Interfaces and Free Boundaries **19** (2017), 351–369
DOI 10.4171/IFB/385

A free boundary problem with log–term singularity

OLIVAINE S. DE QUEIROZ
Universidade Estadual de Campinas, IMECC, Departamento de Matemática, Rua Sérgio Buarque de Holanda, 651, Campinas, SP, Brazil. CEP 13083-859
E-mail: olivaine@ime.unicamp.br

HENRIK SHAHGHOLIAN
Department of Mathematics, Royal Institute of Technology, 10044 Stockholm, Sweden
E-mail: henriksh@math.kth.se

[Received 28 February 2016 and in revised form 23 May 2017]

We study a minimum problem for a non-differentiable functional whose reaction term does not have scaling properties. Specifically we consider the functional

$$\mathcal{J}(v) = \int_\Omega \Big(\frac{|\nabla v|^2}{2} - v^+(\log v - 1)\Big) dx \to \min$$

which should be minimized in some natural admissible class of non-negative functions. Here, $v^+ = \max\{0, v\}$. The Euler–Lagrange equation associated with $\mathcal{J}$ is

$$-\Delta u = \chi_{\{u>0\}} \log u,$$

which becomes singular along the free boundary $\partial\{u > 0\}$. Therefore, the regularity results do not follow from classical methods. Besides, the logarithmic forcing term does not have scaling properties, which are very important in the study of free boundary theory. Despite these difficulties, we obtain optimal regularity of a minimizer and show that, close to every free boundary point, they exhibit a super-characteristic growth like

$$r^2|\log r|.$$

This estimate is crucial in the study of analytic and geometric properties of the free boundary.

2010 Mathematics Subject Classification: Primary 5R35.

Keywords: Free boundary, regularity theory, logarithmic singularity, porosity.

1. Introduction

In this article we address optimal regularity for minimizers of the non-differentiable functional

$$\mathcal{J}(v) = \int_\Omega \left(\frac{|\nabla v|^2}{2} - v^+(\log v - 1)\right) dx \tag{1.1}$$

among functions in the class

$$\mathcal{K}_\varphi := \left\{v \in H^1(\Omega); v = \varphi \text{ on } \partial\Omega\right\},$$

for a fixed $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, $\varphi \geqslant 0$, where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. We are making the convention $\log = \ln$, that is, log is the natural logarithm, which means that $\log s = \log_e s = \ln s$, $s > 0$.

A minimizer u for $\mathcal{J}$ satisfies, at least formally, the following Euler-Lagrange equation:

$$-\Delta u = \chi_{\{u>0\}} \log u \quad \text{in } \Omega, \tag{1.2}$$

where χ_A is the characteristic function of the set A. We call

$$\mathfrak{F}(u) = \partial\{u > 0\} \cap \Omega \tag{1.3}$$

the *free boundary* of u. To the authors knowledge, a variant of Equation (1.2) was studied for the first time in [13]. In that paper existence and regularity properties of a *maximal solution* for a perturbed version of (1.2) with homogeneous Dirichlet boundary condition was studied by an approximating procedure where the singularity was removed, and the problem becomes a regular equation. Since in [13] it was used a sort of variational techniques, the regularity results could be adapted to our setting and any minimizers u of $\mathcal{J}$ should satisfy $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$, for any $0 < \alpha < 1$. Clearly this is not sharp and, hence, not sufficient to study fine analytic and geometric properties of the free boundary $\mathfrak{F}(u)$.

The first step in the study of free boundary problems is the optimal regularity of the solution (see, for instance, [14]). Essentially, sharp regularity implies optimal growth close to the free boundary and it is a consequence of the invariant scale of the equation. The main aspect of Equation (1.2) is that the logarithmic nonlinearity does not have a scale property and this fairly complicates the proof of estimates. Besides, a formal analysis suggests that, close to a free boundary point, that is, close to a point on $\mathfrak{F}(u)$, a solution of (1.2) exhibits a *supercharacteristic growth* like

$$r^2|\log r|. \tag{1.4}$$

In [12] Monneau & Weiss investigated the following obstacle-type problem:

$$-\Delta u = \chi_{\{u>0\}} \quad \text{in } \Omega \subset \mathbb{R}^n. \tag{1.5}$$

Solutions of (1.5) may grow like (1.4) close to some free boundary points and, by the cross–shaped example in [4], they are not necessarily of class $C^{1,1}$. This solution is also *completely unstable* in the sense that its second variation of the energy takes the values $-\infty$ ([12]). Apart from the references already cited, in [3] and [2] the authors introduce a new method for the study of singularities in (1.5). In this spirit, since the second derivative of solution of (1.2) is always unbounded, we call (1.2) a *highly unstable* free boundary problem. We will show that *every* free boundary point exhibits a supercharacteristic growth.

2. Existence and first *a priori* estimates

In this section we prove existence of minimizers for the functional $\mathcal{J}$ in the class $\mathcal{K}_\varphi$. We also show a uniform estimate in $L^\infty(\Omega)$. Let us start with the existence result.

Lemma 2.1 *Let $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, $\varphi \geqslant 0$. Then the functional $\mathcal{J}$ has a minimizer $u \in \mathcal{K}_\varphi$ and $u \geqslant 0$.*

Proof. Notice that the set of functions where we minimize $\mathcal{J}$ is non empty since $|\mathcal{J}(\varphi)| < \infty$. Let $u \in \mathcal{K}_\varphi$ be fixed. Then,

$$\mathcal{J}(u) \geqslant \frac{1}{2}\|\nabla u\|^2_{L^2} + \int_\Omega -u \log u \chi_{\{u>e\}} dx. \tag{2.1}$$

On the other hand, if $0 < a < 1$ is fixed, then $|-u\log u| \leqslant C_1 u^{1+a}$ for $u \geqslant e$ and some constant $C_1 = C_1(a) > 0$. Hölder inequality implies that

$$\left| \int_\Omega -u \log u \chi_{\{u>e\}} dx \right| \leqslant C_1 \|u\|_{L^2}^{1+a}. \tag{2.2}$$

Using (2.2) in (2.1) we get

$$\mathcal{J}(u) \geqslant \frac{1}{2}\|\nabla u\|_{L^2}^2 - C_1\|u\|_{L^2}^{1+a}. \tag{2.3}$$

The Poincaré inequality applied to $u-\varphi$ gives us the following:

$$\frac{1}{2}\|\nabla u\|_{L^2}^2 \geqslant \frac{C_p}{8}\|u-\varphi\|_{L^2}^2 - \frac{1}{2}\|\nabla \varphi\|_{L^2}^2,$$

for some constant $C_p = C_p(\Omega) > 0$. Since $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ we still can estimate

$$\frac{C_p}{8}\|u-\varphi\|_{L^2}^2 \geqslant C_2\|u\|_{L^2}^2 - C_3\|u\|_{L^2} + C_4\|\varphi\|_{L^2}^2.$$

Back to (2.3) we obtain

$$\mathcal{J}(u) \geqslant C_2\|u\|_{L^2}^2 - C_3\|u\|_{L^2} - C_1\|u\|_{L^2}^{1+a} - C_5,$$

where all the constants C_i depend just on n, Ω, a and φ. Thus, taking

$$-C_6 := \min\left\{C_2 t^2 - C_3 t - C_1 t^{1+a}\right\}$$

we will have

$$\mathcal{J}(u) \geqslant -C_6 - C_5$$

for every $u \in \mathcal{K}_\varphi$. Thus, the functional is bounded from bellow.

