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A Note on Hypoellipticity of Degenerate Elliptic Operators

Dedicated to Professor Mutsuhide Matsumura on his 60th birthday

By

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This note concerns C^∞ hypoellipticity for differential operators (in $\boldsymbol{R}^3$) of the form

$$L = D_t^{2m} + f(t) D_x^{2m} + g(t) D_y^{2m}, \quad D_t = -i\partial/\partial t, \quad m = 1, 2, \cdots.$$

Here we assume

(A.1) 1) f and g belong to $C^\infty(-\delta, \delta)$ for some $\delta > 0$,
2) $f(t) > 0$ and $g(t) > 0$ for $t \neq 0$.

Concerning operators closely related to L, criteria for the hypoellipticity have been recently given by several authors See Fediĭ [2], Hoshiro [4,5], Kusuoka and Stroock [6] and Morimoto [7, 8, 9, 10]. In particular, Hoshiro considered the same operator as L with the assumptions (A.1) and

(A.2) both of f and g are non-decreasing in $[0, \delta)$, and non-increasing in $(-\delta, 0]$.

In this note we shall prove the following Theorems 1 and 2, which show that the condition

$$\text{(C)} \qquad \lim_{t\to 0} \mu(t; g) \log f(t) = \lim_{t\to 0} \mu(t; f) \log g(t) = 0$$

is equivalent to the hypoellipticity for L under (A.1) and (A.2), where

$$\begin{aligned} \mu(t; g) &= \max\{g(s)^{1/(2m)} |t-s|\,;\, s \text{ is between 0 and } t\} \\ &= \max\{g(\theta t)^{1/(2m)} (1-\theta)|t|\,;\, 0 \le \theta \le 1\}. \end{aligned}$$

Theorem 1. *If* (A.1) *holds, then* (C) *implies that* L *is hypoelliptic* (*near*

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$t=0$).

Theorem 2. *Let* (A.1) *and* (A.2) *hold. Then* (C) *holds if L is hypoelliptic.*

Notice that no assumption other than (A.1) is required in Theorem 1 whether $m=1$ or not (cf. Theorem 2 of [5] and Theorem 1.1 of [7]).

Example. Let σ be a constant and

$$L = D_t^{2m} + e^{-2m/|t|} D_x^{2m} + \exp[-|t|^{-\sigma} e^{1/|t|}] D_y^{2m}.$$

Then the Theorems show that L is hypoelliptic if and only if $\sigma < 2$.

We get the following Corollary at once (cf. Theorem 8.41 of [6], Theorem 3 of [8] and Proposition 1 of [10]).

Corollary. *Let f satisfy the conditions in* (A.1). *Then the condition*

$$\lim_{t\to 0} t \log f(t) = 0$$

implies that the operator

$$D_t^{2m} + f(t) D_x^{2m} + D_y^{2m}$$

is hypoelliptic. If f satisfies (A.2) *in addition to* (A.1), *then the converse is also true.*

We prove Theorems 1 and 2 in Sections 1 and 2, respectively. Our proofs are modifications of those in [4, 5]. We use the well known integral inequality of Hardy and an interpolation theorem in Sobolev spaces.

§ 1. Proof of Theorem 1

In view of Proposition 2 of [5], it suffices to prove the following: The condition

$$\lim_{t\to 0} \mu(t; g) \log f(t) = 0 \tag{1}$$

together with (A.1) implies that, for every $\varepsilon_0 > 0$, there exists an $N(\varepsilon_0) > 1$ such that

$$(\log \xi)^{2m} \int g(t) |\nu(t)|^2 \, dt \le \varepsilon_0 \left[\int |\nu^{(m)}(t)|^2 \, dt + \xi^{2m} \int f(t) |\nu(t)|^2 \, dt \right] \tag{2}$$

for $\nu \in C_0^m(-\delta, \delta)$ and $\xi \ge N(\varepsilon_0)$.

We may assume g and the derivatives of f are bounded in $(-\delta, \delta)$ (by replacing δ with a smaller one if necessary). Since (2) holds whenever $f(0)>0$, we only consider the case where $f(0)=0$.

