

Varifold solutions of a sharp interface limit of a diffuse interface model for tumor growth

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We discuss the sharp interface limit of a diffuse interface model for a coupled Cahn–Hilliard–Darcy system that models tumor growth when a certain parameter $\varepsilon > 0$, related to the interface thickness, tends to zero. In particular, we prove that weak solutions to the related initial boundary value problem tend to varifold solutions of a corresponding sharp interface model when ε goes to zero.

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1. Introduction

The present contribution is devoted to the study of the relations between a diffuse and a sharp interface Cahn–Hilliard–Darcy model for tumor growth.

The morphological evolution of a growing solid tumor is the result of the dynamics of a complex system that includes many nonlinearly interacting factors, such as cell-cell and cell-matrix adhesion, mechanical stress, cell motility and angiogenesis just to name a few. It is clear that mathematics could make a huge contribution to many areas of experimental cancer investigation since there is now a wealth of experimental data which requires systematic analysis. At the current stage of cancer research, most of the mathematical models are built and developed from the following three perspectives: discrete (microscopic), continuous (macroscopic), and hybrid (micro-macroscopic). Numerous mathematical models have been developed to study various aspects of tumor progression and this has been an area of intense research (see the recent reviews by Fasano et al. [12], Graziano and Preziosi [16], Friedman et al. [14], Bellomo et al. [5], Cristini et al. [7], and Lowengrub et al. [21]). The existing models can be classified into two main categories: continuum models and discrete models. We concentrate on the former ones. This category can be subsequently split in two basic types of models namely the (classical) *sharp* interface models, where the interface between the fluids is modeled as a (sufficiently smooth) surface, and so-called diffuse interface models, where the *sharp* interface is replaced by an interfacial region, where a suitable order parameter (ϕ in what follows) varies smoothly, but with a large gradient between two distinguished values.

The necessity of dealing with multiple interacting constituents has led, in particular, to the consideration of diffuse-interface models based on continuum mixture theory (see, for instance, [8] and references therein). In the diffuse approach, sharp interfaces are replaced by narrow transition layers that arise due to differential adhesive forces among the cell-species. The main advantages of the diffuse interface formulation are:

- it eliminates the need to enforce complicated boundary conditions across the tumor/host tissue and other species/species interfaces that would have to be satisfied if the interfaces were assumed sharp, and
- it eliminates the need to explicitly track the position of interfaces, as is required in the sharp interface framework.

Then, the natural question arises how diffuse and sharp interface models are related if a suitable parameter $\varepsilon > 0$, which measures the width of the diffuse interface, tends to zero. There are already some results on this question, which are based on formally matched-asymptotics calculations (cf. the recent work by Garcke et al. [15]), but so far there are very few mathematically rigorous convergence results (cf. [25]). This is indeed the aim of the present contribution.

The mathematical technique we exploit here consists mainly in considering the known results for Cahn–Hilliard equations by [6] and trying to extend them to the coupled Cahn–Hilliard–Darcy system (first neglecting the nutrient) in the spirit of what Abels et al. (cf. [1] and also [3]) did for a two-phase fluid model. The problem of dealing with a complete tumor-growth model coupling Cahn–Hilliard equation for the tumor phase with a non-zero source, Darcy law for the velocity, and a reaction-diffusion equation for the nutrient (cf., e.g., [10] or [15]) is still open.

Other techniques could also be investigated. For example, recently in [25] the authors exploited Gamma convergence tools for Gradient Flows systems in order to prove the passage from diffuse to sharp interfaces for a variant of a different tumor growth model proposed in [17] (cf. also [18]) where the velocity field is not considered and a coupled Cahn–Hilliard–Reaction–Diffusion system is analyzed. It is worth mentioning that a Gamma-convergence approach cannot be applied to the problem considered in this paper due to the lack of gradient structure of system under consideration.

The initial boundary value problem we are interested in here is indeed the following one:

$$\partial_t \phi - \Delta \mu + \nabla \cdot (u\phi) = 0 \text{ in } \Omega \times (0, \infty), \quad (1.1)$$

$$\mu = -\varepsilon \Delta \phi + \frac{1}{\varepsilon} F'(\phi) \text{ in } \Omega \times (0, \infty), \quad (1.2)$$

$$u = -\nabla P + \mu \nabla \phi \text{ in } \Omega \times (0, \infty), \quad (1.3)$$

$$\nabla \cdot u = 0 \text{ in } \Omega \times (0, \infty), \quad (1.4)$$

$$v \cdot \nabla \phi = v \cdot \nabla \mu = v \cdot u = 0 \text{ on } \partial \Omega \times (0, \infty), \quad (1.5)$$

$$\phi(0) = \phi_{0,\varepsilon} \text{ in } \Omega, \quad (1.6)$$

where Ω is a bounded subset of \mathbb{R}^d with a smooth boundary $\partial \Omega$, v denotes the outward unit normal vector to $\partial \Omega$, F is a double-well potential with minima in -1 and 1 , e.g. $F(r) = \frac{1}{8}(1 - r^2)^2$, and ε is a small positive parameter related to the interface thickness. Moreover, $\phi_{0,\varepsilon}$ is a family of approximating initial data which satisfy a well-preparedness condition (see below). The dynamics of the phase variable ϕ (and of the chemical potential μ) is regulated by the convective Cahn–Hilliard equation (1.1)–(1.2). The velocity field u fulfills the Darcy’s law (1.3) (here P denotes a pressure) including the so-called Korteweg term $\mu \nabla \phi$.

The PDE system we consider here (as well as some generalizations of it) has been already studied from the point of view of existence of solutions, regularity, and long-time behavior in [22] (cf. also [19] and [10] for more general models), while the formal expansion method for the sharp interface limit has been recently performed in [15] again for a more complicated system, where also the nutrient variable and chemotaxis effects have been taken into account.

The matched asymptotic expansion performed in [15] shows, formally, that system (1.1)–(1.6) converges, for $\varepsilon \rightarrow 0$, to the sharp-interface limit problem given by

$$\phi = 1 \text{ in } \Omega^T, \quad (1.7)$$

$$\phi = -1 \text{ in } \Omega^H, \quad (1.8)$$

$$2(-V + u \cdot n) = [\nabla \mu]_H^T \cdot n \text{ on } \Sigma, \quad (1.9)$$

$$\mu = \sigma k \text{ on } \Sigma, \quad (1.10)$$

$$[\mu]_H^T = 0 \text{ on } \Sigma, \quad (1.11)$$

$$-\Delta \mu = 0 \text{ in } \Omega^T \cup \Omega^H, \quad (1.12)$$

$$u = -\nabla P \text{ in } \Omega^T \cup \Omega^H, \quad (1.13)$$

$$\nabla \cdot u = 0 \text{ in } \Omega^T \cup \Omega^H, \quad (1.14)$$

$$[u]_H^T \cdot n = 0 \text{ on } \Sigma, \quad (1.15)$$

$$[P]_H^T = 2\sigma k \text{ on } \Sigma. \quad (1.16)$$

Here *tumor region* Ω^T and the *healthy region* Ω^H are two open and disjoint subset of Ω separated by a smooth interface Σ which moves with *normal velocity* V . Moreover, σ is a constant related to the potential given by $\sigma = \int_{-1}^1 \sqrt{\frac{F(r)}{2}} dr$, k is the mean curvature of Σ , n is the outward unit normal to Σ pointing towards Ω^T , and $[f]_H^T$ denotes the jump of f from Ω^T to Ω^H across the interface Σ . As for the diffuse interface case we close the system with boundary and initial conditions

$$v \cdot u = 0 \text{ on } \partial\Omega \times (0, \infty),$$

$$v \cdot \nabla \mu = 0 \text{ on } \partial\Omega \times (0, \infty),$$

$$\Omega^T(0) = \Omega_0^T,$$

where Ω_0^T is the tumor region at the initial time $t = 0$.

Our goal is to prove the convergence rigorously. More precisely, in the rest of the paper we address the following question: under which assumptions on the potential F do weak solutions of (1.1)–(1.6) converge to weak/generalized solutions of (1.7)–(1.16)? We show that if F satisfies proper growth conditions at infinity, which are fulfilled in particular by the so-called standard double-well potential $F(r) = \frac{1}{8}(1 - r^2)^2$, then the weak solutions of (1.1)–(1.6) converge to the so-called varifold solutions of (1.7)–(1.16), which are defined in the spirit of [6] in Section 3.