Let now $\{v_j\} \subset \mathcal{K}_\varphi$ be a minimizing sequence. We can proceed as in the proof of Theorem 3.1 in [11] to show that, up to a subsequence, there is $u \in H^1(\Omega)$, $u - \varphi \in H_0^1(\Omega)$, such that

$$v_j \to u \text{ weakly in } H^1(\Omega), \quad v_j \to u \text{ in } L^2(\Omega), \quad v_j \to u \text{ a.e. in } \Omega.$$

We also have the existence of a constant $a > 0$ such that

$$v_j^+ \log v_j + a \geqslant 0, \text{ for all } j \in \mathbb{N}.$$

Then, from lower semicontinuity of norms and Fatou's Lemma, we get

$$\int_\Omega |\nabla u|^2 dx \leqslant \liminf_{j\to\infty} \int_\Omega |\nabla v_j| dx,$$
$$\int_\Omega u^+ (\log u - 1) + a dx \leqslant \liminf_{j\to\infty} \int_\Omega v_j^+ (\log v_j - 1) + a dx.$$

Hence,

$$\mathcal{J}(u) \leqslant \liminf_{j\to\infty} \mathcal{J}(v_j),$$

and u is a minimizer. To see that $u \geqslant 0$, we just notice that $\|\nabla u^+\|_{L^2} \leqslant \|\nabla u\|_{L^2}$. □

Let us turn our attention to the L^∞-bound of a minimizer. We should mention that, since the logarithmic nonlinearity changes sign, the estimate is not trivial (as in the case of power-type nonlinearity, which is just a consequence of the Maximum Principle). As in [11], we use the machinery from [10], Chapter 2, Section 5. Here we strongly use the slow growth of the function $s \mapsto \log s$.

We first show a uniform bound in the L^2–norm.

Lemma 2.2 *Let u be a minimizer of (1.1). Then there is a constant $C = C(n, \Omega, \|\varphi\|_{H^1(\Omega)})$ such that*

$$\|u\|_{L^2(\Omega)} \leqslant C.$$

Proof. Since $u \in \mathcal{K}_\varphi$, we can use Poincaré's inequality to obtain:

$$\begin{aligned}\|u\|^2_{L^2(\Omega)} &\leqslant \left(\|u-\varphi\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)}\right)^2\\ &\leqslant \left(C_p\|\nabla u\|_{L^2(\Omega)} + C_p\|\nabla\varphi\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)}\right)^2\\ &\leqslant C_1\|\nabla u\|^2_{L^2(\Omega)} + C_2\|\varphi\|^2_{H_1(\Omega)}.\end{aligned} \tag{2.4}$$

The constants $C_1, C_2 > 0$ depend just on Ω. On the other hand, using that u is a minimizer we get

$$\frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} \leqslant \int_{\{u\geqslant e\}} u\log u\,dx + \mathcal{J}(\varphi).$$

We recall that, given $0 < a < 1$, there exists a constant $C_3 = C_3(a) > 0$ such that $\log u \leqslant C_3 u^a$, for $u \geqslant e$. Thus, applying Hölder inequality twice we have the following:

$$\begin{aligned}\frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} &\leqslant C_3\int_{\{u\geqslant e\}} u u^a\,dx + \mathcal{J}(\varphi)\\ &\leqslant C_3|\Omega|^{\frac{1-a}{2}}\|u\|_{L^2(\Omega)}\|u\|^a_{L^2(\Omega)} + \mathcal{J}(\varphi).\end{aligned} \tag{2.5}$$

Here and afterwards, $|E|$ denotes the Lebesgue measure of the set E.

Using (2.5) in (2.4) we obtain

$$\|u\|^2_{L^2(\Omega)} \leqslant C_4\|u\|^{1+a}_{L^2(\Omega)} + C_5,$$

for $C_i = C_i(n, a, \Omega, \varphi)$. Since $2 > 1 + a$, we necessarily have

$$\|u\|_{L^2(\Omega)} \leqslant C(n, a, \Omega, \varphi),$$

which proves the lemma. □

Lemma 2.3 *Let u be a minimizer of (1.1). Then there is a constant $C = C(n, \Omega, \varphi) > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} \leqslant C.$$

Proof. Let $u \in \mathcal{K}_\varphi$ be a fixed minimizer. For each $N \in \mathbb{N}$ we define

$$u_N := \begin{cases} u \text{ if } 0 \leqslant u \leqslant N,\\ N \text{ if } u > N.\end{cases}$$

and also $A_N := \{x \in \Omega; u(x) > N\}$. We are going to show that, for $N \geqslant N_0$ large enough (depending on $\|\varphi\|_{L^\infty(\Omega)}$), the following estimate holds:

$$\|\nabla u\|^2_{L^2(A_N)} \leqslant C\|u\|^{2a}_{L^2(A_N)}|A_N|^{1-\frac{2}{n}+\varepsilon}, \tag{2.6}$$

where

$$\varepsilon = \frac{4-an}{n}, \quad a < \frac{4}{n}.$$

Since $\|u\|^{2a}_{L^2(\Omega)} \leqslant C$ from Lemma 2.2 and

$$\|u\|_{L^1(A_{N_0})} \leqslant C\|u\|_{L^2(\Omega)}$$

by Hölder inequality, the boundedness will follow from Lemma 5.2 in Chapter 2 of [10].

