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Some results on anisotropic fractional mean curvature flows

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We show the consistency of a threshold dynamics type algorithm for the anisotropic motion by fractional mean curvature, in the presence of a time dependent forcing term. Beside the consistency result, we show that convex sets remain convex during the evolution, and the evolution of a bounded convex set is uniquely defined.

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1. Introduction

In this paper we study the evolution of a hypersurface by anisotropic fractional mean curvature with the addition of a time-dependent forcing term. Such nonlocal evolutions have been first considered in [4, 20], where existence and comparison of weak solutions is proved, by suitably adapting the viscosity theory to (geometric) nonlocal equations. These results have been later extended in [12] to more general (yet translation–invariant) equations. We point out that an existence and uniqueness result for smooth solutions is still not available, even if some results in this direction can be found in [27].

In [10], the authors prove the convergence to the (isotropic) motion by fractional mean curvature of a threshold dynamics scheme, analogous to the one introduced in [26] in the local case. They also consider more singular cases, where the limiting equation is a local mean curvature flow (this would correspond to the cases $s \in [1, 2)$ below). A related result is found in [13]: there the authors study the asymptotic behaviour of an equation modelling dislocations (involving kernels similar to the ones we introduce below (2.2), but for $s = 1$), and recover in the limit a local anisotropic curvature flow.

In this paper we extend the main result in [10] in the nonlocal cases to the anisotropic case, and to the presence of an external driving force. More precisely, we consider a slightly modified scheme defined by an anisotropic convolution followed by a thresholding, in the spirit of what was proposed in [22] in the local case, and we show the convergence of the scheme to a viscosity solution of a geometric equation, at least when this solution is unique (which is generally the case [20]). The limit equation is a flow by anisotropic fractional curvature with a forcing term. Such curvature corresponds to the first variation of an anisotropic fractional perimeter of the form introduced in [25].

We then prove that our scheme is convexity preserving, so that as a consequence also the limit geometric evolution preserves convexity. This is a well-known property of the (anisotropic) mean curvature flow (see [1, 6, 18, 19]), but was not previously known in the fractional case, both isotropic and anisotropic. Let us detail a bit this point. The discrete motion we consider roughly consists in selecting a level set of the function obtained by means of the convolution of a given set with a given function: the Green kernel of the limiting PDE we want to approach. Thus to get that convexity is preserved during the scheme, we need to show that the convolution of our kernel with a convex set is a level set convex function. This in turn can be done by applying some classical results in convex geometry (see Section 4.) Eventually, once this point is settled, we deduce that convex evolutions are necessarily unique.

The plan of the article is as follows: in Section 2 we introduce the geometric flow, the discrete approximation scheme and we recall some definitions, in particular that of viscosity solution (previously introduced in [4, 12, 20]). In Section 3 we establish the convergence of the scheme to a viscosity solution. This is done in Theorem 3.1 and Proposition 3.5. In Section 4, building upon known results on convex bodies [17], we show that the discrete scheme preserves the convexity of a set, and, as a consequence, also the level set equation results convexity preserving. In Section 5, and in particular in Proposition 5.1, we show that the limit motion can be obtained by alternating curvature motions without forcing term, and evolutions with the forcing term only. This technical result allows to estimate easily the relative evolution of two sets with different forcing terms. Thanks to this estimate, we can deduce in Section 6 the geometric uniqueness of convex evolutions. Eventually, in Section 7, we state some final considerations and open problems.

2. Preliminaries and the time-discrete scheme

2.1 *The scheme and the limit equation*

Let $\mathfrak{n} : \mathbb{R}^N \to \mathbb{R}$ be a norm (that is a convex, even, one-homogeneous function), with in particular

$$\underline{c}|x| \leqslant \mathfrak{n}(x) \leqslant \overline{c}|x| \tag{2.1}$$

for every $x \in \mathbb{R}^N$, where $\overline{c}$ and $\underline{c}$ are suitable positive constants. Given $s \in (0,1)$ and $h > 0$, we let throughout the paper $\sigma_h = h^{\frac{s}{1+s}}$ and define the kernels

$$P(x) := \frac{1}{1+\mathfrak{n}(x)^{N+s}} \qquad \text{and} \qquad P_h(x) := \frac{1}{\sigma_h^{N/s}} P\left(\frac{x}{\sigma_h^{1/s}}\right) \quad \text{for } h > 0\,, \tag{2.2}$$

so that $\sigma_h^{-1} P_h$ converges to $\mathfrak{n}^{-(N+s)}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$, as $h \to 0$.

For a measurable set $E \subset \mathbb{R}^N$ and $g \in C^0(\mathbb{R}_+)$, where $\mathbb{R}_+ := [0, +\infty)$, we consider the scheme

$$T_{g(nh),h}(E) := \left\{P_h * (\chi_E - \chi_{E^c}) > g(nh) h^{\frac{s}{1+s}}\right\}. \tag{2.3}$$

Given a closed set E_0, we wish to study the limit, as $h \to 0$, of the iterates

$$T_{g(nh),h} T_{g((n-1)h),h} \ldots T_{g(h),h}(E_0). \tag{2.4}$$

This scheme is a nonlocal variant of the celebrated Merriman–Bence–Osher scheme [26], in a form which has been studied in [10] in the context of fractional curvature flows, and in [22] (see also [11, 24]) in the context of convolution-generated motions with a forcing term. The limit, as we shall see, satisfies the equation of a nonlocal anisotropic mean curvature flow with forcing term, which we will introduce below.

Adopting the notation of [10], as we will do in the whole paper, we define inductively a function $u_h : \mathbb{R}^N \times \{nh\}_{n\in\mathbb{N}} \to \mathbb{R}$ as follows:

$$u_h(\cdot, 0) = \widetilde{\chi}_{E_0} := \chi_{E_0} - \chi_{\mathbb{R}^N\setminus E_0},$$
$$u_h(\cdot, (n+1)h) = \widetilde{\chi}_{\left\{P_h * u_h(\cdot,nh) \geq g(nh)h^{\frac{s}{1+s}}\right\}}.$$

The function u_h is then extended to $\mathbb{R}^N \times \mathbb{R}_+$ by letting $u_h(x,t) = u_h(x, [t/h]h)$ for $t \geqslant 0$, where $[\cdot]$ denotes the integer part.

When $\mathfrak{N} = |\cdot|$ and $g = 0$, it is proved in [10] that as $h \to 0$, u_h converges to the geometric solution of the *fractional curvature flow* defined in [20], at least when no "fattening" occurs. We shall extend this result to a more general setting, that is, with arbitrary norm $\mathfrak{N}$ and a time varying (continuous) forcing term g. The equation which is solved in the limit is a "level-set" equation (an equation which describes the geometric motion of the level sets of a function) which must be understood in the viscosity sense, the precise definition will be given in Section 2.2 below. In our setting the limit solves the following level-set equation:

$$\partial_t u = \mathfrak{A}(Du)|Du| \left(-\kappa_s\left(x, \{u \geqslant u(x,t)\}\right) + g(t)\right), \tag{2.5}$$

where for $p \neq 0$,

$$\mathfrak{A}(p) = \left(2\int_{p^\perp} P(y)\, d\mathcal{H}^{N-1}(y)\right)^{-1} \tag{2.6}$$

and for a smooth set E, the anisotropic fractional mean curvature at $x \in \partial E$ is given by

$$-\kappa_s(x, E) := \int_{\mathbb{R}^N} \frac{\chi_E(y) - \chi_{E^c}(y)}{\mathfrak{N}(y-x)^{N+s}}\, dy \tag{2.7}$$

(where here the "–" sign is so that convex sets have a nonnegative curvature). Here, as in the rest of the paper, we denote with $D\cdot$ the spatial derivative. This singular integral can be given a meaning, and shown to be finite for $C^{1,1}$ sets, see [20].

Following [4, 20], in order to define the right notion of solution we need to introduce the following integral functionals, which extend the definition of the curvature of the level set of a function $v : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$:

$$\overline{I}_A[v](x,t) = \int_A \Big(\chi^+\big(v(y+x,t) - v(x,t)\big) - \chi^-\big(v(y+x,t) - v(x,t)\big)\Big) P(y)\, dy,$$
$$\underline{I}_A[v](x,t) = \int_A \Big(\chi_+\big(v(y+x,t) - v(x,t)\big) - \chi_-\big(v(y+x,t) - v(x,t)\big)\Big) P(y)\, dy,$$

with the notation $\chi^+ = \chi_{[0,\infty)}$, $\chi^- = \chi_{(-\infty,0)}$, $\chi_+ = \chi_{(0,\infty)}$, $\chi_- = \chi_{(-\infty,0]}$.

REMARK 2.1 If $\varphi \in L^\infty(\mathbb{R}^N \times [0,\infty)) \cap C^2(B_\delta(x,t))$ for some $(x,t) \in \mathbb{R}^N \times [0,\infty)$, then the functions $\overline{I}_{B_\delta(x)}[\varphi]$ and $\underline{I}_{B_\delta(x)}[\varphi]$ are pointwise continuous outside the set $\{D\varphi = 0\}$, in the sense that if $D\varphi(x,t) \neq 0$, then

$$\lim_{(y,s)\to(x,t)} \overline{I}_{B_\delta(y)}[\varphi](y,s) = \overline{I}_{B_\delta(x)}[\varphi](x,t), \quad \lim_{(y,s)\to(x,t)} \underline{I}_{B_\delta(y)}[\varphi](y,s) = \underline{I}_{B_\delta(x)}[\varphi](x,t).$$

If φ is just upper semicontinuous (respectively lower semicontinuous), then $\overline{I}_{B_\delta(\cdot)}[\varphi]$ is upper semicontinuous (respectively $\underline{I}_{B_\delta(\cdot)}[\varphi]$ is lower semicontinous). Moreover, if φ_k is a sequence of functions pointwisely converging to φ, then

$$\limsup_{k\to\infty} \overline{I}_A[\varphi_k] \leqslant \overline{I}_A[\varphi], \qquad \liminf_{k\to\infty} \underline{I}_A[\varphi_k] \geqslant \underline{I}_A[\varphi]$$

for every set $A \subseteq \mathbb{R}^N$.