The paper is organized as follows: in Section 2 we introduce some notation and preliminaries we need in the rest of the paper. In Section 3 we state our assumptions on the data and the main result of the paper together with the notion of solutions. Finally, in the last two Sections 4, 5 we prove the main Theorem 4 by establishing suitable a-priori estimates (independent of ε) on the solution to (1.1)–(1.6) leading to the passage to the limit as $\varepsilon \rightarrow 0$.

2. Preliminaries and notation

In this section we fix the notation and recall some known facts about functions of bounded variation and varifolds.

Given $\Omega \subset \mathbb{R}^d$ a bounded set with a smooth boundary, $d, N \in \mathbb{N}$, X a Banach space with separable dual space X^* , we use the following notations for these functional spaces.

- $L^p(\Omega)$ and $L^p(\Omega, X)$, for $p \in [1, \infty]$, denote the standard Lebesgue spaces for scalar and X valued functions, respectively.
- $C_0(\Omega, \mathbb{R}^N)$ is the closure of compactly-supported continuous functions $f : \Omega \rightarrow \mathbb{R}^N$, in the supremum norm.
- $C_0^k(\Omega)$, $k \in \mathbb{N} \cup \{\infty\}$ is the set of k -times-differentiable compactly-supported functions.
- $C^k(\bar{\Omega})$, $k \in \mathbb{N} \cup \{\infty\}$ is the set of k -times-differentiable functions such that all derivatives have a continuous extension on $\bar{\Omega}$.
- $C_{0,\text{div}}^\infty(\Omega) = \{f \in C_0^\infty(\Omega) : \nabla \cdot f = 0\}$ and $L_{\text{div}}^2(\Omega) = \overline{C_{0,\text{div}}^\infty(\Omega)}^{L^2(\Omega)}$.
- $L_{\text{loc}}^p(0, \infty; X)$ for $p \in [1, \infty)$ denotes the space of all measurable functions $f : (0, \infty) \rightarrow X$ such that $f \in L^p(0, t; X)$ for all $t > 0$.
- $M(\Omega; \mathbb{R}^N)$ for $N \in \mathbb{N}$, denotes the space of all finite \mathbb{R}^N -valued Radon measures. $M(\Omega; \mathbb{R}) =: M(\Omega)$.
- $\text{BV}(\Omega)$ is the space of functions of bounded variations.
- $L_{\omega^*}^\infty(\Omega; X^*)$ denotes the space of all functions $f : \Omega \rightarrow X^*$ that are weakly* measurable and essentially bounded.

Given $f \in \text{BV}(\Omega)$ we denote by Df its distributional gradient and by $|Df|$ the Radon measure generated by

$$|Df|(A) = \sup_{Y \in C_0(A; \mathbb{R}^d); |Y| \leq 1} \int_A f \nabla \cdot Y \, dx, \quad \text{for all } A \text{ open in } \Omega.$$

Moreover, one can show (cf., e.g., [13]) that there exists a $|Df|$ -measurable unit vector valued function n such that $Df = n|Df|$, $|Df|$ -a.e.. We recall that

$$\text{BV}(\Omega) = \{f \in L^1(\Omega) : Df \in M(\Omega; \mathbb{R}^d)\}$$

and

$$\|f\|_{\text{BV}(\Omega)} = \|f\|_{L^1(\Omega)} + \|Df\|_{M(\Omega; \mathbb{R}^d)} = \|f\|_{L^1(\Omega)} + |Df|(\bar{\Omega}).$$

Let E be a set in Ω . If the characteristic function χ_E belongs to $\text{BV}(\Omega)$, then we say that E has *finite perimeter* and we denote $D\chi_E = n_E|D\chi_E|$. Note that, if ∂E is smooth, then n_E is the unit inward normal to ∂E . Moreover, we recall that there exists a separable Banach space X such that its dual space coincide with $\text{BV}(\Omega)$, (cf. [4]). As a consequence the space $L_{\omega^*}^\infty(0, s; \text{BV}(\Omega)) = (L^1(0, s; X))^*$ is well defined.

Let now

$$P = S^{d-1}/\{v, -v\}$$

be the set of unit normals of unoriented $(d-1)$ -dimensional hyperplanes in \mathbb{R}^d . A *varifold* V is a Radon measure on $\Omega \times P$. We define the mass measure $\|V\|$ as the Radon measure on Ω given by

$$\|V\|(A) = \int \int_{A \times P} dV(x, p) \quad \text{for all } A \text{ open in } \Omega.$$

The first variation δV of a varifold V is the linear functional on $C_0^1(\Omega; \mathbb{R}^d)$ defined by

$$\langle \delta V, Y \rangle := \int \int_{\Omega \times P} \nabla Y : (I - p \otimes p) \, dV(x, p) \quad \text{for all } Y \in C_0^1(\Omega; \mathbb{R}^d)$$

and its mean curvature vector H (wherever it exists) is a $\|V\|$ -measurable vector-valued function on Ω defined by

$$-\langle \delta V, Y \rangle = \langle \|V\|, H \cdot Y \rangle = \int_{\Omega} (Y(x) \cdot H(x)) \, d\|V\|(x) \quad \text{for all } Y \in C_0^1(\Omega; \mathbb{R}^d).$$

3. Assumptions and main results

In this section we introduce the main assumptions on the problem data and the statement of the main results.

Let the potential F be such that $F \in C^3(\mathbb{R})$, $F(\pm 1) = 0$, and $F(r) > 0$ if $r \neq \pm 1$. Moreover, let the constants $c_0, C_{c_0} > 0$, $p \geq 4$ exist such that $F''(r) \geq C_{c_0}|r|^{p-2}$ for all r such that $|r| \geq 1 - c_0$. An example of potential F satisfying the above assumption is the classical double-well potential $F(r) = \frac{1}{8}(1 - r^2)^2$.

REMARK 1 Note that the same conditions with $p \geq 3$ are assumed in [1], where the authors consider the sharp interface limit of a Cahn–Hilliard equation coupled with a Navier–Stokes equation, instead of the Darcy’s law (1.3), for the velocity field. Here we need stronger coercivity assumptions on F as solutions u to the Darcy’s law (1.3) are, in general, less regular than solutions to the Navier–Stokes equation.

We also assume uniform boundedness of the initial energy. More precisely, let $\phi_{0,\varepsilon} \in H^1(\Omega) \cap L^p(\Omega)$ be such that there exists a positive constant E_0 satisfying

$$E_\varepsilon(\phi_{0,\varepsilon}) \leq E_0, \tag{3.1}$$

where the energy functional E_ε is defined by

$$E_\varepsilon(\phi) = \int_{\Omega} \left(\varepsilon \frac{1}{2} |\nabla \phi|^2 + \frac{1}{\varepsilon} F(\phi) \right) dx. \tag{3.2}$$

Finally, we ask the initial tumor mass to be independent of ε , namely

$$\bar{\phi}_{0,\varepsilon} = \frac{1}{|\Omega|} \int_{\Omega} \phi_{0,\varepsilon} dx = m_0 \in (-1, 1).$$

Before stating our main result, let us rigorously define solutions to system (1.1)–(1.6) and system (1.7)–(1.16).

