Interfaces and Free Boundaries **20** (2018), 165–210
DOI 10.4171/IFB/400

Bubbles and droplets in a singular limit of the FitzHugh–Nagumo system

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[Received 17 November 2016 and in revised form 30 January 2018]

The FitzHugh–Nagumo system gives rise to a geometric variational problem, when its parameters take values in a particular range. A stationary set of the variational problem satisfies an Euler–Lagrange equation that involves the curvature of the boundary of the set and a nonlocal term that inhibits unlimited growth and spreading. The nonlocal term is the solution of a non-homogeneous modified Helmholtz equation with the characteristic function of the stationary set as the inhomogeneous term. Two types of stationary sets are studied: disc shaped stationary sets in the plane, termed bubbles, and unions of perturbed small discs, termed droplet assemblies, in bounded domains. A complete description of the existence and the stability of bubbles is established. Depending on the parameters, there may be zero, one, two, or even three bubbles. Droplet assemblies are constructed by a reduction argument. Each droplet in a stationary assembly is close in size and shape to the corresponding bubble in the plane. The locations of the droplets in the assembly are determined by the Green's function of the modified Helmholtz equation.

2010 Mathematics Subject Classification: Primary 49J40, 33C10; Secondary 92C15, 35K57.

Keywords: FitzHugh–Nagumo system, standing pulse, singular limit, nonlocal geometric variational problem, bubbles, droplet assemblies.

1. Introduction

Reaction-diffusion systems serve as models for studying complex patterns in several fields of sciences [20, 24]. Many patterns emerge from homogeneous media that are destabilized by a spatial modulation; for instance stripe and spot patterns frequently appear in chemical reactions. These patterns are robust in the sense that they are stable and exist for a wide range of parameters; see Turing [41]. The existence of localized structures such as fronts and pulses has been established [5, 6, 10, 28] . Depending on the system parameters and the initial conditions, such localized structures may stay at rest or propagate with a dynamically stabilized velocity.

Typically localized pulses result from a nonlinear balance between dissipation and diffusion. In the events of small diffusion coefficient coupled with bistable reaction terms, these pulses can exhibit sharp transitions so that in the limit they form regions with discontinuous jumps. Two types of stable standing pulses will be investigated in this paper: round discs and assemblies of small, approximately round discs.

The FitzHugh–Nagumo system is a reaction-diffusion system that exhibits such phenomena [21, 30]. It is of the activator-inhibitor type and can be written as

$$u_t = D_1 \Delta u - f(u) - v, \tag{1.1}$$

$$\tau v_t = D_2 \Delta v + \delta u - \gamma v, \tag{1.2}$$

where all the parameters are positive and $f(u) = u(u-\beta)(u-1)$ with $0 < \beta < 1/2$. The diffusion coefficients of the respective species, D_1 and D_2, are of different scales in studying pattern formations. The existence of standing and traveling fronts and pulses for (1.1)–(1.2) in suitable parameter ranges on infinite lines or cylindrical domains was obtained by Chen and Choi [5, 6], Chen et al. [7], Chen, Kung and Morita [10], and Chen and Tanaka [11]. Stability of standing pulse solutions was justified by the index theory (Chen and Hu [8, 9]). A standing pulse solution can be regarded as a dissipative soliton, if it is stable against small perturbations and such pulse solutions act as basic module for more complex structures.

Some early results on the stationary FitzHugh–Nagumo system can be found in Klaasen and Troy [26], Klaasen and Mitidieri [25], de Figueiredo and Mitidieri [18], and Reinecke and Sweers [33, 34]. More recent years have seen studies of steady states of (1.1)–(1.2) where the component u has one or more peaks on the interior or the boundary of D; see Dancer and Yan [15–17], Wei and Winter [42], and Ren and Wei [37]. In these recent papers the diffusion coefficients are chosen as $D_1 = \epsilon^2$ and $D_2 = 1$, and ϵ must be small.

Some of these papers deal with spike solutions, also called spot solutions. If w is the positive radially symmetric solution of $\Delta w - w(w-\beta)(w-1) = 0$ on $\mathbb{R}^N$ with $\lim_{|y|\to\infty} w(y) = 0$, then for a spike solution (u, v), the component u has the form $u(x) \approx \sum_{k=1}^{K} w(\frac{x-\xi_k}{\epsilon})$ with the ξ_k's being the locations of the spikes. For instance, it was proved in [42] that a spike solution exists if the ξ_k's are carefully placed close to each other with pairwise distance being of the order $\epsilon \log \frac{1}{\epsilon}$.

For a spike solution, if $\epsilon \to 0$, a spike shrinks to its center point ξ_k, a set of zero Lebesgue measure in $\mathbb{R}^N$. We are interested in a different type of solutions to the FitzHugh–Nagumo system which as $\epsilon \to 0$, concentrate on sets of positive measure. Bubbles and droplet assemblies are such limiting sets studied in this paper. Spike solutions are unstable with respect to a natural energy functional, (1.13), of the FitzHugh–Nagumo system, essentially because the radially symmetric w mentioned above is unstable with respect to the PDE that w satisfies. On the contrary, the limiting structures that we study here can be stable. This allows the possibility of stable solutions corresponding to small ϵ in the FitzHugh–Nagumo systems.