Notice that

$$u = u_N \text{ in } A_N^c \cap \Omega, \quad u = N \text{ in } A_N.$$

Since u is a minimizer we obtain

$$\begin{aligned}\int_{A_N} \frac{1}{2}|\nabla u|^2 dx &= \int_{A_N} \frac{1}{2}\left(|\nabla u|^2 - |\nabla u_N|^2\right) dx = \int_{\Omega} \frac{1}{2}\left(|\nabla u|^2 - |\nabla u_N|^2\right) dx \\ &\leqslant \int_{A_N} -N(\log N - 1) + u(\log u - 1) dx \\ &\leqslant \int_{A_N} (u-N)^+ \log u dx,\end{aligned} \tag{2.7}$$

where we have used the Mean Value Theorem. Notice that this estimate holds as far as $N \geqslant N_0 > e$. Now we choose a constant $C_1 = C_1(a) > 0$ such that $\log u \leqslant Cu^a$ for $u \geqslant N_0$. The value $0 < a < 1$ will be fixed latter. Then, using Hölder inequality,

$$\int_{A_N} (u-N)^+ \log u dx \leqslant C_1\|u^a\|_{L^2(A_N)}\|(u-N)^+\|_{L^2(A_N)}.$$

Applying Hölder once more and Sobolev inequality we have

$$\begin{aligned}\|(u-N)^+\|_{L^2(A_N)} &\leqslant \|(u-N)^+\|_{L^{2^*}(A_N)}|A_N|^{\frac{1}{2}-\frac{1}{2^*}} \\ &= \|(u-N)^+\|_{L^{2^*}(\Omega)}|A_N|^{\frac{1}{2}-\frac{1}{2^*}} \\ &\leqslant \|\nabla u\|_{L^2(A_N)}|A_N|^{\frac{1}{2}-\frac{1}{2^*}},\end{aligned}$$

where 2^* is the critical Sobolev exponent. Furthermore,

$$\|u^a\|_{L^2(A_N)} \leqslant \|u\|^a_{L^2(A_N)}|A_N|^{\frac{1-a}{2}}.$$

Hence,

$$\int_{A_N} (u-N)^+ \log u dx \leqslant C_1\|u\|^a_{L^2(A_N)}\|\nabla u\|_{L^2(A_N)}|A_N|^{1-\frac{a}{2}-\frac{1}{2^*}}. \tag{2.8}$$

Now we use (2.8) in (2.7) followed by Young's inequality applied for $\delta > 0$ to obtain

$$\frac{1}{2}\|\nabla u\|^2_{L^2(A_N)} \leqslant \frac{C_2}{4\delta}\|u\|^{2a}_{L^2(A_N)}|A_N|^{2-a-\frac{2}{2^*}} + \delta\|\nabla u\|^2_{L^2(A_N)}.$$

Taking $\delta = 1/4$ we finally get

$$\|\nabla u\|^2_{L^2(A_N)} \leqslant 4C_2\|u\|^{2a}_{L^2(A_N)}|A_N|^{1-\frac{2}{n}+\varepsilon},$$

with

$$\varepsilon = \frac{4-an}{n},$$

which is strictly positive if we choose $a < 4/n$. Thus, we have obtained (2.6) and finished the proof of the lemma. □

Let us finish this section showing that a minimizer is a sort of subsolution.

Lemma 2.4 *If u is a minimizer of (1.1), then u satisfies*

$$-\Delta u \leqslant \chi_{\{u>0\}} \log u \quad \text{in } \Omega \tag{2.9}$$

in the weak sense.

Proof. Given $\varphi \in C_0^\infty(\Omega)$ and $0 < \varepsilon \leqslant 1$, we compute

$$\begin{aligned} 0 &\leqslant \mathfrak{J}(u-\varepsilon\varphi) - \mathfrak{J}(u) \\ &= -\varepsilon\int_\Omega \langle\nabla u, \nabla\varphi\rangle dx + \frac{\varepsilon^2}{2}\int_\Omega |\nabla\varphi|^2 dx + \int_\Omega u^+ L(u,0,0)dx - \int_\Omega (u-\varepsilon\varphi)^+ L(u,\varepsilon,\varphi)dx, \end{aligned} \tag{2.10}$$

where, to simplify the notation,

$$L(u,\varepsilon,\varphi) = \log(u-\varepsilon\varphi)^+ - 1.$$

Now,

$$\begin{aligned} &\int_\Omega u^+ L(u,0,0)dx - \int_\Omega (u-\varepsilon\varphi)^+ L(u,\varepsilon,\varphi)dx \\ &\qquad = \int_{\{u>0\}} u^+ L(u,0,0)dx - \int_{\{u>\varepsilon\varphi\}} (u-\varepsilon\varphi)^+ L(u,\varepsilon,\varphi)dx \\ &\qquad = \int_{\{u>0\}} u^+ L(u,0,0)dx - \int_{\{u>\varepsilon\varphi\}} uL(u,\varepsilon,\varphi)dx + \varepsilon\int_{\{u>\varepsilon\varphi\}} \varphi L(u,\varepsilon,\varphi)dx \\ &\qquad \leqslant \int_{\{u>0\}} u^+ L(u,0,0)dx - \int_{\{u>\varepsilon\varphi\}} uL(u,\varepsilon,\varphi)dx + \varepsilon\int_{\{u>\varepsilon\varphi\}} \varphi\log(u-\varepsilon\varphi)dx. \end{aligned}$$

Notice that the difference

$$\int_{\{u>0\}} u^+ L(u,0,0)dx - \int_{\{u>\varepsilon\varphi\}} uL(u,\varepsilon,\varphi)dx$$

goes to zero as $\varepsilon \to 0^+$ by the Dominated Convergence Theorem. Back to (2.10) we see that

$$\int_\Omega \langle \nabla u, \nabla\varphi\rangle dx - \frac{\varepsilon}{2}\int_\Omega |\nabla\varphi|^2 dx \leqslant \int_{\{u>\varepsilon\varphi\}} \varphi \log(u-\varepsilon\varphi)dx < \int_{\{u>\varepsilon\varphi\}} \varphi \log u dx. \tag{2.11}$$

The result now follows if we send $\varepsilon \to 0^+$ in (2.11). □

The following lemma shows that the right hand side of (2.9) is in L^1_{loc}.

Lemma 2.5 *If u is a minimizer of (1.1), then $\chi_{\{u>0\}} \log u \in L^1_{\mathrm{loc}}(\Omega)$.*

Proof. Let $K \subset\subset \Omega$ be an arbitrary compact set and $\zeta \in C_0^\infty(\Omega)$ with $0 \leqslant \zeta \leqslant 1$, $\zeta = 1$ in K be a cut-off function. For any small $\delta > 0$, since u is bounded, we have

$$\begin{aligned}
\int_{K\cap\{\delta\leqslant u\leqslant 1\}} |\log u| dx &\leqslant \int_{K\cap\{\delta\leqslant u\}} -\log u dx + C_1\\
&\leqslant -\int_{K\cap\{u>0\}} \log u dx + C_1\\
&\leqslant -\int_{K\cap\{u>0\}} \zeta \log u dx - \int_{(\Omega\setminus K)\cap\{0<u\leqslant 1\}} \zeta \log u dx + C_1\\
&\leqslant -\int_{\Omega\cap\{u>0\}} \zeta \log u dx + C_2\\
&\leqslant -\int_\Omega \langle\nabla u, \nabla\zeta\rangle dx + C_2 < \infty.
\end{aligned}$$

Taking the limit as $\delta \to 0^+$ and using Fatou Lemma we obtain the estimate. □

3. Sharp estimates close to the free boundary

In this section we prove the supercharacteristic growth (1.4) of a minimizer. The main idea is to use Harnack type inequality in order to control the growth of the solution close to the free boundary (where the log does not change sign), from above. The control of the growth from bellow is obtained with a modification of a classical argument using the Maximum Principle and a nice subsolution.