REMARK 2.2 Observe that if $\varphi \in C^2(\mathbb{R}^N \times [0,\infty))$ and the level set $\{\varphi(\cdot,t) = \varphi(x,t)\}$ is not critical, then for any A, $\overline{I}_A[\varphi](x,t) = \underline{I}_A[\varphi](x,t)$ and we can denote

$$I[\varphi](x,t) = \overline{I}_{\mathbb{R}^N}[\varphi](x,t) = -\kappa_s\big(x, \{\varphi(\cdot,t) \geqslant \varphi(x,t)\}\big).$$

2.2 *Viscosity solutions*

The precise meaning of a solution of Equation (2.5) is given by one the following equivalent definitions (see [4, 20] and [12]) of *viscosity solutions*:

DEFINITION 2.3 A locally bounded upper semicontinuous function u is a viscosity subsolution of (2.5) if for all $\varphi \in C^2(\mathbb{R}^N \times (0,\infty))$, at any maximum point (x,t) of $u - \varphi$, then

$$\begin{cases} \partial_t\varphi(x,t) \leqslant \mathfrak{a}\big(D\varphi(x,t)\big)\big|D\varphi(x,t)\big|\,\big(-\kappa_s\big(x,\{\varphi \geqslant \varphi(x,t)\}\big) + g(t)\big) \\ \qquad\qquad \text{if } D\varphi(x,t) \neq 0 \text{ and } \varphi(x,t) \text{ is not a critical value of } \varphi, \\ \partial_t\varphi(x,t) \leqslant 0 \quad \text{if } D\varphi(x,t) = 0. \end{cases} \tag{2.8}$$

A locally bounded lower semicontinuous function u is a viscosity supersolution if $-u$ is a viscosity subsolution with forcing term $-g$. A solution is a function whose upper semicontinuous envelope is a subsolution, while its lower semicontinuous envelope is a supersolution.

DEFINITION 2.4 A locally bounded upper semicontinuous function u is a viscosity subsolution of (2.5) if for all $\varphi \in C^2(\mathbb{R}^N \times (0,\infty))$, at any maximum point (x,t) of $u - \varphi$ in a ball $B_\delta(x,t)$, it holds

$$\begin{cases} \partial_t\varphi(x,t) \leqslant \mathfrak{a}\big(D\varphi(x,t)\big)\big|D\varphi(x,t)\big|\,\big(\overline{I}_{B_\delta(x)}[\varphi](x,t) + \overline{I}_{\mathbb{R}^N\setminus B_\delta(x)}[u](x,t) + g(t)\big) \\ \qquad\qquad \text{if } D\varphi(x,t) \neq 0, \\ \partial_t\varphi(x,t) \leqslant 0 \quad \text{otherwise}. \end{cases} \tag{2.9}$$

A locally bounded lower semicontinuous function u is a viscosity supersolution of (2.5) if for all $\varphi \in C^2(\mathbb{R}^N \times (0,\infty))$ at any minimum point (x,t) of $u - \varphi$ and for any ball $B_\delta(x,t)$ it holds

$$\begin{cases} \partial_t\varphi(x,t) \geqslant \mathfrak{a}\big(D\varphi(x,t)\big)\big|D\varphi(x,t)\big|\,\big(\underline{I}_{B_\delta(x)}[\varphi](x,t) + \underline{I}_{\mathbb{R}^N\setminus B_\delta(x)}[u](x,t) + g(t)\big) \\ \qquad\qquad \text{if } D\varphi(x,t) \neq 0, \\ \partial_t\varphi(x,t) \geqslant 0 \quad \text{otherwise}. \end{cases} \tag{2.10}$$

A solution is a function whose upper semicontinuous envelope is a subsolution, while its lower semicontinuous envelope is a supersolution.

Observe that in the definition above, one can take δ arbitrarily small; moreover as usual, we may equivalently assume that the maximum (resp. minimum) points are strict.

DEFINITION 2.5 Let $C \subset \mathbb{R}^N$ and g a continuous function and define for $\eta > 0$ $d_C^\eta = -\eta \vee (\eta \wedge (\mathrm{dist}(x, C) - \mathrm{dist}(x, C^c)))$, that is d^η is the signed distance function to ∂C truncated at the levels $\pm\eta$. We say that a family of sets $\{C(t)\}_{t>0}$ is a flow for the geometric equation $\mathfrak{a}^{-1} v = -\kappa_s + g$ starting from C if for all $t \geqslant 0$, $C(t) = \{x \in \mathbb{R}^N : u(x,t) > 0\}$, or if for all $t \geqslant 0$, $C(t) = \{x \in \mathbb{R}^N : u(x,t) \geqslant 0\}$, where u solves (2.5) in the sense of Definition 2.4, with initial data $u(\cdot, 0) = d^\eta$.

REMARK 2.6 It turns out that in this case, $\chi_{\{u>0\}}$ is a subsolution, while $\chi_{\{u\geqslant 0\}}$ is a supersolution, in the sense of Definition 2.4. Moreover, it is well known [20] that the equation in Definitions 2.3 and 2.4 is *geometric*, meaning that if we replace the initial condition with any function u_0 with the same level sets $\{u > 0\}$ and $\{u \geqslant 0\}$, the evolution $C(t)$ remains the same.

Existence and comparison (uniqueness) results for evolutions defined by the equivalent Definitions 2.3 and 2.4 are provided in [20]. It follows, as usual, that given a bounded uniformly continuous initial data u_0, and denoting $u(x,t)$ the solution with $u(\cdot, 0) = u_0$, then, starting from almost all (but a countable number, at most) of the level sets $C = \{u_0 > s\}$ there exists a unique flow $C(t) = \{u(\cdot, t) > s\}$, in the sense of Definition 2.5.

3. Convergence of the discrete flows

3.1 *Main result*

The scope of this section is to prove the following result, which is a variant of the main result in [10]. The only differences are that:

1. we introduce an anisotropy and a forcing term in the spirit of [22];
2. we simplify part of the argument, in particular when estimating the "mobility" $\mathfrak{a}(p)$;
3. we estimate in a separate subsection (Sec. 3.2) the evolution of balls, yielding a then simpler argument to show consistency in flat regions, or that the initial condition is not lost in the limit;
4. we give a proof (Sec. 3.4) of the convexity of the mobility.

Theorem 3.1 *Let u_0 be a bounded, uniformly continuous function, $\Omega = \{u_0 > 0\}$, $\Omega_t^- = \{x \in \mathbb{R}^N : u(x,t) > 0\}$ and $\Omega_t^+ = \{x \in \mathbb{R}^N : u(x,t) \geqslant 0\}$, where u is a viscosity solution of (2.5) with initial data u_0. Then*

$$\begin{cases} \liminf_* u_h(x,t) := \liminf_{y\to x, nh\to t} u_h(y, nh) = 1 \ \textit{in}\ \Omega_t^-, \\ \limsup^* u_h(x,t) := \limsup_{y\to x, nh\to t} u_h(y, nh) = -1 \ \textit{in}\ \mathbb{R}^N \setminus \Omega_t^+. \end{cases}$$

REMARK 3.2 Under the assumptions of Theorem 3.1, suppose that u is such that $\partial\{(x,t) : u(x,t) > 0\} = \partial\{(x,t) : u(x,t) \geqslant 0\} =: \Gamma_t$. Then we have

$$F_h := \cup_n \partial\big\{u_h(\cdot, nh) = h^{\frac{s}{1+s}} g(nh)\big\} \times \{nh\} \ \to\ \cup_{t\geqslant 0} \Gamma_t \times \{t\} \qquad \text{as } n \to \infty,$$

in the Kuratowski sense.

3.2 *The speed of balls*

A useful intermediate result (which illustrates why the scale in (2.2) is the right one) is a control on the (bounded) speed at which balls decrease with the discrete flow. Let

$$\overline{k} = \max_{x \in \partial B_1} \kappa_s(x, B_1) \tag{3.1}$$

be the maximal curvature of the unit ball. Then for any $R > 0$ and $x \in \partial B_R$, by a change of variable of the form $y' = y/R$ we get that

$$\kappa_s(x, B_R) = -\int \frac{\chi_{B_R}(y) - \chi_{B_R^c}(y)}{\mathfrak{N}(y)^{N+s}} dy = -\frac{1}{R^s} \int \frac{\chi_{B_1}(y') - \chi_{B_1^c}(y')}{\mathfrak{N}(y')^{N+s}} dy' \leqslant \frac{\overline{k}}{R^s}.$$

This suggests that the motion of a ball should be at most governed by $\dot{R} \sim -\overline{k}/R^s - \|g\|_\infty$, yielding an extinction time of order $\sim R^{1+s}$ for R small. We check now this is indeed the case for the discrete scheme.

We start with the following lemma which estimates the speed of the scheme applied to the unit ball:

Lemma 3.3 *There exists $\overline{m}$, depending only $\mathfrak{N}$, and $h_0 > 0$ (depending on $\mathfrak{N}$ and $\|g\|_\infty$) such that if $h < h_0$, $P_h * (\chi_{B(0,1)} - \chi_{B(0,1)^c}) \geqslant \|g\|_\infty h^{\frac{s}{1+s}}$ in $B(0, 1 - ch)$ where $c = (\overline{k} + \|g\|_\infty)/\overline{m}$.*

Proof. We let $e = (1, 0, \dots, 0)$ a unit vector and denote $B = B(e, 1)$. We recall

$$P_h(x) = \frac{1}{h^{N/(1+s)}} \frac{1}{1 + \mathfrak{N}\left(\frac{x}{h^{1/(1+s)}}\right)^{N+s}} = h^{\frac{s}{1+s}} \frac{1}{h^{\frac{N+s}{1+s}} + \mathfrak{N}(x)^{N+s}},$$

so that

$$\lim_{h \to 0} h^{\frac{-s}{1+s}} P_h * (\chi_B - \chi_{B^c})(0) = -\kappa_s(0, \partial B).$$

More precisely, it stems from the convexity of B that this limit is, in fact, an infimum. Indeed, using the symmetry of $\mathfrak{N}$, one sees that the convolution $h^{-\frac{s}{1+s}} P_h * (\chi_B - \chi_{B^c})(0)$ is given by

$$-\int_{\mathbb{R}^N \setminus (B \cup (-B))} \frac{dx}{h^{\frac{N+s}{1+s}} + \mathfrak{N}(x)^{N+s}}$$

which is monotone in h, and it follows

$$h^{-\frac{s}{1+s}} P_h * (\chi_B - \chi_{B^c})(0) \geqslant -\overline{k} \tag{3.2}$$

for any $h > 0$.

