that $u^\varepsilon$ satisfies (3.2) for all $w \in C^1(\overline{Q_{\varepsilon,T}})$ with $w(T) = 0$ in $\Omega_\varepsilon(T)$. Since $\Psi_\varepsilon$ is smooth on $\overline{\Omega_\varepsilon(0)} \times [0, T]$, a function

$$W(X, t) := w\big(\Psi_\varepsilon(X, t), t\big), \quad (X, t) \in \overline{\Omega_\varepsilon(0)} \times [0, T]$$

is in $C^1(\overline{\Omega_\varepsilon(0)}\times[0,T])$ and satisfies $W(T)=0$ in $\Omega_\varepsilon(0)$. Hence we can substitute it for $W$ in (3.9) and integrate by parts with respect to $t$ to get (3.8). By changing variables $X=\Psi_\varepsilon^{-1}(x,t)$ in (3.8), we obtain (3.2).

Next we prove the energy estimate (3.14). By the change of variables $x=\Psi_\varepsilon(X,t)$ we have

$$\int_{\Omega_\varepsilon(t)} \left|u^\varepsilon(x,t)\right|^2 dx = \int_{\Omega_\varepsilon(0)} \left|U^\varepsilon(X,t)\right|^2 \left|\det\nabla\Psi_\varepsilon(X,t)\right| dX,$$

$$\int_{\Omega_\varepsilon(t)} \left|\nabla u^\varepsilon(x,t)\right|^2 dx = \int_{\Omega_\varepsilon(0)} \left|\left[\nabla\Psi_\varepsilon^{-1}\left(\Psi_\varepsilon(X,t),t\right)\right]^T \nabla U^\varepsilon(X,t)\right|^2 \left|\det\nabla\Psi_\varepsilon(X,t)\right| dX$$

for all $t\in[0,T]$. Hence the inequalities (3.3) yield

$$\|u^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(t))} \leqslant c\|U^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(0))}, \qquad \|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(t))} \leqslant c\|\nabla U^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(0))}$$

with a constant $c>0$ independent of $\varepsilon$. By these inequalities and (3.10), we obtain (3.14) and thus $u^\varepsilon\in L^2_{H^1(\varepsilon)}$. Hence $u^\varepsilon$ is a variational solution to ($H_\varepsilon$).

Finally, the uniqueness of a variational solution to ($H_\varepsilon$) follows from that of a function given by Theorem 3.3. The proof is complete. □

REMARK 3.5 Let $u^\varepsilon$ be the unique variational solution to ($H_\varepsilon$) with initial data $u_0^\varepsilon\in L^2(\Omega_\varepsilon(0))$. Then it immediately follows from (3.14) that

$$\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}} \leqslant c\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}, \tag{3.15}$$

where $c>0$ is a constant independent of $u_0^\varepsilon$, $u^\varepsilon$, and $\varepsilon$. We will use this inequality in Section 6.

## 4. Basic function spaces on evolving surfaces

In this section, we define function spaces on the space-time manifold $S_T$ introduced by Olshanskii, Reusken, and Xu [19] and give their properties. These spaces will give an appropriate variational formulation of a limit equation on $\Gamma(t)$ we will derive in Section 6. All results in this section are originally obtained in [19] for the three-dimensional case. They can be easily extended for arbitrary dimensions and we give proofs of them for the readers' convenience.

For each fixed $T>0$, we define a function space $H_T$ and an inner product on $H_T$ as

$$\begin{aligned} H_T &:= \left\{v\in L^2(S_T)\mid \nabla_{\Gamma(t)}v\in L^2(S_T)\right\},\\ (v_1,v_2)_{H_T} &:= \int_0^T\int_{\Gamma(t)} \left\{v_1(y,t)v_2(y,t)+\nabla_{\Gamma(t)}v_1(y,t)\cdot\nabla_{\Gamma(t)}v_2(y,t)\right\} d\mathcal{H}^{n-1}(y)\,dt. \end{aligned} \tag{4.1}$$

This inner product induces the norm $\|\cdot\|_{H_T}$ that is equivalent to the one induced by the inner product $\int_{S_T}\{v_1(\sigma)v_2(\sigma)+\nabla_{\Gamma(t)}v_1(\sigma)\cdot\nabla_{\Gamma(t)}v_2(\sigma)\}\,d\mathcal{H}^n(\sigma)$, since the identity

$$\int_0^T\int_{\Gamma(t)} f(y,t)\,d\mathcal{H}^{n-1}(y)\,dt = \int_{S_T} f(\sigma)(1+|v_\Gamma^N(\sigma)|^2)^{-1/2}\,d\mathcal{H}^n(\sigma) \tag{4.2}$$

holds and $v_\Gamma^N$ is bounded on $S_T$. This identity is stated in [19] without proof. We give the proof of (4.2) in Appendix B for the readers' convenience. If $T_1<T_2$, then $H_{T_2}$ is continuously embedded into $H_{T_1}$ just by restricting elements of $H_{T_2}$ on $S_{T_1}$.

Next we define an auxiliary space. Let $H^1(\Gamma_0) := \{V \in L^2(\Gamma_0) \mid \nabla_{\Gamma_0} V \in L^2(\Gamma_0)\}$ with the inner product $(V_1, V_2)_{H^1(\Gamma_0)} := \int_{\Gamma_0} (V_1 V_2 + \nabla_{\Gamma_0} V_1 \cdot \nabla_{\Gamma_0} V_2)\, d\mathcal{H}^{n-1}$, where $\nabla_{\Gamma_0}$ is the tangential gradient on $\Gamma_0$. Then we define a Hilbert space $\widehat{H}_T$ as

$$\widehat{H}_T := L^2\big(0, T; H^1(\Gamma_0)\big), \quad (V_1, V_2)_{\widehat{H}_T} := \int_0^T \big(V_1(t), V_2(t)\big)_{H^1(\Gamma_0)}\, dt$$

and let $\|\cdot\|_{\widehat{H}_T}$ denote the norm of $\widehat{H}_T$ induced by the inner product $(\cdot,\cdot)_{\widehat{H}_T}$.

Let $\Phi(\cdot, t)\colon \Gamma_0 \to \Gamma(t)$ be the flow map of $V_\Gamma$ and $\Phi^{-1}(\cdot, t)$ be its inverse mapping (see Section 2). For a function $V$ on $\Gamma_0 \times (0, T)$, we define a function $v = LV$ on $S_T$ as

$$v(y, t) := V\big(\Phi^{-1}(y, t), t\big), \quad (y, t) \in S_T. \tag{4.3}$$

Also, for a function $v$ on $S_T$, we define a function $V = L^{-1}v$ on $\Gamma_0 \times (0, T)$ as

$$V(Y, t) := v\big(\Phi(Y, t), t\big), \quad (Y, t) \in \Gamma_0 \times (0, T).$$

Clearly $L$ and $L^{-1}$ are linear mappings and satisfy $L^{-1}(LV) = V$ and $L(L^{-1}v) = v$ for all functions $V$ on $\Gamma_0 \times (0, T)$ and $v$ on $S_T$.

**Lemma 4.1** *The linear mapping $L$ given by (4.3) defines an isomorphism between $\widehat{H}_T$ and $H_T$.*

A short proof is given in [19]. We give a detailed proof in Appendix B for the completeness.

Let $C_0^1(S_T)$ be the space of all functions in $C^1(S_T)$ with compact support in $S_T$. That is, each function in $C_0^1(S_T)$ vanishes near $t = 0$ and $t = T$.

**Lemma 4.2** *The space $H_T$ is a Hilbert space and $C_0^1(S_T)$ is dense in $H_T$.*

*Proof.* Since $\widehat{H}_T$ is a Hilbert space, Lemma 4.1 implies that $H_T$ is a Hilbert space. Also, since $C_0^1(\Gamma_0 \times (0, T))$ is dense in $\widehat{H}_T$ (see [19, Lemma 3.1]) and $C_0^1(S_T)$ includes $L[C_0^1(\Gamma_0 \times (0, T))]$, Lemma 4.1 again implies that $C_0^1(S_T)$ is dense in $H_T$. □

The space $H_T$ is continuously embedded into $L^2(S_T)$. Moreover, $H_T$ is dense in $L^2(S_T)$ since it includes a dense subspace $C_0^1(S_T)$ of $L^2(S_T)$. Hence we have continuous and dense embeddings $H_T \hookrightarrow L^2(S_T) \hookrightarrow H_T'$, where $H_T'$ is the dual space of $H_T$.

For $v \in C^1(S_T)$, we define its (strong) material derivative $\partial^\bullet v$ as

$$\partial^\bullet v\big(\Phi(Y, t), t\big) := \frac{d}{dt}\big(v(\Phi(Y, t), t)\big), \quad (Y, t) \in \Gamma_0 \times (0, T). \tag{4.4}$$

From the Leibniz formula (see [4, Lemma 2.2])

$$\frac{d}{dt}\int_{\Gamma(t)} v\, d\mathcal{H}^{n-1} = \int_{\Gamma(t)} (\partial^\bullet v + v\, \mathrm{div}_{\Gamma(t)} V_\Gamma)\, d\mathcal{H}^{n-1}, \quad v \in C^1(S_T),$$

we have the integration by parts identity

$$\begin{aligned}\int_0^T \int_{\Gamma(t)} &(v_2 \partial^\bullet v_1 + v_1 \partial^\bullet v_2 + v_1 v_2\, \mathrm{div}_{\Gamma(t)} V_\Gamma)\, d\mathcal{H}^{n-1}\, dt \\ &= \int_{\Gamma(T)} v_1(T) v_2(T)\, d\mathcal{H}^{n-1} - \int_{\Gamma(0)} v_1(0) v_2(0)\, d\mathcal{H}^{n-1}\end{aligned} \tag{4.5}$$

for all $v_1, v_2 \in C^1(S_T)$. Based on this identity, we define the weak material derivative of $v \in H_T$ as a functional $\partial^\bullet v$ such that

$$\langle \partial^\bullet v, \psi \rangle_T := -\int_0^T \int_{\Gamma(t)} (v\, \partial^\bullet \psi + v\psi\, \mathrm{div}_{\Gamma(t)} V_\Gamma)\, d\mathcal{H}^{n-1}\, dt, \quad \psi \in C_0^1(S_T). \tag{4.6}$$

If $v \in C^1(S_T)$, then its weak material derivative agrees with the strong one. We set

$$\|\partial^\bullet v\|_{H_T'} := \sup_{\psi \in C_0^1(S_T)\setminus\{0\}} \frac{\langle \partial^\bullet v, \psi \rangle_T}{\|\psi\|_{H_T}}, \quad v \in H_T.$$

If $\|\partial^\bullet v\|_{H_T'}$ is finite for some $v \in H_T$, then $\partial^\bullet v$ can be extended to a bounded linear functional on $H_T$ because $C_0^1(S_T)$ is dense in $H_T$ (see Lemma 4.2). In this case, we say that $\partial^\bullet v$ is in $H_T'$ and we define a function space $W_T$ and its norm as

$$W_T := \{v \in H_T \mid \partial^\bullet v \in H_T'\}, \quad \|v\|_{W_T} := \left(\|v\|_{H_T}^2 + \|\partial^\bullet v\|_{H_T'}^2\right)^{1/2}. \tag{4.7}$$

For $T_1 < T_2$, the space $W_{T_2}$ is continuously embedded into $W_{T_1}$ since $C_0^1(S_{T_1}) \subset C_0^1(S_{T_2})$ and $H_{T_2}$ is continuously embedded into $H_{T_1}$.

To investigate properties of $W_T$, we define an auxiliary Hilbert space and its norm as

$$\widehat{W}_T := \{V \in \widehat{H}_T \mid \partial_t V \in \widehat{H}_T'\}, \quad \|V\|_{\widehat{W}_T} := \left(\|V\|_{\widehat{H}_T}^2 + \|\partial_t V\|_{\widehat{H}_T'}^2\right)^{1/2}.$$

Here $\widehat{H}_T'$ is the dual space of $\widehat{H}_T$ and $\partial_t V$ is the weak time derivative of $V \in \widehat{H}_T$ defined as

$$[\partial_t V, \Psi]_T := -\int_0^T \int_{\Gamma_0} V\, \partial_t \Psi\, d\mathcal{H}^{n-1}\, dt, \quad \Psi \in C_0^1\big(\Gamma_0 \times (0,T)\big),$$

and we say $\partial_t V \in \widehat{H}_T'$ if $\|\partial_t V\|_{\widehat{H}_T'} := \sup_{\Psi \in C_0^1(\Gamma_0 \times (0,T))\setminus\{0\}} [\partial_t V, \Psi]_T / \|\Psi\|_{\widehat{H}_T}$ is finite.

**Lemma 4.3** *The linear mapping $L$ given by (4.3) defines an isomorphism between $\widehat{W}_T$ and $W_T$.*

A proof for the three-dimensional case is given in [19] and easily extended for arbitrary dimensions. We give a complete proof in Appendix B for the readers' convenience.

Lemma 4.3 shows that $W_T$ has similar properties to those of $\widehat{W}_T$.

**Lemma 4.4** *The space $W_T$ is a Hilbert space and $C^1(S_T)$ is dense in $W_T$. Moreover, the trace operator $v \mapsto v(t)$ from $C^1(S_T)$ into $L^2(\Gamma(t))$ for each $t \in [0,T]$ can be extended to a bounded linear operator from $W_T$ to $L^2(\Gamma(t))$ and there exists a constant $c > 0$ such that*

$$\max_{t \in [0,T]} \|v(t)\|_{L^2(\Gamma(t))} \le c\|v\|_{W_T}$$

*for all $v \in W_T$.*

*Proof.* Since $\widehat{W}_T$ is a Hilbert space, Lemma 4.3 implies that $W_T$ is a Hilbert space. For the rest of the proof, see [19, Theorem 3.6]. □

Finally we show an integration by parts formula which we will use in Section 6.

**Lemma 4.5** *For all $v_1, v_2 \in W_T$, we have*

$$\langle \partial^\bullet v_1, v_2\rangle_T + \langle \partial^\bullet v_2, v_1\rangle_T + \int_0^T \int_{\Gamma(t)} v_1 v_2 \,\mathrm{div}_{\Gamma(t)} V_\Gamma \, d\mathcal{H}^{n-1}\, dt = \int_{\Gamma(T)} v_1(T)v_2(T)\, d\mathcal{H}^{n-1} - \int_{\Gamma_0} v_1(0)v_2(0)\, d\mathcal{H}^{n-1}. \quad (4.8)$$

Note that, by Lemma 4.4, $v_i(0)$ and $v_i(T)$, $i = 1, 2$, are well-defined as functions in $L^2(\Gamma_0)$ and $L^2(\Gamma(T))$, respectively.

*Proof.* For $v \in C^1(S_T)$, its weak material derivative agrees with the strong one. Thus we have

$$\langle \partial^\bullet v, \psi\rangle_T = \int_0^T \int_{\Gamma(t)} (\partial^\bullet v)\psi \, d\mathcal{H}^{n-1}\, dt, \quad \psi \in C_0^1(S_T).$$

Moreover, since $C_0^1(S_T)$ is dense in $H_T$ (see Lemma 4.2), the above equality holds for all $\psi \in H_T$ and thus (4.8) follows from (4.5) when $v_1, v_2 \in C^1(S_T)$. Since $C^1(S_T)$ is dense in $W_T$ (see Lemma 4.4), a density argument shows that (4.8) holds for general $v_1, v_2 \in W_T$. □

## 5. Average operator

### 5.1 *Definition and basic properties of the average operator*

In this section we define and investigate a weighted average operator. Lemma 5.6 and Lemma 5.13 are fundamental to derivation of a limit equation of ($H_\varepsilon$) in Section 6. Other results in this section are also useful themselves.

For $(y,t) \in \overline{S_T}$, let $\kappa_1(y,t), \dots, \kappa_{n-1}(y,t)$ be the principal curvatures of $\Gamma(t)$ at $y$ (see [11, Section 14.6]). We define a function $J$ on $\overline{S_T} \times (-\delta, \delta)$ as

$$J(y,t,\rho) := \prod_{i=1}^{n-1} \{1 - \rho\kappa_i(y,t)\}, \quad (y,t) \in \overline{S_T},\ \rho \in (-\delta, \delta).$$

Here $\delta > 0$ is a half of the width of the tubular neighborhood $N(t)$ of $\Gamma(t)$, which is independent of $t \in [0,T]$ (see Section 2). The function $J$ is the Jacobian appearing in the transformation formula

$$\int_{\Omega_\varepsilon(t)} u(x)\, dx = \int_{\Gamma(t)} \int_{\varepsilon g_0(y,t)}^{\varepsilon g_1(y,t)} u\big(y + \rho\nu(y,t)\big) J(y,t,\rho)\, d\rho\, d\mathcal{H}^{n-1}(y) \quad (5.1)$$

for a function $u$ on $\Omega_\varepsilon(t)$ with each fixed $t \in [0,T]$, see (14.98) in [11]. This formula can be viewed as a co-area formula. Based on this formula, we define a weighted average operator $M_\varepsilon$ as follows.

DEFINITION 5.1 For a function $u$ on $Q_{\varepsilon,T}$, we define its weighted average $M_\varepsilon u$ as

$$M_\varepsilon u(y,t) := \frac{1}{\varepsilon g(y,t)} \int_{\varepsilon g_0(y,t)}^{\varepsilon g_1(y,t)} u\big(y + \rho\nu(y,t), t\big) J(y,t,\rho)\, d\rho, \quad (y,t) \in S_T. \quad (5.2)$$

We use the same notation $M_\varepsilon u$ for the average of a function $u$ on $\Omega_\varepsilon(t)$ with each fixed $t \in [0,T]$.

Before starting to derive properties of the average operator, we give inequalities which we use throughout Section 5 and Section 6. Since $\kappa_1, \dots, \kappa_{n-1}$ are smooth on $\overline{S_T}$, they are bounded on $\overline{S_T}$ along with their derivatives. Hence, by taking $\delta > 0$ sufficiently small, we may assume that there exists a constant $c > 0$ such that

$$c^{-1} \leqslant 1 - \rho\kappa_i(y,t) \leqslant c \quad \text{for all} \quad (y,t) \in \overline{S_T},\ \rho \in (-\delta,\delta),\ i = 1,\dots,n-1. \tag{5.3}$$

Then $J$ is smooth and bounded on $\overline{S_T} \times (-\delta,\delta)$ along with its derivatives and satisfies

$$c^{-1} \leqslant J(y,t,\rho) \leqslant c \quad \text{for all} \quad (y,t) \in \overline{S_T},\ \rho \in (-\delta,\delta). \tag{5.4}$$

Moreover, since $J(y,t,\rho)$ is of the form

$$J(y,t,\rho) = 1 - \rho\sum_{i=1}^{n-1}\kappa_i(y,t) + \rho^2 P\big(\kappa_1(y,t),\dots,\kappa_{n-1}(y,t),\rho\big),$$

where $P(z)$ is a polynomial in $z = (z_1,\dots,z_n) \in \mathbb{R}^n$, we have

$$\big|1 - J(y,t,\rho)\big| \leqslant c\varepsilon, \quad \big|\nabla_{\Gamma(t)} J(y,t,\rho)\big| \leqslant c\varepsilon, \quad \left|\frac{\partial J}{\partial\rho}(y,t,\rho)\right| \leqslant c \tag{5.5}$$

for all $(y,t) \in \overline{S_T}$ and $\rho \in (\varepsilon g_0(y,t), \varepsilon g_1(y,t))$ with a constant $c > 0$ independent of $\varepsilon$.

Now let us derive properties of the average operator $M_\varepsilon$. For a function $u$ on $Q_{\varepsilon,T}$, we set

$$u^\sharp(y,t,\rho) := u\big(y + \rho\nu(y,t),t\big), \quad (y,t) \in S_T,\ \rho \in \big(\varepsilon g_0(y,t), \varepsilon g_1(y,t)\big). \tag{5.6}$$

For simplicity, we omit arguments of functions unless we need to specify them. For example, the co-area formula (5.1) is referred to as

$$\int_{\Omega_\varepsilon(t)} u\,dx = \int_{\Gamma(t)}\int_{\varepsilon g_0}^{\varepsilon g_1} u^\sharp J\,d\rho\,d\mathcal{H}^{n-1}.$$

Throughout the rest of this subsection and the next subsection, we fix $t \in [0,T]$ and omit it. For example, we refer to $\Gamma(t)$ as $\Gamma$. Also, let $c$ denote a general positive constant independent of $t$.

