

Rigidity and stability of spheres in the Helfrich model

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The Helfrich functional, denoted by \mathcal{H}^{c_0} , is a mathematical expression proposed by Helfrich (1973) for the natural free energy carried by an elastic phospholipid bilayer. Helfrich theorises that idealised elastic phospholipid bilayers minimise \mathcal{H}^{c_0} among all possible configurations. The functional integrates a spontaneous curvature parameter c_0 together with the mean curvature of the bilayer and constraints on area and volume, either through an inclusion of osmotic pressure difference and tensile stress or otherwise. Using the mathematical concept of embedded orientable surface to represent the configuration of the bilayer, one might expect to be able to adapt methods from differential geometry and the calculus of variations to perform a fine analysis of bilayer configurations in terms of the parameters that it depends upon. In this article we focus upon the case of spherical red blood cells with a view to better understanding spherocytes and spherocytosis. We provide a complete classification of spherical solutions in terms of the parameters in the Helfrich model. We additionally present some further analysis on the rigidity and stability of spherocytes.

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1. Introduction

Motivated by Hooke's law, Helfrich (1973) proposed

$$f_c = \frac{k_c(H - c_0)^2}{2} + \bar{k}K$$

as the energy per unit area of a lipid bilayer or membrane. The constants k_c and \bar{k} are the bending moduli. It is argued in Mutz and Helfrich (1990); Duwe et al. (1990) that k_c is small and positive. We shall see shortly that the exact value of \bar{k} is not important for our investigations here (in Seifert (1997) it is even set to zero). The spontaneous curvature c_0 on the other hand is a critical component of the model. It was found, based on experimental data of Evans and Fung (1972), to approximately

satisfy $c_0 = -0.74 \mu m^{-1}$ under the assumptions that the membrane is a typical human erythrocyte and normal physiological conditions are in place (see [Deuling and Helfrich \(1976a,b\)](#)). For more details on values for these and other parameters we refer to the Remark after Theorem 1.

Supposing the membrane is represented by a smooth isometric embedding $f : \Sigma \rightarrow \mathbb{R}^3$ of a two-dimensional closed differentiable manifold Σ , this gives rise to the energy functional

$$\begin{aligned} \hat{\mathcal{H}}^{c_0}(f) &= \frac{k_c}{2} \int_{\Sigma} (H - c_0)^2 d\mu + 2\bar{k}\pi\chi(\Sigma) \\ &= \frac{k_c}{2} \int_{\Sigma} H^2 d\mu - k_c c_0 \int_{\Sigma} H d\mu + \frac{k_c c_0^2}{2} \text{Area } f + 2\bar{k}\pi\chi(\Sigma), \end{aligned}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ and we have used the Gauss-Bonnet theorem. The Euler characteristic $\chi(\Sigma)$ is a topological invariant, satisfying for example $\chi(\Sigma) = 2 - 2g$, where g is the genus of Σ . The genus counts the number of holes in the surface. In cases where the bilayer is topologically spherical, we have $\text{genus}(\Sigma) = 0$. This includes for example stomatocytes, discocytes, spherocytes, echinocytes, and so on (see [Mohandas and Gallagher \(2008\)](#)). As such cells form our primary interest in this paper, we shall work from now on in the topological class of $g = 0$ and $\chi(\Sigma) = 2$. The notation $\text{Area } f$ denotes the area of $(\Sigma, f^*\langle \cdot, \cdot \rangle)$ as a Riemannian manifold, where $f^*\langle \cdot, \cdot \rangle$ is the pullback via the embedding f of the standard metric $\langle \cdot, \cdot \rangle$ on \mathbb{R}^3 , the dot product. That is,

$$\text{Area } f = \int_{\Sigma} d\mu = \int_{\Sigma} \sqrt{\det((\partial_i f, \partial_j f))} dx.$$

Our motivation for the study of the Helfrich model is in connection with spherocytosis, a disorder of the membrane of human red blood cells that causes them to be spherical (spherocytes) as opposed to the standard biconcave disk shape (discocytes). Spherocytes break down faster than discocytes, and as they have a lower surface area than discocytes, (in fact by the isoperimetric problem, spherocytes are in this sense the worst configuration possible) patients with spherocytosis suffer from severe anemia ([Chasis et al., 1988](#); [Perrotta et al., 2008](#); [Svetina and Žekš, 1989](#)). Additionally, the spleen sometimes mistakes otherwise healthy spherocytes for damaged non-functional cells, and destroys them. This leads to haemolytic anemia, and can be fatal ([Perrotta et al., 2008](#)). Up to now, the only known treatment is a (often partial) splenectomy ([Abdullah et al., 2009](#); [Hassoun and Palek, 1996](#); [Perrotta et al., 2008](#); [Rescorla et al., 2007](#)), which comes with a lifetime of medication, and other complications. Spherocytosis is the most common form of inheritable anemia in people of northern European ancestry ([Perrotta et al., 2008](#)).

The cytoskeleton of a human red blood cell is inhomogeneous and sheet-like, with a lipid bilayer and supporting network of proteins. Although there remain many open questions regarding the dynamical forces at play in the cytoskeleton of a human red blood cell (see [Steck \(1989\)](#)), it is in a sense self-organising so as to minimise certain costs (see [Pozrikidis \(2005\)](#)). The basic idea of the Helfrich model is that this cost can be measured in an idealised setting by the Helfrich functional. Our goal is to better understand the appearance of spherocytes *in the model*. In particular, we study spherical solutions of the Euler-Lagrange equation for critical points of \mathcal{H}^{c_0} , giving first a complete classification of parameter ranges that allow spherocytes (Theorem 1.5) and second some first steps into rigidity and stability analysis of spherical solutions (Theorema 2, 4, 8, 9 and Corollaries 5 and 6). Since the parameters c_0 , λ , and p are in principle measurable, we may in the long-term be able to influence them, and in so doing discourage the formation of spherocytes.

We now seek to study embeddings $f : \Sigma \rightarrow \mathbb{R}^3$ that *minimise* the Helfrich functional. These represent, in a model sense, the biomembranes that we wish to investigate.

In the language of the calculus of variations, the problem is then as follows.

PROBLEM (P1) Suppose Σ is a closed differentiable 2-manifold with genus zero. Let c_0, S_0 and V_0 be fixed positive constants. Minimise $\hat{\mathcal{H}}^{c_0}(f)$ in the class of smooth embeddings $f : \Sigma \rightarrow \mathbb{R}^3$ subject to the constraints

$$\text{Area } f = S_0 \quad \text{and} \quad \text{Vol } f = V_0. \tag{P1.1}$$

That is, find an embedding $f_0 : \Sigma \rightarrow \mathbb{R}^3$ such that $\text{Area } f_0 = S_0, \text{Vol } f_0 = V_0$, and

$$\hat{\mathcal{H}}^{c_0}(f_0) \leq \hat{\mathcal{H}}^{c_0}(f) \tag{1.1}$$

for any other smooth embedding f of Σ .

REMARK 1.1 A candidate embedding f_0 which achieves the global energy minimum is called a *solution*. It is not unique. The constraints and the functional $\hat{\mathcal{H}}$ are invariant under reparametrisation as well as rigid motions in \mathbb{R}^3 .

The variational problem (P1) is the classical formulation suggested in [Helfrich \(1973\)](#); [Deuling and Helfrich \(1976a,b\)](#). A solution $\hat{f} : \Sigma \rightarrow \mathbb{R}^3$ will satisfy the aforementioned Euler-Lagrange equation

$$k_c(\Delta H + H|A^o|^2) + 2k_c c_0 K - \left(\frac{k_c c_0^2}{2} + s_0\right)H - v_0 = 0. \tag{1.2}$$

where $s_0, v_0 \in \mathbb{R}$ are Lagrange multipliers (see [Capovilla et al. \(2003\)](#)). In the above we have used $\Delta = g^{ij} \nabla_i \nabla_j$ to denote the Laplace-Beltrami operator and $A^o = A - \frac{1}{2}gH$, where g is the metric induced by \hat{f} , to denote the tracefree part of the second fundamental form A . For more details on our notation we refer the reader to Section 2.

We emphasise that s_0 and v_0 in (1.2) are abstract mathematical constants; they have no physical meaning. Their role is to ensure that the restrictions (P1.1) are satisfied by \hat{f} . They do not represent any physical force in the original formulation.

It is possible to derive an expression similar to (1.2) for the shape of a biomembrane where constants with a possible physical meaning appear in a manner identical to s_0 and v_0 . This can be achieved via the inclusion of the osmotic pressure difference p and tensile stress λ in the expression for the free energy of a closed bilayer. This slightly different approach has by now become quite common – see for example [Black \(2013\)](#); [Tu \(2006\)](#) and [Voinova \(2006\)](#). Indeed, [Deuling and Helfrich \(1976a\)](#) argued that the Lagrange multipliers s_0 and v_0 above essentially play these roles. This leads to the alternative functional

$$\begin{aligned} \mathcal{H}^{c_0}(f) &= \frac{k_c}{2} \int_{\Sigma} (H - c_0)^2 d\mu + \lambda \text{Area } \Sigma + p \text{Vol } \Sigma + 2\bar{k}\pi\chi(\Sigma) \\ &= \frac{k_c}{2} \int_{\Sigma} H^2 d\mu - k_c c_0 \int_{\Sigma} H d\mu + \left(\frac{k_c c_0^2}{2} + \lambda\right) \text{Area } \Sigma + p \text{Vol } \Sigma + 2\bar{k}\pi\chi(\Sigma). \end{aligned}$$

It is the functional \mathcal{H}^{c_0} above that we study in this paper. For clarity, we restate the minimisation problem associated to this functional below.

PROBLEM (P2) Suppose Σ is a closed differentiable 2-manifold with genus zero. Let c_0 , p and λ be fixed constants. Minimise $\mathcal{H}^{c_0}(f)$ in the class of smooth embeddings $f : \Sigma \rightarrow \mathbb{R}^3$. That is, find an embedding $f_0 : \Sigma \rightarrow \mathbb{R}^3$ such that

$$\mathcal{H}^{c_0}(f_0) \leq \mathcal{H}^{c_0}(f) \quad (1.3)$$

for any other smooth embedding f of Σ .