Then, we need to estimate $DP_h * (\chi_B - \chi_{B^c})$ near ∂B. In fact it is enough to have an estimate for $x = \bar{t}e$ with $|\bar{t}| \lesssim c(h)$ for some $c(h) \gg h$. A simple analysis shows that the scaling of $\bar{t}$ should be between h and $h^{1/(1+s)} \gg h$, in what follows we therefore consider $\bar{t} = \tau h^{\frac{1+s/2}{1+s}}$ for $-1 \leqslant \tau \leqslant 1$. One has

$$G := h^{\frac{-s}{1+s}} e \cdot DP_h * (\chi_B - \chi_{B^c})(\bar{t}e) = 2 \int_{\partial B} \frac{-\nu_B \cdot e}{h^{\frac{N+s}{1+s}} + \mathfrak{N}(\bar{t}e - x)^{N+s}} d\mathcal{H}^{N-1}(x).$$

With the change of variable $x = yh^{1/(1+s)}$, we have, denoting $h^{-1/(1+s)}B$ the ball $B(h^{-1/(1+s)}e, h^{-1/(1+s)})$,

$$G = -2\int_{\partial(h^{\frac{-1}{1+s}}B)} \frac{h^{\frac{N-1}{1+s}}\nu_B\cdot e}{h^{\frac{N+s}{1+s}}(1+\mathfrak{N}(\tau h^{\frac{s/2}{1+s}}e-y)^{N+s})}d\mathcal{H}^{N-1}(y)$$
$$= -\frac{2}{h}\int_{\partial(h^{\frac{-1}{1+s}}B)} \frac{\nu_B\cdot e}{1+\mathfrak{N}(\tau h^{\frac{s/2}{1+s}}e-y)^{N+s}}d\mathcal{H}^{N-1}(y).$$

Given $R>0$ let $B^h := B(0, Rh^{-(N-1)/((N+s)(1+s))})$: first observe that if $y\notin B^h$,

$$|\tau h^{\frac{s/2}{1+s}}e-y| \geqslant Rh^{-\frac{N-1}{(N+s)(1+s)}} - h^{\frac{s/2}{1+s}} \geqslant \frac{R}{2}h^{-\frac{N-1}{(N+s)(1+s)}}$$

if h is small enough, so that

$$\left|2\int_{\partial(h^{\frac{-1}{1+s}}B)\setminus B^h} \frac{\nu_B\cdot e}{1+\mathfrak{N}(\tau h^{\frac{s/2}{1+s}}e-y)^{N+s}}d\mathcal{H}^{N-1}(y)\right| \leqslant h^{-\frac{N-1}{1+s}}\frac{1}{1+\underline{c}\big(\frac{R}{2}h^{-\frac{N-1}{(N+s)(1+s)}}\big)^{N+s}}$$
$$\leqslant \frac{1}{h^{\frac{N-1}{1+s}}+\underline{c}\big(\frac{R}{2}\big)^{N+s}}$$

which can be made arbitrarily small by choosing R large enough. On the other hand, if $y\in B^h\cap\partial(h^{-1/(1+s)}B)$, as $h^{-1/(1+s)} \gg Rh^{-(N-1)/((N+s)(1+s))}$ (indeed $1/(1+s)-(N-1)/((N+s)(1+s)) = (N+s-N+1)/((N+s)(1+s)) = 1/(N+s) > 0$), if h is small enough one has $\nu_B\cdot e\leqslant -1/2$, which yields

$$-2\int_{\partial(h^{\frac{-1}{1+s}}B)\cap B^h} \frac{\nu_B\cdot e}{1+\mathfrak{N}(\tau h^{\frac{s/2}{1+s}}e-y)^{N+s}}d\mathcal{H}^{N-1}(y)$$
$$\geqslant \int_{\partial(h^{\frac{-1}{1+s}}B)\cap B^h}\frac{d\mathcal{H}^{N-1}(y)}{1+\overline{c}\big|\tau h^{\frac{s/2}{1+s}}e-y\big|^{N+s}} \to \int_{\{y\cdot e=0\}}\frac{d\mathcal{H}^{N-1}(y)}{1+\overline{c}|y|^{N+s}} =: 2\overline{m} > 0$$

as $h\to 0$. All in all, with an appropriate choice of R, we see that there exists h_0 (depending on $\mathfrak{N}$ and $\|g\|_\infty$ but not, in fact, on the particular point we have chosen on ∂B) such that if $h<h_0$, one has, recalling (3.2)

$$h^{\frac{-s}{1+s}}P_h*(\chi_B-\chi_{B^c})(0)\geqslant -\overline{k},$$
$$h^{\frac{-s}{1+s}}e\cdot DP_h*(\chi_B-\chi_{B^c})(\overline{t}e)\geqslant \frac{\overline{m}}{h}$$

for $|\overline{t}|\leqslant h^{\frac{1+s/2}{1+s}}$. Hence if $-\overline{k}+\overline{t}\overline{m}/h\geqslant\|g\|_\infty$ and $|\overline{t}|\leqslant h^{\frac{1+s/2}{1+s}} = h\cdot h^{-(s/2)/b(1+s)}$, we have $P_h(\chi_B-\chi_{B^c})(\overline{t}e)\geqslant h^{-s/(1+s)}\|g\|_\infty$. Choosing $\overline{t}=h(\|g\|_\infty+\overline{k})/\overline{m}$, and possibly reducing h_0, we check that indeed $|\overline{t}|\leqslant h^{\frac{1+s/2}{1+s}}$ and the inequality holds. We will show in Corollary 4.3 that the level sets of $P_h(\chi_B-\chi_{B^c})$ are all convex, so that the thesis of the Lemma holds. □

Thanks to a simple scaling argument, we find that given $R>0$, if h is small enough, $P_h*(\chi_{B(0,R)}-\chi_{B(0,R)^c})\geqslant\|g\|_\infty h^{\frac{s}{1+s}}$ in $B(0,R-ch)$ with now $c=(\overline{k}/R^s+\|g\|_\infty)/\overline{m}$. As a result, we have the following corollary:

Corollary 3.4 *If $E_0 = B(x_0, R)$, then for h small enough, $u_h(\cdot, nh) \geqslant \widetilde{\chi}_{B(x_0,R/2)}$ as long as*

$$nh \leqslant \frac{\overline{m}R^{1+s}}{2(R^s\|g\|_\infty + 2^s\overline{k})}. \tag{3.3}$$

In particular, if R is small, one has $u_h(x_0, nh) = 1$ for $nh \lesssim R^{1+s}$.

3.3 *The consistency result*

The main difficulty to prove convergence is a consistency result. The strategy of the proof follows [3, 10], with some slight simplification. The important point is to show that $\overline{u} := \limsup^* u_h$ and $\underline{u} := \liminf_* u_h$ are respectively viscosity supersolution and subsolution of (2.5).

Proposition 3.5 *The functions $\liminf_* u_h(x,t)$ and $\limsup^* u_h(x,t)$ defined in Theorem 3.1 are, respectively, a supersolution and a subsolution for (2.5).*

Once this consistency result is settled down, the proof of Theorem 3.1 easily follows: we first notice that $\liminf_* u_h(x,t)$ and $\limsup_* u_h(x,t)$ take only values in $\{\pm 1\}$. Thus to conclude we only have to recall (see [5]) that the maximal upper semicontinuous subsolution and minimal upper semicontinuous supersolution of (2.5) are given by $\chi_{(-\infty,0]}(u) - \chi_{(0,\infty)}(u)$ and $\chi_{(-\infty,0)}(u) - \chi_{[0,\infty)}(u)$, where u is a solution of (2.5). The fact that the initial data is taken easily follows by comparison, using the results of Section 3.2. This immediately entails the statement of Theorem 3.1.

We pass now to the proof of Proposition 3.5.

Proof of Proposition 3.5. A first observation is that, as an easy consequence of Corollary 3.4, the functions $\overline{u}(x,0) = \widetilde{\chi}_{\overline{\Omega}}(x)$ and $\underline{u}(x,0) = \widetilde{\chi}_{\overline{\Omega}}(x)$, in other words, they satisfy the required initial data.

Let us fix $(x_0, t_0) \in \mathbb{R}^N \times (0, +\infty)$ and φ and assume that (x_0, t_0) is a point of maximum of $u - \varphi$. Since $\overline{u}$ takes values ± 1 and it is upper semicontinuous , if $\overline{u}(x_0, t_0) = -1$ then it is constant in a neighborhood of (x_0, t_0) and thus $|D\overline{u}| = |D\varphi| = |\partial_t \varphi| = 0$ and so (2.9) trivially holds. The same assertion holds if (x_0, t_0) is an interior point of $\{\overline{u} = 1\}$. So we can suppose that $\overline{u}(x_0, t_0) = 1$ and (x_0, t_0) is a boundary point of $\{\overline{u} = 1\}$. In this case, replacing first φ with $\varphi_\eta(x,t) + \eta(|x - x_0|^2 + |t - t_0|^2)$, we define for each $h > 0$ small enough the point

$$(x_h, n_h h) = \operatorname{argmax}_{\mathbb{R}^N \times \mathbb{N}} u_h^* - \varphi_\eta. \tag{3.4}$$

The main inequality will be proved for the function φ_η, however one can easily show (thanks to Remark 2.1) that it then follows for φ when one sends $\eta \to 0$, hence in the sequel we will drop the index η and write simply φ.

Up to passing to a (not relabeled) subsequence, we can suppose that $(x_h, n_h h)$ converge to a point (x_1, t_1). In this case, by the regularity of φ and since $\overline{u}$ is upper semicontinuous we have that

$$\overline{u}(x_1, t_1) - \varphi(x_1, t_1) \geqslant \limsup_{h \to 0} u^*(x_h, n_h h) - \varphi(x_h, n_h h) \geqslant \overline{u}(x_0, t_0) - \varphi(x_0, t_0)$$

and thus $(x_1, t_1) = (x_0, t_0)$ since the latter is a strict global maximum of $u - \varphi$. Here we denoted by j^* the upper semicontinuous envelope of a function j.

