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On the shape of the free boundary of variational inequalities with gradient constraints

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In this article we derive an estimate on the number of local maxima of the free boundary of the minimizer of

$$I[v] := \int_U \frac{1}{2}|Dv|^2 - \eta v\, dx,$$

subject to the pointwise gradient constraint

$$(|D_1 v|^q + |D_2 v|^q)^{\frac{1}{q}} \leqslant 1.$$

This also gives an estimate on the number of connected components of the free boundary. In addition, we further study the free boundary when U is a polygon with some symmetry.

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1. Introduction

Variational inequalities with gradient constraints, has been an active area of study, inspired by problems in Physics and engineering. An important example among them is the famous elastic-plastic torsion problem, which is the problem of minimizing the functional

$$I[v] := \int_U \frac{1}{2}|Dv|^2 - \eta v\, dx,$$

over the set

$$W := \{v \in H_0^1(U) \mid |Dv| \leqslant 1 \text{ a.e.}\}.$$

Here U is a bounded open domain in $\mathbb{R}^2$, and $\eta > 0$ is a constant. This problem is equivalent to finding $u \in W$ that satisfies the variational inequality

$$\int_U Du \cdot D(v - u) - \eta(v - u)\, dx \geqslant 0 \qquad \text{for all } v \in W.$$

Brezis and Stampacchia [2] proved the $W^{2,p}$ regularity for the elastic-plastic torsion problem. Caffarelli and Rivière [6] obtained its optimal $W^{2,\infty}$ regularity. Gerhardt [13] proved $W^{2,p}$ regularity for the solution of a quasilinear variational inequality subject to the same constraint as in the elastic-plastic torsion problem. Jensen [16] proved $W^{2,p}$ regularity for the solution of a linear variational inequality subject to a C^2 convex gradient constraint. Choe and Shim [8, 9] proved $C^{1,\alpha}$ regularity for the solution to a quasilinear variational inequality subject to a C^2 convex gradient constraint, and allowed the operator to be degenerate of the p-Laplacian type.

Recently, there has been new interest in these type of problems. Hynd and Mawi [15] studied fully nonlinear equations with convex gradient constraints, which appear in stochastic singular control. They obtained $W^{2,p}$ regularity in general, and $W^{2,\infty}$ regularity with some extra assumptions. De Silva and Savin [10] obtained C^1 regularity for the minimizer of some nonsmooth convex functionals subject to a gradient constraint in two dimensions, arising in the study of random surfaces. Here, the constraint is a convex polygon; so it is not strictly convex in contrast to other studies.

In addition to the works on the regularity of the elastic-plastic torsion problem, Caffarelli and Rivière [4, 5], Caffarelli and Friedman [3], Friedman and Pozzi [12], and Caffarelli et al. [7], have worked on the regularity and the shape of its free boundary, i.e. the boundary of the set $\{|Du| < 1\}$. These works can also be found in [11]. In [17, 18], we extended some of these results, both the regularity of the solution and its free boundary, to the more general case where the functional is unchanged but the constraint is given by the q-norm

$$(|D_1 v|^q + |D_2 v|^q)^{\frac{1}{q}} \leqslant 1.$$

In this work, we continue this study and generalize some other parts of the above works. Especially, we extend the results of Friedman and Pozzi [12] regarding the number of local maxima of the free boundary attached to a line segment of ∂U. As a result, we derive an estimate on the number of connected components of the free boundary attached to a line segment of ∂U.

An interesting consequence of our result is that when U is a convex polygon, there is at most one connected component of the free boundary attached to each side of ∂U. At the end, we generalize the reflection method of Caffarelli et al. [7] to our problem. This method gives some extra information about the free boundary parts attached to the sides of a polygon, when the polygon has some symmetry. Although we only obtain these results in special cases, since the q-norms are not as symmetric as the Euclidean norm.

A motivation for our study was to fill the gap between the known results mentioned above, and the still open questions regarding the regularity and the shape of the free boundary of the above-mentioned problem about random surfaces.

Now, let us introduce the problem in more detail. Let $U \subset \mathbb{R}^2$ be a simply connected bounded open set whose boundary is a simple closed Jordan curve consisting of arcs $S_1, \cdots, S_m$ that are $C^{k,\alpha}$ ($k \geqslant 3$, $\alpha > 0$) or analytic up to their endpoints, satisfying Assumption 1 below. We denote by $V_i := \bar{S}_i \cap \bar{S}_{i+1}$ the vertices of ∂U, and we assume that all the vertices are *nonreentrant* corners i.e., their opening angle is less than π.

As before, let

$$I[v] = \int_U \frac{1}{2}|Dv|^2 - \eta v \, dx \qquad \text{with } \eta > 0. \tag{1.1}$$

Let u be the minimizer of I over

$$K := \{v \in H_0^1(U) \mid \gamma_q(Dv) \leqslant 1 \text{ a.e.}\}. \tag{1.2}$$

Where γ_q is the q-norm on $\mathbb{R}^2$

$$\gamma_q((x_1, x_2)) := \left(|x_1|^q + |x_2|^q\right)^{\frac{1}{q}}.$$

Since $u^+ = \max(u, 0)$ is also a minimizer, we must have $u \geqslant 0$. Now as shown in [17, Section 2] we know that u is also the minimizer of I over

$$\tilde{K} := \{v \in H_0^1(U) \mid v(x) \leqslant d_p(x, \partial U) \text{ a.e.}\}. \tag{1.3}$$

Here $p = \frac{q}{q-1}$ is the dual exponent to q, and d_p is the metric associated to γ_p. We also assume that $1 < q \leqslant 2$, so $2 \leqslant p < \infty$.

Let us mention that the difficulty with the case $q > 2$, is that γ_p for $p < 2$ is not C^2 over $\mathbb{R}^2 - \{0\}$. This prevents many of the arguments here and in [17, 18] from working. Although our recent work [19] on the regularity of the minimizers, opens the door into investigating this case and many more general convex constraints, especially those that are not strictly convex. But obtaining the regularity of the free boundary, which is a prerequisite to understanding its shape, is still in progress.

1.1 *Preliminaries*

Next, we summarize some of the results obtained in [17, 18]. It has been proved in [17, Section 4] that $u \in C^{1,1}_{\text{loc}}(U) = W^{2,\infty}_{\text{loc}}(U)$. Also by [18, Lemma 2], we have the equalities

$$E := \{x \in U \mid u(x) < d_p(x, \partial U)\} = \{x \in U \mid \gamma_q(Du(x)) < 1\},$$

and

$$P := \{x \in U \mid u(x) = d_p(x, \partial U)\} = \{x \in U \mid \gamma_q(Du(x)) = 1\}.$$

The first region is called the *elastic* region and the second one is called the *plastic* region. It is easy to see that if $x \in P$ and $y \in \partial U$ is one of the p-closest points to x on the boundary, then the segment between x and y lies inside P. In addition, we have $\Delta u = -\eta$ over E, and $\Delta u \geqslant -\eta$ a.e. over U. Consequently, by the strong maximum principle we have $u > 0$ in U.

The complement of the largest open set over which $d_p(x) := d_p(x, \partial U)$ is $C^{1,1}$, is called the *p-ridge* and is denoted by R_p. It has been shown [18, Theorem 4] that R_p consists of those points in U with more than one p-closest point on ∂U, and those other points x at which $d_p(x) = \frac{1}{\kappa_p(y)}$ (we define $\kappa_p(y)$ below). One nice property of the p-ridge is that the p-closest point on ∂U varies continuously in $\bar{U} - R_p$ (see [18, Lemma 6]). Also by [18, Theorems 2 and 5], $R_p \subset E$, and outside R_p, d_p is as smooth as ∂U, provided that ∂U satisfies

Assumption 1 We assume that at the points where the normal to one of the S_i's is parallel to one of the coordinate axes, the curvature of S_i is small. In the sense that, if we have $(s + a_0, b(s))$ as a nondegenerate $C^{k,\alpha}$ ($k \geqslant 3$, $0 < \alpha < 1$) parametrization of S_i around $y_0 := (a_0, b(0))$, and $b'(0) = 0$; then we assume b' goes fast enough to 0 so that $b'(s) = c(s)|c(s)|^{p-2}$, where $c(0) = 0$, and c is $C^{k-1,\alpha}$. Note that y_0 can be one of the endpoints of S_i.

