

On the propagation of a periodic flame front by an Arrhenius kinetic

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We consider the propagation of a flame front in a solid periodic medium. The model is governed by a free boundary system in which the front's velocity depends on the temperature via an Arrhenius kinetic. We show the existence of travelling wave solutions and consider their homogenization as the period tends to zero. The main difficulty lies in the degeneracy of the Arrhenius function which requires an a priori lower bound of the propagation's speed. We next analyze the curvature effects on the homogenization and obtain a continuum of limiting waves parametrized by the ratio “curvature coefficient/period.” Remarkable features are the monotonicity of the speed with respect to the “curvature regime,” together with the explicit computations of the minimal and maximal speeds. We finally identify the asymptotic expansion of the heterogeneous front's profile with respect to the period.

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1. Introduction

We investigate the propagation of a flame front in a solid heterogeneous medium $\mathbb{R}_x \times \mathbb{R}_y$. Throughout $Y > 0$ is a fixed period and we consider a solid with horizontal striations which are Y -periodic in y . The fresh region is assumed to be a hypograph $\{x < \xi(y, t)\}$ with a temperature $T = T(x, y, t)$. The flame front $\{x = \xi(y, t)\}$ is assumed to propagate to the left. A typical representation is given in Figure 1, which shows the propagation through a medium consisting of a periodic superposition of two layers.

We consider the model where the evolution of (ξ, T) is governed by the free boundary system

$$\begin{cases} b T_t - \operatorname{div}(a \nabla T) = 0, & x < \xi(y, t), t > 0, \\ \xi_t + R(y, T) \sqrt{1 + \xi_y^2} = \mu \frac{\xi_{yy}}{1 + \xi_y^2}, & x = \xi(y, t), t > 0, \end{cases} \quad (1.1)$$

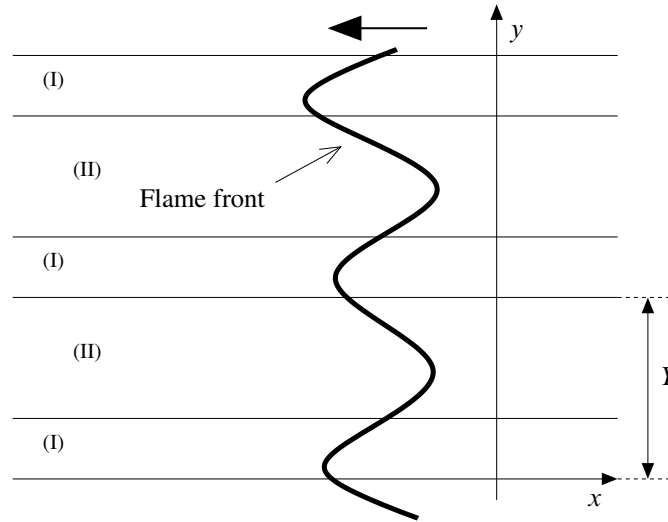


FIG. 1. Periodic superposition of two materials (I) and (II)

subject to the boundary conditions

$$\begin{cases} a \frac{\partial T}{\partial \nu} = g V_n, & x = \xi(y, t), \\ T(x, y, t) \rightarrow 0, & \text{as } x \rightarrow -\infty, \end{cases} \quad (1.2)$$

where $\nu = \frac{(1, -\xi_y)}{\sqrt{1+\xi_y^2}}$ is the outward unit normal and V_n is the normal velocity of the front. Throughout “div,” “ ∇ ” and “ $\frac{\partial}{\partial \nu}$ ” denote the divergence, gradient and normal derivative operators, respectively. The second equation of (1.1) just states that the front propagates with a normal velocity V_n given by

$$V_n = -R(y, T) - \mu \kappa,$$

where κ is the mean curvature and μ is a positive *curvature coefficient*. This latter coefficient is related to *surface tension*¹ effects. The propagation is not just a geometric one (as it was the case for example in [6, 7, 12]) because here the *combustion rate* depends also on the temperature at the front, that is $R = R(y, T)$. This dependence is typically given by an *Arrhenius kinetic* of the form:

$$R = A e^{-\frac{E}{T}}, \quad (1.3)$$

where A is a *prefactor* and E is related to the *activation energy*. These parameters can depend on the layers which means that $A = A(y)$ and $E = E(y)$. As far as the other data are concerned, $a = a(y)$ represents the *thermal diffusivity*, $b = b(y)$ the *heat capacity*, and $g = g(y)$ represents a fraction of the total *heat release* and which serves to heat the solid thus making the combustion self-sustained. Finally, note that the downstream boundary condition in (1.2) corresponds to a normalization to

¹ Rigorously speaking, as here we have a solid/gas interface, we must rather talk of *interface energy* instead of *surface tension* which is usually used for liquid interfaces.

zero of the temperature far from the front. For more details about this model see [3, 6, 7, 12] and the references therein.

From now on we assume that each parameter $f = a, b, g$ satisfies:

- (A) The function $f : \mathbb{R} \rightarrow \mathbb{R}$ is measurable and Y -periodic.
 (B) There are constants $f_M \geq f_m > 0$ such that

$$f_M \geq f(y) \geq f_m \quad \text{for almost each } y \in \mathbb{R}.$$

The parameter R is assumed to be a function from $\mathbb{R} \times \mathbb{R}^+$ into \mathbb{R}^+ such that:

- (C) For almost each $y \in \mathbb{R}$, $T \mapsto R(y, T)$ is continuous and nondecreasing.
 (D) For each $T > 0$, $y \mapsto R(y, T)$ is measurable and Y -periodic.
 (E) There is a constant $R_M > 0$ such that

$$R_M \geq R(y, T) > 0 \quad \text{for almost each } y \in \mathbb{R} \text{ and all } T > 0.$$

- (F) For any $T > 0$, we have:

$$\operatorname{ess\,inf}_{y \in \mathbb{R}} R(y, T) > 0.$$

In the above, \mathbb{R}^+ denotes the interval $(0, +\infty)$. The function R is in particular of Carathéodory's type by (C)–(D). Note that (C)–(F) are satisfied by the typical combustion rate in (1.3) whenever A and E satisfy (A)–(B). This is exactly what occurs for the striated solid medium we consider where all these parameters are constant on the layers. They therefore take just two values for the case presented in Figure 1. Let us point out at this stage that for reasons associated to mathematical analysis, the existing papers usually impose a “positive lower bound to the front's speed” – that is to R in our case –, see for instance [3, 6–8, 11, 12]. In our setting, this might be not satisfactory because the typical R in (1.3) decays towards zero as $T \downarrow 0$. Actually, we will be brought to occasionally prescribe a “slower decay” to R at the neighborhood of $T = 0$, but we will always treat situations where it may go to zero. Moreover a large part of our results work for the Arrhenius law given in (1.3).

Let us continue with some general comments on the literature. Models of similar type are considered in [3, 6, 7, 11, 12] but with slightly different assumptions. In [3], the striations are vertical and the front's profile is a straight line. As a consequence its equation reduces to an ODE. In [6, 7, 11, 12], the propagation is purely geometric (that is to say $R = R(y)$ only) and the analysis concerns the sole front's equation. In the more recent paper [8], a full free boundary system somewhat similar to ours has been studied in the context of solidification process in crystal growth. The front's propagation is governed by a Hamilton–Jacobi equation (where the surface tension effects are neglected). The main purpose of [8] concerns the global in time existence of a solution of the Cauchy problem via the study of the regularity of the expanding front. The front's velocity considered is relaxed regularitywise: It is just Hölder continuous as compared to the natural Lipschitz regularity used for Hamilton–Jacobi equations. In all these references, the speed of propagation is assumed to have a positive lower bound.

In this paper, we focus on the study of travelling wave solutions to (1.1)–(1.2). Our first purpose is to show the existence of such solutions. This comes to looking for fronts and temperatures of the form

$$\xi(y, t) = -ct + v(y) \quad \text{and} \quad T(x, y, t) = u(x + ct, y),$$

where $c > 0$ will be the speed of the wave and v its profile. It is convenient to fix the front through the change of variable $x + c t \mapsto x$. This leads to the problem of finding a triplet (c, v, u) such that

$$\begin{cases} c b u_x - \operatorname{div}(a \nabla u) = 0, & x < v(y), \\ a \frac{\partial u}{\partial v} = \frac{c g}{\sqrt{1+v_y^2}}, & x = v(y), \\ u(x, y) \rightarrow 0, & \text{as } x \rightarrow -\infty, \end{cases} \tag{1.4}$$

and

$$-c + R(y, u) \sqrt{1 + v_y^2} = \mu \frac{v_{yy}}{1 + v_y^2}, \quad x = v(y). \tag{1.5}$$

Note that if there exists a travelling wave, the profile v will be defined up to an additive constant. For simplification and without loss of generality, we will be brought in the course of the analysis to fix this constant.

Let us recall that Equation (1.5) has been the object of a thorough study in [6] but only for $R = R(y)$. The existence of a travelling wave (c, v) is proved with some characterization of the speed c with respect to the curvature coefficient μ . In [7, 12], this analysis is extended to oblique striations and in [11] to almost periodic media. In this paper, we will establish the existence of a nontrivial travelling wave solution for the whole system (1.1)–(1.2). If we consider a general R satisfying (C)–(F), we will need to assume the ratio “period/ μ ” to be small enough. This includes in particular the combustion rate in (1.3). For general ratios we will need to consider a more restrictive class of (degenerate) R , see (3.2).

We unfortunately do not know whether these travelling wave solutions are unique. Nevertheless we can give some precise characterization by considering their homogenization as the period tends to zero. This is the second purpose of this paper. Let us first mention that a similar homogenization problem has been considered in [3] for media with vertical striations. A remarkable difference with our setting is the absence of surface tension effects in [3] (recall that the front’s profile is a straight line in that case). Here we propose an homogenization analysis that will provide information on the curvature effects too. For that we allow the curvature coefficient to depend on the period. This amounts to consider a family of triplets $(c^\varepsilon, v^\varepsilon, u^\varepsilon)$ satisfying:

$$\begin{cases} c^\varepsilon b \left(\frac{y}{\varepsilon}\right) u_x^\varepsilon - \operatorname{div} \left(a \left(\frac{y}{\varepsilon}\right) \nabla u^\varepsilon \right) = 0, & x < v^\varepsilon(y), \\ a \left(\frac{y}{\varepsilon}\right) \frac{\partial u^\varepsilon}{\partial v} = \frac{c^\varepsilon g \left(\frac{y}{\varepsilon}\right)}{\sqrt{1+(v_y^\varepsilon)^2}}, & x = v^\varepsilon(y), \\ u^\varepsilon(x, y) \rightarrow 0, & \text{as } x \rightarrow -\infty, \end{cases} \tag{1.6}$$

and

$$-c^\varepsilon + R \left(\frac{y}{\varepsilon}, u^\varepsilon \right) \sqrt{1 + (v_y^\varepsilon)^2} = \mu(\varepsilon) \frac{v_{yy}^\varepsilon}{1 + (v_y^\varepsilon)^2}, \quad x = v^\varepsilon(y), \tag{1.7}$$

for some given 1-periodic parameters a, b , etc., so that ε is the new period of the medium. We will show that the triplet $(c^\varepsilon, v^\varepsilon, u^\varepsilon)$ converges towards some (c^0, v^0, u^0) , as $\varepsilon \downarrow 0$, if and only if the ratio $\frac{\mu(\varepsilon)}{\varepsilon}$ converges towards some $\lambda \in [0, +\infty]$. This will define a continuum of homogenized waves parametrized by λ for which several characteristics will be easier to compute such as the speed, the temperature at the front, etc.

As far as the speed is concerned, we will show that the map

$$\lambda \mapsto c^0 = c^0(\lambda)$$

is (smooth and) decreasing. Also, the minimal and maximal speeds are achieved (only) at the regimes $\lambda = +\infty$ and 0 , respectively. For both of these regimes we will derive an explicit formula for c^0 . Surprisingly for $\lambda = 0$, c^0 is not given by some mean of the combustion rate. We shall see for instance that, for the typical model (1.3), it reads

$$c^0(0) = \operatorname{ess\,sup}_z A(z) e^{-E(z) \frac{\bar{b}}{\bar{g}}},$$

where \bar{b} and \bar{g} denote the mean values of b and g , respectively. This suggests that the respective width of each layer has no influence in that regime as compared of course to the other intrinsic parameters of the material. This original feature was already observed in [6] for a pure geometric propagation. Here we somewhat extend this observation for the full system “front-temperature.” Note finally that once the speed is known, the temperature u^0 will be an explicit exponential entirely determined by c^0 .

As far as the front’s profile is concerned, we will need a more careful analysis to get some interesting information. It indeed turns out that the homogenized profile is always a straight line (normalized to zero). We will then analyze the microscopic oscillations of the front’s profile by establishing an asymptotic expansion of the form:

$$v^\varepsilon(y) = \varepsilon w\left(\frac{y}{\varepsilon}\right) + o(\varepsilon),$$

where w will be some corrector. We will show that this corrector is entirely determined by λ and satisfies a pure geometric equation of the form considered in [6, 7, 11, 12]. As a byproduct, our analysis thus provides some relationship between these works and the whole system “front-temperature.” In the limiting regime $\lambda = +\infty$, we will show that the corrector equals zero everywhere so that the front is “almost a straight line” at the microscopic level too. To get more information in that case, we will then establish a second-order expansion of the form:

$$v^\varepsilon(y) = \varepsilon^2 Q\left(\frac{y}{\varepsilon}\right) + o(\varepsilon^2),$$

where we will identify the profile Q too. The identification of w in the other limiting regime $\lambda = 0$ is more difficult and remains open. A more detailed discussion will be done in Section 7.

To conclude, let us emphasize once more that we consider a general speed of propagation where R can degenerate to zero, while this speed is generally assumed positively lower bounded in the literature, see for instance [8] and the references therein on systems. Hence, our first main contribution may be stated as follows:

There exists a travelling wave solution to (1.1)–(1.2) even for possibly degenerate combustion rate R .

Moreover, our homogenization analysis provides valuable information about the influence of the curvature on the propagation. If we should select only one result, we would probably keep the monotonicity of the propagation’s speed. It can be compared with the recent analysis of [10] concerning the effects of strain. Here, we give somewhat a similar analysis but with the curvature. Our second main contribution may thus be stated roughly speaking as follows:

The curvature or surface tension effects slow down the propagation in heterogeneous media.

Let us now give comments on the proofs. The existence of travelling wave solutions will be based on the Schauder's fixed point theorem. The key estimates will be some a priori lower bounds on the temperature at the front. For the homogenization, we will give a rather simple proof of convergence, thanks to the a priori observation that v^0 and u^0 do not depend on y . In the regime $\lambda = 0$, the identification of c^0 will necessitate to call for classical elliptic regularity results. This will be possible even for degenerate Arrhenius functions, thanks to the preceding lower bounds. The proof of the monotonicity of the speed, which will call for the implicit function theorem, will be a bit more complicated.

Let us finally give some more references on related topics. Following the inhomogeneity considered, one can as well obtain pulsating travelling fronts, that is where the speed of the travelling wave is no longer a constant but is periodic in time. The preceding references [7, 11, 12] are closely related to that subject. For a rather complete study in the framework of reaction advection diffusion equations, see [2]. Finally, for a survey on travelling waves, be it in homogeneous, periodic or heterogeneous random media, see [14].

The rest of this paper is organized as follows. Section 2 is devoted to some preliminaries on the equations of the front and temperature considered separately. Section 3 is devoted to the existence of a travelling wave solution to (1.1)–(1.2), see Theorems 3.4 and 3.8. The homogenization analysis starts in Section 4, see Theorems 4.2 and 4.5. The qualitative analysis of the speed is done in Section 5, see Theorem 5.1. The asymptotic expansions of the front's profile are given in Section 6, see Theorems 6.1, 6.3 and 6.5. A synthesis and some open questions are proposed in Section 7. For the sake of clarity, the technical or standard proofs are postponed in appendices together with a list of the main specific notations.

2. Preliminaries: Notations and first results

The existence of a travelling wave solution will be proved with the help of the Schauder's fixed point theorem by successively freezing the temperature and the front. In this section we focus on the frozen problems. All along this section, $Y > 0$ is a given fixed period and $\mu > 0$ a given fixed parameter.

2.1 Front's well-posedness

We first consider (1.5) for a fixed temperature. This comes therefore to finding (c, v) which solves

$$\begin{cases} -c + H(y)\sqrt{1 + v_y^2} = \mu \frac{v_{yy}}{1 + v_y^2}, \\ v(y + Y) = v(y), \end{cases} \quad (2.1)$$

for (almost) every $y \in \mathbb{R}$, where H is an arbitrary given function. Here is a result from [6].

Theorem 2.1 *Let $H : \mathbb{R} \rightarrow \mathbb{R}$ be measurable and Y -periodic with*

$$H_m \leq H \leq H_M \quad \text{almost everywhere on } \mathbb{R},$$

for some positive constants H_m and H_M . Then there exists $(c, v) \in \mathbb{R} \times W^{2,\infty}(\mathbb{R})$ which satisfies (2.1) almost everywhere. The speed c is unique and the profile v is unique up to an additive constant. Moreover,

$$H_m \leq c \leq H_M \quad \text{and} \quad \|v_y\|_\infty \leq \sqrt{\frac{H_M^2}{H_m^2} - 1}. \tag{2.2}$$

2.2 Temperature’s well-posedness

Now we suppose that the profile v of the front is given and we introduce the notations

$$\Omega := \{(x, y) \in \mathbb{R}^2 : x < v(y)\}, \quad \Gamma := \{(x, y) \in \mathbb{R}^2 : x = v(y)\},$$

as well as the corresponding restrictions to one period,

$$\Omega_\# := \Omega \cap \{0 < y < Y\} \quad \text{and} \quad \Gamma_\# := \Gamma \cap \{0 < y < Y\}.$$

For simplicity, we do not specify the dependence on v .

Periodic Sobolev’s spaces. We proceed by defining the functional framework that we will need. We use the subscript “#” for spaces of functions which are Y -periodic in y . Hereafter we consider the Hilbert spaces

$$L_\#^2(\Omega) := \left\{ u \in L_{loc}^1(\overline{\Omega}) : u \text{ is } Y\text{-periodic in } y \text{ and } \int_{\Omega_\#} u^2 < +\infty \right\},$$

$$H_\#^1(\Omega) := \left\{ u \in L_\#^2(\Omega) : \nabla u \in (L_\#^2(\Omega))^2 \right\},$$

endowed with the following norms and semi-norm:

$$\|u\|_{L_\#^2(\Omega)} := \left(\frac{1}{Y} \int_{\Omega_\#} u^2 \right)^{\frac{1}{2}}, \quad |u|_{H_\#^1(\Omega)} := \left(\frac{1}{Y} \int_{\Omega_\#} |\nabla u|^2 \right)^{\frac{1}{2}}$$

and $\|u\|_{H_\#^1(\Omega)} := \left(\|u\|_{L_\#^2(\Omega)}^2 + |u|_{H_\#^1(\Omega)}^2 \right)^{\frac{1}{2}}.$

Extension operator. We next define the extension of u to \mathbb{R}^2 , which we will use throughout, by “reflection” as follows:

$$\text{ext}(u)(x, y) := \begin{cases} u(x, y), & x < v(y), \\ u(2v(y) - x, y), & x > v(y). \end{cases} \tag{2.3}$$

Recall that $\text{ext} : H_\#^1(\Omega) \rightarrow H_\#^1(\mathbb{R}^2)$ is linear and bounded with

$$\|\text{ext}(u)\|_{H_\#^1(\mathbb{R}^2)} \leq C (\|v_y\|_\infty) \|u\|_{H_\#^1(\Omega)}; \tag{2.4}$$

see [1].² For brevity, $\text{ext}(u)$ will be simply denoted by u .