By the definition of the points $(x_h, n_h h)$ and since $u_h^* \in \{\pm 1\}$ it is easy to show that

$$u_h^*(x, nh) \leqslant \operatorname{sign}^*\big(\varphi(x, n_h h) - \varphi(x_h, n_h h)\big).$$

We recall now that

$$u_h(\cdot, n_h h) = \widetilde{\chi}_{\{P_h * u_h(\cdot,(n_h-1)h) > g((n_h-1)h)h^{\frac{s}{1+s}}\}} \leqslant \widetilde{\chi}_{\{P_h * u_h^*(\cdot,(n_h-1)h) > g((n_h-1)h)h^{\frac{s}{1+s}}\}}$$

so that (since the right-hand side of the previous inequality is upper semicontinuous)

$$u_h^*(\cdot, n_h h) \leqslant \widetilde{\chi}_{\{P_h * u_h^*(\cdot,(n_h-1)h) > g((n_h-1)h)h^{\frac{s}{1+s}}\}}.$$

By computing the previous inequality in $x = x_h$, where $u_h^*(\cdot, n_h h)$ takes the value 1, we obtain

$$1 = u_h^*(x_h, n_h h) \leqslant \widetilde{\chi}_{\{P_h * u_h^*(x_h,(n_h-1)h) > g((n_h-1)h)h^{\frac{s}{1+s}}\}} \leqslant 1,$$

that is,

$$0 \leqslant P_h * u_h^*\big(\cdot, (n_h-1)h\big)(x_h) - g\big((n_h-1)h\big)h^{\frac{s}{1+s}}.$$

Since $u_h^* = \pm 1$ and $u_h^*(x_h, n_h h) = 1$, the previous inequality can be written as

$$\begin{aligned} 0 \leqslant \int_{\mathbb{R}^N} \bigg(&\chi^+\Big(u_h^*\big(y + x_h, (n_h-1)h\big) - u_h^*(x_h, n_h h)\Big) \\ &- \chi^-\Big(\Big(u_h^*\big(y + x_h, (n_h-1)h\big)\Big) - u_h^*(x_h, n_h h)\Big)\bigg) P_h(y)\,dy \\ &- g\big((n_h-1)h\big)h^{\frac{s}{1+s}}. \end{aligned} \tag{3.5}$$

The idea is now to estimate the right-hand side of (3.5) by means of several terms, converging to the difference between the right-hand side of (2.10) and its left-hand side. We define $t_h = n_h h$, $\varphi_h(y, s) = \varphi(x_h + y, s) - \varphi(x_h, t_h)$. By (3.4) we get that for every $A \subseteq \mathbb{R}^N$ it holds

$$\big\{y : u_h^*(y + x_h, t_h - h) \geqslant u_h^*(x_h, t_h)\big\} \cap A \subseteq \big\{y : \varphi_h(y, t_h - h) \geqslant 0\big\} \cap A, \tag{3.6}$$

and

$$\big\{y : \varphi_h(y, t_h - h) < 0\big\} \cap A \subseteq \big\{y : u_h^*(y + x_h, t_h - h) < u_h^*(x_h, t_h)\big\} \cap A. \tag{3.7}$$

Given a small number $\gamma > 0$, if δ is small enough, then

$$\begin{aligned} &\varphi_h(\cdot, t_h - h) \leqslant \varphi_h(\cdot, t_h) - (\partial_t \varphi(x_0, t_0) - \gamma)h, \\ &|D\varphi(x_0, t_0)| - \gamma \leqslant |D\varphi(\cdot, t_h)| \leqslant |D\varphi(x_0, t_0)| + \gamma \end{aligned} \tag{3.8}$$

inside $B_\delta(0)$.

The analysis is now split into two main parts, depending whether $|D\varphi(x_0, t_0)| = 0$ or not.

3.3.1 Step 1. Case $|D\varphi(x_0, t_0)| \neq 0$. We begin by writing (3.5) as

$$0 \leqslant I + h^{\frac{s}{1+s}} g\big((n_h - 1)h\big) \tag{3.9}$$

and then

$$0 \leqslant I + h^{\frac{s}{1+s}} g(t_h - h) \leqslant II + III + h^{\frac{s}{1+s}} g(t_h - h) \tag{3.10}$$

where II and III, implicitly depending on h and on δ, are given by

$$\begin{aligned} II = \int_{B_\delta(0)^c} \Big(&\chi^+\big(u_h^*(y + x_h, t_h - h) - u_h^*(x_h, t_h)\big) \\ &- \chi^-\big(u_h^*(y + x_h, t_h - h) - u_h^*(x_h, t_h)\big)\Big) P_h(y)\, dy, \end{aligned} \tag{3.11}$$

$$III = \int_{B_\delta(0)} \Big(\chi^+\big(\varphi_h(y, t_h - h)\big) - \chi^-\big(\varphi_h(y, t_h - h)\big)\Big) P_h(y)\, dy. \tag{3.12}$$

The inequality (3.10) follows from the fact that $\chi^+ - \chi^-$ is a non-decreasing function and by (3.6)-(3.7) (with $A = B_\delta(0)$).

We claim that

$$\limsup_{h\to 0} h^{-\frac{s}{1+s}} II \leqslant \overline{I}_{B_\delta(x_0)^c}[u](x_0, t_0). \tag{3.13}$$

Indeed by the definition of P_h, we have that P_h/σ_h converges (in $L^1((B_\delta(0))^c)$ as well as $L^\infty((B_\delta(0))^c)$) to the anisotropic fractional kernel $\mathfrak{N}^{-(N+s)}$. This, together with the fact that $\chi^+ - \chi^-$ is u.s.c., and the fact that $\limsup_* u_h^* - u_h^*(x_h, t_h) = \bar{u} - \bar{u}(x_0, t_0)$, implies (3.13).

Let us divide again III as

$$III \leqslant IV + 2V \tag{3.14}$$

with

$$IV = \int_{B_\delta(0)} \Big(\chi^+\big(\varphi_h(y, t_h)\big) - \chi^-\big(\varphi_h(y, t_h)\big)\Big) P_h(y)\, dy \tag{3.15}$$

and

$$\begin{aligned} V &= \int_{B_\delta(0)} \Big(\chi^+\Big(\varphi_h(y, t_h) - \big(\partial_t\varphi(x_0, t_0) - \gamma\big)h\Big) - \chi^+\big(\varphi_h(y, t_h)\big)\Big)\, P_h(y)\, dy \\ &= -\int_{B_\delta(0)} \Big(\chi^-\Big(\varphi_h(y, t_h) - \big(\partial_t\varphi(x_0, t_0) - \gamma\big)h\Big) - \chi^-\big(\varphi_h(y, t_h)\big)\Big)\, P_h(y)\, dy \end{aligned} \tag{3.16}$$

(using $\chi^+ + \chi^- = 1$). It is immediate to see that (3.14) follows by adding and subtracting the integral defined in (3.16) and using (3.8).

We aim to prove now that

$$\limsup_{h\to 0} h^{-\frac{s}{1+s}} IV \leqslant \overline{I}_{B_\delta(x_0, t_0)}[\varphi](x_0, t_0), \tag{3.17}$$

and

$$\limsup_{h\to 0} h^{-\frac{s}{1+s}} V \leqslant |D\varphi(x_0, t_0)|^{-1} \mathfrak{A}^{-1}\big(D\varphi(x_0, t_0)\big)\big(\gamma - \partial_t\varphi(x_0, t_0)\big). \tag{3.18}$$

The proof of those two latter statements is slightly more involved. To prove the first, we begin as in [10] with the following simple lemma.

Lemma 3.6 *If $|D\varphi(x_0,t_0)| \neq 0$, there exists a constant C such that for any $r>0$, it holds*

$$\int_{\partial B_r} \Big(\chi^+\big(\varphi_h(y,t_h)\big) - \chi^-\big(\varphi_h(y,t_h)\big)\Big) d\mathcal{H}^{N-1}(y) \leqslant C r^N. \tag{3.19}$$

Proof of Lemma 3.6. By the trivial estimate

$$\int_{\partial B_r} \Big(\chi^+\big(\varphi_h(y,t_h)\big) - \chi^-\big(\varphi_h(y,t_h)\big)\Big)\, d\mathcal{H}^{N-1}(y) \leqslant 2r^{N-1},$$

is clear that we have to show the statement of the lemma only for r small. Up to a rotation of the coordinates, we can suppose that $D\varphi(x_h,t_h)/|D\varphi(x_h,t_h)| = e_1$. Since φ is a regular function we have that in a sufficiently small neighborhood of 0

$$\big|\varphi_h(y,t_h)\big| \leqslant \big|D\varphi(x_y,t_h)\big| y_1 + \big|D^2\varphi(x_h,t_h)\big|\, |y|^2.$$

This implies that the set of integration in the left-hand side of (3.19) is contained in the set $\{y = (y_1,y') \in \partial B_r : |y_1| \leqslant c|y'|^2\}$ whose measure can be estimated by $c r^{N-2} r^2 \sim r^N$ (indeed, on the complement, the part where $\varphi_h > 0$ compensates exactly the part where $\varphi_h < 0$). Hence (3.19) holds. Notice that if r and h are small enough, the constant in (3.19) depends only on (the dimension and) $D\varphi(x_0,t_0)$, $D^2\varphi(x_0,t_0)$. □

As a consequence we get that inequality (3.17) holds true. Indeed by using polar coordinates we get that

$$|\sigma_h^{-1} IV| \leqslant C_N \int_0^\delta (\sigma_h^2 + r^2)^{-\frac{N+s}{2}} U_h(r)\, dr \tag{3.20}$$

with

$$U_h(r) = \int_{\partial B_r} \Big(\chi^+\big(\varphi_h(y)\big) - \chi^-\big(\varphi_h(y)\big)\Big)\, d\mathcal{H}^{N-1}(y).$$

By means the previous lemma, we get that $U_h(r) \leqslant C r^N$ and we can apply Lebesgue's Convergence Theorem and conclude that the (superior) limit of $\sigma_h^{-1} III$ is exactly the right-hand side of (3.17).