DEFINITION 2 (Weak solutions to (1.1)–(1.6)) We call $(\phi_\varepsilon, \mu_\varepsilon, u_\varepsilon)$ a weak solution to system (1.1)–(1.6) if these functions belong to the regularity class:

$$\begin{aligned} \phi_\varepsilon &\in C^0([0, \infty); H^1(\Omega)) \cap L^2_{\text{loc}}(0, \infty; H^2(\Omega)) \cap H^1_{\text{loc}}(0, \infty; L^2(\Omega)), \\ \mu_\varepsilon &\in L^2_{\text{loc}}(0, \infty; L^2(\Omega)), \quad \nabla \mu_\varepsilon \in L^2(0, \infty; L^2(\Omega)), \\ u_\varepsilon &\in L^2(0, \infty; L^2_{\text{div}}(\Omega)), \end{aligned}$$

and the following integral identities hold:

$$\int_0^t \int_{\Omega} (\phi_\varepsilon \partial_t \psi + \phi_\varepsilon u_\varepsilon \cdot \nabla \psi - \nabla \mu_\varepsilon \cdot \nabla \psi) \, dx ds = \int_{\Omega} \phi_\varepsilon(t) \psi(t) \, dx - \int_{\Omega} \phi_{0,\varepsilon} \psi(0) \, dx, \tag{3.3}$$

for all $\psi \in C_0^\infty([0, t] \times \Omega)$, $t > 0$, and

$$\frac{d}{dt} E_\varepsilon(\phi_\varepsilon) + \int_{\Omega} |\nabla \mu_\varepsilon|^2 \, dx + \int_{\Omega} |u_\varepsilon|^2 \, dx = 0, \tag{3.4}$$

where

$$\mu_\varepsilon = -\Delta \phi_\varepsilon + \frac{1}{\varepsilon} F'(\phi_\varepsilon) \text{ a.e. in } \Omega \times [0, \infty), \tag{3.5}$$

$$u_\varepsilon = -\nabla P + \mu_\varepsilon \nabla \phi_\varepsilon \text{ a.e. in } \Omega \times [0, \infty), \tag{3.6}$$

$$\nu \cdot \nabla \phi_\varepsilon = 0 \text{ a.e. on } \partial\Omega \times [0, \infty), \tag{3.7}$$

$$E_\varepsilon(\phi_\varepsilon) = \int_{\Omega} e_\varepsilon(\phi_\varepsilon) \, dx = \int_{\Omega} \varepsilon \frac{1}{2} |\nabla \phi_\varepsilon|^2 + \frac{1}{\varepsilon} F(\phi_\varepsilon) \, dx \text{ a.e. in } [0, \infty). \tag{3.8}$$

DEFINITION 3 (Varifold solutions to (1.7)–(1.16)) Let Ω_0^T be a set of finite perimeter. Then, (u, Ω^T, μ, V) is called a *varifold solution* to (1.7)–(1.16) if the following conditions are satisfied:

1. $u \in L^2(0, \infty; L^2(\Omega))$, $\mu \in L^2_{\text{loc}}(0, \infty; L^2(\Omega))$, $\nabla \mu \in L^2(0, \infty; L^2(\Omega))$.
2. Ω^T can be decomposed as $\Omega^T = \cup_{t \geq 0} (\Omega_t^T \times \{t\})$, where Ω_t^T is a measurable subset of Ω .
Furthermore,

$$\chi_{\Omega^T} \in C([0, \infty); L^1(\Omega)) \cap L^\infty_w(0, \infty; \text{BV}(\Omega))$$

and $|\Omega_t^T| = |\Omega_0^T|$ for all $t \geq 0$.

3. V is a Radon measure on $\bar{\Omega} \times P \times (0, \infty)$ such that $V = V^t dt$ where V^t is a Radon measure on $\bar{\Omega} \times P$ for almost all $t \in (0, \infty)$. Moreover, for a.a. $t \in (0, \infty)$, V^t admits the representation

$$\int_{\bar{\Omega} \times P} \psi(x, p) \, dV^t(x, p) = \sum_{i=1}^d \int_{\bar{\Omega}} b_i^t(x) \psi(x, p_i^t(x)) \, d\lambda^t(x) \tag{3.9}$$

for all $\psi \in C(\bar{\Omega} \times P)$, some Radon measure λ^t on $\bar{\Omega}$, and some λ^t -measurable functions b_i^t , p_i^t with values in \mathbb{R} and P respectively such that

$$0 \leq b_i^t \leq 1, \sum_{i=1}^d b_i^t \geq 1, \sum_{i=1}^d p_i^t \otimes p_i^t = I \quad \lambda^t\text{-a.e.},$$

and

$$\frac{|D\chi_{\Omega^T}|}{\lambda^t} \leq \frac{1}{2\sigma}. \tag{3.10}$$

4. For every $t > 0$ and every $\psi \in C_0^\infty([0, t] \times \Omega)$,

$$\int_0^t \int_{\Omega} [2\chi_{\Omega_s^T} \partial_t \psi - \nabla \mu \nabla \psi + 2\chi_{\Omega_s^T} u \cdot \nabla \psi] \, dx ds = \int_{\Omega} 2\chi_{\Omega_t^T} \psi(t) \, dx - \int_{\Omega} 2\chi_{\Omega_0^T} \psi(0) \, dx. \tag{3.11}$$

5. For every $t > 0$ and every $Y \in C_0^1(\Omega, \mathbb{R}^d)$,

$$-\langle D\chi_{\Omega_t^T}, \mu Y \rangle = \int_{\Omega} \chi_{\Omega_t^T} \nabla \cdot (\mu Y) dx = \frac{1}{2} \langle \delta V^t, Y \rangle. \tag{3.12}$$

6. For every $0 \leq \tau < t$,

$$\lambda^t(\bar{\Omega}) + \int_{\tau}^t \int_{\Omega} |\nabla \mu|^2 dx ds + \int_{\tau}^t \int_{\Omega} |u|^2 dx ds \leq \lambda^{\tau}(\bar{\Omega}). \tag{3.13}$$

7. For every $t > 0$ and every $\varphi \in C_{0,\text{div}}^{\infty}(\Omega)$, we have

$$\int_0^t \int_{\Omega} u \varphi dx ds = \int_0^t \int_{\Sigma_s} 2\mu \varphi dS ds, \tag{3.14}$$

where $\Sigma_t = \partial\Omega_t^T \setminus \partial\Omega$.

Let us postpone some remarks and comments on the definition of solutions and state our main result.

Theorem 4 (Sharp interface limit) *Let the above assumptions be satisfied. Then, there exists a sequence $\varepsilon \rightarrow 0$ such that the following holds.*

1. There exists $\Omega^T = \cup_{t \geq 0} (\Omega_t^T \times \{t\}) \subset \Omega \times [0, \infty)$ such that

$$\phi_{\varepsilon} \rightarrow -1 + 2\chi_{\Omega^T} \text{ a.e. in } \Omega \times [0, \infty) \text{ and in } C^{\frac{1}{17}}([0, t]; L^2(\Omega)) \text{ for any } t \geq 0. \tag{3.15}$$

2. There exists $\mu \in L_{\text{loc}}^2(0, \infty, ; L^2(\Omega))$ such that $\nabla \mu \in L^2(0, \infty, ; L^2(\Omega))$ and

$$\mu_{\varepsilon} \rightarrow \mu \text{ weakly in } L_{\text{loc}}^2(0, \infty, ; H^1(\Omega)).$$

3. There exists $u \in L^2(0, \infty; L_{\text{div}}^2(\Omega))$ such that

$$u_{\varepsilon} \rightarrow u \text{ weakly in } L^2(0, \infty; L_{\text{div}}^2(\Omega)).$$

4. There exist a Radon measure λ and measures λ_{ij} , $i, j \in \{1, \dots, d\}$, on $\bar{\Omega} \times [0, \infty)$ such that

$$e_{\varepsilon}(\phi_{\varepsilon}) dx dt \rightarrow \lambda \text{ as a Radon measure } \bar{\Omega} \times [0, \infty), \tag{3.16}$$

i.e., weakly star in $M(\Omega, \mathbb{R})$,

$$\varepsilon \partial_{x_i} \phi_{\varepsilon} \partial_{x_j} \phi_{\varepsilon} dx dt \rightarrow \lambda_{ij} \text{ as a measure on } \bar{\Omega} \times [0, \infty), \tag{3.17}$$

for $i, j \in \{1, \dots, d\}$,

where $e_{\varepsilon}(\phi_{\varepsilon})$ denotes the energy density:

$$e_{\varepsilon}(\phi_{\varepsilon}) = \varepsilon \frac{1}{2} |\nabla \phi_{\varepsilon}|^2 + \frac{1}{\varepsilon} F(\phi_{\varepsilon}).$$

5. There exists a Radon measure $V = V^t dt$ on $\bar{\Omega} \times P \times [0, \infty)$ such that (u, Ω^T, μ, V) is a Varifold solution of (1.7)–(1.16) in the sense of Definition 3, with $d\lambda^t(x)dt = d\lambda(x, t)$ (where λ^t as in (3.13) and λ as in (3.16)) and with $\sigma = \int_{-1}^1 \sqrt{\frac{F(r)}{2}} dr$. Moreover,

$$\int_0^t \langle \delta V^s, Y \rangle ds = \int_0^t \int_{\Omega} \nabla Y : [d\lambda(x, s)I - (d\lambda_{ij}(x, s))_{d \times d}] \tag{3.18}$$

for all Y in $C_0^1(\Omega \times [0, t]; \mathbb{R}^d)$ and for all $t > 0$.