3.1 *Controlling the growth of local averages*

The goal of this section is to prove estimates related with the growth of averages of minimizers. As a consequence, we obtain the Harnack type inequality in the next section. Roughly speaking, we show that minimizers with large averages in some ball are positive at the center (with some exact control). This idea was first introduced by Alt and Phillips in [1] and then explored in [13] in order to show $C^{1,\alpha}$ regularity for (1.2). The lack of scale complicates considerably the achievement of the result in our situation. An important step is the construction of a local subsolution allowed by the next general lemma. This should be compared with Lemma 3.1 in [13].

Lemma 3.1 *Let $u \in H^{1/2}(\partial B_r)$ and $0 < r \leqslant r_0$, for some $r_0 > 0$ to be fixed. There are constants $\theta_0, c_1, c_0 > 0$, both depending only on n and r_0, such that, if*

$$\theta \geqslant \theta_0 \quad \text{and} \quad ⨍_{\partial B_r} u d\sigma \geqslant c_0\theta,$$

then we can find a function $w \in H^1(B_r)$ *satisfying*

$$\begin{cases} -\Delta w \leqslant \log w - \theta \text{ in } B_r, \\ w(x) \geqslant c_1 \delta(x) \fint_{\partial B_r} u d\sigma, \\ w = u \text{ on } \partial B_r, \end{cases}$$

where $\delta(x) = \operatorname{dist}(x, \partial B_r)$. *Besides, there exists a constant* $C > 0$, *depending just on* n, r_0 *and* $\|u\|_{L^\infty(B_{r_0})}$, *such that*

$$\|w\|_{L^\infty(B_r)} \leqslant C.$$

REMARK 3.2 In Lemma 3.1 from [13] the construction of u is possible once you have a lower bound like $C_1(C_2 + \theta)$ to the average at the boundary. Here we obtain a sharp result in the sense that c_0 is a universal constant. This is crucial in the study of the sharp regularity. Although the proof is very similar to the one in [13], we give all the details here to show the control of the constants.

Proof of Lemma 3.1. Let H be the harmonic extension of u in B_r, that is, H solves the following equation:

$$\begin{cases} -\Delta H = 0 & \text{in } B_r, \\ H = u & \text{on } \partial B_r. \end{cases}$$

From [7], Lemma 9.1, we have

$$H(x) \geqslant C\delta(x)m, \text{ for all } x \in B_r, \tag{3.1}$$

where $m := \fint_{\partial B_r} u d\sigma$ and $\delta(x) = \operatorname{dist}(x, \partial B_r)$. The constant C does not depend on r, but just on n. For a fixed exponent $0 < \gamma < 1$, let v be the solution of

$$\begin{cases} -\Delta v = \delta^{-\gamma}(x) & \text{in } B_r, \\ v = 0 & \text{on } \partial B_r. \end{cases}$$

Then, it is known that $v \in C^{1,1-\gamma}(\overline{B}_r)$ (see the proof of Theorem 1.1 in [8] or Lemma 2.1 in [6]). Therefore, there is a constant $M > 0$ depending just on γ and n such that

$$v(x) \leqslant M\delta(x) \text{ in } B_r. \tag{3.2}$$

We set

$$w := H - kv,$$

where

$$k := \frac{Cm}{2M}.$$

From (3.2) and (3.1) we have

$$w(x) \geqslant \delta(x)\,(Cm - kM) = \delta(x)\frac{Cm}{2}. \tag{3.3}$$

Taking into account the definition of m, we obtain the bound from bellow for w. Let us check that w is a subsolution. Since

$$-\log w \leqslant w^{-\gamma} \quad \text{for any} \quad 0 < \gamma < 1, \tag{3.4}$$

we just need to show that

$$\Delta w \geqslant w^{-\gamma} + \theta,$$

if $m \geqslant c_0\theta$, $\theta \geqslant \theta_0$, for θ_0 and c_0 sufficiently large.

We set $c_1 = C/2$ and take a constant $a_0 > 0$ which will be fixed soon. Assuming that $m \geqslant c_0\theta$, using (3.3) and the definition of k we estimate:

$$\begin{aligned} w^{-\gamma}(x) + a_0\theta &\leqslant \delta^{-\gamma}(x)\left(\frac{Cm}{2}\right)^{-\gamma} + a_0\theta \\ &\leqslant \delta^{-\gamma}(x)\left(\left(\frac{Cc_0\theta}{2}\right)^{-\gamma} + a_0\theta\right) \\ &\leqslant \delta^{-\gamma}(x)\left(\left(\frac{Cc_0}{2}\right)^{-\gamma}\theta^{-\gamma} + a_0\frac{2kM}{Cc_0}\right). \end{aligned}$$

Defining

$$a_0 = \frac{Cc_0}{4M},$$

we have

$$w^{-\gamma}(x) + a_0\theta \leqslant \delta^{-\gamma}(x)\left(\left(\frac{Cc_0}{2}\right)^{-\gamma}\theta_0^{-\gamma} + \frac{k}{2}\right).$$

Now, we first fix c_0 such that $a_0 \geqslant 1$ and notice that it depends just on n and γ. Then we choose θ_0 sufficiently large so that

$$\left(\frac{Cc_0}{2}\right)^{-\gamma}\theta_0^{-\gamma} + \frac{k}{2} \leqslant k.$$

Hence:

$$w^{-\gamma}(x) + \theta \leqslant w^{-\gamma}(x) + a_0\theta \leqslant k\delta^{-\gamma}(x) = \Delta w(x) \quad \text{in } B_r.$$

Thus, from (3.4),

$$\Delta w(x) \geqslant -\log w(x) + \theta,$$

and the lemma is proved. ☐

In the next lemma we justify in what sense the function w in Lemma 3.1 is a local subsolution.

Lemma 3.3 *Given $\theta > 0$, let us define the modified functional*

$$\mathfrak{J}_\theta(v) = \int_{B_1}\left(\frac{|\nabla v|^2}{2} - v^+(\log v - 1) + \theta v\right)dx,$$

for $v \in H_0^1(B_1) + \varphi$, for some $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, $\varphi \geqslant 0$. Let u be a minimizer of $\mathfrak{J}_\theta$, $u = \varphi$ on ∂B_1. Assume also that w is the function from Lemma 3.1 satisfying $w = u$ on ∂B_1. There is a constant $\theta_0 = \theta_0(n, \varphi) > 0$ such that, if $\theta \geqslant \theta_0$, then $w \leqslant u$ in B_1.

REMARK 3.4 Existence and *a priori* bounds for minimizers of $\mathcal{J}_\theta$ follow exactly as in Section 2.