We pass to the proof of (3.18). We only follow partially [10] for this estimate. Let us first observe that

$$\chi^+\Big(\varphi_h(y,t_h) - \big(\partial_t\varphi(x_0,t_0) - \gamma\big)h\Big) - \chi^+\big(\varphi(y,t_h)\big) = \begin{cases} -\chi_{\{0\leqslant \varphi(\cdot,t_h) < (\partial_t\varphi-\gamma)h\}}(y) & \text{if } \partial_t\varphi - \gamma \geqslant 0, \\ \chi_{\{(\partial_t\varphi-\gamma)h \leqslant \varphi(\cdot,t_h) < 0\}}(y) & \text{if } \partial_t\varphi - \gamma \leqslant 0. \end{cases}$$

Assuming for instance that $\partial_t\varphi - \gamma > 0$, we obtain (denoting simply by B_r the ball $B_r(0)$)

$$V = -\int_{\{0\leqslant\varphi(\cdot,t_h)<(\partial_t\varphi-\gamma)h\}\cap B_\delta} P_h(y)\, dy = \int_{\{0\leqslant\varphi(\sigma_h^{1/s}\cdot,t_h)<(\partial_t\varphi-\gamma)h\}\cap B_{\delta/\sigma_h^{1/s}}} P(y)\, dy,$$

hence (using the co-area formula)

$$\begin{aligned} V &= -\int_0^{(\partial_t\varphi-\gamma)h} d\tau \int_{\partial\{\varphi(\sigma_h^{1/s}\cdot,t_h)\geqslant\tau\}\cap B_{\delta/\sigma_h^{1/s}}} \frac{P(y)}{\sigma_h^{1/s}|D\varphi(\sigma_h^{1/s}y,t_h)|} d\mathcal{H}^{N-1}(y) \\ &\leqslant -\frac{h}{\sigma_h^{1/s}} \frac{\partial_t\varphi(x_0,t_0) - \gamma}{|D\varphi(x_0,t_0)| + \gamma} \int_0^1 d\tau \int_{\partial\{\varphi(\sigma_h^{1/s}\cdot,t_h)\geqslant h\alpha\tau\}\cap B_{\delta/\sigma_h^{1/s}}} P(y) d\mathcal{H}^{N-1}(y) \end{aligned}$$

where we have denoted, for short, $\alpha = \partial_t \varphi - \gamma$, and used again (3.8). To sum up,

$$\sigma_h^{-1} V \leqslant -\frac{\partial_t \varphi(x_0, t_0) - \gamma}{|D\varphi(x_0, t_0)| + \gamma} \int_0^1 d\tau \int_{\partial\{\varphi(\sigma_h^{1/s}\cdot, t_h) \geqslant h\alpha\tau\} \cap B_{\delta/\sigma_h^{1/s}}} P(y) d\mathcal{H}^{N-1}(y). \tag{3.21}$$

Possibly reducing δ, and assuming that the Nth coordinate is along the vector $e_N = D\varphi(x_0, t_0)/|D\varphi(x_0, t_0)|$, for h small enough we can represent the level surface $\partial\{\varphi_h(\cdot, t_h) \geqslant h\ell\}$ by a graph $\{(y', f_{\ell,h}(y')) : y' \in B'_\delta\} \cap B_\delta$ (where B'_r denotes the $(N-1)$-dimensional ball in $e_N^\perp$ with center 0 and radius r), with $f_{\ell,h}$ which goes uniformly (in both y' and ℓ), in C^2 norm as $h \to 0$, to the function f_0 representing in the same way the surface $\partial\{y \in B_\delta : \varphi(x_0 + y, t_0) \geqslant \varphi(x_0, t_0)\}$, and which is such that $D' f_0(0) = 0$. We observe moreover that $f_{\ell,h}(0) = h\ell/|D\varphi(x_0, t_0)| + o(h)$. We denote $D_{\ell,h} \subseteq B'_\delta$ the set of points y' such that $(y', f_{\ell,h}(y')) \in B_\delta$.

Now, we have that

$$\begin{aligned}\int_{\partial\{\varphi(\sigma_h^{1/s}\cdot, t_h) \geqslant h\alpha\tau\} \cap B_{\sigma_h^{-1/s}\delta}} & P(y) d\mathcal{H}^{N-1}(y) \\ &= \int_{\sigma_h^{-1/s} D_{\alpha\tau,h}} P\left(y', \sigma_h^{-1/s} f_{\alpha\tau,h}(\sigma_h^{1/s} y')\right) \sqrt{1 + |D' f_{\alpha\tau,h}(\sigma_h^{1/s} y')|^2} dy'.\end{aligned}$$

Given $R > 0$, we split the last integral into an integral in B'_R and an integral in $\sigma_h^{-1/s} D_{\alpha\tau,h} \setminus B'_R$: clearly the latter is controlled uniformly by cR^{-1-s} as the gradients of the functions $f_{\alpha\tau,h}$ are uniformly bounded. We now try to express the limit, as $h \to 0$, of

$$\int_{B'_R} P\left(y', \sigma_h^{-1/s} f_{\alpha\tau,h}(\sigma_h^{1/s} y')\right) \sqrt{1 + |D' f_{\alpha\tau,h}(\sigma_h^{1/s} y')|^2} dy'.$$

Observe that if $|y'| \leqslant R$ and h is small enough,

$$\sigma_h^{-1/s} f_{\alpha\tau,h}(\sigma_h^{1/s} y') \approx \frac{\alpha\tau h}{\sigma_h^{1/s} |D\varphi(x_0, t_0)|} + |D' f_{\alpha\tau,h}(\theta_h y')||y'|$$

for some $\theta_h \in [0, \sigma_h^{1/s}]$. Using $h/\sigma_h^{1/s} = \sigma_h$, we find that $\sigma_h^{-1/s} f_{\alpha\tau,h}(\sigma_h^{1/s} y') \to 0$ uniformly (in τ and $y' \in B(0, R')$) as $h \to 0$. We deduce (using again that $D' f_{\alpha\tau,h}(\sigma_h^{1/s} y') \to 0$)

$$\int_{B'_R} P\left(y', \sigma_h^{-1/s} f_{\alpha\tau,h}(\sigma_h^{1/s} y')\right) \sqrt{1 + |D' f_{\alpha\tau,h}(\sigma_h^{1/s} y')|^2} dy' \to \int_{B'_R} P(y', 0) dy'$$

uniformly in τ as $h \to 0$. Hence returning to (3.21), we find, for any R,

$$\limsup_{h \to 0} \sigma_h^{-1} V \leqslant -\frac{\partial_t \varphi(x_0, t_0) - \gamma}{|D\varphi(x_0, t_0)| + \gamma} \int_0^1 d\tau \left(\int_{B'_R} P(y', 0) dy' + \frac{c}{R^{1+s}} \right)$$

and sending $R \to \infty$ we deduce

$$\limsup_{h \to 0} \sigma_h^{-1} V \leqslant -\frac{\partial_t \varphi(x_0, t_0) - \gamma}{|D\varphi(x_0, t_0)| + \gamma} \int_{D\varphi(x_0, t_0)^\perp} P(y) d\mathcal{H}^{N-1}(y).$$

On the other hand, if $\partial_t\varphi(x_0,t_0)-\gamma<0$, the same proof will show

$$\limsup_{h\to 0}\sigma_h^{-1}V \leqslant -\frac{\partial_t\varphi(x_0,t_0)-\gamma}{|D\varphi(x_0,t_0)|-\gamma}\int_{D\varphi(x_0,t_0)^\perp} P(y)\,d\mathcal{H}^{N-1}(y).$$

Together with (3.17) we deduce

$$\limsup_{h\to 0}\sigma_h^{-1}III \leqslant \overline{I}_{B_\delta(x_0,t_0)}[\varphi](x_0,t_0) - 2\frac{\partial_t\varphi(x_0,t_0)-\gamma}{|D\varphi(x_0,t_0)|\pm\gamma}\int_{D\varphi(x_0,t_0)^\perp} P(y)\,d\mathcal{H}^{N-1}(y)$$

and using (3.13), we get

$$\limsup_{h\to 0}\sigma_h^{-1}I \leqslant \overline{I}_{B_\delta(x_0,t_0)^c}[u](x_0,t_0) + \overline{I}_{B_\delta(x_0,t_0)}[\varphi](x_0,t_0) - \frac{\partial_t\varphi(x_0,t_0)-\gamma}{|D\varphi(x_0,t_0)|\pm\gamma}\mathcal{R}\big(D\varphi(x_0,t_0)\big)^{-1},$$

where $\mathcal{R}$ is defined in (2.6). Since g is continuous, $g(t_h-h)\to g(t_0)$ as $h\to 0$ and we obtain (2.9) by taking the limsup in (3.9) and sending then $\gamma\to 0$.

3.3.2 Step 2. Case $|D\varphi(x_0,t_0)|=0$. A first classical observation (see, for instance, [3]) is that in this case one can "decouple" the test function φ as the sum of a function of x and a function of t. Indeed, since (x_0,t_0) is a critical point of $t\mapsto\varphi(x,t)-\partial_t\varphi(x_0,t_0)(t-t_0)$, there exists $\mu>0$ such that, near (x_0,t_0), $\varphi(x,t)\leqslant\psi(x,t):=\varphi(x_0,t_0)+\partial_t\varphi(x_0,t_0)(t-t_0)+\mu(|x-x_0|^2+|t-t_0|^2)/2$. Then as before, (x_0,t_0) is a strict local maximum of $u^*-\psi$. For $n\geqslant 1$, let (x_n,t_n), $t_n<t_0$, be a local maximum of $u^*(x,t)-\psi(x,t)-1/(n(t_0-t))$. Such a maximum exists since for any x, $u^*(x,t)-\psi(x,t)-1/(n(t_0-t))$ diverges to $-\infty$ as $t\to t_0^+$ and $t\to-\infty$. Moreover is easy to see that $(x_n,t_n)\to(x_0,t_0)$ as $n\to\infty$. Assume first that

$$D\psi(x_n,t_n)=\mu(x_n-x_0)\neq 0$$

for infinitely many n. Then we have, thanks to the Step 1,

$$\begin{aligned}\partial_t\varphi(x_n,t_n)+\mu(t_n-t_0)+\frac{1}{n(t_0-t_n)^2} &\leqslant \mu|x_n-x_0|\mathcal{R}(x_n-x_0)\big(I[|x-x_0|^2/2](x_n)+g(t)\big)\\ &\sim C\mu|x_n-x_0|(C|x_n-x_0|^{-s}+\|g\|_\infty)\to 0\end{aligned}$$

as $n\to\infty$, since the s-curvature of a ball of radius r is of order r^{-s}. We deduce (2.9).