Also we require $c'(0)$ to be small enough so that $1 - c'(0)d_p(\cdot)$ does not vanish at the points inside U that have y_0 as the only p-closest point on ∂U.

It is easy to show that there is a p-circle inside U that touches ∂U only at y_0 (see the proof of Theorem 2.3 below). We will call these points the degenerate points of Assumption 1. Note that we modified this assumption to be slightly different than what appeared in [18], to emphasize that we require this assumption to also hold at the endpoints of the arcs $S_1, \cdots, S_m$.

By [18, Theorem 3], away from the p-ridge we have

$$\Delta d_p(x) = \frac{-(p-1)\tau_p(y)\kappa_p(y)}{1-\kappa_p(y)d_p(x)}. \tag{1.4}$$

Here $y \in \partial U$ is the p-closest point to x, and if $(a(\cdot), b(\cdot))$ is a parametrization of ∂U around y,

$$\kappa_p := \frac{a'b'' - b'a''}{(p-1)|a'|^{\frac{p-2}{p-1}}|b'|^{\frac{p-2}{p-1}}(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{p+1}{p}}}$$

is the p-*curvature*, and

$$\tau_p := \frac{(|a'|^{\frac{2}{p-1}} + |b'|^{\frac{2}{p-1}})|a'b'|^{\frac{p-2}{p-1}}}{(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{2p-2}{p}}}$$

is another reparametrization invariant quantity. Note that at the degenerate points of Assumption 1, we have $\lim \kappa_p = c'(0)$ and $\tau_p = 0$. Let us also record here that outside R_p, y is a $C^{k-1,\alpha}$ function of x, and

$$Dd_p(x) = \frac{\nu(y)}{\gamma_q(\nu(y))}, \tag{1.5}$$

where ν is the inward normal to ∂U (see the proof of Theorem 2 of [18]). Note that nonreentrant corners can not be the p-closest point on ∂U to any point inside U.

Let $y = f(s)$ $(0 \leqslant s \leqslant L)$ be a parametrization of ∂U. Then it has been proved [18, page 15] that the *free boundary*, $\Gamma := \partial E \cap U$, can be parametrized by $f(s)+\delta(s)\mu(s)$. Here $\delta : [0, L] \to \mathbb{R}$ is a continuous and nonnegative function, and $\mu(s)$ is the unique direction at $f(s)$ along which points inside U have $f(s)$ as the p-closest point on ∂U. μ is called the inward p-*normal*, and is given by the formula

$$\mu := \frac{1}{\left(|\nu_1|^{\frac{p}{p-1}} + |\nu_2|^{\frac{p}{p-1}}\right)^{\frac{1}{p}}}(\mathrm{sgn}(\nu_1)|\nu_1|^{\frac{1}{p-1}}, \mathrm{sgn}(\nu_2)|\nu_2|^{\frac{1}{p-1}}), \tag{1.6}$$

where as before $\nu = (\nu_1, \nu_2)$ is the inward normal to ∂U at $f(s)$. Furthermore by [18, Theorem 12], we know that Γ is a $C^{k-1,\alpha}$ curve with no cusps, if the part of ∂U that parametrizes it is $C^{k,\alpha}$. Also, $\delta \equiv 0$ in a neighborhood of nonreentrant corners, since it has been shown [18, Theorem 14] that nonreentrant corners have an elastic neighborhood in U. Note that on Γ we have $u = d_p$ and $Du = Dd_p$, since $u - d_p$ attains its maximum there.

Also note that the above characterization of the free boundary implies that E is a simply connected domain bounded by a simple closed Jordan curve.

Finally, we briefly comment on the case that some vertices V_i are *reentrant* corners, i.e., their opening angle is greater than π. The main difference that these corners have with nonreentrant ones, is that they are the p-closest point on ∂U to some points inside U. In fact, if we denote by μ_{i1}, μ_{i2} the inward p-normals to respectively S_i, S_{i+1} at V_i, then the points in U between μ_{i1}, μ_{i2} and close to V_i have V_i as the only p-closest point on ∂U. We denote this set of points by U_i. Note that U_i is an open subset of U.

It is obvious that d_p is analytic on U_i. (Except at the points that lie on a line passing through the corner and parallel to one of the coordinate axes. At theses points the regularity of d_p depends on the integer part of p, but it is at least C^2.) The p-ridge is characterized as before, and is inside the elastic region (see [18, Section 4]). The other difference is that on μ_{ij}, we can only say that d_p is $C^{1,1}$ at the points where $d_p \neq \frac{1}{\kappa_p}$. Furthermore, the free boundary is an analytic curve inside U_i (again, except possibly at a finite number of points). Here, δ is a function of the angle between μ_{i1} and the segment connecting V_i to the free boundary.

2. Global regularity

First, let us give a global regularity result not mentioned in [17, 18].

Theorem 2.1 *When all the vertices of ∂U are nonreentrant corners, we have $u \in C^{1,\alpha}(\bar{U})$ for some $\alpha > 0$. If ∂U has no corners, the conclusion holds for all $\alpha \in (0, 1)$.*

Proof. Note that by the gradient constraint we have $u \in W^{1,\infty}(U) = C^{0,1}(\bar{U})$. Furthermore

$$\Delta u = \begin{cases} -\eta & \text{in } E \\ \Delta d_p & \text{a.e. in } P. \end{cases}$$

Also note that by Assumption 1, κ_p is bounded on ∂U. Thus $1 - \kappa_p d_p \to 1$ uniformly, as we approach ∂U. Also, $1 - \kappa_p d_p > 0$ on P by an argument similar to [18, Lemma 7], so it has a positive minimum there. In addition, $\tau_p \kappa_p$ is bounded on ∂U as we assumed that S_i's are smooth up to their endpoints. Hence Δd_p is bounded on P. Thus Δu is bounded there too. Therefore as $u \in C^0(\bar{U})$, we can apply the Calderon–Zygmund estimate and conclude that u is in $W^{2,s}$ for any $s \in (1, \infty)$, around any $C^{1,1}$ portion of ∂U. Thus u is in $C^{1,\alpha}$ around points in the interior of S_i's, for any $\alpha \in (0, 1)$. (Consult Theorem 9.15 of [14]. Note that we need to multiply u by a smooth bump function with support around some smooth part of ∂U, and use the fact that $u, Du, \Delta u$ are bounded.) As nonreentrant corners have an elastic neighborhood in U, around them we have $\Delta u = -\eta$. Now as u vanishes on ∂U, we can apply the results of [1] to deduce that u is in $C^{1,\alpha}$ for some $\alpha > 0$ around these corners. □

REMARK 2.2 This theorem does not hold when some of the vertices are reentrant corners. Although Du remains bounded as we approach a reentrant corner, it is not necessarily continuous there.

Let us also give an interesting consequence of Assumption 1.

Theorem 2.3 *Every smooth point of ∂U has a U-neighborhood that does not intersect the p-ridge R_p.*

Proof. The reason is that, locally around smooth points, ∂U has *uniform interior p-circle property*. This means that for any smooth point $y_0 \in \partial U$ and any $y \in \partial U$ close enough to y_0, there is a p-circle inside U whose boundary touches ∂U only at y, and its p-radius is independent of y. This implies that close to y_0, no point of U has more than one p-closest point on ∂U. Also, as κ_p is bounded on ∂U by Assumption 1 and smoothness of S_i's up to their endpoints, $1 - \kappa_p d_p \neq 0$ near the boundary. Thus we get the result.

