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Weak solutions to thin-film equations with contact-line friction

MARIA CHIRICOTTO
Institut für Angewandte Mathematik, Universität Heidelberg,
Im Neuenheimer Feld 294, 69120 Heidelberg, Germany
E-mail: chiricotto@math.uni-heidelberg.de

LORENZO GIACOMELLI
SBAI Department, Sapienza University of Rome, Via Antonio Scarpa 16, 00161 Roma, Italy
E-mail: lorenzo.giacomelli@sbai.uniroma1.it

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We consider the thin-film equation with a prototypical contact-line condition modeling the effect of frictional forces at the contact line where liquid, solid, and air meet. We show that such condition, relating flux with contact angle, naturally emerges from applying a thermodynamic argument due to Weiqing Ren and Weinan E [Commun. Math. Sci. 9 (2011), 597–606] directly into the framework of lubrication approximation. For the resulting free boundary problem, we prove global existence of weak solutions, as well as global existence and uniqueness of approximating solutions which satisfy the contact line condition pointwise. The analysis crucially relies on new contractivity estimates for the location of the free boundary.

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1. Introduction and main result

1.1 *Thin-film equations*

Thin-film equations are fourth-order degenerate parabolic equations whose simplest form, in one space dimension, reads as

$$h_t + (m(h)h_{xxx})_x = 0 \quad \text{on } \{h > 0\}, \tag{1.1}$$

where $m \in C([0,\infty))$ with $m(0) = 0$ and $m(h) > 0$ for $h > 0$. Equation (1.1) describes the height h of a thin layer or droplet of a Newtonian fluid on a flat and perfectly smooth horizontal solid substrate in the regime of *lubrication approximation* ([42]; see [28, 38] for its rigorous justification in a related model). In this case m has the typical form $m(h) = h^3 + b^{3-n}h^n$, the parameters $n > 0$ and $b \geqslant 0$ being related to the slip condition imposed at the liquid-solid interface: in particular, $n = 2$ corresponds to a Navier slip condition and $n = 3$ (or $b = 0$) corresponds to a no-slip condition. The case $m(h) = h$ may also be seen as the lubrication approximation of the two-dimensional Hele-Shaw flow in half-space [28].

Throughout the paper we consider for simplicity a symmetric droplet configuration, i.e., h even and $\{h > 0\} = (-s(t), s(t))$. Since (1.1) is of fourth order, at least two conditions are to be imposed at $x = s(t)$: the most natural ones are

$$h(t, s(t)) = 0, \tag{1.2a}$$

$$\lim_{x\to s(t)^-} \frac{m(h)}{h} h_{xxx} = \dot{s}, \tag{1.2b}$$

the former defining $x = s(t)$ itself, the latter representing the kinematic condition that the interface moves with the fluid. It is conjectured for $n \geqslant 3$, and proved for $n \geqslant 7/2$ [2, 6], that the support of solutions to (1.1)–(1.2) remains the same for all times, i.e., $s(t) = s(0)$. However, when $n < 3$ (i.e., when slippage is allowed) the support is expected to change with time, leading to a genuine free boundary problem for which a third condition is required. Most of the literature so far has focused on the so-called complete wetting regime, which amounts to prescribing $h_x(t, s(t)) = 0$; for this problem, global existence of weak solutions and their qualitative properties [1–7, 13–19, 23, 27, 29–33, 35] as well as local-in-time well-posedness [21, 22, 24, 25], have been widely investigated. When a constant non-zero contact angle is prescribed, the theory of global weak solutions is more limited [8, 39, 43], whereas that of local well-posedness is as well developed [36–38].

The aim of this work is to discuss global weak solutions to a class of contact-line conditions which model the effect of frictional forces at the contact line. The prototypical case in this class reads as follows:

$$\dot{s}(t) = \frac{d}{2}\left((h_x^2 - 2hh_{xx}) - \theta_S^2\right)|_{x=s(t)}, \tag{1.3}$$

where $d > 0$, the superposed dot denotes the time derivative, and the notation $f|_{x=s(t)}$ denotes the limit of f as $x \to s(t)^-$. We will now argue that (1.3) stands as the appropriate analogue, in lubrication approximation, of the prototypical contact-line condition proposed by Ren and E in [44] and derived by the same authors in [45] at the level of Navier-Stokes equations.

1.2 *The Ren-E contact-line condition in lubrication approximation*

Let us preliminarily review, in the simplest possible setting, the derivation provided in [45] of the analogue of (1.3) in the framework of Navier-Stokes equations. Consider a Newtonian liquid on a flat solid surface surrounded by a gas. Let Ω denote the region occupied by the liquid and let γ, γ_{SL}, and γ_{SG} denote the surface tensions at the liquid/gas (Γ), solid/liquid (Γ_{SL}), and solid/gas (Γ_{SG}) interfaces, respectively. On introducing a *static contact angle*, $\Theta_S \in [0, \pi]$, defined by

$$\gamma \cos\Theta_S := \begin{cases} \gamma_{SG} - \gamma_{SL} & \text{if } |\gamma_{SG} - \gamma_{SL}| < \gamma & \text{(partial wetting)}, \\ 1 & \text{if } \gamma_{SG} - \gamma_{SL} \geqslant \gamma & \text{(complete wetting)}, \\ -1 & \text{if } \gamma_{SG} - \gamma_{SL} \leqslant -\gamma & \text{(complete dewetting)}, \end{cases}$$

and on neglecting molecular interaction potentials, the total energy of the system may be written as the sum of kinetic and surface energies:

$$\mathcal{E}[\mathbf{u}] = \frac{\rho}{2}\int_\Omega |\mathbf{u}|^2 \mathrm{d}\Omega - \gamma\cos\Theta_S|\Gamma_{SL}| + \gamma|\Gamma|, \tag{1.4}$$

where ρ is the liquid's density and $\mathbf{u}$ is the velocity field within Ω. Using the Navier-Stokes equations,

$$\rho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla p + \eta\nabla\cdot(\nabla\mathbf{u} + \nabla^T\mathbf{u}), \quad \nabla\cdot\mathbf{u} = 0$$

(η is the liquid's viscosity) and the interfacial conditions

$$p + \gamma\kappa = \eta(\nabla\mathbf{u} + \nabla^T\mathbf{u})\mathbf{n}\cdot\mathbf{n}, \quad (\nabla\mathbf{u} + \nabla^T\mathbf{u})\mathbf{n}\cdot\mathbf{t} = 0 \quad \text{at } \Gamma, \quad \mathbf{u}\cdot\mathbf{e}_3 = 0 \quad \text{at } \Gamma_{SL}$$

(κ, $\mathbf{n}$, $\mathbf{t}$, and $\mathbf{e}_3$ are the curvature of Γ, the unit normal and tangential vector at Γ, and the unit normal to the solid, respectively), a formal calculation gives

$$\frac{\mathrm{d}\mathcal{E}[\mathbf{u}(t)]}{\mathrm{d}t} = -\eta\int_\Omega |\nabla\mathbf{u}|^2 \mathrm{d}\Omega + \int_{\Gamma_{SL}} ((\nabla\mathbf{u} + \nabla^T\mathbf{u})\mathbf{n}\cdot\mathbf{t})u_s \mathrm{d}\Gamma_{SL} + \gamma(\cos\Theta - \cos\Theta_S)u_\ell,$$

where u_s is the scalar velocity tangential to the solid at Γ_{SL}, u_ℓ is the outward normal velocity of the contact line, $\Gamma \cap \Gamma_{SL}$, and Θ is the dynamic contact angle, i.e., the angle formed by Γ and Γ_{SL} at $\Gamma \cap \Gamma_{SL}$. In an isothermal framework, the second law of thermodynamics simply requires $\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} \leqslant 0$. The constitutive relations (of *local* type) which are compatible with it are therefore

$$((\nabla\mathbf{u} + \nabla^T\mathbf{u})\mathbf{n}\cdot\mathbf{t})|_{\Gamma_{SL}} = f_s(u_s) \quad \text{with} \quad u_s f_s(u_s) \leqslant 0 \tag{1.5}$$

and

$$\gamma(\cos\Theta - \cos\Theta_S) = f_\ell(u_\ell) \quad \text{with} \quad u_\ell f_\ell(u_\ell) \leqslant 0. \tag{1.6}$$

In the prototypical case of linear constitutive relations, (1.5) reduces to the Navier slip condition, i.e.

$$((\nabla\mathbf{u} + \nabla^T\mathbf{u})\mathbf{n}\cdot\mathbf{t})|_{\Gamma_{SL}} = -\frac{1}{B}u_s$$

with B a slip lengthscale, whereas (1.6) yields

$$\gamma(\cos\Theta - \cos\Theta_S) = -\frac{1}{D}u_\ell \tag{1.7a}$$

with $1/D$ a friction coefficient at the contact line. When contributions coming from the disjoining pressure are also considered (see (11) in [46]), in complete wetting ($\Theta_S = 0$) (1.7a) is modified as follows:

$$\gamma(\cos\Theta - 1) = -\frac{1}{D}\max\{u_\ell, 0\} \quad \text{if } \Theta_S = 0. \tag{1.7b}$$

As argued in [11, 12, 46], in lubrication approximation (1.7a) turns into

$$\dot{s}(t) = \frac{d}{2}(h_x^2 - \theta_S^2)|_{x=s(t)}, \tag{1.8a}$$

where θ_S is a rescaled static contact angle and $1/d$ is a rescaled friction coefficient. Taking also disjoining pressure into account, in complete wetting (1.8a) is modified as follows:

$$\max\{\dot{s}(t), 0\} = \frac{d}{2}h_x^2|_{x=s(t)} \quad \text{if } \theta_S = 0. \tag{1.8b}$$

It should be noted that the right-hand sides in (1.8) differ from that in (1.3) by a single term, $hh_{xx}|_{x=s(t)}$. We will now show that this term naturally emerges when applying the same basic principle –consistency with the second law of thermodynamics in the isothermal case– directly at the level of lubrication theory. We refer to the final paragraphs of this section for a comparison between (1.3) and (1.8).

At leading order in lubrication approximation, kinetic energy is negligible. After a normalization (and taking our symmetry assumption into account), surface energy reads as

$$\mathcal{E}[h(t)] = \frac{1}{2}\int_{-s(t)}^{s(t)} (h_x^2 + \theta_s^2)\mathrm{d}x \tag{1.9}$$

(see, e.g., [8, 26]). We note that

$$h \text{ even} \quad \Rightarrow \quad h_x|_{x=0} = h_{xxx}|_{x=0} = 0, \tag{1.10}$$

$$h(t, s(t)) \overset{(1.2)_1}{=} 0 \quad \Rightarrow \quad h_t|_{x=s(t)} = -h_x|_{x=s(t)}\dot{s}(t), \tag{1.11}$$

$$\lim_{x\to s(t)^-} m(h)h_{xx}h_{xxx} \overset{(1.2)_2}{=} \dot{s} \lim_{x\to s(t)^-} hh_{xx}. \tag{1.12}$$

We compute:

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{E}[h(t)]}{\mathrm{d}t} &= \dot{s}\left(\theta_S^2 + h_x^2|_{x=s(t)}\right) + 2\int_0^{s(t)} h_x h_{xt}\,\mathrm{d}x \\
&= \dot{s}\left(\theta_S^2 + h_x^2|_{x=s(t)}\right) + 2[h_x h_t]_0^{s(t)} - 2\int_0^{s(t)} h_t h_{xx}\,\mathrm{d}x \\
&\overset{(1.10),(1.11)}{=} \dot{s}\left(\theta_S^2 - h_x^2|_{x=s(t)}\right) + 2[m(h)h_{xx}h_{xxx}]_0^{s(t)} - 2\int_0^{s(t)} m(h)h_{xxx}^2\,\mathrm{d}x \\
&\overset{(1.10),(1.12)}{=} \dot{s}\left(\theta_S^2 - h_x^2 + 2hh_{xx}\right)|_{x=s(t)} - 2\int_0^{s(t)} m(h)h_{xxx}^2\,\mathrm{d}x.
\end{aligned} \tag{1.13}$$

Consistency with isothermal thermodynamics thus gives

$$(\theta_S^2 - h_x^2 + 2hh_{xx})|_{x=s(t)} = \tilde{f}_\ell(\dot{s}) \quad \text{with} \quad \tilde{f}_\ell(\dot{s})\dot{s} \leqslant 0, \tag{1.14}$$

leading to (1.3) in the prototypical case of a linear constitutive relation. In turn, it follows from (1.13) and (1.3) that

$$\frac{\mathrm{d}\mathcal{E}[h(t)]}{\mathrm{d}t} = -\frac{2}{d}\dot{s}^2 - \int_{-s(t)}^{s(t)} m(h)h_{xxx}^2\mathrm{d}x. \tag{1.15}$$

The first term on the right-hand side of (1.15) represents energy dissipation due to friction *at* the contact line and does not appear in the standard case of constant – zero or non-zero – contact angle conditions. Along the analysis, we will exploit this new feature in order to control $s(t)$.

Let us now compare (1.3) and (1.8a). It is elementary to see that the additional term $(hh_{xx})|_{x=s(t)}$ vanishes whenever $h_x(t,\cdot)$ is left-continuous at $x = s(t)$ (in particular, it is zero on traveling wave solutions (1.1)–(1.2), see, e.g., [9–11]). Hence, (1.3) and (1.8a) in fact coincide on functions which are C^1 in space up to the free boundary, a regularity which is implicitly assumed in the derivation of (1.7a) and (1.8a). Therefore (1.3) and (1.8a) have the same physical meaning and the additional term in (1.3) appears only in view of less stringent regularity assumptions. However, its presence yields the dissipation relation (1.15), which is the key for the main results of this paper.