A solution to (P2) will satisfy the Euler-Lagrange equation (1.2) with tensile stress λ substituted for s_0 and the osmotic pressure difference p substituted for v_0 :

$$\mathbf{H}^{c_0}(f) := k_c(\Delta H + H|A^o|^2) + 2k_c c_0 K - \left(\frac{k_c c_0^2}{2} + \lambda\right)H - p = 0. \quad (1.4)$$

REMARK 1.2 In problems (P1) and (P2), the spontaneous curvature c_0 is stated to be a fixed constant. From a physical perspective, this is not the case, since it is known that the spontaneous curvature c_0 has the units of one of the principal curvatures of the membrane f . In particular, given a membrane $f : \Sigma \rightarrow \mathbb{R}^3$ with spontaneous curvature c_0 , a dilated membrane ρf has spontaneous curvature $\frac{c_0}{\rho}$. We may further assume that the spontaneous curvature is invariant under rigid motions in \mathbb{R}^3 , i.e. translations and rotations. Since it is agreed in the literature that due to the homogeneity of the membrane the spontaneous curvature does not vary based on position, we could assume that

$$c_0 = \int_{\Sigma} F[f, \vec{v}] d\mu$$

for an operator F and a vector of parametric functions $\vec{v}(p) = (\vec{v} \circ f)(p)$ where $\vec{v} = (v_1(x), \dots, v_M(x))$, $x \in \mathbb{R}^3$, $m \in \mathbb{N}$. The role of the vector \vec{v} would be to incorporate ambient information into the determination of the spontaneous curvature. A similar procedure was enacted in Wheeler (2015) for a model of strings in space influenced by ambient forces. The known behaviour of c_0 under dilation and rigid motions would translate to the operator F being invariant under rigid motions and homogeneous of degree -3 .

Unfortunately this appears to be the most that is known. It is not clear from the literature how exactly the spontaneous curvature depends upon the embedding f . This is important, since the existence of one or more solutions to problems (P1) and (P2) and the qualitative properties such solutions possess would depend critically on the structure of F . Discovering new properties and further information on the nature of F is an important open problem in the field.

REMARK 1.3 It is physically reasonable to require that the energy of a biomembrane not depend on scale – this amounts to the requirement that $\mathcal{H}^{c_0}(f) = \mathcal{H}^{c_0}(\rho f)$. Taking into account the scaling of the measure $d\mu$, the mean curvature H , and the volume, the units of c_0 , λ and p should be $H^2 d\mu$, $(d\mu)^{-1}$ and $(d\text{Vol})^{-1}$ respectively. In terms of ρ , this is ρ^{-1} , ρ^{-2} and ρ^{-3} . The parameters k_c and \bar{k} should be scale invariant, or, the behaviour of λ and p under scaling should incorporate information on how k_c and \bar{k} scale. A precise formulation of the Helfrich model taking into account such scale invariance does not yet appear to be available, although it is implicit in the Lagrange multipliers of (P1).

REMARK 1.4 From an analysis perspective, questions on existence and regularity of solutions to (1.4) must be investigated. For the Willmore functional, where $c_0 = \lambda = p = 0$, this is a

venerable topic. Bryant (1984) classified all closed solutions through a duality method. A landmark contribution in existence was made by Simon (1993). Remarkable progress on regularity issues was made by Rivière (2008), who decoupled (1.4) into two second order systems and studied weak solutions. Both Bernard and Rivière (2013) and Kuwert and Schätzle (2004) made important contributions to the understanding of point singularities. The Willmore conjecture, proposed by Willmore (1965), was recently resolved by Marques and Neves (2014). Work on the Willmore functional continues to be a very active area, with recent progress made on quantisation (Bernard and Riviere, 2014), the gradient flow (Kuwert and Schätzle, 2001, 2002), and boundary value problems (Alessandrini and Kuwert, 2014; Dall’Acqua, 2012; Dall’Acqua et al., 2013; Deckelnick and Grunau, 2009). There are many other works besides those mentioned here – the literature on analysis of the Willmore functional is vast. Work on intermediate functionals, both from a numerical and theoretical standpoint, has also been active – the workshop (Garcke et al., 2008) and articles (Abels et al., 2014; Barrett et al., 2008) are an excellent resource on this. For the full Helfrich functional, many of these issues remain open¹ and form important questions that future research should address.

Analysis of solutions to (1.2) is quite involved. Here our reduced focus allows us to pin down the influence of the spontaneous curvature c_0 on solutions to (1.4) for any value of c_0 . The classification theorem is as follows. It is proved in Section 3. Note that we use below and throughout the paper $S_r : \mathbb{S}^2 \rightarrow \mathbb{R}^3$ to denote the standard embedding of a sphere with (typically) unspecified centre.

Theorem 1.5 *Suppose $f : \Sigma \rightarrow \mathbb{R}^3$ is a closed, smooth, embedded orientable surface in the same topological class as a sphere. If $f(\Sigma) = S_r(\Sigma)$, and f is critical for the Helfrich functional \mathcal{H}^{c_0} , then one of the following must hold:*

- (a) For $c_0 = 0$:
 - (i) $c_0 = \lambda = p = 0$, in which case f may be a sphere of any radius;
 - (ii) $p\lambda < 0$, in which case f must be the unique critical sphere with radius $r = -\frac{2\lambda}{p}$;
- (b) If $c_0 \neq 0$, set $u := \frac{\lambda}{k_c c_0} + \frac{c_0}{2}$ and $v := \frac{2p}{k_c c_0}$. Then:
 - (iii) $v = -u^2$, $u > 0$, in which case f must be the unique critical sphere with radius $r = \frac{u}{2}$;
 - (iv) $u \geq 0$ and $v \geq 0$, in which case f must be the unique critical sphere with radius $r = \frac{1}{2}(u + \sqrt{u^2 + v})$;
 - (v) $u \geq 0$ and $v \in (-u^2, 0)$, in which f may be either of the two critical spheres with radii $r_{\pm} = \frac{1}{2}(u \pm \sqrt{u^2 + v})$;
 - (vi) $u < 0$ and $v > 0$, in which case f must be the unique critical sphere with radius $r = \frac{1}{2}(u + \sqrt{u^2 + v})$.

REMARK 1.6 (Experimental determination of parameters) While it is possible to obtain experimental values for c_0 for vesicles (as in the work of Evans and Fung (1972) mentioned earlier, see also Bassereau et al. (2014)), there currently exist no direct measurements of the spontaneous curvature of red blood cells. One may reasonably guess that $c_0 \neq 0$ as the distribution of the phospholipid types between the two leaflets of the bilayer is asymmetric (see Mohandas and Gallagher (2008)).

¹ A partial answer to the existence and regularity question can be found in Choksi and Veneroni (2013).

Although the spontaneous curvature should realistically depend on position $c_0 = c_0(x)$ where $x \in \Sigma$ (one possible choice of function is to use the mean curvature of a ‘resting shape’, see [Pozrikidis \(2005\)](#); [Krüger \(2012\)](#) for example), we take the view here that c_0 is constant. [Pozrikidis \(2005\)](#) suggests that an appropriate choice for the spontaneous curvature of normal red blood cells is $c_0 = -0.62 \mu\text{m}^{-1}$, very similar to that reported by [Deuling and Helfrich \(1976a,b\)](#), where a range of values that confirm observed experimental data is given, from $-0.56 \mu\text{m}^{-1}$ to $-1.94 \mu\text{m}^{-1}$. Later the value of $-0.74 \mu\text{m}^{-1}$ was settled upon.

It may be interesting to note that if one considers a sphere as the resting shape for red blood cells then these two approaches align, but this only works for positive spontaneous curvature. We have taken c_0 to be constant, allowing positive and negative values, for simplicity. Allowing c_0 to depend on position is an important topic for future work.

The bending modulus k_c can be measured experimentally, although reported values for healthy human red blood cells vary in the range $0.2 - 9.0 \times 10^{-19}$ J (see [Freund \(2014\)](#) and [Guido and Tomaiuolo \(2009\)](#) for example). It is reported in [Guckenberger and Gekle \(2017\)](#) that most simulations are being performed for k_c between 2 and 4×10^{-19} J.

In [Steigmann et al. \(2003\)](#) a similar model to ours here is studied. First, we must repeat that both λ and p should be considered as functions, rather than constants, and here we have treated them as such only for simplicity. The osmotic pressure difference p for a human red blood cell has been argued by [Deuling and Helfrich \(1976a,b\)](#) to be small when approaching spherocytes, but of indeterminate sign. When p is relatively large, discrete rotational symmetries in addition to reflection tend to appear in equilibria, whereas for negative p all discrete symmetries, including reflection symmetry, appear to be lost (see [Deuling and Helfrich \(1976a\)](#)). For the tensile stress λ [Deuling and Helfrich \(1976a\)](#) show that λ should be of the order $-p/k_c$, and since k_c is positive, this means that λ should have sign opposite to that of p . In general p and λ are expected to depend on the equilibrium configuration; it may not be reasonable to prescribe them a-priori.

Regarding the parameters p and λ , [Steigmann et al. \(2003\)](#) state “*Solution of the shape equation is complicated by the fact that the values of the parameters λ and p are not known at the outset.*” They continue to note that in practice, these parameters are tweaked after simulation until the desired constraints on area and volume are satisfied. Their dependence on area and volume is not clear, although, in some cases it is possible to identify when they penalise or reward area and volume growth or decay, see the discussion in [Capovilla et al. \(2002\)](#) on surface tension for example.

There exists a heuristic ‘rule’ that Lagrange multipliers are equal to the derivative of the functional under constraint with respect to the quantity being constrained. This would imply that λ is the derivative of $\int_{\Sigma} (H - c_0)^2 d\mu$, evaluated at extrema, with respect to Area Σ . Similarly, p would be the derivative of $\int_{\Sigma} (H - c_0)^2 d\mu$, evaluated at extrema, with respect to Vol Σ . However such a derivative is not well-defined, as almost nothing is known about the manifold of equilibria of the functional $\int_{\Sigma} (H - c_0)^2 d\mu$; furthermore it seems unlikely that such a derivative could ever be well-defined.

This reveals one core philosophical difference between (P1) and (P2): For (P1), the Lagrange multipliers may never enjoy a physical interpretation, whereas for (P2) the parameters λ and p remain open to interpretation, leaving hope that they may be physically relevant and measurable.

There is no consensus among the literature even on the values of the heavily studied bending modulus, or spontaneous curvature. One may view our results here, that focus on spherical critical points for the Helfrich functional \mathcal{H}^{c_0} , as further informing the discussion on realistic values for λ and p .