² Note that $C (\|v_y\|_\infty)$ does not depend on the period Y , which is immediate by direct computations.

Trace operator. The front Γ will be endowed with its superficial measure, that is for any $\varphi \in C_c(\Gamma)$,

$$\int_{\Gamma} \varphi := \int_{\mathbb{R}} \varphi(v(y), y) \sqrt{1 + v_y^2(y)} \, dy.$$

For brevity, we will denote by $u|_{\Gamma}$ or simply u the trace of $u \in H_{\#}^1(\Omega)$ on Γ , see [1]. Recall that $u \in H_{\#}^1(\Omega) \mapsto u|_{\Gamma} \in H_{\#}^{\frac{1}{2}}(\Gamma)$ is well-defined linear and bounded, so that the function

$$w : y \in \mathbb{R} \mapsto u(v(y), y) \in \mathbb{R}$$

is well-defined (up to some negligible set) and belongs to $H_{\#}^{\frac{1}{2}}(\mathbb{R})$ with

$$\|w\|_{H_{\#}^{\frac{1}{2}}(\mathbb{R})} \leq C(Y, \|v_y\|_{\infty}) \|u\|_{H_{\#}^1(\Omega)}.$$

Well-posedness. We can now state the well-posedness of the temperature.

DEFINITION 2.2 (Variational solutions) Assume (A)–(B) and consider $c \in \mathbb{R}$ and $v \in W_{\#}^{1,\infty}(\mathbb{R})$. We say that u is a *variational solution* to (1.4) if

$$\begin{cases} u \in H_{\#}^1(\Omega), \\ \int_{\Omega_{\#}} (c b u_x w + a \nabla u \nabla w) = \int_{\Gamma_{\#}} \frac{c g}{\sqrt{1+v_y^2}} w, \quad \forall w \in H_{\#}^1(\Omega). \end{cases} \tag{2.5}$$

REMARK 2.3 Note that u is a variational solution to (1.4) if and only if

$$\begin{cases} u \in H_{\#}^1(\Omega), \\ \int_{\Omega} (c b u_x \varphi + a \nabla u \nabla \varphi) = \int_{\Gamma} \frac{c g}{\sqrt{1+v_y^2}} \varphi, \quad \forall \varphi \in C_c^1(\mathbb{R}^2). \end{cases} \tag{2.6}$$

The proof of the above remark is standard and postponed in Appendix A.1 for completeness.

Theorem 2.4 Assume (A)–(B) and let $c > 0$ and $v \in W_{\#}^{1,\infty}(\mathbb{R})$. Then there exists a unique variational solution $u \in H_{\#}^1(\Omega)$ to (1.4). Moreover, this solution satisfies

$$|u|_{H_{\#}^1(\Omega)} \leq \frac{2 c g_M^2}{a_m b_m}, \tag{2.7}$$

and for almost every $(x, y) \in \Omega$,

$$\frac{g_m a_m}{a_M b_M} e^{c \frac{b_M}{a_m} (x - \|v\|_{\infty})} \leq u(x, y) \leq \frac{g_M a_M}{a_m b_m} e^{c \frac{b_m}{a_M} (x + \|v\|_{\infty})}. \tag{2.8}$$

The proof of this result is rather standard. The main ideas are given below. For completeness sake, the details are postponed in Appendix A.2.

Sketch of the proof. It suffices to use general variational methods (see [4] for instance). The main difficulty to apply the Lax–Milgram’s theorem is the lack of a Poincaré inequality (because the domain is not bounded and there is no u -term). But, this can be compensated by the bounds in (2.8). To show these bounds, it suffices to take sub- and supersolutions of the form $(x, y) \mapsto C_1 e^{C_2 x}$. Notice finally that (2.7) is obtained by choosing u as a test function while taking advantage of the sign of the convection term $c b u_x$ (after integration). \square

2.3 Stability

Let us continue with further properties that will be needed later. It deals with the passage to the limit in a sequence of problems of the form

$$\begin{cases} c_n b(u_n)_x - \operatorname{div}(a \nabla u_n) = 0, & x < v_n(y), \\ a \frac{\partial u_n}{\partial v} = \frac{c_n g}{\sqrt{1+(v_n)_y^2}}, & x = v_n(y), \\ u_n(x, y) \rightarrow 0, & \text{as } x \rightarrow -\infty, \end{cases} \quad (2.9)$$

$$-c_n + H_n(y) \sqrt{1+(v_n)_y^2} = \mu \frac{(v_n)_{yy}}{1+(v_n)_y^2}, \quad y \in \mathbb{R}. \quad (2.10)$$

For a technical reason, we will need to fix the front’s profile by assuming for instance that it has a zero mean value. This will be done without loss of generality, since the solution of (2.10) is unique up to an additive constant. For brevity, we will denote throughout by $\bar{f} := \frac{1}{Y} \int_0^Y f$ the mean value of any Y -periodic function $f : \mathbb{R} \rightarrow \mathbb{R}$.

Here is a compactity result.

Lemma 2.5 (Compactness) *Let us assume that (A)–(B) hold and that for each $n \in \mathbb{N}$,*

$$\begin{cases} H_n \in L^\infty(\mathbb{R}), \\ c_n > 0, v_n \in W_\#^{2,\infty}(\mathbb{R}), \\ u_n \in H_\#^1(\Omega_n), \\ v_n \text{ satisfies (2.10) almost everywhere with } \bar{v}_n = 0, \\ u_n \text{ is a variational solution to (2.9),} \end{cases}$$

where $\Omega_n := \{x < v_n(y)\}$. Then, if

$$0 < \inf_n \operatorname{ess\,inf}_y H_n(y) \leq \sup_n \operatorname{ess\,sup}_y H_n(y) < +\infty, \quad (2.11)$$

there exists $(H, c, v, u) \in L^\infty(\mathbb{R}) \times \mathbb{R}_+ \times W_\#^{2,\infty}(\mathbb{R}) \times H_\#^1(\mathbb{R}^2)$ such that

$$\begin{cases} H_n \rightharpoonup H \text{ in } L^\infty(\mathbb{R}) \text{ weak-}\star, \\ c_n \rightarrow c, v_n \rightharpoonup v \text{ in } W^{2,\infty}(\mathbb{R}) \text{ weak-}\star, \\ u_n \rightharpoonup u \text{ weakly in } H_\#^1(\mathbb{R}^2), \end{cases} \quad (2.12)$$

up to some subsequence.

REMARK 2.6 (i) The limit in (2.12) has to be understood for u_n extended to \mathbb{R}^2 by reflection, see (2.3).

(ii) The convergence in (2.12) implies that $v_n \rightarrow v$ and $u_n \rightarrow u$ strongly in $W^{1,\infty}(\mathbb{R})$ and $L_\#^2(\mathbb{R}^2)$, respectively.

Proof. In this proof, the letter C denotes various constants independent of n . By (2.11) and Theorem 2.1(2.2), $\{c_n\}_n$ is bounded and $\|(v_n)_y\|_\infty \leq C$. This leads to $\|v_n\|_\infty \leq C$ because $\bar{v}_n = 0$. Moreover, $\|(v_n)_{yy}\|_\infty \leq C$ by (2.10). By Theorem 2.4(2.7) and (2.8), we deduce that $\|u_n\|_{H_\#^1(\Omega_n)} \leq C$ and, after extending u_n to \mathbb{R}^2 , $\|u_n\|_{H_\#^1(\mathbb{R}^2)} \leq C$ by (2.4). The proof is complete by standard weak compactness theorems. \square

Let us now proceed by giving the stability results whose proofs are postponed in Appendix A.3.

Lemma 2.7 (Stability) *Let (H, c, v, u) be given by Lemma 2.5. Then (c, v, u) is a solution of (1.4) and (2.1), that is*

$$\left\{ \begin{array}{l} v \text{ satisfies (2.1) almost everywhere,} \\ \text{and } u|_{\Omega} \text{ is a variational solution to (1.4).} \end{array} \right.$$

Lemma 2.8 (Strong convergence of the traces) *Let (H, c, v, u) be as in the preceding lemmas. We then have $u_n(v_n(y), y) \rightarrow u(v(y), y)$ for almost every $y \in \mathbb{R}$ (up to the subsequence considered in the preceding lemmas or one of its subsequences).*

3. Existence of a travelling wave solution

Let us now look for a solution to (1.4)–(1.5). We continue to use the notations of the preceding section. In particular, we still do not specify the dependence in v of the sets $\Omega = \{x < v(y)\}$ and $\Gamma = \{x = v(y)\}$ to simplify. Moreover $Y > 0$ is a given fixed period and $\mu > 0$ a given fixed coefficient all along this section too.

DEFINITION 3.1 (Travelling wave solution) *Assume (A)–(F). A triplet (c, v, u) is said to be a travelling wave solution to (1.1)–(1.2) if*

- (i) $c \in \mathbb{R}, v \in W_{\#}^{2,\infty}(\mathbb{R})$,
- (ii) $u \in H_{\#}^1(\Omega), u \geq 0$,
- (iii) u is a variational solution to (1.4) and v satisfies (1.5) almost everywhere.

We first deal with the case where in addition to (C)–(F) the parameter R is also bounded from below by some positive constant R_m , that is to say:

$$R(y, T) \geq R_m > 0 \quad \text{for almost each } y \in \mathbb{R} \text{ and all } T > 0. \tag{3.1}$$

We will next discuss the more general case where R can go to zero.

3.1 The case of nondegenerate R

Theorem 3.2 *Assume (A)–(F) and (3.1). Then there exists a travelling wave solution (c, v, u) to (1.1)–(1.2) with a positive speed c .*

Proof. The idea is to look for a fixed point of some appropriate function $\Phi : \mathcal{C} \rightarrow \mathcal{C}$.

Let us first choose the set

$$\mathcal{C} := \{H \in L_{\#}^{\infty}(\mathbb{R}) : R_m \leq H \leq R_M\}.$$

We will use the Schauder–Tikhonov’s fixed point theorem thus needing this set to be convex and compact. It is clearly convex and to get the compactity we simply consider the $L^{\infty}(\mathbb{R})$ weak- \star topology on \mathcal{C} .

Let us now choose Φ . Given $H \in \mathcal{C}$, we can apply successively Theorems 2.1 and 2.4. We find that there exists a triplet (c, v, u) solution of (1.4) and (2.1). This triplet, which of course depends on H , is unique under the additional condition that $\bar{v} = 0$. We can then define the function

$$\begin{array}{lcl} \Phi : & \mathcal{C} & \rightarrow \mathcal{C} \\ & H & \mapsto \Phi(H) : y \mapsto R(y, u(v(y), y)). \end{array}$$

Now it only remains to show that Φ is continuous. Note that, rigorously, we should also verify that Φ is well-defined. This means that two arbitrary almost everywhere representatives of H should give us two almost everywhere equal measurable functions $\Phi(H)$. This is in fact quite standard because R is a Carathéodory function by (C)–(D). The detailed verification of the well-definition of Φ is thus left to the reader.

Let us then continue by proving that Φ is continuous for the $L^\infty(\mathbb{R})$ weak- \star topology. We can argue with sequences because this topology is metrizable on bounded sets (such as \mathcal{C}). Let thus $\mathcal{C} \ni H_n \rightharpoonup H$ in $L^\infty(\mathbb{R})$ weak- \star . By the construction above,

$$0 < R_m \leq \inf_n \operatorname{ess\,inf} H_n(y) \leq \sup_n \operatorname{ess\,sup} H_n(y) \leq R_M < +\infty.$$

We can then apply Lemma 2.5. This compactity result implies that (H_n, c_n, v_n, u_n) converges to some (\tilde{H}, c, v, u) in the sense of (2.12) (and up to some subsequence). Note that (c_n, v_n, u_n) is the unique triplet associated to H_n as above. Note also that $\tilde{H} = H$ by uniqueness of the limit of H_n . Moreover, the triplet (c, v, u) satisfies (1.4) and (2.1) with our given H , thanks to the stability result of Lemma 2.7. Using then Lemma 2.8 and (C), we deduce that $\Phi(H_n) \rightarrow \Phi(H)$ almost everywhere. As this sequence is bounded, the convergence holds also in $L^\infty(\mathbb{R})$ weak- \star , up to some subsequence. To conclude the convergence of the whole sequence, we apply this reasoning by starting from any arbitrary subsequence of H_n . We deduce that Φ is continuous, since the limit of the obtained converging subsubsequence is always the same, that is $\Phi(H)$.

Finally the Schauder–Tikhonov’s theorem gives us a fixed point $\Phi(H) = H$, whose associated triplet (c, v, u) is a travelling wave solution. Since R is assumed bounded from below by $R_m > 0$, the positivity of c is ensured by Theorem 2.1. \square

3.2 More general R

Now we want to deal with the case where R may go to zero at $T = 0$. For technical reasons, we will restrict to parameters with the following prescribed behavior at zero:

$$\lim_{T \downarrow 0} \left\{ |\ln T| \operatorname{ess\,inf}_{y \in \mathbb{R}} R(y, T) \right\} = +\infty. \tag{3.2}$$

This is for instance the case if $\operatorname{ess\,inf}_y R(y, T) \sim \frac{C}{|\ln T|^\alpha}$ as $T \downarrow 0$ for some $\alpha \in (0, 1)$ and some $C > 0$.

We start by giving a result which establishes an a priori positive lower bound for the temperature at the front.

Lemma 3.3 *Assume (A)–(F) and let (c, v, u) be a travelling wave solution to (1.1)–(1.2) with a positive speed c . If in addition (3.2) holds true, then*

$$u \geq \min \left\{ T > 0 : \ln(T) \operatorname{ess\,inf}_y R(y, T) \geq -C(1 + Y) \right\} > 0$$

almost everywhere on Γ , for some constant $C = C(a_m, g_m, a_M, b_M, R_M) \geq 0$.

Proof. It is sufficient to consider the case where $\bar{v} = 0$ (otherwise one can always consider another travelling wave solution of the same problem with the triplet $(c, \tilde{v}, \tilde{u})$, where $\tilde{v}(y) := v(y) - \bar{v}$ and $\tilde{u}(x, y) := u(x + \bar{v}, y)$).

Set $u_m := \text{ess inf}_\Gamma u$. By (C), we have

$$\text{ess inf}_y R(y, u(v(y), y)) \geq \text{ess inf}_y R(y, u_m).$$

Note that, at this stage, we could have $u_m = 0$. But, as $c > 0$, Theorem 2.4 implies that

$$u_m \geq \frac{g_m a_m}{a_M b_M} e^{-2c \frac{b_M}{a_m} \|v\|_\infty}.$$

The boundedness of v then ensures that $u_m > 0$ and a fortiori so is $\text{ess inf}_y R(y, u_m)$ by (F). Now we can apply Theorem 2.1 to get

$$c \leq R_M \quad \text{and} \quad \|v_y\|_\infty \leq \sqrt{\frac{R_M^2}{\text{ess inf}_y R(y, u_m)^2} - 1}.$$

Since $\bar{v} = 0$, $\|v\|_\infty \leq Y \|v_y\|_\infty$ and

$$u_m \geq \frac{g_m a_m}{a_M b_M} e^{-2c \frac{b_M}{a_m} Y \sqrt{\frac{R_M^2}{\text{ess inf}_y R(y, u_m)^2} - 1}} \geq \frac{g_m a_m}{a_M b_M} e^{-2R_M \frac{b_M}{a_m} Y \frac{R_M}{\text{ess inf}_y R(y, u_m)}}.$$

Taking the logarithm,

$$\ln u_m \geq -C - \frac{C Y}{\text{ess inf}_y R(y, u_m)},$$

for some constant C having the dependence stated in the lemma. Multiplying by $\text{ess inf}_y R(y, u_m)$ and using the fact that $0 < \text{ess inf}_y R(y, u_m) \leq R_M$, we deduce that

$$\ln(u_m) \text{ess inf}_y R(y, u_m) \geq -C R_M - C Y. \quad \square$$

We are now ready to give the analogous of Theorem 3.2 under the more general Assumption (3.2).

Theorem 3.4 *Assume (A)–(F) and (3.2). There then exists a travelling wave solution (c, v, u) to (1.1)–(1.2) with a positive speed c .*

Proof. Let us consider $R_n := \max\{R, \frac{1}{n}\}$. We can apply Theorem 3.2 to get the existence of some nontrivial travelling wave solution (c_n, v_n, u_n) with this truncated parameter. By Lemma 3.3 and (3.2), $\text{ess inf}_{\Gamma_n} u_n \geq \gamma_n$ for

$$\gamma_n := \min\{T > 0 : \ln(T) \text{ess inf}_y R_n(y, T) \geq -C(1 + Y)\} > 0,$$

where C is independent of n . But as $R_n \geq R$, we have

$$\gamma_n \geq \gamma := \min\{T > 0 : \ln(T) \text{ess inf}_y R(y, T) \geq -C(1 + Y)\} > 0,$$

for all n . Thanks to (F), we can now choose n_0 large enough such that

$$\frac{1}{n_0} \leq \text{ess inf}_y R(y, \gamma).$$

This gives us that

$$R_{n_0}(y, u_{n_0}(v_{n_0}(y), y)) = R(y, u_{n_0}(v_{n_0}(y), y)) \quad \text{for almost every } y \in \mathbb{R}.$$

In particular, the triplet $(c, v, u) := (c_{n_0}, v_{n_0}, u_{n_0})$ is a solution to (1.4)–(1.5) and the proof is complete. \square

REMARK 3.5 The proof suggests that the assumption (3.2) can be slightly relaxed. The key result was indeed the lower bound of Lemma 3.3. Assumption (3.2) has only been used to imply that

$$\min \{T > 0 : \ln(T) \operatorname{ess\,inf}_y R(y, T) \geq -C(1 + Y)\} > 0,$$

where $C = C(a_m, g_m, a_M, b_M, R_M)$. It thus suffices to directly assume that this minimum is positive. This would be for instance the case if

$$\operatorname{ess\,inf}_y R(y, T) \sim \frac{\tilde{C}}{|\ln T|} \quad \text{as } T \downarrow 0,$$

for some $\tilde{C} > C(1 + Y)$.

3.3 The case of small periods

We finally consider the case of small Y or more precisely of large values of the ratio $\frac{\mu}{Y}$. For that case, we will show the existence of a nontrivial travelling wave solution without the preceding assumption (3.2). It includes in particular the typical model (1.3). Let us start with an estimate on v_y .

Lemma 3.6 *Let $H \in L^\infty(\mathbb{R})$ be Y -periodic, nonnegative, and assume that the pair $(c, v) \in \mathbb{R}^+ \times W_{\#}^{2,\infty}(\mathbb{R})$ satisfies Equation (2.1) almost everywhere. Then we have the estimate: $\|\arctan(v_y)\|_\infty \leq \frac{2cY}{\mu}$.*

Proof. Let us define $f(y) := H(y)\sqrt{1 + v_y^2(y)}$ and $F(y) := \mu \arctan(v_y(y))$. These functions are Y -periodic and satisfy

$$F'(y) = f(y) - c,$$

thanks to Equation (2.1). Integrating over one period, we first deduce that $c = \overline{f}$. Integrating then over one arbitrary interval (y_*, y) , such that $F(y_*) = 0$, we deduce that

$$\|F\|_\infty \leq \int_0^Y |F'| \leq 2cY$$

(since $f \geq 0$ and $\int_0^Y f = cY$ by what precedes). The proof is complete by the definition of F . \square

Let us now give a new lower bound for the temperature at the front.