REMARK 5 Let us now comment on the notion of solutions introduced in Definition 2 and Definition 3.

1. The weak formulation (3.3) is derived by testing (1.1) with some $\psi \in C_0^\infty([0, t] \times \Omega)$, integrating by parts in time and space, and using the boundary and the initial conditions. We remark that, as $\phi_\varepsilon \in H^1(0, t; L^2(\Omega))$, relation (3.3) can be equivalently rewritten as

$$\int_{\Omega} (-\partial_t \phi_\varepsilon \psi + \phi_\varepsilon u_\varepsilon \cdot \nabla \psi - \nabla \mu_\varepsilon \cdot \nabla \psi) dx = 0 \text{ a.e. in } (0, t), \quad \phi_\varepsilon(0) = \phi_{0,\varepsilon}.$$

2. The energy identity (3.2) can be formally obtained by testing equation (1.1) by μ_ε and (1.2) by $\partial_t \phi_\varepsilon$, comparing the two, integrating by parts (taking into account the boundary conditions) and using (1.3).

3. As stated in Theorem 4, $\lambda^t(\bar{\Omega})$ is the limit of the energy functional $E_\varepsilon(\phi_\varepsilon(t))$ as $\varepsilon \rightarrow 0$. The energy functional for the sharp interface problem is instead given by the interfacial energy: $2\sigma|\chi_{\Omega^T}|(\Omega)$. A natural question is how the two relate. Modica and Mortola [24] and Sternberg [26] proved that the functional E_ε converge to $2\sigma|\chi_{\Omega^T}|(\Omega)$ in the Gamma-convergence sense with respect to the topology of $L^1(\Omega)$. As a consequence of this result and of convergence (3.15), we have that

$$\lambda^t(\bar{\Omega}) = \lim_{\varepsilon \rightarrow 0} E_\varepsilon(\phi_\varepsilon(t)) = \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(\phi_\varepsilon(t)) \geq 2\sigma|\chi_{\Omega^T}|(\Omega). \tag{3.19}$$

A second approach to obtain inequality (3.19) is the following. Consider the relation

$$\varepsilon \int_{\Omega} |\nabla \phi_\varepsilon|^2 dx = \int_{\Omega} e_\varepsilon(\phi_\varepsilon) dx + \int_{\Omega} \xi_\varepsilon(\phi_\varepsilon) dx, \tag{3.20}$$

where the *discrepancy density* ξ_ε is given by $\xi_\varepsilon(\phi_\varepsilon) = \varepsilon/2 |\nabla \phi_\varepsilon|^2 - 1/\varepsilon F(\phi_\varepsilon)$. We will prove that the discrepancy measure is nonpositive in the limit $\varepsilon \rightarrow 0$, namely $\int_{\Omega} (\xi_\varepsilon(\phi_\varepsilon))^+ dx \rightarrow 0$ as $\varepsilon \rightarrow 0$ (see Lemma 7). This yields, by passing to the limit as $\varepsilon \rightarrow 0$ in (3.20), inequality (3.19). Note that, in general, it is not possible to prove equality in (3.19) even in the simpler case $u = 0$, (cf. Section 2.4 of [6]). For example, a strict inequality holds true in case the initial data develop a so-called *phantom interface*, i.e.,

$$2|D\chi_{\Omega_0^T}|(\Omega) = |D \lim_{\varepsilon \rightarrow 0} \phi_{0,\varepsilon}(t)|(\Omega) < \liminf_{\varepsilon \rightarrow 0} |D\phi_{0,\varepsilon}|(\Omega).$$

However, in the case $u = 0$, under some additional assumptions, e.g., radial symmetry of the solutions [6] or limit *equipartition of the energy*: $\int_{\Omega} (\xi_\varepsilon(\phi_\varepsilon)) dx \rightarrow 0$ (which holds true if $d \leq 3$) [20], it is possible to show equality in (3.19). Let us mention that the techniques used in [20] strongly rely on the gradient-flow structure of equation (1.1)–(1.2) in the case $u = 0$. Thus, it seems hard to generalize that result to the system under consideration.

4. Using the definition of V (3.9), we have that

$$dV^t(x, p) = \sum_{i=1}^d b_i^t(x) \delta_{p_i^t(x)} d\lambda^t(x).$$

Thus, by definition of mass measure of a varifold and as a consequence of the properties of b_i^t , we get

$$d \|V^t\| (x) = \sum_{i=1}^d b_i^t(x) d\lambda^t(x) \geq d\lambda^t(x).$$

Let H^t denote the mean curvature vector of V^t . Then, by definition, we have, for all $Y \in C_0^1(\Omega; \mathbb{R}^d)$,

$$\begin{aligned} -\langle \delta V^t, Y \rangle &= \langle \|V^t\|, H \cdot Y \rangle = \int_{\Omega} H(x) \cdot Y(x) d \|V^t\| (x) \\ &= \int_{\Omega} 2\sigma m H(x) \cdot Y(x) |D\chi_{\Omega_i^T}|(x) dx, \end{aligned} \tag{3.21}$$

where

$$m := \frac{d \|V^t\| (x)}{2\sigma |D\chi_{\Omega_i^T}|(x) dx}. \tag{3.22}$$

Note that the two measures $d \|V^t\| (x)$ and $|D\chi_{\Omega_i^T}|(x) dx$ are absolutely continuous one with respect to the other as a consequence of relation (3.10). Moreover, $m \geq 1$. Furthermore, using formula (3.12), we have

$$\begin{aligned} -\langle \delta V^t, Y \rangle &= -2 \int_{\Omega} \chi_{\Omega_i^T} \nabla \cdot (\mu Y) dx = 2 \int_{\Omega} \mu n_{\Omega_i^T} \cdot Y(x) |D\chi_{\Omega_i^T}|(x) dx \\ &= \int_{\Omega} \frac{\mu}{m\sigma} n_{\Omega_i^T} \cdot Y(x) d \|V^t\| (x), \end{aligned} \tag{3.23}$$

where $n_{\Omega_i^T}$ is the unit vector associated with $D\chi_{\Omega_i^T}$ defined as in Section 2. Comparing (3.21) and (3.23), we deduce

$$\frac{\mu}{m} n_{\Omega_i^T} = \sigma H.$$

Multiplying by $n_{\Omega_i^T}$, as $|n_{\Omega_i^T}| = 1$, we get

$$\frac{\mu}{m} = \sigma n_{\Omega_i^T} \cdot H = \sigma k.$$

Here $k := n_{\Omega_i^T} \cdot H$ is the so-called generalized mean curvature. As $m \geq 1$, we have that

$$\mu = m\sigma k \geq \sigma k \text{ on } \Sigma.$$

Thus, relation (1.10) is satisfied up to a multiplicative constant $m \geq 1$ (if $m = 1$, we get equation (1.10)). We remark that, in general, it is not possible to show $m = 1$ even in the simpler case $u = 0$ (cf Section 2.4 of [6]). This is related to a possible gap between the limit of the energy and the energy of the limit problem as already discussed above. Indeed, under some growth assumptions on λ^t , it is possible to show that $\lambda^t = \|V^t\|$ (cf. [6]). In this case, thanks to (3.22), we have

$$\lambda^t = 2\sigma m |D\chi_{\Omega_i^T}| \geq 2\sigma |D\chi_{\Omega_i^T}|, \tag{3.24}$$

which is a quantitative version of inequality (3.19). In particular, this shows that, in the case $\lambda^t = \|V^t\|$, equality in (3.19) and relation $m = 1$ are equivalent.