**Lemma 5.2** *If $v \in L^2(\Gamma)$, then its constant extension $\overline{v}$ in the normal direction of $\Gamma$ is in $L^2(\Omega_\varepsilon)$. Moreover, there exists a constant $c > 0$ independent of $\varepsilon$ such that*

$$\|\overline{v}\|_{L^2(\Omega_\varepsilon)} \leqslant c\varepsilon^{1/2}\|v\|_{L^2(\Gamma)}. \tag{5.7}$$

*Proof.* By the co-area formula (5.1) and (5.4),

$$\|\overline{v}\|^2_{L^2(\Omega_\varepsilon)} = \int_\Gamma\int_{\varepsilon g_1}^{\varepsilon g_0} |v|^2 J\,d\rho\,d\mathcal{H}^{n-1} \leqslant c\int_\Gamma \varepsilon g|v|^2\,d\mathcal{H}^{n-1} \leqslant c\varepsilon\|v\|^2_{L^2(\Gamma)}.$$

Thus (5.7) follows. □

**Lemma 5.3** *If $u \in L^2(\Omega_\varepsilon)$, then $M_\varepsilon u \in L^2(\Gamma)$ and*

$$\|M_\varepsilon u\|_{L^2(\Gamma)} \leqslant c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)} \tag{5.8}$$

*with a constant $c > 0$ independent of $\varepsilon$.*

*Proof.* By Hölder's inequality, (5.4), (2.1), and the co-area formula (5.1),

$$\int_\Gamma |M_\varepsilon u|^2\, d\mathcal{H}^{n-1} \leqslant \int_\Gamma (\varepsilon g)^{-2}\left(\int_{\varepsilon g_0}^{\varepsilon g_1} J\, d\rho\right)\left(\int_{\varepsilon g_0}^{\varepsilon g_1} |u^\sharp|^2 J\, d\rho\right) d\mathcal{H}^{n-1}$$
$$\leqslant c\int_\Gamma (\varepsilon g)^{-1}\int_{\varepsilon g_0}^{\varepsilon g_1} |u^\sharp|^2 J\, d\rho\, d\mathcal{H}^{n-1} \leqslant c\varepsilon^{-1}\int_{\Omega_\varepsilon} |u|^2\, dx.$$

Thus (5.8) follows. □

By Lemma 5.2 and Lemma 5.3, the constant extension of $M_\varepsilon u$ in the normal direction of $\Gamma$ is in $L^2(\Omega_\varepsilon)$ for all $u \in L^2(\Omega_\varepsilon)$. Let us estimate the difference between $u$ and $M_\varepsilon u$ in $L^2(\Omega_\varepsilon)$.

**Lemma 5.4** *There exists a constant $c > 0$ independent of $\varepsilon$ such that*

$$\left\| u - \overline{M_\varepsilon u} \right\|_{L^2(\Omega_\varepsilon)} \leqslant c\varepsilon \|u\|_{H^1(\Omega_\varepsilon)} \tag{5.9}$$

*for all $u \in H^1(\Omega_\varepsilon)$. Here $\overline{M_\varepsilon u}$ is the constant extension of $M_\varepsilon u$ in the normal direction of $\Gamma$.*

*Proof.* For $y \in \Gamma$ and $\rho \in (\varepsilon g_0(y), \varepsilon g_1(y))$, we set

$$I_1(y,\rho) = \frac{1}{\varepsilon g(y)}\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \left\{u^\sharp(y,\rho) - u^\sharp(y,r)\right\} dr,$$
$$I_2(y) = \frac{1}{\varepsilon g(y)}\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} u^\sharp(y,r)\{1 - J(y,r)\}\, dr.$$

Then we have $u^\sharp(y,\rho) - M_\varepsilon u(y) = I_1(y,\rho) + I_2(y)$. Let us estimate $I_1$ and $I_2$. Since

$$|u^\sharp(y,\rho) - u^\sharp(y,r)| = \left|\int_r^\rho \frac{d}{d\eta}\Big(u\big(y + \eta\nu(y)\big)\Big)\, d\eta\right|$$
$$= \left|\int_r^\rho \nu(y)\cdot\nabla u\big(y + \eta\nu(y)\big)\, d\eta\right| \leqslant \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \left|(\nabla u)^\sharp(y,\eta)\right| d\eta$$

for all $\rho, r \in (\varepsilon g_0(y), \varepsilon g_1(y))$ and the right-hand side of the above inequality is independent of $r$,

$$|I_1(y,\rho)| \leqslant \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |(\nabla u)^\sharp(y,\eta)|\, d\eta.$$

On the other hand, by (2.1) and (5.5) we have

$$|I_2(y)| \leqslant c\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |u^\sharp(y,r)|\, dr.$$

These inequalities and Hölder's inequality yield

$$\left|u^\sharp(y,\rho) - M_\varepsilon u(y)\right| \leqslant \left|I_1(y,\rho)\right| + \left|I_2(y)\right| \leqslant c\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \Big(\left|u^\sharp(y,r)\right| + \left|(\nabla u)^\sharp(y,r)\right|\Big)\, dr$$
$$\leqslant c\left(\varepsilon g(y)\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (|u^\sharp(y,r)|^2 + |(\nabla u)^\sharp(y,r)|^2)\, dr\right)^{1/2}.$$

Here the last term is independent of $\rho$. Hence by the co-area formula (5.1) and (5.4) we obtain

$$\begin{aligned}\|u-\overline{M_\varepsilon u}\|^2_{L^2(\Omega_\varepsilon)} &= \int_\Gamma \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \left|u^\sharp(y,\rho)-M_\varepsilon u(y)\right|^2 J(y,\rho)\,d\rho\,d\mathcal{H}^{n-1}(y)\\ &\leqslant c\int_\Gamma \{\varepsilon g(y)\}^2 \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \left(\left|u^\sharp(y,r)\right|^2+\left|(\nabla u)^\sharp(y,r)\right|^2\right) dr\,d\mathcal{H}^{n-1}(y)\\ &\leqslant c\varepsilon^2\int_\Gamma \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \left(\left|u^\sharp(y,r)\right|^2+\left|(\nabla u)^\sharp(y,r)\right|^2\right) J(y,r)\,dr\,d\mathcal{H}^{n-1}(y)\\ &= c\varepsilon^2\|u\|^2_{H^1(\Omega_\varepsilon)}.\end{aligned}$$

Thus (5.9) follows. □

### 5.2 *Tangential gradient of the average operator*

In this subsection, we investigate relations between the usual gradient operator in $\Omega_\varepsilon$ and the tangential gradient operator on $\Gamma$. We first establish estimates for the gradient of the constant extension of a function on $\Gamma$ in the normal direction of $\Gamma$.

**Lemma 5.5** *If $v \in H^1(\Gamma)$, then its constant extension $\bar{v}$ in the normal direction of $\Gamma$ is in $H^1(\Omega_\varepsilon)$. Moreover, there exists a constant $c > 0$ independent of $\varepsilon$ such that*

$$\left\|\nabla\bar{v}\right\|_{L^2(\Omega_\varepsilon)} \leqslant c\varepsilon^{1/2}\left\|\nabla_\Gamma v\right\|_{L^2(\Gamma)}, \quad \left\|\nabla\bar{v}-\overline{\nabla_\Gamma v}\right\|_{L^2(\Omega_\varepsilon)} \leqslant c\varepsilon^{3/2}\left\|\nabla_\Gamma v\right\|_{L^2(\Gamma)}. \tag{5.10}$$

*Proof.* The first inequality of (5.10) and Lemma 5.2 imply $\bar{v} \in H^1(\Omega_\varepsilon)$ for all $v \in H^1(\Gamma)$. The inequalities (5.10) follow from the co-area formula (5.1), (5.4), and the inequalities

$$\left|\nabla\bar{v}\big(y+\rho\nu(y)\big)\right| \leqslant c\left|\nabla_\Gamma v(y)\right|, \quad \left|\nabla\bar{v}\big(y+\rho\nu(y)\big)-\nabla_\Gamma v(y)\right| \leqslant c\varepsilon\left|\nabla_\Gamma v(y)\right| \tag{5.11}$$

for all $y \in \Gamma$ and $\rho \in (\varepsilon g_0(y), \varepsilon g_1(y))$. We prove (5.11) in Appendix C. Here we give the main idea for the proof. We fix each $y_0 \in \Gamma$. By a rotation of coordinates, we can take a smooth function $f: U \to \mathbb{R}$ with an open set $U$ in $\mathbb{R}^{n-1}$ such that $\Gamma$ is described as the graph of $f$ near $y_0$ and

$$\nabla' f(s_0) = 0, \quad (\nabla')^2 f(s_0) = \mathrm{diag}\big[\kappa_1(y_0), \dots, \kappa_{n-1}(y_0)\big],$$

where $y_0 = (s_0, f(s_0))$ with $s_0 \in U$ and $\nabla'$ is the gradient in $s \in \mathbb{R}^{n-1}$ (see [11, Section 14.6]). Then (5.11) at $y_0$ is proved by direct calculations under this local coordinate system. □

Next we approximate an $H^1$-bilinear form on $\Omega_\varepsilon$ by that on $\Gamma$ with the tangential gradient of the weighted average of a function on $\Omega_\varepsilon$.

**Lemma 5.6** *For $u \in C^\infty(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)$ and $\varphi \in H^1(\Gamma)$, let*

$$I^1_\varepsilon(u,\varphi) := \int_{\Omega_\varepsilon} \nabla u \cdot \nabla\bar{\varphi}\,dx - \varepsilon\int_\Gamma g\nabla_\Gamma M_\varepsilon u \cdot \nabla_\Gamma\varphi\,d\mathcal{H}^{n-1}. \tag{5.12}$$

*Then there exists a constant $c > 0$ independent of $u$, $\varphi$, and $\varepsilon$ such that*

$$\left|I^1_\varepsilon(u,\varphi)\right| \leqslant c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\nabla_\Gamma\varphi\|_{L^2(\Gamma)}. \tag{5.13}$$

REMARK 5.7 The bilinear form $I_\varepsilon^1(u,\varphi)$ is well-defined for $u \in C^\infty(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)$ and $\varphi \in H^1(\Gamma)$, since $\overline{\varphi} \in H^1(\Omega_\varepsilon)$ by Lemma 5.5 and $M_\varepsilon u$ is smooth on $\Gamma$ and thus in $H^1(\Gamma)$ by the compactness of $\Gamma$. We will observe later that $I_\varepsilon^1(u,\varphi)$ is well-defined and (5.13) holds for all $u \in H^1(\Omega_\varepsilon)$ and $\varphi \in H^1(\Gamma)$, see Remark 5.9.

*Proof of Lemma 5.6.* By the co-area formula (5.1) we have $I_\varepsilon^1(u,\varphi) = \int_\Gamma I(y)\,d\mathcal{H}^{n-1}(y)$, where

$$I(y) := \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (\nabla u)^\sharp(y,\rho)\cdot(\nabla\overline{\varphi})^\sharp(y,\rho)J(y,\rho)\,d\rho - \varepsilon g(y)\nabla_\Gamma M_\varepsilon u(y)\cdot\nabla_\Gamma\varphi(y).$$

Here we used the notation (5.6). Suppose that there is a constant $c > 0$ independent of $\varepsilon$ such that

$$\left|I(y)\right| \leq c\varepsilon\left|\nabla_\Gamma\varphi(y)\right|\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)}\left(\left|u^\sharp(y,\rho)\right| + \left|(\nabla u)^\sharp(y,\rho)\right|\right)d\rho \tag{5.14}$$

for all $y \in \Gamma$. Then, by (5.14), Hölder's inequality, and (5.4) we have

$$\begin{aligned}
\left|I_\varepsilon^1(u,\varphi)\right| &\leq c\varepsilon\int_\Gamma|\nabla_\Gamma\varphi|\int_{\varepsilon g_0}^{\varepsilon g_1}\left(\left|u^\sharp\right| + \left|(\nabla u)^\sharp\right|\right)d\rho\,d\mathcal{H}^{n-1} \\
&\leq c\varepsilon\left(\int_\Gamma|\nabla_\Gamma\varphi|^2\,d\mathcal{H}^{n-1}\right)^{1/2}\left\{\int_\Gamma\left(\int_{\varepsilon g_0}^{\varepsilon g_1}(|u^\sharp| + |(\nabla u)^\sharp|)\,d\rho\right)^2 d\mathcal{H}^{n-1}\right\}^{1/2} \\
&\leq c\varepsilon\|\nabla_\Gamma\varphi\|_{L^2(\Gamma)}\left(\int_\Gamma\varepsilon g\int_{\varepsilon g_0}^{\varepsilon g_1}(|u^\sharp|^2 + |(\nabla u)^\sharp|^2)J\,d\rho\,d\mathcal{H}^{n-1}\right)^{1/2} \\
&\leq c\varepsilon^{3/2}\|\nabla_\Gamma\varphi\|_{L^2(\Gamma)}\|u\|_{H^1(\Omega_\varepsilon)}.
\end{aligned}$$

Hence (5.13) holds. The inequality (5.14) is proved by direct calculations under the local coordinate system we took in the proof of Lemma 5.5. We give a complete proof in Appendix C. □

Lemma 5.6 gives an estimate for the $L^2(\Gamma)$-norm of $\nabla_\Gamma M_\varepsilon u$ for $u \in H^1(\Omega_\varepsilon)$.

**Lemma 5.8** *If $u \in H^1(\Omega_\varepsilon)$, then $M_\varepsilon u \in H^1(\Gamma)$ and*

$$\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)} \leq c\varepsilon^{-1/2}\|u\|_{H^1(\Omega_\varepsilon)} \tag{5.15}$$

*with a constant $c > 0$ independent of $\varepsilon$.*

*Proof.* First, we show (5.15) for all $u \in C^\infty(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)$. For such $u$, its average $M_\varepsilon u$ is smooth on $\Gamma$ and thus in $H^1(\Gamma)$ by the compactness of $\Gamma$. We substitute $M_\varepsilon u$ for $\varphi$ in (5.12), (5.13) to get

$$\int_\Gamma g|\nabla_\Gamma M_\varepsilon u|^2\,d\mathcal{H}^{n-1} = \varepsilon^{-1}\left(\int_{\Omega_\varepsilon}\nabla u\cdot\nabla\overline{M_\varepsilon u}\,dx - I_\varepsilon^1(u, M_\varepsilon u)\right),$$

$$\left|I_\varepsilon^1(u, M_\varepsilon u)\right| \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)}.$$

Hence, by (2.1), Hölder's inequality, and (5.10) we obtain

$$\begin{aligned}\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)}^2 &\leq c\int_\Gamma g|\nabla_\Gamma M_\varepsilon u|^2\,d\mathcal{H}^{n-1}\\ &\leq c\varepsilon^{-1}\Big(\|\nabla u\|_{L^2(\Omega_\varepsilon)}\|\nabla\overline{M_\varepsilon u}\|_{L^2(\Omega_\varepsilon)}+\big|I_\varepsilon^1(u,M_\varepsilon u)\big|\Big)\\ &\leq c\varepsilon^{-1}(\varepsilon^{1/2}+\varepsilon^{3/2})\|u\|_{H^1(\Omega_\varepsilon)}\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)}\\ &\leq c\varepsilon^{-1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)}\end{aligned}$$

and thus (5.15) follows when $u \in C^\infty(\Omega_\varepsilon)\cap H^1(\Omega_\varepsilon)$. Since $\Omega_\varepsilon$ is bounded, $C^\infty(\Omega_\varepsilon)\cap H^1(\Omega_\varepsilon)$ is dense in $H^1(\Omega_\varepsilon)$, see [10, Section 5.3.2] for the proof. Hence a density argument together with Lemma 5.3 yields that $M_\varepsilon u \in H^1(\Gamma)$ and (5.15) holds for all $u \in H^1(\Omega_\varepsilon)$. □

REMARK 5.9 By Lemma 5.5 and Lemma 5.8, the bilinear form $I_\varepsilon^1(u,\varphi)$ given by (5.12) is well-defined for all $u \in H^1(\Omega_\varepsilon)$ and $\varphi \in H^1(\Gamma)$. Moreover, since $C^\infty(\Omega_\varepsilon)\cap H^1(\Omega_\varepsilon)$ is dense in $H^1(\Omega_\varepsilon)$, a density argument implies that (5.13) also holds for all $u \in H^1(\Omega_\varepsilon)$ and $\varphi \in H^1(\Gamma)$.

### 5.3 *Material derivative of the average operator*

Now let us return to the evolving surface $\Gamma(t)$. Recall the function spaces $L^2_{H^1(\varepsilon)}$ and $H_T$ given by (3.1) and (4.1), respectively. By Lemma 5.3 and Lemma 5.8 we immediately get the next lemma.

**Lemma 5.10** *If $u \in L^2_{H^1(\varepsilon)}$, then $M_\varepsilon u \in H_T$ and*

$$\|M_\varepsilon u\|_{H_T} \leq c\varepsilon^{-1/2}\|u\|_{L^2_{H^1(\varepsilon)}}$$

*with a constant $c > 0$ independent of $\varepsilon$.*

Lemma 5.10 enables us to consider the weak material derivative of $M_\varepsilon u \in H_T$ for $u \in L^2_{H^1(\varepsilon)}$. Our goal in this subsection is to give a relation between the weak time derivative of $u$ and the weak material derivative of $M_\varepsilon u$. To this end, we show an auxiliary statement about the material derivative of a function on $S_T$.

**Lemma 5.11** *Let $\varphi \in C^1(S_T)$ and $\overline{\varphi}$ be its constant extension in the normal direction of $\Gamma(t)$. Then*

$$\partial^\bullet\varphi\big(p(x,t),t\big) = \partial_t\overline{\varphi}(x,t) + \{V_\Gamma\big(p(x,t),t\big) + a(x,t)\}\cdot\nabla\overline{\varphi}(x,t) \tag{5.16}$$

*holds for all $(x,t) \in N_T$ with a vector field $a : N_T \to \mathbb{R}^n$ given by*

$$a(x,t) := d(x,t)\Big\{\partial_t\nu\big(p(x,t),t\big) + \nabla\nu\big(p(x,t),t\big)V_\Gamma\big(p(x,t),t\big)\Big\}. \tag{5.17}$$

*Here $\nabla\nu := (\partial\nu_i/\partial x_j)_{i,j}$ is the gradient matrix of $\nu$.*

*Proof.* For $X \in N(0)$ and $t \in (0,T)$ we set

$$\Psi(X,t) := \Phi\big(p(X,0),t\big) + d(X,0)\nu\Big(\Phi\big(p(X,0),t\big),t\Big),$$

where $\Phi(\cdot,t)\colon \Gamma_0 \to \Gamma(t)$ is the flow map of $V_\Gamma$ (see Section 2). By the definition of the constant extension $\overline{\varphi}$ and the formula $p(\Psi(X,t),t) = \Phi(p(X,0),t)$ we have

$$\overline{\varphi}\big(\Psi(X,t),t\big) = \varphi\Big(\Phi\big(p(X,0),t\big),t\Big)$$

for all $X \in N(0)$ and $t \in (0,T)$. Differentiating both sides with respect to $t$ and observing that each $x \in N(t)$ is represented as $x = \Psi(X,t)$ with a unique $X \in N(0)$, we get the formula (5.16). For detailed calculations, see Appendix C. □

REMARK 5.12 Let $\varphi \in C^1(S_T)$. Since $p(y,t) = y$ and $d(y,t) = 0$ for all $(y,t) \in S_T$, we have

$$\partial^\bullet \varphi = \partial_t \overline{\varphi} + V_\Gamma \cdot \nabla\overline{\varphi} = \partial_t\overline{\varphi} + v_\Gamma^N \nu \cdot \nabla\overline{\varphi} + V_\Gamma^T \cdot \nabla_{\Gamma(t)}\overline{\varphi} \quad \text{on} \quad S_T$$

by Lemma 5.11. Here the last equality follows from the fact that $V_\Gamma^T$ is tangent to $\Gamma(t)$. Based on this equality, the material derivative operator acting on functions on $\Gamma(t)$ is formally represented as $\partial^\bullet = \partial_t + V_\Gamma \cdot \nabla = \partial_t + v_\Gamma^N \nu \cdot \nabla + V_\Gamma^T \cdot \nabla_{\Gamma(t)}$.