Hence we are reduced to the case where $D\psi(x_n,t_n)=0$ for all n large enough, which implies $x_n=x_0$. We argue by contradiction supposing that $\partial\varphi_t(x_0,t_0)>0$. For any x and $t<t_n$, by maximality of (x_n,t_n) for the function $u^*(x,t)-\psi(x,t)+1/n(t-t_n)$ we have

$$\begin{aligned}&u^*(x,t)-\varphi(x_0,t_0)-\partial_t\varphi(x_0,t_0)(t-t_0)-\frac{\mu}{2}(|t-t_0|^2+|x-x_0|^2)-\frac{1}{n(t_0-t)}\\ &\qquad\leqslant u^*(x_0,t_n)-\varphi(x_0,t_0)-\partial_t\varphi(x_0,t_0)(t_n-t_0)-\frac{\mu}{2}|t_n-t_0|^2-\frac{1}{n(t_0-t_n)}\end{aligned}$$

so that

$$\begin{aligned}u^*(x,t)-u^*(x_0,t_n) &\leqslant -\partial_t\varphi(x_0,t_0)(t_n-t)-\mu(t_n-t)(t-t_0)\\ &\qquad -\frac{1}{n(t_0-t_n)}+\frac{1}{n(t_0-t)}+\frac{\mu}{2}|x-x_0|^2\\ &\leqslant \big(-\partial_t\varphi(x_0,t_0)+\mu(t_0-t)\big)(t_n-t)+\frac{\mu}{2}|x-x_0|^2.\end{aligned}$$

It follows that $u^*(x,t) = -1$ if $|x - x_0| < r(t) := \sqrt{\partial_t\varphi(x_0,t_0)(t_n - t)/\mu}$, provided n is large enough and t is close enough to t_n so that $-\partial_t\phi(x_0,t_0) + \mu(t_0 - t) \leqslant -\partial_t\varphi(x_0,t_0)/2$. It also follows that $u^*(x_0,t_n) = 1$.

Now Corollary 3.4 yields that $u^*(x_0, t+\tau) = -1$ for $\tau \leqslant \gamma/(4\bar{k})r(t)^{1+s} \sim (t_n - t)^{\frac{1+s}{2}}$, so that if t is close enough to t_n this is true up to $\tau = t_n - t$, a contradiction. □

3.4 *Convexity of the mobility*

The equation which is solved in the limit can be written as

$$\partial_t u = \Phi(Du)\Big(-\kappa_s\big(x, \{u \geqslant u(x)\}\big) + g(t)\Big),$$

with $\Phi(p)$, the (inverse of the) *mobility*, is the one-homogeneous function $|p|\mathfrak{A}(p)$ (which from now on we will often denote by Φ). If Φ is convex, this law precisely states that the boundary of the level sets of u evolve with the speed $-\kappa_s + g$, where $-\kappa_s$ is the fractional curvature and g the forcing term, in the direction of the "Φ-normal" $\partial\Phi^\circ(\nu)$, where ν is the normal to the level set, and Φ° the polar of Φ (or dual norm).

We will show that Φ is indeed a convex (and obviously even, one-homogeneous) function, hence a norm.

Lemma 3.7 *The* 1*-homogeneous function*

$$\Phi(p) := |p|\left(2\int_{p^\perp}\frac{dx}{1+\mathfrak{N}(x)^{N+s}}\right)^{-1}$$

is a convex function in $\mathbb{R}^N$.

Proof. Consider $p^0, p^1 \in \mathbb{R}^N$: without loss of generality, we assume that $\mathrm{vect}\{p^0,p^1\} = \mathrm{vect}\{e_1,e_2\}$ (where $(e_i)_{i=1}^N$ is the canonical basis). We denote $x = (x_1,x_2,x') \in \mathbb{R}^N$, and for $p = (p_1,p_2,0) \in \mathrm{vect}\{p^0,p^1\}$, $Rp = (-p_2,p_1,0)$. One can check that for such a p,

$$\Phi(p) = \left(\int_{\mathbb{R}^{N-2}} dx' \int_{-\infty}^{+\infty} dz \frac{1}{a + \mathfrak{N}(x' + zRp)^{N+s}}\right)^{-1}.$$

Then one computes, performing successively the changes of variables $\xi' = x'/z$ and $t = \mathfrak{N}(\xi' + Rp)z$,

$$\begin{aligned}\Phi(p) &= \left(\int_{\mathbb{R}^{N-2}} dx' \int_{-\infty}^{+\infty} dz \frac{1}{1 + z^{N+s}\mathfrak{N}(\frac{x'}{z} + Rp)^{N+s}}\right)^{-1}\\ &= \left(\int_{\mathbb{R}^{N-2}} d\xi' \int_{-\infty}^{+\infty} dz \frac{z^{N-2}}{1 + z^{N+s}\mathfrak{N}(\xi' + Rp)^{N+s}}\right)^{-1}\\ &= \left(\int_{\mathbb{R}^{N-2}} \frac{d\xi'}{\mathfrak{N}(\xi' + Rp)^{N-1}} \int_{-\infty}^{+\infty} dt \frac{t^{N-2}}{1 + t^{N+s}}\right)^{-1},\end{aligned}$$

so that

$$\Phi(p) = C(N,s)\left(\int_{\mathbb{R}^{N-2}} \frac{d\xi'}{\mathfrak{N}(\xi' + Rp)^{N-1}}\right)^{-1} \tag{3.22}$$

for some constant $C(N,s)$. Now, consider $\lambda \in [0,1]$ and the functions (assuming $p_1, p_0, \lambda p_1 + (1-\lambda)p_0 \neq 0$)

$$h(x') = \frac{1}{\mathfrak{N}\Big(x' + R\big(\lambda p_1 + (1-\lambda)p_0\big)\Big)^{N-1}},$$
$$f(x') = \frac{1}{\mathfrak{N}(x' + Rp_1)^{N-1}}, \quad g(x') = \frac{1}{\mathfrak{N}(x' + Rp_0)^{N-1}}.$$

Then, using the convexity of $\mathfrak{N}$, we have for all x', y',

$$h\big(\lambda x' + (1-\lambda y')\big) \geqslant M_{-1/(N-1)}\big(f(x'), g(y'), \lambda\big)$$

where M_p is defined in [17, p. 368] by

$$M_p(a,b,\lambda) = \big(\lambda a^p + (1-\lambda)b^p\big)^{1/p}. \tag{3.23}$$

Thanks to Borell–Braskamp–Lieb's inequality [17, Thm. 10.1, (38)] we have

$$\int_{\mathbb{R}^{N-1}} h(x')dx' \geqslant M_q\left(\int_{\mathbb{R}^{N-1}} f(x')dx', \int_{\mathbb{R}^{N-1}} g(x')dx', \lambda\right)$$

for

$$q = \frac{-1/(N-1)}{-\frac{N-2}{N-1} + 1} = -1.$$

But using (3.22), this precisely boils down to

$$\Phi\big(\lambda p_1 + (1-\lambda)p_0\big) \leqslant \lambda\Phi(p_1) + (1-\lambda)\Phi(p_0).$$

$\square$

4. Evolution of convex sets

In this section we show that the during the flow the convexity of a set is preserved. The main result in this direction is contained in Lemma 4.3, where it is shown that in each step of the discrete approximation, the convexity is preserved. Such a Lemma is actually a consequence of a series of non-trivial results in convex geometry. We begin by recalling such results. All the above definitions and results can be found, together with a comprehensive list of references, in the survey [17].

DEFINITION 4.1 We say that a function $f \in L^1(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$ is a $p-$concave function if it is $\log-$concave when $p = 0$ and, if $p \neq 0$, for every $x, y \in \mathbb{R}^N$ it holds

$$f\big((1-\lambda)x + \lambda y\big) \geqslant M_p\big(\lambda, f(y), f(x)\big)$$

where M_p is defined in (3.23). Equivalently, f is $p-$concave if f^p is convex, for p negative, and f^p is concave, for p positive.

For our scopes we will need the following result which is a consequence of [14, Theorem 3.16], [17, Corollary 11.2] (see the discussion at page 379 of [17]).

Lemma 4.2 *Let f be a p-concave function with $p > -1/N$ and K a convex body (that is a convex set with non-empty interior). Then the function*

$$h(x) = f * \chi_K(x)$$

is $(1/Np + 1)$-concave, and as a consequence, h is level set convex.

Corollary 4.3 *If E is convex then for any $h > 0$, $c \in \mathbb{R}$, the set $T_{c,h}(E)$ (defined in (2.3)) is convex.*

Proof. We only have to notice that the function P_h is continuous, integrable over $\mathbb{R}^N$, and that it is $-1/(N + s)$-concave. The latter property follows by a direct inspection. □

Corollary 4.4 *Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function. Let u_0 be a regular function such that all level sets $\{u_0 > s\}$ are convex. If u is the solution of (2.5) with initial data u_0, then the level sets $\{u(\cdot, t) > s\}$ are convex.*

Proof. This follows from the fact that, thanks to Theorem 3.1 and Remark 2.6, (almost) all the level sets of $u(t)$ can be obtained as limits of the scheme 2.3, which preserves convexity. □

5. A splitting result

The goal of this section is to show that the motion with forcing term can be obtained by alternating free curvature motions and evolutions with the forcing term only. This is a variant of the famous results of Trotter and Kato for nonlinear semigroups, which was first implemented in the viscosity solutions setting probably in [28]. Our setting is closer to [2] where second order parabolic equations are considered. A consequence of this result will be an elementary proof of how the distance between two sets evolve by the forced curvature flow (as this distance increases by unforced mean curvature flow, and its evolution is trivial for sets evolving with constant speeds), see Prop. 6.2 below.

Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function. For a fixed $\varepsilon > 0$ consider the sets $A_\varepsilon = \cup_{n \geqslant 0}(2n\varepsilon, (2n+1)\varepsilon]$ and $B_\varepsilon = (0, \infty) \setminus A_\varepsilon$. Let, for $t > 0$, $p \in \mathbb{R}^N$ and $I \in \mathbb{R}$,

$$F_\varepsilon(t, p, I) := 2\chi_{A_\varepsilon}(t)\Phi(p)c_\varepsilon(t) + 2\chi_{B_\varepsilon}(t)\Phi(p)I, \tag{5.1}$$

where c_ε is the piecewise constant function defined by

$$c_\varepsilon(t) = \frac{1}{2\varepsilon}\int_{2n\varepsilon}^{(2n+2)\varepsilon} g(\tau)\, d\tau,$$

if $t \in [2n\varepsilon, (2n+2)\varepsilon]$, and where $\Phi(p) = \mathfrak{a}(p)|p|$ is the mobility (see Section 3.4). We let also

$$F(t, p, I) := \Phi(p)\big(I + g(t)\big), \tag{5.2}$$

and we observe that the function $t \mapsto \int_0^t (F_\varepsilon(\tau, p, I) - F(\tau, p, I))d\tau$ goes locally uniformly to 0 as $\varepsilon \to 0$ (for fixed p, I).