Lemma 3.7 *Assume (A)–(F) and let (c, v, u) be a travelling wave solution to (1.1)–(1.2) with a positive speed c . If in addition $\frac{\mu}{Y} > \frac{4R_M}{\pi}$, then*

$$u \geq \frac{g_m a_m}{a_M b_M} e^{-\frac{2R_M b_M Y}{a_M} \tan\left(\frac{2R_M Y}{\mu}\right)} > 0 \quad \text{almost everywhere on } \Gamma.$$

Proof. By the estimate (2.8) of Theorem 2.4 and the upper bound $c \leq R_M$ given by Theorem 2.1,

$$\operatorname{ess\,inf}_\Gamma u \geq \frac{g_m a_m}{a_M b_M} e^{-\frac{2R_M b_M Y}{a_m} \|v_y\|_\infty}.$$

The proof is complete by applying the previous lemma. □

Here is finally our existence result for large ratio $\frac{\mu}{Y}$.

Theorem 3.8 *Let us assume that (A)–(F) hold together with the following condition:*

$$\frac{\mu}{Y} > \frac{4 R_M}{\pi}.$$

Then there exists a solution (c, v, u) to (1.4) and (1.5) with a positive speed c .

The proof is exactly the same as for Theorem 3.4, but this time we use Lemma 3.7 instead of Lemma 3.3 to bound the temperature at the front from below.

4. Homogenization

We will now be interested in the ε -dependent free boundary problem

$$\begin{cases} c^\varepsilon b\left(\frac{y}{\varepsilon}\right) u_x^\varepsilon - \operatorname{div}\left(a\left(\frac{y}{\varepsilon}\right) \nabla u^\varepsilon\right) = 0, & x < v^\varepsilon(y), \\ a\left(\frac{y}{\varepsilon}\right) \frac{\partial u^\varepsilon}{\partial v} = \frac{c^\varepsilon g\left(\frac{y}{\varepsilon}\right)}{\sqrt{1+(v_y^\varepsilon)^2}}, & x = v^\varepsilon(y), \\ u^\varepsilon(x, y) \rightarrow 0, & \text{as } x \rightarrow -\infty, \end{cases} \quad (4.1)$$

and

$$-c^\varepsilon + R\left(\frac{y}{\varepsilon}, u^\varepsilon\right) \sqrt{1+(v_y^\varepsilon)^2} = \mu \frac{v_{yy}^\varepsilon}{1+(v_y^\varepsilon)^2}, \quad x = v^\varepsilon(y), \quad (4.2)$$

where $\mu > 0$ is a fixed curvature coefficient and a, b, g, R , are fixed 1-periodic parameters assumed to satisfy (A)–(F) (thus with $Y = 1$). The new parameter ε is the period of the medium. Note that the normal v depends on ε too (which is not specified in (4.1)–(4.2) for simplicity). The purpose of this section is to find the limit of $(c^\varepsilon, v^\varepsilon, u^\varepsilon)$ as $\varepsilon \downarrow 0$.

To avoid confusion with the preceding notations, the new fresh region, front, etc., will be denoted differently. More precisely, we will denote by

$$\Omega^\varepsilon := \{(x, y) : x < v^\varepsilon(y)\} \quad \text{and} \quad \Gamma^\varepsilon := \{(x, y) : x = v^\varepsilon(y)\},$$

respectively the fresh region and the flame front, and by

$$\Omega_{\text{per}}^\varepsilon := \Omega^\varepsilon \cap \{0 < y < \varepsilon\} \quad \text{and} \quad \Gamma_{\text{per}}^\varepsilon := \Gamma^\varepsilon \cap \{0 < y < \varepsilon\},$$

the corresponding restrictions to one period, just as in Section 2.2. Likewise the ε -periodic (in y) Sobolev’s spaces will be denoted by $L^2_{\text{per}}(\Omega^\varepsilon)$ and $H^1_{\text{per}}(\Omega^\varepsilon)$.

Our main convergence results are stated in the subsection below and their proofs are postponed in the next subsection.

4.1 *Main results*

We start by recalling the definition of travelling wave solutions in this new setting.

DEFINITION 4.1 Let $\varepsilon, \mu > 0$ and assume (A)–(F) with $Y = 1$. Then the triplet $(c^\varepsilon, v^\varepsilon, u^\varepsilon)$ is a solution to (4.1)–(4.2) if

- (i) $c^\varepsilon \in \mathbb{R}, v^\varepsilon \in W_{\text{per}}^{1,\infty}(\mathbb{R})$,
- (ii) $u^\varepsilon \in H_{\text{per}}^1(\Omega^\varepsilon), u^\varepsilon \geq 0$,
- (iii) v^ε satisfies (4.2) almost everywhere and

$$\int_{\Omega_{\text{per}}^\varepsilon} (c^\varepsilon b^\varepsilon u_x^\varepsilon w + a^\varepsilon \nabla u^\varepsilon \nabla w) = \int_{\Gamma_{\text{per}}^\varepsilon} \frac{c^\varepsilon g^\varepsilon}{\sqrt{1 + v_y^\varepsilon}} w \quad \forall w \in H_{\text{per}}^1(\Omega^\varepsilon)$$

(where $f^\varepsilon(y) = f(y/\varepsilon)$ for $f = a, b, g$).

Here is our first result.

Theorem 4.2 Let $\mu > 0$ and assume (A)–(F) with $Y = 1$. Let us then consider a family of solutions to (4.1)–(4.2) of the form $\{(c^\varepsilon, v^\varepsilon, u^\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0]}$ and such that

$$\bar{v}^\varepsilon = 0 \quad \forall \varepsilon \in (0, \varepsilon_0], \tag{4.3}$$

for some $\varepsilon_0 > 0$. Then:

$$\begin{cases} \lim_{\varepsilon \downarrow 0} c^\varepsilon = c^0 & \text{in } \mathbb{R}, \\ \lim_{\varepsilon \downarrow 0} v^\varepsilon = v^0 & \text{uniformly on } \mathbb{R}, \\ \lim_{\varepsilon \downarrow 0} u^\varepsilon \mathbf{1}_{\Omega^\varepsilon} = u^0 \mathbf{1}_{x < 0} & \text{in } L_{\text{loc}}^p(\mathbb{R}_y; L^p(\mathbb{R}_x)), \quad \forall p \in [1, +\infty), \end{cases}$$

where

$$c^0 = \int_0^1 R\left(z, \frac{\bar{g}}{b}\right) dz$$

and (v^0, u^0) are given by:

$$v^0 = 0 \quad \text{and} \quad u^0(x) = \frac{\bar{g}}{b} \exp\left(\frac{c^0 \bar{b}}{a} x\right) \quad \text{for } x < 0. \tag{4.4}$$

Moreover, if we consider the extensions to \mathbb{R}^2 , then u^ε converges to u^0 for $p = +\infty$ too. More precisely, we have:

$$\lim_{\varepsilon \downarrow 0} \text{ext}(u^\varepsilon) = \text{ext}(u^0) \quad \text{in } L^\infty(\mathbb{R}^2),$$

where

$$\text{ext}(u^\varepsilon)(x, y) := \begin{cases} u^\varepsilon(x, y), & x < v^\varepsilon(y), \\ u^\varepsilon(2v^\varepsilon(y) - x, y), & x > v^\varepsilon(y), \end{cases} \tag{4.5}$$

and $\text{ext}(u^0)(x) := \frac{\bar{g}}{b} \exp\left(-\frac{c^0 \bar{b}}{a} |x|\right)$.

REMARK 4.3 (i) The existence of the family of triplets $\{(c^\varepsilon, v^\varepsilon, u^\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0]}$ is guaranteed by Theorem 3.8 for ε_0 small enough.

- (ii) The extensions above are defined just as in (2.3), but with respect to the respective fresh regions $\{x < v^\varepsilon(y)\}$ and $\{x < 0\}$. As before, we will simply use the letters u^ε and u^0 to denote these extensions. This means that throughout $u^0(x) = \frac{\bar{g}}{b} \exp(-\frac{c^0 \bar{b}}{a} |x|)$ for all $x \in \mathbb{R}$.
- (iii) In the above the convergence holds for the “whole family” of triplets and not only for some particular subsequence.

REMARK 4.4 (The homogenized system) After homogenization, the front’s profile, which becomes planar, reduces to

$$\Gamma^0 = \{(x, y) : x = v^0 = 0\}$$

and the fresh region to the half plane

$$\Omega^0 = \{(x, y) : x < 0\}$$

(here (4.3) allows to fix the front’s profile and get Γ^0 at the limit). The temperature u^0 of the fresh region, which becomes independent of y , is given by the solution of the one dimensional problem

$$\begin{cases} c^0 \bar{b} u_x^0 - \bar{a} u_{xx}^0 = 0, & x < 0, \\ \bar{a} u_x^0 = c^0 \bar{g}, & x = 0, \\ u^0(x) \rightarrow 0, & \text{as } x \rightarrow -\infty, \end{cases} \tag{4.6}$$

which is the homogenized version of (1.6). Finally the speed c^0 is given by the equation

$$-c^0 + \int_0^1 R(z, u^0(0)) dz = 0,$$

which is the homogenized version of (1.7).

Let us now analyze the effects of the curvature on the propagation. For that we allow μ to depend on ε . Next we assume that the limit

$$\lambda = \lim_{\varepsilon \rightarrow 0} \frac{\mu(\varepsilon)}{\varepsilon} \tag{4.7}$$

exists and we propose to run the analysis following the different values of λ . We will technically need to assume in addition that

$$\text{either } \lambda > \frac{4 R_M}{\pi} \text{ or } \lim_{T \downarrow 0} \{ |\ln T| \operatorname{ess\,inf}_z R(z, T) \} = +\infty. \tag{4.8}$$

This means that for a small curvature regime λ , the combustion rate R will be allowed to degenerate but not too much (that is we will use the second assumption). We will also need to assume that:

$$\text{The function } T \in \mathbb{R}^+ \mapsto \operatorname{ess\,sup}_z R(z, T) \text{ is continuous at } T = \frac{\bar{g}}{b}. \tag{4.9}$$

Note that this function is continuous for all T if considering the typical combustion rate in (1.3) (with A and E bounded). During the proof, the continuity at $T = \frac{\bar{g}}{b}$ only will be sufficient. This particular value will correspond to the constant value of the homogenized temperature at the front.

Here is our second and last convergence result.

Theorem 4.5 For each $\varepsilon > 0$, let $\mu(\varepsilon) > 0$ be such that the limit λ in (4.7) exists in $[0, +\infty]$. Assume next that (A)–(F) and (4.8)–(4.9) hold with $Y = 1$. Let us then consider a family of solutions to (4.1)–(4.2) of the form $\{(c^\varepsilon, v^\varepsilon, u^\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0]}$ with

$$\bar{v}^\varepsilon = 0 \quad \forall \varepsilon \in (0, \varepsilon_0],$$

for some $\varepsilon_0 > 0$. Then:

(i) There exists $c^0 = c^0(\lambda) > 0$ such that

$$\lim_{\varepsilon \downarrow 0} (c^\varepsilon, v^\varepsilon, u^\varepsilon) = (c^0, v^0, u^0),$$

where (v^0, u^0) is defined as in Theorem 4.2 and the limits are taken in the same sense.

(ii) This speed c^0 is uniquely determined as follows:

1. If $\lambda = +\infty$ then $c^0 = \int_0^1 R\left(z, \frac{g}{b}\right) dz$.
2. If $\lambda \in (0, +\infty)$ then c^0 is the unique real such that the equation

$$-c^0 + R\left(z, \frac{g}{b}\right) \sqrt{1 + w_z^2} = \lambda \frac{w_{zz}}{1 + w_z^2}$$

admits an almost everywhere 1-periodic solution $w \in W^{2,\infty}(\mathbb{R}_z)$.

3. If $\lambda = 0$ then $c^0 = \text{ess sup}_z R\left(z, \frac{g}{b}\right)$.

REMARK 4.6 (i) Such a family $\{(c^\varepsilon, v^\varepsilon, u^\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0]}$ exists provided that ε_0 is small enough, thanks now to (4.8) and Theorems 3.4 or 3.8.

(ii) Here also the whole family of triplets converges and this is in fact equivalent to the convergence of the whole family of ratios, see Remark 5.2.

(iii) Theorem 4.2 is a particular case of Theorem 4.5 with $\lambda = +\infty$. Actually in that case, we will see during the proof that the convergence of v^ε towards v^0 holds in fact in $W^{1,\infty}$.

(iv) During the proof, we will technically need the assumption (4.9) only in the regime where $\lambda = 0$.

REMARK 4.7 (The corrector) The well-definition of c^0 in the item (ii2) is a consequence of Theorem 2.1. We also know that w is unique up to an additive constant. The w such that $\bar{w} = 0$ appears as a corrector in relation with the ansatz $v^\varepsilon(y) \approx \varepsilon w\left(\frac{y}{\varepsilon}\right)$.

REMARK 4.8 (The homogenized speed) Here again $v^0 = 0$ and u^0 is deduced from c^0 through the formula in (4.4). The knowledge of the mapping

$$\lambda \mapsto c^0 = c^0(\lambda)$$

then entirely determines the homogenized triplets parametrized by λ . A qualitative analysis of this mapping will be done in the next section.

4.2 Proofs

Let us prove Theorems 4.2 and 4.5. We proceed by giving a few lemmas and we start with some bounds uniform in small ε .

Lemma 4.9 *Let the assumptions of Theorem 4.5 hold. Then there are some positive constants c_m , c_M and C such that for all ε small enough,*

$$c_m \leq c^\varepsilon \leq c_M, \quad \|v_y^\varepsilon\|_\infty \leq C \quad \text{and} \quad \|u^\varepsilon\|_\infty \leq C.$$

Proof. We first claim that there is some constant $u_m > 0$ such that for all ε small enough,

$$u^\varepsilon \geq u_m \quad \text{at the front} \quad \{y = v^\varepsilon(x)\}. \tag{4.10}$$

The important assumption to show this claim is (4.8). If the first condition holds, that is $\lambda > \frac{4R_M}{\pi}$, we take any $\lambda > \lambda_0 > \frac{4R_M}{\pi}$ and we choose $\tilde{\varepsilon}_0 > 0$ sufficiently small such that

$$\frac{\mu(\varepsilon)}{\varepsilon} \geq \lambda_0 \quad \forall \varepsilon \in (0, \tilde{\varepsilon}_0]$$

(recall that $\frac{\mu(\varepsilon)}{\varepsilon} \rightarrow \lambda$ as $\varepsilon \downarrow 0$). We then apply Lemma 3.7 to Problem (4.1)–(4.2). This implies that

$$u^\varepsilon \geq \frac{g_m a_m}{a_M b_M} e^{-\frac{2R_M b_M \varepsilon}{a_M} \tan\left(\frac{2R_M \varepsilon}{\mu(\varepsilon)}\right)} > 0,$$

at the front, with the bounds a_m, g_m , etc., from (A)–(F). Notice indeed that all the ε -dependent parameters in (4.1)–(4.2) satisfy these assumptions with the same bounds. To get (4.10), it thus suffices to take

$$u_m = \frac{g_m a_m}{a_M b_M} e^{-\frac{2R_M b_M \tilde{\varepsilon}_0}{a_M} \tan\left(\frac{2R_M}{\lambda_0}\right)} > 0.$$

In the case where the second condition holds in (4.8), the reasoning is the same but we apply Lemma 3.3 instead. This completes the proof of (4.10).

Now use (C), (E) and (F) to infer that the combustion rate

$$y \mapsto R\left(\frac{y}{\varepsilon}, u^\varepsilon(v^\varepsilon(y), y)\right),$$

in Equation (4.2), is positively bounded from below and above (by some constants independent of small ε). Applying then Theorem 2.1 (2.2) and Theorem 2.4 (2.8) completes the proof. \square

At this stage, we only know that u^ε is in $L^\infty \cap H^1$. But, we can get more regularity as below by applying some classical results for elliptic PDEs.

Lemma 4.10 *Let the assumptions of Theorem 4.5 hold. Then the (extension of the) temperature u^ε is Hölder continuous on \mathbb{R}^2 , uniformly in small ε .*

Sketch of the proof. Recall that u^ε is a variational solution of an elliptic Neuman problem with Lipschitz boundary, see (4.1). It is then Hölder continuous up to the boundary by [13]. To get estimates uniform in small ε , it suffices to inject the bounds of Lemma 4.9 in the a priori estimates of [13]. \square

Note that the reader might need to go inside the proofs of [13] themselves in order to check the details; indeed, we did not find any selfcontained statement of these a priori estimates. For that reason, the rigorous computations are postponed in Appendix A.4 for the reader’s convenience. Let us continue with a compacity result.

Lemma 4.11 *Let us assume the hypotheses of Theorem 4.5. Then there exist $c_0 > 0$, $v^0 \in C_b(\mathbb{R})$ and $u^0 \in C_b(\mathbb{R}^2)$, such that*

$$\begin{cases} c^\varepsilon \rightarrow c^0 \text{ in } \mathbb{R}, \\ v^\varepsilon \rightarrow v^0 \text{ locally uniformly on } \mathbb{R}, \\ u^\varepsilon \rightarrow u^0 \text{ locally uniformly on } \mathbb{R}^2, \end{cases}$$

as $\varepsilon \downarrow 0$. This convergence holds more precisely at least along some sequence $\{\varepsilon_n\}_n$ converging to zero.

Proof. This is an immediate consequence of the preceding lemmas and Ascoli–Arzéla theorem. \square

We proceed by giving some properties on v^0 and u^0 .

Lemma 4.12 *Let $(v^0, u^0) \in C_b(\mathbb{R}) \times C_b(\mathbb{R}^2)$ be given by the preceding lemma. Then $v^0 = 0$ and u^0 does not depend on y , that is $u^0 = u^0(x)$.*

Proof. Let $a < b$ and let us first show that $u^0(x, a) = u^0(x, b)$ for any $x \in \mathbb{R}$. Define

$$b_\varepsilon := a + \lfloor \frac{b-a}{\varepsilon} \rfloor \varepsilon \quad \forall \varepsilon > 0,$$

where throughout the symbol “ $\lfloor \cdot \rfloor$ ” is used for the lower integer part. By periodicity $u^\varepsilon(x, b_\varepsilon) = u^\varepsilon(x, a)$ and we also know that b_ε converges to b , as $\varepsilon \downarrow 0$. Let us pass to the limit in the last equality by using the uniform convergence of u^ε towards u^0 (holding at least along the sequence given in Lemma 4.11). We get that

$$u^0(x, b) = \lim_{\varepsilon \downarrow 0} u^\varepsilon(x, b_\varepsilon) = \lim_{\varepsilon \downarrow 0} u^\varepsilon(x, a) = u^0(x, a),$$

which completes the proof that $u^0 = u^0(x)$. We proceed in the same way to show that v^0 does not depend on y . In particular, v^0 is a constant which is necessarily zero, since $\bar{v}^\varepsilon = 0$ (see the assumptions of Theorem 4.5). \square

Lemma 4.13 *Let $u^0 \in C_b(\mathbb{R})$, $u^0 = u^0(x)$, be given by the preceding lemmas. Then u^0 is even, tends to zero at infinity, and $u^\varepsilon \rightarrow u^0$ uniformly on \mathbb{R}^2 and in $L^p_{\text{loc}}(\mathbb{R}_y, L^p(\mathbb{R}_x))$ for all $p \in [1, +\infty)$ (along the sequence given by Lemma 4.11).*

Proof. Any local uniform limit u^0 of u^ε is clearly even in x , because of the choice of the extension (4.5). To show the other properties, it suffices to verify that

$$|u^\varepsilon(x, y)| \leq C e^{-C|x|} \quad \forall x, y \in \mathbb{R}^2, \tag{4.11}$$

for some positive constant C independent of small ε . To show (4.11), we apply Theorem 2.4(2.8) to Problem (4.1). This implies that

$$0 \leq u^\varepsilon(x, y) \leq \frac{gM aM}{a_m b_m} e^{c^\varepsilon \frac{b_m}{aM} (x + \|v^\varepsilon\|_\infty)} \quad \text{on } \{x \leq v^\varepsilon(y)\};$$

recall indeed that, under the assumption of Theorem 4.5, the ε -dependent parameters of (4.1) satisfy (A)–(B) with these same bounds. Inequality (4.11) thus follows from Lemma 4.9 (and the choice of the extension (4.5)). \square

We can now identify u^0 .