To prove the property, first assume that $y_0 = (a_0, b_0)$ is a degenerate point of Assumption 1, and around it we can parametrize ∂U by

$$s \mapsto (s + a_0, b(s)).$$

Where $b(0) = b_0$, $b'(0) = 0$, and $b(s) = c(s)|c(s)|^{p-2}$ for some smooth enough function c. We assume that U is above ∂U around y_0.

Let s_1 be close to 0, and consider $y_1 = (s_1 + a_0, b(s_1))$ near y_0. Then, $\frac{(-c(s_1),1)}{(1+|c(s_1)|^p)^{\frac{1}{p}}}$ is the p-normal at y_1. Consider the p-circle with p-radius r and center $(a_1 + a_0, b_1)$, where $a_1 := s_1 - \frac{rc(s_1)}{(1+|c(s_1)|^p)^{\frac{1}{p}}}$ and $b_1 := b(s_1) + \frac{r}{(1+|c(s_1)|^p)^{\frac{1}{p}}}$. We will show that this p-circle which passes through y_1, is above ∂U near y_1. Let

$$\alpha(s) := -(r^p - |s - a_1|^p)^{\frac{1}{p}} + b_1 - b(s).$$

It is enough to show that α is positive around s_1. Note that $\alpha(s_1) = 0$.

For this to happen, it suffices to show that

$$\alpha'(s) = \frac{(s - a_1)|s - a_1|^{p-2}}{(r^p - |s - a_1|^p)^{\frac{p-1}{p}}} - c(s)|c(s)|^{p-2}$$

is positive after s_1 and negative before it. But as the map $s \mapsto s|s|^{p-2}$ is increasing, we just need to show that

$$\beta(s) := \frac{s - a_1}{(r^p - |s - a_1|^p)^{\frac{1}{p}}} - c(s)$$

has the same property. As $\beta(s_1) = 0$, it is sufficient to show that

$$\beta'(s) = \frac{1}{(r^p - |s - a_1|^p)^{\frac{1}{p}}} + \frac{|s - a_1|^p}{(r^p - |s - a_1|^p)^{\frac{p+1}{p}}} - c'(s)$$

is positive.

Choose r small enough so that $c'(s) < \frac{1}{2r}$ for $|s| \leqslant 2\epsilon_0$, where ϵ_0 is very small compared to r. Then for any s_1 with $|s_1| < \epsilon_0$, we have $\beta'(s) > \frac{1}{2r}$ for $|s - s_1| \leqslant \epsilon_0$. Thus $\alpha(s) > 0$ for $0 < |s - s_1| \leqslant \epsilon_0$. Now inside the p-circle with p-radius r, we take a p-circle with p-radius r_1 that passes through y_1. Let $|s_1| \leqslant \frac{1}{4}\epsilon_0$. We can take r_1 to be small enough (independently from y_1), so that this smaller p-circle has a positive distance from $\partial U - \{(s + a_0, b(s)) \mid |s| < \frac{1}{2}\epsilon_0\}$. Hence the smaller p-circle is inside U, and this is what we wanted to prove.

Now assume that y_0 is a nondegenerate point. Then due to the inverse function theorem, we can find a parametrization for ∂U around y_0 of the form

$$s \mapsto \big(s + a_0, b(s)\big).$$

This time $b'(s) \neq 0$ for s small, so we can define the smooth function $c(s) := \frac{b'(s)}{|b'(s)|}|b'(s)|^{\frac{1}{p-1}}$. Hence $b' = c|c|^{p-2}$ and we can repeat the above argument. □

REMARK 2.4 When $p \neq 2$, this theorem is false without Assumption 1. A simple example is a disk, whose p-ridge is the union of its two diagonals parallel to the coordinate axes.

REMARK 2.5 An important consequence of this theorem is that d_p is at least C^1 up to smooth points of ∂U. The reason is that d_p, Dd_p are uniformly continuous on a U-neighborhood of these points.

3. Plastic components attached to line segments of boundary

We start with a lemma about the level sets of a function satisfying an elliptic equation in some region of the plane.

Lemma 3.1 *Let $U \subset \mathbb{R}^2$ be a bounded simply connected domain whose boundary is a simple closed Jordan curve. Suppose $u \in C^2(U) \cap C(\bar{U})$ is a nonconstant function satisfying*

$$Lu := -a^{ij} D^2_{ij} u + b^i D_i u = 0.$$

Where L is a uniformly elliptic operator with continuous coefficients. Then the closure of every level set of u in U, intersects ∂U.

Furthermore, when $L = -\Delta$, the closure of every connected component of any level set of u in U, intersects ∂U.

Proof. Let $S := \{x \in U \mid u(x) = c\}$ be a nonempty level set, and suppose to the contrary that $\bar{S} \subset U$. Then as both $\bar{S}$ and ∂U are compact, their distance, 2ε, is positive. For any $y \in \partial U$, let $U_\varepsilon(y)$ be the connected component of $B_\varepsilon(y) \cap U$ that has y on its boundary. First, note that there is at most one such component since ∂U is a simple Jordan curve. Second, on any $U_\varepsilon(y)$, u is either greater than c or less than c. The reason is that if both happen, u must take the value c in $U_\varepsilon(y)$ which is impossible.

Now suppose that for some $y_0 \in \partial U$ we have $u < c$ on $U_\varepsilon(y_0)$. We claim that the same thing happens for every $y \in \partial U$. Let

$$A := \{y \in \partial U \mid u < c \text{ on } U_\varepsilon(y)\}.$$

Obviously A is open in ∂U. But it is also closed, since if for $y \in \partial U$ we have $y_i \to y$ for some sequence $y_i \in A$, then for large enough i we have $y \in \overline{U_\varepsilon(y_i)}$. Thus as A is nonempty and ∂U is connected we have $A = \partial U$. This implies that $u \leqslant c$ on ∂U. But in that case, the strong maximum principle implies that u is constant, which is a contradiction.

Now suppose $L = -\Delta$. Then u is analytic inside U, and its level sets are locally, several analytic arcs emanating from a point. Suppose, $S_1 \subset U$ is a connected component of S, and $\bar{S}_1 \subset U$. Then as S_1 is a maximal connected subset of S, we have $S_1 = \bar{S}_1$. Thus S_1 is compact. Hence S_1 has a positive distance from ∂U. It also has a positive distance from $\overline{S - S_1}$. The reason is that if $S_1 \cap \overline{S - S_1} \neq \emptyset$, then there is a sequence in $S - S_1$ converging to a point in S_1, which is also in U. But this implies that, that sequence belongs to one of the analytic arcs emanating from that point. This means, that sequence belongs to S_1, which is a contradiction.

Therefore, we can enclose S_1 by a simple closed Jordan curve inside U that still has a positive distance from $\bar{S}_1$, and leaves $\overline{S - S_1}$ outside. We can argue as before and get a contradiction, noting that as u is analytic, it can not be constant on this new domain. □

Now we return to our free boundary problem.

DEFINITION 3.2 Remember that f parametrizes ∂U, μ is the inward p-normal to ∂U given by (1.6); and δ is a nonnegative function so that $f + \delta\mu$ parametrizes the free boundary when $\delta > 0$. Suppose $\delta(s) > 0$ for $s \in (a, b)$ and $\delta(a) = \delta(b) = 0$, then we call the set

$$\{f(s) + t\mu(s) \mid s \in [a, b],\ t \in [0, \delta(s)]\}$$

a *plastic component*.

Note that there are at most countably many plastic components.

The following theorem is a stronger version of a result proved in [18, Theorem 15], which states that the number of plastic components attached to a closed line segment of ∂U is finite. Here we do not require the line segment to be a proper subset of an open line segment of ∂U. We also give some details of the proof that are not presented in [18]. For this theorem, we can allow U to have several holes homeomorphic to a disk, and not be simply connected. We can also allow the corners to be reentrant.