Let $u_0 : \mathbb{R}^N \to \mathbb{R}$ a bounded uniformly continuous function and $u_\varepsilon : \mathbb{R}^N \times [0, \infty)$ be the function constructed as follows. We let $u_\varepsilon(\cdot, 0) = u_0$ and for each n, define $u(\cdot, t)$ on $(n\varepsilon, (n+1)\varepsilon]$ as the (unique) viscosity solution, starting from $u_\varepsilon(n\varepsilon)$, of

$$\partial_t u_\varepsilon = F_\varepsilon\Big(t, Du_\varepsilon, -\kappa_s\big(x, \{u_\varepsilon \geqslant u_\varepsilon(x,t)\}\big)\Big). \tag{5.3}$$

The idea of this equation is that it alternates, at a frequency $1/\varepsilon$, between the fractional curvature flow (without forcing) and the forced flow (without curvature). This will allow us later on to study separately the influence of both terms. More precisely, we split the time into the disjoint sets A_ε and B_ε where we ask u_ε to follow alternatively the two simpler flows, one with only the forcing term, and the other with the free fractional curvature flow. This leads precisely to (5.3). Proposition 5.1 below guarantees the convergence of this scheme to the curvature flow with forcing as ε converges to 0.

It is easy to see that u_ε remains bounded and spatially uniformly continuous, moreover it is classical that it is also uniformly continuous in time (see for instance [20]). Hence up to a subsequence, we may assume that it converges uniformly, as $\varepsilon \to 0$, to a continuous limit $u(x,t)$. We will show that u is the solution of (2.5). Since this limit is independent of the chosen subsequence, it yields the following proposition.

Proposition 5.1 *Let u_ε, u be respectively the solutions of* (5.3), (2.5)*, with initial datum u_0. Then $u_\varepsilon \to u$, as $\varepsilon \to 0$, locally uniformly in $\mathbb{R}^N \times [0,\infty)$.*

Proof. We just need to show that u, limit of a subsequence of (u_ε), satisfies (2.8). The proof is based on a convergence result of Barles [2], based the theory of L^1-viscosity solution [8, 21]. We adapt it to our nonlocal setting, and simplify significantly the argument, as we do not wish to show a very general convergence result for nonlocal geometric motions (even if this could be of independent interest).

Consider φ and $(\bar{x},\bar{t})$ a strict global maximum of $u-\varphi$, and assume first $D\varphi(\bar{x},\bar{t}) \neq 0$ and $\varphi(\bar{x},\bar{t})$ is not a critical value of φ. As in [2], we introduce

$$\psi_\varepsilon(t) := F_\varepsilon\Big(t, D\varphi(\bar{x},\bar{t}), -\kappa_s\big(\bar{x}, \{\varphi \geqslant \varphi(\bar{x},\bar{t})\}\big)\Big) - F\Big(t, D\varphi(\bar{x},\bar{t}), -\kappa_s\big(\bar{x}, \{\varphi \geqslant \varphi(\bar{x},\bar{t})\}\big)\Big)$$

which is such that $\int_0^t \psi_\varepsilon(\tau)d\tau \to 0$ uniformly. Hence, $u_\varepsilon(x,t) - \int_0^t \psi_\varepsilon(\tau)d\tau \to u(x,t)$ locally uniformly, and one can find points $(x_\varepsilon, t_\varepsilon)$ of global maximum of

$$u_\varepsilon(x,t) - \int_0^t \psi_\varepsilon(\tau)d\tau - \varphi(x,t),$$

such that $(x_\varepsilon,t_\varepsilon) \to (\bar{x},\bar{t})$ as $\varepsilon \to 0$. If $t/\varepsilon \notin \mathbb{N}$, one deduces that

$$\partial_t\varphi(x_\varepsilon,t_\varepsilon) + \psi_\varepsilon(t_\varepsilon) \leqslant F_\varepsilon\Big(t_\varepsilon, D\varphi(x_\varepsilon,t_\varepsilon), -\kappa_s\big(x_\varepsilon, \{\varphi \geqslant \varphi(x_\varepsilon,t_\varepsilon)\}\big)\Big), \tag{5.4}$$

observing in particular that since $(x_\varepsilon,t_\varepsilon) \to (\bar{x},\bar{t})$, the value $\varphi(x_\varepsilon,t_\varepsilon)$ can be assumed to be noncritical.[1] If $t_\varepsilon/\varepsilon \in \mathbb{N}$, classical arguments for parabolic semigroups show that (5.4) still holds, if one takes for ψ_ε and F_ε their left limit, see for instance [23]. It follows

$$\begin{aligned}\partial_t\varphi(x_\varepsilon,t_\varepsilon) \leqslant{}& F\Big(t_\varepsilon, D\varphi(\bar{x},\bar{t}), -\kappa_s\big(\bar{x}, \{\varphi \geqslant \varphi(\bar{x},\bar{t})\}\big)\Big) + \\ &\Big[F_\varepsilon\Big(t_\varepsilon, D\varphi(x_\varepsilon,t_\varepsilon), -\kappa_s\big(x_\varepsilon, \{\varphi \geqslant \varphi(x_\varepsilon,t_\varepsilon)\}\big)\Big) - F_\varepsilon\Big(t_\varepsilon, D\varphi(\bar{x},\bar{t}), -\kappa_s\big(\bar{x}, \{\varphi \geqslant \varphi(\bar{x},\bar{t})\}\big)\Big)\Big].\end{aligned} \tag{5.5}$$

[1] Strictly speaking, there could be critical points of the corresponding level sets, however these points tend to infinity as $\varepsilon \to 0$, and do not alter significantly the value of the integrals defining κ_s.

As

$$\lim_{\varepsilon\to 0} D\varphi(x_\varepsilon,t_\varepsilon) = D\varphi(\bar{x},\bar{t}) \quad \text{and} \quad \lim_{\varepsilon\to 0} \kappa_s\big(x_\varepsilon,\{\varphi \geqslant \varphi(x_\varepsilon,t_\varepsilon)\}\big) = \kappa_s\big(\bar{x},\{\varphi \geqslant \varphi(\bar{x},\bar{t})\}\big),$$

the error term in square brackets in (5.5) vanishes in the limit and it follows

$$\partial_t\varphi(\bar{x},\bar{t}) \leqslant F\Big(t, D\varphi(\bar{x},\bar{t}), -\kappa_s\big(\bar{x},\{\varphi \geqslant \varphi(\bar{x},\bar{t})\}\big)\Big),$$

which is (2.8).

If on the other hand $D\varphi(\bar{x},\bar{t}) = 0$, then the proof that $\partial_t\varphi(\bar{x},\bar{t}) \leqslant 0$ is identical to Step 2 in the proof of Proposition 3.5, provided one can first estimate the speed at which balls evolve under the equation (5.3), which is of the same order as in Corollary 3.4. □

6. Geometric uniqueness in the convex case

In this section we show that if the initial set is bounded and convex, then the fattening phenomenon can not occur and the evolution is unique. The proof is based on [7, Theorem 4.9], and follows from a (simple) estimate of the distance between two evolutions with different forcing terms. In the rest of the paper, we will always consider in $\mathbb{R}^N$ the distance $\mathrm{dist}_{\Phi^\circ}$ induced by the norm $\Phi^\circ(x) := \sup\{\xi\cdot x : \Phi(\xi) \leqslant 1\}$, polar of Φ. Hence we will drop the subscript and write dist instead of $\mathrm{dist}_{\Phi^\circ}$. Similarly, we will write $d_C^\eta = -\eta \vee (\eta \wedge (\mathrm{dist}_{\Phi^\circ}(x,C) - \mathrm{dist}_{\Phi^\circ}(x,C^c)))$.

Lemma 6.1 *Let $C_1 \subseteq C_2$ two sets and let $C_1(t)$ and $C_2(t)$ be the evolutions of the flow $v_i = \Phi(\nu)c_i$, with c_i two constants, starting from C_1 and C_2 respectively. That is, $C_i(t) = \{u_i \geqslant t\}$ where u_i is the solution of*

$$\begin{cases} \partial_t u_i = c_i\Phi(Du_i),\\ u_i(x,0) = d_{C_i}^\eta(x). \end{cases}$$

Then the function

$$\delta(t) := \mathrm{dist}_{\Phi^\circ}\big(\partial C_1(t), \partial C_2(t)\big)$$

satisfies

$$\delta(t) \geqslant \delta(0) - t(c_2 - c_1),$$

for every $0 \leqslant t \leqslant T_S := \sup\{\tau \geqslant 0 : \delta(\tau) > 0\}$ (i.e., until the first contact time).

Proof. We consider first the case where c_1 and c_2 are not positive. We recall that, by the Hopf–Lax formula for the Hamiltonian $H_i(p) = |c_i|\Phi(p)$, the solution of the system

$$\begin{cases} \partial_t u_i(x,t) + |c_i|\Phi\big(Du(x,t)\big) = 0,\\ u_i(x,0) = d_{C_i}^\eta(x), \end{cases}$$

with $i = 1, 2$, is given by (see for instance [16])

$$u_i(x,t) = \inf_{y\in\mathbb{R}^N}\left\{d_{C_i}^\eta(x) + tH_i^*\left(\frac{x-y}{t}\right)\right\},$$

where H_i^* denotes the Legendre–Fenchel transform of the function H_i, given by

$$H_i^*(q) = \begin{cases} 0 & \text{if } \Phi^\circ(q) \leqslant |c_i|,\\ +\infty & \text{else.} \end{cases}$$

Thus

$$u_i(x,t) = \inf_{y:\Phi^\circ(y-x)\leqslant|c_i|t} d^\eta_{C_i}(x).$$

Since $\delta(0) > 0$ we can suppose that t is such that $C_1(t) \subset C_2(0)$. We have that

$$\{x : u_i(x,t) > 0\} = \{C_i + B_{\Phi^\circ}(0, |c_i|t)\}^c. \tag{6.1}$$

Indeed if $\xi \in \{x : u_i(x,t) > 0\}$ then ξ can not belong to $C_i(t) + B_{\Phi^\circ}(0, |c_i|t)$, otherwise there would exist $z \in \partial C_i$ with $\Phi^\circ(z-\xi) \leqslant |c_i|t$ and thus

$$0 = \mathrm{dist}(z, \partial C_i) \geqslant u_i(\xi,t) > 0.$$

On the other hand it is immediate to verify that if $\mathrm{dist}(\xi, C_i) \leqslant |c_i|t$ then $u(\xi,t) \leqslant 0$.