Bending forces in Helfrich’s spontaneous curvature model have been recently surveyed by [Guckenberger and Gekle \(2017\)](#), where one may find many further details and simulations in the above directions.

The question of when a critical point for the functional \mathcal{H}^{c_0} is a sphere is much more delicate. We present the following result, which is a straightforward consequence of the uniqueness of embedded CMC surfaces (the Hopf theorem) and the isoperimetric inequality.

Theorem 1.7 *Spheres are the unique global minimisers of the energy \mathcal{H}^{c_0} among closed surfaces in the same topological class as a sphere with volume fixed at $\text{Vol } \Sigma = \frac{32\pi}{3c_0^3}$, if c_0, λ, p satisfy:*

$$3\lambda + \frac{p}{c_0} = 0. \tag{1.5}$$

This result requires the very restrictive assumptions that the volume be fixed at a level that includes the sphere with radius $\frac{1}{c_0}$, and the parameters of the functional satisfy (1.5). Relaxing these conditions can be achieved by introducing a closeness assumption.

Such a closeness assumption has been used by [McCoy and Wheeler \(2013\)](#) in the case of zero spontaneous curvature. In [McCoy and Wheeler \(2013\)](#), the functional

$$\tilde{\mathcal{H}}_{\lambda_1, \lambda_2}^{c_0}(f) = \frac{1}{4} \int_{\Sigma} (H - c_0)^2 d\mu + \lambda_1 \text{Area } \Sigma + \lambda_2 \text{Vol } \Sigma$$

was studied. This is far from $\hat{\mathcal{H}}^{c_0}$ but is on the other hand quite close to the functional \mathcal{H}^{c_0} featuring in problem (P2), with many properties common to both $\hat{\mathcal{H}}^{c_0}$ and \mathcal{H}^{c_0} . The difference between the two is given by:

$$\mathcal{H}^{c_0}(f) - 2k_c \tilde{\mathcal{H}}_{\frac{\lambda}{2k_c}, \frac{p}{2k_c}}^{c_0}(f) = 2\bar{k}\pi\chi(\Sigma),$$

which is constant. For the case where the bilayer is topologically spherical we have $\chi(\Sigma) = 2$ and

$$\mathcal{H}^{c_0}(f) - (2k_c \tilde{\mathcal{H}}_{\frac{\lambda}{2k_c}, \frac{p}{2k_c}}^{c_0}(f) + 4\bar{k}\pi) = 0. \tag{1.6}$$

That is, the functionals \mathcal{H}^{c_0} and $\tilde{\mathcal{H}}^{c_0}$, up to taking special choices of the parameters λ_1 and λ_2 , differ by a constant. The variational properties of these functionals are therefore equivalent; only the numerical energy of shapes is altered. The following theorem is known:

Theorem 1.8 (Theorem 1 in [McCoy and Wheeler \(2013\)](#)) *Let $f : \Sigma \rightarrow \mathbb{R}^3$ be a closed, smooth, embedded orientable surface in the same topological class as a sphere. Suppose additionally that*

$$\int_{\Sigma} |A^{\sigma}|^2 d\mu < \varepsilon_1 \quad \text{and} \quad \lambda \geq 0,$$

where ε_1 is an explicit universal constant. Then, if f is critical for the Helfrich functional \mathcal{H}^0 , it is a standard round sphere.

In this paper, we are concerned primarily with non-zero spontaneous curvature. In addition to Theorem 1.5 and Theorem 1.7 above we give the following, which extends the main theorem of [McCoy and Wheeler \(2013\)](#) for closed surfaces to the case of non-vanishing spontaneous curvature.

Theorem 1.9 Let $f : \Sigma \rightarrow \mathbb{R}^3$ be a closed, smooth, embedded orientable surface. Assume that f is a Helfrich surface, that is,

$$\mathbf{H}^{c_0}(f) = 0. \quad (1.7)$$

There exist universal constants c_1, c_2, c_3, c_4 such that if

$$\int_{\Sigma} |A^o|^2 d\mu \leq \min \left\{ \frac{1}{2c_1c_3}, \frac{1}{2c_0^2c_2(\text{Area } \Sigma)} \right\},$$

$$c_0^2 + 2\frac{\lambda}{k_c} = 2c_0u \geq 0,$$

and

$$c_0^2(\text{Area } \Sigma) < \frac{1}{c_4},$$

then $f(\Sigma) = S_r(\Sigma)$ is a standard round sphere.

REMARK 1.10 The quantity $Q = c_0^2(\text{Area } \Sigma)$ identified in the proof is dimensionless, as c_0 has the units of the mean curvature, and $\text{Area } \Sigma$ has the units of $d\mu$. Therefore $c_0^2(\text{Area } \Sigma)$ has the units of $H^2 d\mu$, and is scale-invariant.

REMARK 1.11 The constants c_1, \dots, c_4 are universal, but the area of Σ is not. One corollary that interprets Theorem 1.9 is:

Corollary 1.12 Let $A_0 \in (0, \infty)$. Consider the class

$$\mathfrak{F} = \{f : \Sigma \rightarrow \mathbb{R}^3 : f \text{ is a smooth closed embedded orientable surface with } \text{Area } \Sigma < A_0\}.$$

Suppose $c_0^2 + 2\frac{\lambda}{k_c} = 2c_0u \geq 0$, and $c_0^2A_0 < \frac{1}{c_4}$. There exists a constant $\delta > 0$ depending only on A_0 such that any Helfrich surface $f \in \mathfrak{F}$ with $\|A^o\|_2 < \delta$ is a standard round sphere.

REMARK 1.13 Theorem 1.9 holds for a range of the parameters c_0, λ and p . This range is larger than the range that parameters that spheres are critical for, see Theorem 1.5 for a full classification. For parameters outside this range, we therefore have a *reverse energy gap* phenomenon:

Corollary 1.14 Assume the hypotheses of Corollary 1.12 and in addition that the parameters c_0, λ , and p satisfy one of

- (i) $c_0 = 0$ and $p\lambda \geq 0$ with one of p, λ non-zero;
- (ii) $c_0 \geq 0, u \geq 0$ and $v < -u^2$;
- (iii) $c_0 \leq 0, u \leq 0$ and $v \leq 0$.

There exists an absolute constant $C > 0$ depending only on A_0 such that any Helfrich surface satisfies

$$\int_{\Sigma} |A^o|^2 d\mu > C.$$

The above results indicate *rigidity* of the sphere. The question of *stability* of the spherocytes is also important, since, in a patient with spherocytosis, the spherocytes do not regularly become singular; they are instead stable and nominally functional, despite being regularly destroyed by the spleen. This behaviour is not typical for the Helfrich model; indeed, we expect that generically

perturbed spherocytes revert after a perturbation (if they are ever formed at all), to a standard discocyte shape. In general, after a perturbation acts upon a biomembrane, there is no guarantee that the bilayer will return to a global minimum. It may focus instead on a stable local minimum, or become singular.

In order to illustrate the general setting, here is a result for the model case where $c_0 = \lambda = p = 0$ and we are dealing with the Willmore functional and closed Willmore surfaces. We say that a smooth isometrically embedded surface $f : \Sigma \rightarrow \mathbb{R}^3$ is weakly mean convex if $H(x) \geq 0$ for all $x \in \Sigma$, and mean convex if $H(x) > 0$ for all $x \in \Sigma$. If $K \geq 0$ then weak mean convexity becomes weak convexity and mean convexity becomes convexity.²

Proposition 1.15 (Mean convex closed Willmore surfaces) *Consider a smoothly embedded closed weakly mean convex orientable surface $f : \Sigma \rightarrow \mathbb{R}^3$. Suppose that $c_0 = \lambda = p = 0$. Then if f is critical for the Helfrich functional \mathcal{H}^{c_0} , it must be a sphere.*

Proposition 1.15 yields a stability statement in the following sense. Let $S_r : \Sigma \rightarrow \mathbb{R}^3$ be a standard sphere centred at the origin with radius r . Then it is critical for the Helfrich functional with $c_0 = \lambda = p = 0$. Consider, for some smooth function $\psi : \Sigma \rightarrow \mathbb{R}$, the perturbed surface $\eta : \Sigma \rightarrow \mathbb{R}^3$, $\eta(x) = S_r(x) + v(x)\psi(x)$, where v is a smooth choice of outward-pointing normal vector. Note that we assume a-priori that the perturbed map η is an isometric embedding, which in turn restricts the function ψ .

Let us impose further that the perturbed surface η is again critical for the Helfrich functional with the given parameters. We ask ourselves the question:

Question. Under which conditions will the perturbed surface η be a sphere?

All such perturbations are termed *mild*. Note that all mild perturbations of standard round spheres have $\psi(x) = \psi_0$ where $\psi_0 \in (-r, \infty)$. Proposition 1.15 informs us that, in this case, any perturbation which leaves η at least mean convex is *mild*.

Section 5 contains some initial stability analysis for spherocytes in the Helfrich model. We summarise these results as follows.

Theorem 1.16 *Let $S_r : \Sigma \rightarrow \mathbb{R}^3$ be the standard embedding of a sphere with radius r . Consider a perturbed surface $\eta : \Sigma \rightarrow \mathbb{R}^3$, $\eta(x) = S_r(x) + v(x)\psi(x)$, with $\psi : \Sigma \rightarrow \mathbb{R}$ a smooth function. Assume that η is critical for the Helfrich functional \mathcal{H}^{c_0} .*

The perturbation ψ is mild (that is, η is a sphere) in the following cases:

- (i) ($c_0 = 0$) Any perturbation ψ such that η is weakly mean convex and λ, p are such that the average of the mean curvature over the perturbed surface η is equal to $-p/\lambda$;
- (ii) ($c_0 \geq 0, \lambda \leq -k_c \frac{c_0^2}{2}, p \leq 0$) Any perturbation ψ such that η is weakly convex;
- (iii) ($c_0 \geq 0, \lambda \leq -k_c \frac{c_0^2}{2}, p \leq -k_c c_0 a_0^2$) Any perturbation ψ such that η is weakly mean convex and on the perturbed surface the inequality $|A^o|^2(x) \leq a_0^2$, for some $a_0 \in (0, \infty)$, holds.
- (iv) ($c_0 \geq 0, p \leq -k_c (c_0 a_0^2 + \frac{1}{2c_0} (\frac{c_0^2}{2} + \frac{\lambda}{k_c})^2)$) Any perturbation ψ such that on the perturbed surface the inequality $|A^o|^2(x) \leq a_0^2$, for some $a_0 \in (0, \infty)$, holds.