5. From (1.10) and (1.16), we easily deduce

$$[P]_H^T = 2\mu \text{ on } \Sigma. \tag{3.25}$$

Equation (3.14) is obtained by multiplying equation (1.13) by some $\varphi \in C_{0,\text{div}}^\infty(\Omega)$, integrating by parts, and using (3.25), and the boundary conditions:

$$\begin{aligned} \int_0^t \int_{\Omega_s^T \cup \Omega_s^H} u \cdot \varphi \, dx \, ds &= - \int_0^t \int_{\Omega_s^T \cup \Omega_s^H} \nabla P \cdot \varphi \, dx \, ds \\ &= \int_0^t \int_{\Sigma_s} [P]_H^T \cdot \varphi \, dS \, ds = \int_0^t \int_{\Sigma_s} 2\mu \cdot \varphi \, dS \, ds. \end{aligned}$$

6. Equation (3.12) together with (3.18) imply

$$\int_{\Omega} 2\chi_{\Omega_t^T} \nabla \cdot (\mu(t)Y) \, dx = \int_{\Omega} \nabla Y : [d\lambda(x,t)I - (d\lambda_{ij}(x,t))_{d \times d}] \text{ for a.a. } t > 0.$$

This relation can be obtained by passing to the limit $\varepsilon \rightarrow 0$ in formula (5.2). Therefore, equation (3.12) stands as a reformulation of identity (3.5) and of condition $\mu_\varepsilon \in \frac{\delta E_\varepsilon}{\delta \phi}$ (here $\frac{\delta E_\varepsilon}{\delta \phi}$ denotes the first variation of E_ε) in the limit $\varepsilon \rightarrow 0$.

7. Inequality (3.13) has the meaning of energy dissipation inequality. It is obtained starting from (4.1) and by passing to the liminf as $\varepsilon \rightarrow 0$. We remark that equality does not hold in general. Indeed, we have just weak convergence for $\nabla \mu_\varepsilon$ and u_ε .

8. Note that, for all $\psi \in C_0^\infty([0, t] \times \Omega)$, we have $\int_0^t \int_{\Omega^T \cup \Omega^H} \partial_t \psi \, dx \, ds = \int_{\Omega^T \cup \Omega^H} \psi(t) \, dx - \int_{\Omega^T \cup \Omega^H} \psi(0) \, dx$ and $\int_0^t \int_{\Omega^T \cup \Omega^H} u_\varepsilon \cdot \nabla \psi \, dx \, ds = 0$. Thus, the weak formulation of the diffuse interface problem (3.3) is equivalent to

$$\begin{aligned} \int_0^t \int_{\Omega^T \cup \Omega^H} \{ (\phi_\varepsilon + 1) \partial_t \psi + (\phi_\varepsilon + 1) u_\varepsilon \cdot \nabla \psi - \nabla \mu_\varepsilon \cdot \nabla \psi \} \, dx \, ds \\ = \int_{\Omega^T \cup \Omega^H} (\phi_\varepsilon(t) + 1) \psi(t) \, dx - \int_{\Omega^T \cup \Omega^H} (\phi_{0,\varepsilon} + 1) \psi(0) \, dx. \end{aligned} \tag{3.26}$$

By passing to the limit as $\varepsilon \rightarrow 0$ and using the convergence results of Theorem 4, one gets the weak formulation of the sharp interface problem (3.11). Moreover, equation (3.11) can be formally deduced as follows. Test equation (1.12) with some $\psi \in C_0^\infty([0, t] \times \Omega)$, multiply (1.14) by $(-1 + 2\chi_{\Omega^T})\psi$, and take the sum getting

$$0 = \int_0^t \int_{\Omega^T \cup \Omega^H} \{ \Delta \mu \psi - (-1 + 2\chi_{\Omega^T}) \nabla \cdot u \psi \} \, dx \, ds.$$

By integrating by parts and using equation (1.9) and the boundary conditions on μ and u , one obtains

$$\begin{aligned} 0 &= \int_0^t \int_{\Omega^T \cup \Omega^H} \{ (-\nabla \mu) \nabla \psi + (-1 + 2\chi_{\Omega^T}) u \cdot \nabla \psi \} \, dx \, ds \\ &\quad + \int_0^t \int_{\Sigma} ([\nabla \mu]_H^T \cdot n - 2u \cdot n) \psi \, dS \, ds \\ &= \int_0^t \int_{\Omega^T \cup \Omega^H} \{ (-\nabla \mu) \nabla \psi + (-1 + 2\chi_{\Omega^T}) u \cdot \nabla \psi \} \, dx \, ds + \int_0^t \int_{\Sigma} (-2V) \psi \, dS \, ds. \end{aligned} \tag{3.27}$$

Interpreting V as the velocity describing the evolution of the interface Σ , we intuitively and formally have

$$\int_{\Sigma} 2V\psi dS = \int_{\Omega^T \cup \Omega^H} \partial_t(-1 + 2\chi_{\Omega^T})\psi dx.$$

By (formally) integrating by parts this relation and substituting into (3.27) and using relations $\int_{\Omega^T \cup \Omega^H} u \cdot \nabla \psi dx = 0$ and $\int_0^t \int_{\Omega^T \cup \Omega^H} \partial_t \psi dx ds = \int_{\Omega^T \cup \Omega^H} \psi(t) dx - \int_{\Omega^T \cup \Omega^H} \psi(0) dx$, we get (3.11). This suggests that condition (3.11) encodes equations (1.9), (1.12), (1.14), and the boundary conditions.

4. A priori estimates

In this section we derive some uniform-in- ε estimates for solutions $(u_\varepsilon, \phi_\varepsilon, \mu_\varepsilon)$ to system (1.1)–(1.6). In what follows C will denote a positive constant independent of ε which possibly varies even within the same line.

Let $(u_\varepsilon, \phi_\varepsilon, \mu_\varepsilon)$ be a solution to system (1.1)–(1.6). Integrating identity (3.4) over $[\tau, t]$ and recalling well preparedness of initial data (3.1),

$$E_\varepsilon(\phi_\varepsilon(t)) + \int_\tau^t \int_{\Omega} |\nabla \mu_\varepsilon|^2 dx ds + \int_\tau^t \int_{\Omega} |u_\varepsilon|^2 dx ds = E_\varepsilon(\phi_\varepsilon(\tau)) \leq E_\varepsilon(\phi_{0,\varepsilon}) \leq C. \quad (4.1)$$

Thus, recalling the definition of the energy functional

$$E_\varepsilon(\phi_\varepsilon) = \int_{\Omega} \left(\varepsilon \frac{1}{2} |\nabla \phi_\varepsilon|^2 + \frac{1}{\varepsilon} F(\phi_\varepsilon) \right) dx,$$

we have

$$\int_{\Omega} \varepsilon \frac{1}{2} |\nabla \phi_\varepsilon(t)|^2 dx + \frac{1}{\varepsilon} F(\phi_\varepsilon(t)) + \int_\tau^t \int_{\Omega} |\nabla \mu_\varepsilon|^2 dx ds + \int_\tau^t \int_{\Omega} |u_\varepsilon|^2 dx ds \leq C. \quad (4.2)$$

By using p -growth of F for large ϕ and positivity of $F''(\pm 1)$, we get that $F(\phi) \geq \frac{1}{C}(|\phi| - 1)^2$ for all $\phi \in \mathbb{R}$. In particular, by using again p -growth of F , we deduce the estimates:

$$\|\nabla \mu_\varepsilon\|_{L^2(0,\infty;L^2(\Omega))} \leq C, \quad (4.3)$$

$$\|u_\varepsilon\|_{L^2(0,\infty;L^2(\Omega))} \leq C, \quad (4.4)$$

$$\|\varepsilon^{1/2} \nabla \phi_\varepsilon\|_{L^\infty(0,\infty;L^2(\Omega))} \leq C, \quad (4.5)$$

$$\int_{\Omega} |\phi_\varepsilon(t)|^p dx \leq C \text{ for all } t \geq 0, \quad (4.6)$$

$$\int_{\Omega} (|\phi_\varepsilon(t)| - 1)^2 dx \leq \varepsilon C \text{ for all } t \geq 0. \quad (4.7)$$

Following [6], we define

$$W(\phi) = \int_{-1}^{\phi} \sqrt{2\tilde{F}(r)} dr, \text{ where } \tilde{F}(r) = \min \left\{ F(r), \max_{z \in [-1,1]} F(z) + r^2 \right\}$$

and

$$w_\varepsilon(x, t) = W(\phi_\varepsilon(x, t)) \text{ for a.a. } (x, t) \in \Omega \times (0, \infty).$$

Note that by definition $F(r) = \tilde{F}(r)$ for all $r \in [-1, 1]$. By applying the Young inequality, we easily estimate

$$\int_\Omega |\nabla w_\varepsilon| dx = \int_\Omega \sqrt{2\tilde{F}(\phi_\varepsilon)} |\nabla \phi_\varepsilon| dx \leq E_\varepsilon(\phi_\varepsilon) \leq C. \tag{4.8}$$