Lemma 4.14 *Let $c^0 > 0$ and $u^0 \in C_b(\mathbb{R})$, $u_0 = u_0(x)$, be given by the preceding lemmas. Then for all $x \in \mathbb{R}$,*

$$u^0(x) = \frac{\bar{g}}{b} \exp\left(-\frac{c^0 \bar{b} |x|}{a}\right),$$

and where the speed c^0 will be identified later.

This is a consequence of the passage to the limit as $\varepsilon \downarrow 0$, in the temperature's equation, followed by the integration of the limiting problem. In fact, since u^0 does not depend on y , it is sufficient to use only x -dependent test functions during this passage to the limit. This roughly speaking allows to argue with weak-strong convergences only. The details are postponed in Appendix A.3 for completeness.

Let us now identify c^0 . We will homogenize the front's equation (4.2) (with $\mu = \mu(\varepsilon)$ for Theorem 4.5). To do so, it will be convenient to rewrite (4.2) as

$$-c^\varepsilon + R(z, u^\varepsilon(v^\varepsilon(\varepsilon z), \varepsilon z)) \sqrt{1 + (w_z^\varepsilon)^2} = \frac{\mu(\varepsilon)}{\varepsilon} \frac{w_{zz}^\varepsilon}{1 + (w_z^\varepsilon)^2}, \tag{4.12}$$

thanks to the change of variables $z = \frac{y}{\varepsilon}$ and

$$w^\varepsilon(z) := \frac{v^\varepsilon(\varepsilon z)}{\varepsilon}. \tag{4.13}$$

We shall see later that the limit w of w^ε is such that $v^\varepsilon(y) \approx \varepsilon w\left(\frac{y}{\varepsilon}\right)$. Here are some technical properties that will be needed to identify c^0 .

Lemma 4.15 *Let $\{\varepsilon_n\}_n$ be the sequence given by Lemma 4.11. Then:*

- (i) *The function $R(\cdot, u^{\varepsilon_n}(v^{\varepsilon_n}(\varepsilon_n \cdot), \varepsilon_n \cdot)) \rightarrow R\left(\cdot, \frac{\bar{g}}{b}\right)$ in $L^1(0, 1)$ as $n \rightarrow +\infty$.*
- (ii) *The sequence $\{w^{\varepsilon_n}\}_n$ is bounded in $W^{1,\infty}(\mathbb{R})$.*
- (iii) *The latter sequence is bounded in $W^{2,\infty}(\mathbb{R})$ if $\lambda = \lim_{\varepsilon \downarrow 0} \frac{\mu(\varepsilon)}{\varepsilon} > 0$.*

Proof. Let us start with (i). By Lemma 4.11,

$$u^{\varepsilon_n}(v^{\varepsilon_n}(\varepsilon_n z), \varepsilon_n z) \rightarrow u^0(0)$$

uniformly in z ; recall indeed that $v^0 = 0$ and $u^0 = u^0(x)$ by Lemma 4.12. By the identification of u^0 in Lemma 4.14, we also know that $u^0(0) = \frac{\bar{g}}{b}$. Thus by (C),

$$R(z, u^{\varepsilon_n}(v^{\varepsilon_n}(\varepsilon_n z), \varepsilon_n z)) \rightarrow R\left(z, \frac{\bar{g}}{b}\right)$$

for almost every $z \in \mathbb{R}$. Using in addition that R is bounded by (E), the dominated convergence theorem implies the desired convergence in (i).

For the second item, we will apply Theorem 2.1 to Equation (4.12). Due to the uniform convergence of $u^{\varepsilon_n}(v^{\varepsilon_n}(\varepsilon_n z), z)$ to $u^0(0)$, we know that for all z and sufficiently large values of n , we have for example

$$u^{\varepsilon_n}(v^{\varepsilon_n}(\varepsilon_n z), z) \geq \frac{u^0(0)}{2} = \frac{\bar{g}}{2b} > 0.$$

The assumptions (C) and (F) then imply that

$$\operatorname{ess\,inf}_z R(z, u^{\varepsilon_n}(v^{\varepsilon_n}(\varepsilon_n z), \varepsilon_n z)) \geq \operatorname{ess\,inf}_z R\left(z, \frac{\bar{g}}{2b}\right) > 0.$$

Hence, using also the other bound

$$\operatorname{ess\,sup}_z R(z, u^{\varepsilon_n}(v^{\varepsilon_n}(\varepsilon_n z), \varepsilon_n z)) \leq R_M$$

from (E), Estimate (2.2) implies that the sequence of derivatives $\{w_z^{\varepsilon_n}\}_n$ is bounded by some C in $L^\infty(\mathbb{R})$. To get some bound on the antiderivatives, we note that each w^{ε_n} is 1-periodic with $\int_0^1 w^{\varepsilon_n} = \frac{1}{\varepsilon_n^2} \int_0^{\varepsilon_n} v^{\varepsilon_n} = 0$ thanks to (4.13). This implies that $\|w^{\varepsilon_n}\|_\infty \leq \|w_z^{\varepsilon_n}\|_\infty \leq C$ and the proof of the item (ii) is complete.

The item (iii) immediately follows from the latter item and Equation (4.12). □

We can now prove Theorems 4.2 and 4.5. We will denote again the sequence $\{\varepsilon_n\}_n$ given by Lemma 4.11, or any of its subsequences, simply by $\{\varepsilon\}$.

Proof of Theorem 4.2. By the preceding lemmas, we have already proved the convergence of v^ε and u^ε towards the desired limits (at least along the above sequence). It therefore remains to identify c^0 .

For this sake, let us integrate Equation (4.12) between 0 and 1 (thus with μ fixed here). We get

$$-c^\varepsilon + \int_0^1 R(z, u^\varepsilon(v^\varepsilon(\varepsilon z), \varepsilon z)) \sqrt{1 + (w_z^\varepsilon)^2} \, dz = \frac{\mu}{\varepsilon} [\arctan(w_z^\varepsilon)]_0^1.$$

As the right-hand side vanishes due to the 1-periodicity of w^ε , we have

$$c^\varepsilon = \int_0^1 R(z, u^\varepsilon(v^\varepsilon(\varepsilon z), \varepsilon z)) \sqrt{1 + (w_z^\varepsilon)^2} \, dz. \tag{4.14}$$

Note then that $\|w_z^\varepsilon\|_\infty = \|v_y^\varepsilon\|_\infty$ and recall from Lemma 3.6 that

$$\|v_y^\varepsilon\|_\infty \leq \tan\left(\frac{2c^\varepsilon \varepsilon}{\mu}\right).$$

This implies that $w_z^\varepsilon \rightarrow 0$ uniformly as $\varepsilon \downarrow 0$. Therefore in the limit $\varepsilon \downarrow 0$, (4.14) and Lemma 4.15(i) lead to

$$c^\varepsilon \rightarrow \int_0^1 R\left(z, \frac{\bar{g}}{b}\right) dz.$$

This identifies the limiting speed c^0 .

To conclude, we have established the convergence of the triplet $(c^\varepsilon, v^\varepsilon, u^\varepsilon)$ towards

$$c^0 = \int_0^1 R\left(z, \frac{\bar{g}}{b}\right) dz, \quad v^0 = 0 \quad \text{and} \quad u^0 = u^0(x) = \frac{\bar{g}}{b} \exp\left(-\frac{c^0 \bar{b} |x|}{a}\right),$$

along some sequence $\{\varepsilon_n\}_n$ converging to zero. But as the limiting triplet is uniquely determined, the convergence holds for the whole family $\varepsilon \downarrow 0$. This implies the desired result and completes the proof. □

Proof of Theorem 4.5. Here too it remains to identify c^0 . The new difficulty is that we may no longer have the strong convergence of w_z^ε towards zero. This will complicate the passage to the limit in (4.14). Note also that, as above, the identification of c^0 will automatically gives us the convergence of the whole family as $\varepsilon \downarrow 0$. We thus continue to argue along some particular sequence (simply denoted by $\{\varepsilon\}$).

1. *The case $\lambda = +\infty$.* This is the only case for which w_z^ε still strongly converges towards zero. Indeed by (4.13) and the estimate of Lemma 3.6, we have

$$\|w_z^\varepsilon\|_\infty = \|v_y^\varepsilon\|_\infty \leq \tan\left(\frac{2c^\varepsilon \varepsilon}{\mu(\varepsilon)}\right), \tag{4.15}$$

and as $\frac{\varepsilon}{\mu(\varepsilon)}$ goes to zero as $\varepsilon \downarrow 0$, the preceding arguments still apply to give $c^0 = \int_0^1 R\left(z, \frac{\bar{g}}{b}\right) dz$.

2. *The case $\lambda \in (0, +\infty)$.* In that case we need to identify the limit w of w^ε , defined in (4.13), and the equation it satisfies. We have seen in Lemma 4.15(iii) that the function w^ε remains bounded in $W^{2,\infty}(\mathbb{R})$ as $\varepsilon \downarrow 0$ (at least along the sequence given by Lemma 4.11). By Ascoli–Arzela theorem, it then converges towards some w in $W^{1,\infty}(\mathbb{R})$ (up to some subsequence). This limit w is necessarily in $W^{2,\infty}(\mathbb{R})$ and is also 1-periodic. To identify the equation in w , let us rewrite (4.12) as

$$w_{zz}^\varepsilon = \frac{\varepsilon}{\mu(\varepsilon)} \left\{ 1 + (w_z^\varepsilon)^2 \right\} \left\{ -c^\varepsilon + R\left(z, u^\varepsilon(v^\varepsilon(\varepsilon z), \varepsilon z)\right) \sqrt{1 + (w_z^\varepsilon)^2} \right\}.$$

Using Lemma 4.15(i), we infer that $w_{zz}^\varepsilon(\cdot)$ converges in $L^1(0, 1)$ towards

$$\frac{1}{\lambda} \left\{ 1 + w_z^2(\cdot) \right\} \left\{ -c^0 + R\left(\cdot, \frac{\bar{g}}{b}\right) \sqrt{1 + w_z^2(\cdot)} \right\}.$$

But this function is necessarily $w_{zz}(\cdot)$ by uniqueness of the distributional limit. Hence w satisfies

$$-c^0 + R\left(z, \frac{\bar{g}}{b}\right) \sqrt{1 + w_z^2} = \lambda \frac{w_{zz}}{1 + w_z^2} \quad \text{almost everywhere.}$$

We then conclude by Theorem 2.1 which gives the existence of a unique real c^0 such that the above equation admits a 1-periodic solution $w \in W^{2,\infty}(\mathbb{R})$. Note that the c^0 thus identified will depend on λ .

3. *The case $\lambda = 0$.* We must show that $c^0 = \text{ess sup}_z R\left(z, \frac{\bar{g}}{b}\right)$. Let us start by applying Theorem 2.1(2.2) to the front’s equation (4.2). We get

$$c^\varepsilon \leq \text{ess sup}_{y \in \mathbb{R}} R\left(\frac{y}{\varepsilon}, u^\varepsilon(v^\varepsilon(y), y)\right).$$

Using (C), we infer that

$$c^\varepsilon \leq \text{ess sup}_{z \in \mathbb{R}} R(z, T^\varepsilon),$$

with $T^\varepsilon := \max_y u^\varepsilon(v^\varepsilon(y), y)$. Recalling that $c^\varepsilon \rightarrow c^0$ and $u^\varepsilon(v^\varepsilon(\cdot), \cdot) \rightarrow u^0(0) = \frac{\bar{g}}{b}$ uniformly on \mathbb{R} , we have $T^\varepsilon \rightarrow \frac{\bar{g}}{b}$ and thus

$$c^0 \leq \limsup_{T \rightarrow \frac{\bar{g}}{b}} \left\{ \text{ess sup}_z R(z, T) \right\}.$$

We conclude that $c^0 \leq \text{ess sup}_z R(z, \frac{\bar{g}}{b})$ by using (4.9).

To prove the inequality in the other direction, we consider Equation (4.12). Given any nonnegative $\varphi \in C_c^\infty(\mathbb{R})$, we multiply (4.12) by φ and integrate the right-hand side by parts. We get that

$$\int_{\mathbb{R}} \left\{ -c^\varepsilon + R(z, u^\varepsilon(v^\varepsilon(\varepsilon z), \varepsilon z)) \sqrt{1 + (w_z^\varepsilon)^2} \right\} \varphi \, dz = -\frac{\mu(\varepsilon)}{\varepsilon} \int_{\mathbb{R}} \varphi_z \arctan(w_z^\varepsilon) \, dz.$$

From Lemma 4.15 and the fact that $\frac{\mu(\varepsilon)}{\varepsilon} \rightarrow 0$ as $\varepsilon \downarrow 0$, we easily deduce that

$$c^0 \int_{\mathbb{R}} \varphi \geq \int_{\mathbb{R}} R\left(z, \frac{\bar{g}}{b}\right) \varphi(z) \, dz,$$

at the limit. Since the nonnegative test function φ is arbitrary,

$$c^0 \geq \text{ess sup}_z R\left(z, \frac{\bar{g}}{b}\right)$$

and the proof is complete. □

5. Monotonicity of the homogenized speed

In this section, we consider the qualitative analysis of the speed c^0 as defined by Theorem 4.5. Let us recall that it depends on $\lambda = \lim_{\varepsilon \downarrow 0} \frac{\mu(\varepsilon)}{\varepsilon}$. Our main result will be that $\lambda \mapsto c^0(\lambda)$ is monotonous.

5.1 Main result

For brevity, we will denote $c^0(\lambda)$ simply by $c(\lambda)$ all along this section. Its derivative in λ will be denoted by $c'(\lambda)$. The gradient of the corrector $w = w(z)$ given by Theorem 4.5 will be denoted by $h = w_z$. This function satisfies the problem:

$$\begin{cases} -c + \mathcal{R}(z) \sqrt{1 + h^2} = \lambda \frac{h_z}{1 + h^2}, \\ h(z + 1) = h(z), \\ \int_0^1 h(t) \, dt = 0, \end{cases} \tag{5.1}$$

for almost every $z \in \mathbb{R}$, where hereafter

$$\mathcal{R}(z) := R\left(z, \frac{\bar{g}}{b}\right).$$

Under the assumptions of Theorem 4.5, we have:

$$\text{The function } \mathcal{R} \in L^\infty(\mathbb{R}) \text{ is 1-periodic and positively lower bounded.} \tag{5.2}$$

This is the only property that we will need. Let us recall that it is sufficient to define the mapping

$$\lambda \in \mathbb{R}^+ \mapsto (c(\lambda), h(\cdot, \lambda)) \in \mathbb{R} \times W_\#^{1,\infty}(\mathbb{R}), \tag{5.3}$$

whose image is the unique $(c, h) \in \mathbb{R} \times W^{1,\infty}(\mathbb{R})$ solution of (5.1), see Theorem 2.1. Here is the main result of this section.

Theorem 5.1 *Assume (5.2). Then $\lambda \mapsto c(\lambda)$ (defined as above) is C^∞ and nonincreasing. At the limits, it satisfies:*

$$\lim_{\lambda \downarrow 0} c(\lambda) = \operatorname{ess\,sup}_z \mathcal{R}(z) \quad \text{and} \quad \lim_{\lambda \rightarrow +\infty} c(\lambda) = \int_0^1 \mathcal{R}(z) \, dz.$$

More precisely $c'(\lambda) < 0$ for all $\lambda > 0$ as soon as \mathcal{R} is not a constant.

REMARK 5.2 As a corollary of the above theorem, the speed $c^0 = c^0(\lambda)$ of Theorem 4.5 defines a continuous map of the form:

$$\lambda \in [0, +\infty] \mapsto c^0(\lambda) \in \left[\int_0^1 R \left(z, \frac{\bar{g}}{b} \right) dz, \operatorname{ess\,sup}_z R \left(z, \frac{\bar{g}}{b} \right) \right].$$

This map is smooth in $(0, +\infty)$, and it is decreasing and bijective whenever R is not constant. In that case, it is in particular injective and the convergence stated in Theorem 4.5 holds if and only if the limit $\lambda = \lim_{\varepsilon \downarrow 0} \frac{\mu(\varepsilon)}{\varepsilon}$ exists.

The limit as $\lambda \downarrow 0$ has been established in [6]. To the best of our knowledge, the other properties are new. The monotonicity happens to be the most difficult to obtain. In the course of the proof, we will call for the implicit function theorem which will give us enough regularity to justify the computations. The rest of this section is devoted to the proof of Theorem 5.1.

5.2 Proof

We will start by stating a version of the implicit function theorem in Banach spaces which we will use.

Let then E_1, E_2 and F be some given Banach spaces, \mathcal{O} an open subset of $E_1 \times E_2$, and $\phi : \mathcal{O} \subseteq E_1 \times E_2 \rightarrow F$ a function. Also recall that:

DEFINITION 5.3 The function ϕ is *differentiable* at (x_1, x_2) if there is a bounded linear map $T : E_1 \times E_2 \rightarrow F$ such that for all $(h_1, h_2) \in E_1 \times E_2$,

$$\phi(x_1 + h_1, x_2 + h_2) = \phi(x_1, x_2) + T(h_1, h_2) + o(h_1, h_2)$$

where $\frac{\|o(h_1, h_2)\|_F}{\|(h_1, h_2)\|_{E_1 \times E_2}} \rightarrow 0$ as $\|(h_1, h_2)\|_{E_1 \times E_2} \rightarrow 0$.

In that case T is unique and defined as the differential of ϕ at (x_1, x_2) . The partial differential with respect to x_1 and x_2 are defined as the maps $T_1 : h_1 \mapsto T(h_1, 0)$ and $T_2 : h_2 \mapsto T(0, h_2)$. Throughout we shall denote them by

$$d\phi(x_1, x_2) := T \quad \text{and} \quad d_{x_i} \phi(x_1, x_2) := T_i,$$

respectively. To avoid confusion between (x_1, x_2) and (h_1, h_2) , we shall use the standard notation

$$d\phi(x_1, x_2) \cdot (h_1, h_2) := d\phi(x_1, x_2)(h_1, h_2) \in F$$

(with similar notations for the partial differentials). This defines a map

$$d\phi : E_1 \times E_2 \rightarrow \mathcal{L}(E_1 \times E_2, F),$$

where

$$\mathfrak{L}(E_1 \times E_2, F) := \{T : E_1 \times E_2 \rightarrow F \text{ linear and bounded}\}.$$

For further details, see for instance [5]. In the implicit function theorem below, we shall use the following notation:

$$\text{Isom}(E_2, F) := \{T \in \mathfrak{L}(E_2, F) \text{ such that } T \text{ is bijective}\}.$$

Here is the theorem.