Proof of Lemma 3.3. Let $z = \max\{u, w\}$. Then z is an admissible function and, by the minimality,

$$0 \geqslant \mathcal{J}_\theta(u) - \mathcal{J}_\theta(z).$$

We compute:

$$\begin{aligned}\mathcal{J}_\theta(u) - \mathcal{J}_\theta(z) =& \frac{1}{2}\int_\Omega (z-u)\Delta(u+z)dx + \int_\Omega z^+(\log z - 1) - u^+(\log u - 1) + \theta(u-z)dx \\ \geqslant& \frac{1}{2}\int_{\{z>u>0\}} (z-u)(-\log u - \log z + 2\theta)dx + \frac{1}{2}\int_{\{z>u=0\}} (-z\log z + z\theta)dx \\ &+ \int_{\{z>u>0\}} z(\log z - 1) - u(\log u - 1) + \theta(u-z)dx.\end{aligned}$$

Let us define, for $s \geqslant t$, the auxiliary functions:

$$\psi(s) = t(\log t - 1) - s(\log s - 1) + \theta(s-t),$$
$$\phi(s) = \frac{1}{2}(s-t)(-\log t - \log s + 2\theta).$$

Notice that $\phi(t) = \psi(t) = 0$ and $\phi'(s) \geqslant \psi'(s) - 1/2$. Hence,

$$\phi(s) - \psi(s) \geqslant \frac{1}{2}(t-s).$$

Thus,

$$\begin{aligned}\mathcal{J}_\theta(u) - \mathcal{J}_\theta(z) &\geqslant \int_{\{z>u>0\}} \phi(z)dx + \frac{1}{2}\int_{\{z>u=0\}} (-z\log z + z\theta)dx + \int_\Omega -\psi(z)dx \\ &\geqslant +\frac{1}{2}\int_{\{z>u=0\}} (-z\log z + z\theta)dx + \frac{1}{2}\int_{\{z>u>0\}} u - z dx \\ &\geqslant +\frac{1}{2}\int_{\{z>u=0\}} (-z\log z + z(\theta-1))\,dx + \frac{1}{2}\int_{\{z>u>0\}} u dx.\end{aligned}$$

Notice that z is *a priori* bounded in L^∞ for a constant depending just on boundary data, since w and u satisfy this property. Therefore, we can take θ_0 large enough such that, if $\theta \geqslant \theta_0$ and $z > 0$, then

$$-z\log z + z(\theta-1) \geqslant z(-\log C + \theta_0 - 1) > 0,$$

and this implies

$$\mathcal{J}_\theta(u) - \mathcal{J}_\theta(z) > 0.$$

But this is a contradiction, which gives us $u = z$. □

We are in position to prove the main result of this section.

Lemma 3.5 *Let u be a minimizer of* (1.1) *in $B_r(x_0)$, $x_0 \in \Omega$, r and r_0 be fixed satisfying $r \in]0, r_0[$, with r_0 depending only on n and φ. There are constants $c_0, c_1 > 0$, depending just on n, r_0 and φ such that, if*

$$\fint_{\partial B_r(x_0)} u d\sigma \geqslant 2c_0 r^2 |\log r|,$$

then,

$$u(x) \geqslant c_1 r^{-1} \operatorname{dist}\big(x, \partial B_r(x_0)\big) \fint_{\partial B_r(x_0)} u\, d\sigma, \text{ for almost every } x \in B_r(x_0).$$

Proof. Translating to the origin we may assume $x_0 = 0$. Then we scale u as

$$u_r(y) := r^{-2} u(ry), \quad y \in B_1.$$

Using polar coordinates and change of variables we have

$$\fint_{\partial B_1} u_r d\sigma = r^{-2} r^{-n+1} \int \partial B_r u d\sigma = r^{-2} \fint_{\partial B_r} u d\sigma.$$

On the other hand, after some calculus we see that,

$$\mathcal{J}(u) = r^{n+2} \mathcal{J}_\theta(u_r), \quad \theta = -\log r^2,$$

where $\mathcal{J}$ is minimized in B_r and $\mathcal{J}_\theta$ is minimized in B_1. In particular, u_r is a minimum of $\mathcal{J}_\theta$. Let us fix c_0 from Lemma 3.1 and then r_0 small enough such that $\theta = -\log r^2$ satisfies Lemma 3.3. Notice that r_0 depends only on n and φ. Applying Lemma 3.1 with $u = u_r$ in B_1 we obtain w such that, from Lemma 3.3,

$$u_r(y) \geqslant w(y) \geqslant c_1 \delta(y) \fint_{\partial B_1} u_r d\sigma, \tag{3.5}$$

if

$$r^{-2} \fint_{\partial B_r} u d\sigma \geqslant c_0(-\log r^2).$$

Scaling back to B_r the inequality (3.5) we obtain the result. □

3.2 *Harnack type inequality*

The first lemma of this section states that a minimizer is continuous on its positivity set.

Lemma 3.6 *Let u be local minimizer of (1.1). Then the set $\{u > 0\}$ is open and, up to redefinition on a set of zero measure, $u \in C(\Omega)$.*

Proof. Using Lemma 3.5 we just need to adapt Corollary 3.6 in [13] and the proof is the same. □

Now we prove the Harnack type inequality.

Theorem 3.7 *There are positive constants c_0, c_1 and r_0 depending only on n and φ, such that, if u is a local minimizer of (1.1) in $B_r \subset \Omega$, $r \leqslant r_0$, and*

$$\fint_{\partial B_r} u d\sigma \geqslant 2c_0 r^2 |\log r|,$$

then

$$\sup_{B_{r/4}} u \leqslant C(\inf_{B_{r/4}} u + r^{1+\mu}), \quad 0 < \mu < 1,$$

for some constant $C = C(n, \Omega, \varphi, \mu) > 0$. *In addition, if*

$$C \inf_{B_{r/4}} u \leqslant \frac{1}{2},$$

then, for every small $\rho > 0$ *such that* $B_\rho \subset \{u > 0\}$ *we have*

$$\sup_{x \in B_\rho} u \leqslant C \inf_{x \in B_\rho} u.$$

Proof. First we recall that, from Lemma 3.5, we have

$$u(x) \geqslant C_1 \fint_{\partial B_r} u \, d\sigma, \quad x \in B_{r/4}. \tag{3.6}$$

We split u as $u = u_1 + u_2$, where

$$\begin{cases} -\Delta u_1 = \chi_{\{0<u\leqslant 1\}} \log u & \text{in } B_r, \\ \quad u_1 = u & \text{on } \partial B_r, \end{cases}$$

and

$$\begin{cases} -\Delta u_2 = \chi_{\{u>1\}} \log u & \text{in } B_r, \\ \quad u_2 = 0 & \text{on } \partial B_r. \end{cases}$$

Since the function u_1 is subharmonic, we have

$$u_1(x) \leqslant C_2 \fint_{\partial B_r} u_1 \, d\sigma = C_2 \fint_{\partial B_r} u \, d\sigma.$$

Furthermore, $u_2 \in C^{1,\mu}(\overline{B_r})$ and estimates for Hölder continuous functions (see [13, p. 313]) imply that

$$u_2(x) \leqslant C_3 r^{1+\mu}.$$

Then, using (3.6) we obtain

$$\sup_{B_{r/4}} u \leqslant C_2 \fint_{\partial B_r} u_1 \, d\sigma + C_3 r^{1+\mu} \leqslant C_4 \inf_{B_{r/4}} u + C_3 r^{1+\mu},$$

and the first part of the lemma is proved.