In the case $\theta_S = 0$, the difference between (1.3) and (1.8b) for receding droplets ($\dot{s} < 0$) is due to the fact that, as we mentioned, (1.8b) is obtained in [46] by taking the disjoining pressure into account, whereas only surface energy is considered here. Under the energetic assumptions of [46], we expect that (1.3) will take the form

$$\max\{\dot{s}, 0\} = \frac{d}{2}\big((h_x^2 - 2hh_{xx})\big)|_{x=s(t)} \quad \text{if } \theta_S = 0.$$

1.3 *Problem and main results*

We translate (1.1)–(1.3) onto a fixed domain $I = (-1, 1)$ by a simple change of variable:

$$y = \frac{x}{s(t)} \in I := (-1, 1), \quad v(t, y) = h\big(t, ys(t)\big). \tag{1.16}$$

Furthermore, we relax the defining condition (1.2a) to

$$v(t, 1) = \varepsilon \geqslant 0 \quad \text{for all } t \geqslant 0. \tag{1.17}$$

Taking symmetry into account, (1.1)–(1.3) turns into the following:

$$v_t - \frac{\dot{s}}{s} y v_y + \frac{1}{s^4}\big(m(v)v_{yyy}\big)_y = 0, \; v > 0, \; v \text{ even} \quad \text{in } (0, T) \times I \tag{1.18a}$$

$$v = \varepsilon, \tag{1.18b}$$

$$\dot{s} = \frac{m(v)}{v}\frac{v_{yyy}}{s^3} \quad \text{at } (0, T) \times \{y = 1\} \tag{1.18c}$$

$$\dot{s} = \frac{d}{2s^2}\left(v_y^2 - 2vv_{yy} - s^2\theta_S^2\right) \quad \text{at } (0, T) \times \{y = 1\} \tag{1.18d}$$

and the surface energy functional (1.9) is replaced by

$$E[v] = \frac{1}{2}\int_I \left(\frac{v_y^2}{s} + s\theta_S^2\right) \mathrm{d}y. \tag{1.19}$$

We let

$$\{v > 0\}_T := \{(t, y) \in \operatorname{dom}(v) : \; t < T, \; v(t, y) > 0\}, \quad \{v > 0\} := \{v > 0\}_\infty,$$

and we denote by $\langle \cdot, \cdot \rangle_I$ the duality pairing between $(H^1(I))'$ and $H^1(I)$. The definition of weak solutions to (1.18) is nowadays standard:

DEFINITION 1.1 Let $\varepsilon \geqslant 0$ and assume that

$$m \in C^0(\mathbf{R}) \cap C^1(\mathbf{R} \setminus \{0\}), \; m(v) > 0 \text{ for all } v \neq 0, \text{ and } \liminf_{v \to \pm\infty} m(v) > 0; \tag{1.20}$$

$$s_0 > 0, \; v_0 \in H^1(I; [0, \infty)) \text{ even, and } v_0(1) = 0. \tag{1.21}$$

A pair of functions $(s, v) \in H^1_{loc}([0, \infty); (0, \infty)) \times C([0, \infty) \times \overline{I}; [0, \infty))$ is a weak solution to (1.18) with data $(s_0, v_0 + \varepsilon)$ if:

(i) $v \in L^\infty_{\text{loc}}([0, \infty); H^1(I))$, $v_t \in L^2_{loc}([0, \infty); (H^1(I))')$;

(ii) $v_{yyy} \in L^2_{\text{loc}}(\{v > 0\})$ and $\sqrt{m(v)}v_{yyy} \in L^2(\{v > 0\})$;

(iii) for all $T > 0$ and all $\varphi \in L^2((0,T); H^1(I))$,

$$\int_0^T \langle v_t, \varphi\rangle_I \, dt = \int_0^T \int_I \frac{\dot{s}}{s}(y\varphi)_y v \, dx dt + \int_0^T \int_I \frac{1}{s^4} m(v) v_{yyy} \varphi_y \, dx dt; \tag{1.22}$$

(iv) $s(0) = s_0$, $v|_{t=0} = v_0 + \varepsilon$, $v|_{y=1} = \varepsilon$, and v is even;

(v) v dissipates E in the sense that

$$E[v(T)] + \frac{2}{d}\int_0^T \dot{s}^2 dt + \iint_{\{v>0\}_T} \frac{1}{s^5} m(v) v_{yyy}^2 \, dx dt \leqslant E[v_0] \quad \text{for all } T > 0. \tag{1.23}$$

Note that the kinematic condition (1.18c) is encoded into the weak formulation as a natural boundary condition. Indeed, formally, for all φ such that $\varphi|_{y=-1} = 0$ we have

$$\int_I \left(\frac{\dot{s}}{s} y v_y - \frac{1}{s^4}\left(m(v)v_{yyy}\right)_y\right)\varphi$$
$$= \frac{1}{s}\left(\varphi\left(\dot{s}v - \frac{m(v)v_{yyy}}{s^3}\right)\right)|_{y=1} - \int_I \left(\frac{\dot{s}}{s} v(y\varphi)_y - \frac{1}{s^4} m(v) v_{yyy}\varphi_y\right). \tag{1.24}$$

Note also that (1.23) is the (time-integrated) counterpart of (1.15) in the new variables (1.16), with equality replaced by inequality.

The first main result is the existence of a weak solution to (1.18) for a class of mobilities which includes the relevant case of $m(v) = |v|^3 + b^{3-n}|v|^n$:

$$m_0 \text{ satisfies (1.20) and } m_0(h) \sim Ch^n \text{ as } h \to 0 \text{ for some } n \in (0,\infty) \text{ and } C > 0. \tag{1.25}$$

It reads as follows:

Theorem 1.2 *Assume that $\varepsilon = 0$, s_0, v_0 satisfy (1.21), and $m = m_0$ satisfies (1.25). Then there exists a pair of functions (s, v) which solves (1.18) with data (s_0, v_0) in the sense of Definition 1.1.*

The main limitation of Theorem 1.2 is that the contact-line condition (1.18d) is encoded only very weakly, in the form of the energy inequality (1.23). This limitation is removed in the second main result, which we introduce now. As is customary for thin-film equations, the weak solution in Theorem 1.2 is obtained as limit of a sequence of solutions to approximating problems in which:

- the initial datum is raised by a height ε, thus ensuring initial positivity;
- the mobility m is replaced by one which degenerates sufficiently strongly at $v = 0$, thus preserving positivity:

$$m_\varepsilon(v) := \frac{v^4 m_0(v)}{\varepsilon m_0(v) + v^4}. \tag{1.26}$$

Under these assumptions, we are able to construct unique approximating solutions $(s_\varepsilon, v_\varepsilon)$ in which v_ε *is positive and* $(s_\varepsilon, v_\varepsilon)$ *satisfy the contact-line condition* (1.18d):

Theorem 1.3 *Assume that $\varepsilon > 0$, s_0, v_0 satisfy (1.21), and m_0 satisfies (1.25). Let $m = m_\varepsilon$ be defined by (1.26). Then there exists a pair (s, v) such that: (s, v) solves (1.18) with data $(s_0, v_0 + \varepsilon)$ in the sense of Definition 1.1, $v > 0$ in $[0,\infty) \times \overline{I}$, and (1.18d) holds in $L^2((0,\infty))$. Furthermore, (s, v) is unique in this class.*

At this stage, we are not able to prove that the weak solution in Theorem 1.2 is positive a.e. in $(0,T)\times I$. In this respect, it is important to notice that, even for the well-known case of a zero-contact angle condition and for $n \geqslant 2$, in the y-formulation (1.18a) the standard local version of entropy estimates [2, 6] would not yield a.e. positivity in $(-1,1)$. This is because in the standard local version of entropy estimates the support of test functions is fixed in the x-variable, that is, receding in the y-variable when s increases. This points to the necessity of a refinement of standard entropy estimates, localized in such a way that the test function "follows" the free-boundary.

In the proofs of Theorem 1.2 and Theorem 1.3, it is essential that we work with the contact-line condition (1.3) (in the form of (1.18d)) rather than with (1.8). Indeed, as we discussed in Section 1.2, (1.8) does not guarantee dissipativity of E unless the product hh_{xx} is assumed to vanish at the contact line, whereas (1.3) *guarantees dissipativity of E without assuming any a-priori regularity of the solution*. A refinement of standard entropy estimates would also permit to prove that the weak solution constructed in Theorem 1.2 belongs to $C^1(\overline{I})$ for almost every t, implying that the additional term hh_{xx} in (1.3) can eventually be ignored.

A further open question of course concerns local-in-time well-posedness for (1.1)–(1.3). The main difficulty in reproducing the method in [21, 22, 24, 25, 36–38] is that, while the contact-line condition $h_x|_{x=s(t)}$ =constant is linear and (scaling-wise) of lower order, (1.3) is nonlinear and of higher order (through (1.2b), it depends on the trace of the third derivative for a fourth-order problem).

This work stands as a first investigation of nonconstant (and nonlinear) contact-line conditions for thin-film equations. Its main merit is the construction of unique approximating solutions which satisfy such condition pointwise. To this aim, some technical novelties are introduced, the most relevant one being an H^1-contractivity estimate for the free boundary, $s(t)$ (see §1.4 below). We believe that this approach may be used to treat more general contact-line conditions which relate the speed of the contact line to (derivatives of) h. In fact, granted the aforementioned refinement of entropy estimates, we believe that it might also yield improvements in the theory of global weak solutions.

1.4 *Plan of the proofs*

As is customary in this framework, our argument is based on a multi-step approximating procedure. As we said, a solution to (1.18) with $m = m_0$ will be obtained as limit of solutions to (1.18) with $m = m_\varepsilon$. In turn, these solutions will be obtained as limit, as $\delta \to 0$, of problems in which we replace the mobility $m_\varepsilon(v)$, which is itself degenerate as $v \to 0$ and unbounded as $v \to \infty$, by an approximating family of non-degenerate and bounded mobilities $m_{\delta,\varepsilon}$:

$$m_{\delta,\varepsilon}(v) := \delta + \frac{v^4 m_0(v)}{\varepsilon m_0(v) + v^4 + \delta v^4 m_0(v)}. \tag{1.27}$$

In order to obtain global existence for (1.18) with $m = m_{\delta,\varepsilon}$, we first *prescribe* the free boundary $s(t)$ and consider the following problems:

$$(P_s)\begin{cases} v_t - \dfrac{\dot{s}}{s} y v_y + \dfrac{1}{s^4}\big(m(v)v_{yyy}\big)_y = 0 & \text{in } (0,T)\times(0,1) \\ v_y = v_{yyy} = 0 & \text{at } (0,T)\times\{y=0\} \\ v = \varepsilon,\ \dot{s}(t) = \dfrac{m(v)v_{yyy}}{s^3 v} & \text{at } (0,T)\times\{y=1\} \end{cases} \tag{1.28}$$

where indeed the contact-line condition (1.18d) is removed. Here we have encoded the symmetry assumption through the boundary conditions at $y = 0$.

The first new technical issue concerns the existence of solutions to (P_s). Indeed, once s is fixed (i.e., the contact-line condition does not hold), the dissipative structure is lost (compare (1.13)). As a consequence, our existence result for (P_s) is only local in time (see Proposition 2.1 in Section 2).

The second new technical issue, which in fact is the crucial one, is to capture the contact-line condition (1.18d). To this aim, we let $m = m_{\delta,\varepsilon}$ and we look at the mapping

$$s(t) \mapsto \tilde{s}(t) := s_0 + \int_0^t \frac{d}{2s^2}\left(v_y^2 - 2vv_{yy} - s^2\theta_S^2\right)|_{y=1}\mathrm{d}\tau, \quad \text{where } v \text{ solves } (P_s).$$

In Section 3 we will argue that $T > 0$ exists such that

$$\|\dot{\tilde{s}}_1 - \dot{\tilde{s}}_2\|_{L^2((0,T))} \leqslant \frac{1}{2}\|\dot{s}_1 - \dot{s}_2\|_{L^2((0,T))}$$

for any pair s_i and any pair v_i of solutions to (P_{s_i}) (in particular, $s_1 = s_2$ implies $\tilde{s}_1 = \tilde{s}_2$, hence the mapping is well defined). The unique fixed point of this mapping produces a local-in-time pair (s, v) that solves

$$(P)\begin{cases} v_t - \frac{\dot{s}}{s} y v_y + \frac{1}{s^4}\left(m(v)v_{yyy}\right)_y = 0 & \text{in } (0,T)\times(0,1) \\ v_y = v_{yyy} = 0 & \text{at } (0,T)\times\{y=0\} \\ v = \varepsilon,\ \dot{s}(t) = \dfrac{m(v)v_{yyy}}{s^3 v} & \text{at } (0,T)\times\{y=1\} \\ \dot{s} = \frac{d}{2s^2}\left(v_y^2 - 2vv_{yy} - s^2\theta_S^2\right) & \text{at } (0,T)\times\{y=1\} \end{cases} \tag{1.29}$$

with $m = m_{\delta,\varepsilon}$. Since the contact-line condition is recovered, then also the dissipative structure is (cf. (1.13)): therefore the local existence result for (P) can be upgraded to a global existence one (cf. Section 4).

Finally, in Section 5 we prove an entropy-type estimate for solutions to (1.18) with $m = m_{\delta,\varepsilon}$ which is uniform with respect to δ (see Lemma 5.1): this allows to pass to the limit as $\delta \to 0$ obtaining *positive* solutions to (1.18) with $m = m_\varepsilon$ and thus proving Theorem 1.3. Here the new issue is to pass to the limit in the contact-line condition, which requires strong convergence of the trace of v_y and v_{yy}: this is achieved by combining $v|_{y=1} = \varepsilon$ and Hölder estimates into a uniform lower bound on v near $y = 1$, and then using the estimate for the flux. Given Theorem 1.3, we pass to the limit as $\varepsilon \to 0$ in a nowadays usual fashion and complete the proof of Theorem 1.2.