² Note that in higher dimensions this is typically expressed by saying that the second fundamental form is positive semi-definite or positive definite. The given definition is simpler and agrees with this for surfaces.

REMARK 1.17 (Mild perturbations and [Zhong-Can and Helfrich \(1987\)](#)) The above notion of stability, via classification of mild perturbations, is not a-priori closely related to the classical notion of positivity of the second variation of energy. However, earlier studies can be cast in some cases into this framework, as we briefly now explain.

In [Zhong-Can and Helfrich \(1987\)](#) it is shown that given a sufficiently large pressure p , spherocytes may be deformed into one of a family of surfaces associated with l -th order spherical harmonics.

In particular, these critical shapes are not standard round spheres. However, some of them do appear to be convex (and also mean convex), see Figure 6 in [Deuling and Helfrich \(1976a\)](#). Further, they have for sufficiently large pressure lower Helfrich energy than a sphere. This gives us an interesting non-existence result for mild perturbations of spheres.

In the language we use here, this reads:

Theorem ([Zhong-Can and Helfrich \(1987\)](#)) *Let $S_r : \Sigma \rightarrow \mathbb{R}^3$ be the standard embedding of a sphere with radius r . Assume that the pressure p is sufficiently large. There exist smooth perturbations $\psi : \Sigma \rightarrow \mathbb{R}$ with the following properties:*

- *The perturbed surface $\eta : \Sigma \rightarrow \mathbb{R}^3$, $\eta(x) = S_r(x) + v(x)\psi(x)$ is critical for the Helfrich functional \mathcal{H}^{c_0} ;*
- *The perturbed surface η is not congruent to any round sphere, and so the perturbation ψ is **not mild**;*
- *The perturbed surface is convex.*

In particular, this shows that Theorem 1.16 could not hold with large pressure. It is interesting to note that only alternative (i) allows for p to be positive, so long as λ is negative and the average curvature condition is satisfied.

We further note that [Pleiner \(1990\)](#) studied perturbations of the shapes identified by [Zhong-Can and Helfrich \(1987\)](#) for $l = 2$, performing a nonlinear stability analysis. Bifurcations were found at various levels of pressure. Bifurcations were also studied by [Peterson \(1985\)](#). [Safran \(1991\)](#) examined the role played by \bar{k} in the stability of spheres.

Theorem 1.16 highlights the crucial role that convexity plays in the analysis of stability. We observe that weak convexity (that each of the principal curvatures are non-negative at every point) is a much stronger condition than weak mean convexity (that the sum of the principal curvatures is non-negative at every point). Since the value of a_0 may be quite large depending upon the osmotic pressure difference p , it may be that in practice perturbations of spherocytes fall into or near to categories (iii) and (iv) of Theorem 1.16.

Due to this focus on convexity and mean convexity, we suspect that many of the solutions to (P2) are mean convex, irrespective of any stability concerns, that is, without using any knowledge about the solution being a-priori ‘close’ to a sphere in some sense. This is partially confirmed in the last result of our paper:

Theorem 1.18 *Suppose that $c_0 > 0$, $|A^o|^2 \leq a_0^2$, $\lambda \geq k_c(a_0^2 - \frac{c_0^2}{2})$, and $p < -c_0 k_c a_0^2$. Consider a smoothly embedded closed weakly mean convex orientable surface $f : \Sigma \rightarrow \mathbb{R}^3$. Then if f is critical for the Helfrich functional \mathcal{H}^{c_0} , it must be strictly mean convex.*

The paper is organised as follows. Section 2 contains some brief mathematical background required for the calculations and proofs in the later sections. Section 3 contains the proof of

Theorem 1.5 as well as some discussion on minimisers. Section 4 is concerned with rigidity, and contains the proof of Theorem 1.7 and Theorem 1.9. Section 5 is concerned with stability analysis and the proof of Theorem 1.16 and Theorem 1.18.

2. Mathematical background

Let us briefly set notation and describe the mathematical setting in which we work. We are interested in properties of a human red blood cell realised as an infinitely thin two-dimensional shell. Mathematically we represent that as the image of a map $f : \Sigma \rightarrow \mathbb{R}^3$ with the following properties:

- Σ is a smooth closed orientable differentiable manifold of dimension two;
- f is a smooth map with injective first derivative;
- f is a homeomorphism onto its image.

An example of a smooth closed differentiable manifold of dimension two is the sphere \mathbb{S}^2 . Since we are primarily interested in the analysis of possibly spherical red blood cells, this is the primary example to keep in mind. For the second dot point, this is enough to imply that f is an *immersion*, which implies that the tensor g_{ij} with components $(\partial$ here denotes the standard partial derivative)

$$g_{ij} = \langle \partial_i f, \partial_j f \rangle$$

is a Riemannian metric. It is the induced or pullback metric, and sometimes written as $g = f^*\delta$ where δ is the Euclidean metric (the identity matrix). This means that the pair (Σ, g) is a Riemannian manifold, and f is then called an *isometric immersion*. If bullet point three holds, then $f(\Sigma)$ does not have any self-intersections, the map f is injective or one-to-one, and f is upgraded to an *isometric embedding*. The red blood cell as we see it under the microscope is not (Σ, g) , but the image $f(\Sigma)$.

All geometric data can be derived from the map f . At each point p there is a tangent space $T_p M$ and a normal space $N_p M$. Since the codimension of f is one, the normal space is always a line. We choose a canonical global normal vector field ν pointing out from the interior of $f(\Sigma)$.

The curvature is encoded in the second fundamental form, with components A_{ij} given by

$$A_{ij} = \langle \partial_i f, \partial_j \nu \rangle.$$

The Weingarten equation tells us that

$$\partial_i \nu = A_{il} g^{lm} \partial_m f = A_i^m \partial_m f.$$

Coordinate independent curvature quantities that arise in the paper include the mean curvature:

$$H = g^{ij} A_{ij} = \kappa_1 + \kappa_2,$$

(in the above repeated indices are summed over, $g^{ij} = (g^{-1})_{ij}$, and κ_1, κ_2 are the eigenvalues of $A_j^i = g^{ik} A_{kj}$), the Gauss curvature

$$K = \det A_j^i = \kappa_1 \kappa_2,$$

the square of the second fundamental form:

$$|A|^2 = \kappa_1^2 + \kappa_2^2,$$

and the square of the tracefree second fundamental form:

$$|A^o|^2 = \frac{1}{2}(\kappa_1 - \kappa_2)^2.$$

The mean curvature, the unit normal, and the Laplace-Beltrami operator are further related by

$$\Delta f = -Hv = \vec{H}.$$

The rightmost expression is called the *mean curvature vector*.

In Section 3 we use some facts about the standard sphere of radius $r > 0$ embedded in \mathbb{R}^3 . Let us denote by S_r an embedding $S_r : \mathbb{S}^2 \rightarrow \mathbb{R}^3$ that yields a standard round sphere of radius $r > 0$ centred at any point $c \in \mathbb{R}^3$. The image is given by

$$S_r(\mathbb{S}^2) = \{x \in \mathbb{R}^3 : |x - c| = r\},$$

where $|y| = |(y_1, y_2, y_3)| = \sqrt{y_1^2 + y_2^2 + y_3^2}$ denotes the standard length of vectors in \mathbb{R}^3 . An exterior unit normal vectorfield to S_r is given by ($x = f(p)$)

$$v(x) = \frac{x - c}{|x - c|} = \frac{x - c}{r} = \frac{f(p) - c}{r}.$$

From the Weingarten equation we find

$$H = g^{ij} A_{ij} = g_j^i A_i^j = g^{ij} A_i^m g_{mj} = g^{ij} \langle \partial_i v, \partial_j f \rangle = \operatorname{div} v,$$

and so for the sphere of radius r ,

$$H = g^{ij} \langle \partial_i f, \partial_j f \rangle \frac{1}{r} = g^{ij} g_{ij} \frac{1}{r} = \frac{1}{r} \operatorname{Trace} I_2 = \frac{2}{r}.$$

We note that this calculation is invariant under translation: the centre point c of the sphere does not play any role in the curvature. Since the sphere with centre translated back to the origin is $SO(3)$ -invariant, each of the principal curvatures κ_1 and κ_2 are equal and so the above implies $\kappa_1 = \kappa_2 = \frac{1}{r}$. In particular, we have

$$K = \frac{1}{r^2} \quad \text{and} \quad |A^o|^2 = 0.$$

The last condition is necessary and sufficient: if f is a sphere, then $|A^o|^2 = 0$, and if $|A^o|^2 = 0$ and f is closed, then f is a sphere (see Proposition 8.2.9 in Pressley (2010)). This leads one naturally to consider the range of non-zero values that $|A^o|^2$ may take to signify a kind of ‘distance’ from being spherical.

In Section 3 we also use some elementary facts about the calculus of variations and the existence of minimisers. In particular, there the following fact is used.

Lemma 2.1 *Suppose that the functional $\hat{\mathcal{H}}^{c_0}$ (or \mathcal{H}^{c_0}) is unbounded from below for a given choice of parameters c_0, S_0, V_0 (or c_0, λ, p) in the class of smooth embeddings. Then there does not exist a solution to (P1) (or (P2)).*

The proof is straightforward and standard.

Proof. By hypothesis, there exists a sequence $\{f_i\}_{i \in \mathbb{N}}$ of smooth embedded orientable surfaces such that the energy $E(f_i) \rightarrow -\infty$ as $i \rightarrow \infty$. Here we have used E to denote either $\mathcal{H}^{\hat{c}_0}$ or \mathcal{H}^{c_0} . If a solution f_0 to either problem were to exist, then the minimisation condition (1.1) (or (1.3)) would be satisfied. However since f_0 is smooth, it has finite energy, and so

$$E(f_0) > E(f_j)$$

for some j sufficiently large. This is a contradiction. \square

REMARK 2.2 The hypothesis that the functional be unbounded from below only has to hold for one particular sequence.

In Section 5 we need the following elementary result from differential geometry.