We use the "*sew together*" method as in [4].

First, we have

Lemma 1. *The inequality*

$$\int_0^a g(t)|\nu(t)|^2\,dt \leq 4\mu(a;g)^{2m}\int |\nu^{(m)}(t)|^2\,dt \tag{3}$$

holds if $0<a<\delta$, $\nu\in C_0^m(-\delta,\delta)$ *and* $\nu^{(k)}(a)=0$ $(k=0,\cdots,m-1)$. *Similarly, the estimate*

$$\int_{-b}^0 g(t)|\nu(t)|^2\,dt \leq 4\mu(-b;g)^{2m}\int |\nu^{(m)}(t)|^2\,dt$$

holds if $0<b<\delta$, $\nu\in C_0^m(-\delta,\delta)$ *and* $\nu^{(k)}(-b)=0$ $(k=0,\cdots,m-1)$.

Proof. We prove (3). Since

$$\nu(t)=\int_t^a \frac{(-1)^m}{(m-1)!}(s-t)^{m-1}\nu^{(m)}(s)\,ds\,,$$

we have, for $t<a$,

$$(a-t)^{-m}|\nu(t)| \leq (a-t)^{-1}\int_t^a \left|\frac{s-t}{a-t}\right|^{m-1}|\nu^{(m)}(s)|\,ds$$

$$\leq \theta(t)\equiv(a-t)^{-1}\int_t^a |\nu^{(m)}(s)|\,ds\,,$$

thus the left hand side of (3) is estimated above by

$$\int_0^a g(t)(a-t)^{2m}\theta(t)^2\,dt \leq \mu(a;g)^{2m}\int_0^a \theta(t)^2\,dt\,.$$

Therefore the inequality (3) follows from the estimate

$$\int_{-\infty}^a \theta(t)^2\,dt \leq 2^2\int_{-\infty}^a |\nu^{(m)}(t)|^2\,dt\,,$$

which holds by the Hardy's inequality (Theorem 327 of Hardy, Littlewood and Polya [3]). Q.E.D.

Second, in order to prove Lemma 2 below, we use the following one-dimensional interpolation inequality (see, e.g., Adams [1], the proofs of Lemmas 4.10 and 4.12).

Proposition. *There exist a constant C_0 and a positive integer l such that, if I is an (open) interval in $\mathbf{R}$ with the length $|I|\leq\delta$, then the inequality*

$$\int_I |u^{(k)}(t)|^2\,dt\leq C_0|I|^{-2l}\Big[\rho\int_I |u^{(m)}(t)|^2\,dt+\rho^{-k/(m-k)}\int_I |u(t)|^2\,dt\Big]$$

holds for $u\in C^m(I)$, $0\leq k\leq m-1$ and $0<\rho\leq 1$.

Let ϕ be a function belonging to $C^\infty(\mathbf{R})$ such that

$$0\leq\phi\leq 1,\quad \phi(\tau)=1 \quad\text{for}\quad \tau\leq 1,\quad\text{and}\quad \phi(\tau)=0\quad\text{for}\quad \tau\geq 2.$$

Putting

$$\chi(t)=\phi(\xi^\gamma f(t)),\qquad\text{where}\ \ \gamma=m/(lm+m+1),\tag{4}$$

we obtain

Lemma 2. *There exists a constant C such that the estimate*

$$\int |[\chi(t)\nu(t)]^{(m)}|^2\,dt\leq C\Big[\int |\nu^{(m)}(t)|^2\,dt+\xi^{2m}\int f(t)|\nu(t)|^2\,dt\Big]\tag{5}$$

holds for $\xi\geq 1$ and $\nu\in C_0^m(-\delta,\delta)$.