Theorem 3.3 *The number of plastic components attached to a closed line segment of ∂U is finite, if the endpoints of the segment are not reentrant corners, and a neighborhood of each endpoint in the segment either has an elastic neighborhood in U, or belongs to a plastic component.*

Proof. Let the line segment be

$$\lambda_1 := \big\{(x_1, \rho_1 x_1 + \rho_2) \mid a \leqslant x_1 \leqslant b\big\},$$

and assume that U is above the segment. Suppose to the contrary that there are infinitely many plastic components

$$P_i = \big\{(x_1, \rho_1 x_1 + \rho_2) + t(\mu_1, \mu_2) \mid x_1 \in [a_i, b_i]\,,\ t \in \big[0, \delta(x_1)\big]\big\}$$

attached to the line segment. Where $\mu := (\mu_1, \mu_2)$ is the inward p-normal, $b_i \leqslant a_{i+1}$, and as noted before δ is a continuous nonnegative function on $[a, b]$. Let

$$H_i := \max_{x \in [a_i, b_i]} \delta(x).$$

Since $b_i - a_i \to 0$ as $i \to \infty$, we must have $H_i \to 0$. Otherwise a subsequence, H_{n_i} converges to a positive number. By taking a further subsequence we can assume that this subsequence is $\delta(x_{n_i})$, for a sequence x_{n_i} which converges to some $c \in [a, b]$. But this contradicts the continuity of δ at c because $b_{n_i} \to c$ too.

Hence any line $x_2 = \rho_1 x_1 + \rho_2 + \epsilon$ intersects only a finite number, $n(\epsilon)$, of P_i's, and $n(\epsilon) \to \infty$ as $\epsilon \to 0$.

Consider the tilted graph of δ over λ_1. It is in the subset of U consisting of points whose p-closest point on ∂U belongs to λ_1. Since $U - R_p$ is open, the subset of this part of the tilted graph over which $\delta > 0$ has a positive distance from R_p. On the part where $\delta = 0$, we have the same conclusion, noting that R_p has a positive distance from the interior of λ_1. If $\delta > 0$ at the endpoints of λ_1, we can argue as above, and if $\delta = 0$ there, we actually work with a subsegment of λ_1. Thus as the p-closest point on ∂U varies continuously in $U - R_p$, the p-normals to λ_1 are parallel, and λ_1 is compact, the tilted graph of δ attached to λ_1 has a tubular neighborhood in E that does not intersect R_p and consists of points whose p-closest point on ∂U belongs to λ_1.

Consider a piecewise analytic curve γ in this tubular neighborhood, that has no self intersection. The endpoints of γ are on λ_1. We specify the left endpoint of γ, the other one is similar. If the part of λ_1 near its left endpoint has an elastic neighborhood, we start γ slightly to the right of the left endpoint, staying in the elastic region. If the part of λ_1 near its left endpoint belongs to a plastic component, we start γ at the maximum point on the tilted graph of δ on that plastic component, which is on the right of the left endpoint. Even if the maximum happens at the endpoint itself, we have to start γ slightly after the endpoint on the free boundary.

By our construction, γ is close enough to λ_1 so that for points between them, the p-distance to ∂U is the p-distance to λ_1. Thus for those points $d_p(x, \partial U)$ is a function of only $-\rho_1 x_1 + x_2$. Since as proved in [18, page 17], the p-distance to a line is a multiple of the 2-distance to the line, with coefficient depending only on the line and p. Thus for $\zeta := \frac{1}{\sqrt{1+\rho_1^2}}(1, \rho_1)$ we have $D_\zeta d_p = 0$ in this region.

Now let E_0 to be the elastic region enclosed by γ and the tilted graph of δ over λ_1. Let $\epsilon > 0$ be small enough. On every open connected segment of $E_0 \cap \{x_2 = \rho_1 x_1 + \rho_2 + \epsilon\}$ with endpoints on the free boundary of two different P_j's, the function $D_\zeta(u - d_p) = D_\zeta u$ is analytic and changes sign, as $u - d_p$ is zero on the endpoints and negative between them. Let

$$\tilde{c}_i := (c_i, \rho_1 c_i + \rho_2 + \epsilon)\,,\ \tilde{c}_{i+1} := (c_{i+1}, \rho_1 c_{i+1} + \rho_2 + \epsilon)$$

for $c_i < c_{i+1}$ be points close to those endpoints such that

$$D_\zeta u(\tilde{c}_i) < 0\ ,\ \ D_\zeta u(\tilde{c}_{i+1}) > 0.$$

We can also assume that $D_\zeta u \leqslant 0$ on the part of the segment joining $\tilde{c}_i$ to the free boundary, and similarly $D_\zeta u \geqslant 0$ on the part of the segment joining $\tilde{c}_{i+1}$ to the free boundary. Let $\sigma_i(\epsilon)$ be the connected component containing $\tilde{c}_i$, of the level set

$$\big\{y \in E_0 \ \mid\ D_\zeta u(y) = D_\zeta u(\tilde{c}_i)\big\}.$$

Then by Lemma 3.1, the closure of the connected components of the level sets of the harmonic function $D_\zeta u$, will intersect the boundary of its domain E_0. Note that ∂E_0 consists of γ and part of the image of

$$x_1 \mapsto (x_1, \rho_1 x_1 + \rho_2) + \delta(x_1)(\mu_1, \mu_2),$$

hence it is a simple closed Jordan curve, and E_0 is simply connected. (Note that E_0 is simply connected even when U is not, since for all points in it, the p-closest point on ∂U lies on λ_1. Thus no other part of ∂U can be inside E_0.) Also as shown in the introduction, $D_\zeta u$ is continuous on $\overline{E_0}$, as we are away from reentrant corners. Obviously, $D_\zeta u$ is not constant over E_0 too, unless the points $\tilde{c}_i, \tilde{c}_{i+1}$ do not exist, in which case we have at most one plastic component.

We claim that there is a path in $\overline{\sigma_i(\epsilon)}$ that connects $\tilde{c}_i$ to a point on γ. To see this, note that $D_\zeta u = D_\zeta(u - d_p)$ is zero on the free boundary and on the segment λ_1. Hence, $\overline{\sigma_i(\epsilon)}$ must intersect γ. In addition, $D_\zeta u$ is harmonic on a neighborhood of γ. The reason is that locally, $D_\zeta u$ has harmonic continuation across the elastic parts of the segment λ_1, and the free boundary attached to it, since they are analytic curves and $D_\zeta u$ vanishes along them. Thus $D_\zeta u$ is harmonic on a neighborhood of $\overline{\sigma_i(\epsilon)}$. But, the level sets of a harmonic function are locally, the union of several analytic arcs emanating from a vertex. On the other hand, as γ is piecewise analytic, $\overline{\sigma_i(\epsilon)} \cap \gamma$ is a finite set. Hence, $\overline{\sigma_i(\epsilon)}$ is locally path connected, and as it is connected it must be path connected.

Consider an injective path that connects $\tilde{c}_i$ to γ, and its last intersection with the segment joining $\tilde{c}_i$ to the free boundary along the line $x_2 = \rho_1 x_1 + \rho_2 + \epsilon$. Let $\phi_i(\epsilon)$ be the union of the part of the path that connects that last intersection point to γ, and the part of the segment that joins it to the free boundary. Therefore, $\phi_i(\epsilon)$ is a simple Jordan curve, connecting two distinct points of ∂E_0. Hence it disconnects E_0. Since obviously $\phi_{i+1}(\epsilon) \cap \phi_i(\epsilon) = \emptyset$, $D_\zeta u$ must change sign at least $n(\epsilon) - 1$ times along γ. But $n(\epsilon) - 1$ grows to infinity as $\epsilon \to 0$, contradicting the fact that γ is piecewise analytic and $D_\zeta u$ is analytic on a neighborhood of it. $\square$

REMARK 3.4 The only kind of line segments not covered by the above theorem, are those that one of their endpoints is the accumulation point of a family of plastic components. The main difficulty in this case is that, $D_\zeta u$ might not have analytic continuation in a neighborhood of the endpoints of γ. For these segments, we can still apply the above reasoning to their proper subsegments. Since we can choose the curve γ to start and end slightly before and after the endpoints of the subsegment, at new endpoints satisfying one of the conditions of the theorem. This way we can prove that the family of plastic components attached to these subsegments is finite too.