Using Lemma 5.11, we derive an integral formula related to the weak time derivative of a function $u \in L^2_{H^1(\varepsilon)}$ and the weak material derivative of its average $M_\varepsilon u \in H_T$.

**Lemma 5.13** *Let $u \in L^2_{H^1(\varepsilon)}$, $\varphi \in C_0^1(S_T)$, and $\overline{\varphi}$ be its constant extension. Then we have*

$$\begin{aligned}\int_0^T \int_{\Omega_\varepsilon(t)} u\,\partial_t\overline{\varphi}\,dx\,dt &= -\varepsilon\langle \partial^\bullet M_\varepsilon u, g\varphi\rangle_T - \varepsilon\int_0^T\int_{\Gamma(t)} (\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)} V_\Gamma)(M_\varepsilon u)\varphi\,d\mathcal{H}^{n-1}\,dt \\ &\quad - \varepsilon\int_0^T\int_{\Gamma(t)} g(M_\varepsilon u)V_\Gamma^T\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt + I_\varepsilon^2(u,\varphi;T). \end{aligned} \tag{5.18}$$

*Here $I_\varepsilon^2(u,\varphi;T)$ is a residual term that satisfies*

$$\left|I_\varepsilon^2(u,\varphi;T)\right| \leq c\varepsilon^{3/2}\int_0^T \|u(t)\|_{L^2(\Omega_\varepsilon(t))}\|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\,dt \tag{5.19}$$

*with a constant $c > 0$ independent of $u$, $\varphi$, and $\varepsilon$.*

Note that the tangential velocity $V_\Gamma^T$ appears instead of the total velocity $V_\Gamma$ in the third term of the right-hand side of (5.18), see Remark 5.15 below.

*Proof.* By (5.16), we have $\overline{\partial^\bullet\varphi} = \partial_t\overline{\varphi} + (\overline{V_\Gamma} + a)\cdot\nabla\overline{\varphi}$ on $N_T$, where $a$ is the vector field on $N_T$ given by (5.17). Hence if we set

$$I_\varepsilon^2(u,\varphi;T) := -\int_0^T\int_{\Omega_\varepsilon(t)} u\left\{a\cdot\nabla\overline{\varphi} + \overline{V_\Gamma}\cdot\left(\nabla\overline{\varphi} - \overline{\nabla_{\Gamma(t)}\varphi}\right)\right\}dx\,dt,$$

then we have

$$\int_0^T\int_{\Omega_\varepsilon(t)} u\,\partial_t\overline{\varphi}\,dx\,dt = \int_0^T\int_{\Omega_\varepsilon(t)} u\left(\overline{\partial^\bullet\varphi} - \overline{V_\Gamma}\cdot\overline{\nabla_{\Gamma(t)}\varphi}\right)dx\,dt + I_\varepsilon^2(u,\varphi;T). \tag{5.20}$$

Let us compute the first term of the right-hand side of (5.20). By the co-area formula (5.1) and the definition of the weighted average $M_\varepsilon u$ (see (5.2)),

$$\begin{aligned}\int_{\Omega_\varepsilon(t)} u(x,t)\overline{\partial^\bullet\varphi}(x,t)\,dx &= \int_{\Gamma(t)}\int_{\varepsilon g_0(y,t)}^{\varepsilon g_1(y,t)} u\big(y+\rho\nu(y,t),t\big)\partial^\bullet\varphi(y,t)J(y,t,\rho)\,d\rho\,d\mathcal{H}^{n-1}(y)\\ &= \varepsilon\int_{\Gamma(t)} g(y,t)M_\varepsilon u(y,t)\partial^\bullet\varphi(y,t)\,d\mathcal{H}^{n-1}(y)\end{aligned}$$

for all $t\in(0,T)$. On the other hand, since the weak material derivative is given by (4.6),

$$\begin{aligned}\langle\partial^\bullet M_\varepsilon u, g\varphi\rangle_T &= -\int_0^T\int_{\Gamma(t)}\big\{(M_\varepsilon u)\partial^\bullet(g\varphi)+(M_\varepsilon u)g\varphi\operatorname{div}_{\Gamma(t)}V_\Gamma\big\}\,d\mathcal{H}^{n-1}\,dt\\ &= -\int_0^T\int_{\Gamma(t)}\big\{(\partial^\bullet g+g\operatorname{div}_{\Gamma(t)}V_\Gamma)(M_\varepsilon u)\varphi+g(M_\varepsilon u)\partial^\bullet\varphi\big\}\,d\mathcal{H}^{n-1}\,dt.\end{aligned}$$

Thus it follows that

$$\begin{aligned}&\int_0^T\int_{\Omega_\varepsilon(t)} u\,\overline{\partial^\bullet\varphi}\,dx\,dt\\ &\qquad= -\varepsilon\langle\partial^\bullet M_\varepsilon u, g\varphi\rangle_T-\varepsilon\int_0^T\int_{\Gamma(t)}(\partial^\bullet g+g\operatorname{div}_{\Gamma(t)}V_\Gamma)(M_\varepsilon u)\varphi\,d\mathcal{H}^{n-1}\,dt. \qquad (5.21)\end{aligned}$$

Since $V_\Gamma = v_\Gamma^N\nu+V_\Gamma^T$ and $\nu\cdot\nabla_{\Gamma(t)}\varphi=0$, we have $V_\Gamma\cdot\nabla_{\Gamma(t)}\varphi = V_\Gamma^T\cdot\nabla_{\Gamma(t)}\varphi$ on $S_T$. This equality together with the co-area formula (5.1) yields

$$\begin{aligned}&\int_{\Omega_\varepsilon(t)} u(x,t)\overline{V_\Gamma}(x,t)\cdot\overline{\nabla_{\Gamma(t)}\varphi}(x,t)\,dx\\ &\qquad= \int_{\Gamma(t)}\int_{\varepsilon g_0(y,t)}^{\varepsilon g_1(y,t)} u\big(y+\rho\nu(y,t),t\big)V_\Gamma(y,t)\cdot\nabla_{\Gamma(t)}\varphi(y,t)J(y,t,\rho)\,d\rho\,d\mathcal{H}^{n-1}(y)\\ &\qquad= \varepsilon\int_{\Gamma(t)} g(y,t)M_\varepsilon u(y,t)V_\Gamma^T(y,t)\cdot\nabla_{\Gamma(t)}\varphi(y,t)\,d\mathcal{H}^{n-1}(y)\end{aligned}$$

for all $t\in(0,T)$ and thus

$$\int_0^T\int_{\Omega_\varepsilon(t)} u\Big(\overline{V_\Gamma}\cdot\overline{\nabla_{\Gamma(t)}\varphi}\Big)\,dx\,dt = \varepsilon\int_0^T\int_{\Gamma(t)} g(M_\varepsilon u)V_\Gamma^T\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt. \qquad (5.22)$$

Substituting (5.21) and (5.22) for (5.20), we obtain the equality (5.18).

Let us show the inequality (5.19). In (5.17), the first-order derivatives of $\nu$ are bounded on $N_T$ and $V_\Gamma$ is bounded on $S_T$. Hence there is a constant $c>0$ independent of $\varepsilon$ such that

$$|a(x,t)|\leqslant c|d(x,t)|\leqslant c\varepsilon\max_{i=1,2}\sup_{(y,\tau)\in\overline{S_T}}|g_i(y,\tau)|\leqslant c\varepsilon$$

for all $(x,t) \in Q_{\varepsilon,T}$. By this inequality, Hölder's inequality, and (5.10) we obtain

$$\begin{aligned} \left|I_\varepsilon^2(u,\varphi;T)\right| &\leqslant c\int_0^T \|u(t)\|_{L^2(\Omega_\varepsilon(t))}\Big(\varepsilon\|\nabla\overline{\varphi}(t)\|_{L^2(\Omega_\varepsilon(t))} + \big\|\nabla\overline{\varphi}(t) - \overline{\nabla_{\Gamma(t)}\varphi(t)}\big\|_{L^2(\Omega_\varepsilon(t))}\Big)\,dt \\ &\leqslant c\varepsilon^{3/2}\int_0^T \|u(t)\|_{L^2(\Omega_\varepsilon(t))}\|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\,dt. \end{aligned}$$

Thus (5.19) holds. □

REMARK 5.14 If $M_\varepsilon u$ is in the Hilbert space $W_T$ given by (4.7), then the right-hand side of (5.18) is well-defined for $\varphi \in H_T$ since $C_0^1(S_T)$ is dense in $H_T$ (see Lemma 4.2). In particular, we can substitute $M_\varepsilon u$ for $\varphi$ in the right-hand side of (5.18). This fact is essential for derivation of the energy estimate for the weighted average of a variational solution to ($H_\varepsilon$) (see Lemma 6.4). If we replace $M_\varepsilon$ in (5.18) by a usual unweighted average operator

$$\mathfrak{M}_\varepsilon u(y,t) := \frac{1}{\varepsilon g(y,t)}\int_{\varepsilon g_0(y,t)}^{\varepsilon g_1(y,t)} u\big(y+\rho\nu(y,t),t\big)\,d\rho,$$

then the estimate for the residual term becomes

$$\left|I_\varepsilon^2(u,\varphi;T)\right| \leqslant c\varepsilon^{3/2}\int_0^T \|u(t)\|_{L^2(\Omega_\varepsilon(t))}\big(\|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))} + \|\partial^\bullet\varphi(t)\|_{L^2(\Gamma(t))}\big)\,dt.$$

Because of the term $\|\partial^\bullet\varphi(t)\|_{L^2(\Gamma(t))}$ in the above inequality, the right-hand side of (5.18) with $M_\varepsilon$ replaced by $\mathfrak{M}_\varepsilon$ is not well-defined for $\varphi \in H_T$. Therefore we can not derive the energy estimate for the unweighted average of a variational solution to ($H_\varepsilon$).

REMARK 5.15 Let $\Gamma \subset \mathbb{R}^n$ be a closed, connected, and oriented smooth hypersurface. Then, since $\partial\Gamma = \emptyset$, the integral formula (see [25, Section 7.2])

$$\int_\Gamma \operatorname{div}_\Gamma V\,d\mathcal{H}^{n-1} = -\int_\Gamma (V\cdot\nu)H\,d\mathcal{H}^{n-1}$$

holds for smooth vector fields $V\colon\Gamma \to \mathbb{R}^n$. Here $\nu$ is the unit outward normal vector of $\Gamma$ and $H := -\operatorname{div}_\Gamma\nu$ is the mean curvature of $\Gamma$. This formula yields the equality

$$\int_\Gamma V\cdot\nabla_\Gamma\varphi\,d\mathcal{H}^{n-1} = -\int_\Gamma \big\{\operatorname{div}_\Gamma V + (V\cdot\nu)H\big\}\varphi\,d\mathcal{H}^{n-1}$$

for smooth functions $\varphi$ on $\Gamma$. In this equality we decompose $V = v^N\nu + V^T$ into the normal component $v^N := V\cdot\nu$ and the tangential component $V^T := V - (V\cdot\nu)\nu$. Then, since

$$\nu\cdot\nabla_\Gamma\varphi = 0, \quad \operatorname{div}_\Gamma\big(v^N\nu\big) = \nabla_\Gamma v^N\cdot\nu + v^N\operatorname{div}_\Gamma\nu = 0 + v^N\cdot(-H) = -(V\cdot\nu)H,$$

we obtain a usual integration by parts formula

$$\int_\Gamma V^T\cdot\nabla_\Gamma\varphi\,d\mathcal{H}^{n-1} = -\int_\Gamma \varphi\operatorname{div}_\Gamma V^T\,d\mathcal{H}^{n-1}, \tag{5.23}$$

which we will use to recover a limit equation on $\Gamma(t)$ from its variational formulation. This is the reason the tangential velocity $V_\Gamma^T$ appears in (5.18) instead of the total velocity $V_\Gamma$ of $\Gamma(t)$.

## 6. Convergence and characterization of the limit

### 6.1 *Variational formulations of the average of solutions to the heat equation*

Let us return to the initial-boundary value problem ($H_\varepsilon$) of the heat equation. By Theorem 3.4, for every $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ there exists a unique variational solution $u^\varepsilon \in L^2_{H^1(\varepsilon)}$ to ($H_\varepsilon$).

Let $M_\varepsilon$ be the weighted average operator defined in Definition 5.1. Our goal in this subsection is to derive a variational formulation of $M_\varepsilon u^\varepsilon$.

**Lemma 6.1** *Let $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ and $u^\varepsilon \in L^2_{H^1(\varepsilon)}$ be the unique variational solution to ($H_\varepsilon$) given by Theorem 3.4. Then $M_\varepsilon u^\varepsilon \in H_T$ and it satisfies*

$$\langle \partial^\bullet M_\varepsilon u^\varepsilon, g\varphi\rangle_T + \int_0^T \int_{\Gamma(t)} (\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)} V_\Gamma)(M_\varepsilon u^\varepsilon)\varphi \, d\mathcal{H}^{n-1}\,dt \\ + \int_0^T \int_{\Gamma(t)} g\{\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon + (M_\varepsilon u^\varepsilon) V_\Gamma^T\} \cdot \nabla_{\Gamma(t)}\varphi \, d\mathcal{H}^{n-1}\,dt = I_\varepsilon(u^\varepsilon, \varphi; T) \quad (6.1)$$

*for all $\varphi \in C_0^1(S_T)$. Here $I_\varepsilon(u^\varepsilon, \varphi; T)$ is a residual term that satisfies*

$$|I_\varepsilon(u^\varepsilon, \varphi; T)| \leqslant c\varepsilon^{1/2} \int_0^T \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))} \|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\,dt \quad (6.2)$$

*with a constant $c > 0$ independent of $u^\varepsilon$, $\varphi$, and $\varepsilon$.*

*Proof.* Since $u^\varepsilon \in L^2_{H^1(\varepsilon)}$, we have $M_\varepsilon u^\varepsilon \in H_T$ by Lemma 5.10. For each $\varphi \in C_0^1(S_T)$, its constant extension $\overline{\varphi}$ is in $C^1(\overline{Q_{\varepsilon,T}})$ and satisfies $\overline{\varphi}(0) = 0$ in $\Omega_\varepsilon(0)$ and $\overline{\varphi}(T) = 0$ in $\Omega_\varepsilon(T)$. Thus, by substituting $\overline{\varphi}$ for $w$ in the variational formulation (3.2) we obtain

$$\int_0^T \int_{\Omega_\varepsilon(t)} (-u^\varepsilon \partial_t \overline{\varphi} + \nabla u^\varepsilon \cdot \nabla\overline{\varphi})\,dx\,dt = 0. \quad (6.3)$$

Moreover, from Lemma 5.6 and Lemma 5.13 we have

$$\int_0^T \int_{\Omega_\varepsilon(t)} \nabla u^\varepsilon \cdot \nabla\overline{\varphi}\,dx\,dt = \varepsilon \int_0^T \int_{\Gamma(t)} g\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon \cdot \nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt + I_\varepsilon^1(u^\varepsilon, \varphi; T) \quad (6.4)$$

and

$$\int_0^T \int_{\Omega_\varepsilon(t)} u^\varepsilon \partial_t \overline{\varphi}\,dx\,dt \\ = -\varepsilon\langle \partial^\bullet M_\varepsilon u^\varepsilon, g\varphi\rangle_T - \varepsilon \int_0^T \int_{\Gamma(t)} \big(\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)} V_\Gamma\big)(M_\varepsilon u^\varepsilon)\varphi\,d\mathcal{H}^{n-1}\,dt \\ - \varepsilon \int_0^T \int_{\Gamma(t)} g(M_\varepsilon u^\varepsilon) V_\Gamma^T \cdot \nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt + I_\varepsilon^2(u^\varepsilon, \varphi; T), \quad (6.5)$$

where $I_\varepsilon^1(u^\varepsilon, \varphi; T)$ and $I_\varepsilon^2(u^\varepsilon, \varphi; T)$ satisfy

$$\big|I_\varepsilon^k(u^\varepsilon, \varphi; T)\big| \leqslant c\varepsilon^{3/2} \int_0^T \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))} \|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\,dt, \quad k = 1, 2, \quad (6.6)$$

with some constant $c > 0$ independent of $\varepsilon$. Hence, by substituting (6.4) and (6.5) for (6.3) and dividing both sides by $\varepsilon$, we obtain (6.1) with the residual term

$$I_\varepsilon(u^\varepsilon, \varphi; T) := \varepsilon^{-1}\{I_\varepsilon^2(u^\varepsilon, \varphi; T) - I_\varepsilon^1(u^\varepsilon, \varphi; T)\},$$

which satisfies (6.2) because $I_\varepsilon^1(u^\varepsilon, \varphi; T)$ and $I_\varepsilon^2(u^\varepsilon, \varphi; T)$ satisfy (6.6). □

### 6.2 *Estimates for the average $M_\varepsilon u^\varepsilon$ in the space $W_T$*

In this subsection, we estimate $M_\varepsilon u^\varepsilon$ in the Hilbert space $W_T$ given by (4.7).

**Lemma 6.2** *Let $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ and $u^\varepsilon \in L^2_{H^1(\varepsilon)}$ be the unique variational solution to ($H_\varepsilon$) given by Theorem 3.4. Then $M_\varepsilon u^\varepsilon \in W_T$ and there exists a constant $c > 0$ independent of $u^\varepsilon$ and $\varepsilon$ such that*

$$\|\partial^\bullet M_\varepsilon u^\varepsilon\|_{H_T'} \leqslant c\left(\|M_\varepsilon u^\varepsilon\|_{H_T} + \varepsilon^{1/2}\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}}\right). \tag{6.7}$$

*Proof.* Let $\varphi$ be an arbitrary function in $C_0^1(S_T)$. By substituting $g^{-1}\varphi \in C_0^1(S_T)$ for $\varphi$ in (6.1), we obtain $\langle \partial^\bullet M_\varepsilon u^\varepsilon, \varphi\rangle_T = I(u^\varepsilon, \varphi) + I_\varepsilon(u^\varepsilon, g^{-1}\varphi; T)$, where

$$\begin{aligned} I(u^\varepsilon, \varphi) := \int_0^T \int_{\Gamma(t)} &\{g^{-1}(V_\Gamma^T \cdot \nabla_{\Gamma(t)} g - \partial^\bullet g) - \mathrm{div}_{\Gamma(t)} V_\Gamma\}(M_\varepsilon u^\varepsilon)\varphi \, d\mathcal{H}^{n-1}\, dt \\ &- \int_0^T \int_{\Gamma(t)} \{\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon + (M_\varepsilon u^\varepsilon) V_\Gamma^T\} \cdot \nabla_{\Gamma(t)}\varphi \, d\mathcal{H}^{n-1}\, dt \\ &+ \int_0^T \int_{\Gamma(t)} g^{-1}(\nabla_{\Gamma(t)} g \cdot \nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon)\varphi \, d\mathcal{H}^{n-1}\, dt. \end{aligned}$$

Since $g$ and $V_\Gamma$ are smooth on $\overline{S_T}$, they are bounded on $S_T$ along with their derivatives. Moreover, $g^{-1}$ and $V_\Gamma^T$ are bounded on $S_T$. Thus we have $|I(u^\varepsilon, \varphi)| \leqslant c\|M_\varepsilon u^\varepsilon\|_{H_T}\|\varphi\|_{H_T}$ with a constant $c > 0$ independent of $u^\varepsilon$, $\varphi$, and $\varepsilon$. Also, by (6.2),

$$\begin{aligned} |I_\varepsilon(u^\varepsilon, g^{-1}\varphi; T)| &\leqslant c\varepsilon^{1/2} \int_0^T \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))} \|\nabla_{\Gamma(t)}(g^{-1}\varphi)(t)\|_{L^2(\Gamma(t))}\, dt \\ &\leqslant c\varepsilon^{1/2} \int_0^T \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))} \left(\|\varphi(t)\|_{L^2(\Gamma(t))} + \|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\right) dt \\ &\leqslant c\varepsilon^{1/2}\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}}\|\varphi\|_{H_T} \end{aligned}$$

with some $c > 0$ independent of $u^\varepsilon$, $\varphi$, and $\varepsilon$. Hence we obtain

$$\left|\langle \partial^\bullet M_\varepsilon u^\varepsilon, \varphi\rangle_T\right| \leqslant \left|I(u^\varepsilon, \varphi)\right| + \left|I_\varepsilon(u^\varepsilon, g^{-1}\varphi; T)\right| \leqslant c\left(\|M_\varepsilon u^\varepsilon\|_{H_T} + \varepsilon^{1/2}\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}}\right)\|\varphi\|_{H_T}$$

for all $\varphi \in C_0^1(S_T)$, which implies $M_\varepsilon u^\varepsilon \in W_T$ and the inequality (6.7). □

REMARK 6.3 Since $M_\varepsilon u^\varepsilon \in W_T$ and $C_0^1(S_T)$ is dense in $H_T$ (see Lemma 4.2), the equality (6.1) also holds for all $\varphi \in H_T$. Moreover, since $W_{T_1}$ is continuously embedded into $W_{T_2}$ when $T_1 > T_2$, we have $M_\varepsilon u^\varepsilon \in W_\tau$ for each $\tau \in [0, T]$. Hence (6.1) and (6.2) with $T$ replaced by each $\tau \in [0, T]$ are also valid for all $\varphi \in H_\tau$.