In particular, the functions w_ε are uniformly bounded in $L^\infty(0, \infty; W^{1,1}(\Omega))$. We now prove that

$$\begin{aligned} \|w_\varepsilon\|_{C^{\frac{1}{16}}([0, \infty); L^1(\Omega))} &\leq C, \\ \|\phi_\varepsilon\|_{C^{\frac{1}{16}}([0, \infty); L^2(\Omega))} &\leq C. \end{aligned}$$

To this aim let $\rho \in C^\infty(\mathbb{R}^d)$ be any fixed mollifier satisfying

$$0 \leq \rho \leq 1 \text{ in } B_1, \quad \rho = 0 \text{ in } \mathbb{R}^d \setminus B_1, \quad \int_{B_1} \rho dx = 1.$$

For any $\eta_0 > 0$ and any $\eta \in (0, \eta_0]$, we define

$$\phi_\varepsilon^\eta(x, t) = \int_{B_1} \rho(y) \phi_\varepsilon(x - \eta y, t) dy \text{ for all } x \in \Omega, t \geq 0.$$

Here we have assumed that ϕ_ε has been extended to a small neighborhood of Ω as follows: for any $x \notin \Omega$ such that $\text{dist}(x, \Omega) \leq \eta_0$, we define

$$\phi_\varepsilon(S + \eta \nu(S), t) = \phi_\varepsilon(S - \eta \nu(S), t) \text{ for all } S \in \partial\Omega, \eta \in [0, \eta_0], t \geq 0$$

where ν denotes the outward normal to $\partial\Omega$. Note that, by standard properties of mollifiers, we have

$$\|\nabla \phi_\varepsilon^\eta(t)\|_{L^q(\Omega)} \leq C \eta^{-1} \|\phi_\varepsilon(t)\|_{L^q(\Omega)} \leq C \eta^{-1} \text{ for all } 1 < q \leq p. \tag{4.9}$$

and

$$\begin{aligned} \|\phi_\varepsilon^\eta(t) - \phi_\varepsilon(t)\|_{L^2(\Omega)}^2 &\leq \int_\Omega \int_{B_1} |\phi_\varepsilon(x - \eta y, t) - \phi_\varepsilon(x, t)| dx dy \\ &\leq C \int_\Omega \int_{B_1} |w_\varepsilon(x - \eta y, t) - w_\varepsilon(x, t)| dx dy \\ &\leq C \eta \|\nabla w_\varepsilon(t)\|_{L^1(\Omega)} \leq C \eta. \end{aligned} \tag{4.10}$$

Here we have used inequality

$$c_1 |\phi_1 - \phi_2| \leq |W(\phi_1) - W(\phi_2)| \leq c_2 |\phi_1 - \phi_2| (1 + |\phi_1| + |\phi_2|), \tag{4.11}$$

for all $\phi_1, \phi_2 \in \mathbb{R}$ and some positive constant c_1, c_2 , which follows directly from the definition of W . We fix $0 < \tau < t$. Taking the difference of equation (3.3) at time t and the same equation at time τ , and using a density argument

$$\begin{aligned} \int_\Omega \phi_\varepsilon(t) \psi dx - \int_\Omega \phi_\varepsilon(\tau) \psi dx &= \int_\tau^t \int_\Omega (\phi_\varepsilon \partial_t \psi - (\nabla \mu_\varepsilon - u_\varepsilon \phi_\varepsilon) \nabla \psi) dx ds \\ &= \int_\tau^t \int_\Omega (-(\nabla \mu_\varepsilon - u_\varepsilon \phi_\varepsilon) \nabla \psi) dx ds \end{aligned}$$

for all $\psi \in H_0^1(\Omega)$. Choosing $\psi = \phi_\varepsilon^\eta(t) - \phi_\varepsilon^\eta(\tau)$, as it is constant in time, we estimate

$$\begin{aligned}
& \int_{\Omega} (\phi_\varepsilon(t) - \phi_\varepsilon(\tau))(\phi_\varepsilon^\eta(t) - \phi_\varepsilon^\eta(\tau)) dx \\
&= - \int_{\tau}^t \int_{\Omega} (\nabla \mu_\varepsilon(s) - u_\varepsilon(s) \phi_\varepsilon(s)) (\nabla \phi_\varepsilon^\eta(t) - \nabla \phi_\varepsilon^\eta(\tau)) dx ds \\
&\leq \left(\int_{\tau}^t \int_{\Omega} |\nabla \phi_\varepsilon^\eta(t) - \nabla \phi_\varepsilon^\eta(\tau)|^4 dx ds \right)^{\frac{1}{4}} \left(\int_{\tau}^t \int_{\Omega} |\nabla \mu_\varepsilon(s) - u_\varepsilon(s) \phi_\varepsilon(s)|^{\frac{4}{3}} dx ds \right)^{\frac{3}{4}} \\
&\leq C(t - \tau)^{\frac{1}{4}} \sup_{s \in (\tau, t)} \|\nabla \phi_\varepsilon^\eta(t)\|_{L^4(\Omega)} \left(1 + \int_{\tau}^t \|\nabla \mu_\varepsilon(s)\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} ds \right. \\
&\quad \left. + \int_{\tau}^t \|\phi_\varepsilon(s)\|_{L^4(\Omega)}^4 \|u_\varepsilon(s)\|_{L^2(\Omega)}^2 ds \right)^{\frac{3}{4}} \\
&\leq C(t - \tau)^{\frac{1}{4}} \eta^{-1} \left(1 + \|\nabla \mu_\varepsilon(s)\|_{L^2(0, \infty; L^2(\Omega))}^{\frac{4}{3}} \right. \\
&\quad \left. + \|\phi_\varepsilon(s)\|_{L^\infty(0, \infty; L^4(\Omega))}^4 \|u_\varepsilon(s)\|_{L^2(0, \infty; L^2(\Omega))}^2 \right)^{\frac{3}{4}} \\
&\leq C(t - \tau)^{\frac{1}{4}} \eta^{-1}. \tag{4.12}
\end{aligned}$$

Here we used estimates (4.3), (4.4), (4.6) together with $p \geq 4$, and (4.9) for $q = 4$. Let now a, b, c, d be real number such that $a = b + c + d$. Then,

$$a^2 = a(b + c + d) \leq ab + ac + ad \leq ab + \frac{1}{2}a^2 + c^2 + d^2. \tag{4.13}$$

Using (4.13) for $a = \phi_\varepsilon(t) - \phi_\varepsilon(\tau)$, $b = \phi_\varepsilon^\eta(t) - \phi_\varepsilon^\eta(\tau)$, $c = \phi_\varepsilon(t) - \phi_\varepsilon^\eta(t)$, and $d = \phi_\varepsilon(\tau) - \phi_\varepsilon^\eta(\tau)$, and estimates (4.10)–(4.12), we deduce

$$\begin{aligned}
\|\phi_\varepsilon(t) - \phi_\varepsilon(\tau)\|_{L^2(\Omega)}^2 &\leq 2\|\phi_\varepsilon(t) - \phi_\varepsilon^\eta(t)\|_{L^2(\Omega)}^2 + 2\|\phi_\varepsilon(\tau) - \phi_\varepsilon^\eta(\tau)\|_{L^2(\Omega)}^2 \\
&\quad + 2 \int_{\Omega} (\phi_\varepsilon(t) - \phi_\varepsilon(\tau))(\phi_\varepsilon^\eta(t) - \phi_\varepsilon^\eta(\tau)) dx \\
&\leq C(\eta + |t - \tau|^{\frac{1}{4}} \eta^{-1}).
\end{aligned}$$