1.5 *Notations and preliminaries*

The constants $\theta_S \geqslant 0$ and $d > 0$ are fixed throughout the paper. We let

$$\Omega = (0,1), \quad \Omega_T = (0,T)\times\Omega, \quad I = (-1,1), \quad I_T = (0,T)\times I.$$

Since we always integrate with respect to the Lebesgue measure, we omit to specify dy, dt. Finally, we list the interpolation inequalities [20, 40, 41] used in the sequel:

$$\sup_{\Omega} |v_y| \leqslant C_1 \|v_y\|^{3/4}_{L^2(\Omega)} \|v_{yyy}\|^{1/4}_{L^2(\Omega)} \quad \text{if } v_y|_{y=0} = 0, \tag{1.30}$$

$$\sup_{\Omega} |v_y| \leqslant C_1 \|v_y\|^{3/4}_{L^2(\Omega)} \|v_{yyy}\|^{1/4}_{L^2(\Omega)} + C_2 \|v_y\|_{L^2(\Omega)}, \tag{1.31}$$

$$\sup_{\Omega} |v_{yy}| \leqslant C_1 \|v_y\|^{1/4}_{L^2(\Omega)} \|v_{yyy}\|^{3/4}_{L^2(\Omega)} + C_2 \|v_y\|_{L^2(\Omega)}, \tag{1.32}$$

$$\|v_{yy}\|^2_{L^2(\Omega)} \leqslant C_1 \|v_y\|_{L^2(\Omega)} \|v_{yyy}\|_{L^2(\Omega)} + C_2 \|v_y\|^2_{L^2(\Omega)}, \tag{1.33}$$

where $C_1 > 0$ is a universal constant and $C_2 > 0$ depends on the domain.

2. Local existence of solutions to approximating problems with a prescribed free boundary

The main result of the section is the following local existence result:

Proposition 2.1 *Let $\varepsilon, \delta \in (0,1)$, $m \in C^1(\mathbf{R}; [\delta, \delta^{-1}])$, $k > 0$, and $s_m > 0$. Then $T > 0$ (depending on δ, k, and s_m) exists such that for any $v_0 \in H^1(\Omega)$ with $v_0(1) = 0$ and any $s \in H^1((0,T))$ satisfying*

$$\int_0^T \dot{s}^2 \leqslant k^2 \quad \text{and} \quad 0 < s_m \leqslant s(t) \quad \forall\, t \in [0,T] \tag{2.1}$$

there exists a solution v to (P_s) in $(0,T)$ with initial datum $v_0 + \varepsilon$ in the sense that

$$v \in L^\infty\big((0,T); H^1(\Omega)\big) \cap L^2\big((0,T); H^3(\Omega)\big), \quad v_t \in L^2\big((0,T), \big(H^1(\Omega)\big)'\big),$$

$$-\int_0^T \langle v_t, \varphi \rangle_\Omega = \iint_{\Omega_T} \frac{\dot{s}}{s} v (y\varphi)_y - \iint_{\Omega_T} \frac{1}{s^4} m(v) v_{yyy} \varphi_y \tag{2.2}$$

for all $\varphi \in L^2((0,T); H^1(\Omega))$, and

$$v|_{t=0} = v_0 + \varepsilon \text{ in } H^1(\Omega),\ v_y|_{y=0} = 0 \text{ in } L^2((0,T)),\ \text{and}\ \ v|_{y=1} = \varepsilon \text{ in } L^2((0,T)).$$

REMARK 2.2 Since $v \in L^2((0,T); H^3(\Omega))$ and $s \geqslant s_m > 0$ in $[0,T]$, $\chi_{(t_1,t_2)}(t) v_{yy}(t,y)/s(t)$ is an admissible test function in (2.2), and (with two integrations by parts)

$$\int_{t_1}^{t_2} \int_\Omega \frac{\dot{s}}{s^2} v (y v_{yy})_y = -\frac{1}{2} \int_{t_1}^{t_2} \frac{\dot{s}}{s^2} (v_y^2 - 2 v v_{yy})|_{y=1} + \frac{1}{2} \int_{t_1}^{t_2} \int_\Omega \frac{\dot{s}}{s^2} v_y^2.$$

Since $v \in L^\infty((0,T); H^1(\Omega))$ and $v_t \in L^2((0,T), (H^1(\Omega))')$, we have $v \in C([0,T]; H^1(\Omega))$; since $v_y|_{y=0} = (v - \varepsilon)|_{y=1} = 0$,

$$-\int_{t_1}^{t_2} \langle v_t, \frac{v_{yy}}{s} \rangle_\Omega = \int_\Omega \frac{v_y^2}{2s} \bigg|_{t=t_1}^{t=t_2} + \frac{1}{2} \int_{t_1}^{t_2} \int_\Omega \frac{\dot{s}}{s^2} v_y^2.$$

Therefore the following relation is satisfied as an equality:

$$\frac{1}{2} \int_\Omega \left(\frac{v_y^2}{s} + s\theta_S^2 \right) \bigg|_{t=t_1}^{t=t_2} = -\frac{1}{2} \int_{t_1}^{t_2} \frac{\dot{s}}{s^2} (v_y^2 - 2 v v_{yy} - s^2 \theta_S^2)|_{y=1} - \int_{t_1}^{t_2} \int_\Omega \frac{1}{s^5} m(v) v_{yyy}^2.$$

When both s and v_0 are assumed to be smooth, (P_s) in fact possesses global and classical solutions:

Lemma 2.3 *Under the assumption of Proposition 2.1, for any $T > 0$, any $s \in C^{1+1/8}([0,T])$ satisfying (2.1), and any $\tilde{v}_0 \in C^{4+1/2}(\Omega)$ satisfying*

$$\tilde{v}_{0y}|_{y=0} = \tilde{v}_{0yyy}|_{y=0} = 0, \tilde{v}_0|_{y=1} = \varepsilon,\ \tilde{v}_{0yyy}|_{y=1} = \frac{\varepsilon s(0)^3 \dot{s}(0)}{m(\varepsilon)}, \tag{2.3}$$

there exists a unique solution $v \in C^{1+1/8,4+1/2}_{loc}([0,T] \times \overline{\Omega})$ to (P_s) such that $v|_{t=0} = \tilde{v}_0$.

Lemma 2.3 follows from an application of classical linear theory [47] and is proved in the Appendix. In order to pass from Lemma 2.3 to Proposition 2.1, some a-priori estimates are needed:

Lemma 2.4 *Under the assumptions Proposition 2.1, $T > 0$ and $C > 0$ (depending on δ, k, and s_m) exist such that: for any $s \in C^{1+1/8}([0,T])$ satisfying (2.1) and any $\tilde{v}_0 \in C^{4+1/2}(\Omega)$ satisfying (2.3), the solution v in Lemma 2.3 satisfies*

$$\sup_{t \leqslant T} \int_\Omega \frac{v_y^2}{2s} + \iint_{\Omega_T} \frac{\delta}{s^5} v_{yyy}^2 \leqslant 2\left(1 + \int_\Omega \frac{\tilde{v}_{0y}^2}{2s_0}\right), \tag{2.4}$$

$$\|v_t\|_{L^2((0,T);(H^1(\Omega))')} \leqslant C\left(1 + \int_\Omega \frac{\tilde{v}_{0y}^2}{2s_0}\right)^{1/2}. \tag{2.5}$$

Proof. We write $a \lesssim b$ when $C \geqslant 1$, independent of δ, k, and s_m, exists such that $a \leqslant Cb$. (Though unnecessary, we keep track of constants' dependence on δ, k, and s_m in order to make estimates more transparent). Since v satisfies (P_s), with integrations by parts we obtain

$$\begin{aligned}
\int_\Omega \frac{v_y^2}{2s}\bigg|_0^t &\overset{(1.28)_2,(1.28)_3}{=} -\iint_{\Omega_t} \frac{\dot{s}}{2s^2} v_y^2 - \iint_{\Omega_t} \frac{1}{s} v_{yy} v_t \\
&\overset{(1.28)_1}{=} -\iint_{\Omega_t} \frac{\dot{s}}{2s^2} v_y^2 - \iint_{\Omega_t} \frac{\dot{s}}{2s^2} y\,(v_y^2)_y + \iint_{\Omega_t} \frac{1}{s^5}\left(m(v) v_{yyy}\right)_y v_{yy} \\
&= -\int_0^t \frac{\dot{s}}{2s^2}\left[y v_y^2\right]_{y=0}^{y=1} + \int_0^t \frac{1}{s^5}\left[m(v) v_{yyy} v_{yy}\right]_{y=0}^{y=1} - \iint_{\Omega_t} \frac{1}{s^5} m(v) v_{yyy}^2 \\
&\overset{(1.28)_2,(1.28)_3}{=} -\int_0^t \frac{\dot{s}}{2s^2}(v_y^2 - 2v v_{yy})|_{y=1} - \iint_{\Omega_t} \frac{1}{s^5} m(v) v_{yyy}^2,
\end{aligned}$$

hence (since $m \geqslant \delta$)

$$L(T) := \sup_{t \leqslant T} \int_\Omega \frac{v_y^2}{2s} + \iint_{\Omega_T} \frac{\delta}{s^5} v_{yyy}^2 \leqslant L(0) + \int_0^T \left|\frac{\dot{s}}{2s^2}(v_y^2 - 2v v_{yy})|_{y=1}\right|. \tag{2.6}$$

We use (1.30) and (1.32) to estimate the boundary term in (2.6): since $v|_{y=1} = \varepsilon \leqslant 1$, we obtain

$$
\begin{aligned}
L(T) - L(0) \lesssim \int_0^T \frac{|\dot{s}|}{2s^2} \left(\int_\Omega v_y^2 \right)^{3/4} \left(\int_\Omega v_{yyy}^2 \right)^{1/4} \\
+ \int_0^T \frac{|\dot{s}|}{2s^2} \left(\int_\Omega v_y^2 \right)^{1/8} \left(\int_\Omega v_{yyy}^2 \right)^{3/8} + \int_0^T \frac{|\dot{s}|}{2s^2} \left(\int_\Omega v_y^2 \right)^{1/2} \\
=: I_1 + I_2 + I_3. \qquad (2.7)
\end{aligned}
$$

By Hölder inequality and (2.1), we have

$$
\begin{aligned}
I_1 &\leqslant \frac{1}{\delta^{1/4}} \left(\int_0^T \dot{s}^2 \right)^{1/2} \left(\int_0^T \left(\int_\Omega \frac{v_y^2}{2s} \right)^{3/2} \left(\int_\Omega \frac{\delta}{s^5} v_{yyy}^2 \right)^{1/2} \right)^{1/2} \\
&\leqslant \frac{k}{\delta^{1/4}} \left(\sup_{t \leqslant T} \int_\Omega \frac{v_y^2}{2s} \right)^{3/4} \left(\int_0^T \left(\int_\Omega \frac{\delta}{s^5} v_{yyy}^2 \right)^{1/2} \right)^{1/2} \\
&\leqslant \frac{kT^{1/4}}{\delta^{1/4}} \left(\sup_{t \leqslant T} \int_\Omega \frac{v_y^2}{2s} \right)^{3/4} \left(\iint_{\Omega_T} \frac{\delta}{s^5} v_{yyy}^2 \right)^{1/4} \leqslant \frac{kT^{1/4}}{\delta^{1/4}} L(T). \qquad (2.8)
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
I_2 &\leqslant \frac{1}{\delta^{3/8}} \left(\int_0^T \dot{s}^2 \right)^{1/2} \left(\int_0^T \left(\int_\Omega \frac{v_y^2}{2s} \right)^{1/4} \left(\int_\Omega \frac{\delta}{s^5} v_{yyy}^2 \right)^{3/4} \right)^{1/2} \\
&\leqslant \frac{k}{\delta^{3/8}} \left(\sup_{t \leqslant T} \int_\Omega \frac{v_y^2}{2s} \right)^{1/8} \left(\int_0^T \left(\int_\Omega \frac{\delta}{s^5} v_{yyy}^2 \right)^{3/4} \right)^{1/2} \leqslant \frac{kT^{1/8}}{\delta^{3/8}} \big(L(T) \big)^{1/2} \qquad (2.9)
\end{aligned}
$$

and

$$
I_3 \leqslant \frac{1}{s_m^{3/2}} \left(\int_0^T \dot{s}^2 \right)^{1/2} T^{1/2} \left(\sup_{t \leqslant T} \int_\Omega \frac{v_y^2}{2s} \right)^{1/2} \leqslant \frac{kT^{1/2}}{s_m^{3/2}} \big(L(T) \big)^{1/2}. \qquad (2.10)
$$

Collecting (2.8), (2.9), and (2.10) in (2.7), we obtain (2.4) by choosing T sufficiently small.

The estimate (2.5) on the time derivative follows from

$$
\left| \iint_{\Omega_T} v_t \varphi \right| \leqslant \left| \iint_{\Omega_T} \frac{\dot{s}}{s} y v_y \varphi \right| + \left| \iint_{\Omega_T} \frac{1}{s^4} m(v) v_{yyy} \varphi_y \right| = J_1 + J_2,
$$

estimating

$$\begin{aligned} J_1 &\leqslant \frac{1}{s_m^{1/2}} \int_0^T |\dot{s}| \left(\left(\int_\Omega \frac{v_y^2}{2s} \right)^{1/2} \left(\int_\Omega \varphi^2 \right)^{1/2} \right) \\ &\overset{(2.4)}{\lesssim} \frac{1}{s_m^{1/2}} \left(1 + \int_\Omega \frac{\tilde{v}_{0y}^2}{2s_0} \right)^{1/2} \left(\int_0^T \dot{s}^2 \right)^{1/2} \left(\iint_{\Omega_T} \varphi^2 \right)^{1/2} \\ &\overset{(2.1)}{\leqslant} \frac{k}{s_m^{1/2}} \left(1 + \int_\Omega \frac{\tilde{v}_{0y}^2}{2s_0} \right)^{1/2} \|\varphi\|_{L^2((0,T);H^1(\Omega))}, \end{aligned}$$

and (since $m \leqslant \delta^{-1}$)

$$\begin{aligned} J_2 &\leqslant \frac{1}{\delta} \iint_{\Omega_T} \frac{1}{s^4} |v_{yyy}\varphi_y| \leqslant \frac{1}{(\delta s_m)^{3/2}} \left(\iint_{\Omega_T} \varphi_y^2 \right)^{1/2} \left(\iint_{\Omega_T} \frac{\delta}{s^5} v_{yyy}^2 \right)^{1/2} \\ &\overset{(2.4)}{\lesssim} \frac{1}{(\delta s_m)^{3/2}} \left(1 + \int_\Omega \frac{\tilde{v}_{0y}^2}{2s_0} \right)^{1/2} \|\varphi\|_{L^2((0,T);H^1(\Omega))}. \end{aligned}$$