Lemma 2.3 *Suppose $f : \Sigma \rightarrow \mathbb{R}^3$ is a smooth embedded orientable surface containing the origin. Then there exists at least one point where the mean curvature of f is strictly positive.*

Proof. Consider the function $|f|^2$. Since Σ is closed, $|f|^2$ achieves a global maximum on Σ . At this point the Hessian of $|f|^2$ is non-positive. We compute

$$\text{Hess}_{ij}|f|^2 = \nabla_i \nabla_j |f|^2 = 2g_{ij} + \langle f, \nabla_i \nabla_j f \rangle.$$

Tracing the above with g , at a maximum we have

$$0 \geq \Delta |f|^2 = 4 - \langle f, \nu \rangle H. \quad (2.1)$$

Since ν is an outward-pointing unit normal and the origin is contained in the interior of $f(\Sigma)$, at a global maximum of $|f|^2$ we have $\langle f, \nu \rangle = c > 0$. Therefore we conclude from (2.1) that

$$H \geq \frac{4}{\langle f, \nu \rangle}$$

at a global maximum of $|f|^2$. \square

3. Spherical solutions

Let $S_r : \mathbb{S}^2 \rightarrow \mathbb{R}^3$ be the embedding of the standard sphere with radius r and centre at the origin. Recalling the basic properties of spheres explained in Section 2, the Euler-Lagrange equation (1.4) evaluated at S_r is quadratic in $1/r$:

$$2k_c c_0 r^{-2} - \left(\frac{k_c c_0^2}{2} + \lambda \right) 2r^{-1} - p = 0. \quad (3.1)$$

If $c_0 \neq 0$ (we deal with the case $c_0 = 0$ in Case 0 below) then (3.1) is equivalent to

$$r^{-2} - \left(\frac{c_0}{2} + \frac{\lambda}{k_c c_0} \right) r^{-1} - \frac{p}{2k_c c_0} = 0. \quad (3.2)$$

We set r_{\pm} to be the roots, if they exist, of this quadratic; that is

$$r_{\pm} = \frac{1}{2} \left(\frac{\lambda}{k_c c_0} + \frac{c_0}{2} \pm \sqrt{\left(\frac{c_0}{2} + \frac{\lambda}{k_c c_0} \right)^2 + \frac{2p}{k_c c_0}} \right). \quad (3.3)$$

Given that S_r exists only for $r > 0$, there is at least one spherical critical point of the Helfrich energy if

$$\frac{\lambda}{k_c c_0} + \frac{c_0}{2} + \sqrt{\left(\frac{c_0}{2} + \frac{\lambda}{k_c c_0}\right)^2 + \frac{2p}{k_c c_0}} > 0. \tag{3.4}$$

Let us set

$$u = \frac{\lambda}{k_c c_0} + \frac{c_0}{2}, \quad \text{and} \quad v = \frac{2p}{k_c c_0}.$$

For any spherical solution to exist, the argument of the square root in Equation (3.4) must be non-negative, that is,

$$v \geq -u^2. \tag{3.5}$$

We separate now into three cases:

Case 0 : $c_0 = 0$

Case 1 : $u = -u^2$

Case 2 : $u > -u^2$.

3.1 Case 0: $c_0 = 0$

In this case the problem greatly simplifies: Formula (3.1) reads

$$2\lambda r^{-1} + p = 0.$$

If $\lambda = p = 0$ then we are dealing with the degenerate case of the Willmore functional and any sphere S_r of any radius is a critical point (in fact a minimiser, see Willmore (1965)).

If $\lambda = 0$ and $p \neq 0$ then there does not exist a spherical solution. Otherwise, we find

$$r = -\frac{2\lambda}{p}.$$

For the right hand side to be positive, we require λ and p to be of opposite signs. If this is the case the sphere $S_{-2\lambda/p}$ is the (only) spherical solution.

We summarise this in the following lemma.

Lemma 3.1 (Resolution of Case 0) *Suppose $f : \Sigma \rightarrow \mathbb{R}^3$ is a closed, smooth, embedded orientable surface in the same topological class as a sphere. Suppose $c_0 = 0$. If $f(\Sigma) = S_r(\Sigma)$ and f is critical for the Helfrich functional \mathcal{H}^{c_0} , then one of the following must hold:*

- (i) $\lambda = p = 0$, in which case any $r \in (0, \infty)$ is possible; or
- (ii) $p\lambda < 0$, in which case $r = -\frac{2\lambda}{p}$.

REMARK 3.2 (Minimisers I) If we work under the additional assumption that f is in fact a minimiser, then we may refine the parameters above and in later lemmata throughout this section. In particular, the functional reads

$$\begin{aligned} \mathcal{H}^{c_0}(S_r) &= \frac{k_c}{2} \int_{\Sigma} (H - c_0)^2 d\mu + \lambda \text{Area } \Sigma + p \text{Vol } \Sigma + 2\bar{k}\pi\chi(\Sigma) \\ &= 4k_c\pi - 8k_c c_0\pi r + 4\left(\lambda + \frac{k_c c_0^2}{2}\right)\pi r^2 + \frac{4p\pi}{3}r^3 + 4\bar{k}\pi. \end{aligned}$$

If $p < 0$, then $\mathcal{H}^{c_0}(S_r) \rightarrow -\infty$ as $r \rightarrow \infty$. In light of Lemma 2.1 this implies that no minimiser exists, spherical or otherwise. Adding the hypothesis that f solves (P2) and thus is a minimiser would remove the possibility that $p < 0$ in alternative (ii), leaving only $\lambda < 0$. It removes the possibility that $p < 0$ in later lemmata as well.

One must note that this argument succeeds in producing anything simply due to the lack of scaling in our interpretation of the Helfrich model. As noted earlier, c_0 , λ and p should be functions of the configuration $f : \Sigma \rightarrow \mathbb{R}^3$, and as such should depend on scale. Therefore one should not take this remark and the others (Minimisers II – IV) to be anything more than purely mathematical observations.

3.2 Case 1: $v = -u^2$

In this case the problem again greatly simplifies: The square root in (3.3) drops out and so $r_+ = r_-$. Further, inequality (3.4) simplifies to $u > 0$. Then by (3.3) we have $r = \frac{u}{2}$.

The lemma for this case is as follows.

Lemma 3.3 (Resolution of Case 1) *Suppose $f : \Sigma \rightarrow \mathbb{R}^3$ is a closed, smooth, embedded orientable surface in the same topological class as a sphere. If $f(\Sigma) = S_r(\Sigma)$, $v = -u^2$, and f is critical for the Helfrich functional \mathcal{H}^{c_0} , then $u > 0$ and*

$$r = \frac{u}{2} = \sqrt{\frac{-p}{2k_c c_0}}.$$

REMARK 3.4 (Minimisers II) If f is a minimiser then by the previous remark $p \geq 0$. Then since $v < 0$ in this case we must have $c_0 < 0$. Also, as $u > 0$ this implies $\lambda < 0$. That is, the only remaining possibility that a sphere is minimising here requires $p \geq 0$, $c_0 < 0$ and $\lambda < 0$.

3.3 Case 2: $v > -u^2$

In this case inequality (3.5) must be satisfied for us to have at least one spherical solution. That is,

$$u + \sqrt{u^2 + v} > 0.$$

We further separate into the two subcases where $u \geq 0$ and $u < 0$.

3.3.1 Case 2.1: $u \geq 0$. In this case we automatically have $r_+ > 0$ and so one critical sphere exists. For a second to exist we require by (3.3)

$$\begin{aligned} u - \sqrt{u^2 + v} &> 0, \text{ or} \\ u^2 &> u^2 + v, \text{ since } u > 0 \end{aligned}$$

which implies that $v < 0$, or $v \in (-u^2, 0)$.

3.3.2 *Case 2.2: $u < 0$.* In this case we have at most one critical sphere. Since $u = -|u|$, for this single sphere to exist we require

$$\begin{aligned} u + \sqrt{u^2 + v} &> 0, \text{ or} \\ -|u| &> -\sqrt{u^2 + v}, \text{ which implies} \\ v &> 0. \end{aligned}$$

We summarise the results of Case 2 in the following lemma.

Lemma 3.5 *Suppose $f : \Sigma \rightarrow \mathbb{R}^3$ is a closed, smooth, embedded orientable surface in the same topological class as a sphere. If $f(\Sigma) = S_r(\Sigma)$, $v > -u^2$, and f is critical for the Helfrich functional \mathcal{H}^{c_0} , then one of the following must hold:*

- (i) $u \geq 0$ and $v \geq 0$, in which case f must be the unique critical sphere with radius $r = \frac{1}{2}(u + \sqrt{u^2 + v})$; or
- (ii) $u \geq 0$ and $v \in (-u^2, 0)$, in which f may be either of the two critical spheres with radii $r_{\pm} = \frac{1}{2}(u \pm \sqrt{u^2 + v})$; or
- (iii) $u < 0$ and $v > 0$, in which case f must be the unique critical sphere with radius $r = \frac{1}{2}(u + \sqrt{u^2 + v})$.

REMARK 3.6 (Minimisers III) Supposing that f is a minimiser rules out $p < 0$ as before. In (ii) it is dramatic: $v < 0$ and so we must have $c_0 < 0$. As $u \geq 0$, this implies $\lambda < 0$. For (i) it is not as useful since $v \geq 0$ implies $c_0 \geq 0$, but this doesn't combine with $u \geq 0$ to yield a sign condition on λ ; it only implies $\lambda \geq -\frac{k_c c_0^2}{2}$. For (iii) we do obtain a sign restriction, since $v > 0$ implies $c_0 > 0$, and so $u < 0$ implies $\lambda < 0$.

Proof of Theorem 1.5. Combine Lemmata 3.1–3.5. □

Although the theorem details many possible circumstances under which spherical critical surfaces exist, there is a three-dimensional family of parameters and so one should interpret this as roughly stating that spherical biomembranes may only occur in very special situations.

We expect that several of the spherocytes identified by Theorem 1.5 have high energy and are unstable. For a more rigorous interpretation of this intuition see the remarks after Lemmata 3.1–3.5.

Therefore we do not expect to observe these spherocytes often in live experiments. Some work in identifying stability properties is presented in Sections 4 and 5. In these sections, we see that deformations which preserve a notion of convexity, or do not alter the magnitude of the difference of the principal curvatures in too great a manner, are mild. The work in Section 4 shows that there exist deformations of spherocytes that do not preserve convexity that are also mild. (This is because a non-convex embedded sphere may nevertheless satisfy $\|A^o\|_2^2 < \varepsilon$.)