Theorem 5.4 (see for instance [5]) *Let ϕ be as above and $(x_1^0, x_2^0) \in \mathcal{O}$ be such that*

$$\phi(x_1^0, x_2^0) = 0.$$

Let us also assume that $\phi \in C^1(\mathcal{O})$ with

$$d_{x_2}\phi(x_1^0, x_2^0) \in \text{Isom}(E_2, F). \tag{5.4}$$

Then:

- (i) *There are open sets $U \subseteq E_1$ and $V \subseteq E_2$ and a function $\varphi : U \rightarrow V$ such that $(x_1^0, x_2^0) \in U \times V \subseteq \mathcal{O}$ and*

$$[\phi(x_1, x_2) = 0 \Leftrightarrow x_2 = \varphi(x_1)] \quad \forall (x_1, x_2) \in U \times V.$$

- (ii) *Moreover $\varphi \in C^1(U)$ and*

$$d\varphi(x_1^0) = -[d_{x_2}\phi(x_1^0, x_2^0)]^{-1} \circ d_{x_1}\phi(x_1^0, x_2^0).$$

In the sequel we shall use (ii) to compute $c'(\lambda)$. Before we need to give a few lemmas. Let us first precise the Banach spaces and the function ϕ which we will take in the frame of our problem. To this end, we consider $(\lambda, (c, h))$ as free variables living in the following Banach spaces:

$$E_1 := \mathbb{R}_\lambda \quad \text{and} \quad E_2 := \mathbb{R}_c \times E,$$

where

$$E := \left\{h \in W_{\#}^{1,\infty}(\mathbb{R}_z) \text{ such that } \int_0^1 h(z) dz = 0\right\}. \tag{5.5}$$

(For the sake of clarity, we have added some subscripts to the real space \mathbb{R} in order to remember which variable is considered.) We then define the function ϕ as:

$$\begin{aligned} \phi : E_1 \times E_2 &\rightarrow F \\ (\lambda, (c, h)) &\mapsto \phi(\lambda, (c, h)) : z \mapsto -c + \mathcal{R}(z)\sqrt{1 + h^2(z)} - \lambda \frac{h_z(z)}{1+h^2(z)}, \end{aligned} \tag{5.6}$$

with the arrival space

$$F := L_{\#}^{\infty}(\mathbb{R}_z).$$

All these spaces are endowed with their usual norms: The absolute value for E_1 , the $\|\cdot\|_{W^{1,\infty}}$ -norm for E , etc. Our first lemma is a computation of the partial differential of ϕ with respect to the (c, h) -variable.

Lemma 5.5 *Assume (5.2). Then the function ϕ as defined in (5.6) is C^∞ . Moreover for all $\lambda \in E_1$, $(c, h) \in E_2$ and $(\mathcal{C}, \mathcal{H}) \in E_2$,*

$$d_{c,h}\phi(\lambda, (c, h)) \cdot (\mathcal{C}, \mathcal{H}) = -\mathcal{C} + \widetilde{\mathcal{R}} \widetilde{\mathcal{H}} - \lambda \widetilde{\mathcal{H}}_z,$$

where $\widetilde{\mathcal{R}} = \mathcal{R} h \sqrt{1 + h^2}$ and $\widetilde{\mathcal{H}} = \frac{\mathcal{H}}{1+h^2}$.

Proof. The function ϕ is smooth as composition of smooth functions – notice that $h \in W^{1,\infty} \mapsto h_z \in L^\infty$ is smooth as it is a bounded and linear map. To differentiate ϕ , we consider the useful formula:

$$d_{c,h}\phi(\lambda, (c, h)) \cdot (\mathcal{C}, \mathcal{H}) = \lim_{t \downarrow 0} \frac{\phi(\lambda, (c + t \mathcal{C}, h + t \mathcal{H})) - \phi(\lambda, (c, h))}{t}.$$

Recall that here: c and \mathcal{C} are reals; h and \mathcal{H} are functions in $W^{1,\infty}_\#(\mathbb{R}_z)$; and the limit has to be taken in F that is to say strongly in $L^\infty_\#(\mathbb{R}_z)$. Let us set $I = I(z)$ equal to the quotient above and let us compute its limit in $L^\infty_\#(\mathbb{R}_z)$. Setting

$$F(r) := \sqrt{1 + r^2} \quad \text{and} \quad G(r) := \frac{1}{1 + r^2}$$

for any real r , we can write

$$\phi(\lambda, (c, h)) = -c + \mathcal{R} F(h) - \lambda G(h) h_z.$$

Injecting this in I and rearranging the terms, we get

$$I = -\mathcal{C} + \mathcal{R} \frac{F(h + t \mathcal{H}) - F(h)}{t} - \lambda \frac{G(h + t \mathcal{H})(h + t \mathcal{H})_z - G(h) h_z}{t}.$$

Now it is easy to see that the quotient in F goes to $F'(h) \mathcal{H} = \frac{h}{\sqrt{1+h^2}} \mathcal{H}$ as $t \downarrow 0$ and in $L^\infty_\#(\mathbb{R}_z)$. As concerning the quotient in G , by rewriting it as

$$h_z \frac{G(h + t \mathcal{H}) - G(h)}{t} + G(h + t \mathcal{H}) \mathcal{H}_z,$$

we see that its limit is $h_z G'(h) \mathcal{H} + G(h) \mathcal{H}_z = \{G(h) \mathcal{H}\}_z$. Finally in the limit $t \downarrow 0$, we get

$$d_{c,h}\phi(\lambda, (c, h)) \cdot (\mathcal{C}, \mathcal{H}) = -\mathcal{C} + \frac{\mathcal{R} h}{\sqrt{1 + h^2}} \mathcal{H} - \lambda \left\{ \frac{\mathcal{H}}{1 + h^2} \right\}_z.$$

This is the desired formula with $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{H}}$ defined as in the lemma. □

Our next lemma will serve to verify Condition (5.4) of Theorem 5.4.

Lemma 5.6 *Assume (5.2) and let $\lambda_0 > 0$, $c_0 = c(\lambda_0)$ and $h_0(z) = h(z, \lambda_0)$ be as defined by (5.3). Then $(c_0, h_0) \in E_2$ and for any $f \in F$, there is a unique $(\mathcal{C}, \mathcal{H}) \in E_2$ such that*

$$d_{c,h}\phi(\lambda_0, (c_0, h_0)) \cdot (\mathcal{C}, \mathcal{H}) = f.$$

Moreover \mathcal{C} is given by:

$$\mathcal{C} = - \frac{\int_0^1 f(z) \exp\left(\frac{1}{\lambda_0} \int_z^1 \tilde{\mathcal{R}}_0(t) dt\right) dz}{\int_0^1 \exp\left(\frac{1}{\lambda_0} \int_z^1 \tilde{\mathcal{R}}_0(t) dt\right) dz} \tag{5.7}$$

where $\tilde{\mathcal{R}}_0 = \mathcal{R} h_0 \sqrt{1 + h_0^2}$.

Note that one can also give an explicit formula for \mathcal{H} even if we will not need it in our case.

Proof. It is clear that $(c_0, h_0) \in E_2$ as it satisfies (5.1). Let now $f \in F = L^\infty(\mathbb{R}_z)$ and consider the problem of finding $(\mathcal{C}, \mathcal{H}) \in E_2 = \mathbb{R}_c \times E$ such that

$$d_{c,h}\phi(\lambda_0, (c_0, h_0)) \cdot (\mathcal{C}, \mathcal{H}) = f.$$

Assume first that such a pair exists and let us show that it is entirely determined by some explicit formulas. Recall that \mathcal{H} will belong to E (see (5.5)) so that $\mathcal{H} \in W_\#^{1,\infty}(\mathbb{R}_z)$ and

$$\int_0^1 \mathcal{H} = 0. \tag{5.8}$$

By Lemma 5.5 we also have that

$$-\mathcal{C} + \tilde{\mathcal{R}}_0 \tilde{\mathcal{H}} - \lambda_0 \tilde{\mathcal{H}}_z = f \quad \text{almost everywhere in } \mathbb{R}_z,$$

where $\tilde{\mathcal{R}}_0 = \mathcal{R} h_0 \sqrt{1 + h_0^2}$ and $\tilde{\mathcal{H}} = \frac{\mathcal{H}}{1+h_0^2}$. Note that $\tilde{\mathcal{H}}$ is Lipschitz because so is \mathcal{H} . By the variation of the constant method, $\tilde{\mathcal{H}}$ is necessarily of the form:

$$\tilde{\mathcal{H}}(z) = C \exp\left(\frac{1}{\lambda_0} \int_0^z \tilde{\mathcal{R}}_0(t) dt\right) - \frac{1}{\lambda_0} \int_0^z (f(t) + \mathcal{C}) \exp\left(\frac{1}{\lambda_0} \int_t^z \tilde{\mathcal{R}}_0(s) ds\right) dt, \tag{5.9}$$

for some constant C . Since \mathcal{H} is 1-periodic, so is $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}(0) = \tilde{\mathcal{H}}(1)$. This leads to

$$C = C \exp\left(\frac{1}{\lambda_0} \int_0^1 \tilde{\mathcal{R}}_0(t) dt\right) - \frac{1}{\lambda_0} \int_0^1 (f(t) + \mathcal{C}) \exp\left(\frac{1}{\lambda_0} \int_t^1 \tilde{\mathcal{R}}_0(s) ds\right) dt. \tag{5.10}$$

To continue we claim that:

Lemma 5.7 *We have $\int_0^1 \tilde{\mathcal{R}}_0 = 0$.*

Indeed $\tilde{\mathcal{R}}_0 = \mathcal{R} h_0 \sqrt{1 + h_0^2} = c_0 h_0 + \lambda_0 \frac{h_0}{1+h_0^2} (h_0)_z$ by the ODE in (5.1), so that

$$\int_0^1 \tilde{\mathcal{R}}_0 = c_0 \int_0^1 h_0 + \frac{\lambda_0}{2} \int_0^1 \{\ln(1 + h_0^2)\}_z = 0$$

(due to the two last conditions in (5.1)). This completes the proof of the intermediate lemma. Injecting it into (5.10) therefore leads to

$$\mathcal{C} \int_0^1 \exp\left(\frac{1}{\lambda_0} \int_t^1 \tilde{\mathcal{R}}_0(s) ds\right) dt = - \int_0^1 f(t) \exp\left(\frac{1}{\lambda_0} \int_t^1 \tilde{\mathcal{R}}_0(s) ds\right) dt.$$

This implies that \mathcal{C} is uniquely determined by the desired formula (5.7). To get the uniqueness of \mathcal{H} , we rewrite (5.8) as $\int_0^1 \widetilde{\mathcal{H}}(1 + h_0^2) = 0$ and inject this into (5.9). We find that C is uniquely determined and so are therefore $\widetilde{\mathcal{H}}$ and $\mathcal{H} = \widetilde{\mathcal{H}}(1 + h_0^2)$. This completes the proof of the uniqueness of $(\mathcal{C}, \mathcal{H})$.

Conversely, if we take $(\mathcal{C}, \mathcal{H})$ defined by the preceding formulas, the same arguments allow to show that $(\mathcal{C}, \mathcal{H}) \in E_2$ and $d_{c,h}\phi(\lambda_0, (c_0, h_0)) \cdot (\mathcal{C}, \mathcal{H}) = f$. This proves the existence of the pair $(\mathcal{C}, \mathcal{H})$ and completes the proof of the lemma. \square

We can now apply the implicit function theorem to get the result below.

Lemma 5.8 *Assume (5.2). Then the map $\lambda \mapsto c(\lambda)$ is C^∞ in $(0, +\infty)$. Moreover for all $\lambda_0 > 0$,*

$$c'(\lambda_0) = - \frac{\int_0^1 \frac{(h_0)_z}{1+h_0^2}(z) \exp\left(\frac{1}{\lambda_0} \int_z^1 \widetilde{\mathcal{R}}_0(t) dt\right) dz}{\int_0^1 \exp\left(\frac{1}{\lambda_0} \int_z^1 \widetilde{\mathcal{R}}_0(t) dt\right) dz} \tag{5.11}$$

where $h_0(z) = h(z, \lambda_0)$ and $\widetilde{\mathcal{R}}_0 = \mathcal{R} h_0 \sqrt{1 + h_0^2}$.

Proof. Let us consider the open set $\mathcal{O} := \mathbb{R}_\lambda^+ \times E_2 \subset E_1 \times E_2$, $c_0 = c(\lambda_0)$, and let us apply Theorem 5.4 to $\phi : \mathcal{O} \subset E_1 \times E_2 \rightarrow F$ at the point $(\lambda_0, (c_0, h_0)) \in \mathcal{O}$. Let us recall that (c_0, h_0) satisfies (5.1). Hence $\phi(\lambda_0, (c_0, h_0)) = 0$ in $F = L^\infty_\#(\mathbb{R}_z)$ since by the choice of ϕ in (5.6), this equality is equivalent to the ODE in (5.1). We also know that $d_{c,h}\phi(\lambda_0, (c_0, h_0)) \in \text{Isom}(E_2, F)$ by Lemma 5.6. Theorem 5.4 then gives us the existence of the implicit function $\varphi : U \rightarrow V$, with U an open set containing λ_0 . In our case, φ is C^∞ because so is ϕ , see for instance [5].

To continue, we claim that for all $\lambda \in U$, $\varphi(\lambda) = (c(\lambda), h(\cdot, \lambda))$. This is a consequence of Theorem 5.4(i). Indeed, the pair $(c, h) := \varphi(\lambda) \in V \subset E_1 \times E_2$ satisfies $\phi(\lambda, (c, h)) = 0$ in F . This means that h solves the ODE in (5.1). Moreover, since h belongs to E (E defined by (5.5)) we have that: $h \in W^{1,\infty}(\mathbb{R}_z)$, h is 1-periodic and $\int_0^1 h = 0$. Hence (c, h) satisfies all the conditions in (5.1), which completes the proof of the claim by the uniqueness of such a pair.

Having verified the preceding claim, we can define the speed $c(\lambda)$ through the implicit function φ . To do so, we consider the projection

$$\Pi_c : (c, h) \in E_2 = \mathbb{R}_c \times E \mapsto c \in \mathbb{R},$$

which gives us that $c(\lambda) = \Pi_c(\varphi(\lambda))$ for all $\lambda \in U$. This projection is C^∞ as a bounded and linear map. Thus the composition $c(\cdot) = (\Pi_c \circ \varphi)(\cdot)$ is C^∞ in U . We have shown that $\lambda \mapsto c(\lambda)$ is C^∞ in some neighborhood U of λ_0 . The regularity then holds on all \mathbb{R}^+ , because $\lambda_0 > 0$ is arbitrarily taken.

It remains to show (5.11). By the chain rule for differentials, see [5], we have that

$$d(\Pi_c \circ \varphi)(\lambda_0) = d\Pi_c(\varphi(\lambda_0)) \circ d\varphi(\lambda_0) \in \mathcal{L}(\mathbb{R}_\lambda, \mathbb{R}_c).$$

Since the variable λ is real, $d(\Pi_c \circ \varphi)(\lambda_0) \cdot 1 = (\Pi_c \circ \varphi)'(\lambda_0)$ and we get

$$c'(\lambda_0) = (\Pi_c \circ \varphi)'(\lambda_0) = d\Pi_c(\varphi(\lambda_0)) \cdot (d\varphi(\lambda_0) \cdot 1).$$

As Π_c is linear continuous, $d\Pi_c = \Pi_c$ everywhere in E_2 . Hence

$$c'(\lambda_0) = \Pi_c(d\varphi(\lambda_0) \cdot 1)$$

which, thanks to Theorem 5.4(ii), gives

$$c'(\lambda_0) = -\Pi_c \left\{ [d_{c,h}\phi(\lambda_0, (c_0, h_0))]^{-1} \cdot (d_\lambda\phi(\lambda_0, (c_0, h_0)) \cdot 1) \right\}.$$

If we denote by f the function $d_\lambda\phi(\lambda_0, (c_0, h_0)) \cdot 1 \in F$, the above formula means that $c'(\lambda_0) = -\mathcal{C}$ where \mathcal{C} is the speed of the unique pair $(\mathcal{C}, \mathcal{H}) \in E_2 = \mathbb{R}_c \times E$ solution of

$$d_{c,h}\phi(\lambda_0, (c_0, h_0)) \cdot (\mathcal{C}, \mathcal{H}) = f.$$

By (5.7), we then deduce that

$$c'(\lambda_0) = \frac{\int_0^1 f(z) \exp\left(\frac{1}{\lambda_0} \int_z^1 \tilde{\mathcal{R}}_0(t) dt\right) dz}{\int_0^1 \exp\left(\frac{1}{\lambda_0} \int_z^1 \tilde{\mathcal{R}}_0(t) dt\right) dz}.$$

It only remains to compute $f = d_\lambda\phi(\lambda_0, (c_0, h_0)) \cdot 1$. Here again the variable λ is real and f is the usual partial derivative:

$$f = \phi'_\lambda(\lambda_0, (c_0, h_0)) = \lim_{\lambda \rightarrow \lambda_0} \frac{\phi(\lambda, (c_0, h_0)) - \phi(\lambda_0, (c_0, h_0))}{\lambda - \lambda_0}$$

(the limit being in $F = L^\infty(\mathbb{R}_z)$). Recalling the definition of ϕ given by (5.6), we end up with $f = -\frac{(h_0)_z}{1+h_0^2}$ and the proof is complete. \square

REMARK 5.9 We claim that the mapping

$$\lambda \in \mathbb{R}^+ \mapsto h(\cdot, \lambda) \in W_\#^{1,\infty}(\mathbb{R}_z)$$

is also C^∞ . Indeed we have seen above that $\varphi(\lambda) = (c(\lambda), h(\cdot, \lambda))$, so that we can copy the arguments used to get the regularity of the speed by considering this time the projection $\Pi_h : (c, h) \mapsto h$.

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. It remains to prove that $c'(\lambda_0)$ is nonpositive, for any $\lambda_0 > 0$, and that $\lim_{\lambda \rightarrow +\infty} c(\lambda) = \int_0^1 \mathcal{R}(z) dz$.

Let us start with the first claim. By (5.11), $c'(\lambda_0)$ has the same sign as the numerator term

$$J := -\int_0^1 \frac{(h_0)_z}{1+h_0^2}(z) \exp\left(\frac{1}{\lambda_0} \int_z^1 \tilde{\mathcal{R}}_0(t) dt\right) dz.$$

An integration by parts gives

$$J = \frac{1}{\lambda_0} \int_0^1 \arctan(h_0) \tilde{\mathcal{R}}_0(z) \exp\left(\frac{1}{\lambda_0} \int_z^1 \tilde{\mathcal{R}}_0(t) dt\right) dz - \left[\arctan(h_0) \exp\left(\frac{1}{\lambda_0} \int_z^1 \tilde{\mathcal{R}}_0(t) dt\right) \right]_{z=0}^{z=1}.$$

The second term vanishes since h_0 is 1-periodic and $\int_0^1 \widetilde{\mathcal{R}}_0 = 0$ by Lemma 5.7. Recalling that $\widetilde{\mathcal{R}}_0 = \mathcal{R} h_0 \sqrt{1 + h_0^2}$, we obtain

$$J = -\frac{1}{\lambda_0} \int_0^1 \left(\mathcal{R} h_0 \arctan(h_0) \sqrt{1 + h_0^2} \right) (z) \exp \left(\frac{1}{\lambda_0} \int_z^1 \widetilde{\mathcal{R}}_0(t) dt \right) dz.$$

Now since $\arctan(h_0)$ has the same sign as h_0 and $\mathcal{R} \geq 0$ by (5.2), we have $J \leq 0$. This proves that $\lambda \mapsto c(\lambda)$ is nonincreasing.