Proof. We have by the Leibniz rule

$$|(\chi\nu)^{(m)}|^2\leq C_1[|\nu^{(m)}|^2+\sum_{j=1}^m|\chi^{(j)}\nu^{(m-j)}|^2],\tag{6}$$

where C_1 depends only on m. Notice that, if $1\leq j\leq m$, the function $\chi^{(j)}\nu^{(m-j)}$ vanishes on the outside of the open set

$$I(\xi)=\{t\in(-\delta,\delta);\ \xi^\gamma f(t)>1\},\tag{7}$$

and $I(\xi)$ is disjoint union of (at most) countably many open intervals $I_p=(\alpha_p,\beta_p)$ (contained in $(0,\delta)$ or $(-\delta,0)$) with $\xi^\gamma f(\alpha_p)=1$ or $\xi^\gamma f(\beta_p)=1$. Since $\phi^{(k)}(\tau)$ ($k>0$) vanishes to infinite order at $\tau=1$, we have $|\phi^{(k)}(\tau)|\leq C_2|\tau-1|^l$ for $\tau\geq 1$ and $1\leq k\leq m$, where C_2 is independent of τ and k. Therefore, letting η_p be α_p or β_p with $\xi^\gamma f(\eta_p)=1$, we have , for $t\in I_p$ and $1\leq j\leq m$,

$$\begin{aligned}|\chi^{(j)}(t)|&\leq\text{const.}\ \xi^{\gamma j}\max_{1\leq k\leq m}|\phi^{(k)}(\xi^\gamma f(t))|\\&\leq\text{const.}\ \xi^{\gamma j}C_2|\xi^\gamma f(t)-\xi^\gamma f(\eta_p)|^l\\&\leq C_3\xi^{\gamma j+\gamma l}|t-\eta_p|^l\leq C_3\xi^{\gamma(j+l)}|I_p|^l\end{aligned}$$

with a constant C_3 independent of ξ, p and t. Hence

$$\int |\chi^{(j)}\nu^{(m-j)}|^2\,dt \le C_3^2\xi^{2\gamma(j+l)} \sum_p |I_p|^{2l} \int_{I_p} |\nu^{(m-j)}|^2\,dt\,.$$

The above Proposition and (7) imply that the right hand side of the last inequality is estimated above by

$$C_3^2\xi^{2\gamma(j+l)}C_0\left[\rho\int_{I(\xi)} |\nu^{(m)}|^2\,dt + \rho^{-(m-j)/j}\xi^\gamma \int_{I(\xi)} f|\nu|^2\,dt\right]$$

for $\rho\in(0,1]$. Putting $\rho=\xi^{-2\gamma(j+l)}$, we obtain (5) by the definition of γ and (6). Q.E.D.

Now we can prove (2). Let $a=a(\xi)=\sup\{t\in(0,\delta);\ \xi^\gamma f(t)\le 2\}$, $-b=-b(\xi)=\inf\{t\in(-\delta,0);\ \xi^\gamma f(t)\le 2\}$, where γ is the number as in (4). Then, a and b tend to 0 as $\xi\to\infty$, and, since $\xi=(2/f(a))^{1/\gamma}=(2/f(-b))^{1/\gamma}$, (1) implies that for every $\varepsilon>0$ there exists an $N=N_\varepsilon>1$ such that

$$4[\mu(-b;g)^{2m}+\mu(a;g)^{2m}](\log\xi)^{2m}\le\varepsilon \qquad \text{for} \quad \xi\ge N.$$

For arbitrary $\nu\in C_0^m(-\delta,\delta)$, we put $\nu_1=\chi\nu$ and $\nu_2=(1-\chi)\nu$, where χ is the function defined by (4). Since the support of ν_1 is contained in $[-b,a]$, we can apply Lemma 1 to ν_1. Hence

$$(8)\qquad \begin{aligned}(\log\xi)^{2m}\int g(t)|\nu_1(t)|^2\,dt &\le \varepsilon\int |\nu_1^{(m)}(t)|^2\,dt\\ &\le \varepsilon C\left[\int |\nu^{(m)}(t)|^2\,dt+\xi^{2m}\int f(t)|\nu(t)|^2\,dt\right]\end{aligned}$$

by Lemma 2. Since g is bounded, $(\log\xi)^{2m}g(t)\le\varepsilon\xi^{2m-\gamma}$ for $\xi\ge N$ provided N is sufficiently large. Furthermore ν_2 vanishes on the outside of the set $I(\xi)$ defined in (7), and, accordingly, the inequality $|\nu_2(t)|\le|\nu(t)|$ yields the estimate

$$(9)\qquad (\log\xi)^{2m}\int g(t)|\nu_2(t)|^2\,dt \le \varepsilon\xi^{2m}\int f(t)|\nu(t)|^2\,dt\,.$$