□

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. Let $T > 0$ as in Lemma 2.4. Since $s \in H^1((0,T))$ satisfies (2.1), its constant extension to $[0,2T]$, denoted by $\tilde{s}$, satisfies (2.1) in $(0,2T)$ and $\tilde{s} \leqslant s_M < \infty$. Pick $\tilde{s}_n \in C^1([0,2T])$ such that $\tilde{s}_n(0) = s(0)$, $\tilde{s}_n \in [s_m, s_M]$, and

$$\dot{\tilde{s}}_n \to \dot{\tilde{s}} \quad \text{in } L^2((0,2T)) \quad \text{and} \quad \tilde{s}_n \to \tilde{s} \quad \text{uniformly in } [0,2T] \quad \text{as } n \to \infty,$$

Let $\tilde{v}_{0n}$ be a sequence of initial data satisfying the assumptions of Lemma 2.3 and such that $\tilde{v}_{0n} \to v_0 + \varepsilon$ in $H^1(\Omega)$. Let v_n be the solution to $(P_{\tilde{s}_n})$ with $\tilde{v}_0 = \tilde{v}_{0n}$ obtained in Lemma 2.3. Since $\tilde{s}_n \leqslant s_M$, it follows from (2.4) and (2.5) that

$$\frac{1}{s_M} \sup_{t \leqslant T} \int_\Omega v_{ny}^2 + \frac{\delta}{s_M^5} \iint_{\Omega_T} v_{nyyy}^2 + \|v_{nt}\|^2_{L^2((0,T);(H^1(\Omega))')} \leqslant C$$

with C independent of n. The remainder of the proof (i.e., the passage to the limit as $n \to \infty$) is standard and we omit it, referring to the proof of Theorem 1.2 below for details (in a more complex case). □

3. A fixed point result

In this and in the following sections we look at solutions to (P) in the following sense:

DEFINITION 3.1 Let $s_0 > 0$ and $v_0 \in H^1(\Omega)$ such that $v_0|_{y=1} = 0$. A pair $(s, v) \in H^1((0,T);(0,\infty)) \times L^\infty((0,T);H^1(\Omega))$ is a solution to (P) in $(0,T)$ with data $(s_0, v_0 + \varepsilon)$

if $v \in L^2((0,T);H^3(\Omega))$, $v_t \in L^2((0,T),(H^1(\Omega))')$,

$$\int_0^T \langle v_t,\varphi\rangle_\Omega = -\iint_{\Omega_T}\frac{\dot s}{s}v(y\varphi)_y + \iint_{\Omega_T}\frac{1}{s^4}m(v)v_{yyy}\varphi_y \quad \text{for all } \varphi \in L^2\big((0,T);H^1(\Omega)\big), \tag{3.1}$$

$$\dot s = \frac{d}{2s^2}\left(v_y^2 - 2vv_{yy} - s^2\theta_S^2\right)|_{y=1} \quad \text{in } L^2((0,T)), \tag{3.2}$$

$s(0) = s_0$, $v|_{t=0} = v_0 + \varepsilon$ in $H^1(\Omega)$, $v_y|_{y=0} = 0$ in $L^2((0,T))$, $v|_{y=1} = \varepsilon$ in $L^2((0,T))$.

In this section we prove local-in-time existence of solutions for $\varepsilon > 0$, $\delta > 0$:

Proposition 3.2 *Let $\varepsilon,\delta \in (0,1)$, m_0 satisfying (1.25), and $m = m_{\delta,\varepsilon}$ defined by (1.27). Then for any $s_0 > 0$ and any $v_0 \in H^1(\Omega)$ such that $v_0|_{y=1} = 0$ there exists $T > 0$ such that (P) has a solution (s,v) in $(0,T)$ with data $(s_0, v_0+\varepsilon)$ in the sense of Definition 3.1.*

Proof. Fix $s_m \in (0,\frac{s_0}{2}]$, let $k \geqslant 1$, let $T_* > 0$ be the time identified in Proposition 2.1, and let $T \in (0,T_*) \cap (0,1)$. The constants k and T will be chosen later, in this order. We set

$$S_T = \left\{s \in H^1((0,T)) : \|\dot s\|_{L^2((0,T))} \leqslant k,\ s(0) = s_0,\ s \geqslant s_m\right\}. \tag{3.3}$$

Given $s \in S_T$, let v be a solution to (P_s) in $(0,T_*)$ as given in Proposition 2.1. We write $f \lesssim g$, resp. $f \ll g$, if a constant $C \geqslant 1$, independent of k and of T, exists such that $f \leqslant Cg$, resp. $Cf \leqslant g$. We note for later reference that in fact

$$\left.\begin{array}{r}\text{constants depend on } v_0,\, s_m,\, s_0,\, \varepsilon,\, \delta = \inf m_{\delta,\varepsilon}(v),\\ \delta^{-1} = \sup m_{\delta,\varepsilon}(v), \text{ and the Lipschitz constant of } m = m_{\delta,\varepsilon}.\end{array}\right\} \tag{3.4}$$

The a-priori bound (2.4) translates into

$$\sup_{t\leqslant T}\int_\Omega v_y^2 + \iint_{\Omega_T} v_{yyy}^2 \lesssim 1. \tag{3.5}$$

We observe that since $T < 1$,

$$\begin{aligned}\int_0^T v_y^4|_{y=1} &\overset{(1.30)}{\lesssim} \int_0^T \left(\int_\Omega v_{yyy}^2\right)^{1/2}\left(\int_\Omega v_y^2\right)^{3/2}\\ &\leqslant \left(\sup_{t\leqslant T}\int_\Omega v_y^2\right)^{3/2}\left(\iint_{\Omega_T} v_{yyy}^2\right)^{1/2} T^{1/2} &(3.6)\\ &\overset{(3.5)}{\lesssim} T^{1/2} \lesssim 1 &(3.7)\end{aligned}$$

and

$$\begin{aligned}\int_0^T v_{yy}^2|_{y=1} &\overset{(1.32)}{\lesssim} \left(\sup_{t\leqslant T}\int_\Omega v_y^2\right)^{1/4}\left(\iint_{\Omega_T} v_{yyy}^2\right)^{3/4} T^{1/4} + \left(\sup_{t\leqslant T}\int_\Omega v_y^2\right) T\\ &\lesssim \left(\sup_{t\leqslant T}\int_\Omega v_y^2 + \iint_{\Omega_T} v_{yyy}^2\right) T^{1/4} &(3.8)\\ &\overset{(3.5)}{\lesssim} T^{1/4} \lesssim 1. &(3.9)\end{aligned}$$

Hence the function

$$\tilde{s}(t) := s_0 + \int_0^t \frac{d}{2s^2}\left(v_y^2 - 2vv_{yy} - s^2\theta_S^2\right)|_{y=1} \tag{3.10}$$

is well defined in $L^2((0,T))$, and

$$\int_0^T \dot{\tilde{s}}^2 \overset{(3.10)}{\lesssim} 1 + \int_0^T v_y^4|_{y=1} + \int_0^T \varepsilon^2 v_{yy}^2|_{y=1} \overset{(3.7),(3.9)}{\lesssim} 1. \tag{3.11}$$

Choosing k sufficiently large, inequality (3.11) implies that

$$\|\dot{\tilde{s}}\|_{L^2((0,T))} \leqslant k. \tag{3.12}$$

In addition,

$$\sup_{t\leqslant T} |\tilde{s} - s_0| \leqslant T^{1/2}\left(\int_0^T \dot{\tilde{s}}^2\right)^{1/2} \overset{(3.11)}{\lesssim} T^{1/2} \leqslant \frac{s_0}{2} \quad \text{for } T \ll 1. \tag{3.13}$$

Inequality (3.13) implies that $\tilde{s}(t) \geqslant s_0/2 \geqslant s_m$ for all $t \leqslant T$: together with (3.12), this yields

$$\tilde{s} \in S_T. \tag{3.14}$$

We claim that if T is sufficiently small, then

$$\|\dot{\tilde{s}}_1 - \dot{\tilde{s}}_2\|_{L^2((0,T))} \leqslant \tfrac{1}{2}\|\dot{s}_1 - \dot{s}_2\|_{L^2((0,T))} \tag{3.15}$$

for any $s_1, s_2 \in S_T$ and any pair v_1, v_2 of solutions to (P_{s_1}) and (P_{s_2}), respectively, where $\tilde{s}_i$ are defined by (3.10). In particular, $s_1 = s_2$ implies $\tilde{s}_1 = \tilde{s}_2$. Together with (3.14), this implies that the map $F : S_T \to S_T$, $F(s) = \tilde{s}$, is well defined. By (3.15), F is a contraction and its unique fixed point s satisfies (3.2), completing the proof. Thus, the rest of the proof is concerned with showing (3.15).

From now on k is fixed once for all and the symbols $\lesssim$, $\ll$ also include dependence on k. We argue for $t \leqslant T \ll 1$. We let $s = s_1 - s_2$, $\tilde{s} = \tilde{s}_1 - \tilde{s}_2$, and $v = v_1 - v_2$. We note that, since $s(0) = 0$,

$$s^2(t) = \left(\int_0^t |\dot{s}|\right)^2 \lesssim T\int_0^T \dot{s}^2. \tag{3.16}$$

We have

$$\begin{aligned}\int_0^T \dot{\tilde{s}}^2 &\lesssim \int_0^T \left(\frac{v_{1y}^2}{s_1^2} - \frac{v_{2y}^2}{s_2^2} - \frac{v_1 v_{1yy}}{s_1^2} + \frac{v_2 v_{2yy}}{s_2^2}\right)^2 |_{y=1} \\ &\lesssim \int_0^T \left(\frac{v_{1y}^2}{s_1^2} - \frac{v_{2y}^2}{s_2^2}\right)^2 |_{y=1} + \int_0^T \left(\frac{v_1 v_{1yy}}{s_1^2} - \frac{v_2 v_{2yy}}{s_2^2}\right)^2 |_{y=1} \\ &= I_1 + I_2. \end{aligned} \tag{3.17}$$

We estimate I_1:

$$
\begin{aligned}
I_1 &\lesssim \int_0^T \left(\frac{v_{1y}^2}{s_1^2} - \frac{v_{2y}^2}{s_1^2}\right)^2 |_{y=1} + \int_0^T v_{2y}^4|_{y=1} \left(\frac{1}{s_1^2} - \frac{1}{s_2^2}\right)^2 \\
&\overset{(3.3)}{\lesssim} \int_0^T ((v_{1y}^2 + v_{2y}^2)v_y^2)|_{y=1} + \left(\sup_{t\leqslant T} s^2\right) \int_0^T v_{2y}^4|_{y=1} \\
&\overset{(3.7),(3.16)}{\lesssim} T^{1/4}\left(\int_0^T v_y^4|_{y=1}\right)^{1/2} + T^{3/2}\int_0^T \dot{s}^2 \\
&\overset{(3.6)}{\lesssim} \left(\sup_{t\leqslant T}\int_\Omega v_y^2\right)^{3/4}\left(\iint_{\Omega_T} v_{yyy}^2\right)^{1/4} T^{1/2} + T^{3/2}\int_0^T \dot{s}^2. \qquad (3.18)
\end{aligned}
$$

We estimate I_2:

$$
\begin{aligned}
I_2 &= \int_0^T \left(\frac{v v_{1yy}}{s_1^2} + \frac{v_2 v_{yy}}{s_1^2} + v_2 v_{2yy}\left(\frac{1}{s_1^2} - \frac{1}{s_2^2}\right)\right)^2 |_{y=1} \\
&\lesssim \int_0^T \left(\frac{\varepsilon^2 v_{yy}^2}{s_1^4}\right)|_{y=1} + \int_0^T \left(\varepsilon v_{2yy}\left(\frac{1}{s_1^2} - \frac{1}{s_2^2}\right)\right)^2 |_{y=1} \quad (\text{since } v|_{y=1} = 0) \\
&\overset{(3.3)}{\lesssim} \int_0^T v_{yy}^2|_{y=1} + \left(\sup_{t\leqslant T} s^2\right)\int_0^T v_{2yy}^2|_{y=1} \\
&\overset{(3.9),(3.16)}{\lesssim} \int_0^T v_{yy}^2|_{y=1} + T^{5/4}\int_0^T \dot{s}^2 \\
&\overset{(3.8)}{\lesssim} \left(\sup_{t\leqslant T}\int_\Omega v_y^2 + \iint_{\Omega_T} v_{yyy}^2\right) T^{1/4} + T^{5/4}\int_0^T \dot{s}^2. \qquad (3.19)
\end{aligned}
$$

By (3.18) and (3.19), for $T \leqslant 1$ (3.17) turns into

$$
\int_0^T \dot{\bar{s}}^2 \lesssim K(T)T^{1/4} + T^{5/4}\int_0^T \dot{s}^2, \quad \text{where} \quad K(T) := \sup_{t\leqslant T}\int_\Omega v_y^2 + \iint_{\Omega_T} v_{yyy}^2. \qquad (3.20)
$$

In order to bound $K(T)$, we test by v_{yy} in (2.2) and we argue as in Remark 2.2: recalling that $v|_{t=0} = 0$,

$$
\begin{aligned}
&\int_\Omega \frac{v_y^2(t)}{2} + \iint_{\Omega_t} \frac{1}{s_2^4} m(v_2) v_{yyy}^2 \\
&\qquad = \iint_{\Omega_t} \left(\frac{\dot{s}_1}{s_1}v_1 - \frac{\dot{s}_2}{s_2}v_2\right)(y v_{yy})_y - \iint_{\Omega_t}\left(\frac{1}{s_1^4}m(v_1)v_{1yyy} - \frac{1}{s_2^4}m(v_2)v_{2yyy}\right)v_{yyy};
\end{aligned}
$$

integrating by parts, recalling that $v_1|_{y=1} = v_2|_{y=1} = \varepsilon$, and rearranging, we obtain

$$\begin{aligned}\int_\Omega \frac{v_y^2(t)}{2} &+ \iint_{\Omega_t} \frac{1}{s_2^4} m(v_2) v_{yyy}^2 \\ &= \varepsilon \int_0^t \frac{\dot{s}}{s_1} v_{yy}|_{y=1} + \varepsilon \int_0^t \dot{s}_2 \left(\frac{1}{s_1} - \frac{1}{s_2}\right) v_{yy}|_{y=1} \\ &\quad - \iint_{\Omega_t} \frac{\dot{s}}{s_1} y v_{1y} v_{yy} - \iint_{\Omega_t} \dot{s}_2 \left(\frac{1}{s_1} - \frac{1}{s_2}\right) y v_{1y} v_{yy} - \iint_{\Omega_t} \frac{\dot{s}_2}{s_2} y v_y v_{yy} \\ &\quad - \iint_{\Omega_t} \left(\frac{1}{s_1^4} - \frac{1}{s_2^4}\right) m(v_1) v_{1yyy} v_{yyy} - \iint_{\Omega_t} \frac{1}{s_2^4} \big(m(v_1) - m(v_2)\big) v_{1yyy} v_{yyy} \\ &=: R_1(t). \end{aligned} \tag{3.21}$$

Since $m \geqslant \delta$ and $s \geqslant s_m > 0$,

$$K(T) \lesssim \left(\sup_{t \leqslant T} \frac{1}{2} \int_\Omega v_y^2(t) + \iint_{\Omega_T} \frac{1}{s_2^4} m(v_2) v_{yyy}^2 \right) \overset{(3.21)}{=} \sup_{t \leqslant T} R_1(t). \tag{3.22}$$

Assume for a moment that $T \ll 1$ exists such that

$$\sup_{t \leqslant T} R_1(t) \lesssim T^{1/8} \left(K(T) + \int_0^T \dot{s}^2 \right). \tag{3.23}$$

Then, choosing T sufficiently small, (3.22) and (3.23) combine into

$$K(T) \lesssim \int_0^T \dot{s}^2.$$

Plugging this estimate into (3.20) and choosing T sufficiently small, we conclude that (3.15) holds. The rest of the proof is therefore concerned with showing (3.23).