Let us immediately show that this map is decreasing if \mathcal{R} is not constant. In that case, the ODE in (5.1) implies that h_0 is not identically equal to zero. Hence $J < 0$ – since $\mathcal{R} > 0$ by (5.2) – and this completes the proof that $c'(\lambda_0) < 0$ if \mathcal{R} is not trivial.

Let us end with the limit of $c(\lambda)$ as $\lambda \rightarrow +\infty$. Recall that

$$c(\lambda) = \int_0^1 \mathcal{R}(z) \sqrt{1 + h^2(z, \lambda)} dz, \tag{5.12}$$

after integrating the equation in (5.1). Recall now that by (5.2), we have $\mathcal{R}_M \geq \mathcal{R} \geq \mathcal{R}_m > 0$ for some constants. Then $c \leq \mathcal{R}_M$ and

$$\max_{z \in \mathbb{R}} |h(z, \lambda)| \leq \sqrt{\frac{\mathcal{R}_M^2}{\mathcal{R}_m^2} - 1}$$

by Theorem 2.1. Using again the ODE in (5.1),

$$\max_{z \in \mathbb{R}} |h_z(z, \lambda)| \leq \frac{C}{\lambda}$$

for some C not depending on λ . Since $h(\cdot, \lambda)$ is 1-periodic with $\int_0^1 h(z, \lambda) dz = 0$, an integration gives

$$\max_{z \in \mathbb{R}} |h(z, \lambda)| \leq \frac{C}{\lambda}.$$

This proves that $h(\cdot, \lambda)$ uniformly converges towards zero on \mathbb{R} , as $\lambda \rightarrow +\infty$. Then the fact that $\lim_{\lambda \rightarrow +\infty} c(\lambda) = \int_0^1 \mathcal{R}(z) dz$ is obvious from (5.12). The proof of the theorem is now complete. \square

REMARK 5.10 We have also proved that $h(\cdot, \lambda) \rightarrow 0$ in $W^{1,\infty}(\mathbb{R})$ as $\lambda \rightarrow +\infty$.

6. Asymptotic expansion of the front’s profile

From Section 4, we know that at a macroscopic level, the profile of the front, v^ε , behaves like v^0 which is a constant (normalized to zero). In this section, we propose to analyze its microscopic oscillations by looking at the corrector $w = w(z)$ given by Theorem 4.5. So let us consider for any $\lambda \in (0, +\infty)$, the unique $w \in W^{2,\infty}(\mathbb{R})$ satisfying

$$\begin{cases} -c^0 + R \left(z, \frac{g}{b} \right) \sqrt{1 + w_z^2} = \lambda \frac{w_{zz}}{1 + w_z^2}, \\ w(z + 1) = w(z), \\ \int_0^1 w(t) dt = 0, \end{cases} \tag{6.1}$$

for almost every $z \in \mathbb{R}$.

Theorem 6.1 *Let the assumptions of Theorem 4.5 hold. Then for all $y \in \mathbb{R}$ and $\varepsilon > 0$,*

$$v^\varepsilon(y) = \begin{cases} o(\varepsilon) & \text{if } \lambda = +\infty, \\ \varepsilon w\left(\frac{y}{\varepsilon}\right) + o(\varepsilon) & \text{if } \lambda \in (0, +\infty), \end{cases}$$

where $\frac{o(\varepsilon)}{\varepsilon} \rightarrow 0$ in $L^\infty(\mathbb{R})$ as $\varepsilon \downarrow 0$ and with w given by (6.1).

REMARK 6.2 If $R(\cdot, \frac{\bar{g}}{b})$ is not constant, then w is not trivial. Note also that the more difficult case $\lambda = 0$ will be discussed in Section 7.

Proof. Let us begin with the proof for $\lambda = +\infty$. By (4.15),

$$\|v_y^\varepsilon\|_\infty = O\left(\frac{\varepsilon}{\mu(\varepsilon)}\right) = o_\varepsilon(1)$$

(since $\frac{\mu(\varepsilon)}{\varepsilon} \rightarrow \lambda = +\infty$). Since v^ε is ε -periodic with a zero mean value,

$$\|v^\varepsilon\|_\infty \leq \varepsilon \|v_y^\varepsilon\|_\infty = o(\varepsilon).$$

This completes the proof in that case.

Let us now consider the case $\lambda \in (0, +\infty)$ and set $w^\varepsilon(z) = \frac{v^\varepsilon(\varepsilon z)}{\varepsilon}$ (as in (4.13)). We claim that w^ε converges to w as $\varepsilon \downarrow 0$ strongly in $W^{1,\infty}(\mathbb{R})$. We have in fact established this claim during the proof of Theorem 4.5 but only along some sequence $\varepsilon_n \rightarrow 0$. But, applying this reasoning to any such sequence, as before, we get again the convergence for the whole family, because here also the limit is always the same, that is w defined by (6.1). We thus have:

$$\max_y \left| \frac{d}{dy} \left\{ v^\varepsilon(y) - \varepsilon w\left(\frac{y}{\varepsilon}\right) \right\} \right| = \max_y \left| v_y^\varepsilon(y) - w_z\left(\frac{y}{\varepsilon}\right) \right| = \max_z |w_z^\varepsilon(z) - w_z(z)| = o_\varepsilon(1).$$

Using as above the periodicity of v^ε and w and the fact that they have zero mean values,

$$\max_y \left| v^\varepsilon(y) - \varepsilon w\left(\frac{y}{\varepsilon}\right) \right| \leq \varepsilon \max_y \left| \frac{d}{dy} \left\{ v^\varepsilon(y) - \varepsilon w\left(\frac{y}{\varepsilon}\right) \right\} \right| = o(\varepsilon)$$

and the proof is complete. □

Just as for the speeds, here too we can see that there is a smooth one-to-one correspondence between the correctors w and the curvature regimes λ . More precisely, let us consider the mapping

$$\lambda \in \mathbb{R}^+ \mapsto w(\cdot) = w(\cdot, \lambda) \in W_\#^{2,\infty}(\mathbb{R}_z),$$

defined by (6.1). Then:

Theorem 6.3 *Under the assumptions of Theorem 4.5, $\lambda \mapsto w(\cdot, \lambda)$ is C^∞ , injective, and satisfies:*

$$\lim_{\lambda \rightarrow +\infty} w(\cdot, \lambda) = 0 \quad \text{in } W^{2,\infty}(\mathbb{R}).$$

REMARK 6.4 This is a version of Theorem 5.1 for the correctors. The limit at $\lambda = 0$ will be discussed in the concluding remarks.

Proof. Recall that $w_z(\cdot, \lambda) = h(\cdot, \lambda)$ in (5.1) and use Remark 5.9 to get the C^∞ regularity. For the injectivity, use that $\lambda \mapsto c^0 = c^0(\lambda)$ is injective by Theorem 5.1. For the limit at $+\infty$, use Remark 5.10. \square

Our last result considers a fixed μ as in Theorem 4.2. In that case, we can identify the second order term of the expansion. To get an expansion in L^∞ , we will need to assume in addition that:

$$\text{The function } T \in \mathbb{R}^+ \mapsto R(\cdot, T) \in L^\infty(\mathbb{R}) \text{ is continuous at } T = \frac{\bar{g}}{b}. \tag{6.2}$$

Theorem 6.5 *Let the assumptions of Theorem 4.2 hold. Then for all $y \in \mathbb{R}$ and $\varepsilon > 0$, we have*

$$v^\varepsilon(y) = \varepsilon^2 Q\left(\frac{y}{\varepsilon}\right) + o(\varepsilon^2),$$

where $Q \in W^{2,\infty}(\mathbb{R}_z)$ is the unique (almost everywhere) solution of

$$\begin{cases} Q_{zz}(z) = \frac{1}{\mu} \left\{ R\left(z, \frac{\bar{g}}{b}\right) - \int_0^1 R\left(t, \frac{\bar{g}}{b}\right) dt \right\}, \\ Q(z+1) = Q(z), \quad \text{and} \\ \int_0^1 Q = 0, \end{cases}$$

and where $\lim_{\varepsilon \downarrow 0} \frac{o(\varepsilon^2)}{\varepsilon^2} = 0$ in $L^p_{\text{loc}}(\mathbb{R})$ for any $p \in [1, +\infty)$. If in addition (6.2) holds then the latter limit holds in $L^\infty(\mathbb{R})$.

REMARK 6.6 (i) The limit in $L^p_{\text{loc}}(\mathbb{R})$ has to be understood as follows: For any fixed $p \in [1, +\infty)$ and $r > 0$,

$$\frac{1}{\varepsilon^2} \left(\int_{-r}^r |o(\varepsilon^2)|^p dy \right)^{\frac{1}{p}} \rightarrow 0$$

(the limit being uniform neither in p nor in r).

(ii) The additional assumption (6.2) is satisfied by the combustion rate in (1.3) (provided that A and E are bounded).

(iii) The profile Q is not trivial if $R\left(\cdot, \frac{\bar{g}}{b}\right)$ is not a constant.

Proof. Note that Q is well-defined, because the function

$$z \mapsto R\left(z, \frac{\bar{g}}{b}\right) - \int_0^1 R\left(t, \frac{\bar{g}}{b}\right) dt$$

is 1-periodic with a zero mean value. Let us divide the rest of the proof in two cases.

1. *Expansion in L^p_{loc} for $p \neq +\infty$.* Let us use again $w^\varepsilon(z) = \frac{v^\varepsilon(\varepsilon z)}{\varepsilon}$ (as in (4.13)). We can rewrite its equation (4.12) as

$$\frac{\mu}{\varepsilon} w^\varepsilon_{zz} = F_\varepsilon(z) := -c^\varepsilon(1 + (w^\varepsilon_z)^2) + R\left(z, u^\varepsilon(v^\varepsilon(\varepsilon z), \varepsilon z)\right)(1 + (w^\varepsilon_z)^2)^{\frac{3}{2}}.$$

We claim that for any $p \in [1, +\infty)$

$$F_\varepsilon(\cdot) \rightarrow R\left(\cdot, \frac{\bar{g}}{b}\right) - \int_0^1 R\left(t, \frac{\bar{g}}{b}\right) dt \quad \text{in } L^p(0, 1), \tag{6.3}$$

as $\varepsilon \downarrow 0$. To show this claim, recall that $c^\varepsilon \rightarrow \int_0^1 R\left(t, \frac{\bar{g}}{b}\right) dt$ and $\|w_z^\varepsilon\|_\infty \rightarrow 0$, by Theorem 4.2 and (4.15). The case $p = 1$ is thus a consequence of Lemma 4.15(i) (whose result holds for the whole family $\{\varepsilon\}$, thanks to the same arguments). Since R is bounded by R_M , this also implies the convergence for any $p \neq +\infty$ by interpolation.

With (6.3) in hands, we can write that

$$\begin{aligned} \frac{d^2}{dy^2} \left\{ v^\varepsilon(y) - \varepsilon^2 Q\left(\frac{y}{\varepsilon}\right) \right\} &= \frac{w_{zz}^\varepsilon\left(\frac{y}{\varepsilon}\right)}{\varepsilon} - Q_{zz}\left(\frac{y}{\varepsilon}\right) \\ &= \frac{1}{\mu} F_\varepsilon\left(\frac{y}{\varepsilon}\right) - \frac{1}{\mu} \left\{ R\left(\frac{y}{\varepsilon}, \frac{\bar{g}}{b}\right) - \int_0^1 R\left(t, \frac{\bar{g}}{b}\right) dt \right\} \end{aligned}$$

and conclude that

$$\frac{1}{\varepsilon} \int_0^\varepsilon \left| \frac{d^2}{dy^2} \left\{ v^\varepsilon(y) - \varepsilon^2 Q\left(\frac{y}{\varepsilon}\right) \right\} \right|^p dy = o_\varepsilon(1)$$

(with $o_\varepsilon(1)$ depending on p). Let us now apply the Poincaré–Wirtinger’s inequality which states that $\int_0^\varepsilon |f|^p \leq C \varepsilon^p \int_0^\varepsilon |f_y|^p$ for any ε -periodic $f \in W^{1,p}(\mathbb{R})$ with a zero mean value (and for some $C = C(p)$). Applying this twice, we get that

$$\frac{1}{\varepsilon} \int_0^\varepsilon \left| v^\varepsilon(y) - \varepsilon^2 Q\left(\frac{y}{\varepsilon}\right) \right|^p dy = \varepsilon^{2p} o_\varepsilon(1).$$

Given then $r > 0$, we take $r_\varepsilon = \varepsilon^{\lceil \frac{r}{\varepsilon} \rceil} \geq r$ (this symbol denoting the upper integer part). We can then rewrite the mean value above over the larger period $2r_\varepsilon$ as follows: $\frac{1}{\varepsilon} \int_0^\varepsilon |\dots|^p dy = \frac{1}{2r_\varepsilon} \int_{-r_\varepsilon}^{r_\varepsilon} |\dots|^p dy$. This easily implies the desired result.

2. *Expansion in L^∞ .* Let us prove that if (6.2) holds then (6.3) holds in $L^\infty(\mathbb{R})$. Set

$$T_\varepsilon := \min_z u^\varepsilon(v^\varepsilon(\varepsilon z), \varepsilon z) \quad \text{and} \quad T^\varepsilon := \max_z u^\varepsilon(v^\varepsilon(\varepsilon z), \varepsilon z).$$

Assumption (C) then gives

$$R(z, T_\varepsilon) \leq R(z, u^\varepsilon(v^\varepsilon(\varepsilon z), \varepsilon z)) \leq R(z, T^\varepsilon)$$

for almost every z . Now recalling that $T_\varepsilon, T^\varepsilon \rightarrow \frac{\bar{g}}{b}$ (again by Theorem 4.2), we have by the assumption (6.2)

$$R(\cdot, u^\varepsilon(v^\varepsilon(\varepsilon \cdot), \varepsilon \cdot)) \rightarrow R\left(\cdot, \frac{\bar{g}}{b}\right) \quad \text{in} \quad L^\infty(\mathbb{R}).$$

This implies (6.3) in $L^\infty(\mathbb{R})$ (since $c^\varepsilon \rightarrow \int_0^1 R\left(t, \frac{\bar{g}}{b}\right) dt$ and $\|w_z^\varepsilon\|_\infty \rightarrow 0$) and the rest of the proof is the same (applying Poincaré–Wirtinger for $p = +\infty$). □

7. Concluding remarks

Let us conclude by a synthesis on the propagation governed by the typical Arrhenius law in (1.3),

$$R(y, T) = A(y) e^{-\frac{E(y)}{T}},$$

with R the combustion rate, T the temperature, A a prefactor and E related to the activation energy. Recall that this R degenerates as T goes to zero. We have established the existence of a travelling wave solution “speed-front-temperature” provided that the ratio “period/curvature coefficient” is small enough, see Theorem 3.8. For large ratios, such waves still exist up to slightly modifying the degeneracy at $T = 0$, see Theorem 3.4. During the homogenization of these waves as the period ε tends to zero, we have allowed the curvature coefficient $\mu = \mu(\varepsilon)$ to depend on ε too (this parameter being related to surface tension effects). Then the limiting speed of propagation c^0 is entirely determined by the value of the curvature regime $\lambda = \lim_{\varepsilon \downarrow 0} \frac{\mu(\varepsilon)}{\varepsilon}$, see Theorem 4.5 and Remark 5.2. This speed $c^0 = c^0(\lambda)$ is decreasing in λ and takes the following minimal and maximal values:

$$c^0(\lambda = +\infty) = \overline{A e^{-E \frac{\bar{b}}{g}}} \quad \text{and} \quad c^0(\lambda = 0) = \operatorname{ess\,sup}_z A(z) e^{-E(z) \frac{\bar{b}}{g}},$$

with b the heat capacity, g the heat release, \bar{b} and \bar{g} their mean values, etc., see Theorem 5.1 and Remark 5.2. Here the constant $\frac{\bar{g}}{b} > 0$ is the limiting temperature at the front. Finally the profile v^ε of the heterogeneous flame front satisfies:

$$v^\varepsilon(y) = \varepsilon w\left(\frac{y}{\varepsilon}\right) + o(\varepsilon), \tag{7.1}$$

where $w = w(z)$ is a corrector entirely determined by λ , see Theorem 6.1. At the macroscopic level, the front’s profile is a straight line (normalized to zero without loss of generality). At the microscopic level, its oscillations are entirely described by w which solves the geometric equation

$$-c_0(\lambda) + A(z) e^{-E(z) \frac{\bar{b}}{g}} \sqrt{1 + w_z^2} = \lambda \frac{w_{zz}}{1 + w_z^2}. \tag{7.2}$$

If μ is fixed, then the profile w is also a straight line and

$$v^\varepsilon(y) = \varepsilon^2 Q\left(\frac{y}{\varepsilon}\right) + o(\varepsilon^2),$$

where the profile Q is given by:

$$Q(z) = \frac{1}{\mu} \left\{ P(z) - z \overline{P_z} - \overline{(P - z \overline{P_z})} \right\}$$

where

$$P(z) := \int_0^z \int_0^t \left(A(s) e^{-E(s) \frac{\bar{b}}{g}} - c^0(+\infty) \right) ds dt,$$

see Theorem 6.5.

We end up with an open question. So far we have been able to give ansatz of the front’s profile for all values of λ , including $+\infty$, but not for $\lambda = 0$ where the expansion (7.1) is not clear. This question is related to the passage to the limit in (7.2) as $\lambda \downarrow 0$. By the result of [6], it is known that some sequence converges towards a viscosity solution of the Hamilton–Jacobi equation

$$-c + A(z) e^{-E(z) \frac{\bar{b}}{g}} \sqrt{1 + w_z^2} = 0$$

(with $c = c^0(\lambda = 0)$). Unfortunately, the convergence of the whole family is not clear because this solution is not unique (even up to the addition of a constant). The problem is that w_z^2 is unique but not w_z . The identification of the first-order term in (7.1) for $\lambda = 0$ is thus open and probably difficult (to the best of our knowledge).