Now, assuming that

$$C \inf_{B_{r/4}} u \leqslant \frac{1}{2},$$

we see from the first part that, if $\rho > 0$ is small, then

$$\sup_{B_\rho} u \leqslant \frac{1}{2} + C_3 \rho^{1+\mu} \leqslant 1.$$

Hence, $u_2 = 0$ and u is subharmonic in B_ρ. As in the first part, this implies the result. □

REMARK 3.8 The constant C from Theorem 3.7 depends also on $\mu > 0$. But this does not create a problem since we shall fix it when we use it.

REMARK 3.9 From the continuity of the minimizers we see that $\{u > 0\}$ is open. Then the hypothesis $C \inf_{B_{r/4}} u \leqslant 1/2$ is always true for a ball centered closed to a free boundary point and with sufficiently small r.

We now present the first consequence of Theorem 3.7.

Lemma 3.10 *Let u be a minimizer of (1.1) in B_r such that*

$$C \inf_{B_{r/4}} u \leqslant \frac{1}{2},$$

with the constant $C > 0$ as in Theorem (3.7). Then there is a constant $r_0 > 0$ depending only on n, such that, if $r \leqslant r_0$, then

$$\sup_{Br/4} u \leqslant C(\inf_{B_{r/4}} u + r^2|\log r|).$$

In particular, if $x_0 \in \partial\{u > 0\} = \mathfrak{F}(u)$ and B_r is centered in x_0, then

$$u(x) \leqslant C|x - x_0|^2 \big|\log|x - x_0|\big|, \quad x \in B_{r/4}(x_0).$$

Proof. Suppose first that

$$\fint_{\partial B_r} u d\sigma \geqslant 2c_0 r^2|\log r|, \tag{3.7}$$

with c_0 the constant of Theorem 3.7. Then,

$$\sup_{B_{r/4}} u \leqslant C \inf_{B_{r/4}} u, \tag{3.8}$$

for $r \leqslant r_0$ sufficiently small.

On the other hand, if (3.7) is not true for a certain $r > 0$, we split $u = u_1 + u_2$ as in the proof of Theorem 3.7 in this particular ball to obtain

$$\sup_{B_{r/4}} u \leqslant C_2 \fint_{\partial B_r} u_1 d\sigma + C_3 r^{1+\mu} = C_3 \fint_{\partial B_r} u d\sigma + C_3 r^{1+\mu} \leqslant C_4 r^2|\log r| + C_3 r^{1+\mu}.$$

Decreasing r_0 if necessary we obtain that $u \leqslant 1$ in B_r. Here we see that the smallness of r_0 depends only on C, which depends only in n, φ and Ω. Again, as in the proof of Theorem 3.7 we obtain that u is subharmonic, which means:

$$u(x) \leqslant C \fint_{\partial B_r} u d\sigma \leqslant C r^2|\log r|, \quad x \in B_{r/4}, \tag{3.9}$$

for a constant C depending only on the quantities already mentioned.

Combining (3.8) and (3.9) we have

$$\sup_{B_{r/4}} u \leqslant C\big(\inf_{B_{r/4}} u + r^2|\log r|\big).$$

For the last part, assuming $x_0 \in \mathfrak{F}(u)$ we have $\inf_{B_{r/4}} u = 0$. Taking $x \in \{u > 0\}$ and $\rho = |x - x_0|$, ρ sufficiently small, we have

$$u(x) \leqslant \sup_{B_\rho} u \leqslant C\rho^2 |\log \rho| = C|x - x_0|^2 \big|\log|x - x_0|\big|.$$

This is the desired result. □

3.3 *Sharp regularity and non-degeneracy*

The purpose of this section is to establish optimal Log–Lipschitz regularity for the gradient of a minimizer.

Theorem 3.11 *Let u be a minimizer of (1.1) and $\Omega' \subset\subset \Omega$. There exist $r_0 > 0$ and a constant $C > 0$ depending both only on* $\operatorname{dist}(\Omega', \partial\Omega)$, *$\varphi$ and n such that, if $x \in \Omega'$ with* $\operatorname{dist}(x, \partial\{u > 0\}) \leqslant r_0$, *then*

$$\big|\nabla u(x)\big|^2 \leqslant C u(x) \left(\log \frac{1}{u(x)} - 1\right). \tag{3.10}$$

In particular, if $d(x) = \operatorname{dist}(x, \partial\{u > 0\})$, then

$$\big|\nabla u(x)\big| \leqslant C d(x) \log \frac{1}{d(x)}, \tag{3.11}$$

if $x \in \Omega'$ and $d(x) \leqslant r_0$.

Proof. We denote by C_i, $i = 1, \ldots, 7$, constants depending only on the quantities in the hypothesis. From Lemma 3.10 we know that we can find $r_1 > 0$ such that, if $\operatorname{dist}(x, \partial\{u > 0\}) \leqslant r_1$ then $u(x) \leqslant 1/2$. We then fix

$$r_2 := \min\big\{r_1, \operatorname{dist}(\Omega', \partial\Omega)\big\}$$

and $x \in \Omega' \cap \{u > 0\}$ with $\operatorname{dist}(x, \partial\{u > 0\}) \leqslant r_2$. Recalling that u is continuous we have, by compactness of $\partial\{u > 0\} \cap \overline{\Omega}'$, the existence of a point $x_0 \in \partial\{u > 0\}$ such that

$$|x - x_0| = \operatorname{dist}\big(x, \partial\{u > 0\}\big).$$

Now we decrease again r_2. Let r_3 be fixed small enough such that, whenever we have $\operatorname{dist}(x, \partial\{u > 0\}) \leqslant r_3$, then

$$-\log|x - x_0| \geqslant 0, \quad -C|x - x_0|\log|x - x_0| \leqslant \frac{1}{e} \quad \text{and} \quad |x - x_0| \leqslant 1/2,$$

where $C > 0$ is the constant from Lemma 3.10. Notice that r_3 can be fixed independent of x. Then let $r_0 := \min\{r_2, r_3\}$ and assume $\operatorname{dist}(x, \partial\{u > 0\}) \leqslant r_0$.