**Lemma 6.4** *Let $u_0^\varepsilon$ and $u^\varepsilon$ be as in Lemma 6.2. Then there exists a constant $c > 0$ independent of $u_0^\varepsilon$, $u^\varepsilon$, and $\varepsilon$ such that the energy estimate*

$$\|M_\varepsilon u^\varepsilon(\tau)\|^2_{L^2(\Gamma(\tau))} + \int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\,dt \leqslant c\big(\|M_\varepsilon u_0^\varepsilon\|^2_{L^2(\Gamma_0)} + \varepsilon\|u_0^\varepsilon\|^2_{L^2(\Omega_\varepsilon(0))}\big) \tag{6.8}$$

*holds for all $\tau \in [0, T]$.*

*Proof.* As we mentioned in Remark 6.3, the equality (6.1) holds with $T$ replaced by each $\tau \in [0, T]$. Hence, by substituting $g^{-1}M_\varepsilon u^\varepsilon \in H_\tau$ for $\varphi$ in (6.1) with $T$ replaced by $\tau$, we obtain

$$\begin{aligned}
&\langle \partial^\bullet M_\varepsilon u^\varepsilon, M_\varepsilon u^\varepsilon\rangle_\tau + \int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\,dt \\
&+ \int_0^\tau \int_{\Gamma(t)} \{g^{-1}(\partial^\bullet g - V_\Gamma^T \cdot \nabla_{\Gamma(t)} g) + \mathrm{div}_{\Gamma(t)} V_\Gamma\}|M_\varepsilon u^\varepsilon|^2\,d\mathcal{H}^{n-1}\,dt \\
&\qquad + \int_0^\tau \int_{\Gamma(t)} M_\varepsilon u^\varepsilon (V_\Gamma^T - g^{-1}\nabla_{\Gamma(t)} g)\cdot \nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon\,d\mathcal{H}^{n-1}\,dt = I_\varepsilon(u^\varepsilon, g^{-1}M_\varepsilon u^\varepsilon; \tau).
\end{aligned}$$

Moreover, from (4.8) with $T$ replaced by $\tau$,

$$\begin{aligned}
&\langle \partial^\bullet M_\varepsilon u^\varepsilon, M_\varepsilon u^\varepsilon\rangle_\tau = \\
&-\frac{1}{2}\int_0^\tau \int_{\Gamma(t)} |M_\varepsilon u^\varepsilon|^2 \,\mathrm{div}_{\Gamma(t)} V_\Gamma\,d\mathcal{H}^{n-1}\,dt + \frac{1}{2}\|M_\varepsilon u^\varepsilon(\tau)\|^2_{L^2(\Gamma(\tau))} - \frac{1}{2}\|M_\varepsilon u^\varepsilon(0)\|^2_{L^2(\Gamma_0)}.
\end{aligned}$$

Applying this equality and the relation $u^\varepsilon(0) = u_0^\varepsilon$ in $L^2(\Omega_\varepsilon(0))$ (see Theorem 3.4) to the above equality, we have

$$\begin{aligned}
\frac{1}{2}\|M_\varepsilon u^\varepsilon(\tau)\|^2_{L^2(\Gamma(\tau))} &+ \int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\,dt \\
&= \frac{1}{2}\|M_\varepsilon u_0^\varepsilon\|^2_{L^2(\Gamma_0)} + I_1(\tau) + I_2(\tau) + I_\varepsilon(u^\varepsilon, g^{-1}M_\varepsilon u^\varepsilon; \tau),
\end{aligned} \tag{6.9}$$

where

$$\begin{aligned}
I_1(\tau) &:= -\frac{1}{2}\int_0^\tau \int_{\Gamma(t)} \{2g^{-1}(\partial^\bullet g - V_\Gamma^T \cdot \nabla_{\Gamma(t)} g) + \mathrm{div}_{\Gamma(t)} V_\Gamma\}|M_\varepsilon u^\varepsilon|^2\,d\mathcal{H}^{n-1}\,dt, \\
I_2(\tau) &:= -\int_0^\tau \int_{\Gamma(t)} M_\varepsilon u^\varepsilon (V_\Gamma^T - g^{-1}\nabla_{\Gamma(t)} g)\cdot \nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon\,d\mathcal{H}^{n-1}\,dt.
\end{aligned}$$

Since $g$ and $V_\Gamma$ are smooth on $\overline{S_T}$, they are bounded on $S_T$ along with their derivatives. Also, $g^{-1}$ and $V_\Gamma^T$ are bounded on $S_T$. Thus it follows that

$$\begin{aligned}
|I_1(\tau)| &\leqslant c\int_0^\tau \|M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\,dt, \\
|I_2(\tau)| &\leqslant c\int_0^\tau \|M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}\|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}\,dt.
\end{aligned} \tag{6.10}$$

On the other hand, the inequality (6.2) with $T$ replaced by $\tau$ yields

$$
\begin{aligned}
&|I_\varepsilon(u^\varepsilon, g^{-1}M_\varepsilon u^\varepsilon;\tau)| \\
&\leqslant c\varepsilon^{1/2}\int_0^\tau \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))}\|\nabla_{\Gamma(t)}(g^{-1}M_\varepsilon u^\varepsilon)(t)\|_{L^2(\Gamma(t))}\,dt \\
&\leqslant c\varepsilon^{1/2}\int_0^\tau \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))}\big(\|M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}+\|\nabla_{\Gamma(t)}M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}\big)\,dt. \qquad (6.11)
\end{aligned}
$$

Thus, by applying (6.10) and (6.11) to (6.9), we obtain

$$
\begin{aligned}
&\frac{1}{2}\|M_\varepsilon u^\varepsilon(\tau)\|_{L^2(\Gamma(\tau))}^2+\int_0^\tau \|\nabla_{\Gamma(t)}M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}^2\,dt \\
&\qquad\leqslant \frac{1}{2}\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)}^2+\frac{1}{2}\int_0^\tau \|\nabla_{\Gamma(t)}M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}^2\,dt \\
&\qquad\qquad + c\int_0^\tau \big(\|M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}^2+\varepsilon\|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))}^2\big)\,dt.
\end{aligned}
$$

We multiply both sides by two and subtract $\int_0^\tau \|\nabla_{\Gamma(t)}M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}^2\,dt$ to get

$$
\begin{aligned}
&\|M_\varepsilon u^\varepsilon(\tau)\|_{L^2(\Gamma(\tau))}^2+\int_0^\tau \|\nabla_{\Gamma(t)}M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}^2\,dt \\
&\qquad\leqslant \|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)}^2+c\int_0^\tau \big(\|M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}^2+\varepsilon\|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))}^2\big)\,dt.
\end{aligned}
$$

Hence Gronwall's inequality implies

$$
\|M_\varepsilon u^\varepsilon(\tau)\|_{L^2(\Gamma(\tau))}^2+\int_0^\tau \|\nabla_{\Gamma(t)}M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}^2\,dt \leqslant c\big(\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)}^2+\varepsilon\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}}^2\big)
$$

for all $\tau \in [0,T]$, and we obtain (6.8) by applying (3.15) to the second term of the right-hand side of the above inequality. □

**Lemma 6.5** *Let $u_0^\varepsilon$ and $u^\varepsilon$ be as in Lemma 6.2. Then there exists a constant $c > 0$ independent of $u_0^\varepsilon$, $u^\varepsilon$, and $\varepsilon$ such that*

$$
\|M_\varepsilon u^\varepsilon\|_{W_T} \leqslant c\big(\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)}+\varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\big). \qquad (6.12)
$$

*Proof.* It follows from (6.8) that

$$
\|M_\varepsilon u^\varepsilon\|_{H_T} \leqslant c\big(\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)}+\varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\big).
$$

Moreover, by applying this inequality and (3.15) to (6.7) we have

$$
\|\partial^\bullet M_\varepsilon u^\varepsilon\|_{H_T'} \leqslant c\big(\|M_\varepsilon u^\varepsilon\|_{H_T}+\varepsilon^{1/2}\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}}\big) \leqslant c\big(\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)}+\varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\big).
$$

Thus we obtain (6.12). □

6.3 *Limit equation on evolving surfaces and weak convergence of $M_\varepsilon u^\varepsilon$*

Assume that $I_\varepsilon(u^\varepsilon,\varphi;T)=0$ holds for all $\varphi\in C_0^1(S_T)$ and $v=M_\varepsilon u^\varepsilon$ is independent of $\varepsilon$ in the variational formulation (6.1). Then $v$ satisfies

$$\langle\partial^\bullet v, g\varphi\rangle_T+\int_0^T\int_{\Gamma(t)}(\partial^\bullet g+g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)v\,\varphi\,d\mathcal{H}^{n-1}\,dt \\ +\int_0^T\int_{\Gamma(t)}g\big(\nabla_{\Gamma(t)}v+vV_\Gamma^T\big)\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt=0 \quad (6.13)$$

for all $\varphi\in C_0^1(S_T)$. In addition we assume that $v$ is sufficiently smooth. Since vector fields $gvV_\Gamma^T$ and $g\nabla_{\Gamma(t)}v$ are tangent to $\Gamma(t)$ for each $t\in[0,T]$, we can apply the integration by parts formula (5.23) to obtain

$$\langle\partial^\bullet v, g\varphi\rangle_T+\int_0^T\int_{\Gamma(t)}\Big\{(\partial^\bullet g+g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)v-\mathrm{div}_{\Gamma(t)}\big[g(\nabla_{\Gamma(t)}v+vV_\Gamma^T)\big]\Big\}\varphi\,d\mathcal{H}^{n-1}\,dt=0.$$

Since this equality holds for all $\varphi\in C_0^1(S_T)$, we conclude that $v$ satisfies

$$\partial^\bullet(gv)+(g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)v-\mathrm{div}_{\Gamma(t)}\big[g(\nabla_{\Gamma(t)}v+vV_\Gamma^T)\big]=0 \quad\text{on}\quad S_T.$$

This is the limit equation of $(H_\varepsilon)$. To justify the above argument, we employ a variational framework introduced by Olshanskii, Reusken, and Xu [19].

DEFINITION 6.6 Let $v_0\in L^2(\Gamma_0)$. A function $v\in W_T$ is said to be a *variational solution* to the initial value problem

$$\begin{cases}\partial^\bullet(gv)+(g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)v-\mathrm{div}_{\Gamma(t)}\big[g(\nabla_{\Gamma(t)}v+vV_\Gamma^T)\big]=0 & \text{on}\quad S_T,\\ v(0)=v_0 & \text{on}\quad \Gamma_0,\end{cases}\qquad (H_0)$$

if $v$ satisfies (6.13) for all $\varphi\in H_T$ and $v(0)=v_0$ in $L^2(\Gamma_0)$.

Note that the condition $v(0)=v_0$ in $L^2(\Gamma_0)$ makes sense for $v\in W_T$ by Lemma 4.4.

REMARK 6.7 Suppose that $v\in W_T$ is a variational solution to $(H_0)$. Then we have $v\in W_\tau$ for each $\tau\in[0,T]$ since $W_T$ is continuously embedded into $W_\tau$. Moreover, by taking test functions $\varphi$ from $C_0^1(S_\tau)$ we observe that $v$ is a variational solution to $(H_0)$ with $T$ replaced by $\tau$.

We first prove the uniqueness of a variational solution to the initial value problem $(H_0)$.

**Lemma 6.8** *For each $v_0\in L^2(\Gamma_0)$, there is at most one variational solution to $(H_0)$.*

*Proof.* Since $(H_0)$ is linear, it is sufficient to show that if $v\in W_T$ is a variational solution to $(H_0)$ with zero initial data then $v=0$.

Let $v$ be a variational solution to $(H_0)$ with $v(0)=0$ in $L^2(\Gamma_0)$. For each $\tau\in[0,T]$, we substitute $g^{-1}v\in H_\tau$ for $\varphi$ in (6.13) with $T$ replaced by $\tau$ and compute as in the proof of Lemma 6.4 (replace $M_\varepsilon u^\varepsilon$ by $v$ and omit $I_\varepsilon(u^\varepsilon,\varphi;\tau)$). Then we have

$$\|v(\tau)\|_{L^2(\Gamma(\tau))}^2+\int_0^\tau\|\nabla_{\Gamma(t)}v(t)\|_{L^2(\Gamma(t))}^2\,dt\leqslant\|v(0)\|_{L^2(\Gamma_0)}^2+c\int_0^\tau\|v(t)\|_{L^2(\Gamma(t))}^2\,dt.$$

Since $v(0) = 0$ in $L^2(\Gamma_0)$, the above inequality yields

$$\|v(\tau)\|^2_{L^2(\Gamma(\tau))} \leqslant \int_0^\tau \|v(t)\|^2_{L^2(\Gamma(t))}\,dt.$$

Hence by Gronwall's inequality we obtain $v(\tau) = 0$ in $L^2(\Gamma(\tau))$ for all $\tau \in [0, T]$. □

Now let us show that $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ converges weakly in $W_T$ and that the limit is a unique variational solution to the initial value problem ($H_0$).

**Theorem 6.9** *Let $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ and $u^\varepsilon \in L^2_{H^1(\varepsilon)}$ be the unique variational solution to ($H_\varepsilon$) given by Theorem 3.4. Suppose that the following two conditions are satisfied:*

(a) *There exist constants $c > 0$ and $\gamma \in (0, 1/2)$ such that $\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))} \leqslant c\varepsilon^{-\gamma}$ for all $\varepsilon > 0$.*
(b) *There exists $v_0 \in L^2(\Gamma_0)$ such that $\{M_\varepsilon u_0^\varepsilon\}_\varepsilon$ converges weakly to $v_0$ in $L^2(\Gamma_0)$ as $\varepsilon \to 0$.*

*Then $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ converges weakly in $W_T$ as $\varepsilon \to 0$. Moreover, the weak limit $v \in W_T$ of $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ is the unique variational solution to ($H_0$) with initial data $v_0$.*

*Proof.* By the condition (b), $\{M_\varepsilon u_0^\varepsilon\}_\varepsilon$ is bounded in $L^2(\Gamma_0)$. From this fact, the inequality (6.12), and the condition (a) it follows that

$$\|M_\varepsilon u^\varepsilon\|_{W_T} \leqslant c\big(\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\big) \leqslant c(1 + \varepsilon^{-\gamma+1/2}) \leqslant c \tag{6.14}$$

with some constant $c > 0$ independent of $\varepsilon$. Here the last inequality follows from the condition $\gamma \in (0, 1/2)$. Hence $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ is bounded in the Hilbert space $W_T$ and there exist $v \in W_T$ and a sequence $\{\varepsilon_k\}_k$ of positive numbers with $\lim_{k\to\infty} \varepsilon_k = 0$ such that $\{M_{\varepsilon_k} u^{\varepsilon_k}\}_k$ converges weakly to $v$ in $W_T$ as $k \to \infty$.

Let us show that $v$ is the unique variational solution to ($H_0$) with initial data $v_0$. First we show that $v$ satisfies the variational formulation (6.13) for all $\varphi \in H_T$. To this end, we return to the variational formulation (6.1) of $M_{\varepsilon_k} u^{\varepsilon_k}$:

$$\begin{aligned}&\langle \partial^\bullet M_{\varepsilon_k} u^{\varepsilon_k}, g\varphi\rangle_T + \int_0^T\int_{\Gamma(t)} (\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)} V_\Gamma)(M_{\varepsilon_k} u^{\varepsilon_k})\varphi\,d\mathcal{H}^{n-1}\,dt\\ &\quad + \int_0^T\int_{\Gamma(t)} g\{\nabla_{\Gamma(t)} M_{\varepsilon_k} u^{\varepsilon_k} + (M_{\varepsilon_k} u^{\varepsilon_k})V_\Gamma^T\}\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt = I_{\varepsilon_k}(u^{\varepsilon_k}, \varphi; T).\end{aligned} \tag{6.15}$$

Let $k \to \infty$ in (6.15). Since $\{M_{\varepsilon_k} u^{\varepsilon_k}\}_k$ converges weakly to $v$ in $W_T$ as $k \to \infty$ and $g$, $V_\Gamma$ are bounded on $S_T$ along with their derivatives, the left-hand side of (6.15) converges to

$$\begin{aligned}&\langle \partial^\bullet v, g\varphi\rangle_T + \int_0^T\int_{\Gamma(t)} (\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)} V_\Gamma)v\varphi\,d\mathcal{H}^{n-1}\,dt\\ &\qquad + \int_0^T\int_{\Gamma(t)} g\big(\nabla_{\Gamma(t)} v + vV_\Gamma^T\big)\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt.\end{aligned}$$

On the other hand, it follows from (6.2) and (3.15) that

$$\begin{aligned}|I_{\varepsilon_k}(u^{\varepsilon_k}, \varphi; T)| &\leqslant c\varepsilon_k^{1/2}\int_0^T \|u^{\varepsilon_k}(t)\|_{H^1(\Omega_{\varepsilon_k}(t))}\|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\,dt\\ &\leqslant c\varepsilon_k^{1/2}\|u^{\varepsilon_k}\|_{L^2_{H^1(\varepsilon_k)}}\|\varphi\|_{H_T} \leqslant c\varepsilon_k^{1/2}\|u_0^{\varepsilon_k}\|_{L^2(\Omega_{\varepsilon_k}(0))}\|\varphi\|_{H_T}\end{aligned}$$

with a constant $c > 0$ independent of $\varepsilon_k$. This inequality and the condition (a) imply that

$$|I_{\varepsilon_k}(u^{\varepsilon_k}, \varphi; T)| \leqslant c\varepsilon_k^{-\gamma+1/2}\|\varphi\|_{H_T} \to 0 \quad \text{as} \quad k \to \infty, \tag{6.16}$$

since $\gamma \in (0, 1/2)$ and $c$ is independent of $\varepsilon_k$. Hence $v$ satisfies (6.13) for all $\varphi \in H_T$.