Appendix A. Technical or standard proofs

A.1 Proof of the claim in Remark 2.3

Proof that “(2.5) \Rightarrow (2.6)”. Let $\varphi \in C_c^1(\mathbb{R}^2)$ and define a partition of unity $\{\theta_i\}_{i \in \mathbb{Z}}$ such that for all $i \in \mathbb{Z}$,

$$\begin{cases} 0 \leq \theta_i \in C_c^1(\mathbb{R}), \\ \text{supp } \theta_i \subset (\frac{i}{2} Y, \frac{i+2}{2} Y), \\ \theta_{i+1}(\cdot) = \theta_i(\cdot - \frac{Y}{2}), \\ \sum_{i \in \mathbb{Z}} \theta_i = 1. \end{cases}$$

Note that the sum at any value is locally taken only on two consecutive indices. Let us define

$$\varphi_i(x, y) := \varphi(x, y) \theta_i(y),$$

so that $\varphi = \sum_{i \in \mathbb{Z}} \varphi_i$. By linearity, it suffices to prove (2.6) for each φ_i . So let $i \in \mathbb{Z}$ and consider $w \in C^1(\mathbb{R}^2)$ such that $w = \varphi_i$ on $\{\frac{i}{2} Y < y < \frac{i+2}{2} Y\}$ and extended to $y \in \mathbb{R}$ by periodicity. It is clear that $w|_{\Omega} \in H_{\#}^1(\Omega)$. In particular, it can be chosen in (2.5), which exactly gives (2.6). \square

Proof that “(2.6) \Rightarrow (2.5)”. Conversely, let us consider $w \in H_{\#}^1(\Omega)$. Extending it by reflexion if necessary, we can consider that $w \in H_{\#}^1(\mathbb{R}^2)$. Let us define $w_i \in H^1(\mathbb{R}^2)$ by

$$w_i(x, y) := w(x, y) \theta_i(y).$$

By density of $C_c^1(\mathbb{R}^2)$ in $H^1(\mathbb{R}^2)$, (2.6) holds true for each w_i whenever it is so for test functions. Since $\text{supp } w_i \subset (\frac{i}{2} Y, \frac{i+2}{2} Y)$, we get:

$$\int_{\Omega \cap \{0 < y < \frac{3}{2} Y\}} (c b u_x (w_0 + w_1) + a \nabla u \nabla (w_0 + w_1)) = \int_{\Gamma \cap \{0 < y < \frac{3}{2} Y\}} \frac{c g}{\sqrt{1 + v_y^2}} (w_0 + w_1). \tag{A1}$$

Let us rewrite these terms as integrals over $\{0 < y < Y\}$. If $\frac{Y}{2} < y < Y$, then we use that $w = \sum_{i \in \mathbb{Z}} w_i = w_0 + w_1$ (for such y). We get:

$$\int_{\Omega \cap \{\frac{Y}{2} < y < Y\}} (c b u_x (w_0 + w_1) + a \nabla u \nabla (w_0 + w_1)) = \int_{\Omega \cap \{\frac{Y}{2} < y < Y\}} (c b u_x w + a \nabla u \nabla w). \tag{A2}$$

For the remaining y , we use that u is Y -periodic and $w_1(\cdot) = w_{-1}(\cdot - Y)$ to show that

$$\int_{\Omega \cap \{Y < y < \frac{3}{2} Y\}} (c b u_x w_1 + a \nabla u \nabla w_1) = \int_{\Omega \cap \{0 < y < \frac{Y}{2}\}} (c b u_x w_{-1} + a \nabla u \nabla w_{-1}).$$

Using in addition that $w_0 = 0$ on $\{Y < y < \frac{3}{2} Y\}$ and $w_1 = 0$ on $\{0 < y < \frac{Y}{2}\}$, we deduce that

$$\begin{aligned} & \int_{\Omega \cap (\{0 < y < \frac{Y}{2}\} \cup \{Y < y < \frac{3}{2} Y\})} (c b u_x (w_0 + w_1) + a \nabla u \nabla (w_0 + w_1)) \\ &= \int_{\Omega \cap \{0 < y < \frac{Y}{2}\}} (c b u_x (w_0 + w_{-1}) + a \nabla u \nabla (w_0 + w_{-1})) \\ &= \int_{\Omega \cap \{0 < y < \frac{Y}{2}\}} (c b u_x w + a \nabla u \nabla w). \end{aligned} \tag{A3}$$

The last line is obtained from similar arguments as for (A2). Adding (A2) and (A3), we conclude that

$$\int_{\Omega \cap \{0 < y < \frac{3}{2} Y\}} (c b u_x (w_0 + w_1) + a \nabla u \nabla (w_0 + w_1)) = \int_{\Omega \cap \{0 < y < Y\}} (c b u_x w + a \nabla u \nabla w).$$

We show in the same way that

$$\int_{\Gamma \cap \{0 < y < \frac{3}{2} Y\}} \frac{c g}{\sqrt{1 + v_y^2}} (w_0 + w_1) = \int_{\Gamma \cap \{0 < y < Y\}} \frac{c g}{\sqrt{1 + v_y^2}} w.$$

We thus complete the proof of (2.5) from (A1). □

A.2 Proof of Theorem 2.4: Further details

In the preceding sections, we have only sketched this proof by referring to standard variational arguments [4]. Let us be more precise in this appendix for completeness sake.

Given $\alpha > 0$, we introduce the following auxiliary problem:

$$\begin{cases} \text{find } u_\alpha \in H_\#^1(\Omega) \text{ such that} \\ a_\alpha(u_\alpha, w) = l(w) \quad \forall w \in H_\#^1(\Omega), \end{cases} \tag{A4}$$

where

$$\begin{aligned} a_\alpha(u_\alpha, w) &:= \frac{\alpha}{Y} \int_{\Omega_\#} u_\alpha w + \frac{1}{Y} \int_{\Omega_\#} (c b (u_\alpha)_x w + a \nabla u_\alpha \nabla w), \\ l(w) &:= \frac{1}{Y} \int_{\Gamma_\#} \frac{c g}{\sqrt{1 + v_y^2}} w. \end{aligned}$$

Recall that the L^2 -norm and H^1 -semi-norm for periodic spaces have been defined by

$$\|w\|_{L_\#^2(\Omega)} = \left(\frac{1}{Y} \int_{\Omega_\#} w^2 \right)^{\frac{1}{2}} \quad \text{and} \quad |w|_{H_\#^1(\Omega)} = \left(\frac{1}{Y} \int_{\Omega_\#} |\nabla w|^2 \right)^{\frac{1}{2}}.$$

The proof of Theorem 2.4 will follow from the passage to the limit as $\alpha \downarrow 0$. Let us first give a few lemmas. We start by a well-posedness result.

Lemma A.1 *Under the assumptions of Theorem 2.4, Problem (A4) admits a unique solution, for each $\alpha > 0$.*

Proof. To use the Lax–Milgram’s theorem, it suffices to check the coercivity of a_α which follows from:

$$\frac{1}{Y} \int_{\Omega_\#} (c b w_x w + a |\nabla w|^2) = \underbrace{\frac{1}{Y} \int_{\Gamma_\#} c b \frac{w^2}{2}}_{\geq 0} + \underbrace{\frac{1}{Y} \int_{\Omega_\#} a |\nabla w|^2}_{\geq a_m |w|_{H_\#^1(\Omega)}^2} \quad \forall w \in H_\#^1(\Omega). \tag{A5} \quad \square$$

We proceed by a nonnegativity lemma.

Lemma A.2 Assume the hypotheses of Theorem 2.4 and let $\alpha \geq 0$ and u be such that

$$\begin{cases} u \in H^1_{\#}(\Omega), \\ a_{\alpha}(u, w) \geq 0 \quad \forall w \in H^1_{\#}(\Omega) \text{ with } w \geq 0. \end{cases}$$

Then $u \geq 0$.

Proof. Let us take $w = u^- := -\min\{u, 0\}$ and show that $u^- = 0$. Since $\nabla u^- = -\mathbf{1}_{u < 0} \nabla u$, we have:

$$\begin{aligned} -\alpha \|u^-\|_{L^2_{\#}(\Omega)}^2 - \frac{1}{Y} \int_{\Omega_{\#}} a |\nabla u^-|^2 &\geq \frac{1}{Y} \int_{\Omega_{\#}} c b u_x^- u^- \\ &= \frac{1}{Y} \int_{\Gamma_{\#}} c b \frac{(u^-)^2}{2} \geq 0 \end{aligned}$$

This completes the proof even for $\alpha = 0$ since u^- is (square) integrable as x approaches $-\infty$. \square

We finally give some estimates on u_{α} .

Lemma A.3 Let $\alpha > 0$ and u_{α} be given by Lemma A.1. Then

$$|u_{\alpha}|^2_{H^1_{\#}(\Omega)} \leq \frac{2c g_M^2}{a_m b_m}$$

and for almost every $(x, y) \in \Omega$,

$$\frac{g_M a_m}{a_M b_M} e^{c \frac{b_M}{a_m}(x - \|v\|_{\infty})} \leq u_{\alpha}(x, y) \leq \frac{g_M a_M}{a_m b_m} e^{c \frac{b_m}{a_M}(x + \|v\|_{\infty})}.$$

Proof. If we take $w = u_{\alpha}$ in (A5) and use the inequality

$$l(u_{\alpha}) = a_{\alpha}(u_{\alpha}, u_{\alpha}) \geq \frac{1}{Y} \int_{\Omega_{\#}} (c b (u_{\alpha})_x u_{\alpha} + a |\nabla u_{\alpha}|^2),$$

we infer that

$$\begin{aligned} \frac{c b_m}{2} \frac{1}{Y} \int_0^Y u_{\alpha}^2(v(y), y) dy &\leq l(u_{\alpha}) \\ &\leq \frac{c g_M}{Y} \int_0^Y u_{\alpha}(v(y), y) dy \leq c g_M \left(\frac{1}{Y} \int_0^Y u_{\alpha}^2(v(y), y) dy \right)^{\frac{1}{2}}, \end{aligned}$$

so that $\left(\frac{1}{Y} \int_0^Y u_{\alpha}^2(v(y), y) dy \right)^{\frac{1}{2}} \leq \frac{2g_M}{b_m}$ and $l(u_{\alpha}) \leq \frac{2c g_M^2}{b_m}$. Using again (A5), we get the first desired estimate $|u_{\alpha}|^2_{H^1_{\#}(\Omega)} \leq \frac{a(u_{\alpha}, u_{\alpha})}{a_m} = \frac{l(u_{\alpha})}{a_m} \leq \frac{2c g_M^2}{a_m b_m}$.

For the remaining estimate, we look for sub and supersolutions of the form $\tilde{u}(x, y) := C_1 e^{C_2 x}$. Let us prove the upper bound (the proof for the lower bound being similar). Taking $C_2 = \frac{c b_m}{a_M} \geq 0$, $C_1 = \frac{c g_M}{a_m C_2} e^{C_2 \|v\|_{\infty}} \geq 0$, easy computations show that

$$\begin{cases} \alpha \tilde{u} + c b \tilde{u}_x - \operatorname{div}(a \nabla \tilde{u}) \geq 0 & \text{in } \Omega, \\ a \frac{\partial \tilde{u}}{\partial \nu} \geq \frac{c g}{\sqrt{1+v_y^2}} & \text{on } \Gamma. \end{cases}$$

In particular, $\tilde{u} \in C^\infty(\mathbb{R}^2) \cap H_\#^1(\Omega)$ satisfies the following variational inequalities:

$$a_\alpha(\tilde{u}, w) \geq l(w) \quad \forall w \in H_\#^1(\Omega) \text{ such that } w \geq 0.$$

We complete the proof by applying Lemma A.2 to $u = \tilde{u} - u_\alpha$. □

We can now prove the theorem.

Proof of Theorem 2.4. The uniqueness is an immediate consequence of Lemma A.2 (valid for $\alpha = 0$). Moreover, by Lemma A.3, $\{u_\alpha\}_{\alpha>0}$ is bounded in $H_\#^1(\Omega)$. Hence, by the weak-compactness theorem, u_α weakly converges to some u in $H_\#^1(\Omega)$ as $\alpha \downarrow 0$ (and up to a subsequence). It is then standard to pass to the limit in (A4) and get a solution of (1.4). This solution satisfies the desired estimates (2.7) and (2.8), since so does u_α (by Lemma A.3). □

A.3 Proofs of Lemmas 2.7, 2.8 and 4.14

Proof of Lemma 2.7. Given $\varphi \in C_c^1(\mathbb{R}^2)$, we have

$$\int_{\Omega_n} (c_n b (u_n)_x \varphi + a \nabla u_n \nabla \varphi) = \int_{\mathbb{R}} c_n g(y) \varphi(v_n(y), y) dy,$$

that is

$$\int_{\mathbb{R}^2} (c_n (u_n)_x b \varphi \mathbf{1}_{\Omega_n} + a \mathbf{1}_{\Omega_n} \nabla \varphi \nabla u_n) = \int_{\mathbb{R}} c_n g(y) \varphi(v_n(y), y) dy. \tag{A6}$$

Let us pass to the limit in (A6) (along the subsequence given by Lemma 2.5).

We claim that $\mathbf{1}_{\Omega_n} \rightarrow \mathbf{1}_\Omega$ almost everywhere on \mathbb{R}^2 . Indeed, for all $x \neq v(y)$, we have $x \neq v_n(y)$ whenever n is sufficiently large, since $v_n(y) \rightarrow v(y)$. This shows that the convergence holds for all $(x, y) \notin \Gamma = \{x = v(y)\}$. This proves our claim, since the two-dimensional Lebesgue measure of this Lipschitz graph is zero.

We deduce that $b \varphi \mathbf{1}_{\Omega_n} \rightarrow b \varphi \mathbf{1}_\Omega$ and $a \mathbf{1}_{\Omega_n} \nabla \varphi \rightarrow a \mathbf{1}_\Omega \nabla \varphi$ strongly in $L^2(\mathbb{R}^2)$, by the dominated convergence theorem. Moreover, $\nabla u_n \rightharpoonup \nabla u$ weakly in $(L_\#^2(\mathbb{R}^2))^2$ since $u_n \rightharpoonup u$ weakly in $H_\#^1(\mathbb{R}^2)$. It is then standard to pass to the limit in (A6) and deduce that

$$\int_{\Omega} (c b u_x \varphi + a \nabla u \nabla \varphi) = \int_{\Gamma} \frac{c g}{\sqrt{1 + v_y^2}} \varphi,$$

for all $\varphi \in C_c^1(\mathbb{R}^2)$. To pass to the limit in the boundary term, we have simply used the uniform convergence of v_n towards v . This proves that $u|_{\Omega}$ is a variational solution to (1.4).

To pass to the limit in (2.10), we consider $\varphi \in C_c(\mathbb{R})$ and write that

$$\int_{\mathbb{R}} \left(-c_n + H_n \sqrt{1 + (v_n)_y^2}\right) \varphi = \mu \int_{\mathbb{R}} (v_n)_{yy} \frac{\varphi}{1 + (v_n)_y^2}.$$

Since $(v_n)_y \rightarrow v_y$ uniformly, the $L^\infty(\mathbb{R})$ weak- \star convergences of H_n and $(v_n)_{yy}$ are sufficient to pass to the limit. We get

$$\int_{\mathbb{R}} \left(-c + H \sqrt{1 + v_y^2}\right) \varphi = \mu \int_{\mathbb{R}} \frac{v_{yy}}{1 + v_y^2} \varphi$$

for all $\varphi \in C_c(\mathbb{R})$, which completes the proof. □

Proof of Lemma 2.8. Let $w_n(y) := u_n(v_n(y), y)$ and recall that

$$\|w_n\|_{H_{\#}^{\frac{1}{2}}(\mathbb{R})} \leq C(Y, \|(v_n)_y\|_{\infty}) \|u_n\|_{H_{\#}^1(\Omega_n)},$$

as noticed in Section 2.2. By the bounds of the proof of Lemma 2.5, it follows that $\{w_n\}_n$ is bounded in $H_{\#}^{\frac{1}{2}}(\mathbb{R})$. Hence, the compact embedding of $H_{\#}^{\frac{1}{2}}(\mathbb{R})$ into $L_{\#}^2(\mathbb{R})$ implies that w_n converges to some \tilde{w} strongly in $L_{\#}^2(\mathbb{R})$ and almost everywhere, up to some subsequence (chosen as stated in the lemma). It remains to show that $\tilde{w}(y) = u(v(y), y)$ almost everywhere. By the Gauss–Green formula, we have for any $\varphi \in C_c^1(\mathbb{R}^2)$,

$$\int_{\mathbb{R}} w_n(y) \varphi(v_n(y), y) \, dy = \int_{\Omega_n} (u_n \varphi)_x \, dx \, dy = \int_{\mathbb{R}^2} (\mathbf{1}_{\Omega_n} \varphi (u_n)_x + \mathbf{1}_{\Omega_n} \varphi_x u_n) \, dx \, dy.$$

Arguing as in the proof of Lemma 2.7, $\mathbf{1}_{\Omega_n} \rightarrow \mathbf{1}_{\Omega}$ almost everywhere and passing to the limit in the above equation implies that

$$\begin{aligned} \int_{\mathbb{R}} \tilde{w}(y) \varphi(v(y), y) \, dy &= \int_{\mathbb{R}^2} (\mathbf{1}_{\Omega} \varphi u_x + \mathbf{1}_{\Omega} \varphi_x u) \, dx \, dy \\ &= \int_{\Omega} (u \varphi)_x \, dx \, dy = \int_{\mathbb{R}} u(v(y), y) \varphi(v(y), y) \, dy. \end{aligned}$$

We complete the proof by taking test functions of the form $\varphi(x, y) = \theta(x) \psi(y)$, with $\theta(x) = 1$ when $|x| \leq \|v\|_{\infty}$. □

Proof of Lemma 4.14. Let us pass to the limit in the equation (4.1) satisfied by u^ε (along the sequence given by Lemma 4.11). Recall that we have

$$\int_0^\varepsilon \int_{-\infty}^{v^\varepsilon(y)} (c^\varepsilon b^\varepsilon u_x^\varepsilon w + a^\varepsilon \nabla u^\varepsilon \nabla w) \, dx \, dy = \int_{x=v^\varepsilon(y), 0 < y < \varepsilon} \frac{c^\varepsilon g^\varepsilon}{\sqrt{1+v_y^\varepsilon}} w,$$

for all $w \in H_{\text{per}}^1(\Omega^\varepsilon)$, see Definition 4.1. By the periodicity of u^ε (and w), we can rewrite this equality as

$$\int_0^{l_\varepsilon} \int_{-\infty}^{v^\varepsilon(y)} c^\varepsilon b^\varepsilon u_x^\varepsilon w \, dx \, dy + \int_0^{l_\varepsilon} \int_{-\infty}^{v^\varepsilon(y)} a^\varepsilon \nabla u^\varepsilon \nabla w \, dx \, dy = \int_{x=v^\varepsilon(y), 0 < y < l_\varepsilon} \frac{c^\varepsilon g^\varepsilon}{\sqrt{1+v_y^\varepsilon}} w,$$

where $l_\varepsilon = \lfloor \frac{1}{\varepsilon} \rfloor \varepsilon$. Note that $l_\varepsilon \rightarrow 1$ as $\varepsilon \downarrow 0$. It is in this latter equation that we are going to pass to the limit by proceeding term by term. Note that we can restrict ourselves to w of the form $w(x, y) = \varphi(x)$ with an arbitrary $\varphi \in C_c^\infty(\mathbb{R})$. This is motivated by the fact that the limit u^0 is already known to be independent of y , see Lemma 4.12. We thus have to pass to the limit in the equation:

$$\begin{aligned} 0 &= \int_0^{l_\varepsilon} \int_{-\infty}^{v^\varepsilon(y)} c^\varepsilon b \left(\frac{y}{\varepsilon}\right) u_x^\varepsilon \varphi \, dx \, dy \\ &\quad + \int_0^{l_\varepsilon} \int_{-\infty}^{v^\varepsilon(y)} a \left(\frac{y}{\varepsilon}\right) u_x^\varepsilon \varphi_x \, dx \, dy \\ &\quad - \int_0^{l_\varepsilon} c^\varepsilon g \left(\frac{y}{\varepsilon}\right) \varphi(v^\varepsilon(y)) \, dy \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{A7}$$