4. The number of local maxima of plastic components

Next, we are going to give an estimate on the number of local maxima of each plastic component attached to a flat part of the boundary of U. Let us remind that ∂U consists of smooth arcs $S_1, \cdots, S_m$. We set

$$\lambda := \bar{S}_1 = \big\{(x_1, \rho_1 x_1 + \rho_2) \mid 0 \leqslant x_1 \leqslant b\big\}$$

to be a flat part of ∂U, and $\Lambda := \bar{S}_2 \cup \cdots \cup \bar{S}_m$; so we have $\partial U = \lambda \cup \Lambda$. As before, we denote the vertices of ∂U by $V_i = \bar{S}_i \cap \bar{S}_{i+1}$. In this section we assume that all the corners of ∂U are nonreentrant.

We assume that some U-neighborhood of λ lies in $\{x_2 > \rho_1 x_1 + \rho_2\}$. Let $y = f(s)$ be a parametrization of ∂U for $0 \leqslant s \leqslant L$, with $f(0) = (b, \rho_1 b + \rho_2) = V_0$ and $f(s_1) = (0, \rho_2) = V_1$.

We know that along ∂U the p-distance function d_p is differentiable, except at the points $f(s_j) = V_j$. Let $\nu(s) = (\nu_1(s), \nu_2(s))$ for $s \neq s_j$ be the inward normal to ∂U at $f(s)$ with $\gamma_q(\nu(s)) = 1$. Also let

$$\zeta = \frac{1}{\sqrt{1 + \rho_1^2}}(1, \rho_1) \tag{4.1}$$

be the unit vector along the line segment S_1. Then by (1.5) and continuity of Dd_p, we have

$$D_\zeta d_p\big(f(s)\big) = \nu(s) \cdot \zeta.$$

ASSUMPTION 2 The set $\{s \in [0, L] - \{s_j\} \mid \nu(s) \cdot \zeta = 0\}$ consists of a finite number of points, and a finite number of intervals.

Therefore $\nu \cdot \zeta$ changes sign a finite number of times. Let

$$\begin{aligned} k :=\ &\text{The number of times } \nu \cdot \zeta \\ &\text{changes sign from positive to negative on the interval } [s_1, L]. \end{aligned} \tag{4.2}$$

Remember that μ is the inward p-normal to ∂U given by (1.6); and δ is a nonnegative function so that $f(s) + \delta(s)\mu(s)$ for $s \neq s_j$, parametrizes the free boundary when $\delta > 0$. Note that d_p is $C^{1,\alpha}$ around these points even if $\delta(s) = 0$.

Since $f(s)$ is the unique p-closest point on ∂U to $f(s) + \delta(s)\mu(s)$ when $s \neq s_j$, by (1.5) we have

$$D_\zeta d_p\big(f(s) + \delta(s)\mu(s)\big) = \nu(s) \cdot \zeta.$$

Now consider the function

$$\begin{aligned} &u_1(s) := D_\zeta u\big(f(s) + \delta(s)\mu(s)\big) \qquad s \neq s_j \\ &u_1(s_j) := 0. \end{aligned} \tag{4.3}$$

Note that u_1 is continuous at s_j's. The reason is that $Du(f(s_j)) = 0$ by continuity of Du there, and the fact that the directional derivatives of u vanish in two directions at $f(s_j)$.

Lemma 4.1 *$u_1(s)$ has the same sign as $\nu(s) \cdot \zeta$ for $s \neq s_j$, where ν is the inward normal to ∂U and ζ, u_1 are given by (4.1), (4.3) respectively.*

Proof. Since on the free boundary $Du = Dd_p$, we have

$$u_1(s) = \nu(s) \cdot \zeta$$

when $\delta(s) > 0$.

Consider a point s_0 different than s_j's, with $\nu(s_0) \cdot \zeta > 0$. If $\delta(s_0) > 0$ then obviously $u_1(s_0) > 0$ too. If $\delta(s_0) = 0$ but $s_0 = \lim s_k$ where $\delta(s_k) > 0$, then by continuity we still have $u_1(s_0) = \nu(s_0) \cdot \zeta > 0$. And finally, if neither of these happen at s_0, then $\delta \equiv 0$ on a neighborhood of s_0. This means that some U-neighborhood of $f(s_0)$ is elastic. Thus in that neighborhood we have

$$-\Delta u = \eta > 0.$$

As $u > 0$ in U and $u = 0$ on ∂U, the strong maximum principle (actually the Hopf's lemma used in its proof) implies that

$$\nu(s_0) \cdot \zeta \, D_\zeta u\big(f(s_0)\big) + \nu(s_0) \cdot \xi \, D_\xi u\big(f(s_0)\big) = D_\nu u\big(f(s_0)\big) > 0. \tag{4.4}$$

Here ξ is a unit vector orthogonal to ζ. On the other hand, u is constant along ∂U, therefore its tangential derivative vanishes, i.e.

$$-\nu(s_0) \cdot \xi \, D_\zeta u\big(f(s_0)\big) + \nu(s_0) \cdot \zeta \, D_\xi u\big(f(s_0)\big) = 0. \tag{4.5}$$

Now using this and the fact that $\nu(s_0) \cdot \zeta > 0$, we can rewrite (4.4) to get

$$\left[\nu(s_0) \cdot \zeta + \frac{[\nu(s_0) \cdot \xi]^2}{\nu(s_0) \cdot \zeta}\right] D_\zeta u\big(f(s_0)\big) > 0.$$

Hence $u_1(s_0) = D_\zeta u(f(s_0)) > 0$ as desired. When $\nu(s_0) \cdot \zeta < 0$, we can repeat the above arguments to deduce that $u_1(s_0) < 0$ too.

When $\nu(s_0) \cdot \zeta = 0$, we can still deduce that $u_1(s_0) = 0$. The only difference with the above argument is that when $\delta \equiv 0$ on a neighborhood of s_0, we have to use (4.5) to get the result, noting that $\nu(s_0) \cdot \xi \neq 0$ when $\nu(s_0) \cdot \zeta = 0$. □

It should be noted that $u_1(s) = 0$ for $s \in [0, s_1]$.

DEFINITION 4.2 Remember that f parametrizes ∂U, μ is the inward p-normal to ∂U given by (1.6); and δ is a nonnegative function so that $f + \delta\mu$ parametrizes the free boundary when $\delta > 0$. Also, u_1 is defined by (4.3). The points of the form $f(s) + \delta(s)\mu(s)$ for which $u_1(s) = 0$, will be called *flat points*. By Assumption 2 and the above lemma, the set of flat points consists of a finite number of points, and a finite number of arcs which we call *flat intervals*.

Consider the harmonic function $D_\zeta u$ over the elastic region E. $D_\zeta u$ has harmonic continuation to a neighborhood of each interior point of a flat interval, if around that point either $\delta > 0$ or $\delta \equiv 0$. The reason is that for a flat interval we have $\nu \cdot \zeta \equiv 0$ over the part of ∂U attached to it. Hence that part of ∂U is a line segment in the ζ direction. Thus the flat interval which is either this line segment, or a free boundary attached to it, is in both cases an analytic curve.

Lemma 4.3 *Let $x_0 \in E$ be a point where $D_\zeta u(x_0) = 0$, where E is the elastic region and ζ is given by (4.1). There exists a simple Jordan curve $\{t \mapsto \gamma(t)\,;\, t \in \mathbb{R}\}$ in E passing through x_0, along which $D_\zeta u = 0$. Furthermore,*

$$\lim_{t\to-\infty} \gamma(t), \qquad \lim_{t\to+\infty} \gamma(t)$$

exist, are different, and belong to ∂E.