Next we show that $v(0) = v_0$ in $L^2(\Gamma_0)$. Let $\eta \in C^\infty([0,T])$ satisfy $\eta(0) = 1$ and $\eta(T) = 0$. We take an arbitrary $\varphi_0 \in C^\infty(\Gamma_0)$ and set $\varphi(y,t) := \varphi_0(\Phi^{-1}(y,t))\eta(t)$ for $(y,t) \in \overline{S_T}$, where $\Phi^{-1}(\cdot,t)$ is the inverse mapping of the flow map $\Phi(\cdot,t)\colon \Gamma_0 \to \Gamma(t)$ (see Section 2). Due to the smoothness of $\Phi^{-1}$, the function $\varphi$ is smooth on $\overline{S_T}$ and thus $\varphi \in W_T$. Moreover, it satisfies $\varphi(0) = \varphi_0$ on $\Gamma_0$ and $\varphi(T) = 0$ on $\Gamma(T)$. Substituting $g^{-1}\varphi$ for $\varphi$ in (6.13) and (6.15), we have

$$\langle \partial^\bullet v, \varphi\rangle_T + \int_0^T \int_{\Gamma(t)} (g^{-1}\partial^\bullet g + \mathrm{div}_{\Gamma(t)} V_\Gamma) v\,\varphi\, d\mathcal{H}^{n-1}\, dt + \int_0^T \int_{\Gamma(t)} g\big(\nabla_{\Gamma(t)} v + vV_\Gamma^T\big)\cdot \nabla_{\Gamma(t)}(g^{-1}\varphi)\, d\mathcal{H}^{n-1}\, dt = 0$$

and

$$\langle \partial^\bullet M_{\varepsilon_k} u^{\varepsilon_k}, \varphi\rangle_T + \int_0^T \int_{\Gamma(t)} (g^{-1}\partial^\bullet g + \mathrm{div}_{\Gamma(t)} V_\Gamma)(M_{\varepsilon_k} u^{\varepsilon_k})\varphi\, d\mathcal{H}^{n-1}\, dt + \int_0^T \int_{\Gamma(t)} g\big\{\nabla_{\Gamma(t)} M_{\varepsilon_k} u^{\varepsilon_k} + (M_{\varepsilon_k} u^{\varepsilon_k})V_\Gamma^T\big\}\cdot \nabla_{\Gamma(t)}(g^{-1}\varphi)\, d\mathcal{H}^{n-1}\, dt = I_{\varepsilon_k}(u^{\varepsilon_k}, g^{-1}\varphi; T).$$

Since $\varphi$, $v$, and $M_{\varepsilon_k} u^{\varepsilon_k}$ are in $W_T$, we can apply the identity (4.8) to get

$$\langle \partial^\bullet v, \varphi\rangle_T = -\langle \partial^\bullet \varphi, v\rangle_T - \int_{\Gamma_0} v(0)\varphi_0\, d\mathcal{H}^{n-1} - \int_0^T \int_{\Gamma(t)} v\varphi\, \mathrm{div}_{\Gamma(t)} V_\Gamma\, d\mathcal{H}^{n-1}\, dt$$

and the same identity for $M_{\varepsilon_k} u^{\varepsilon_k}$. Here we used the conditions $\varphi(0) = \varphi_0$ on $\Gamma_0$ and $\varphi(T) = 0$ on $\Gamma(T)$. Thus we have

$$-\langle \partial^\bullet \varphi, v\rangle_T + \int_0^T \int_{\Gamma(t)} g^{-1}(\partial^\bullet g) v\varphi\, d\mathcal{H}^{n-1}\, dt + \int_0^T \int_{\Gamma(t)} g\big(\nabla_{\Gamma(t)} v + vV_\Gamma^T\big)\cdot \nabla_{\Gamma(t)}(g^{-1}\varphi)\, d\mathcal{H}^{n-1}\, dt = \int_{\Gamma_0} v(0)\varphi_0\, d\mathcal{H}^{n-1} \tag{6.17}$$

and

$$-\langle \partial^\bullet \varphi, M_{\varepsilon_k} u^{\varepsilon_k}\rangle_T + \int_0^T \int_{\Gamma(t)} g^{-1}(\partial^\bullet g)(M_{\varepsilon_k} u^{\varepsilon_k})\varphi\, d\mathcal{H}^{n-1}\, dt + \int_0^T \int_{\Gamma(t)} g\big\{\nabla_{\Gamma(t)} M_{\varepsilon_k} u^{\varepsilon_k} + (M_{\varepsilon_k} u^{\varepsilon_k})V_\Gamma^T\big\}\cdot \nabla_{\Gamma(t)}(g^{-1}\varphi)\, d\mathcal{H}^{n-1}\, dt = \int_{\Gamma_0} (M_{\varepsilon_k} u_0^{\varepsilon_k})\varphi_0\, d\mathcal{H}^{n-1} + I_{\varepsilon_k}(u^{\varepsilon_k}, g^{-1}\varphi; T). \tag{6.18}$$

Let $k \to \infty$ in (6.18). Since $\{M_\varepsilon u_0^\varepsilon\}_\varepsilon$ converges weakly to $v_0$ in $L^2(\Gamma_0)$ as $\varepsilon \to 0$,

$$\lim_{k\to\infty} \int_{\Gamma_0} M_{\varepsilon_k} u_0^{\varepsilon_k} \varphi_0 \, d\mathcal{H}^{n-1} = \int_{\Gamma_0} v_0 \varphi_0 \, d\mathcal{H}^{n-1}.$$

Moreover, since $\{M_{\varepsilon_k} u^{\varepsilon_k}\}_k$ converges weakly to $v$ in $W_T$ as $k \to \infty$ and (6.16) holds with $\varphi$ replaced by $g^{-1}\varphi$, both sides of (6.18) converge to

$$\begin{aligned} -\langle \partial^\bullet \varphi, v\rangle_T + \int_0^T \int_{\Gamma(t)} g^{-1}(\partial^\bullet g) v \varphi \, d\mathcal{H}^{n-1} \, dt \\ + \int_0^T \int_{\Gamma(t)} g\big(\nabla_{\Gamma(t)} v + v V_\Gamma^T\big) \cdot \nabla_{\Gamma(t)}(g^{-1}\varphi) \, d\mathcal{H}^{n-1} \, dt \\ = \int_{\Gamma_0} v_0 \varphi_0 \, d\mathcal{H}^{n-1}. \end{aligned} \tag{6.19}$$

Comparing (6.17) and (6.19), we obtain

$$\int_{\Gamma_0} v(0)\varphi_0 \, d\mathcal{H}^{n-1} = \int_{\Gamma_0} v_0 \varphi_0 \, d\mathcal{H}^{n-1} \quad \text{for all} \quad \varphi_0 \in C^\infty(\Gamma_0).$$

Since $C^\infty(\Gamma_0)$ is dense in $L^2(\Gamma_0)$, it follows that $v(0) = v_0$ in $L^2(\Gamma_0)$. Hence $v$ is the unique variational solution to ($H_0$) with initial data $v_0$. Here the uniqueness follows from Lemma 6.8.

Finally, using the boundedness of $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ in $W_T$ (see (6.14)) and the uniqueness of a variational solution to ($H_0$) (see Lemma 6.8), we can prove by contradiction that the full sequence $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ converges weakly to $v$ in $W_T$ as $\varepsilon \to 0$. The argument is standard and thus we omit the details. □

**Corollary 6.10** *For every $v_0 \in L^2(\Gamma_0)$, there exists a unique variational solution to ($H_0$).*

*Proof.* For each $\varepsilon > 0$, we define a function $u_0^\varepsilon$ on $\Omega_\varepsilon(0)$ as

$$u_0^\varepsilon(X) := \frac{v_0\big(p(X,0)\big)}{J\big(p(X,0), 0, d(X,0)\big)}, \quad X \in \Omega_\varepsilon(0).$$

Clearly $M_\varepsilon u_0^\varepsilon = v_0$ holds on $\Gamma_0$. Moreover, by the co-area formula (5.1) and (5.4) we have

$$\begin{aligned} \|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))} &= \left( \int_{\Gamma_0} \int_{\varepsilon g_0(Y,0)}^{\varepsilon g_1(Y,0)} |v_0(Y)|^2 J(Y,0,\rho)^{-1} \, d\rho \, d\mathcal{H}^{n-1}(Y) \right)^{1/2} \\ &\leqslant c \left( \int_{\Gamma_0} \varepsilon g(Y,0) |v_0(Y)|^2 \, d\mathcal{H}^{n-1}(Y) \right)^{1/2} \leqslant c\varepsilon^{1/2} \|v_0\|_{L^2(\Gamma_0)} \end{aligned}$$

with a constant $c > 0$ independent of $\varepsilon$. Hence $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ and $u_0^\varepsilon$, $v_0$ satisfy the conditions (a) and (b) of Theorem 6.9. Thus the corollary follows from Theorem 3.4 and Theorem 6.9. □

REMARK 6.11 Let $H = -\mathrm{div}_{\Gamma(t)} \nu$ be the mean curvature of $\Gamma(t)$. Since the material derivative operator is formally of the form $\partial^\bullet = \partial_t + v_\Gamma^N \nu \cdot \nabla + V_\Gamma^T \cdot \nabla_{\Gamma(t)}$ (see Remark 5.12) and the formula $\mathrm{div}_{\Gamma(t)}(v_\Gamma^N \nu) = -v_\Gamma^N H$ holds (see Remark 5.15), the limit equation ($H_0$) is formally equivalent to

$$\partial^\circ(gv) - v_\Gamma^N H g v - \mathrm{div}_{\Gamma(t)}(g \nabla_{\Gamma(t)} v) = 0 \quad \text{on} \quad S_T.$$

Here $\partial^\circ = \partial_t + v_\Gamma^N \nu \cdot \nabla$ is the normal time derivative (see [2, 3, 5]). This equation depends on $v_\Gamma^N$, $\nu$, and $H$, which represent the geometric motion of $\Gamma(t)$. On the other hand, it is independent of the tangential velocity $V_\Gamma^T$, which represents advection along $\Gamma(t)$. Hence, as we mentioned in Section 1, the evolution of the limit $v$ given by Theorem 6.9 is not affected by advection along $\Gamma(t)$, but the geometric motion of $\Gamma(t)$.

### 6.4 *Estimates for the difference between solutions to the heat equation and the limit equation*

Let us estimate the difference between variational solutions to ($H_\varepsilon$) and ($H_0$). For a function $v$ on $S_T$, let $\overline{v}$ be its constant extension in the normal direction of $\Gamma(t)$. For a function $u$ on $Q_{\varepsilon,T}$, we set

$$\|u\|_{L^2(Q_{\varepsilon,T})} := \left(\int_0^T \int_{\Omega_\varepsilon(t)} |u|^2 \, dx \, dt\right)^{1/2}.$$

**Theorem 6.12** *Let $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ and $u^\varepsilon \in L^2_{H^1(\varepsilon)}$ be a unique variational solution to ($H_\varepsilon$). Also, let $v_0 \in L^2(\Gamma_0)$ and $v \in W_T$ be a unique variational solution to ($H_0$). Then there exists a constant $c > 0$ independent of $u_0^\varepsilon$, $u^\varepsilon$, $v_0$, $v$, and $\varepsilon$ such that*

$$\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} \leqslant c\left(\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \varepsilon^{3/2}\|v_0\|_{L^2(\Gamma_0)}\right). \tag{6.20}$$

*In particular, for each $\alpha \in [0, 3/2)$ we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{-\alpha}\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} = 0 \quad \textit{provided} \quad \lim_{\varepsilon\to 0} \varepsilon^{-\alpha}\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} = 0.$$

We first estimate the difference between $M_\varepsilon u^\varepsilon$ and $v$ in the space $W_T$.

**Lemma 6.13** *Let $u_0^\varepsilon$, $u^\varepsilon$, $v_0$, and $v$ be as in Theorem 6.12. Then there exists a constant $c > 0$ independent of $u_0^\varepsilon$, $u^\varepsilon$, $v_0$, $v$, and $\varepsilon$ such that*

$$\|M_\varepsilon u^\varepsilon - v\|_{W_T} \leqslant c\left(\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}\right). \tag{6.21}$$

*In particular, if $\lim_{\varepsilon\to 0}\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} = 0$ and $\lim_{\varepsilon\to 0}\varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))} = 0$, then $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ converges strongly to $v$ in $W_T$.*

*Proof.* For each $\tau \in [0, T]$, we subtract both sides of (6.13) with $T$ replaced by $\tau$ from those of (6.1). Then we have

$$\begin{aligned}\left\langle \partial^\bullet(M_\varepsilon u^\varepsilon - v), g\varphi\right\rangle_\tau &+ \int_0^\tau \int_{\Gamma(t)} (\partial^\bullet g + g \operatorname{div}_{\Gamma(t)} V_\Gamma)(M_\varepsilon u^\varepsilon - v)\varphi \, d\mathcal{H}^{n-1} \, dt \\ &+ \int_0^\tau \int_{\Gamma(t)} g\left\{\nabla_{\Gamma(t)}(M_\varepsilon u^\varepsilon - v) + (M_\varepsilon u^\varepsilon - v)V_\Gamma^T\right\} \cdot \nabla_{\Gamma(t)}\varphi \, d\mathcal{H}^{n-1} \, dt = I_\varepsilon(u^\varepsilon, \varphi; \tau)\end{aligned}$$

for all $\varphi \in H_\tau$. Hence, by calculating as in the proof of Lemma 6.2, Lemma 6.4, and Lemma 6.5 (replace $M_\varepsilon u^\varepsilon$ by $M_\varepsilon u^\varepsilon - v$), we obtain (6.21). □

*Proof of Theorem* 6.12. First we show the inequality

$$\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} \leqslant c\varepsilon^{1/2}\left(\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\right). \tag{6.22}$$

To this end, we use the triangle inequality

$$\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} \leqslant \big\|u^\varepsilon - \overline{M_\varepsilon u^\varepsilon}\big\|_{L^2(Q_{\varepsilon,T})} + \big\|\overline{M_\varepsilon u^\varepsilon} - \overline{v}\big\|_{L^2(Q_{\varepsilon,T})}$$

and estimate the right-hand side of the above inequality. By (5.9) and (3.15), we have

$$\big\|u^\varepsilon - \overline{M_\varepsilon u^\varepsilon}\big\|_{L^2(Q_{\varepsilon,T})} \leqslant c\varepsilon\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}} \leqslant c\varepsilon\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}$$

with a constant $c > 0$ independent of $\varepsilon$. On the other hand, by (5.7) and (6.21),

$$\big\|\overline{M_\varepsilon u^\varepsilon} - \overline{v}\big\|_{L^2(Q_{\varepsilon,T})} \leqslant c\varepsilon^{1/2}\|M_\varepsilon u^\varepsilon - v\|_{H_T} \leqslant c\varepsilon^{1/2}\big(\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}\big).$$

Hence (6.22) follows.

Next we estimate the right-hand side of (6.22) to get (6.20). We use the notation

$$(u_0^\varepsilon)^\sharp(Y,\rho) := u_0^\varepsilon(Y + \rho\nu(Y,0)), \quad Y \in \Gamma_0,\ \rho \in \big(\varepsilon g_0(Y,0), \varepsilon g_1(Y,0)\big),$$

and omit the variables $Y$, $\rho$, and $t = 0$. We set

$$I_1 := \frac{1}{\varepsilon g}\int_{\varepsilon g_0}^{\varepsilon g_1} \{(u_0^\varepsilon)^\sharp - v_0\} J\, d\rho, \quad I_2 := \frac{v_0}{\varepsilon g}\int_{\varepsilon g_0}^{\varepsilon g_1} (J-1)\, d\rho.$$

Then $M_\varepsilon u_0^\varepsilon - v_0 = I_1 + I_2$ on $\Gamma_0$. By Hölder's inequality and (2.1), (5.4), we have

$$|I_1|^2 \leqslant \frac{1}{\varepsilon g}\int_{\varepsilon g_0}^{\varepsilon g_1} \big|(u_0^\varepsilon)^\sharp - v_0\big|^2 J^2\, d\rho \leqslant c\varepsilon^{-1}\int_{\varepsilon g_0}^{\varepsilon g_1} \big|(u_0^\varepsilon)^\sharp - v_0\big|^2 J\, d\rho.$$

On the other hand, (5.5) yields $|I_2| \leqslant c\varepsilon|v_0|$. Hence

$$\begin{aligned}
\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)}^2 &\leqslant c\int_{\Gamma_0} \big(|I_1|^2 + |I_2|^2\big)\, d\mathcal{H}^{n-1} \\
&\leqslant c\int_{\Gamma_0} \left(\varepsilon^{-1}\int_{\varepsilon g_0}^{\varepsilon g_1} |(u_0^\varepsilon)^\sharp - v_0|^2 J\, d\rho + \varepsilon^2|v_0|\right) d\mathcal{H}^{n-1} \\
&= c\big(\varepsilon^{-1}\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))}^2 + \varepsilon^2\|v_0\|_{L^2(\Gamma_0)}^2\big).
\end{aligned}$$

Here we used the co-area formula (5.1) in the last equality. The above inequality is equivalent to

$$\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} \leqslant c\big(\varepsilon^{-1/2}\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \varepsilon\|v_0\|_{L^2(\Gamma_0)}\big). \tag{6.23}$$

Moreover, by the triangle inequality and (5.7),

$$\begin{aligned}
\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))} &\leqslant \|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \|\overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} \\
&\leqslant \|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + c\varepsilon^{1/2}\|v_0\|_{L^2(\Gamma_0)}.
\end{aligned} \tag{6.24}$$

Finally, by applying (6.23) and (6.24) to (6.22), we obtain

$$\begin{aligned}
\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} &\leqslant c\varepsilon^{1/2}\big(\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\big) \\
&\leqslant c\varepsilon^{1/2}\big((\varepsilon^{-1/2} + \varepsilon^{1/2})\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \varepsilon\|v_0\|_{L^2(\Gamma_0)}\big) \\
&\leqslant c\big(\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \varepsilon^{3/2}\|v_0\|_{L^2(\Gamma_0)}\big)
\end{aligned}$$

with a constant $c > 0$ independent of $\varepsilon$. Hence (6.20) holds. □

## A. Heuristic derivation of the limit equation

Let us give a heuristic derivation of the limit equation (1.1) from ($H_\varepsilon$) when $\Omega_\varepsilon(t)$ is of the form $\Omega_\varepsilon(t) = \{x \in \mathbb{R}^n \mid -\varepsilon < d(x,t) < \varepsilon\}$. In this case, the unit outward normal vector field $\nu_\varepsilon$ of $\partial\Omega_\varepsilon(t)$ and the outer normal velocity $v_\varepsilon^N$ of $\partial\Omega_\varepsilon(t)$ are of the form

$$\nu_\varepsilon(x,t) = \pm\nu(p(x,t),t), \quad v_\varepsilon^N(x,t) = \pm v_\Gamma^N(p(x,t),t), \quad (x,t) \in \partial_\ell Q_{\varepsilon,T},$$

according to $d(x,t) = \pm\varepsilon$ (double-sign corresponds). Thus we start from the heat equation

$$\partial_t u^\varepsilon(x,t) - \Delta u^\varepsilon(x,t) = 0, \quad (x,t) \in Q_{\varepsilon,T}$$

with the boundary condition

$$\nu(p(x,t),t)\cdot\nabla u^\varepsilon(x,t) + v_\Gamma^N(p(x,t),t)u^\varepsilon(x,t) = 0, \quad (x,t) \in \partial_\ell Q_{\varepsilon,T}. \tag{A.1}$$

To derive the limit equation, we make the following assumptions:

(1) The signed distance $d(x,t)$ of $x \in \Omega_\varepsilon(t)$ is negligible ($d(x,t) \approx 0$), although the quantity $\varepsilon^{-1}d(x,t)$ is not negligible.
(2) The relation $v_\Gamma^N(p(x,t),t) \approx -\partial_t d(x,t)$ holds for all $(x,t) \in Q_{\varepsilon,T}$.
(3) The boundary condition (A.1) also holds in the noncylindrical domain $Q_{\varepsilon,T}$.