1. *The first term.* We have

$$I_1 = - \int_{\mathbb{R}^2} c^\varepsilon b \left(\frac{y}{\varepsilon}\right) u^\varepsilon \varphi_x \mathbf{1}_{x < v^\varepsilon(y), 0 < y < l_\varepsilon} dx dy + \int_{\mathbb{R}} c^\varepsilon b \left(\frac{y}{\varepsilon}\right) u^\varepsilon(v^\varepsilon(y), y) \varphi(v^\varepsilon(y)) \mathbf{1}_{0 < y < l_\varepsilon} dy,$$

thanks to an integration by parts in x and a rewriting of the obtained integrals with indicator functions. From classical results and the preceding lemmas, we have:

$$\begin{cases} b \left(\frac{\cdot}{\varepsilon}\right) \rightharpoonup \bar{b} \text{ in } L^\infty(\mathbb{R}) \text{ weak-}\star, \\ c^\varepsilon \rightarrow c^0, v^\varepsilon \rightarrow 0 \text{ uniformly on } \mathbb{R}, \\ \text{and } u^\varepsilon \rightarrow u^0 \text{ uniformly on } \mathbb{R}^2, \end{cases}$$

as $\varepsilon \downarrow 0$. In particular, we also have:

$$\begin{cases} \mathbf{1}_{x < v^\varepsilon(y), 0 < y < l_\varepsilon} \rightarrow \mathbf{1}_{x < 0, 0 < y < 1} \text{ in } L^1_{\text{loc}}(\mathbb{R}_x \times \mathbb{R}_y), \\ \mathbf{1}_{0 < y < l_\varepsilon} \rightarrow \mathbf{1}_{0 < y < 1} \text{ in } L^1_{\text{loc}}(\mathbb{R}_y), \\ \text{and } u^\varepsilon(v^\varepsilon(\cdot), \cdot) \rightarrow u^0(0) \text{ uniformly on } \mathbb{R}. \end{cases}$$

To show the above convergence for the indicator functions, it suffices to use again the arguments of the proof of Lemma 2.7. Let us now pass to the limit in I_1 , which is possible by weak-strong convergence arguments. We get:

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} I_1 &= - \int_{-\infty}^0 \int_0^1 c^0 \bar{b} u^0(x) \varphi_x(x) dy dx + \int_0^1 c^0 \bar{b} u^0(0) \varphi(0) dy \\ &= - \int_{-\infty}^0 c^0 \bar{b} u^0(x) \varphi_x(x) dx + c^0 \bar{b} u^0(0) \varphi(0). \end{aligned}$$

2. *The second and third terms.* One can verify that the same reasoning leads to

$$\lim_{\varepsilon \downarrow 0} I_2 = - \int_{-\infty}^0 \bar{a} u^0(x) \varphi_{xx}(x) dx + \bar{a} u^0(0) \varphi_x(0)$$

and

$$\lim_{\varepsilon \downarrow 0} I_3 = - \int_0^1 c^0 \bar{g} \varphi(0) dy = -c^0 \bar{g} \varphi(0).$$

3. *Conclusion.* Finally in the limit $\varepsilon \downarrow 0$, Equation (A7) becomes

$$\int_{-\infty}^0 (c^0 \bar{b} u^0 \varphi_x + \bar{a} u^0 \varphi_{xx}) dx = c^0 (\bar{b} u^0(0) - \bar{g}) \varphi(0) + \bar{a} u^0(0) \varphi_x(0) \tag{A8}$$

for all $\varphi \in C_c^\infty(\mathbb{R})$. We recognize the weak formulation of Problem (4.6) and, from that, it is quite standard to identify u^0 . Let us give some details for completeness.

By taking φ with compact support in $\{x < 0\}$ in (A8), we obtain that

$$c^0 \bar{b} u_x^0 - \bar{a} u_{xx}^0 = 0 \quad \text{for } x < 0$$

(in the distribution sense). Thus $u^0(x) = C \exp\left(\frac{c^0 \bar{b} x}{\bar{a}}\right) + \tilde{C}$ for all $x < 0$. Using that u^0 is continuous on \mathbb{R} , even, and tends to zero at infinity, see Lemma 4.13, we infer that $\tilde{C} = 0$ and

$$u^0(x) = C \exp\left(\frac{c^0 \bar{b} |x|}{\bar{a}}\right) \quad \forall x \in \mathbb{R}.$$

Injecting this formula in (A8) and integrating by parts,

$$\int_{-\infty}^0 \underbrace{\left(c^0 \bar{b} u_x^0 - \bar{a} u_{xx}^0\right)}_{=0} \varphi \, dx + \bar{a} \underbrace{u_x^0(0)}_{=C \frac{c^0 \bar{b}}{\bar{a}}} \varphi(0) = c^0 \bar{g} \varphi(0),$$

for all $\varphi \in C_c^\infty(\mathbb{R})$. This implies that $C = \frac{\bar{g}}{\bar{b}}$ and completes the proof. □

A.4 Proof of Lemma 4.10: The ideas of [13]

In the preceding sections, we have only sketched this proof by referring to the computations used in the proofs of [13]. Let us be more precise in this appendix for completeness sake.

To simplify the notations, let us consider the original system instead of the ε -dependent problem (4.1)–(4.2). Here is the a priori estimate that we can get with the ideas of [13].

Lemma A.4 *Let c_M and C be some positive constants. Assume that (c, v, u) is a travelling solution to (1.1)–(1.2) such that (A)–(F) hold with*

$$0 < c \leq c_M, \quad \|v_y\|_\infty \leq C \quad \text{and} \quad \|u\|_\infty \leq C.$$

Then the (extension of the) temperature u is Hölder continuous with

$$|u(x, y) - u(\tilde{x}, \tilde{y})| \leq \tilde{C} (|x - \tilde{x}|^\alpha + |y - \tilde{y}|^\alpha) \quad \forall (x, y), (\tilde{x}, \tilde{y}) \in \mathbb{R}^2,$$

for some positive constants \tilde{C} and α depending only on a_m, a_M, b_M, g_M and the preceding given constants c_M and C .

It is clear that this result and Lemma 4.9 imply Lemma 4.10. It thus suffices to prove the result above. The idea of [13] consists in identifying an elliptic PDE for the extension of u in order to apply the (local) De Giorgi–Nash–Moser’s theorem. The following lemma identifies the equation satisfied by the extension of u to the whole space.

Lemma A.5 *Let (c, v, u) be as in the preceding lemma. Let us define, for almost every $(x, y) \in \mathbb{R}^2$,*

$$\begin{aligned} \tilde{b}(x, y) &:= b(y) \times \begin{cases} c, & x < v(y), \\ -c, & x > v(y), \end{cases} \\ A(x, y) &:= a(y) \times \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & x < v(y), \\ \begin{pmatrix} 1 + 4v_y^2 & 2v_y \\ 2v_y & 1 \end{pmatrix}, & x > v(y), \end{cases} \\ \tilde{g}(x, y) &:= g(y) \times \begin{cases} -c, & x < v(y), \\ c, & x > v(y). \end{cases} \end{aligned}$$

Then $u \in H_{\text{loc}}^1(\mathbb{R}^2)$ satisfies:

$$\int_{\mathbb{R}^2} (\tilde{b} u_x \varphi + \langle A \nabla u, \nabla \varphi \rangle) = - \int_{\mathbb{R}^2} \tilde{g} \varphi_x \quad \forall \varphi \in C_c^1(\mathbb{R}^2),$$

where $\langle \cdot, \cdot \rangle$ is the inner product.

REMARK A.6 In other words, the extension of u is a variational solution of

$$\tilde{b} u_x - \operatorname{div}(A \nabla u) = \tilde{g}_x \quad \text{in } \mathbb{R}^2, \tag{A9}$$

with measurable and bounded coefficients \tilde{b} , A and \tilde{g} .

Proof. Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$\phi(x, y) := (2v(y) - x, y).$$

Then ϕ is a C^1 bijection, since v is $W^{2,\infty}$, and $\phi^{-1} = \phi$. Moreover

$$u \circ \phi = u$$

(u being extended to \mathbb{R}^2 by (2.3)). Taking the gradient, it follows that

$$(\operatorname{Jac} \phi)^t \nabla u \circ \phi = \nabla u,$$

where

$$\operatorname{Jac} \phi := \begin{pmatrix} -1 & 2v_y \\ 0 & 1 \end{pmatrix}$$

is the Jacobian matrix of ϕ and $(\operatorname{Jac} \phi)^t$ its transpose. But, it is easy to see that $[(\operatorname{Jac} \phi)^t]^{-1} = (\operatorname{Jac} \phi)^t$ so that

$$\nabla u \circ \phi = (\operatorname{Jac} \phi)^t \nabla u \quad \text{so that} \quad u_x \circ \phi = -u_x. \tag{A10}$$

Let us introduce a new test function

$$\psi := \varphi \circ \phi^{-1}.$$

By the same computations as above,

$$\nabla\psi \circ \phi = (\text{Jac } \phi)^t \nabla\varphi. \tag{A11}$$

But $\psi \in C_c^1(\mathbb{R}^2)$ can be used as a test function in (2.6), that is

$$\int_{x < v(y)} (c b u_x \psi + a \nabla u \nabla \psi) \underbrace{(x, y)}_{=\phi(\phi^{-1}(x, y))} dx dy = \int_{\mathbb{R}} c g(y) \psi(v(y), y) dy.$$

Let us change the variable by $\phi^{-1}(x, y) \mapsto (x, y)$ in the first integral. We have the new element of integration $|\text{Jac } \phi| dx dy = dx dy$, since $|\text{Jac } \phi| = 1$, and the new domain $\phi^{-1}(\{x < v(y)\}) = \{x > v(y)\}$. Hence,

$$\int_{x > v(y)} (c b u_x \psi + a \nabla u \nabla \psi)(\phi(x, y)) dx dy = \int_{\mathbb{R}} c g(y) \varphi(v(y), y) dy,$$

where we have used in addition that $\psi = \varphi$ at the front $\{x = v(y)\}$ to rewrite the second integral. By (A10) and (A11), we conclude that

$$\begin{aligned} \int_{x > v(y)} (-c b u_x \varphi + \langle a (\text{Jac } \phi)^t \nabla u, (\text{Jac } \phi)^t \nabla \varphi \rangle) (x, y) dx dy \\ = \int_{\mathbb{R}} c g(y) \varphi(v(y), y) dy = - \int_{x > v(y)} c g(y) \varphi_x(x, y) dx dy. \end{aligned}$$

Using that $(\text{Jac } \phi) (\text{Jac } \phi)^t = \begin{pmatrix} 1 + 4 v_y^2 & 2 v_y \\ 2 v_y & 1 \end{pmatrix}$, we obtain:

$$\int_{x > v(y)} (\tilde{b} u_x \varphi + \langle A \nabla u, \nabla \varphi \rangle) = - \int_{x > v(y)} \tilde{g} \varphi_x,$$

where \tilde{b} , A and \tilde{g} are defined as in the lemma. To complete the proof, it suffices to choose φ in (2.6), thus getting the remaining equality for $x < v(y)$, and then sum up the result with the equality above. \square

Lemma 4.10 will now be a consequence of [9, Theorem 8.24]. Let us first check that the coefficients \tilde{b} and A satisfy the assumption required in [9, pp. 177–178].

Lemma A.7 *Let (c, v, u) , \tilde{b} and A be as in the preceding lemmas. Then for almost every $(x, y) \in \mathbb{R}^2$ and all $\xi \in \mathbb{R}^2$,*

- (i) $\langle A(x, y) \xi, \xi \rangle \geq \lambda |\xi|^2$,
- (ii) $|A(x, y)| \leq \Lambda$,
- (iii) $\lambda^{-1} |\tilde{b}(x, y)| \leq v$,

for some positive constants λ , Λ and v depending only on a_m , a_M , b_M and the constants c_M , C (given in Lemma A.4).

Proof. All the items are easy to prove except may be (i). To show this property, it suffices to find a lower bound $\lambda > 0$ of the eigenvalues of $A = A(x, y)$. If $x < v(y)$, we have

$A = a(y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and any constant $\lambda \leq a_m$ will work. Let us thus focus on the other case where $A = a(y) \begin{pmatrix} 1 + 4v_y^2 & 2v_y \\ 2v_y & 1 \end{pmatrix}$. Simple computations show that:

- If $v_y = 0$, then A has a double eigenvalue

$$\lambda_0 = a(y),$$

- and if $v_y \neq 0$, then it has two single eigenvalues

$$\lambda_1 = a(y) \left(1 + 2v_y^2 + 2\sqrt{v_y^2(1+v_y^2)} \right) \quad \text{and} \quad \lambda_2 = \lambda_1^{-1}.$$

The existence of λ then follows from the bound on v_y assumed in Lemma A.4. □

We can now prove the theorem.

Proof of Theorem 4.10. Let $(x_0, y_0) \in \mathbb{R}^2$ be arbitrary and consider the balls

$$\Omega := B((x_0, y_0), 1) \quad \text{and} \quad \Omega' := B((x_0, y_0), 1/2).$$

Let us apply [9, Theorem 8.24] to Equation (A9) satisfied by $u \in H^1(\Omega)$ as stated in Lemma A.5; note that we are using the notation of [9] for simplicity, so that Ω is not just the fresh region here. We choose $q = +\infty$, since \tilde{g} is bounded, and note that the distance $d' = 1/2$ between Ω' and the boundary of Ω does not depend on (x_0, y_0) . The estimate stated in [9, Theorem 8.24] then reads:

$$|u(x, y) - u(\tilde{x}, \tilde{y})| \leq \tilde{C} \times (\|u\|_{L^2(\Omega)} + \lambda^{-1} \|\tilde{g}\|_\infty) \times (|x - \tilde{x}|^\alpha + |y - \tilde{y}|^\alpha)$$

for all $x, \tilde{x}, y, \tilde{y} \in \overline{B((x_0, y_0), 1/2)}$ and for some constants $\tilde{C} = \tilde{C}(\Lambda/\lambda, \nu) \geq 0$ and $\alpha = \alpha(\Lambda/\lambda, \nu) > 0$, where λ, Λ and ν come from Lemma A.7. Let us recall that these three constants have the desired dependences (as stated in Lemma A.4). Moreover, by the definition of \tilde{g} in Lemma A.5, $\|u\|_{L^2(\Omega)} + \lambda^{-1} \|\tilde{g}\|_\infty \leq \tilde{C}$ for another eventual larger constant \tilde{C} having the same dependences. This completes the proof since all the constants above have the desired dependences and in particular do not depend on the arbitrary (x_0, y_0) . □

Appendix B. Main notations

(Essentially by order of the first occurrence in the paper.)

\mathbb{R}^+	set of positive reals (excluding 0)
Y	period used for the existence of travelling waves
Ω	fresh region $\{x < v(y)\}$ (for a given Y -periodic v)
Γ	position of front $\{x = v(y)\}$
$\Omega_\#$	$\Omega \cap \{0 < y < Y\}$
$\Gamma_\#$	$\Gamma \cap \{0 < y < Y\}$

$L_{\#}^p, H_{\#}^1$, etc.	spaces of functions Y -periodic in y
\overline{f}	mean value $\frac{1}{Y} \int_0^Y f$ of a Y -periodic $f = f(y)$
ε	period used for the homogenization process
Ω^ε	fresh region $\{x < v^\varepsilon(y)\}$
Γ^ε	position of the front $\{x = v^\varepsilon(y)\}$
$\Omega_{\text{per}}^\varepsilon$	$\Omega^\varepsilon \cap \{0 < y < \varepsilon\}$
$\Gamma_{\text{per}}^\varepsilon$	$\Gamma^\varepsilon \cap \{0 < y < \varepsilon\}$
$L_{\text{per}}^p, H_{\text{per}}^1$, etc.	spaces of functions ε -periodic in y
\overline{f}	mean value of an ε -periodic $f = f(y)$
$\lfloor \cdot \rfloor, \lceil \cdot \rceil$	lower and upper integer parts
$w^\varepsilon(z) = \frac{v^\varepsilon(\varepsilon z)}{\varepsilon}$	rescaled front
$c(\lambda)$	shorthand notation of $c^0(\lambda)$ in Section 5
$h(z) = w_z(z) = h(z, \lambda)$	solution of Equation (5.1)
$\mathcal{R}(z) = R(z, \frac{z}{b})$	combustion rate of Equation (5.1)
$d_{x_i} \phi$	partial differential in Banach spaces
$d_{x_i} \phi(x_1, x_2) \cdot h_i$	partial differential at (x_1, x_2) in the direction h_i
$\mathcal{L}(E, F)$	space of bounded and linear maps
$\text{Isom}(E, F)$	space of isomorphisms of Banach spaces
$o_\varepsilon(1), o(\varepsilon), O(\varepsilon)$, etc.	usual Landau notations

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References

1. Adams, R. A., *Sobolev spaces*. Pure and Applied Mathematics, Vol. 65, Academic Press, New York-London, 1975. [Zb10314.46030](#) [MR0450957](#)
2. Berestycki, H. & Hamel, F., Front propagation in periodic excitable media. *Comm. Pure Appl. Math.* **55** (2002), 949–1032. [Zb11024.37054](#) [MR1900178](#)
3. Brauner, C. M., Fife, P. C., Namah, G. & Schmidt-Lainé, C., Propagation of a combustion front in a striated solid medium: a homogenization analysis. *Quart. Appl. Math.* **51** (1993), 467–493. [Zb10803.35009](#) [MR1233525](#)
4. Brézis, H., *Analyse fonctionnelle : Théorie et Applications*. Collection Mathématiques Appliquées pour la Maîtrise, Masson, Paris, 1983. [Zb10511.46001](#) [MR0697382](#)
5. Cartan, H., *Calcul différentiel*. Hermann, Paris, 1967. [Zb10156.36102](#) [MR0231303](#)
6. Chen, X. & Namah, G., Wave propagation under curvature effects in a heterogeneous medium. *Appl. Anal.* **64** (1997), 219–233. [Zb10878.35009](#) [MR1460080](#)
7. Chen, X. & Namah, G., Periodic travelling wave solutions of a parabolic equation: A monotonicity result. *J. Math. Anal. Appl.* **275** (2002), 804–820. [Zb11072.35539](#) [MR1943781](#)

8. Chen, X. & Jiang, H., Regularity of expanding front and its application to solidification/melting in undercooled liquid/superheated solid. *Interfaces Free Bound.* **15** (2013), 281–322. [Zb11286.35271](#) [MR3148594](#)
9. Gilbarg, D. & Trudinger, N., *Elliptic partial differential equations of second order*. Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001. [Zb11042.35002](#) [MR1814364](#)
10. Gao, H., Strain induced slowdown of front propagation in random shear flow via analysis of G-equations. *Proc. Amer. Math. Soc.* **144** (2016), 3063–3076. [Zb106573154](#) [MR3487236](#)
11. Lou, B. & Chen, X., Traveling waves of a curvature flow in almost periodic media. *J. Differential Equations* **247** (2009), 2189–2208. [Zb11182.35073](#) [MR2561275](#)
12. Namah, G. & Roquejoffre, J.-M., Convergence to periodic fronts in a class of semilinear parabolic equations. *NoDEA Nonlinear Differential Equations Appl.* **4** (1997), 521–536. [Zb10887.35070](#) [MR1485736](#)
13. Nittka, S., Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains. *J. Differential Equations* **251** (2011), 860–880. [Zb11225.35077](#) [MR2812574](#)
14. Xin, J., Front propagation in heterogeneous media. *SIAM Rev.* **42** (2000), 161–230. [Zb10951.35060](#) [MR1778352](#)