We estimate each summand in $R_1(t)$. We have

$$\begin{aligned}\int_0^T \left| \frac{\dot{s}}{s_1} v_{yy}|_{y=1} \right| &\overset{(3.3)}{\lesssim} \left(\int_0^T \dot{s}^2 \right)^{1/2} \left(\int_0^T v_{yy}^2|_{y=1} \right)^{1/2} \\ &\overset{(3.8)}{\lesssim} T^{1/8} \left(\int_0^T \dot{s}^2 \right)^{1/2} K(T)^{1/2}, \end{aligned} \tag{3.24}$$

$$\begin{aligned}\int_0^T \left| \dot{s}_2 \left(\frac{1}{s_1} - \frac{1}{s_2}\right) v_{yy}|_{y=1} \right| &\overset{(3.3)}{\lesssim} \left(\sup_{t \leqslant T} |s| \right) \left(\int_0^T \dot{s}_2{}^2 \right)^{1/2} \left(\int_0^T v_{yy}^2|_{y=1} \right)^{1/2} \\ &\overset{(3.16),(3.8)}{\lesssim} T^{5/8} \left(\int_0^T \dot{s}^2 \right)^{1/2} K(T)^{1/2}. \end{aligned} \tag{3.25}$$

We note that

$$\iint_{\Omega_T} v_{yy}^2 \overset{(1.33)}{\lesssim} \int_0^T \left(\int v_y^2\right)^{1/2} \left(\int v_{yyy}^2\right)^{1/2} + \iint_{\Omega_T} v_y^2$$
$$\overset{(3.20)}{\lesssim} (T^{1/2} + T)K(T) \lesssim T^{1/2}K(T). \tag{3.26}$$

Therefore

$$\iint_{\Omega_T} \left|\frac{\dot{s}_2}{s_2} y v_y v_{yy}\right| \overset{(3.3)}{\lesssim} \left(\int_0^T \dot{s}_2^2\right)^{1/2} \left(\int_0^T \left(\int_\Omega v_y^2 \int_\Omega v_{yy}^2\right)\right)^{1/2}$$
$$\overset{(3.3),(3.26)}{\lesssim} T^{1/4}K(T). \tag{3.27}$$

By analogous arguments

$$\iint_{\Omega_T} \left|\frac{\dot{s}}{s_1} y v_{1y} v_{yy}\right| \overset{(3.3),(3.5),(3.26)}{\lesssim} T^{1/4} \left(\int_0^T \dot{s}^2\right)^{1/2} K(T)^{1/2}, \tag{3.28}$$

$$\iint_{\Omega_T} \left|\dot{s}_2 \left(\frac{1}{s_1} - \frac{1}{s_2}\right) y v_{1y} v_{yy}\right| \overset{(3.3),\,(3.5),\,(3.26)}{\lesssim} T^{1/4} \left(\sup_{t\leqslant T} |s|\right) K(T)^{1/2}$$
$$\overset{(3.16)}{\lesssim} T^{3/4} \left(\int_0^T \dot{s}^2\right)^{1/2} K(T)^{1/2}, \tag{3.29}$$

and

$$\iint_{\Omega_T} \left|\left(\frac{1}{s_1^4} - \frac{1}{s_2^4}\right) m(v_1) v_{1yyy} v_{yyy}\right| \overset{(3.3)}{\lesssim} \left(\sup_{t\leqslant T} |s|\right) \left(\iint_{\Omega_T} v_{1yyy}^2\right)^{1/2} \left(\iint_{\Omega_T} v_{yyy}^2\right)^{1/2}$$
$$\overset{(3.5),(3.16)}{\lesssim} T^{1/2} \left(\int_0^T \dot{s}^2\right)^{1/2} K(T)^{1/2}. \tag{3.30}$$

Only the last term on the right-hand side of (3.21) remains to be estimated. Since m is Lipschitz continuous,

$$\iint_{\Omega_T} \left|\frac{1}{s_1^4}\big(m(v_1) - m(v_2)\big) v_{yyy} v_{2yyy}\right| \tag{3.31}$$
$$\overset{(3.3)}{\lesssim} \sup_{t\leqslant T, y\in\Omega} |m(v_1) - m(v_2)| \left(\iint_{\Omega_T} v_{2yyy}^2\right)^{1/2} \left(\iint_{\Omega_T} v_{yyy}^2\right)^{1/2}$$
$$\overset{(3.5)}{\lesssim} \left(\sup_{t\leqslant T, y\in\Omega} |v|\right) K(T)^{1/2}. \tag{3.32}$$

Assume for a moment that

$$\sup_{t\leqslant T} \int_\Omega v^2 \lesssim T^{1/2} \left(\int_0^T \dot{s}^2 + K(T)\right). \tag{3.33}$$

Then, by interpolation and Young's inequality,

$$\sup_{t\leqslant T, y\in\Omega}|v|^2 \leqslant \sup_{t\leqslant T}\left(\left(\int_\Omega v^2\right)^{1/2}\left(\int_\Omega v_y^2\right)^{1/2}+\int_\Omega v^2\right)$$
$$\lesssim \sup_{t\leqslant T}\left(T^{1/4}\int_\Omega v_y^2+\frac{1}{T^{1/4}}\int_\Omega v^2\right) \overset{(3.33)}{\lesssim} T^{1/4}\left(\int_0^T \dot{s}^2+K(T)\right). \tag{3.34}$$

Combining (3.32) and (3.34), we obtain

$$\iint_{\Omega_T}\left|\frac{1}{s_1^4}(m(v_1)-m(v_2))v_{yyy}v_{2yyy}\right| \overset{(3.34)}{\lesssim} T^{1/8}\left(K(T)+\int_0^T \dot{s}^2\right). \tag{3.35}$$

Inserting (3.24), (3.25), (3.27)–(3.30), and (3.35) into (3.22), we obtain (3.23). Therefore, the remainder of the proof is concerned with the proof of (3.33).

We use v as test function in (2.2), obtaining (after a rearrangement)

$$\int_\Omega \frac{v^2(t)}{2} = \iint_{\Omega_t}\frac{\dot{s}}{s_1}yvv_{1y}+\iint_{\Omega_t}\dot{s}_2\left(\frac{1}{s_1}-\frac{1}{s_2}\right)yvv_{1y}+\iint_{\Omega_t}\frac{\dot{s}_2}{s_2}yvv_y$$
$$+\iint_{\Omega_t}\left(\frac{1}{s_1^4}-\frac{1}{s_2^4}\right)m(v_1)v_{1yyy}v_y+\iint_{\Omega_t}\frac{1}{s_2^4}\big(m(v_1)-m(v_2)\big)v_{1yyy}v_y$$
$$+\iint_{\Omega_t}\frac{1}{s_2^4}m(v_2)v_{yyy}v_y =: R_2(t). \tag{3.36}$$

We now estimate $R_2(t)$ in a similar fashion as for $R_1(t)$:

$$\iint_{\Omega_T}\left|\frac{\dot{s}}{s_1}yvv_{1y}\right| \overset{(3.3)}{\lesssim} \int_0^T|\dot{s}|\left(\int_\Omega v^2\int_\Omega v_{1y}^2\right)^{1/2}$$
$$\overset{(3.5)}{\lesssim} T^{1/2}\left(\sup_{t\leqslant T}\int_\Omega v^2\right)^{1/2}\left(\int_0^T\dot{s}^2\right)^{1/2} \tag{3.37}$$

and, similarly,

$$\iint_{\Omega_T}\left|\dot{s}_2\left(\frac{1}{s_1}-\frac{1}{s_2}\right)yvv_{1y}\right| \overset{(3.3),(3.5)}{\lesssim} T^{1/2}\left(\sup_{t\leqslant T}\int_\Omega v^2\right)^{1/2}\left(\sup_{t\leqslant T}|s|\right)$$
$$\overset{(3.16)}{\lesssim} T\left(\sup_{t\leqslant T}\int_\Omega v^2\right)^{1/2}\left(\int_0^T\dot{s}^2\right)^{1/2}, \tag{3.38}$$

$$\iint_{\Omega_T}\left|\frac{\dot{s}_2}{s_2}yvv_y\right| \overset{(3.3)}{\lesssim} T^{1/2}\left(\sup_{t\leqslant T}\int_\Omega v^2\right)^{1/2}K(T)^{1/2}, \tag{3.39}$$

$$\iint_{\Omega_T} \left| \left(\frac{1}{s_1^4} - \frac{1}{s_2^4} \right) m(v_1) v_{1yyy} v_y \right| \overset{(3.3)}{\lesssim} T^{1/2} \left(\sup_{t \leqslant T} |s| \right) \left(\iint_{\Omega_T} v_{1yyy}^2 \right)^{1/2} \left(\sup_{t \leqslant T} \int_\Omega v_y^2 \right)^{1/2}$$
$$\overset{(3.5),(3.16)}{\lesssim} T \left(\int_0^T \dot{s}^2 \right)^{1/2} K(T)^{1/2}, \tag{3.40}$$

and

$$\iint_{\Omega_T} \left| \frac{1}{s_2^4} m(v_2) v_{yyy} v_y \right| \overset{(3.3)}{\lesssim} T^{1/2} K(T). \tag{3.41}$$

Finally, since

$$\sup_{y \in \Omega} |v| \leqslant \underbrace{v|_{y=1}}_{=0} + \int_y^1 v_y \leqslant \left(\int_\Omega v_y^2 \right)^{1/2} \tag{3.42}$$

we estimate

$$\iint_{\Omega_T} \left| \frac{1}{s_2^4} (m(v_1) - m(v_2)) v_{1yyy} v_y \right| \overset{(3.3),(3.5)}{\lesssim} T^{1/2} \left(\sup_{t \leqslant T, y \in \Omega} |v| \right) \left(\sup_{t \leqslant T} \int_\Omega v_y^2 \right)^{1/2}$$
$$\overset{(3.42)}{\lesssim} T^{1/2} K(T). \tag{3.43}$$

Taking the sup with respect to t in (3.36), using (3.37)–(3.41) and (3.43), and absorbing on the left-hand side, we obtain (3.33). □

4. A-priori estimates and global existence for the approximating problems

We can now exploit the dissipative structure of the problem, obtaining the following a-priori bounds.

Lemma 4.1 *Let $\varepsilon, \delta \in (0,1)$, m_0 satisfying (1.25), $m = m_{\delta,\varepsilon}$ defined by (1.27), $s_0 > 0$, and $v_0 \in H_0^1(\Omega)$ non-negative and such that $v_0|_{y=1} = 0$. Then a positive constant C, depending only on $\|v_0\|_{H^1}$ and s_0, exists such that any solution (s, v) to (1.18) in $(0, T)$ in the sense of Definition 3.1 satisfies for all $y, y_1, y_2 \in \Omega$ and all $t, t_1, t_2 \in [0, T)$:*

$$\int_\Omega s(t) v(t) = \int_\Omega s_0 (v_0 + \varepsilon) \quad \textit{for all } t \in (0, T), \tag{4.1}$$

$$\sup_{t \leqslant T} \int_\Omega v_y^2(t) + \int_0^T \int_\Omega m(v) v_{yyy}^2 \leqslant C, \tag{4.2}$$

$$\frac{1}{d} \int_0^T \dot{s}^2 = \int_0^T \frac{d}{4s^4} \left(v_y^2 - 2v v_{yy} - s^2 \theta_S^2 \right)^2 |_{y=1} \leqslant C, \tag{4.3}$$

$$C^{-1} \leqslant s(t) \leqslant \begin{cases} C & \textit{if } \theta_S > 0 \\ C(1 + \sqrt{t}) & \textit{if } \theta_S = 0, \end{cases} \tag{4.4}$$

$$\|v_t\|_{L^2((0,T);(H^1(\Omega))')} \leqslant C, \tag{4.5}$$

$$|v(t, y_1) - v(t, y_2)| \leqslant C |y_1 - y_2|^{1/2}, \tag{4.6}$$

$$|v(t_1, y) - v(t_2, y)| \leqslant C\left(|t_1 - t_2|^{1/8} + |t_1 - t_2|^{1/6}\right), \tag{4.7}$$

$$|v(t, y)| \leqslant C. \tag{4.8}$$

Proof. Let (s, v) be a solution to (P) in $(0, T)$ in the sense of Definition 3.1, and let $t \in (0, T)$. Choosing $\varphi = s$ as test function in (3.1), we immediately infer (4.1). Choosing $\varphi = -\frac{v_{yy}}{s}$ as test function in (3.1) and arguing as in Remark 2.2, we obtain

$$\frac{1}{2}\int_\Omega \left(\frac{v_y^2}{s} + s\theta_S^2\right)\Bigg|_0^t \overset{(3.2)}{=} -\frac{1}{d}\int_0^t \dot{s}^2 - \iint_{\Omega_t} \frac{1}{s^5} m(v) v_{yyy}^2. \tag{4.9}$$