These assumptions come from the smallness of the width $2\varepsilon$ of $\Omega_\varepsilon(t)$. Taking the third assumption into account, we consider the two equations

$$\partial_t u^\varepsilon(x,t) - \Delta u^\varepsilon(x,t) = 0, \tag{A.2}$$
$$\nu(p(x,t),t)\cdot\nabla u^\varepsilon(x,t) + v_\Gamma^N(p(x,t),t)u^\varepsilon(x,t) = 0 \tag{A.3}$$

for $(x,t) \in Q_{\varepsilon,T}$. Recall that each $x \in \Omega_\varepsilon(t)$ is represented as

$$x = p(x,t) + d(x,t)\nu\big(p(x,t),t\big), \quad \nabla d(x,t) = \nu(x,t) = \nu\big(p(x,t),t\big).$$

First, we consider the gradient matrix of the projection $p(x,t)$ onto $\Gamma(t)$ given by

$$\nabla p = \begin{pmatrix} \partial_1 p_1 & \dots & \partial_n p_1 \\ \vdots & \ddots & \vdots \\ \partial_1 p_n & \dots & \partial_n p_n \end{pmatrix} \quad \text{for} \quad p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}.$$

By differentiating both sides of $x = p(x,t) + d(x,t)\nu(x,t)$ and using $\nabla d(x,t) = \nu(x,t)$, we have

$$I_n = \nabla p(x,t) + \nu(x,t)\otimes\nu(x,t) + d(x,t)\nabla\nu(x,t).$$

According to the assumption (1), the above equality reads

$$\nabla p(x,t) \approx I_n - \nu(x,t)\otimes\nu(x,t) = I_n - \nu\big(p(x,t),t\big)\otimes\nu\big(p(x,t),t\big). \tag{A.4}$$

We define a function $v\colon S_T \times (-1,1) \to \mathbb{R}$ as

$$v(y,t,r) := u^\varepsilon\big(y + \varepsilon r\nu(y,t),t\big), \quad (y,t) \in S_T,\ r \in (-1,1).$$

Then $u^\varepsilon$ is represented by $v$ as

$$u^\varepsilon(x,t) = v\big(p(x,t), t, \varepsilon^{-1}d(x,t)\big), \quad (x,t) \in Q_{\varepsilon,T}. \tag{A.5}$$

For abbreviation, we write $p$ and $d$ for $p(x,t)$ and $d(x,t)$ in arguments of functions unless we would like to emphasize them. For example, we write $\nu(p,t)$ for $\nu(p(x,t),t)$ and $v(p,t,\varepsilon^{-1}d)$ for $v(p(x,t),t,\varepsilon^{-1}d(x,t))$. By the chain rule of differentiation we have

$$\nabla u^\varepsilon(x,t) = \big[\nabla p(x,t)\big]^T \nabla v(p,t,\varepsilon^{-1}d) + \varepsilon^{-1}\partial_r v(p,t,\varepsilon^{-1}d)\nabla d(x,t).$$

By (A.4) and $\nabla d(x,t) = \nu(p(x,t),t)$, this equality reads

$$\nabla u^\varepsilon(x,t) \approx \nabla_{\Gamma(t)} v(p,t,\varepsilon^{-1}d) + \varepsilon^{-1}\partial_r v(p,t,\varepsilon^{-1}d)\nu(p,t). \tag{A.6}$$

Here we abused the definition of the tangential gradient $\nabla_{\Gamma(t)} = (I_n - \nu \otimes \nu)\nabla$. Applying (A.6) to (A.3) and observing that $\nu(p,t) \cdot \nabla_{\Gamma(t)} v(p,t,\varepsilon^{-1}d) = 0$, we obtain

$$\varepsilon^{-1}\partial_r v(p,t,\varepsilon^{-1}d) \approx -v_\Gamma^N(p,t)v(p,t,\varepsilon^{-1}d) \tag{A.7}$$

and thus (A.6) becomes

$$\nabla u^\varepsilon(x,t) \approx \nabla_{\Gamma(t)} v(p,t,\varepsilon^{-1}d) - v_\Gamma^N(p,t)v(p,t,\varepsilon^{-1}d)\nu(p,t). \tag{A.8}$$

Next we compute $\Delta u^\varepsilon = \operatorname{div} \nabla u^\varepsilon$. For a vector field $F$ on $\Omega_\varepsilon(t)$ with each fixed $t \in [0,T]$,

$$\begin{aligned}\operatorname{div} F(x) &= \operatorname{trace}\big[\nabla F(x)\big] \\ &= \operatorname{trace}\big[\{I_n - \nu(x,t) \otimes \nu(x,t)\}\nabla F(x)\big] + \operatorname{trace}\big[\nu(x,t) \otimes \nu(x,t)\nabla F(x)\big] \\ &= \operatorname{div}_{\Gamma(t)} F(x) + \nu(x,t) \cdot \partial_\nu F(x)\end{aligned}$$

holds since $\nu \otimes \nu$ is a projection matrix onto the $\nu$-direction. Hence we have

$$\operatorname{div}\big[\nabla_{\Gamma(t)} v(p,t,\varepsilon^{-1}d)\big] = \operatorname{div}_{\Gamma(t)}\big[\nabla_{\Gamma(t)} v(p,t,\varepsilon^{-1}d)\big] + \nu(x,t) \cdot \partial_\nu\big[\nabla_{\Gamma(t)} v(p,t,\varepsilon^{-1}d)\big].$$

Moreover, since $p(x + h\nu(x,t), t) = p(x,t)$ and $d(x + h\nu(x,t), t) = d(x,t) + h$ for sufficiently small $h \in \mathbb{R}$, it follows that

$$\nabla_{\Gamma(t)} v\Big(p\big(x + h\nu(x,t), t\big), t, \varepsilon^{-1}d\big(x + h\nu(x,t), t\big)\Big) = \nabla_{\Gamma(t)} v(p(x,t), t, \varepsilon^{-1}d(x,t) + \varepsilon^{-1}h)$$

and thus

$$\partial_\nu\big[\nabla_{\Gamma(t)} v(p,t,\varepsilon^{-1}d)\big] = \varepsilon^{-1}\partial_r\big[\nabla_{\Gamma(t)} v(p,t,\varepsilon^{-1}d)\big]$$

by the formula $\partial_\nu f(x) = \lim_{h\to 0}\{f(x + h\nu(x,t)) - f(x)\}/h$ for functions $f$ on $\Omega_\varepsilon(t)$ with fixed $t \in [0,T]$. Hence we obtain

$$\operatorname{div}\big[\nabla_{\Gamma(t)} v(p,t,\varepsilon^{-1}d)\big] = \operatorname{div}_{\Gamma(t)}\big[\nabla_{\Gamma(t)} v(p,t,\varepsilon^{-1}d)\big] + \varepsilon^{-1}\nu(x,t) \cdot \partial_r\big[\nabla_{\Gamma(t)} v(p,t,\varepsilon^{-1}d)\big].$$

Similarly we have

$$
\begin{aligned}
&\operatorname{div}\big[v_\Gamma^N(p,t)v(p,t,\varepsilon^{-1}d)\nu(p,t)\big] \\
&\quad = \operatorname{div}_{\Gamma(t)}\big[v_\Gamma^N(p,t)v(p,t,\varepsilon^{-1}d)\nu(p,t)\big] + \nu(x,t)\cdot\big\{\varepsilon^{-1}v_\Gamma^N(p,t)\partial_r v(p,t,\varepsilon^{-1}d)\nu(p,t)\big\} \\
&\quad \approx \operatorname{div}_{\Gamma(t)}\big[v_\Gamma^N(p,t)v(p,t,\varepsilon^{-1}d)\nu(p,t)\big] - \big\{v_\Gamma^N(p,t)\big\}^2 v(p,t,\varepsilon^{-1}d).
\end{aligned}
$$

Here the last approximation follows from $\nu(x,t) = \nu(p(x,t),t)$ and (A.7). Hence, by (A.8),

$$
\begin{aligned}
\Delta u^\varepsilon(x,t) \approx{}& \operatorname{div}_{\Gamma(t)}\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big] + \varepsilon^{-1}\nu(x,t)\cdot\partial_r\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big] \\
&- \operatorname{div}_{\Gamma(t)}\big[v_\Gamma^N(p,t)v(p,t,\varepsilon^{-1}d)\nu(p,t)\big] + \big\{v_\Gamma^N(p,t)\big\}^2 v(p,t,\varepsilon^{-1}d). \qquad \text{(A.9)}
\end{aligned}
$$

On the other hand, we differentiate both sides of (A.5) with respect to $t$ to get

$$
\partial_t u^\varepsilon(x,t) = \partial_t p(x,t)\cdot\nabla v(p,t,\varepsilon^{-1}d) + \partial_t v(p,t,\varepsilon^{-1}d) + \varepsilon^{-1}\partial_t d(x,t)\partial_r v(p,t,\varepsilon^{-1}d).
$$

To this equality we apply (A.7) and

$$
\partial_t p(x,t) = -\partial_t d(x,t)\nu(x,t) - d(x,t)\partial_t\nu(x,t) \approx v_\Gamma^N(p,t)\nu(p,t),
$$

where the last approximation follows from the assumptions (1), (2), and $\nu(x,t) = \nu(p(x,t),t)$. Then we have

$$
\partial_t u^\varepsilon(x,t) \approx v_\Gamma^N(p,t)\nu(p,t)\cdot\nabla v(p,t,\varepsilon^{-1}d) + \partial_t v(p,t,\varepsilon^{-1}d) + \big\{v_\Gamma^N(p,t)\big\}^2 v(p,t,\varepsilon^{-1}d). \qquad \text{(A.10)}
$$

Substituting (A.9) and (A.10) for the equation (A.2), we obtain

$$
\begin{aligned}
&\partial_t v(p,t,\varepsilon^{-1}d) + v_\Gamma^N(p,t)\nu(p,t)\cdot\nabla v(p,t,\varepsilon^{-1}d) + \operatorname{div}_{\Gamma(t)}\big[v_\Gamma^N(p,t)v(p,t,\varepsilon^{-1}d)\nu(p,t)\big] \\
&\qquad - \operatorname{div}_{\Gamma(t)}\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big] - \varepsilon^{-1}\nu(x,t)\cdot\partial_r\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big] = 0.
\end{aligned}
$$

Now let us make an additional assumption: the function $v(y,t,r)$ is independent of the variable $r$. Then, the above equation reads

$$
\begin{aligned}
&\partial_t v(y,t) + v_\Gamma^N(y,t)\nu(y,t)\cdot\nabla v(y,t) + \operatorname{div}_{\Gamma(t)}\big[v_\Gamma^N(y,t)v(y,t)\nu(y,t)\big] \\
&\qquad - \operatorname{div}_{\Gamma(t)}\big[\nabla_{\Gamma(t)}v(y,t)\big] = 0
\end{aligned}
$$

with $y = p(x,t) \in \Gamma(t)$. Finally we observe that

$$
\operatorname{div}_{\Gamma(t)}(v_\Gamma^N v\nu) = \nabla_{\Gamma(t)}(v_\Gamma^N v)\cdot\nu + v_\Gamma^N v\operatorname{div}_{\Gamma(t)}\nu = 0 + v_\Gamma^N v\cdot(-H) = -v_\Gamma^N Hv,
$$

where $H = -\operatorname{div}_{\Gamma(t)}\nu$ is the mean curvature of $\Gamma(t)$, to obtain

$$
\partial_t v(y,t) + v_\Gamma^N(y,t)\nu(y,t)\cdot\nabla v(y,t) - v_\Gamma^N(y,t)H(y,t)v(y,t) - \Delta_{\Gamma(t)}v(y,t) = 0
$$

for $(y,t) \in S_T$. This is the limit equation (1.1) we mentioned in Section 1.

## B. Elementary facts on integrals over evolving surfaces

In this appendix we give complete proofs of several facts on integrals over evolving surfaces which are essentially known or easily proved but there is no detailed proof for the readers' convenience. We first show the transformation formula (4.2).

*Proof of* (4.2). By a localization argument with a partition of unity of $S_T$, it is sufficient to show

$$\int_I \int_{\mu_t(U)} f(y,t)\,d\mathcal{H}^{n-1}(y)\,dt = \int_{\zeta(U\times I)} f(\sigma)\bigl(1+\bigl|v_\Gamma^N(\sigma)\bigr|^2\bigr)^{-1/2}\,d\mathcal{H}^n(\sigma), \tag{B.1}$$

where $I$ is an open interval in $(0,T)$, $U$ is an open set in $\mathbb{R}^{n-1}$, $\mu_t\colon U\to\Gamma(t)$ is a smooth local parametrization of $\Gamma(t)$ for each $t\in I$, and $\zeta\colon U\times I\to S_T$ is given by $\zeta(s,t)=(\mu_t(s),t)$. Moreover, by rotating coordinates and taking $I$ sufficiently small, we may assume that there exists a smooth function $h$ on $U\times I$ such that $\mu_t(s)=(s,h(s,t))$ for all $s\in U$ and $t\in I$. Then $\zeta(s,t)=(s,h(s,t),t)$ and the outward normal velocity $v_\Gamma^N$ of $\Gamma(t)$ is given by

$$v_\Gamma^N(\mu_t(s),t) = \frac{\partial_t h(s,t)}{\sqrt{1+|\nabla' h(s,t)|^2}}, \quad (s,t)\in U\times I. \tag{B.2}$$

Here $\nabla'$ is the gradient in $s\in\mathbb{R}^{n-1}$ and we assume that the $n$-th component of the normal $\nu$ is positive on $\zeta(U\times I)$. For $t\in I$ the Riemannian metric on $\Gamma(t)$ is locally given by

$$\frac{\partial\mu_t}{\partial s_i}(s)\cdot\frac{\partial\mu_t}{\partial s_j}(s) = \delta_{ij} + \frac{\partial h}{\partial s_i}(s,t)\frac{\partial h}{\partial s_j}(s,t), \quad s\in U,\ i,j=1,\dots,n-1,$$

where $\delta_{ij}$ is the Kronecker delta. Hence the left-hand side of (B.1) is

$$\int_I\int_{\mu_t(U)} f(y,t)\,d\mathcal{H}^{n-1}(y)\,dt = \int_I\int_U f\bigl(\mu_t(s),t\bigr)\sqrt{1+|\nabla' h(s,t)|^2}\,ds\,dt. \tag{B.3}$$

On the other hand, since the Riemannian metric on $S_T$ is locally given by

$$\frac{\partial\zeta}{\partial s_i}(s,t)\cdot\frac{\partial\zeta}{\partial s_j}(s,t) = \delta_{ij} + \frac{\partial h}{\partial s_i}(s,t)\frac{\partial h}{\partial s_i}(s,t),$$
$$\frac{\partial\zeta}{\partial s_i}(s,t)\cdot\frac{\partial\zeta}{\partial t}(s,t) = \frac{\partial h}{\partial s_i}(s,t)\frac{\partial h}{\partial t}(s,t), \quad \frac{\partial\zeta}{\partial t}(s,t)\cdot\frac{\partial\zeta}{\partial t}(s,t) = 1+\left|\frac{\partial h}{\partial t}(s,t)\right|^2$$

for $s\in U$, $t\in I$, and $i,j=1,\dots,n-1$, the right-hand side of (B.1) is

$$\int_{\zeta(U\times I)} f(\sigma)\bigl(1+\bigl|v_\Gamma^N(\sigma)\bigr|^2\bigr)^{-1/2}\,d\mathcal{H}^n(\sigma)$$
$$= \int_{U\times I} f\bigl(\mu_t(s),t\bigr)\bigl(1+\bigl|v_\Gamma^N\bigl(\mu_t(s),t\bigr)\bigr|^2\bigr)^{-1/2}\sqrt{\det A(s,t)}\,ds\,dt. \tag{B.4}$$

Here $A$ is a matrix of the form

$$A = \begin{pmatrix} I_{n-1}+\nabla' h\otimes\nabla' h & \partial_t h\nabla' h \\ \partial_t h(\nabla' h)^T & 1+|\partial_t h|^2 \end{pmatrix},$$

where $(\nabla' h)^T$ is the transpose of the column vector $\nabla' h$. By elementary row operations we have

$$\begin{aligned}
\det A &= \det\begin{pmatrix} I_{n-1} + \{1 - |\partial_t h|^2/(1+|\partial_t h|^2)\}\nabla' h \otimes \nabla' h & 0 \\ \partial_t h (\nabla' h)^T & 1 + |\partial_t h|^2 \end{pmatrix} \\
&= (1+|\partial_t h|^2) \det\left[ I_{n-1} + \left(1 - \frac{|\partial_t h|^2}{1+|\partial_t h|^2}\right) \nabla' h \otimes \nabla' h \right] \\
&= (1+|\partial_t h|^2) \left\{ 1 + \left(1 - \frac{|\partial_t h|^2}{1+|\partial_t h|^2}\right) |\nabla' h|^2 \right\} \\
&= \left(1 + \frac{|\partial_t h|^2}{1+|\nabla' h|^2}\right)(1+|\nabla' h|^2).
\end{aligned}$$

Hence, by (B.2),

$$\det A(s,t) = \left(1 + \left|v_\Gamma^N(\mu_t(s),t)\right|^2\right)\left(1 + \left|\nabla' h(s,t)\right|^2\right).$$

Substituting this for the right-hand side of (B.4) and applying Fubini's theorem, we get the right-hand side of (B.3) and thus conclude that (B.1) holds. □

Next we give complete proofs of Lemma 4.1 and Lemma 4.3. Before starting to prove, let us construct a partition of unity of $\Gamma(t)$ by that of $\Gamma_0$. Since $\Gamma_0$ is compact, we can take a finite family $\{U_k\}_{k=1}^N$ of open sets in $\mathbb{R}^{n-1}$ and smooth local parametrizations $\mu_0^k\colon U_k \to \Gamma_0$, $k = 1,\dots,N$ such that $\{\mu_0^k(U_k)\}_{k=1}^N$ is an open covering of $\Gamma_0$. Let $\{\psi_0^k\}_{k=1}^N$ be a partition of unity of $\Gamma_0$ subordinate to the covering $\{\mu_0^k(U_k)\}_{k=1}^N$. For $k = 1,\dots,N$ and $t \in [0,T]$ we set

$$\mu_t^k(s) := \Phi\big(\mu_0^k(s),t\big), \quad s \in U_k, \quad \psi_t^k := \psi_0^k \circ \mu_0^k \circ (\mu_t^k)^{-1}, \tag{B.5}$$

where $\Phi(\cdot,t)\colon \Gamma_0 \to \Gamma(t)$ is the flow map of $V_\Gamma$ (see Section 2). Then for each $k = 1,\dots,N$ the mapping $\mu_t^k\colon U_k \to \Gamma(t)$ is a local parametrization of $\Gamma(t)$ and $\{\mu_t^k(U_k)\}_{k=1}^N$ is an open covering of $\Gamma(t)$. Moreover, $\{\psi_t^k\}_{k=1}^N$ is a partition of unity of $\Gamma(t)$ subordinate to $\{\mu_t^k(U_k)\}_{k=1}^N$. We use these partitions of unity to localize integrals over $\Gamma(t)$.