Since $|\nu(t)|^2\le 2|\nu_1(t)|^2+2|\nu_2(t)|^2$, adding (8) and (9) ("*sewing together*"), we have (2). Q.E.D.

§ 2. Proof of Theorem 2

Let (C) be violated. We consider the case where $\mu(t;g)\log f(t)$ does not converge to 0 (the other case is treated similarly). Then $f(0)=0$. There exist an $\varepsilon>0$ and sequences s_n, t_n ($n=1, 2, \cdots$) such that s_n is between 0 and t_n,

$$(10)\qquad g(s_n)|t_n-s_n|^{2m}|\log f(t_n)|^{2m}\ge\varepsilon$$

and $t_n\to 0$ as $n\to\infty$.

We prove that L becomes non-hypoelliptic by modifying the proof in [4] slightly. In fact, consider the eigenvalue problem in the interval $(-\delta, \delta)$

$$P(\xi)v(t) \equiv [D_t^{2m}+\xi^{2m}f(t)]v(t) = \lambda g(t)v(t),$$
$$v^{(k)}(-\delta) = v^{(k)}(\delta) = 0, \qquad k = 0, \cdots, m-1,$$

where ξ is a real valued parameter. Here λ is regarded as an eigenvalue. Let $\lambda(\xi)$ be the smallest positive eigenvalue and $v=v(t;\xi)$ the corresponding eigenfunction such that $\|v(\cdot\,;\xi)\|^2 \equiv \int_{-\delta}^{\delta} |v(t;\xi)|^2\,dt=1$. Then we have

$$\begin{aligned}\lambda(\xi) &= (P(\xi)v(\cdot\,;\xi),\ v(\cdot\,;\xi))/(gv(\cdot\,;\xi),\ v(\cdot\,;\xi))\\ &= \inf\{(P(\xi)u, u)/(gu, u);\ u\in C_0^\infty(-\delta, \delta),\ u\not\equiv 0\}\end{aligned}$$

(the infimum of the Rayleigh ratio), where $(u, v)=\int_{-\delta}^{\delta} u(t)\,\overline{v(t)}\,dt$. Let ξ_n be $f(t_n)^{-1/(2m)}$, which tends to $+\infty$ as $n\to\infty$, and let J_n be the interval (s_n, t_n) (or (t_n, s_n)). Then $\xi_n^{2m}f(t)\le 1$ and $g(t)\ge g(s_n)$ for $t\in J_n$ by (A.2). Thus

$$\begin{aligned}\lambda(\xi_n) &\le g(s_n)^{-1}\cdot\inf\{[\|u^{(m)}\|^2+\|u\|^2]/\|u\|^2;\ u\in C_0^\infty(J_n),\ u\not\equiv 0\}\\ &\le g(s_n)^{-1}\,[\text{const. }|t_n-s_n|^{-2m}+1]\le \text{const. }|\log f(t_n)|^{2m}\end{aligned}$$

(for large n) by (10). Hence $\lambda(\xi_n)\le$ const. $(\log\xi_n)^{2m}$. Let us put $\kappa_n = [(-1)^{m+1}\lambda(\xi_n)]^{1/(2m)}$ (Re $\kappa_n>0$). Then $|\kappa_n|\le$ const. $\log\xi_n$. The rest of the proof is quite similar to that of [4], with the function u_n in (2.8) of [4] replaced by $v(t;\xi_n)\exp(i\xi_n x+\kappa_n y)$. Thus we omit it here. Q.E.D.

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