Since $s(t) > 0$ as long as the solution is defined, we also have

$$v(y, t) = \varepsilon + \int_1^y v_y \leqslant \varepsilon + \sqrt{s(t)}\left(\int_0^1 \frac{v_y^2}{s(t)}\right)^{1/2} \overset{(4.9)}{\leqslant} \varepsilon + C\sqrt{s}, \tag{4.10}$$

On the other hand,

$$s(t)\big(\varepsilon + C\sqrt{s(t)}\big) \overset{(4.10)}{\geqslant} s(t)\int_\Omega v \overset{(4.1)}{=} s_0 \int_\Omega (v_0 + \varepsilon),$$

which implies the lower bound in (4.4). Using such lower bound and (3.2) in (4.9) yields (4.2), (4.3), and the upper bound in (4.4). Estimate (4.6) follows from (4.2); estimate (4.8) follows from (4.6) and $v|_{y=1} = \varepsilon$; estimate (4.5) follows from (4.1)–(4.4), (4.8), and the weak formulation (3.1). The proof of (4.7) is also standard, and we reproduce it only in order to motivate the $|t_1 - t_2|^{1/6}$-contribution. Consider a non-negative cut-off function $\varphi \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}(\varphi) \subset (-2, 2)$ and $\int_\mathbb{R} \varphi(s)\, \mathrm{d}s = 1$, let $\bar{y} \in \Omega$ and let $\varphi_\eta(y) = \eta^{-1}\varphi(\eta^{-1}(y - \bar{y}))$, $\eta > 0$. We have

$$\begin{aligned}\big|v(t_2, \bar{y}) - v(t_1, \bar{y})\big| \leqslant \int_\Omega \varphi_\eta(y)\big|v(t_2, \bar{y}) - v(t_2, y)\big| + \left|\int_\Omega \varphi_\eta(y)\big(v(t_2, y) - v(t_1, y)\big)\right| \\ + \int_\Omega \varphi_\eta(y)\big|v(t_1, y) - y(t_1, \bar{y})\big| =: I_1 + I_2 + I_3.\end{aligned} \tag{4.11}$$

For the first and the third term we have

$$I_1 + I_3 \overset{(4.6)}{\leqslant} C\int_\Omega \varphi_\eta(y)|\bar{y} - y|^{1/2}\, \mathrm{d}y \leqslant C\eta^{1/2}. \tag{4.12}$$

For the second term we have

$$\begin{aligned} I_2 &= \left|\int_\Omega \varphi_\eta(y)\big(v(t_2, y) - v(t_1, y)\big)\right| = \left|\int_{t_1}^{t_2}\int_\Omega \varphi_\eta v_t\right| \\ &\overset{(3.1)}{\leqslant} \left|\int_{t_1}^{t_2}\int_\Omega \frac{\dot{s}}{s} v(y\varphi_\eta)_y\right| + \left|\int_{t_1}^{t_2}\int_\Omega \varphi_{\eta y}\frac{1}{s^4} m(v) v_{yyy}\right| =: I_2' + I_2''.\end{aligned} \tag{4.13}$$

We note that

$$\begin{aligned} I_2' &\overset{(4.4)}{\leqslant} C\left(\int_{t_1}^{t_2} \dot{s}^2\right)^{1/2}\left(\int_{t_1}^{t_2}\left(\int_\Omega (|\varphi_\eta| + |\varphi_{\eta y}|)v\right)^2\right)^{1/2} \\ &\overset{(4.3),(4.8)}{\leqslant} C|t_1 - t_2|^{1/2}\eta^{-2}|\operatorname{supp}(\varphi_\eta)| \leqslant C|t_1 - t_2|^{1/2}\eta^{-1}\end{aligned} \tag{4.14}$$

and

$$\begin{aligned} I_2'' &\overset{(4.4)}{\leqslant} C\left(\int_{t_1}^{t_2}\int_\Omega m(v)v_{yyy}^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_\Omega m(v)\varphi_{\eta y}^2\right)^{1/2} \\ &\overset{(4.2),(4.8)}{\leqslant} C|t_1-t_2|^{1/2}\eta^{-2}|\operatorname{supp}(\varphi_\eta)|^{1/2} = C|t_1-t_2|^{1/2}\eta^{-3/2}. \end{aligned} \tag{4.15}$$

Combining (4.12)–(4.15) in (4.11), we obtain

$$\left|v(t_2,\bar y)-v(t_1,\bar y)\right| \leqslant C\left(|t_1-t_2|^{1/2}\eta^{-3/2}+|t_1-t_2|^{1/2}\eta^{-1/2}+\eta^{1/2}\right). \tag{4.16}$$

Minimizing the right-hand side of (4.16) with respect to η yields (4.7). □

Global existence of solutions for $\delta,\varepsilon>0$ is an immediate consequence of Lemma 4.1.

Proposition 4.2 *Let $\varepsilon,\delta\in(0,1)$, m_0 satisfying (1.25), and $m=m_{\delta,\varepsilon}$ defined by (1.27). For any $s_0>0$ and any non-negative $v_0\in H^1(\Omega)$ such that $v_0|_{y=1}=0$, there exists a pair (s,v) which solves (P) with data $(s_0,v_0+\varepsilon)$ in $(0,T)$ for all $T<\infty$ in the sense of Definition 3.1. Furthermore, (s,v) satisfies estimates (4.1)–(4.8).*

Proof. By Proposition 3.2, there exists a pair (s,v) which solves (P) in the sense of Definition 3.1 up to a maximal time $T>0$. If by contradiction $T<\infty$, by (4.4), (4.2) and (4.8) we may find a subsequence $t_n\to T$ such that $v(t_n,y)\rightharpoonup v_T(y)$ in $H^1(\Omega)$. We may therefore apply Proposition 3.2 with initial datum $v_T(y)$ and $s_T(0)=s(T)$, obtaining a solution $(s_T(t),v_T)$ in $\Omega_{T'}$ for some $T'>0$. But then

$$\tilde s(t)=\begin{cases} s(t) & t<T\\ s_T(t-T) & t\in(T,T+T')\end{cases}\qquad \tilde v(t,y)=\begin{cases} v(t,y) & t<T\\ v_T(t-T,y) & t\in(T,T+T')\end{cases}$$

would solve (P) in $(0,T+T')$, in contradiction with the maximality of T. □

5. Entropy estimates and proof of the main results

In this section we prove the main results. We first pass to the limit as $\delta\to 0$, obtaining *positive* solutions to (P) with $m=m_\varepsilon$ and thus proving Theorem 1.3. Crucial to this aim is the following entropy-type estimate:

Lemma 5.1 *Let $0<\varepsilon,\delta\ll 1$, m_0 satisfying (1.25), $m=m_{\delta,\varepsilon}$ defined by (1.27), $s_0>0$, and $v_0\in H^1(\Omega)$ non-negative and such that $v_0(1)=0$. Let (s,v) be a global solution to Problem (P) with initial data $(s_0,v_0+\varepsilon)$, as given by Proposition 4.2. Then positive constants $C\geqslant 1$ and $C_\varepsilon\geqslant 1$ exist such that*

$$\sup_{t\leqslant T}\int_0^1 G_{\delta,\varepsilon}(v(t))+C^{-1}\iint_{\Omega_T} v_{yy}^2\leqslant C_\varepsilon(1+T)\quad \text{for all } T<\infty, \tag{5.1}$$

where

$$G_{\delta,\varepsilon}(\tau)=\int_\tau^A\int_{\tau'}^A \frac{1}{m_{\delta,\varepsilon}(\tau'')}\,\mathrm{d}\tau''\mathrm{d}\tau'\quad \text{with } A>\|v\|_\infty.$$

Note that A is independent of δ and ε in view of (4.8).

Proof. The proof follows the lines of [2, 6], with a few additional efforts in order to control the boundary terms. For notational convenience, we let $G = G_{\delta,\varepsilon}$ and $m = m_{\delta,\varepsilon}$. Since $G'' = 1/m$, using $sG'(v)$ as test function in (2.2) we obtain

$$s\int_0^1 G(v)\Big|_0^t = \int_0^t \langle v_t, sG'(v)\rangle_\Omega + \iint_{\Omega_t} \dot{s}G(v)$$
$$= -\iint_{\Omega_t} \dot{s}v(yG'(v))_y + \iint_{\Omega_t} \frac{1}{s^3} m(v)v_{yyy}\big(G'(v)\big)_y + \iint_{\Omega_t} \dot{s}G(v)$$
$$= -\int_0^t \dot{s}\left[vyG'(v)\right]_{y=0}^{y=1} + \iint_{\Omega_t} \dot{s}v_y yG'(v) + \iint_{\Omega_t} \frac{1}{s^3} v_y v_{yyy} + \iint_{\Omega_t} \dot{s}G(v),$$

hence

$$s\int_0^1 G(v)\Big|_0^t = -\varepsilon\int_0^t \dot{s}G'(\varepsilon) + \int_0^t \dot{s}\left[yG(v)\right]_{y=0}^{y=1} + \int_0^t \frac{1}{s^3}\left[v_y v_{yy}\right]_{y=0}^{y=1} - \iint_{\Omega_t} \frac{1}{s^3} v_{yy}^2$$
$$= \int_0^t \left(\dot{s}\left[G(\varepsilon) - \varepsilon G'(\varepsilon)\right] + \frac{1}{s^3}\left|(v_y v_{yy})|_{y=1}\right|\right) - \iint_{\Omega_t} \frac{1}{s^3} v_{yy}^2.$$

It follows easily from (1.25) (the assumption on m_0) and (1.27) (the definition of m) that $m(\varepsilon) \geqslant \frac{\varepsilon^n}{2}$ for all $\delta, \varepsilon \ll 1$. Then

$$G(\varepsilon) = \int_\varepsilon^A \int_\tau^A \frac{\mathrm{d}r}{m(r)} \leqslant \frac{(A-\varepsilon)^2}{m(\varepsilon)} \leqslant \frac{C}{\varepsilon^n} \quad \text{and} \quad |G'(\varepsilon)| = \left|\int_\varepsilon^A \frac{\mathrm{d}r}{m(r)}\right| \leqslant \frac{A-\varepsilon}{m(\varepsilon)} \leqslant \frac{C}{\varepsilon^n}.$$

Therefore, recalling (4.3) and (4.4), we obtain

$$\sup_{t\in(0,T)} \int_\Omega G\big(v(t)\big) + C^{-1}\iint_{\Omega_T} v_{yy}^2 \leqslant \int_\Omega G(v_0) + C_\varepsilon T^{1/2} + C\int_0^T \left|(v_y v_{yy})|_{y=1}\right|. \tag{5.2}$$

In order to estimate the boundary term in (5.2), we need a local interpolation estimate. Since $v|_{y=1} = \varepsilon$, we have

$$|v(t,y) - \varepsilon| \overset{(4.6)}{\leqslant} C|1-y|^{1/2} \leqslant \frac{\varepsilon}{2} \qquad \text{for all } y \in I_\varepsilon := \big(1 - (2C)^{-2}\varepsilon^2, 1\big).$$

Therefore

$$\int_0^t \int_{I_\varepsilon} v_{yyy}^2 \leqslant C_\varepsilon \iint_{\Omega_t} m(v)v_{yyy}^2 \overset{(4.2)}{\leqslant} C_\varepsilon, \tag{5.3}$$

and by interpolation we obtain

$$\int_0^T \sup_{y\in I_\varepsilon} \left(v_y^2 + v_{yy}^2\right) \overset{(1.31),(1.32)}{\lesssim} \int_0^T \left(C\int_{I_\varepsilon} v_{yyy}^2 + C_\varepsilon \int_{I_\varepsilon} v_y^2\right) \overset{(4.2),(5.3)}{\leqslant} C_\varepsilon(1+T).$$

Plugging this bound into (5.2) and since $\varepsilon \leqslant v_0 + \varepsilon \leqslant C$, we obtain (5.1). □

Passing to the limit as $\delta \to 0$ we prove the following result:

Proposition 5.2 *Let* $0 < \varepsilon \ll 1$, m_0 *satisfying* (1.25)*, and* $m = m_\varepsilon$ *defined by* (1.26)*. For any* $s_0 > 0$ *and any non-negative* $v_0 \in H^1(\Omega)$ *such that* $v_0|_{y=1} = 0$ *there exists a solution* (s, v) *to Problem* (P) *with data* $(s_0, v_0+\varepsilon)$ *in* $(0, T)$ *for all* $T > 0$ *in the sense of Definition* 3.1*. Furthermore* $v > 0$ *in* $\overline{\Omega}_T$ *and* v *satisfies estimates* (4.1)–(4.8)*.*

Proof. Let (s_δ, v_δ) be a global solution to (P) with $m = m_{\delta,\varepsilon}$ as given by Proposition 4.2, and let $T > 0$. In view of (4.6)–(4.8), the Ascolì-Arzelà theorem allows to select a subsequence (still indexed by δ) such that

$$v_{\delta,\varepsilon} \to v \ \text{ in } \ C^{\frac{1}{8},\frac{1}{2}}\big([0,T]\times\bar{\Omega}\big) \quad \text{as} \ \ \delta \to 0. \tag{5.4}$$

The right-hand side of (5.1) is uniformly bounded with respect to δ. Passing to the limit in (5.1) and using lower semi-continuity we see that (5.1) holds. In particular

$$\sup_{t\leqslant T}\int_0^1 G_{0,\varepsilon}\big(v(t)\big) < \infty. \tag{5.5}$$

Since $G_{0,\varepsilon}(v) \sim v^{-2}$ as $v \to 0$, (5.4) and (5.5) imply that $v > 0$ in $\overline{\Omega}_T$ for all $T > 0$. The remainder of the proof is nowadays standard and we omit it, referring to the proof of Theorem 1.2 for details. □

We are now ready to prove Theorem 1.3 and Theorem 1.2.