From Lemma 3.10 we have

$$u(x) \leqslant -C|x - x_0|^2 \log|x - x_0| \leqslant \frac{1}{e}|x - x_0| \leqslant 1/2.$$

Thus,

$$-\log u(x) \geqslant -\log|x - x_0| + 1$$

and also

$$\frac{u(x)}{-C\big(\log u(x)+1\big)} \leqslant |x-x_0|^2.$$

Let us define

$$\rho := \left(\frac{u(x)}{-\sqrt{2}C\big(\log u(x)+1\big)}\right)^{1/2}. \tag{3.12}$$

Then

$$\rho \leqslant \frac{|x-x_0|}{2},$$

which means that $B_\rho(x) \subset \{u>0\} \cap \Omega$. By the Harnack inequality from Theorem 3.7 we have

$$0 < \sup_{B_{\rho/4}(x)} u \leqslant C_1 \inf_{B_{\rho/4}(x)} u \leqslant C_1 u(x).$$

As in [1], from elliptic estimates we know that

$$|\partial_i u(y)| \leqslant C_2\Big(\rho \sup_{B_{\rho/4}(x)} |\log u| + \rho^{-1} \sup_{B_{\rho/4}(x)} u\Big), \quad y \in B_{\rho/8}(x), \quad i=1,\dots,n. \tag{3.13}$$

But, from the definition of ρ and Harnack inequality,

$$\rho^{-1} \sup_{B_{\rho/4}(x)} u \leqslant C_3\left(u(x)\Big(\log\frac{1}{u(x)}-1\Big)\right)^{1/2}.$$

On the other hand, since $s \mapsto -\log s$ is decreasing, the Harnack inequality once more implies the following:

$$\rho \sup_{B_{\rho/4}(x)} |\log u| = -\rho \log(\inf_{B_{\rho/4}} u) \leqslant -\rho\log\left(\frac{u(x)}{C}\right) = -\rho\log u(x) + \rho C_4,$$

for some $C_4>0$. Again using the definition of ρ :

$$\begin{aligned}\rho \sup_{B_{\rho/4}(x)} |\log u| &\leqslant C_5\left(u(x)\left(\log\frac{1}{u(x)}-1\right)\right)^{1/2}\frac{1-\log u(x)}{-1-\log u(x)}\\ &\leqslant C_6\left(u(x)\left((\log\frac{1}{u(x)}-1\right)\right)^{1/2}.\end{aligned}$$

Back to (3.13) we obtain

$$|\partial_i u(y)|^2 \leqslant C_7 u(x)\left(\log\frac{1}{u(x)}-1\right), \quad y \in B_{\rho/8}(x), \quad i=1,\dots,n.$$

Applying this inequality for x we obtain (3.10).

Now we prove (3.11). Notice first that the function $s \mapsto -s\log s$ is increasing for $0<s\leqslant s_0$, for some $s_0>0$. Then choose $r_0>0$ such that, if $d(x)\leqslant r_0$ then

$$Cd^2(x)\log\frac{1}{d(x)} \leqslant s_0,$$

where $C > 0$ is the constant from Lemma 3.10. Using (3.10) and again Lemma 3.10 we estimate

$$
\begin{aligned}
|\nabla u(x)|^2 &\leqslant C u(x) \log \frac{1}{u(x)} \\
&\leqslant -C_8 d^2(x) \left(\log \frac{1}{d(x)}\right)^2 \left(\frac{1}{-\log d(x)} - 2 - \frac{\log\left(-\log d(x)\right)}{\log d(x)}\right) \\
&\leqslant C_9 d^2(x) \left(\log \frac{1}{d(x)}\right)^2 .
\end{aligned}
$$

This finishes the proof. □

REMARK 3.12 The gradient estimate (3.10) does make sense only when $u(x)$ is small, that is, close to the free boundary. Lemma 3.11 should be compared with Lemma 8.5 of [12]. But the proof there uses potential-theoretic arguments and the fact that $\Delta u \in L^\infty(\Omega)$ is crucial. We do not have this hypothesis here.

Corollary 3.13 *Let u be a minimizer of (1.1), $\Omega' \subset\subset \Omega$ and $x_0 \in \mathcal{F}(u) \cap \Omega'$. There exist $r_0 > 0$ and a constant $C > 0$ depending both only on* $\operatorname{dist}(\Omega', \partial\Omega)$, *$\varphi$ and n such that*

$$
|\partial_{ij} u(x)| \leqslant C\left(|\log u(x) + 1|\right), \quad \text{in } B_r(x_0) \cap \{u > 0\}, \quad 0 < r < r_0. \tag{3.14}
$$

Proof. If $x \in B_r(x_0) \cap \Omega'$, then u is smooth in a neighborhood of x and, as before,

$$
\left|\partial_{ij} u(x)\right| \leqslant C \left(\rho \sup |\Delta \partial_j u| + \rho^{-1} \sup |\partial_j u|\right),
$$

with $\rho > 0$ as in (3.12). Using the definition of ρ (for some convenient C) and the equation that $\partial_j u$ satisfies on $B_r(x_0) \cap \{u > 0\}$ we obtain the estimate. □

We prove now the non-degeneracy result.

Lemma 3.14 *Let u be a minimizer of $\mathcal{J}$ in $\mathcal{K}_\varphi$ and $x_0 \in \partial\{u > 0\} \cap \Omega$. Then we have the inequality*

$$
\sup_{\partial B_r(x_0)} u \geqslant C r^2 |\log r|,
$$

for some constant $C > 0$ depending only n, provided $B_r(x_0) \subset\subset \Omega$ and $0 < r < r_0$, for some r_0 depending only on Ω and φ.

Proof. From Lemma 3.10 we have that

$$
\sup_{B_r(x_0)} u \leqslant C r^2 |\log r|, \quad 0 < r < r_0,
$$

for a certain $r_0 > 0$ depending only on the quantities in the hypothesis. Decreasing r_0 if necessary we can assume

$$
C r |\log r| \leqslant 1.
$$

Thus,

$$
\sup_{B_r(x_0)} u \leqslant r.
$$

Defining $\lambda_r := -\log r$ we have, in the weak sense,

$$-\Delta u = \log u \leqslant -\lambda_r.$$

By Caffarelli's non-degeneracy lemma (see [5] or [14, Section 3.1]), we obtain

$$\sup_{\partial B_r(x_0)} u \geqslant C r^2 \lambda_r = C r^2 |\log r|,$$

for a constant C depending only on dimension. □

As a corollary we have the non-degeneracy of the gradient.

Corollary 3.15 *Under the same hypothesis of Lemma 3.14 the following inequality holds:*

$$\sup_{B_r(x_0)} |\nabla u| \geqslant C r |\log r|,$$

for a positive constant C depending only on dimension n.

Proof. Choose $y \in B_r(x_0)$ with $u(y) > 0$. Applying Mean Value Theorem to u restricted to the line segment connecting x_0 and y we obtain

$$\big|u(x_0) - u(y)\big| \leqslant \big|\nabla u(z)\big| |x_0 - y|,$$

for some z on the line segment. It follows that

$$\big|\nabla u(z)\big| \geqslant \frac{u(y)}{|x_0 - y|} \geqslant u(y)/r.$$

Taking the sup on $B_r(x_0)$ we obtain the result from the non-degeneracy of u. □

4. Porosity of the free boundary

In this section we start the study of fine properties of the free boundary. To formulate our first result in this direction, we recall the definition of porosity. A set $E \subset \mathbb{R}^n$ is called *porous* with *porosity constant* δ, $0 < \delta \leqslant 1$, if there is a $r_1 > 0$ such that, for each $x \in E$ and $0 < r < r_1$ there is a point y such that $B_{\delta r}(y) \subset B_r(x) \setminus E$. The set E is called *locally porous* in an open set Ω if $E \cap K$ is porous (with possibly different porosity constants) for any $K \subset\subset \Omega$. For more details the reader is invited to see Section 3.2 of [14]. Here we follow closely [9].