One exception to this generic instability appears to be the case when $c_0 > 0$ and $r = \frac{2}{c_0}$, so that $H = c_0$ and the curvature integral in \mathcal{H}^{c_0} on spheres with radius r vanishes. This case was earlier identified in Peterson (1989) as having energy independent of c_0 . It is in fact quite special and spheres are fundamental with those parameters: spheres are global minimisers of the energy and thus have the best stability possible.

Theorem 3.7 *Suppose $c_0 > 0$ and $\lambda = p = 0$. Then the unique global minimal solution to (P2) in the class of smooth embedded orientable surfaces is a sphere of radius $\frac{2}{c_0}$.*

Proof. For these choices of c_0 , λ and p the functional reads

$$\mathfrak{H}^{c_0}(f) = \frac{k_c}{2} \int_{\Sigma} (H - c_0)^2 d\mu + 2\bar{k}\pi\chi(\Sigma) = \frac{k_c}{2} \int_{\Sigma} (H - c_0)^2 d\mu + 4\bar{k}\pi. \quad (3.6)$$

The Euler–Lagrange equation is

$$k_c(\Delta H + H|A^o|^2) + 2k_c c_0 K - \frac{k_c c_0^2}{2} H = 0. \quad (3.7)$$

The sphere with $r = \frac{2}{c_0}$ has $H = c_0$ and $K = \frac{c_0^2}{4}$. Further $|A^o| = 0$ and $\Delta H = 0$ on any sphere, so $f = S_r : \Sigma \rightarrow \mathbb{R}^3$ solves (3.7).

This proves that S_r is a critical point for the functional in (3.6). Minimality is easy to see, since the functional is the sum of a non-negative integral and a constant. The sphere S_r has $H = c_0$ and so the integral in (3.6) takes on its lowest possible value: zero.

Let us also prove that up to translation and rotation it is unique. If $f : \Sigma \rightarrow \mathbb{R}^3$ is any smooth embedded orientable surface, then

$$\frac{k_c}{2} \int_{\Sigma} (H - c_0)^2 d\mu = 0 \implies H = c_0.$$

A classical theorem of [Aleksandrov \(1962\)](#) tells us that any embedded orientable surface with constant mean curvature must be a sphere, with radius $r = \frac{2}{H}$, which is exactly what we wanted. \square

In the next section below we extend this reasoning using additionally the isoperimetric inequality to allow some cases where λ and p do not vanish.

4. Rigidity

4.1 Proof of Theorem 1.7

Let $f : \Sigma \rightarrow \mathbb{R}^3$ be a closed smooth surface in the same topological class as a sphere. We combine two famous results on the uniqueness of the standard round sphere. The first is the Hopf theorem. This states that:

$$\text{If } f \text{ is an embedded closed CMC surface, then it must be a standard round sphere.} \quad (4.1)$$

The second is the isoperimetric theorem. It states that:

$$(\text{Area } \Sigma)^3 \geq 36\pi(\text{Vol } \Sigma)^2$$

or

$$\text{Area } \Sigma \geq (36\pi)^{\frac{1}{3}}(\text{Vol } \Sigma)^{\frac{2}{3}} \quad (4.2)$$

with equality if and only if f is a standard round sphere.

Fix $\text{Vol } \Sigma = \frac{32\pi}{3c_0^3}$. For $f \in \mathcal{F}$, using the above and condition (1.5), we estimate the energy by

$$\begin{aligned} (\mathcal{H}^{c_0}(f) - 4\bar{k}\pi) &= \frac{k_c}{2} \int_{\Sigma} (H - c_0)^2 d\mu + \lambda \text{Area } \Sigma + p \text{Vol } \Sigma \\ &\geq \left[\lambda(36\pi)^{\frac{1}{3}} + p \left(\frac{4\pi}{3c_0^3} \right)^{\frac{1}{3}} \right] \text{Vol } \Sigma^{\frac{2}{3}} \\ &\geq \left[3\lambda + \frac{p}{c_0} \right] \left(\frac{4\pi}{3} \right)^{\frac{1}{3}} \text{Vol } \Sigma^{\frac{2}{3}} \\ &\geq 0 \end{aligned}$$

with equality only when f is a standard round sphere with the given prescribed volume, that is, any sphere with radius $\frac{2}{c_0}$.

4.2 Proof of Theorem 1.9

Note. In this section the constant $c \in \mathbb{R}$ may change from line to line, referring to an absolute constant with each usage.

We begin by writing Corollary 5 in McCoy and Wheeler (2013) in our notation for closed surfaces and with the choice $\gamma \equiv 1$:

Lemma 4.1 *Let $f : \Sigma \rightarrow \mathbb{R}^3$ be a closed, smooth, embedded orientable surface. Then*

$$\begin{aligned} \int_{\Sigma} (|\nabla_{(2)} A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^\circ|^2) d\mu + \left(c_0^2 + 2 \frac{\lambda}{k_c} \right) \int_{\Sigma} |\nabla H|^2 d\mu \\ \leq c \int_{\Sigma} \mathbf{H}^{c_0}(f) \cdot \Delta H d\mu + c_1 \int_{\Sigma} (|A^\circ|^6 + |A^\circ|^2 |\nabla A^\circ|^2) d\mu \\ + c c_0 \int_{\Sigma} (\Delta H) |A^\circ|^2 d\mu - c \frac{c_0}{2} \int_{\Sigma} (\Delta H) H^2 d\mu, \end{aligned}$$

where c and c_1 are absolute constants.

Now we work to estimate the additional terms involving the spontaneous curvature on the right hand side.

Lemma 4.2 *Let $f : \Sigma \rightarrow \mathbb{R}^3$ be a closed, smooth, embedded orientable surface. Then*

$$\begin{aligned} c c_0 \int_{\Sigma} (\Delta H) |A^\circ|^2 d\mu \leq \frac{1}{2} \int_{\Sigma} |\nabla_{(2)} A|^2 d\mu \\ + c_2 c_0^2 \cdot (\text{Area } \Sigma) \|A^\circ\|_2^2 \int_{\Sigma} |\nabla_{(2)} A|^2 + |A|^4 |A^\circ|^2 d\mu. \end{aligned}$$

where c and c_2 are absolute constants.

Proof. Since $H = g^{ij} A_{ij}$ and $\Delta = g^{ij} \nabla_i \nabla_j$, we can use the Cauchy–Schwarz inequality to see that

$$|\Delta H| \leq |\nabla_{(2)} A|, \quad (4.3)$$

where on the left we have the absolute value of the scalar function ΔH and on the right we have the induced norm of the $(0, 4)$ -tensor field with components $\nabla_i \nabla_j A_{kl}$.

Using the inequality $2ab \leq a^2 + b^2$ we find

$$c c_0 \int_{\Sigma} (\Delta H) |A^o|^2 d\mu \leq \frac{1}{2} \int_{\Sigma} |\nabla_{(2)} A|^2 d\mu + \frac{1}{2} c c_0^2 \int_{\Sigma} |A^o|^4 d\mu. \quad (4.4)$$

Applying the Michael–Simon Sobolev inequality (see [Michael and Simon \(1973\)](#)) we find

$$\int_{\Sigma} |A^o|^4 d\mu \leq c \left(\int_{\Sigma} |\nabla A^o| |A^o| + |H| |A^o|^2 d\mu \right)^2.$$

To each term on the right hand side we use the Hölder inequality, yielding

$$\int_{\Sigma} |A^o|^4 d\mu \leq c \|A^o\|_2^2 \int_{\Sigma} |\nabla A^o|^2 d\mu + c \|A^o\|_2^2 \int_{\Sigma} |H|^2 |A^o|^2 d\mu. \quad (4.5)$$

For the first term, we estimate

$$c \|A^o\|_2^2 \int_{\Sigma} |\nabla A^o|^2 d\mu \leq \frac{1}{4 \text{Area } \Sigma} \|A^o\|_2^4 + c \cdot (\text{Area } \Sigma) \|\nabla A^o\|_2^4. \quad (4.6)$$

The divergence theorem and closedness implies

$$\|\nabla A^o\|_2^2 = -\langle A^o, \Delta A^o \rangle_{L^2} \leq \|A^o\|_2 \|\Delta A^o\|.$$

Since $A^o = A - \frac{1}{2}gH$, estimate (4.3) implies $|\Delta A^o| \leq 2|\nabla_{(2)} A|$. Hölder's inequality implies that $\|A^o\|_2^4 \leq (\text{Area } \Sigma) \|A^o\|_4^4$. Combining these with (4.6), we find

$$c \|A^o\|_2^2 \int_{\Sigma} |\nabla A^o|^2 d\mu \leq \frac{1}{4} \|A^o\|_4^4 + c \cdot (\text{Area } \Sigma) \|A^o\|_2^2 \|\nabla_{(2)} A\|_2^2. \quad (4.7)$$

Now for the second term on the right hand side of (4.5), we estimate similarly

$$\begin{aligned} c \|A^o\|_2^2 \int_{\Sigma} |H|^2 |A^o|^2 d\mu &\leq \frac{1}{4} \|A^o\|_4^4 + c \cdot (\text{Area } \Sigma) \left(\int_{\Sigma} |H|^2 |A^o|^2 d\mu \right)^2 \\ &\leq \frac{1}{4} \|A^o\|_4^4 + c \cdot (\text{Area } \Sigma) \|A^o\|_2^2 \int_{\Sigma} |A|^4 |A^o|^2 d\mu. \end{aligned} \quad (4.8)$$

Now we combine (4.7), (4.8) with (4.5) and absorb to conclude

$$\int_{\Sigma} |A^o|^4 d\mu \leq c \cdot (\text{Area } \Sigma) \|A^o\|_2^2 \|\nabla_{(2)} A\|_2^2 + c \cdot (\text{Area } \Sigma) \|A^o\|_2^2 \int_{\Sigma} |A|^4 |A^o|^2 d\mu. \quad (4.9)$$

Together with the estimate (4.4) this finishes the proof. \square

Proposition 4.3 *Let $f : \Sigma \rightarrow \mathbb{R}^3$ be a closed, smooth, embedded orientable surface. There exist universal constants c_1, c_2, c_3, c_4 such that if*

$$\int_{\Sigma} |A^o|^2 d\mu \leq \min \left\{ \frac{1}{2c_1 c_3}, \frac{1}{2c_0^2 c_2 (\text{Area } \Sigma)} \right\} := \varepsilon_2 \quad (4.10)$$

then

$$\begin{aligned} & \int_{\Sigma} (|\nabla_{(2)}A|^2 + |A|^2|\nabla A|^2 + |A|^4|A^o|^2) d\mu + \left(c_0^2 + 2\frac{\lambda}{k_c}\right) \int_{\Sigma} |\nabla H|^2 d\mu \\ & \leq c \int_{\Sigma} \mathbf{H}^{c_0}(f) \cdot \Delta H d\mu + c_4 c_0^2 (\text{Area } \Sigma) \int_{\Sigma} |\nabla_{(2)}A|^2 + |A|^2|\nabla A|^2 d\mu, \end{aligned}$$

where c is an absolute constant.