Choosing $\eta = |t - \tau|^{\frac{1}{8}}$, we get

$$\|\phi_\varepsilon\|_{C^{\frac{1}{16}}([0, \infty); L^2(\Omega))} \leq C$$

and, recalling (4.11),

$$\begin{aligned}
\|w_\varepsilon(t) - w_\varepsilon(\tau)\|_{L^1(\Omega)} &\leq \|\phi_\varepsilon(t) - \phi_\varepsilon(\tau)\|_{L^2(\Omega)}^2 (C + \|\phi_\varepsilon(t)\|_{L^2(\Omega)}^2 + \|\phi_\varepsilon(\tau)\|_{L^2(\Omega)}^2) \\
&\leq C(t - \tau)^{\frac{1}{16}},
\end{aligned}$$

which implies

$$\|w_\varepsilon\|_{C^{\frac{1}{16}}([0, \infty); L^1(\Omega))} \leq C.$$

Starting from the elliptic equation $\mu_\varepsilon = -\varepsilon\Delta\phi_\varepsilon + \frac{1}{\varepsilon}F(\phi_\varepsilon)$, it is possible to derive uniform estimates for μ_ε and for the discrepancy density

$$\xi_\varepsilon(\phi_\varepsilon) = \frac{\varepsilon}{2}|\nabla\phi_\varepsilon|^2 - \frac{1}{\varepsilon}F(\phi_\varepsilon).$$

Lemma 6 [6, Lemma 3.4] *There exist positive constants C and ε_0 such that for every t and $\varepsilon \in (0, \varepsilon_0)$ the following holds*

$$\|\mu_\varepsilon(t)\|_{H^1(\Omega)} \leq C (E_\varepsilon(t) + \|\nabla\mu_\varepsilon(t)\|_{L^2(\Omega)}).$$

In particular, for every $s > 0$ there exists a positive constant $C(s)$, such that $\|\mu_\varepsilon(t)\|_{L^2(0,s;L^2(\Omega))} \leq C(s)$.

Lemma 7 [6, Theorem 3.6] *There exist a positive constant $\eta_0 \in (0, 1]$ and continuous nondecreasing functions $M_1(\eta)$ and $M_2(\eta)$ defined on $[0, \eta_0)$ such that for all $\varepsilon \in (0, \frac{1}{M_1(\eta)})$ and all $t > 0$, we have*

$$\int_0^t \int_\Omega (\xi_\varepsilon(\phi_\varepsilon))^+ dx ds \leq \eta \int_0^t E_\varepsilon(\phi_\varepsilon) dx ds + \varepsilon M_2(\eta) \int_0^t \int_\Omega |\mu_\varepsilon|^2 dx ds.$$

In particular,

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \int_\Omega (\xi_\varepsilon(\phi_\varepsilon))^+ dx ds = 0. \quad (4.14)$$

5. Convergence

Starting from the above uniform estimates, we now deduce some convergence results.

Lemma 8 *For every sequence $\varepsilon \rightarrow 0$, there exists a (not relabeled) subsequence and a nonincreasing function E , such that*

$$E_\varepsilon(\phi_\varepsilon(t)) \rightarrow E(t) \text{ for all } t \geq 0.$$

Proof. Define $E_\varepsilon(t) = E_\varepsilon(\phi_\varepsilon(t))$. Note that $E_\varepsilon(t)$ is uniformly bounded as a consequence of identity (4.1). Furthermore, the sequence $E_\varepsilon(\cdot)$ is uniformly continuous as a consequence of monotonicity, of the energy identity (4.1), and of the uniform bounds of $\nabla\mu_\varepsilon$ and u_ε in $L^2(0, T; L^2(\Omega))$. Thus, the statement of the lemma follows by applying the Ascoli–Arzelà theorem. \square

Lemma 9 *For every sequence $\varepsilon \rightarrow 0$, there exists a (not relabeled) subsequence and a set $\Omega^T \subset \Omega \times [0, \infty)$, such that*

$$\begin{aligned} w_\varepsilon &\rightarrow 2\sigma\chi_{\Omega^T} \text{ a.e. in } \Omega \times [0, \infty) \text{ and in } C^{\frac{1}{17}}([0, t]; L^1(\Omega)) \text{ for all } t > 0, \\ \phi_\varepsilon &\rightarrow -1 + 2\chi_{\Omega^T} \text{ a.e. in } \Omega \times [0, \infty) \text{ and in } C^{\frac{1}{17}}([0, t]; L^2(\Omega)) \text{ for all } t > 0, \\ \mu_\varepsilon &\rightarrow \mu \text{ weakly in } L^2_{\text{loc}}(0, \infty; H^1(\Omega)), \\ u_\varepsilon &\rightarrow u \text{ weakly in } L^2(0, \infty; L^2_{\text{div}}(\Omega)). \end{aligned}$$

Moreover,

$$\begin{aligned} \int_{\Omega} |\chi_{\Omega_t^T} - \chi_{\Omega_\tau^T}| dx &\leq C |t - \tau|^{\frac{1}{8}} \quad \text{for any } 0 \leq \tau < t, \\ |\Omega_t^T| &= |\Omega_0^T| \quad \text{for any } t \geq 0, \quad \chi_{\Omega^T} \in L^\infty(0, \infty; \text{BV}(\Omega)) \quad \text{and} \\ 2\sigma |\text{D}\chi_{\Omega_t^T}|(\Omega) &\leq E(t) \leq E(0). \end{aligned}$$

Proof. As $\|w_\varepsilon\|_{L^\infty(0, \infty; W^{1,1}(\Omega))} + \|w_\varepsilon\|_{C^{\frac{1}{16}}([0, \infty); L^1(\Omega))} \leq C$ and $W^{1,1}(\Omega)$ is compactly embedded in $L^1(\Omega)$, there exists a (not relabeled) sequence $\varepsilon \rightarrow 0$ such that

$$w_\varepsilon \rightarrow w \text{ a.e. in } \Omega \times [0, \infty) \text{ and in } C^{\frac{1}{17}}([0, t]; L^1(\Omega)) \text{ for all } t > 0,$$

for some limit $w \in C^{\frac{1}{17}}([0, t]; L^1(\Omega))$ (cf. [23, Prop. 1.1.4] and [2, Thm 4.4]). Recalling the definition of w_ε and estimate (4.11), we conclude that there exists $\phi \in C^{\frac{1}{17}}([0, t]; L^2(\Omega))$ such that

$$\phi_\varepsilon \rightarrow \phi \text{ a.e. in } \Omega \times [0, \infty) \quad \text{and in } C^{\frac{1}{17}}([0, t]; L^2(\Omega)) \text{ for all } t > 0.$$

As a consequence of estimate (4.7), we deduce

$$\int_{\Omega} (|\phi_\varepsilon| - 1)^2 dx \leq C \int_{\Omega} F(\phi_\varepsilon) dx \leq \varepsilon C.$$

Thus, the limit ϕ takes values in $\{-1, 1\}$. In particular, there exists a set $\Omega^T \subset \Omega \times [0, \infty)$ such that

$$\phi = -1 + 2\chi_{\Omega^T}.$$

Hence, by definition of w_ε and continuity of \tilde{F} , we get

$$w = \int_{-1}^{\phi} \sqrt{2\tilde{F}(r)} dr = 2\sigma \chi_{\Omega^T},$$

where $\sigma = \int_{-1}^1 \sqrt{\frac{1}{2}\tilde{F}(r)} dr = \int_{-1}^1 \sqrt{\frac{1}{2}F(r)} dr$. Here we used the fact that $F(r) = \tilde{F}(r)$ for $r \in [-1, 1]$, which directly follows from the definition of \tilde{F} . Let now $\Omega_t^T = \{x \in \Omega : (x, t) \in \Omega^T\}$. Then, for every $0 \leq \tau < t$, we have

$$\begin{aligned} \int_{\Omega} |\chi_{\Omega_t^T} - \chi_{\Omega_\tau^T}| dx &= \int_{\Omega} |\chi_{\Omega_t^T} - \chi_{\Omega_\tau^T}|^2 dx = \lim_{\varepsilon \rightarrow 0} \frac{1}{4} \int_{\Omega} |\phi_\varepsilon(t) - \phi_\varepsilon(\tau)|^2 dx \\ &\leq C |t - \tau|^{\frac{1}{8}}. \end{aligned}$$

As a consequence of the mass conservation

$$\int_{\Omega} \phi_\varepsilon(t) dx = \int_{\Omega} \phi_0 dx = m_0 |\Omega|,$$

we have

$$|\Omega_t^T| = \int_{\Omega} \chi_{\Omega_t^T} dx = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{\Omega} (\phi_\varepsilon(t) + 1) dx = \frac{m_0 + 1}{2} |\Omega| = |\Omega_0^T|.$$

Moreover, as a consequence of estimate (4.8), we have $|Dw_\varepsilon(t)|(\Omega) = \|\nabla w_\varepsilon(t)\|_{L^1(\Omega)} \leq E_\varepsilon(\phi_\varepsilon(t))$. Taking the liminf for $\varepsilon \rightarrow 0$ and using the lower semicontinuity of the BV norm, we conclude

$$2\sigma|D\chi_{\Omega^T}|(\Omega) \leq |Dw|(\Omega) \leq E(t).$$

Finally, convergences

$$\begin{aligned} \mu_\varepsilon &\rightarrow \mu \text{ weakly in } L^2_{\text{loc}}(0, \infty; H^1(\Omega)), \\ u_\varepsilon &\rightarrow u \text{ weakly in } L^2(0, \infty; L^2(\Omega)) \end{aligned}$$

follow directly from Lemma 6 and estimate (4.1) respectively. □

As a consequence of estimate (4.1) and (4.5), we have that convergences (3.16) and (3.17) hold for some limit measures λ and λ_{ij} . Thus, we proved the convergence results stated in Theorem 4. We now construct the varifold V and show that the limits μ , u , λ , and λ_{ij} solve the sharp-interface problem.