*Proof of Lemma* 4.1. Let $V$ be a function on $\Gamma_0 \times (0,T)$ and $v := LV$. Our goal is to show

$$\begin{aligned}
c_1\|V(t)\|_{L^2(\Gamma_0)} \leqslant \|v(t)\|_{L^2(\Gamma(t))} \leqslant c_2\|V(t)\|_{L^2(\Gamma_0)}, \\
c_1\|\nabla_{\Gamma_0} V(t)\|_{L^2(\Gamma_0)} \leqslant \|\nabla_{\Gamma(t)} v(t)\|_{L^2(\Gamma(t))} \leqslant c_2\|\nabla_{\Gamma_0} V(t)\|_{L^2(\Gamma_0)}
\end{aligned}$$

for all $t \in (0,T)$ with some positive constants $c_1$, $c_2$ independent of $t$. These inequalities yield $c_1\|V\|_{\widehat{H}_T} \leqslant \|v\|_{H_T} \leqslant c_2\|V\|_{\widehat{H}_T}$, which means that $L$ is an isomorphism between $\widehat{H}_T$ and $H_T$. By a localization argument with the partitions of unity given by (B.5), it is sufficient to show that

$$c_1 \int_{\mu_0(Q)} |V(t)|^2 \, d\mathcal{H}^{n-1} \leqslant \int_{\mu_t(Q)} |v(t)|^2 \, d\mathcal{H}^{n-1} \leqslant c_2 \int_{\mu_0(Q)} |V(t)|^2 \, d\mathcal{H}^{n-1}, \tag{B.6}$$

$$c_1 \int_{\mu_0(Q)} |\nabla_{\Gamma_0} V(t)|^2 \, d\mathcal{H}^{n-1} \leqslant \int_{\mu_t(Q)} |\nabla_{\Gamma(t)} v(t)|^2 \, d\mathcal{H}^{n-1} \leqslant c_2 \int_{\mu_0(Q)} |\nabla_{\Gamma_0} V(t)|^2 \, d\mathcal{H}^{n-1} \tag{B.7}$$

for all $t \in (0,T)$ and all $V$ supported in $\mu_0(Q) \times (0,T)$. Here $\mu_0\colon U \to \Gamma_0$ be a smooth local parametrization of $\Gamma_0$ with an open set $U$ in $\mathbb{R}^{n-1}$, $Q$ is a compact subset of $U$, and $\mu_t\colon U \to \Gamma(t)$ is the local parametrization of $\Gamma(t)$ given by $\mu_t(s) := \Phi(\mu_0(s),t)$ for $s \in U$. Note that in this case $v = LV$ is supported in $\bigcup_{t\in(0,T)} \mu_t(Q) \times \{t\}$. Let $\theta_t = (\theta_{t,ij})_{i,j}$ be a matrix given by

$$\theta_{t,ij}(s) := \frac{\partial \mu_t}{\partial s_i}(s) \cdot \frac{\partial \mu_t}{\partial s_j}(s), \quad (s,t) \in U \times [0,T],\ i,j = 1,\dots,n-1, \tag{B.8}$$

and $\theta_t^{-1} = (\theta_t^{ij})_{ij}$ be the inverse matrix of $\theta_t$. By the definition of integrals over hypersurfaces,

$$\int_{\mu_0(Q)} |V(Y,t)|^2 \, d\mathcal{H}^{n-1}(Y) = \int_Q |V(\mu_0(s),t)|^2 \sqrt{\det \theta_0(s)}\, ds,$$
$$\int_{\mu_t(Q)} |v(y,t)|^2 \, d\mathcal{H}^{n-1}(y) = \int_Q |v(\mu_t(s),t)|^2 \sqrt{\det \theta_t(s)}\, ds.$$

Since $\sqrt{\det \theta_t(s)}$ is continuous and does not vanish as a function of $(s,t)$ on the compact set $Q \times [0,T]$, there is a constant $c > 0$ such that

$$c^{-1} \leqslant \sqrt{\det \theta_t(s)} \leqslant c \quad \text{for all} \quad (s,t) \in Q \times [0,T]. \tag{B.9}$$

Moreover, by the definitions of $L$ and $\mu_t$,

$$v(\mu_t(s),t) = V(\Phi^{-1}(\mu_t(s),t),t) = V(\Phi^{-1}(\Phi(\mu_0(s),t),t),t) = V(\mu_0(s),t) \tag{B.10}$$

for all $(s,t) \in U \times [0,T]$. Hence (B.6) follows. Similarly, by (B.9) and the equality

$$\int_{\mu_0(Q)} |\nabla_{\Gamma_0} V(Y,t)|^2 \, d\mathcal{H}^{n-1}(Y) = \int_Q |\nabla_{\Gamma_0} V(\mu_0(s),t)|^2 \sqrt{\det \theta_0(s)}\, ds,$$
$$\int_{\mu_t(Q)} |\nabla_{\Gamma(t)} v(y,t)|^2 \, d\mathcal{H}^{n-1}(y) = \int_Q |\nabla_{\Gamma(t)} v(\mu_t(s),t)|^2 \sqrt{\det \theta_t(s)}\, ds,$$

it is sufficient for (B.7) to show that

$$c_1 |\nabla_{\Gamma_0} V(\mu_0(s),t)|^2 \leqslant |\nabla_{\Gamma(t)} v(\mu_t(s),t)|^2 \leqslant c_2 |\nabla_{\Gamma_0} V(\mu_0(s),t)|^2 \tag{B.11}$$

for all $(s,t) \in Q \times [0,T]$. The tangential gradients $\nabla_{\Gamma_0} V$ and $\nabla_{\Gamma(t)} v$ are locally expressed as (see [6, Section 2.1 and Section 2.2] for example)

$$\nabla_{\Gamma_0} V(\mu_0(s),t) = \sum_{i,j=1}^{n-1} \theta_0^{ij}(s) \frac{\partial}{\partial s_j}(V(\mu_0(s),t)) \frac{\partial \mu_0}{\partial s_i}(s),$$
$$\nabla_{\Gamma(t)} v(\mu_t(s),t) = \sum_{i,j=1}^{n-1} \theta_t^{ij}(s) \frac{\partial}{\partial s_j}(v(\mu_t(s),t)) \frac{\partial \mu_t}{\partial s_i}(s)$$

for $(s,t) \in U \times [0,T]$ and their Euclidean norms are

$$\left|\nabla_{\Gamma_0} V\big(\mu_0(s),t\big)\right|^2 = \sum_{i,j=1}^{n-1} \theta_0^{ij}(s) \frac{\partial}{\partial s_i}\Big(V\big(\mu_0(s),t\big)\Big) \frac{\partial}{\partial s_j}\Big(V\big(\mu_0(s),t\big)\Big),$$
$$\left|\nabla_{\Gamma(t)} v\big(\mu_t(s),t\big)\right|^2 = \sum_{i,j=1}^{n-1} \theta_t^{ij}(s) \frac{\partial}{\partial s_i}\Big(v\big(\mu_t(s),t\big)\Big) \frac{\partial}{\partial s_j}\Big(v\big(\mu_t(s),t\big)\Big).$$

Then, by (B.10), it is sufficient for (B.11) to show

$$c_1\theta_0^{-1}(s)a\cdot a \leqslant \theta_t^{-1}(s)a\cdot a \leqslant c_2\theta_0^{-1}(s)a\cdot a \quad \text{for all} \quad (s,t,a)\in Q\times[0,T]\times\mathbb{R}^{n-1}. \tag{B.12}$$

To this end, we consider a real-valued function

$$F(s,t,a) := \theta_t^{-1}(s)a\cdot a, \quad (s,t,a)\in Q\times[0,T]\times\mathbb{R}^{n-1}.$$

It is continuous on $Q\times[0,T]\times\mathbb{R}^{n-1}$ and satisfies $F(s,t,a)=|B(s,t,a)|^2$, where

$$B(s,t,a) := \sum_{i=1}^{n-1} b_i(s,t,a)\frac{\partial\mu_t}{\partial s_i}(s), \quad b=(b_1,\dots,b_{n-1}) := \theta_t^{-1}(s)a.$$

For $a\neq 0$ we have $b\neq 0$ and thus $B\neq 0$. Hence $F$ does not vanish on the compact set $Q\times[0,T]\times S^{n-2}$, where $S^{n-2}$ is the unit sphere in $\mathbb{R}^{n-1}$. From this fact and the continuity of $F$ there is a constant $c>0$ such that $c^{-1}\leqslant F(s,t,a)\leqslant c$ for all $(s,t,a)\in Q\times[0,T]\times S^{n-2}$ and thus

$$c^{-1}|a|^2 \leqslant \theta_t^{-1}(s)a\cdot a \leqslant c|a|^2 \quad \text{for all} \quad (s,t,a)\in Q\times[0,T]\times\mathbb{R}^{n-1}.$$

This inequality yields (B.12) and we conclude that (B.7) is valid. □

*Proof of Lemma 4.3.* First we give transformation formulas of integrals over $\Gamma_0$ and $\Gamma(t)$. Let $U$ be an open set in $\mathbb{R}^{n-1}$ and $\mu_0\colon U\to\Gamma_0$ be a smooth local parametrization of $\Gamma_0$. Moreover, let $\mu_t\colon U\to\Gamma(t)$ be the local parametrization of $\Gamma(t)$ given by $\mu_t(s):=\Phi(\mu_0(s),t)$. We set

$$\Lambda(\mu_0(s),t) := \sqrt{\frac{\det\theta_t(s)}{\det\theta_0(s)}}, \quad \lambda(\mu_t(s),t) := \sqrt{\frac{\det\theta_0(s)}{\det\theta_t(s)}}, \quad (s,t)\in U\times[0,T],$$

where $\theta_t=(\theta_{t,ij})_{ij}$ is given by (B.8). We can show that the right-hand sides of the above definitions are independent of the choice of the local parametrization $\mu_0$. From this fact and the smoothness assumption on $\Phi$, the functions $\Lambda$ and $\lambda$ are well-defined and smooth on the compact manifolds $\Gamma_0\times[0,T]$ and $\overline{S_T}$, respectively. In particular, they are bounded on $\Gamma_0\times[0,T]$ and $\overline{S_T}$ along with their derivatives. Moreover, by a localization argument with the partitions of unity given by (B.5), we get the integral transformation formulas

$$\int_{\Gamma(t)} v(y,t)\,d\mathcal{H}^{n-1}(y) = \int_{\Gamma_0} V(Y,t)\Lambda(Y,t)\,d\mathcal{H}^{n-1}(Y), \tag{B.13}$$

$$\int_{\Gamma_0} V(Y,t)\,d\mathcal{H}^{n-1}(Y) = \int_{\Gamma(t)} v(y,t)\lambda(y,t)\,d\mathcal{H}^{n-1}(y) \tag{B.14}$$

for all functions $V$ on $\Gamma_0\times(0,T)$ and all $t\in(0,T)$, where $v=LV$.

Now let us prove the statement of Lemma 4.3. For $V\in\widehat{W}_T$ we set $v:=LV$. Then Lemma 4.1 yields $v\in H_T$ and $\|v\|_{H_T}\leqslant c\|V\|_{\widehat{H}_T}$. We next show that $\partial^\bullet v\in H_T'$ and $\|\partial^\bullet v\|_{H_T'}\leqslant c\|V\|_{\widehat{W}_T}$. Let $\psi\in C_0^1(S_T)$. Then $\Psi:=L^{-1}\psi$ is in $C_0^1(\Gamma_0\times(0,T))$ and $\partial^\bullet\psi(\Phi(Y,t),t)=\partial_t\Psi(Y,t)$ for all $Y\in\Gamma_0$. Hence (B.13) yields

$$\begin{aligned}\langle\partial^\bullet v,\psi\rangle_T &= -\int_0^T\int_{\Gamma(t)}(v\,\partial^\bullet\psi + v\psi\,\mathrm{div}_{\Gamma(t)}V_\Gamma)\,d\mathcal{H}^{n-1}\,dt\\ &= -\int_0^T\int_{\Gamma_0}(V\,\partial_t\Psi + V\Psi F)\Lambda\,d\mathcal{H}^{n-1}\,dt,\end{aligned}$$

where $F := L^{-1}(\operatorname{div}_{\Gamma(t)} V_\Gamma) \in C^\infty(\Gamma_0 \times [0,T])$. Moreover, since $\Psi\Lambda \in C_0^1(\Gamma_0 \times (0,T))$,

$$-\int_0^T \int_{\Gamma_0} V\Lambda\,\partial_t\Psi\,d\mathcal{H}^{n-1}\,dt = [\partial_t V, \Psi\Lambda]_T + \int_0^T \int_{\Gamma_0} V\Psi\,\partial_t\Lambda\,d\mathcal{H}^{n-1}\,dt$$

by the definition of the weak time derivative $\partial_t V$. From these formulas and the boundedness of $F$ and $\Lambda$ on $\Gamma_0 \times (0,t)$ along with their derivatives, it follows that

$$\begin{aligned}|\langle\partial^\bullet v, \psi\rangle_T| &= \left|[\partial_t V, \Psi\Lambda]_T + \int_0^T \int_{\Gamma_0} (V\Psi\,\partial_t\Lambda - V\Psi\Lambda F)\,d\mathcal{H}^{n-1}\,dt\right| \\ &\leqslant c(\|\partial_t V\|_{\widehat{H}_T'}\|\Psi\Lambda\|_{\widehat{H}_T} + \|V\|_{\widehat{H}_T}\|\Psi\|_{\widehat{H}_T}) \leqslant c\|V\|_{\widehat{W}_T}\|\psi\|_{H_T}\end{aligned}$$

with a constant $c > 0$ independent of $V$ and $\psi$, which implies $\partial^\bullet v \in H_T'$ and $\|\partial^\bullet v\|_{H_T'} \leqslant c\|V\|_{\widehat{W}_T}$. Hence $v = LV$ is in $W_T$ and $\|v\|_{W_T} \leqslant c\|V\|_{\widehat{W}_T}$ for every $V \in \widehat{W}_T$.

Similarly, by (B.14) and the smoothness of $\lambda$ on $\overline{S_T}$ we can show that $V := L^{-1}v$ is in $\widehat{W}_T$ and $\|V\|_{\widehat{W}_T} \leqslant c\|v\|_{W_T}$ for every $v \in W_T$. Hence $L$ is an isomorphism between $\widehat{W}_T$ and $W_T$. □

## C. Calculations involving the differential geometry of tubular neighborhoods

The purpose of this appendix is to show detailed calculations in the proofs of Lemma 5.5, Lemma 5.6, and Lemma 5.11. We fix $t \in [0,T]$ and omit it until the end of the proof of Lemma 5.6.

The proofs of Lemma 5.5 and Lemma 5.6 involve calculations of the usual gradient in $N$ and the tangential gradient on $\Gamma$ under a local coordinate system. Let $\mu\colon U \to \Gamma$ be a local parametrization of $\Gamma$ with an open set $U$ in $\mathbb{R}^{n-1}$. We set

$$\theta_{ij}(s) := \frac{\partial\mu}{\partial s_i}(s) \cdot \frac{\partial\mu}{\partial s_j}(s), \quad s \in U,\ i,j = 1,\dots,n-1.$$

Then, the tangential gradient of a function $v$ on $\Gamma$ is locally expressed as

$$\nabla_\Gamma v(y) = \sum_{i,j=1}^{n-1} \theta^{ij}(s)\frac{\partial\widetilde{v}}{\partial s_j}(s)\frac{\partial\mu}{\partial s_i}(s), \quad y = \mu(s) \in \mu(U), \tag{C.1}$$

where $\widetilde{v}(s) := v(\mu(s))$ and $\theta^{-1} = (\theta^{ij})_{i,j}$ denotes the inverse matrix of $\theta = (\theta_{ij})_{i,j}$. We define a mapping $M\colon U \times (-\delta,\delta) \to N$ as $M(s,\rho) := \mu(s) + \rho\nu(\mu(s))$ for $(s,\rho) \in U \times (-\delta,\delta)$ and set

$$\Theta_{ij}(s,\rho) := \frac{\partial M}{\partial s_i}(s,\rho) \cdot \frac{\partial M}{\partial s_j}(s,\rho), \quad (s,\rho) \in U \times (-\delta,\delta),\ i,j = 1,\dots,n,$$

where $s_n := \rho$. Then the gradient (in $\mathbb{R}^n$) of a function $u$ on $N$ is locally expressed as

$$\nabla u(x) = \sum_{i,j=1}^{n} \Theta^{ij}(s,\rho)\frac{\partial\widetilde{u}}{\partial s_j}(s,\rho)\frac{\partial M}{\partial s_i}(s,\rho), \quad x = M(s,\rho) \in M\big(U \times (-\delta,\delta)\big), \tag{C.2}$$

where $\widetilde{u}(s,\rho) := u(M(s,\rho))$ and $\Theta^{-1} = (\Theta^{ij})_{i,j}$ is the inverse matrix of $\Theta = (\Theta_{ij})_{i,j}$.

Let $v$ be a function on $\Gamma$ and $\overline{v}$ be its constant extension in the normal direction of $\Gamma$. Then their local representations $\widetilde{v} := v \circ \mu$ and $\widetilde{\overline{v}} := \overline{v} \circ M$ satisfy

$$\widetilde{\overline{v}}(s,\rho) = \overline{v}\Big(p\big(M(s,\rho)\big)\Big) = v\big(\mu(s)\big) = \widetilde{v}(s), \quad (s,\rho) \in U \times (-\delta,\delta).$$

Hereafter we use this fact without mention.

*Proof of Lemma 5.5.* Let $v \in H^1(\Gamma)$. Our goal is to show the inequalities

$$\big|\nabla\overline{v}\big(y+\rho\nu(y)\big)\big| \leq c\big|\nabla_\Gamma v(y)\big|, \quad \big|\nabla\overline{v}\big(y+\rho\nu(y)\big) - \nabla_\Gamma v(y)\big| \leq c\varepsilon\big|\nabla_\Gamma v(y)\big| \tag{C.3}$$

for all $y \in \Gamma$ and $\rho \in (\varepsilon g_0(y), \varepsilon g_1(y))$ with a constant $c > 0$ independent of $y$, $\rho$, and $\varepsilon$. For each fixed $y_0 \in \Gamma$, by a rotation of coordinates we can take an open set $U$ in $\mathbb{R}^{n-1}$ and a local parametrization $\mu\colon U \to \Gamma$ such that $y_0 = \mu(s_0)$ with $s_0 \in U$ and $\mu$ is of the form $\mu(s) = (s, f(s))$ with a smooth function $f$ on $U$ satisfying

$$\nabla' f(s_0) = 0, \quad (\nabla')^2 f(s_0) = \mathrm{diag}[\kappa_1,\dots,\kappa_{n-1}], \tag{C.4}$$

where $\nabla'$ is the gradient in $s \in \mathbb{R}^{n-1}$ and $\kappa_i := \kappa_i(y_0)$ for $i = 1,\dots,n-1$ (see [11, Section 14.6]). We set the direction of $\nu(y_0)$ in the positive direction of the $x_n$-axis to get

$$\nu\big(\mu(s)\big) = \frac{\big(-\nabla' f(s), 1\big)}{\sqrt{1+\big|\nabla' f(s)\big|^2}}, \quad s \in U.$$

Then we have $\nu(y_0) = \nu(\mu(s_0)) = e_n$ and

$$\frac{\partial\mu}{\partial s_i}(s_0) = e_i, \quad \frac{\partial}{\partial s_i}\Big(\nu\big(\mu(s)\big)\Big)\Big|_{s=s_0} = -\kappa_i e_i, \quad i = 1,\dots,n-1 \tag{C.5}$$

by (C.4), where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$. This equality yields

$$\frac{\partial M}{\partial s_i}(s_0,\rho) = (1-\rho\kappa_i)e_i, \quad i = 1,\dots,n-1, \quad \frac{\partial M}{\partial \rho}(s_0,\rho) = \nu\big(\mu(s_0)\big) = e_n. \tag{C.6}$$

Hence we have $\theta(s_0) = I_{n-1}$, $\Theta(s_0,\rho) = \mathrm{diag}[(1-\rho\kappa_1)^2,\dots,(1-\rho\kappa_{n-1})^2, 1]$, and

$$\theta^{-1}(s_0) = I_{n-1}, \quad \Theta^{-1}(s_0,\rho) = \mathrm{diag}[(1-\rho\kappa_1)^{-2},\dots,(1-\rho\kappa_{n-1})^{-2}, 1]. \tag{C.7}$$

Applying (C.5), (C.6), and (C.7) to (C.1) and (C.2) with $u = \overline{v}$, we obtain

$$\nabla_\Gamma v(y_0) = \sum_{i=1}^{n-1} \frac{\partial\widetilde{v}}{\partial s_i}(s_0)e_i, \quad \nabla\overline{v}\big(y_0+\rho\nu(y_0)\big) = \sum_{i-1}^{n-1}(1-\rho\kappa_i)^{-1}\frac{\partial\widetilde{v}}{\partial s_i}(s_0)e_i$$

and thus (5.3) implies that

$$\big|\nabla\overline{v}\big(y_0+\rho\nu(y_0)\big)\big|^2 = \sum_{i=1}^{n-1}(1-\rho\kappa_i)^{-2}\left(\frac{\partial\widetilde{v}}{\partial s_i}(s_0)\right)^2 \leq c\sum_{i=1}^{n-1}\left(\frac{\partial\widetilde{v}}{\partial s_i}(s_0)\right)^2 = \big|\nabla_\Gamma v(y_0)\big|^2,$$

which yields the first inequality of (C.3) with $y$ replaced by $y_0$. Moreover, by (5.3) we have

$$\left|(1-\rho\kappa_i)^{-1}-1\right| = \left|\rho\kappa_i(1-\rho\kappa_i)^{-1}\right| \leqslant c\varepsilon$$

for all $\rho \in (\varepsilon g_0(y_0), \varepsilon g_1(y_0))$ and $i = 1, \dots, n-1$ and thus

$$\left|\nabla\bar{v}\big(y_0+\rho\nu(y_0)\big) - \nabla_\Gamma v(y_0)\right|^2 = \sum_{i=1}^{n-1}\left\{(1-\rho\kappa_i)^{-1}-1\right\}^2\left(\frac{\partial\widetilde{v}}{\partial s_i}(s_0)\right)^2 \leqslant c\varepsilon^2\left|\nabla_\Gamma v(y_0)\right|^2.$$