Proof of Theorem 1.3. Let $(s_\varepsilon, \tilde{v}_\varepsilon)$ be a global solution to (P) as given in Proposition 5.2, and let

$$v_\varepsilon(t,y) = \begin{cases} \tilde{v}_\varepsilon(t,y) & \text{if } \ y \in [0,1] \\ \tilde{v}_\varepsilon(t,-y) & \text{if } \ y \in [-1,0). \end{cases}$$

Note that we have $v_\varepsilon \in L^2_{loc}([0,\infty); H^3(I))$ since $(\tilde{v}_\varepsilon)_y|_{y=0} = 0$. Hence v_ε is a solution to (1.18) with data $(s_0, v_0 + \varepsilon)$ in the sense of Definition 1.1, $v_\varepsilon > 0$ in $[0,\infty) \times \overline{I}$, and (1.18d) holds in $L^2((0,\infty))$. This completes the proof of the existence part of Theorem 1.3.

In order to prove uniqueness, let $(s_{\varepsilon i}, v_{\varepsilon i})$, $i = 1, 2$, be two solution with the same initial data. Since $s_{\varepsilon i}$ and $v_{\varepsilon i}$ are continuous and positive in $[0,\infty)$, resp. $[0,\infty)\times\overline{I}$, we have in particular

$$C^{-1} \leqslant s_{\varepsilon i}(t) \leqslant C, \quad C^{-1} \leqslant m_\varepsilon\big(v_{\varepsilon i}(t,y)\big) \leqslant C, \quad C^{-1} \leqslant m'_\varepsilon\big(v_{\varepsilon i}(t,y)\big) \leqslant C$$

for all $(t, y) \in [0,1]\times\overline{I}$ and $i = 1, 2$, for some $C > 1$. Comparing these bounds with (3.4), we see that that the proof of (3.15) in Lemma 3.2 can be repeated line by line: hence a sufficiently small $T > 0$ exists such that $\|\dot{s}_{\varepsilon 1} - \dot{s}_{\varepsilon 2}\|_{L^2((0,T))} \leqslant \frac{1}{2}\|\dot{s}_{\varepsilon 1} - \dot{s}_{\varepsilon 2}\|_{L^2((0,T))}$, i.e. $s_{\varepsilon 1} = s_{\varepsilon 2}$ in $(0, T)$. Then, testing by $(v_{\varepsilon 1} - v_{\varepsilon 2})_{yy}$ and recalling Remark 2.2, the arguments in [34, Lemma 7.1] can be repeated line by line, yielding $v_{\varepsilon 1} = v_{\varepsilon 2}$ in $(0, T)$. A continuation argument completes the proof. □

Proof of Theorem 1.2. Theorem 1.2 follows form passing to the limit as $\varepsilon \to 0$. Its proof is nowadays standard [2, 6] and we reproduce it for completeness. In the course of the proof C will denote a generic positive constant independent of ε. Fix an arbitrary $T > 0$. In view of (4.6)–(4.8), the Ascolì-Arzelà theorem allows to select a subsequence (still indexed by ε) such that

$$v_\varepsilon \to v \ \text{ in } \ C^{\frac{1}{8},\frac{1}{2}}\big([0,T]\times\bar{I}\big) \quad \text{as} \ \ \varepsilon \to 0. \tag{5.6}$$

Bounds (4.2), (4.4), (4.3), and (4.5) imply, respectively, that (for a subsequence)

$$v_\varepsilon \overset{*}{\rightharpoonup} v \ \text{ in } \ L^\infty\big((0,T); H^1(I)\big) \quad \text{as } \ \varepsilon \to 0, \tag{5.7}$$

$$s_\varepsilon \rightharpoonup s \ \text{ in } \ H^1((0,T)) \quad \text{as } \ \varepsilon \to 0, \tag{5.8}$$

$$s_\varepsilon \to s > 0 \ \text{ uniformly in } \ (0,T) \quad \text{as } \ \varepsilon \to 0, \tag{5.9}$$

$$v_{\varepsilon t} \rightharpoonup v \ \text{ in } \ L^2\big((0,T); \big(H^1(I)\big)'\big) \quad \text{as } \ \varepsilon \to 0. \tag{5.10}$$

In addition, it follows from (4.2) and (5.6) that

$$v_{\varepsilon yyy} \rightharpoonup v_{yyy} \quad \text{in} \quad L^2_{loc}\big(\{v > 0\}_T\big) \quad \text{as } \ \varepsilon \to 0 \tag{5.11}$$

$$\sqrt{m_\varepsilon(v_\varepsilon)}v_{\varepsilon yyy} \rightharpoonup \sqrt{m(v)}v_{yyy} \quad \text{in} \quad L^2\big(\{v > 0\}_T\big) \quad \text{as } \ \varepsilon \to 0. \tag{5.12}$$

We now pass to the limit as $\varepsilon \to 0$ in

$$\int_0^T \langle v_{\varepsilon t}, \varphi\rangle_I + \iint_{\Omega_T} \frac{\dot{s}}{s} v_\varepsilon (y\varphi)_y + \iint_{\Omega_T} \frac{1}{s^4} m_\varepsilon(v_\varepsilon) v_{\varepsilon yyy}\varphi_y = 0 \tag{5.13}$$

for all $\varphi \in L^2((0,T), H^1(I))$. The first two terms in (5.13) are straightforward in view of (5.6)–(5.10). For a fixed $\eta > 0$, we split I_ε as follows:

$$I_\varepsilon = \iint_{\{v \geqslant \eta\}} m_\varepsilon(v_\varepsilon) v_{\varepsilon yyy}\varphi_y + \iint_{\{v < \eta\}} m_\varepsilon(v_\varepsilon) v_{\varepsilon yyy}\varphi_y = I'_\varepsilon + I''_\varepsilon. \tag{5.14}$$

From (5.11) an (5.6) we obtain

$$I'_\varepsilon = \iint_{\{v \geqslant \eta\}} m_\varepsilon(v_\varepsilon) v_{\varepsilon yyy}\varphi_y \overset{\varepsilon \to 0}{\to} \iint_{\{v \geqslant \eta\}} m(v) v_{yyy}\varphi_y. \tag{5.15}$$

By Hölder inequality, and since $v_\varepsilon < 2\eta$ in $\{v < \eta\}$ for $\varepsilon < \varepsilon(\eta)$ sufficiently small, we have

$$\begin{aligned} |I''_\varepsilon| = \left| \iint_{\{v<\eta\}} m_\varepsilon(v_\varepsilon) v_{\varepsilon yyy}\varphi_y \right| &\leqslant C \left(\iint_{\Omega_T} m_\varepsilon(v_\varepsilon) v^2_{\varepsilon yyy} \right)^{1/2} \left(\iint_{\{v<\eta\}} m_\varepsilon(v_\varepsilon)\varphi_y^2 \right)^{1/2} \\ &\overset{(4.2)}{\leqslant} C \left(\sup_{v \in (0,2\eta)} |m_\varepsilon(v)| \right)^{1/2} \left(\iint_{\Omega_T} \varphi_y^2 \right)^{1/2}. \end{aligned}$$

Therefore $\limsup_{\varepsilon \to 0} |I''_\varepsilon| \leqslant o_\eta(1)$ as $\eta \to 0$. Hence, passing to the limit in (5.14) as $\varepsilon \to 0$ and $\eta \to 0$, in this order, we conclude that

$$\iint_{I_T} m_\varepsilon(v_\varepsilon) v_{\varepsilon yyy}\varphi_y \to \iint_{\{v>0\}} m(v) v_{yyy}\varphi_y \overset{(5.12)}{=} \iint_{I_T} m(v) v_{yyy}\varphi_y \quad \text{as } \ \varepsilon \to 0.$$

Finally, the energy inequality (1.23) is an immediate consequence of (4.9) and lower semi-continuity. □

6. Conclusions

We have investigated a nonstandard free boundary condition for thin-film equations, relating flux and dynamic contact-angle at the contact line in a nonlinear way. For the resulting free boundary problem, we have introduced a notion of weak solution and we have proved existence of weak solutions. Notably, we have also obtained existence and uniqueness of approximating solutions which satisfy the free boundary condition point-wise. The main technical novelty is an H^1-contractivity estimate for the position of the free boundary, obtained at the level of approximating solutions. Our analysis leaves a few questions open, the most relevant being a refinement of known entropy estimates, localized so that they "follow" the evolution of the free boundary. Such refinement would yield stronger non-negativity result and would be useful also in the well-studied case of zero-contact angle free boundary condition.

Appendix

Proof of Lemma 2.3. We fix $T > 0$ and proceed in various steps.

Step 1: A linear problem. We take any function

$$a(t,x) \in C^{1/8,1/2}\big([0,T]\times\overline{\Omega};[\delta,\delta^{-1}]\big) \quad \text{such that} \quad a(t,1) = m(\varepsilon) \tag{A.1}$$

and we pass to a linear problem by replacing $m(v)$ with a:

$$\begin{cases} v_t - \dfrac{\dot{s}}{s} y v_y + \dfrac{1}{s^4}(a v_{yyy})_y = 0 & \text{in } (0,T)\times\Omega \\ v_y = v_{yyy} = 0 & \text{at } (0,T)\times\{y=0\} \\ v = \varepsilon,\ v_{yyy} = \frac{s^3\varepsilon\dot{s}}{m(\varepsilon)} & \text{at } (0,T)\times\{y=1\}. \end{cases} \tag{A.2}$$

We denote with C a generic positive constant independent of $\|a\|_{C^{1/2,1/8}([0,T]\times\Omega)}$. Let

$$\tilde{w}_0(y) = s(0)\int_0^y \tilde{v}_0(\xi)\mathrm{d}\xi, \quad w(t,y) = s(t)\int_0^y v(t,\xi)\mathrm{d}\xi. \tag{A.3}$$

One easily checks that if v is a classical solution to (A.2), then w solves

$$w_t - \frac{\dot{s}}{s} y w_y + \frac{1}{s^4} a w_{yyyy} = 0 \quad \text{in } (0,T)\times(0,1) \tag{A.4}$$

with initial datum $\tilde{w}_0$ and

$$w|_{y=0} = w_{yy}|_{y=0} = 0,\ w_y|_{y=1} = \varepsilon s. \tag{A.5}$$

In addition,

$$w|_{y=1} = \tilde{w}_0|_{y=1} = s_0\int_\Omega \tilde{v}_0. \tag{A.6}$$

Indeed,

$$\begin{aligned} w_t|_{y=1} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega s v = \int_\Omega \dot{s} v + \int_\Omega s v_t = \int_\Omega \dot{s} v + \int_\Omega \dot{s} y v_y - \int_\Omega \frac{1}{s^3}(a v_{yyy})_y \\ &= \dot{s} v|_{y=1} - \frac{1}{s^3}(a v_{yyy})|_{y=1} \overset{(A.1),(A.2)}{=} 0. \end{aligned} \tag{A.7}$$

Theorem 4.9 in [47] guarantees the existence of a unique solution w to (A.4)–(A.6) with initial datum $\tilde{w}_0$, such that

$$\|w\|_{C^{1+1/8,4+1/2}([0,T]\times\overline{\Omega})} \leqslant L\Big(\underbrace{\|\tilde{w}_0\|_{C^{4+1/2}(\Omega)}}_{\text{initial datum}} + \underbrace{\|s\|_{C^1([0,T])}}_{\text{boundary datum}}\Big) \tag{A.8}$$

for all $T > 0$, where L depends on the parabolicity coefficient (i.e. on δ and s_m) and on the $C^{1/8,1/2}$-norm of the equation's coefficients (i.e. on $\|s\|_{C^{1+1/8}([0,T])}$ and on $\|a\|_{C^{1/8,1/2}([0,T]\times\Omega)}$): in particular,

$$L \text{ depends on } a \text{ only through } \delta \text{ and } \|a\|_{C^{1/8,1/2}([0,T]\times\Omega)}. \tag{A.9}$$

Undoing the transformation (A.3), i.e. defining $v(t,y) := s(t)w_y(t,y)$, and recalling that $s \geqslant s_m > 0$, we obtain a (distributional) solution v to (A.2) with initial datum $\tilde{v}_0$ such that

$$\|v\|_{C^{1+1/8,3+1/2}([0,T]\times\overline{\Omega})} \leqslant L\,C. \tag{A.10}$$

We note that if a_1 and a_2 satisfy (A.1) and w_i are the corresponding solutions to (A.4)–(A.6), then

$$(w_1-w_2)_t - \frac{\dot{s}}{s}y(w_1-w_2)_y + \frac{1}{s^4}a_1(w_1-w_2)_{yyyy} = \frac{1}{s^4}(a_1-a_2)w_{2yyyy}$$

with zero initial and boundary condition: hence, again by Theorem 4.9 in [47], we have

$$\|w_1-w_2\|_{C^{1+1/8,4+1/2}([0,T]\times\overline{\Omega})} \leqslant L\left\|\frac{1}{s^4}(a_1-a_2)w_{2yyyy}\right\|_{C^{1/8,1/2}((0,T)\times\Omega)} \tag{A.11}$$

with L as in (A.9).