We first note that, for any $0 \leq \tau < t$, we have

$$\int_\tau^t \int_\Omega d\lambda(x, s) = \lim_{\varepsilon \rightarrow 0} \int_\tau^t \int_\Omega e_\varepsilon(\phi_\varepsilon) dx ds = \int_\tau^t E(s) ds.$$

Moreover, λ can be decomposed (in the sense of Radon measures) as follows

$$d\lambda(x, t) = d\lambda^t(x) dt,$$

where $\lambda^t(\bar{\Omega}) = E(t)$ for a.a. $t \in (0, \infty)$. In particular, using relation (4.1) and the weak lower semicontinuity of the norm, we obtain

$$\begin{aligned} \lambda^t(\bar{\Omega}) &= E(t) = \lim_{\varepsilon \rightarrow 0} E_\varepsilon(t) \\ &\leq -\liminf_{\varepsilon \rightarrow 0} \left\{ \int_\tau^t \int_\Omega |\nabla \mu_\varepsilon|^2 dx ds + \int_\tau^t \int_\Omega |u_\varepsilon|^2 dx ds \right\} + \lim_{\varepsilon \rightarrow 0} E_\varepsilon(\phi_\varepsilon(\tau)) \\ &\leq -\int_\tau^t \int_\Omega |\nabla \mu|^2 dx ds - \int_\tau^t \int_\Omega |u|^2 dx ds + E(\tau) = \lambda^\tau(\bar{\Omega}), \end{aligned}$$

which is equivalent to (3.13). Moreover, as a consequence of condition $2\sigma|D\chi_{\Omega^T}|(\Omega) \leq E(t)$ obtained in Lemma 9, we deduce estimate (3.10). Next we study the relation between λ_{ij} and λ . Let $Y, Z \in C(\bar{\Omega} \times [0, t]; \mathbb{R}^d)$ and observe that

$$\int_0^t \int_\Omega Y \cdot (\varepsilon \nabla \phi_\varepsilon \otimes \nabla \phi_\varepsilon) \cdot Z dx ds \leq \int_0^t \int_\Omega |Y||Z|e_\varepsilon(\phi_\varepsilon) dx ds + \int_0^t \int_\Omega |Y||Z|\xi_\varepsilon(\phi_\varepsilon) dx ds.$$

Using Lemma 7, we have that

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \int_\Omega |Y||Z|\xi_\varepsilon(\phi_\varepsilon) dx ds \leq 0.$$

Hence, taking the limit for $\varepsilon \rightarrow 0$, we get

$$\int_0^t \int_\Omega Y \cdot (d\lambda_{ij}(x, s))_{d \times d} \cdot Z \leq \int_0^t \int_\Omega |Y||Z| d\lambda(x, s). \tag{5.1}$$

Thus, λ_{ij} are absolutely continuous with respect of λ in the sense of measures. Consequently, we can define the Radon-Nikodin derivative of λ_{ij} with respect to λ as a λ -measurable function v_{ij} such that

$$d\lambda_{ij}(x, t) = v_{ij}(x, t)d\lambda(x, t) \quad \lambda\text{-a.e.}$$

From formula (5.1), it follows that

$$0 \leq (v_{ij})_{d \times d} \leq I \quad \lambda\text{-a.e.}$$

and that

$$(v_{ij})_{d \times d} = \sum_{i=1}^d c_i v_i \otimes v_i \quad \lambda\text{-a.e.}$$

for some λ -measurable functions c_i and unit vectors $v_i, i = 1, \dots, d$. Moreover, they satisfy

$$0 \leq c_i \leq 1, \quad \sum_{i=1}^d c_i \leq 1, \quad \sum_{i=1}^d v_i \otimes v_i = I.$$

In order to construct the varifold V , we observe that, by multiplying equation

$$\mu_\varepsilon = -\varepsilon \Delta \phi_\varepsilon + \frac{1}{\varepsilon} F'(\phi_\varepsilon)$$

with $Y \cdot \nabla \phi_\varepsilon$ for some $Y \in C^1(\bar{\Omega}; \mathbb{R}^d)$ and integrating over Ω , we get

$$\begin{aligned} \int_{\Omega} Y \cdot \nabla \phi_\varepsilon \mu_\varepsilon dx &= \int_{\Omega} Y \cdot \nabla \phi_\varepsilon \left(-\varepsilon \Delta \phi_\varepsilon + \frac{1}{\varepsilon} F'(\phi_\varepsilon) \right) dx \\ &= - \int_{\Omega} \nabla Y : (e_\varepsilon(\phi_\varepsilon) I - \varepsilon \nabla \phi_\varepsilon \otimes \nabla \phi_\varepsilon) dx + \int_{\partial \Omega} e_\varepsilon(\phi_\varepsilon) Y \cdot \nu dS. \end{aligned} \quad (5.2)$$

By passing to the limit for $\varepsilon \rightarrow 0$ in relation (5.2), we obtain, for every $t > 0$ and $Y \in C_0^1(\Omega; \mathbb{R}^d)$,

$$\begin{aligned} 2 \int_{\Omega} \chi_{\Omega_t^T} \nabla \cdot (\mu(t) Y) dx &= \int_{\Omega} \nabla Y : \left(I - \sum_{i=1}^d c_i(x, t) v_i(x, t) \otimes v_i(x, t) \right) d\lambda^t(x) \\ &= \int_{\Omega} \nabla Y : \sum_{i=1}^d b_i^t(x) (I - v_i(x, t) \otimes v_i(x, t)) d\lambda^t(x) \end{aligned}$$

where the coefficients b_i^t are given by

$$b_i^t(x) = c_i(x, t) + \frac{1}{d-1} \left(1 - \sum_{i=1}^d c_i(x, t) \right).$$

Note that

$$0 \leq b_i^t \leq 1, \quad \sum_{i=1}^d b_i^t \geq 1.$$

Finally, we define $p_i^t \in P$ by

$$p_i^t = \{v_i(x, t), -v_i(x, t)\},$$

V^t as in (3.9), and V by

$$dV(x, t, p) = dV^t(x, p)dt.$$

Moreover, by construction V satisfies conditions (3.12) and (3.18).

Relation (3.11) follows from (3.26) by passing to the limit $\varepsilon \rightarrow 0$, and using the above convergences.

We are only left to show relation (3.14). To this aim, let $\varphi \in C_{0,\text{div}}^\infty(\Omega; \mathbb{R}^d)$. Then, by using relation $u_\varepsilon = -\nabla P + \mu_\varepsilon \nabla \phi_\varepsilon$, for every $t > 0$, we have

$$\int_0^t \int_\Omega u_\varepsilon \varphi dx ds = - \int_0^t \int_\Omega \nabla \mu_\varepsilon \cdot \varphi \phi_\varepsilon dx ds.$$

The above convergence results allow us to pass to the limit for $\varepsilon \rightarrow 0$ getting

$$\int_0^t \int_\Omega u \varphi dx ds = - \int_0^t \int_\Omega \nabla \mu \cdot \varphi (-1 + 2\chi_{\Omega_s^T}) dx ds.$$

Integrating by parts the right-hand side, we obtain

$$\int_0^t \int_\Omega u \varphi dx ds = \int_0^t \int_{\Sigma_s} 2\mu \varphi dS ds.$$

This concludes the proof of Theorem 4.

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