Hence the second inequality of (C.3) with $y$ replaced by $y_0$ is valid. □

To prove Lemma 5.6, we need a differentiation formula of the average operator under a local coordinate system. Let $U$ be an open set in $\mathbb{R}^{n-1}$ and $\mu\colon U \to \Gamma$ be a local parametrization of $\Gamma$. The weighted average of a function $u$ on $\Omega_\varepsilon$ is locally expressed as

$$\widetilde{M_\varepsilon u}(s) = \frac{1}{\varepsilon\widetilde{g}(s)}\int_{\varepsilon\widetilde{g}_0(s)}^{\varepsilon\widetilde{g}_1(s)} \widetilde{u}(s,\rho)\widetilde{J}(s,\rho)\,d\rho, \quad s \in U, \tag{C.8}$$

where $\widetilde{M_\varepsilon u}(s) = M_\varepsilon u(\mu(s))$, $\widetilde{u}(s,\rho) = u(M(s,\rho))$, and

$$\widetilde{J}(s,\rho) := J\big(\mu(s),\rho\big) = \prod_{i=1}^{n-1}\left\{1-\rho\kappa_i\big(\mu(s)\big)\right\}. \tag{C.9}$$

**Lemma C.1** *Let $u \in H^1(\Omega_\varepsilon)$. Then*

$$\begin{aligned}\frac{\partial\widetilde{M_\varepsilon u}}{\partial s_i}(s) &= \frac{1}{\varepsilon\widetilde{g}(s)}\int_{\varepsilon\widetilde{g}_0(s)}^{\varepsilon\widetilde{g}_1(s)}\left\{\frac{\partial\widetilde{u}}{\partial s_i}(s,\rho)\widetilde{J}(s,\rho) + \widetilde{u}(s,\rho)\frac{\partial\widetilde{J}}{\partial s_i}(s,\rho)\right\}d\rho \\ &\quad + \frac{1}{\varepsilon\widetilde{g}(s)}\int_{\varepsilon\widetilde{g}_0(s)}^{\varepsilon\widetilde{g}_1(s)}\left\{\frac{\partial\widetilde{u}}{\partial\rho}(s,\rho)\widetilde{J}(s,\rho) + \widetilde{u}(s,\rho)\frac{\partial\widetilde{J}}{\partial\rho}(s,\rho)\right\}\chi_i(s,\rho)\,d\rho\end{aligned} \tag{C.10}$$

*for all $s \in U$ and $i = 1, \dots, n-1$, where*

$$\chi_i(s,\rho) := \frac{1}{\widetilde{g}(s)}\left\{(\rho-\varepsilon\widetilde{g}_0(s))\frac{\partial\widetilde{g}_1}{\partial s_i}(s) + (\varepsilon\widetilde{g}_1(s)-\rho)\frac{\partial\widetilde{g}_0}{\partial s_i}(s)\right\}. \tag{C.11}$$

*Proof.* For simplicity, we set $\partial_i = \partial/\partial s_i$ and $\partial_\rho = \partial/\partial\rho$. For each $i = 1, \dots, n-1$, we differentiate both sides of (C.8) with respect to $s_i$ to get

$$\partial_i\widetilde{M_\varepsilon u} = \frac{I}{\varepsilon\widetilde{g}} - \frac{\partial_i\widetilde{g}}{\varepsilon(\widetilde{g})^2}\int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\widetilde{u}\widetilde{J}\,d\rho + \frac{1}{\varepsilon\widetilde{g}}\int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\left\{(\partial_i\widetilde{u})\widetilde{J} + \widetilde{u}(\partial_i\widetilde{J})\right\}d\rho, \tag{C.12}$$

where $I = I(s)$ is given by

$$I(s) := \varepsilon\partial_i\widetilde{g}_1(s)\widetilde{u}\big(s,\varepsilon\widetilde{g}_1(s)\big)\widetilde{J}\big(s,\varepsilon\widetilde{g}_1(s)\big) - \varepsilon\partial_i\widetilde{g}_0(s)\widetilde{u}\big(s,\varepsilon\widetilde{g}_0(s)\big)\widetilde{J}\big(s,\varepsilon\widetilde{g}_0(s)\big).$$

Since $I = \left[\widetilde{u}(\rho)\widetilde{J}(\rho)\chi_i(\rho)\right]_{\rho=\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1} = \int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\partial_\rho(\widetilde{u}\widetilde{J}\chi_i)\,d\rho$ and $\partial_\rho\chi_i = \partial_i\widetilde{g}/\widetilde{g}$, we have

$$\frac{I}{\varepsilon\widetilde{g}} = \frac{\partial_i\widetilde{g}}{\varepsilon(\widetilde{g})^2}\int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\widetilde{u}\widetilde{J}\,d\rho + \frac{1}{\varepsilon\widetilde{g}}\int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\left\{(\partial_\rho\widetilde{u})\widetilde{J} + \widetilde{u}(\partial_\rho\widetilde{J})\right\}\chi_i\,d\rho. \tag{C.13}$$

Substituting (C.13) for (C.12), we obtain (C.10). □

*Proof of Lemma* 5.6. As in the proof of Lemma C.1, we write $\partial_i = \partial/\partial s_i$ and $\partial_\rho = \partial/\partial\rho$. Let $u \in C^\infty(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)$, $\varphi \in H^1(\Gamma)$, and

$$I(y) := \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (\nabla u)^\sharp(y,\rho)\cdot(\nabla\overline{\varphi})^\sharp(y,\rho)J(y,\rho)\,d\rho - \varepsilon g(y)\nabla_\Gamma M_\varepsilon u(y)\cdot\nabla_\Gamma\varphi(y).$$

Here we used the notation (5.6). Our goal is to show

$$\left|I(y)\right| \leqslant c\varepsilon\left|\nabla_\Gamma\varphi(y)\right|\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)}\left(\left|u^\sharp(y,\rho)\right| + \left|(\nabla u)^\sharp(y,\rho)\right|\right)d\rho \tag{C.14}$$

for all $y \in \Gamma$ with a constant $c > 0$ independent of $y$ and $\varepsilon$. As in the proof of Lemma 5.5, we fix $y_0 \in \Gamma$ and take a local parametrization $\mu(s) = (s, f(s))$ of $\Gamma$ near $y_0 = \mu(s_0)$, $s_0 \in U$, where $U$ is an open set in $\mathbb{R}^{n-1}$ and $f$ is a smooth function on $U$ satisfying (C.4). We set the direction of $\nu(y_0)$ in the positive direction of the $x_n$-axis. Then by (C.5), (C.6), and (C.7) we have

$$(\nabla u)^\sharp(y_0,\rho) = \sum_{i=1}^{n-1}(1-\rho\kappa_i)^{-1}\partial_i\widetilde{u}(s_0,\rho)e_i + \partial_\rho\widetilde{u}(s_0,\rho)e_n, \quad \nabla_\Gamma M_\varepsilon u(y_0) = \sum_{i=1}^{n-1}\partial_i\widetilde{M_\varepsilon u}(s_0)e_i,$$

$$(\nabla\overline{\varphi})^\sharp(y_0,\rho) = \sum_{i=1}^{n-1}(1-\rho\kappa_i)^{-1}\partial_i\widetilde{\varphi}(s_0)e_i, \quad \nabla_\Gamma\varphi(y_0) = \sum_{i=1}^{n-1}\partial_i\widetilde{\varphi}(s_0)e_i,$$

where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$ and $\kappa_i := \kappa_i(y_0)$, $i = 1,\dots,n-1$. Hereafter we omit the variables $\rho$ and $s_0$ unless we need to specify them. The above equality yields

$$\begin{aligned}(\nabla u)^\sharp(y_0,\rho)\cdot(\nabla\overline{\varphi})^\sharp(y_0,\rho) &= \sum_{i=1}^{n-1}(1-\rho\kappa_i)^{-2}\partial_i\widetilde{u}\,\partial_i\widetilde{\varphi}, \\ \varepsilon g(y_0)\nabla_\Gamma M_\varepsilon u(y_0)\cdot\nabla_\Gamma\varphi(y_0) &= \sum_{i=1}^{n-1}\varepsilon\widetilde{g}\left(\partial_i\widetilde{M_\varepsilon u}\right)\partial_i\widetilde{\varphi}.\end{aligned} \tag{C.15}$$

Moreover, (C.10) implies that

$$\varepsilon\widetilde{g}\left(\partial_i\widetilde{M_\varepsilon u}\right) = \int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\left\{(\partial_i\widetilde{u})\widetilde{J} + \widetilde{u}(\partial_i\widetilde{J}) + (\partial_\rho\widetilde{u})\widetilde{J}\chi_i + \widetilde{u}(\partial_\rho\widetilde{J})\chi_i\right\}d\rho,$$

where $\chi_i$ is given by (C.11), and thus

$$\begin{aligned}&\varepsilon g(y_0)\nabla_\Gamma M_\varepsilon u(y_0)\cdot\nabla_\Gamma\varphi(y_0) \\ &= \int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\widetilde{J}\sum_{i=1}^{n-1}\partial_i\widetilde{u}\,\partial_i\widetilde{\varphi}\,d\rho + \int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\widetilde{u}\sum_{i=1}^{n-1}\partial_i\widetilde{J}\,\partial_i\widetilde{\varphi}\,d\rho + \int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\left\{(\partial_\rho\widetilde{u})\widetilde{J} + \widetilde{u}(\partial_\rho\widetilde{J})\right\}\sum_{i=1}^{n-1}\chi_i\,\partial_i\widetilde{\varphi}\,d\rho.\end{aligned}$$

From this equality and (C.15), we obtain $I(y_0) = I_1 + I_2 + I_3$ with

$$I_1 = \int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\widetilde{J}\sum_{i=1}^{n-1}\left\{(1-\rho\kappa_i)^{-2}-1\right\}\partial_i\widetilde{u}\,\partial_i\widetilde{\varphi}\,d\rho,$$

$$I_2 = -\int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\widetilde{u}\sum_{i=1}^{n-1}\partial_i\widetilde{J}\,\partial_i\widetilde{\varphi}\,d\rho, \quad I_3 = -\int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\left\{(\partial_\rho\widetilde{u})\widetilde{J} + \widetilde{u}(\partial_\rho\widetilde{J})\right\}\sum_{i=1}^{n-1}\chi_i\,\partial_i\widetilde{\varphi}\,d\rho.$$

Let us estimate these integrals. By the definition of $\widetilde{J}$ (see (C.9)), we have

$$\nabla_\Gamma J(y_0,\rho) = \sum_{i=1}^{n-1} \partial_i \widetilde{J}(s_0,\rho) e_i, \quad \sum_{i=1}^{n-1} \partial_i \widetilde{J}(s_0,\rho)\partial_i \widetilde{\varphi}(s_0) = \nabla_\Gamma J(y_0,\rho)\cdot\nabla_\Gamma\varphi(y_0).$$

Hence $I_2$ is of the form

$$I_2 = -\int_{\varepsilon g_0(y_0)}^{\varepsilon g_1(y_0)} u^\sharp(y_0,\rho)\nabla_\Gamma J(y_0,\rho)\cdot\nabla_\Gamma\varphi(y_0)\,d\rho$$

and by applying (5.5) to the right-hand side we obtain

$$|I_2| \leqslant c\varepsilon|\nabla_\Gamma\varphi(y_0)| \int_{\varepsilon g_0(y_0)}^{\varepsilon g_1(y_0)} |u^\sharp(y_0,\rho)|\,d\rho. \tag{C.16}$$

Next we estimate $I_3$. By the definitions of $\widetilde{u}$, $\widetilde{J}$, and $\chi_i$ (see (C.9) and (C.11)),

$$\partial_\rho\widetilde{u}(s_0,\rho) = \nu(y_0)\cdot(\nabla u)^\sharp(y_0,\rho), \quad \partial_\rho\widetilde{J}(s_0,\rho) = \partial_\rho J(y_0,\rho),$$
$$\sum_{i=1}^{n-1}\chi_i(s_0,\rho)\partial_i\widetilde{\varphi}(s_0) = \chi_\varepsilon(y_0,\rho)\cdot\nabla_\Gamma\varphi(y_0),$$

where

$$\chi_\varepsilon(y_0,\rho) := \frac{(\rho-\varepsilon g_0(y_0))\nabla_\Gamma g_1(y_0) + (\varepsilon g_1(y_0)-\rho)\nabla_\Gamma g_0(y_0)}{g(y_0)}.$$

Hence $I_3$ is of the form

$$I_3 = -\int_{\varepsilon g_0(y_0)}^{\varepsilon g_1(y_0)} \chi_\varepsilon(y_0,\rho)\cdot\nabla_\Gamma\varphi(y_0)\big\{\nu(y_0)\cdot(\nabla u)^\sharp(y_0,\rho)J(y_0,\rho) + u^\sharp(y_0,\rho)\partial_\rho J(y_0,\rho)\big\}\,d\rho.$$

Since $\nabla_\Gamma g_0$, $\nabla_\Gamma g_1$ are bounded and $g_1 - g_0 = g$,

$$\big|\chi_\varepsilon(y_0,\rho)\big| \leqslant \frac{\big|\nabla_\Gamma g_0(y_0)\big| + \big|\nabla_\Gamma g_1(y_0)\big|}{g(y_0)}\big\{(\rho-\varepsilon g_0(y_0)) + (\varepsilon g_1(y_0)-\rho)\big\} \leqslant c\varepsilon$$

for all $\rho\in(\varepsilon g_0(y_0),\varepsilon g_1(y_0))$. This inequality together with (5.4) and (5.5) yields

$$|I_3| \leqslant c\varepsilon\big|\nabla_\Gamma\varphi(y_0)\big|\int_{\varepsilon g_0(y_0)}^{\varepsilon g_1(y_0)} \big(\big|u^\sharp(y_0,\rho)\big| + \big|(\nabla u)^\sharp(y_0,\rho)\big|\big)\,d\rho. \tag{C.17}$$

Let us estimate $I_1$. For all $\rho\in(\varepsilon g_0(y_0),\varepsilon g_1(y_0))$ and $i=1,\dots,n-1$, we have

$$\big|(1-\rho\kappa_i)^{-2}-1\big| = \big|\rho\kappa_i(2-\rho\kappa_i)(1-\rho\kappa_i)^{-2}\big| \leqslant c\varepsilon$$

by (5.3). From this inequality, Hölder's inequality, and (5.3),

$$\begin{aligned}\left|\sum_{i=1}^{n-1}\{(1-\rho\kappa_i)^{-2}-1\}\partial_i\widetilde{u}\,\partial_i\widetilde{\varphi}\right| &\leqslant c\varepsilon\left(\sum_{i=1}^{n-1}(\partial_i\widetilde{u})^2\right)^{1/2}\left(\sum_{i=1}^{n-1}(\partial_i\widetilde{\varphi})^2\right)^{1/2}\\ &\leqslant c\varepsilon\left(\sum_{i=1}^{n-1}(1-\rho\kappa_i)^{-2}(\partial_i\widetilde{u})^2\right)^{1/2}\left(\sum_{i=1}^{n-1}(\partial_i\widetilde{\varphi})^2\right)^{1/2}\\ &\leqslant c\varepsilon\left|(\nabla u)^\sharp(y_0,\rho)\right|\left|\nabla_\Gamma\varphi(y_0)\right|.\end{aligned}$$

Using this inequality and (5.4) we obtain

$$|I_1| \leqslant c\varepsilon\left|\nabla_\Gamma\varphi(y_0)\right|\int_{\varepsilon g_0(y_0)}^{\varepsilon g_1(y_0)}\left|(\nabla u)^\sharp(y_0,\rho)\right|d\rho. \tag{C.18}$$

By (C.16), (C.17), and (C.18) we conclude that (C.14) with $y$ replaced by $y_0$ holds. □

Finally we give the complete proof of Lemma 5.11.

*Proof of Lemma* 5.11. Let $\Phi(\cdot,t)\colon\Gamma_0\to\Gamma(t)$ be the flow map of $V_\Gamma$ and $\Phi^{-1}(\cdot,t)$ be its inverse mapping (see Section 2). For $X\in N(0)$ and $t\in(0,T)$ we set

$$\Psi(X,t) := \Phi\big(p(X,0),t\big)+d(X,0)\nu\big(\Phi(p(X,0),t),t\big). \tag{C.19}$$

For each $t\in(0,T)$ the mapping $\Psi(\cdot,t)\colon N(0)\to N(t)$ is a bijection whose inverse is given by

$$\Psi^{-1}(x,t) := \Phi^{-1}\big(p(x,t),t\big)+d(x,t)\nu\Big(\Phi^{-1}\big(p(x,t),t\big),0\Big),\quad (x,t)\in N_T.$$

Let $\varphi\in C^1(S_T)$ and $\overline{\varphi}$ be its constant extension in the normal direction of $\Gamma(t)$. By the definition of $\overline{\varphi}$ and the formula $p(\Psi(X,t),t)=\Phi(p(X,0),t)$ we have

$$\overline{\varphi}\big(\Psi(X,t),t\big)=\varphi\Big(\Phi\big(p(X,0),t\big),t\Big),\quad (X,t)\in N(0)\times(0,T).$$

We differentiate both sides with respect to $t$. The time derivative of the left-hand side is

$$\partial_t\overline{\varphi}\big(\Psi(X,t),t\big)+\partial_t\Psi(X,t)\cdot\nabla\overline{\varphi}\big(\Psi(X,t),t\big).$$

On the other hand, the time derivative of the right-hand side is

$$\partial^\bullet\varphi\Big(\Phi\big(p(X,0),t\big),t\Big)=\partial^\bullet\varphi\Big(p\big(\Psi(X,t),t\big),t\Big)$$

by the definition of the strong material derivative (see (4.4)). Hence

$$\partial^\bullet\varphi\Big(p\big(\Psi(X,t),t\big),t\Big)=\partial_t\overline{\varphi}\big(\Psi(X,t),t\big)+\partial_t\Psi(X,t)\cdot\nabla\overline{\varphi}\big(\Psi(X,t),t\big)$$

for all $(X,t)\in N(0)\times(0,T)$. Substituting $\Psi^{-1}(x,t)$ for $X$ in this equality we further get

$$\partial^\bullet\varphi\big(p(x,t),t\big)=\partial_t\overline{\varphi}(x,t)+\partial_t\Psi\big(\Psi^{-1}(x,t),t\big)\cdot\nabla\overline{\varphi}(x,t) \tag{C.20}$$

for all $(x,t) \in N_T$. Let us show

$$\partial_t \Psi\big(\Psi^{-1}(x,t),t\big) = V_\Gamma\big(p(x,t),t\big) + a(x,t), \tag{C.21}$$

where $a(x,t)$ is given by (5.17). We differentiate both sides of (C.19) with respect to $t$ to get

$$\begin{aligned}\partial_t \Psi(X,t) &= \partial_t \Phi\big(p(X,0),t\big) \\ &\quad + d(X,0)\Big\{\partial_t \nu\Big(\Phi\big(p(X,0),t\big),t\Big) + \nabla\nu\Big(\Phi\big(p(X,0),t\big),t\Big)\partial_t\Phi\big(p(X,0),t\big)\Big\}\end{aligned}$$

for $(X,t) \in N(0) \times (0,T)$. Moreover, since

$$\begin{gathered} d(X,0) = d\big(\Psi(X,t),t\big), \quad \Phi(p(X,0),t) = p\big(\Psi(X,t),t\big), \\ \partial_t \Phi(p(X,0),t) = V_\Gamma\Big(\Phi\big(p(X,0),t\big),t\Big) = V_\Gamma\Big(p\big(\Psi(X,t),t\big),t\Big), \end{gathered}$$

it follows that

$$\begin{aligned}\partial_t \Psi(X,t) &= V_\Gamma\Big(p\big(\Psi(X,t),t\big),t\Big) \\ &\quad + d\big(\Psi(X,t),t\big)\Big\{\partial_t\nu\Big(p\big(\Psi(X,t),t\big),t\Big) + \nabla\nu\Big(p\big(\Psi(X,t),t\big),t\Big)V_\Gamma\Big(p\big(\Psi(X,t),t\big),t\Big)\Big\}.\end{aligned}$$

Substituting $\Psi^{-1}(x,t)$ for $X$ in this equality we obtain (C.21). Finally, the formula (5.16) follows from (C.20) and (C.21). □

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