Step 2: Energy bounds. We now derive uniform bounds on v. It follows from the equation in (A.2) that $v_t \in L^2((0,T);H^1(\Omega)')$. Hence, arguing similarly to Remark 2.2 we see that

$$\int_\Omega \frac{v_y^2}{2s}\bigg|_0^t = -\int_0^t \frac{\dot{s}}{2s^2}(v_y^2 - v v_{yy})|_{y=1} - \iint_{\Omega_t}\frac{1}{s^5}a v_{yyy}^2.$$

Using $v|_{y=1}=\varepsilon$, $s\geqslant s_m$, $s\in C^1([0,T])$, $a\geqslant\delta$, and Young's inequality, we obtain

$$\int_\Omega \frac{v_y^2}{2}\bigg|_0^t + C^{-1}\int_\Omega v_{yyy}^2 \leqslant C\left(v_y^2|_{y=1} + |v_{yy}|_{y=1}|\right) \overset{(1.30),(1.32)}{\leqslant} \frac{C^{-1}}{2}\int_\Omega v_{yyy}^2 + C\int_\Omega v_y^2.$$

Therefore a Gronwall argument yields

$$\sup_{t\in(0,T)}\int_\Omega v_y^2 + \iint_{\Omega_T} v_{yyy}^2 \leqslant C. \tag{A.12}$$

Arguing as in the proof of Lemma 4.1, (A.7) and (A.12) yield uniform Hölder estimates:

$$|v(t,y_1)-v(t,y_2)| \leqslant C|y_1-y_2|^{1/2}, \tag{A.13}$$

$$|v(t_1,y)-v(t_2,y)| \leqslant C\left(|t_1-t_2|^{1/8} + |t_1-t_2|^{1/6}\right), \tag{A.14}$$

$$|v(t,y)| \leqslant C \tag{A.15}$$

for all $y, y_1, y_2 \in \overline{\Omega}$ and all $t, t_1, t_2 \in [0, T]$. It follows from (A.13)–(A.15) that any solution v constructed in Step 1 belongs to

$$\mathbb{B} = \{v : \|v\|_{C^{1/8,1/2}([0,T]\times\overline{\Omega})} \leqslant C, v|_{y=1} = \varepsilon\}.$$

Step 3: Fixed point. Since $m \in C^1(\mathbf{R}; [\delta, \delta^{-1}])$, $a = m(v)$ satisfies (A.1) for all $v \in \mathbb{B}$. Hence we may define the operator

$$\mathbb{B} \ni v \mapsto \mathfrak{A} v := s w_y, \quad \text{where } w \text{ solves (A.4)–(A.6) with } a = m(v).$$

Because of (A.13)–(A.15) and the definition of $\mathbb{B}$, $\mathfrak{A}$ maps $\mathbb{B}$ into itself. Again since $m \in C^1(\mathbf{R})$ with $\delta \leqslant m \leqslant \delta^{-1}$, we have that

$$\|m(v)\|_{C^{1/8,1/2}([0,T]\times\overline{\Omega})} \leqslant C \quad \text{for all } v \in \mathbb{B}. \tag{A.16}$$

Combining (A.16) and (A.8)–(A.9) (with $a = m(v)$), (A.10) and (A.11) turn into

$$\|\mathfrak{A} v\|_{C^{1+1/8,3+1/2}([0,T]\times\overline{\Omega})} \leqslant C, \tag{A.17}$$

respectively

$$\|\mathfrak{A}(v_1 - v_2)\|_{C^{1+1/8,3+1/2}([0,T]\times\overline{\Omega})} \leqslant C \left\| \frac{1}{s^5}\big(m(v_1) - m(v_2)\big)(\mathfrak{A} v_2)_{yyy} \right\|_{C^{1/8,1/2}((0,T)\times\Omega)}. \tag{A.18}$$

It follows from (A.17) that $\mathfrak{A} v \in \mathbb{K} := \left\{v \in \mathbb{B} : \|v\|_{C^{1+1/8,3+1/2}([0,T]\times\overline{\Omega})} \leqslant C\right\}$ for all $v \in \mathbb{B}$. Since $\mathbb{K}$ is a compact and convex subset of $\mathbb{B}$, Schauder's fixed point Theorem guarantees the existence of a fixed point v of $\mathfrak{A}$ in $\mathbb{K}$ provided $\mathfrak{A}$ is continuous, which is what we show now:

$$\begin{aligned}
\|\mathfrak{A}(v_1 - v_2)\|_{C^{1+1/8,3+1/2}([0,T]\times\overline{\Omega})} &\overset{(A.18)}{\leqslant} C \left\| \frac{1}{s^5}\big(m(v_2) - m(v_1)\big)(\mathfrak{A} v_2)_{yyy} \right\|_{C^{1/8,1/2}\left((0,T)\times\overline{\Omega}\right)} \\
&\overset{(A.17)}{\leqslant} C \,\|m(v_2) - m(v_1)\|_{C^{1/8,1/2}((0,T)\times\overline{\Omega})} \\
&\leqslant C \,\|v_2 - v_1\|_{C^{1/8,1/2}((0,T)\times\overline{\Omega})}.
\end{aligned} \tag{A.19}$$

Step 4: Conclusion. The fixed point v of $\mathfrak{A}$ is a distributional solution to (A.2) with $a = m(v)$, initial datum $\tilde{v}_0$, and such that (A.10) holds. Such regularity allows to apply [47, Theorem 4.9] directly to (A.2) with $a = m(v)$: this, together with an argument completely analogous to (A.19), yields the desired uniqueness and regularity of v. □

References

1. Ansini, L. & Giacomelli, L., Doubly nonlinear thin-film equations in one space dimension. *Arch. Rational Mech. Anal.* **173** (2004), 89–131. Zbl1064.76012 MR2073506
2. Beretta, E., Bertsch, M. & Dal Passo, R., Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation, *Arch. Rational Mech. Anal.* **129** (1995), 175–200. Zbl0827.35065 MR1328475
3. Bernis, F., Finite speed of propagation and continuity of the interface for thin viscous flows. *Adv. Differential Equations* **1** (1996), 337–368. Zbl0846.35058 MR1401398

4. Bernis, F., Finite speed of propagation for thin viscous flows when $2 \leqslant n < 3$. *C. R. Acad. Sci. Paris Sér. I Math.* **322** (1996), 1169–1174. Zbl0853.76018 MR1396660
5. Bernis, F. & Friedman, A., Higher order nonlinear degenerate parabolic equations. *J. Differential Equations* **83** (1990), 179–206. Zbl0702.35143 MR1031383
6. Bertozzi, A. L. & Pugh, M., The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions. *Comm. Pure Appl. Math.* **49** (1996), 85–123. Zbl0863.76017 MR1371925
7. Bertsch, M., Dal Passo, R., Garcke, H. & Grün, G., The thin viscous flow equation in higher space dimensions. *Adv. Differential Equations* **3** (1998), 417–440. Zbl0954.35035 MR1751951
8. Bertsch, M., Giacomelli, L. & Karali, G., Thin-film equations with "partial wetting" energy: existence of weak solutions. *Phys. D* **209** (2005), 17–27. Zbl1079.76011 MR2167440
9. Boatto, S., Kadanoff, L. P. & Olla, P., Traveling-wave solutions to thin-film equations. *Phys. Rev. E* **48** (1993), 4423–4431. MR1376986
10. Buckingham, R., Shearer, M. & Bertozzi, A., Thin film traveling waves and the Navier slip condition. *SIAM J. Appl. Math.* **63** (2002), 722–744. Zbl1024.35038 MR1951957
11. Chiricotto, M. & Giacomelli, L., Droplets spreading with contact-line friction: lubrication approximation and traveling wave solutions. *Commun. Appl. Ind. Math.* **2** (2011), e-388, 16 pp. Zbl1329.35358 MR2873623
12. Chiricotto, M. & Giacomelli, L., Scaling laws for droplets spreading under contact-line friction. *Commun. Math. Sci.* **11** (2013), 361–383. Zbl1310.35037 MR3002555
13. Chugunova, M., King, J. R. & Taranets, R. M., Uniqueness of the regular waiting-time type solution of the thin film equation. *Euro. J. Appl. Math.* **23** (2012), 537–554. Zbl1242.76041 MR2944624
14. Chugunova, M., Pugh, M. C. & Taranets, R. M., Nonnegative solutions for a long-wave unstable thin film equation with convection. *SIAM J. Math. Anal.* **42** (2010), 1826–1853. Zbl1222.35098 MR2679597
15. Dal Passo, R., Garcke, H. & Grün, G., On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions. *SIAM J. Math. Anal.* **29** (1998), 321–342. Zbl0929.35061 MR1616558
16. Dal Passo, R., Giacomelli, L. & Grün, G., A waiting time phenomenon for thin film equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **30** (2001),437–463. Zbl1024.35051 MR1895718
17. Dal Passo, R., Giacomelli, L. & Shishkov, A. E., The thin film equation with nonlinear diffusion. *Comm. Partial Differential Equations* **26** (2001), 1509–1557. Zbl1001.35070 MR1865938
18. Fischer, J., Optimal lower bounds on asymptotic support propagation rates for the thin-film equation. *J. Differential Equations* **255** (2013), 3127–3149. Zbl1328.35331 MR3093359
19. Fischer, J., Upper Bounds on Waiting Times for the Thin-Film Equation: The Case of Weak Slippage. *Arch. Rational Mech. Anal.* **211** (2014), 771–818. Zbl1293.35241 MR3158807
20. Gagliardo, E., Ulteriori proprietà di alcune classi di funzioni in più variabili. *Ricerche Mat.* **8** (1959), 24–51. Zbl0199.44701 MR0109295
21. Giacomelli, L., Gnann, M. V., Knüpfer, H. & Otto, F., Well-posedness for the Navier-slip thin-film equation in the case of complete wetting. *J. Differential Equations* **257** (2014), 15–81. Zbl1302.35218 MR3197240
22. Giacomelli, L., Gnann, M. & Otto, F., Regularity of source-type solutions to the thin-film equation with zero contact angle and mobility exponent between 3/2 and 3. *Euro. J. Appl. Math.* **24** (2013), 735–760. Zbl1292.35067 MR3104288
23. Giacomelli, L. & Grün, G., Lower bounds on waiting times for degenerate parabolic equations and systems. *Interfaces Free Bound.* **8** (2006), 111–129. Zbl1100.35058 MR2231254
24. Giacomelli, L. & Knüpfer, H., A free boundary problem of fourth order: classical solutions in weighted Hölder spaces. *Comm. Partial Differential Equations* **35** (2010), 2059–2091. Zbl1223.35208 MR2754079

25. Giacomelli, L., Knüpfer, H. & Otto, F., Smooth zero-contact-angle solutions to a thin-film equation around the steady state. *J. Differential Equations* **245** (2008), 1454–1506. Zbl1159.35039 MR3494407
26. Giacomelli, L. & Otto, F. , Variational formulation for the lubrication approximation of the Hele-Shaw flow. *Calc. Var. Partial Differential Equations* **13** (2001), 377–403. Zbl1086.35004 MR1865003
27. Giacomelli, L. & Otto, F., Droplet spreading: intermediate scaling law by PDE methods. *Comm. Pure Appl. Math.* **55** (2002), 217–254. Zbl1021.76014 MR1865415
28. Giacomelli, L. & Otto, F., Rigorous lubrication approximation. *Interfaces Free Bound.* **5** (2003), 483–529. Zbl1039.76012 MR2031467
29. Giacomelli, L. & Shishkov, A. E., Propagation of support in one-dimensional convected thin-film flow. *Indiana Univ. Math. J.* **54** (2005), 1181–1215. Zbl1088.35050 MR2164423
30. Grün, G., Degenerate parabolic differential equations of fourth order and a plasticity model with non-local hardening. *Z. Anal. Anwendungen* **14** (1995), 541–574. Zbl0835.35061 MR1362530
31. Grün, G., Droplet spreading under weak slippage: the optimal asymptotic propagation rate in the multi-dimensional case. *Interfaces Free Bound.* **4** (2002), 309–323. Zbl1056.35072 MR1914626
32. Grün, G., Droplet spreading under weak slippage—existence for the Cauchy problem. *Comm. Partial Differential Equations* **29** (2004), 1697–1744. Zbl 02171338 MR2105985
33. Grün, G., Droplet spreading under weak slippage: the waiting time phenomenon. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **21** (2004), 255–269. Zbl1062.35012 MR2047357
34. Grün, G. & Rumpf, M., Simulation of singularities and instabilities arising in thin film flow. *Euro. J. Appl. Math.* **12** (2001), 293–320. Zbl0991.76041 MR1936040
35. Hulshof, J. & Shishkov, A. E., The thin film equation with $2 \leqslant n < 3$: finite speed of propagation in terms of the L^1-norm. *Adv. Differential Equations* **3** (1998), 625–642. Zbl0953.35072 MR1665858
36. Knüpfer, H., Well-posedness for the Navier Slip Thin-Film equation in the case of partial wetting. *Comm. Pure Appl. Math.* **64** (2011), 1263–1296. Zbl1227.35146 MR2839301
37. Knüpfer, H., Well-Posedness for a Class of Thin-Film Equations with General Mobility in the Regime of Partial Wetting. *Arch. Rational Mech. Anal.* **218** (2015), 1083–1130. Zbl 06481064 MR3375546
38. Knüpfer, H. & Masmoudi, N., Darcy's Flow with Prescribed Contact Angle: Well-Posedness and Lubrication Approximation. *Arch. Rational Mech. Anal.* **218** (2015), 589–646. Zbl 06481054 MR3375536
39. Mellet, A., The thin film equation with non-zero contact angle: a singular perturbation approach. *Comm. Partial Differential Equations* **40** (2015), 1–39. Zbl1323.35063 MR3268920
40. Nirenberg, L., On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa (3)* **13** (1959), 115–162. Zbl0088.07601 MR0109940
41. Nirenberg, L., An extended interpolation inequality. *Ann. Scuola Norm. Sup. Pisa (3)* **20** (1966), 733–737. Zbl0163.29905 MR0208360
42. Oron, A., Davis, S. H. & Bankoff, S. G., Long-scale evolution of thin liquid films, *Rev. Mod. Phys.* **69** (1997), 931–980.
43. Otto, F., Lubrication approximation with prescribed nonzero contact angle. *Comm. Partial Differential Equations* **23** (1998), 2077–2164. Zbl0923.35211 MR1662172
44. Ren, W. & E, W., Boundary conditions for the moving contact line problem. Phys. Fluids **19** (2007), 022101. Zbl1146.76513
45. Ren, W. & E, W., Derivation of continuum models for the moving contact line problem based on thermodynamic principles. *Comm. Math. Sci.* **9** (2011), 597–606. Zbl05914221 MR2815687
46. Ren, W., Hu, D. & E, W., Continuum models for the contact line problem. *Phys. Fluids* **22** (2010), 102103. Zbl1308.76082
47. Solonnikov, V. A., On boundary value problems for linear parabolic systems of differential equations of general form. *Trudy Mat. Inst. Steklov.* **83** (1965), 3–163. Zbl0164.12502 MR0211083