Proof. We note the following consequence of the Michael–Simon Sobolev inequality, first proven in Lemma 2.5 of [Kuwert and Schätzle \(2002\)](#):

$$\int_{\Sigma} (|\nabla A^o|^2 |A^o|^2 + |A^o|^6) \gamma^4 d\mu \leq c_3 \|A^o\|_2^2 \int_{\Sigma} |\nabla_{(2)}A|^2 + |\nabla A|^2 |A|^2 + |A|^4 |A^o|^2 d\mu. \quad (4.11)$$

By assuming that $\varepsilon_2 \leq \frac{1}{2c_1 c_3}$ we may absorb this term from the right hand side of Lemma 4.1 into the left and multiply through to obtain

$$\begin{aligned} & \int_{\Sigma} (|\nabla_{(2)}A|^2 + |A|^2|\nabla A|^2 + |A|^4|A^o|^2) d\mu + \left(c_0^2 + 2\frac{\lambda}{k_c}\right) \int_{\Sigma} |\nabla H|^2 d\mu \\ & \leq c \int_{\Sigma} \mathbf{H}^{c_0}(f) \cdot \Delta H d\mu + c c_0 \int_{\Sigma} (\Delta H) |A^o|^2 d\mu - c \frac{c_0}{2} \int_{\Sigma} (\Delta H) H^2 d\mu. \end{aligned}$$

Second, by assuming that $\varepsilon_2 \leq \min\{(2c_1 c_3)^{-1}, (2c_0^2 c_2 (\text{Area } \Sigma))^{-1}\}$ and absorbing with the help of Lemma 4.2, we find

$$\begin{aligned} & \int_{\Sigma} (|\nabla_{(2)}A|^2 + |A|^2|\nabla A|^2 + |A|^4|A^o|^2) d\mu + \left(c_0^2 + 2\frac{\lambda}{k_c}\right) \int_{\Sigma} |\nabla H|^2 d\mu \\ & \leq c \int_{\Sigma} \mathbf{H}^{c_0}(f) \cdot \Delta H d\mu - c \frac{c_0}{2} \int_{\Sigma} (\Delta H) H^2 d\mu. \end{aligned}$$

For the last remaining term we integrate by parts and estimate to obtain

$$-c \frac{c_0}{2} \int_{\Sigma} (\Delta H) H^2 d\mu = c c_0 \int_{\Sigma} H |\nabla H|^2 d\mu \leq \frac{1}{2} \int_{\Sigma} |\nabla A|^2 |A|^2 d\mu + c c_0^2 \int_{\Sigma} |\nabla H|^2 d\mu. \quad (4.12)$$

Now the identity (6) from [McCoy and Wheeler \(2013\)](#) implies

$$c c_0^2 \int_{\Sigma} |\nabla H|^2 d\mu \leq 4c c_0^2 \int_{\Sigma} |\nabla^* A^o|^2 d\mu \leq 4c c_0^2 \int_{\Sigma} |\nabla A^o|^2 d\mu.$$

Using the Michael–Simon Sobolev inequality we find

$$\begin{aligned} 4c c_0^2 \int_{\Sigma} |\nabla A^o|^2 d\mu & \leq c c_0^2 \left(\int_{\Sigma} |\nabla_{(2)}A| + |H| |\nabla A| d\mu \right)^2 \\ & \leq c c_0^2 (\text{Area } \Sigma) \int_{\Sigma} |\nabla_{(2)}A|^2 + |A|^2 |\nabla A|^2 d\mu. \end{aligned} \quad (4.13)$$

Absorbing with (4.12), and bounding the extra term as in (4.13), our main estimate reads

$$\begin{aligned} & \int_{\Sigma} (|\nabla_{(2)}A|^2 + |A|^2|\nabla A|^2 + |A|^4|A^o|^2) d\mu + \left(c_0^2 + 2\frac{\lambda}{k_c}\right) \int_{\Sigma} |\nabla H|^2 d\mu \\ & \leq c \int_{\Sigma} \mathbf{H}^{c_0}(f) \cdot \Delta H d\mu + c_4 c_0^2 (\text{Area } \Sigma) \int_{\Sigma} |\nabla_{(2)}A|^2 + |A|^2 |\nabla A|^2 d\mu, \end{aligned} \quad (4.14)$$

where c_4 is an absolute constant. The estimate (4.14) is the statement of the Proposition. \square

In Proposition 4.3 we note the quantity $c_0^2(\text{Area } \Sigma)$; this is dimensionless, as c_0 has the units of the mean curvature, and $\text{Area } \Sigma$ has the units of $d\mu$. Therefore $c_0^2(\text{Area } \Sigma)$ has the units of $H^2 d\mu$, and is scale-invariant.

Corollary 4.4 *Let $f : \Sigma \rightarrow \mathbb{R}^3$ be a closed, smooth, embedded orientable surface. Assume that f is a Helfrich surface, that is,*

$$\mathbf{H}^{c_0}(f) = 0.$$

There exist universal constants c_1, c_2, c_3, c_4 such that if (4.10) holds, $c_0^2 + 2\frac{\lambda}{k_c} \geq 0$, and

$$c_0^2(\text{Area } \Sigma) < \frac{1}{c_4} \quad (4.15)$$

then $f(\Sigma) = S_r(\Sigma)$.

Proof. The smallness condition (1.7) allows us to absorb the second term in (4.14) into the left hand side. This yields

$$\begin{aligned} & \int_{\Sigma} (|\nabla_{(2)}A|^2 + |A|^2|\nabla A|^2 + |A|^4|A^o|^2) d\mu + \left(c_0^2 + 2\frac{\lambda}{k_c}\right) \int_{\Sigma} |\nabla H|^2 d\mu \\ & \leq c \int_{\Sigma} \mathbf{H}^{c_0}(f) \cdot \Delta H d\mu. \end{aligned}$$

The condition $c_0^2 + 2\frac{\lambda}{k_c} \geq 0$, and (1.7) upgrade this to

$$\int_{\Sigma} (|\nabla_{(2)}A|^2 + |A|^2|\nabla A|^2 + |A|^4|A^o|^2) d\mu \leq 0.$$

With smoothness of f , this implies $|A^o| = 0$, and we are done. \square

5. Local stability

We begin with a proof of Proposition 1.15 from the introduction.

Proof of Proposition 1.15. In this case we have

$$\Delta H + H|A^o|^2 = 0.$$

Integrating and using the divergence theorem we have

$$\int_{\Sigma} \Delta H + H|A^o|^2 d\mu = \int_{\Sigma} H|A^o|^2 d\mu = 0.$$

Since the surface is smooth, the mean curvature H and the norm squared of the trace-free second fundamental form $|A^o|^2$ are smooth. Due to the mean convexity hypothesis the only possibility is that $|A^o|^2(x) = 0$ wherever $H(x) > 0$. By Lemma 2.3 there is at least one x_0 where $H(x_0) > 0$. Therefore $|A^o|^2(x_0) = 0$. Now recall the definition of A^o :

$$A_{ij}^o = A_{ij} - \frac{1}{2}g_{ij}H.$$

The Codazzi equation tells us that the tensor $\nabla_i A_{jk}$ is totally symmetric. Therefore, using the summation convention and the definition $H = g^{ij} A_{ij}$

$$\nabla^k A_{kj}^o = \nabla^k A_{kj} - \frac{1}{2}\nabla_j H = \frac{1}{2}\nabla_j H. \tag{5.1}$$

By smoothness, $H(x) > 0$ in a neighbourhood of x_0 . Denote the maximal such neighbourhood by Ω . Suppose that $\partial\Omega \neq \emptyset$. Note that $H(x) = 0$ for all $x \in \partial\Omega$. Now the above argument shows that $|A^o|^2(x) = 0$ on Ω , and so by (5.1) above we have

$$\nabla_j H = \nabla^k A_{kj}^o = 0 \quad \text{on } \Omega$$

for any j . Therefore $H(x) = H(x_0)$ is constant on Ω . However, $H(x) = 0$ on $\partial\Omega$, and therefore H must be discontinuous on $\partial\Omega$. This is a contradiction. Therefore $\partial\Omega = \emptyset$, $\Omega = \Sigma$, and H is constant on all of Σ . Since $f : \Sigma \rightarrow \mathbb{R}^3$ is an embedding, the theorem of Aleksandrov (1962) applies and we conclude that f is a sphere. \square

Proposition 1.15 yields a stability statement in the following sense. Let $S_r : \Sigma \rightarrow \mathbb{R}^3$ be a standard sphere with radius r . Then it solves the problem (P2) with $c_0 = \lambda = p = 0$. Consider, for some smooth function $\psi : \Sigma \rightarrow \mathbb{R}$, the perturbed surface $\eta : \Sigma \rightarrow \mathbb{R}^3$, $\eta(x) = S_r(x) + v(x)\psi(x)$, where v is a smooth choice of normal vector. Let us impose that the perturbed surface η also solves the problem (P2) with the given parameters. We ask ourselves the question: under which conditions on the perturbation ψ will η be a sphere? All such perturbations are termed *mild*. Proposition 1.15 informs us that, in this case, any perturbation which leaves η at least mean convex is *mild*.

As witnessed in Section 3, variations on the parameters λ , p and c_0 induce wild changes in the behaviour of the solutions to (P2). In general, we do not expect spherical solutions to be stable. For certain ranges of these parameters, we are nevertheless able to obtain a result analogous to that of Proposition 1.15.