Let $x_i \in \partial(C_i + B_{\Phi^\circ}(0, |c_i|t))$ be such that $\delta(t) = \Phi^\circ(x_1 - x_2)$ and denote ξ the unique intersection between ∂C_2 and the segment with extrema x_1 and x_2. Let moreover z be the projection of x_1 onto ∂C_1, so that $\Phi^\circ(x_1 - z) = |c_1|t$. We have

$$\begin{aligned}
\delta(t) &= \Phi^\circ(x_1 - x_2)\\
&= \Phi^\circ(x_1 - \xi) + \Phi^\circ(\xi - x_2)\\
&\geqslant |c_2|t + \Phi^\circ(\xi - x_1)\\
&\geqslant |c_2|t + \Phi^\circ(\xi - z) - \Phi^\circ(z - x_1)\\
&= |c_2|t + \delta(0) - |c_1|t\\
&= \delta(0) - (c_2 - c_1)t,
\end{aligned}$$

which is exactly the statement of the lemma. The proof in the case where c_1 and c_2 are positive follows the same lines of the above proof, once we notice that if u solves $\partial_t u - |c|\Phi(Du) = 0$ then $v = -u$ solves $\partial_t v + |c|\Phi(Dv) = 0$. If $c_1 < 0$ and $c_2 > 0$ by similar arguments we get that

$$C_1(t) = \{C_1 + B_{\Phi^\circ}(0, |c_1|t)\}^c, \qquad C_2(t) = \{x : \mathrm{dist}(x, \partial C_2 > c_2 t)\}.$$

Let x_1 and x_2 be points such that $\delta(t) = \Phi^\circ(x_1 - x_2)$ and let $\overline{x_i}$ be the projection of x_i to ∂C_i, $i = 1, 2$. Then we have

$$\begin{aligned}
\delta(t) &= \Phi^\circ(x_1 - x_2)\\
&\geqslant \Phi^\circ(\overline{x_1} - \overline{x_2}) - \Phi^\circ(x_1 - \overline{x_1}) - \Phi^\circ(x_2 - \overline{x_2})\\
&\geqslant \delta(0) - |c_1|t - |c_2|t\\
&= \delta(0) - (c_2 - c_1)t.
\end{aligned}$$

The proof of the case $c_1 > 0$ and $c_2 < 0$ follows by an analogous argument. □

Proposition 6.2 *Let $C_1 \subseteq C_2$ be two sets and let, g_1 and $g_2 : \mathbb{R}_+ \to \mathbb{R}$ two continuous functions and, for $i = 1, 2$, let u_i be the solution of*

$$\begin{cases}\partial_t u_i = \Phi(Du_i)\big(-\kappa_s(x, \{u_i \geqslant u_i(x,t)\}) + g_i(t)\big),\\ u_i(x,0) = d^\eta_{C_i}(x)\end{cases}.$$

Let for $t \geqslant 0$, $i = 1, 2$, $C_i(t) = \{u_i \geqslant 0\}$. *Then the function* $\delta(t) = \operatorname{dist}(\partial C_1, \partial C_2)$ *satisfies*

$$\delta(t) \geqslant \delta(0) - \int_0^t \big(c_2(s) - c_1(s)\big)\, ds$$

for every $0 \leqslant t \leqslant T_S := \sup\{\tau \geqslant 0 : \delta(\tau) > 0\}$.

Proof. Without loss of generality we can assume that the 0-level sets of u_i do not *fatten*, that is, $C_i(t) = \overline{\{u_i > 0\}}$ (otherwise we should consider the τ-level set and then let $\tau \to 0$).

For $i = 1, 2$, let $u_{\varepsilon,i}$ be the functions constructed in Section 5 with $g = g_i$, let $c_{\varepsilon,i}$ be the corresponding piecewise forcing terms, and let $C_\varepsilon^i(t) = \{x : u_{\varepsilon,i}(x,t) \geqslant 0\}$ and $\delta_\varepsilon(t) = \operatorname{dist}(\partial C_\varepsilon^1(t), C_\varepsilon^2(t))$. By Lemma 5.1 we have that $\delta_\varepsilon(t) \to \delta(t)$ for every $t \leqslant T_S$.

Let $t \leqslant T_S$ and let n be the largest integer such that $n\varepsilon < t$. Let us write $\delta_\varepsilon(t)$ as

$$\delta_\varepsilon(t) = \delta_\varepsilon(\varepsilon) + \big[\delta_\varepsilon(2\varepsilon) - \delta_\varepsilon(\varepsilon)\big] + \big[\delta_\varepsilon(3\varepsilon) - \delta_\varepsilon(2\varepsilon)\big] + \cdots + \big[\delta_\varepsilon(t) - \delta_\varepsilon(n\varepsilon)\big].$$

Since the functions $u_{\varepsilon,i}$ solve in $[0,\varepsilon]$ the geometric and translation-invariant equation $\partial_t u_{\varepsilon,i} = 2\Phi(Du_{\varepsilon,i})(-\kappa_s(x, \{u_{\varepsilon,i} \geqslant u_i(x,t)\}))$, the distance between their 0-level sets is nondecreasing, so that $\delta_\varepsilon(\varepsilon) \geqslant \delta_\varepsilon(0)$. Moreover, since the $u_{\varepsilon,i}$'s solve in $(\varepsilon, 2\varepsilon]$ the equation $\partial_t u_{\varepsilon,i} = 2\Phi(Du_{\varepsilon,i})c_{\varepsilon,i}$, by Lemma 6.1 we get that

$$\delta_\varepsilon(2\varepsilon) \geqslant \delta_\varepsilon(\varepsilon) - 2\varepsilon\big(c_{\varepsilon,2}(2\varepsilon) - c_{\varepsilon,1}(2\varepsilon)\big).$$

By iterating this argument we obtain that

$$\begin{cases} \delta_\varepsilon(k\varepsilon) - \delta_\varepsilon\big((k-1)\varepsilon\big) \geqslant 0 & \text{if } k \text{ is odd,} \\ \delta_\varepsilon(k\varepsilon) - \delta_\varepsilon\big((k-1)\varepsilon\big) \geqslant -2\varepsilon\big(c_{\varepsilon,2}(k\varepsilon) - c_{\varepsilon,1}(k\varepsilon)\big) & \text{otherwise.} \end{cases}$$

By summing in k we then get

$$\delta_\varepsilon(n\varepsilon) \geqslant \delta_\varepsilon(0) + 2\varepsilon \sum_{k=1}^{n} \big(c_{\varepsilon,2}(k\varepsilon) - c_{\varepsilon,1}(k\varepsilon)\big) = \delta_\varepsilon(0) + \int_0^{n\varepsilon} \big(g_2(\tau) - g_1(\tau)\big)\, d\tau\,.$$

By passing to the limit as $\varepsilon \to 0$, the thesis follows. □

Thanks to the previous proposition, by reasoning exactly as in the proofs of [7, Theorem 4.9] (which is based in turn on [6, Theorem 8.4]), we get the following corollary:

Corollary 6.3 *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous function. Let* $C_1 \subseteq C_2$ *be two compact convex sets and let* $C_1(t)$ *and* $C_2(t)$ *be the flows for the equation* $\mathfrak{A}^{-1}v = -\kappa_s + g$, *starting from* C_1 *and* C_2 *respectively. Then* $C_1(t) \subseteq C_2(t)$ *for all* $t \geqslant 0$.

Proof. Notice first that, if C_1 has empty interior, then $C_1(t) = \emptyset$ for all $t > 0$, and there is nothing to prove. Therefore, we can assume that C_1 has nonempty interior and that 0 lies in the interior of C_1.

For $\theta > 1$, we let $g_1(t) := g(t)$, $g_2(t) := g(t/\theta^{1+s})/\theta^s$. Notice that the set $\theta C_2(t/\theta^{1+s})$ solves the equation $\mathfrak{A}^{-1}v = -\kappa_s + g(t/\theta^{1+s})/\theta^s$, with initial datum θC_2. Therefore, letting $\delta_\theta(t) := \operatorname{dist}(\partial C_1(t), \theta\partial C_2(r/\theta^{1+s}))$, so that $\delta_\theta(0) > 0$, by Proposition 6.2 we get

$$\delta_\theta(t) \geqslant \delta_\theta(0) - \int_0^t \left(\frac{1}{\theta^s} g\left(\frac{\tau}{\theta^{1+s}}\right) - g(\tau)\right) d\tau$$

until the first contact time. Now, the integral can be estimated by

$$(\theta - 1)\int_0^{t/\theta^{1+s}} g(\tau)d\tau + \int_{t/\theta^{1+s}}^{t} g(\tau)d\tau \leqslant 3t(\theta - 1)\|g\|_\infty,$$

while $\delta_\theta(0) \geqslant c(\theta - 1)$ where c depends only on the C_1 and Φ. Hence, $\delta_\theta(t) \geqslant 0$ as long as $t \leqslant c/(3\|g\|_\infty)$, which does not depend on θ. It follows that $C_1(t) \subseteq C_2(t)$, this as long as C_1 has nonempty interior, which concludes the proof. □

REMARK 6.4 The same proof shows that, in general, a strictly star-shaped domain with respect to a center point x will have a unique evolution for a positive time, as long as no line issued from x becomes tangent to its boundary.

7. Concluding remarks

A natural question is whether one can characterize the sets which evolve homothetically by the anisotropic flow (2.5), with $g = 0$. A way to build such sets could be by first showing existence of evolutions with constant volume (by tuning appropriately the forcing term as in [7, 24]) and then studying their asymptotic limit. Anyway, the characterization of the limiting equilibrium shape seems to be a difficult question, related to the anisotropic fractional isoperimetric problem.

More precisely, it is known (see [25]) that the $\mathfrak{N}-$fractional perimeter converges, as $s \to 1$, to an anisotropic perimeter with a specific anisotropy different from $\mathfrak{N}$, yielding an indication on the behavior of the isoperimetric sets in this limit.

Another natural question is whether the $\mathfrak{N}-$fractional perimeter is decreasing under the limit flow, in absence of the forcing term. This is true in the isotropic case, and can be easily seen by writing the flow in [10] as a minimizing movement scheme as in [15, 24], and is probably true also in our case. However a complete proof would require a thorough study of the properties of the kernel P_h.

In this respect, it would be interesting to extend the analysis in [15, 24] to the fractional case.

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