Proof. Since $D_\zeta u$ is harmonic, its level sets in E are locally, the union of several analytic arcs emanating from a vertex. Consider the family of injective continuous maps from $(-1, 1)$ into the level set of $D_\zeta u$ at x_0, which take zero to x_0. We endow this family with a partial order relation. For f_1, f_2 in the family, we say $f_1 \leqslant f_2$ if

$$f_1((-1, 1)) \subseteq f_2((-1, 1)).$$

Now, we can apply Zorn's lemma to deduce the existence of a maximal map. We only need to check that any increasing chain has an upper bound. Consider such a chain $\{f_\alpha\}$. We claim that each $f_\beta((-1, 1))$ is open in $\bigcup_\alpha f_\alpha((-1, 1))$. Consider a point $f_\beta(t_0)$ in $f_\beta((-1, 1))$, then the level set around it, is the union of several arcs emanating from it, and $f_\beta((t_0-\epsilon, t_0+\epsilon))$ is one of them. Now, none of the sets $f_\alpha((-1, 1)) - f_\beta((t_0 - \epsilon, t_0 + \epsilon))$ can intersect one of these arcs. Since otherwise we have a loop in the level set, which results in $D_\zeta u \equiv 0$ by the maximum principle and simple connectedness of E. This contradiction gives the result.

Therefore $\bigcup_\alpha f_\alpha((-1, 1))$ is the union of countably many of $f_\alpha((-1, 1))$'s, since the topology of $\mathbb{R}^2$ is second countable. Now, by reparametrizing the maps in this countable subchain and gluing them together, we obtain a continuous map from $(-1, 1)$ onto $\bigcup_\alpha f_\alpha((-1, 1))$. The injectivity of this map is easy to show, since if it fails it must fail for one of the maps in the countable subchain too.

Now, consider γ, a maximal simple Jordan curve in the level set $\{D_\zeta u = 0\}$ passing through x_0, parametrized from $-\infty$ to ∞ with $\gamma(0) = x_0$. Since E is bounded, every sequence $t_k \to \infty$ has a subsequence such that $\gamma(t_{k_i}) \to x^*$. If $x^* \in E$, then $\gamma(t_{k_i})$ belongs to one of the arcs in the level set emanating from x^*. Thus, $\gamma((t_{k_i}, \infty))$ coincides with that arc, as the level set around x^* is the union of those arcs, and γ is one to one. Therefore, either γ can be extended beyond x^*, or we get a loop in the level set, which are contradictions. Hence, every such limit must belong to ∂E and be a flat point.

Now suppose that for two sequences $t_k, t_l \to \infty$, we have $\gamma(t_k) \to x^*$ and $\gamma(t_l) \to x'$, where $x^*, x' \in \partial E$. Suppose $x^* \neq x'$ and one of them, say x', belongs to the interior of a flat interval. Then, if $D_\zeta u$ has harmonic continuation in a disk around x', the level set $\{D_\zeta u = 0\}$ is again the union of finitely many arcs emanating from x'. Therefore, γ can not intersect the boundary of that disk an infinite number of times, contradicting our assumption. If $D_\zeta u$ does not have harmonic continuation around x', then $\delta(x') = 0$ and a sequence of plastic components accumulate at x'. In this case, we can find a sequence of points $\gamma(t_{l'})$ at an appropriate distance from $\gamma(t_l)$, such that $\gamma(t_{l'}) \to x''$. Where x'' is in the interior of the same flat interval, and either $\delta(x'') > 0$ or $\delta \equiv 0$ around it. Thus $D_\zeta u$ has harmonic continuation around x'' and we can argue as before.

Thus, if $x^* \neq x'$ then none of them can belong to the interior of a flat interval. Hence they are either isolated flat points or the endpoints of flat intervals. But again, looking at the arcs between $\gamma(t_k)$ and $\gamma(t_l)$ on the image of γ, we see that there are infinitely many limit points on ∂E between x^*, x', which contradicts Assumption 2 and the argument in the previous paragraph. Hence the limits $\lim_{t\to\pm\infty} \gamma(t)$ exist.

Finally, if the two limit points of γ coincide, the strong maximum principle and continuity of Du over $\bar{U}$ imply that $D_\zeta u \equiv 0$ over some domain, and consequently over E, which is a contradiction. □

Lemma 4.4 *The set of level curves of* $\{D_\zeta u = 0\}$ *described in Lemma 4.3 is finite.*

Proof. First, note that any such curve can not have both its endpoints on the same flat interval, since otherwise $D_\zeta u \equiv 0$ on E which is a contradiction. Second, for the same reason, two such curves can not have the same endpoints, or have each of their endpoints on the same flat intervals, or one endpoint the same and the other one on one flat interval.

Therefore, there is at most one such curve, connecting two isolated flat points, or two flat intervals, or an isolated flat point and a flat interval. Hence we get the result. □

REMARK 4.5 A consequence of this lemma is that all the level curves given by Lemma 4.3 are piecewise analytic. The reason is that the singularities of the level curves happen at the zeros of $DD_\zeta u$, and through any such point at least two level curves pass. Thus their number must be finite, as no two level curves can intersect more than once.

Let us fix some notation before proceeding. We denote by λ_E the part of λ with no plastic component attached to it. We also denote by Γ_λ the union of the free boundaries of the plastic components attached to λ. Finally let $\lambda_0 := \bar{\lambda}_E \cup \Gamma_\lambda$. Similarly we define $\Lambda_E, \Gamma_\Lambda$ and Λ_0.

Since all the corners of ∂U are nonreentrant and they have an elastic neighborhood, the number of plastic components attached to λ is finite by Theorem 3.3. We denote these plastic components by

$$P_j := \big\{(x_1, \rho_1 x_1 + \rho_2) + t(\mu_1, \mu_2) \mid a_j \leqslant x_1 \leqslant b_j\,,\, 0 \leqslant t \leqslant \delta(x_1)\big\} \qquad j = 1, 2, \cdots \tau,$$

where (μ_1, μ_2) is the inward p-normal, δ is the function whose tilted graph is the free boundary, and

$$0 < a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant a_\tau < b_\tau < b.$$

Let

$$N_j := \text{ The number of local maxima of } \delta \text{ on the interval } \{a_j \leqslant x_1 \leqslant b_j\}. \tag{4.6}$$

Note that these are strict local maxima since the tilted graph of δ, which is the free boundary, is an analytic curve.

Lemma 4.6 *Consider one of the plastic components P_j described above. Let β be a point of local maximum of $\delta(x_1)$ over $x_1 \in [a_j, b_j]$. Then, there exists a level curve $\{t \mapsto \gamma(t)\,;\, t \in \mathbb{R}\}$ of $\{D_\zeta u = 0\}$ in E with no self intersections, such that*

$$\lim_{t\to-\infty} \gamma(t) = (\beta, \rho_1\beta + \rho_2) + \delta(\beta)(\mu_1, \mu_2) =: \tilde{\beta},$$

and $\gamma(\infty) := \lim_{t\to\infty} \gamma(t)$ belongs to $\Lambda_0 - \lambda_0$. Here E is the elastic region, ζ is given by (4.1), and Λ_0, λ_0 are defined in the paragraph below Remark 4.5.

Proof. The fact that $\gamma(\infty)$ can not belong to λ_0, or γ does not intersect itself, is a consequence of the strong maximum principle as argued before. Now let us show the existence of such a level curve. When $\epsilon > 0$ is small enough, as β is a strict local maximum, we have

$$\big(\beta \pm \epsilon, \rho_1(\beta \pm \epsilon) + \rho_2\big) + \delta(\beta)(\mu_1, \mu_2) \in E.$$

Thus the line

$$t \mapsto \big(\beta + \delta(\beta)\mu_1 + t\,,\ \rho_1\beta + \rho_2 + \delta(\beta)\mu_2 + \rho_1 t\big)$$

is tangent to the free boundary at $\tilde{\beta}$. Hence the unit vector tangent to the free boundary at $\tilde{\beta}$ is ζ.