Proposition 5.1 *Consider a smoothly embedded closed weakly mean convex surface $f : \Sigma \rightarrow \mathbb{R}^3$. Suppose that $c_0 = 0$ and that p, λ are such that the average of the mean curvature of f is equal to $-p/\lambda$. Then if f is critical for the Helfrich functional \mathcal{H}^{c_0} , it must be a sphere.*

Proof. In this case we have

$$\Delta H + \left(|A^o|^2 - \frac{\lambda}{k_c}\right)H - \frac{p}{k_c} = 0.$$

Rearranging, this implies

$$\Delta H + H|A^o|^2 = \frac{\lambda}{k_c}H + \frac{p}{k_c}.$$

Observe that the integral of the right hand side vanishes:

$$\int_{\Sigma} \left(\frac{\lambda}{k_c} H + \frac{p}{k_c} \right) d\mu = k_c^{-1} \left[\lambda \int_{\Sigma} H d\mu + p |\Sigma| \right] = k_c^{-1} \left[\lambda |\Sigma| \frac{-p}{\lambda} + p |\Sigma| \right] = 0.$$

Therefore the proof of Proposition 1.15 goes through analogously in this case. \square

It is possible to extend this integral method in various directions to obtain results specific for narrow choices of the parameters λ , p and c_0 . What we wish to do now is to illustrate a different method that appears more suitable to the case where $c_0 \neq 0$. It has the drawback of requiring either weak convexity (as opposed to weak *mean* convexity above) or a condition on $|A^o|^2$. The method relies on the following standard tool. The statement below is a corollary of the more general theorem proved in Calabi (1958) (see also Hopf (1927)).

Theorem 5.2 (Calabi–Hopf Maximum Principle) *Suppose (Σ, g) is a Riemannian manifold. Consider $u : U \rightarrow \mathbb{R}$ a smooth function defined over the open set $U \subset \Sigma$. If*

$$(\Delta u)(x) \leq 0$$

everywhere in U , and if u attains a local minimum value at some point in U , then u is identically constant in U .

The theorem allows the following pair of corollaries.

Corollary 5.3 *Consider a smoothly embedded closed weakly convex surface $f : \Sigma \rightarrow \mathbb{R}^3$. Suppose that $c_0 \geq 0$, $p \leq 0$, and $\lambda \leq -k_c \frac{c_0^2}{2}$. Then if f is critical for the Helfrich functional \mathcal{H}^{c_0} , it must be a sphere.*

Proof. In this case we have

$$\begin{aligned} \Delta H &= - \left(|A^o|^2 - \frac{c_0^2}{2} - \frac{\lambda}{k_c} \right) H - 2c_0 K + \frac{p}{k_c} \\ &\leq \left(\frac{c_0^2}{2} + \frac{\lambda}{k_c} \right) H - 2c_0 K + \frac{p}{k_c}. \end{aligned}$$

The weak convexity hypothesis means that at every point the principal curvatures κ_1, κ_2 are non-negative. This implies that $-2c_0 K := -2c_0 \kappa_1 \kappa_2 \leq 0$. The mean curvature is also clearly non-negative, and the conditions on c_0, λ, p imply that the entire right hand side is non-positive.

Therefore we have

$$(\Delta H)(x) \leq 0$$

at every point $x \in \Sigma$. Note that Σ is an open set inside Σ , and since it is additionally compact and H is a smooth function on Σ , it must achieve its minimum at some point in Σ . Theorem 5.2 applies, yielding H identically constant, and again Aleksandrov (1962) shows that f must be a standard embedding of the sphere $S_r : \Sigma \rightarrow \mathbb{R}^3$. \square

Corollary 5.4 *Consider a smoothly embedded closed weakly mean convex surface $f : \Sigma \rightarrow \mathbb{R}^3$. Let $a_0 > 0$ be such that*

$$|A^o|^2(x) \leq a_0^2$$

for every $x \in \Sigma$. Suppose that $c_0 \geq 0$, $p \leq -k_c c_0 a_0^2$, and $\lambda \leq -k_c \frac{c_0^2}{2}$. Then if f is critical for the Helfrich functional \mathcal{H}^{c_0} , it must be a sphere.

Proof. Similarly to the proof above, we have

$$\Delta H \leq \left(\frac{c_0^2}{2} + \frac{\lambda}{k_c} \right) H - 2c_0 K + \frac{p}{k_c}. \quad (5.2)$$

The general strategy has not changed – our goal remains to show that under the given hypotheses, the right hand side of the above differential inequality is non-positive. Recall that, in terms of the principal curvatures κ_1, κ_2 , we have:

$$|A|^2 = \kappa_1^2 + \kappa_2^2; \quad H^2 = (\kappa_1 + \kappa_2)^2; \quad |A^o|^2 = \frac{1}{2}(\kappa_1 - \kappa_2)^2; \quad K = \kappa_1 \kappa_2,$$

so that $K = \frac{1}{4}H^2 - \frac{1}{2}|A^o|^2$. Therefore

$$-2c_0 K + \frac{p}{k_c} = -\frac{1}{2}c_0 H^2 + \left(\frac{p}{k_c} + c_0 |A^o|^2 \right) \leq 0,$$

by hypothesis.

The weak mean convexity hypothesis means that $H \geq 0$, and so we have again that the conditions on c_0, λ, p imply that the entire right hand side of (5.2) is non-positive. The proof now continues analogously to that of Corollary 5.3. \square

If p is sufficiently negative, then one may use this maximum principle idea to remove all geometric assumptions apart from embeddedness. We present a final variation of the idea above incorporating this observation.

Theorem 5.5 *Consider a smoothly embedded closed surface $f : \Sigma \rightarrow \mathbb{R}^3$. Let $a_0 > 0$ be such that*

$$|A^o|^2(x) \leq a_0^2$$

for every $x \in \Sigma$. Suppose that $c_0 \geq 0$, and

$$p \leq -k_c \left(c_0 a_0^2 + \frac{1}{2c_0} \left(\frac{c_0^2}{2} + \frac{\lambda}{k_c} \right)^2 \right).$$

Then if f is critical for the Helfrich functional \mathcal{H}^{c_0} , it must be a sphere.

Proof. Using the calculations in the proof above, we factorise

$$\Delta H \leq - \left(\frac{\sqrt{c_0}}{\sqrt{2}} H + \frac{1}{\sqrt{2c_0}} \left(\frac{c_0^2}{2} + \frac{\lambda}{k_c} \right) \right)^2 + \frac{1}{2c_0} \left(\frac{c_0^2}{2} + \frac{\lambda}{k_c} \right)^2 + c_0 a_0^2 + \frac{p}{k_c}.$$

The hypothesis implies that the right hand side is non-positive. The proof now continues analogously to that of Corollary 5.3. \square

Combining these four results gives alternatives (i) through to (iv) of Theorem 1.16.

REMARK 5.6 (Minimisers IV) Proposition 1.15 is for Willmore surfaces, and alternatives (ii)–(iv) require $p < 0$. This means that although a certain class of perturbations are mild, there must be other perturbations that decrease the energy.

Therefore from the perspective of minimisers, the most interesting alternative is (i), in the particular case where $p > 0$ and $\lambda < 0$.

Clearly one may tweak the geometric and parametric conditions that allow the two strategies outlined in the above stability results to go through. It appears that some form of convexity is critical to the argument, and so it is interesting to determine conditions on the parameters under which convexity automatically holds. We finish this section with a demonstration of how one may deduce such a result.

Proof of Theorem 1.16. Rearranging the Euler–Lagrange equation (1.4) we find

$$\Delta H + \left(|A^o|^2 - \frac{c_0^2}{2} - \frac{\lambda}{k_c} \right) H + 2c_0K - \frac{p}{k_c} = 0$$

Suppose a global minimum for H occurs at x . Setting $H = H(x)$, we have

$$\left(|A^o|^2 - \frac{c_0^2}{2} - \frac{\lambda}{k_c} \right) H \leq \frac{p}{k_c} - 2c_0K.$$

Using $2K = \frac{1}{2}H^2 - |A^o|^2$ as in the proof of Corollary 5.4, at x we have

$$\left(|A^o|^2 - \frac{c_0^2}{2} - \frac{\lambda}{k_c} \right) H \leq \frac{p}{k_c} - \frac{c_0}{2}H^2 + c_0|A^o|^2.$$

Setting $a = \frac{c_0}{2}$, $b = |A^o|^2 - \frac{c_0^2}{2} - \frac{\lambda}{k_c}$, and $c = -\frac{p}{k_c} - c_0|A^o|^2$ the above is

$$P(H) = aH^2 + bH + c \leq 0. \tag{5.3}$$

In H , P is a parabola with zero, one, or two real roots.

Zero roots. In this case either P is always positive or always negative. Since $c_0 > 0$, the term aH^2 dominates for large enough H and P will be positive there. This contradicts (5.3), which holds for $H = H(x)$.

One root. In this case P is always non-positive or always non-negative. As with the case above, $c_0 > 0$ implies that for large enough H , P is positive. Therefore H is the unique point where P touches the axis, that is H solves

$$P'(H) = 0 \iff H = -\frac{b}{2a}.$$

By hypothesis $|A^o|^2 \leq a_0^2$, and $\lambda \geq k_c(a_0^2 - \frac{c_0^2}{2})$, so

$$b = |A^o|^2 - \frac{c_0^2}{2} - \frac{\lambda}{k_c} \leq a_0^2 - \frac{c_0^2}{2} - \frac{k_c(a_0^2 - \frac{c_0^2}{2})}{k_c} = 0, \tag{5.4}$$

and thus H is non-negative. Further, $H = 0$ only if $b = 0$, in which case $-4ac = 0$, since $b^2 - 4ac = 0$ when we have only one root. As $a = \frac{c_0}{2} > 0$, this implies that $c = 0$. However our hypothesis $p < -c_0k_c a_0^2$ implies

$$c \geq -\frac{p}{k_c} - c_0a_0^2 > 0, \tag{5.5}$$

a contradiction. Therefore H is strictly positive.

Two roots. In this case P changes sign. There are two roots H_1 and H_2 given by the quadratic formula. We may assume that $H_1 < H_2$. Since $P(H) \leq 0$ and $a > 0$, H will lie in the interval $[H_1, H_2]$. The lower bound H_1 satisfies

$$H_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The computation (5.4) shows that b is non-positive, and (5.5) shows that c is strictly positive. Therefore

$$\begin{aligned} & ac > 0 \\ \implies & b^2 > b^2 - 4ac \\ \implies & -b > \sqrt{b^2 - 4ac}. \end{aligned}$$

Therefore $H_1 > 0$.

We conclude that the mean curvature in each case is strictly positive, and so the minimum of the mean curvature in Σ is strictly positive, as required. \square

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