Proposition 4.1 *Let u be a minimizer of (1.1). Then, for every compact set $K \subset \Omega$ we have that $\partial\{u > 0\} \cap K$ is porous with porosity constant depending only on n, Ω, $\mathrm{dist}(K, \partial\Omega)$ and φ. In particular,*

$$\big|\partial\{u > 0\} \cap K\big| = 0$$

for any $K \subset\subset \Omega$.

Proof. Let $x \in \{u > 0\} \cap K$ and define $d(x) = \mathrm{dist}(x, \partial\{u > 0\})$. We fix $z_x \in \partial\{u > 0\}$ such that $|z_x - x| = d(x)$. Let

$$\tau := \min\big\{r_0, \mathrm{dist}(K, \partial\Omega)\big\}$$

where r_0 is the constant from Lemma 3.10. Then we assume that $d(x) \leqslant \tau/4$. The same lemma gives us

$$u(x) \leqslant C_1|x - z_x|^2 \log \frac{1}{|x - z_x|} = C_1 d^2(x) \log \frac{1}{d(x)}. \tag{4.1}$$

Now let $z \in \partial\{u > 0\} \cap K$. Since $0 < r < \tau/4$, Lemma 3.14 implies the existence of $x_z \in \partial B_r(z)$ such that

$$u(x_z) \geqslant C_2 r^2 \log \frac{1}{r}.$$

Using (4.1) we have

$$C_2 r^2 \log \frac{1}{r} \leqslant u(x_z) \leqslant C_1 d^2(x_z) \log \frac{1}{d(x_z)}. \tag{4.2}$$

On the other hand, the function $s \mapsto -s \log s$ is increasing in the interval $(0, s_0)$ for some s_0. Let

$$r_1 := \min\{\tau/4, s_0\}.$$

Then, if $0 < r < r_1$, we have $d(x_z) \leqslant r < r_1$. Thus,

$$d^2(x_z) \log \frac{1}{d(x_z)} \leqslant d(x_z) r \log \frac{1}{r}$$

Back to (4.2) we see that

$$d(x_z) \geqslant \frac{C_2}{C_1} r.$$

Defining $\delta = C_2/C_1$ we see that $\delta \leqslant 1$. The proof of the proposition now can be done with the same argument of Theorem 3.1 in [9]. □

REMARK 4.2 Once we have local porosity of the free boundary $\partial\{u > 0\}$, we easily obtain the density of $\{u > 0\}$ as in [14]. In fact, for each $x^0 \in \{u > 0\}$ we have

$$\frac{\left|B_r(x^0) \cap \{u > 0\}\right|}{|B_r|} \geqslant \gamma \tag{4.3}$$

for a constant $\gamma > 0$ depending only on n, Ω, $\operatorname{dist}(K, \partial\Omega)$ and φ, provided $B_r(x^0) \subset \Omega$.

Acknowledgments. OSQ was partially supported by CNPq-Brazil and FAPESP/SP-Brazil under the grant 2012/20197-0. H. Shahgholian was partially supported by Swedish Research Council. Both authors would like to thank the referees for their valuable comments and suggestions.

References

1. Alt, H.W. & Phillips, D., A free boundary problem for semilinear elliptic equations. *J. Reine Angew. Math.* **368** (1986), 63–107. Zbl0598.35132 MR0850615
2. Andersson, J., Shahgholian, H. & Weiss, G.S., The singular set of higher dimensional unstable obstacle type problems. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **24** (2013), 123–146. Zbl1302.35459 MR3034585

3. Andersson, J., Shahgholian, H. & Weiss, G. S., Uniform regularity close to cross singularities in an unstable free boundary problem. *Comm. Math. Phys.* **296** (2010) 251–270. Zbl1197.35088 MR2606634
4. Andersson, J. & Weiss, G. S., Cross-shaped and degenerate singularities in an unstable elliptic free boundary problem. *J. Differential Equations.* **228** (2006), 633–640. Zbl1136.35104 MR2289547
5. Caffarelli, L. A., The regularity of free boundaries in higher dimensions. *Acta Math.* **139** (1977), 155–184. Zbl0386.35046 MR0454350
6. Dávila, J., Global regularity for a singular equation and local H^1 minimizers of a nondifferentiable functional. *Commun. Contemp. Math.* **6** (2004), 165–193. Zbl1048.35022 MR2048779
7. Dávila, J. & Montenegro, M., Nonlinear problems with solutions exhibiting a free boundary on the boundary. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22** (2005), 303–330. Zbl1083.35139 MR2136246
8. Gui, C. & Lin, F. H.: Regularity of an elliptic problem with a singular nonlinearity. *Proc. Roy. Soc. Edinburgh Sect. A.* **123** (1993), 1021–1029. Zbl0805.35032 MR1263903
9. Karp, L., Kilpeläinen, T., Petrosyan, A. & Shahgholian, H., On the porosity of free boundaries in degenerate variational inequalities. *J. Differential Equations* **164** (2000), 110–117. Zbl0956.35054 MR1761419
10. Ladyzhenskaya, O. A. & Ural'tseva, N. N., *Linear and Quasilinear Elliptic Equations.* Mathematics in Science and Engineering, 46, Academic Press, New York, 1968. Zbl0164.13002 MR0244627
11. Leitão, R., de Queiroz, O. S. & Teixeira, E. V., Regularity for degenerate two-phase free boundary problems. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **32** (2015), 741–762. Zbl06476998 MR3390082
12. Monneau, R. & Weiss, G. S., An unstable elliptic free boundary problem arising in solid combustion. *Duke Math. J.* **136** (2007), 321–341. Zbl1119.35123 MR2286633
13. Montenegro, M. & de Queiroz, O. S., Existence and regularity to an elliptic equation with logarithmic nonlinearity. *J. Differential Equations.* **246** (2009), 482–511. Zbl1155.35045 MR2468725
14. Petrosyan, A., Shahgholian, H. and Uraltseva, N., *Regularity of free boundaries in obstacle-type problems.* Graduate Studies in Mathematics, **136**. American Mathematical Society, Providence, RI, 2012. Zbl1254.35001 MR2962060
15. Yang, R., Optimal regularity and nondegeneracy of a free boundary problem related to the fractional Laplacian. *Arch. Ration. Mech. Anal.* **208** (2013), 693–723. Zbl1282.35422 MR3048593