Now, since $Du = Dd_p = \nu$ on the free boundary, we have $D_\nu u = |\nu|^2$ there. As ν is constant along λ, the derivative of $D_\nu u$ vanishes along the part of the free boundary that contains $\tilde{\beta}$. The same is true about the derivative of $D_\zeta u$ along that part of the free boundary, as $D_\zeta u$ is constant zero there. Therefore we have

$$D_\zeta D_\zeta u(\tilde{\beta}) = 0$$
$$D_\zeta D_\nu u(\tilde{\beta}) = 0.$$

But $D_\zeta u$ has harmonic continuation in a neighborhood of $\tilde{\beta}$, so if $x_i \in E$ converge to $\tilde{\beta}$, we have

$$D_\nu D_\zeta u(\tilde{\beta}) = \lim D_\nu D_\zeta u(x_i) = \lim D_\zeta D_\nu u(x_i) = D_\zeta D_\nu u(\tilde{\beta}) = 0.$$

Thus $DD_\zeta u(\tilde{\beta}) = 0$. Hence the level set of $D_\zeta u$ at $\tilde{\beta}$ must be the union of at least four arcs emanating from $\tilde{\beta}$ making equal angles with each other. Thus, there is at least one level curve starting at $\tilde{\beta}$ that remains in E. Now, similarly to Lemma 4.3, we can extend this level curve until it hits ∂E. □

REMARK 4.7 The conclusion of the above lemma is also true when β is a point of local minimum with $\delta(\beta) > 0$.

Now we state our main result in this section.

Theorem 4.8 *Let N_j, k be given by (4.6), (4.2) respectively. Then, each N_j is finite and*

$$N := \sum_{j=1}^{\tau} N_j \leqslant k.$$

Proof. Note that Lemmata 4.4 and 4.6 imply that each N_j is finite. Because no level curve of $\{D_\zeta u = 0\}$ can have both its endpoints on λ_0, as otherwise we have $D_\zeta u \equiv 0$.

Now consider the finite set of level curves $\tilde{\gamma}_i$ given by Lemma 4.3, that have both their endpoints on Λ_0. Let $\hat{\gamma}_j$'s be the parts of the other level curves that have both endpoints on $\tilde{\gamma}_i$'s, or one endpoint at them and the other one on Λ_0. Note that two level curves can not intersect at more than one point. Thus the number of $\hat{\gamma}_j$'s is finite. Also note that two level curves with one endpoint on λ_0 can not intersect.

Denote by E_1 the component of $E - \{\tilde{\gamma}_i, \hat{\gamma}_j\}$ which is attached to λ_0. The boundary of E_1 consists of λ_0 and part of Λ_0 together with parts of some $\tilde{\gamma}_i$'s and $\hat{\gamma}_j$'s. Let

$$\Lambda_1 := \overline{\partial E_1 - \lambda_0}.$$

Note that by our construction, any level curve in E_1 given by Lemma 4.3 must have one endpoint on λ_0. Let $\gamma_1, \cdots \gamma_N$ be the level curves given by Lemma 4.6, numbered as we move from V_1 to V_0. Then one endpoint of each γ_i is a strict local maximum point on the tilted graph of δ over λ which we call it β_i, and the other endpoint is on Λ_1 which we call it τ_i. Let $\mathfrak{D}_1, \cdots, \mathfrak{D}_{N+1}$ be the components of $E_1 - \{\gamma_i\}$. Note that $\bar{\gamma}_i \cap \bar{\gamma}_j = \emptyset$ when $i \neq j$.

Consider $\mathfrak{D}_i$, whose boundary consists of γ_{i-1}, γ_i and parts of λ_0, Λ_1, which we denote the latter two by $\lambda_{0i}, \Lambda_{1i}$. Note that γ_0, γ_{N+1} are empty. Suppose $1 < i < N+1$. First we claim that $D_\zeta u$ must change sign along Λ_{1i}. Otherwise we have for example $D_\zeta u \geqslant 0$ there. As $D_\zeta u$ vanishes on the other parts of $\partial\mathfrak{D}_i$, maximum principle implies

$$D_\zeta u > 0 \qquad \text{in } \mathfrak{D}_i .$$

Since near λ_0 we have $D_\zeta d_p = 0$, we get

$$D_\zeta(d_p - u) < 0$$

near λ_0 in $\mathfrak{D}_i$. This implies that δ is strictly increasing along the subset of λ_{0i} over which $\delta > 0$. To see this, just look at the behavior of $d_p - u$ on segments in the ζ direction starting on the free boundary. Hence we get a contradiction with β_{i-1} being a strict local maximum. Note that this argument also shows that $D_\zeta u$ must be positive on part of Λ_{11}, and negative on part of Λ_{1N+1}.

Let $i \neq 1, N+1$. Then consider the finite set of level curves of $D_\zeta u = 0$ in $\mathfrak{D}_i$. These level curves have one endpoint on λ_{0i} and one endpoint on Λ_{1i}, and do not intersect each other. Consider the one closest to γ_{i-1}, and let $\tilde{\mathfrak{D}}$ be the subdomain of $\mathfrak{D}_i$ that they enclose. Then $D_\zeta u$ must have one sign on $\tilde{\mathfrak{D}}$, since it can not vanish there, as there is no further level curve inside $\tilde{\mathfrak{D}}$. Thus we must have $D_\zeta u < 0$ on $\tilde{\mathfrak{D}}$. Since otherwise we get as before that δ is strictly increasing near and on the right of β_{i-1}, contradicting the fact that it is a local maximum. Hence $D_\zeta u$ must be negative on some part of Λ_1 near and on the right of τ_{i-1}. Similarly, $D_\zeta u$ must be positive on some part of Λ_1 near and on the left of τ_i.

Therefore, $D_\zeta u$ must change sign from positive to negative along Λ_1 at least N times. Finally note that as $D_\zeta u$ vanishes on $\tilde{\gamma}_i$'s and $\hat{\gamma}_j$'s, these sign changes are actually happening along Λ_0. Thus by Lemma 4.1 we get the desired result. ☐

We immediately get the following

Theorem 4.9 *Suppose U is a convex polygon. Then for any side S_j of ∂U there is at most one plastic component attached to it. Furthermore, the plastic component is given by*

$$\big\{ f(s) + t\mu(s) \;\mid\; s \in (a_j, b_j)\,,\; t \in \big(0, \delta(s)\big) \big\},$$

where $s_{j-1} < a_j < b_j < s_j$; and as before f parametrizes ∂U with $f((s_{j-1}, s_j)) = S_j$, μ is the inward p-normal, and $f + \delta\mu$ parametrizes the free boundary. In addition, there is $c_j \in (a_j, b_j)$ such that $\delta(s)$ is strictly increasing for $s \in (a_j, c_j)$ and strictly decreasing for $s \in (c_j, b_j)$.

Proof. Let ζ_j be the unit vector in the S_j direction. We only need to notice that since U is a convex polygon, $\nu \cdot \zeta_j$ is zero on at most one S_i for $i \neq j$. Thus it changes sign from positive to negative exactly once, and we have $k = 1$. Hence Theorem 4.8 gives the first part of the theorem. The second part of the theorem follows from analyticity of the tilted graph of δ. ☐

5. Reflection method

In this section we give an example of how to apply the reflection method used by Caffarelli et al. [7] to our problem. Let U be the rectangle

$$\big\{ (x_1, x_2) \;\mid\; |x_1| < a\,,\; |x_2| < b \big\}.$$

By Theorem 4.9, symmetry of γ_p, and symmetry of U, there are four plastic components

$$
\begin{aligned}
&P_1: \ |x_1| \leqslant \alpha\ ,\ -b \leqslant x_2 \leqslant -b + \phi(x_1),\\
&P_2: \ |x_2| \leqslant \beta\ ,\ -a \leqslant x_1 \leqslant -a + \psi(x_1),\\
&P_3: \ \text{the reflection of } P_1 \text{ with respect to the } x_1 \text{ axis},\\
&P_4: \ \text{the reflection of } P_2 \text{ with respect to the } x_2 \text{ axis}.
\end{aligned}
$$

Here ϕ, ψ are even functions. Let ρ be the reflection with respect to the bisector of ∂U at $(-a, -b)$, i.e.,

$$x_2 = x_1 + a - b.$$

Thus

$$\rho(x_1, x_2) = (x_2 - a + b, x_1 + a - b).$$

Theorem 5.1 *If $b < a$, then $\rho(P_2) \subset P_1$.*

Proof. Let

$$\mathfrak{D} := E \cap \big\{(x_1, x_2) \ |\ x_2 < x_1 + a - b\ ,\ -a < x_1 < -a + 2b\big\}.$$

Consider the function

$$w(x) := u\big(\rho(x)\big) - u(x)$$

in $\mathfrak{D}$. Since $\Delta u \geqslant -\eta$, and $\Delta u(x) = -\eta$ for $x \in E$, we have

$$\Delta w \geqslant 0$$

in $\mathfrak{D}$, noting that Laplacian is invariant under reflections.

The boundary of $\mathfrak{D}$ consists of parts of the lines $x_2 = -b$, $x_2 = x_1 + a - b$, $x_1 = -a + 2b$, and parts of $\Gamma_1, \Gamma_3, \Gamma_4$. Here Γ_i is the free boundary attached to P_i. Note that some of these parts can be empty. Also note that Γ_2 is on the other side of the line $x_2 = x_1 + a - b$, so it does not intersect $\partial\mathfrak{D}$.

Since u vanishes on ∂U, and ρ takes $x_2 = -b$ to $x_1 = -a$, $w = 0$ on it. The same is true on the line $x_2 = x_1 + a - b$, as it is fixed by ρ. Also as ρ takes $x_1 = -a + 2b$ to $x_2 = b$, for x on it we have

$$w(x) = 0 - u(x) \leqslant 0.$$

If $x \in \Gamma_1$ then $u(x) = d_p(x)$. But $d_p(\rho(x)) \leqslant d_p(x)$, since due to the symmetry of γ_p, $\rho(x)$ has the same p-distance to $x_1 = -a$ as x has to $x_2 = -b$. Thus

$$w(x) = u\big(\rho(x)\big) - d_p(x) \leqslant u\big(\rho(x)\big) - d_p\big(\rho(x)\big) \leqslant 0.$$

We can argue similarly when $x \in \Gamma_3$, noting that ρ decreases the p-distance to $x_2 = b$ over $\mathfrak{D}$. Finally when $x \in \Gamma_4$, we get the same result noting that the p-distance of $\rho(x)$ to $x_2 = b$ is less than the p-distance of x to $x_1 = a$, when $x \in \mathfrak{D}$.

Therefore, by the strong maximum principle

$$w(x) < 0 \qquad x \in \mathfrak{D}.$$

Note that if $w \equiv 0$ on $\bar{\mathfrak{D}}$, then we must have $u = 0$ on $x_1 = -a + 2b$ inside U, which is impossible.

Now suppose there is $x \in P_2$ such that $\rho(x) \notin P_1$. Then $\rho(x) \in \mathfrak{D}$. Thus

$$w\big(\rho(x)\big) = u(x) - u\big(\rho(x)\big) = d_p(x) - u\big(\rho(x)\big).$$

But $d_p(\rho(x)) \leqslant d_p(x)$ as the p-distance of $\rho(x)$ to $x_2 = -b$ equals the p-distance of x to $x_1 = -a$. Hence

$$0 > w\big(\rho(x)\big) \geqslant d_p\big(\rho(x)\big) - u\big(\rho(x)\big),$$

which contradicts $u \leqslant d_p$. □

REMARK 5.2 Since γ_p is not invariant under arbitrary reflections, the more general results proved in [7] using reflections does not necessarily hold here. Although some special cases can be proved similarly to the above, for example when a bisector of a triangle is parallel to one of the coordinate axes.

References

1. Azzam, A., Smoothness properties of bounded solutions of Dirichlet's problem for elliptic equations in regions with corners on the boundary. *Canad. Math. Bull.* **23** (1980), 213–226. Zbl0448.35022 MR0576102
2. Brezis, H. & Stampacchia, G., Sur la régularité de la solution d'inéquations elliptiques. *Bull. Soc. Math. France* **96** (1968), 153–180. Zbl0165.45601 MR0239302
3. Caffarelli, L. A. & Friedman, A., The free boundary for elastic-plastic torsion problems. *Trans. Amer. Math. Soc.* **252** (1979), 65–97. Zbl0426.35033 MR534111
4. Caffarelli, L. A. & Rivière, N. M., Smoothness and analyticity of free boundaries in variational inequalities. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **3** (1976), no. 2, 289–310. Zbl0363.35009 MR0412940
5. Caffarelli, L. A. & Rivière, N. M., The smoothness of the elastic-plastic free boundary of a twisted bar. *Proc. Amer. Math. Soc.* **63** (1977), 56–58. Zbl0362.35003 MR0521411
6. Caffarelli, L. A. & Rivière, N. M., The Lipschitz character of the stress tensor, when twisting an elastic plastic bar. *Arch. Rational Mech. Anal.* **69** (1979), 31–36. Zbl0399.73044 MR513957
7. Caffarelli, L. A., Friedman, A., & Pozzi, G., Reflection methods in the elastic-plastic torsion problem. *Indiana Univ. Math. J.* **29** (1980), 205–228. Zbl0462.58017 MR563207
8. Choe, H. J. & Shim, Y. S., On the variational inequalities for certain convex function classes. *J. Differential Equations* **115** (1995), 325–349. Zbl0809.49007 MR1310935
9. Choe, H. J. & Shim, Y. S., Degenerate variational inequalities with gradient constraints. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **22** (1995), 25–53. Zbl0830.49005 MR1315349
10. De Silva, D. & Savin, O., Minimizers of convex functionals arising in random surfaces. *Duke Math. J.* **151** (2010), 487–532. Zbl1204.35080 MR2605868
11. Friedman, A., *Variational Principles and Free-Boundary Problems*. Pure and Applied Mathematics. John Wiley & Sons, Inc., New York (1982). Zbl0564.49002 MR679313
12. Friedman, A. & Pozzi, G., The free boundary for elastic-plastic torsion problems. *Trans. Amer. Math. Soc.* **257** (1980), 411–425. Zbl0459.35093 MR552267
13. Gerhardt, C., Regularity of solutions of nonlinear variational inequalities with a gradient bound as constraint. *Arch. Rational Mech. Anal.* **58** (1975), 309–315. Zbl0338.49009 MR0385296
14. Gilbarg, D. & Trudinger, N. S., *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer-Verlag, Berlin (2001). Zbl1042.35002 MR1814364
15. Hynd, R. & Mawi, H., On Hamilton–Jacobi–Bellman equations with convex gradient constraints. *Interfaces and Free Boundaries* **18** (2016), 291–315. Zbl06659246 MR3563780

16. Jensen, R., Regularity for elastoplastic type variational inequalities. *Indiana Univ. Math. J.* **32** (1983), 407–423. MR697646
17. Safdari, M., The regularity of some vector-valued variational inequalities with gradient constraints. To appear in: *Comm. on Pure and Applied Analysis*.
18. Safdari, M., The free boundary of variational inequalities with gradient constraints. *Nonlinear Analysis: Theory, Methods & Applications* **123–124** (2015), 1–22. Zbl1321.49018 MR3353792
19. Safdari, M., Optimal regularity for variational problems with nonsmooth non-strictly convex gradient constraints. *preprint*, arXiv:1602.03425v2