The aim of this paper is to study a singular limit of the FitzHugh–Nagumo system on $\mathbb{R}^2$ or on a bounded domain D in $\mathbb{R}^2$. It is a geometric variational problem that contains a nonlocal interaction term. This term functions as an inhibitor that prevents unlimited growth or spreading. There are two parameters in the problem: $\alpha > 0$ and $\sigma > 0$. The energy functional takes the form

$$J_D(\Omega) = P_D(\Omega) - \alpha|\Omega| + \frac{\sigma}{2}\int_\Omega N_D(\Omega)\,dx. \tag{1.3}$$

In (1.3) Ω is a measurable subset of D and $|\Omega|$ is its Lebesgue measure, so the admissible set of

J_D is

$$\mathcal{A} = \{\Omega \subset D : \ \Omega \text{ is Lebesgue measurable, } |\Omega| < \infty\}. \tag{1.4}$$

Denote the perimeter of Ω in D by $P_D(\Omega)$. In the case that Ω is of class C^1, $P_D(\Omega)$ is the length of the part of the boundary of Ω that is inside D; namely the length of $\partial\Omega \cap D$. One calls $\partial\Omega \cap D$ the interface of Ω because it separates Ω from $D\backslash\Omega$. For a general subset Ω of D,

$$P_D(\Omega) = \sup\left\{\int_\Omega \operatorname{div} g(x)\, dx : \ g \in C_0^1(D, \mathbb{R}^n), \ |g(x)| \leqslant 1 \ \forall x \in D\right\}. \tag{1.5}$$

In (1.5), $|g(x)|$ is the geometric norm of the vector $g(x)$.

The integral term in (1.3) is most novel and from there the nonlocality of the problem comes. In this term N_D is an operator that assigns each Ω the solution of the following non-homogeneous modified Helmholtz equation:

$$-\Delta N_D(\Omega) + N_D(\Omega) = \chi_\Omega \text{ in } D; \ \ \partial_\nu N_D(\Omega) = 0 \text{ on } \partial D. \tag{1.6}$$

Here ∂_ν is the outward normal derivative. In the case $D = \mathbb{R}^2$,

$$N_{\mathbb{R}^2}(\Omega)(x) = \int_\Omega \frac{1}{2\pi} K_0(|x-y|)\, dy \tag{1.7}$$

where K_0 is the zero-order modified Bessel function of the second kind; see [3, page 598, table 9.5]. If D is bounded, then with some smoothness condition on D (say $C^{2,\alpha}$ according to [22, section 6.7]) N_D is well defined.

A classical stationary set of J_D has a C^2 interface and satisfies the Euler–Lagrange equation

$$K(\partial\Omega \cap D) - \alpha + \sigma N_D(\Omega) = 0 \text{ on } \partial\Omega \cap D \tag{1.8}$$

where K denotes the curvature, by the first variation formula (4.7) or (5.22). For the scenarios studied in this paper, the boundary of Ω does not touch that of D so that $\partial\Omega \cap D = \partial\Omega$.

We now discuss the connection between (1.3) and the FitzHugh–Nagumo system (1.1)–(1.2). Since in the function $f(u) = u(u-\beta)(u-1)$, $\beta \in (0, 1/2)$, there exists $A > 0$ such that $\int_{r_L}^{r_R} (f(u) + A)\, du = 0$ where r_L and r_R are respectively the smalleset and the largest of the three zeros of $f(u) + A$. Introduce $\bar{u}$ so that $u = B\bar{u} - C$ where $B = r_R - r_L > 0$ and $C = -r_L > 0$. Then $f(u) + A = E\bar{u}\big(\bar{u} - \frac{1}{2}\big)(\bar{u} - 1)$ for some suitable constant $E > 0$. With $\bar{u}$ and v as variables, steady states of (1.1)–(1.2) satisfy

$$D_1 B \Delta\bar{u} - E\bar{u}\Big(\bar{u} - \frac{1}{2}\Big)(\bar{u} - 1) + A - v = 0; \ \ D_2 \Delta v + \delta(B\bar{u} - C) - \gamma v = 0.$$

Next introduce a new space variable $\bar{x}$ so that $x\sqrt{\frac{\gamma}{D_2}} = \bar{x}$. Taking Laplacians with respect to $\bar{x}$ changes D_1 to $\frac{\gamma D_1}{D_2}$ and D_2 to γ. If we set $\bar{v} = \frac{\gamma}{\delta B}(v + \frac{\delta C}{\gamma})$, then the above equations become

$$\frac{\gamma D_1 B}{E D_2} \Delta\bar{u} - \bar{u}\Big(\bar{u} - \frac{1}{2}\Big)(\bar{u} - 1) + \frac{A + \frac{\delta C}{\gamma}}{E} - \frac{\delta B}{E\gamma}\bar{v} = 0; \ \ \Delta\bar{v} - \bar{v} + \bar{u} = 0.$$

Dropping the bars over u and v, we arrive at the system

$$-\epsilon^2 \Delta u + u\Big(u - \frac{1}{2}\Big)(u - 1) - \epsilon\alpha + \epsilon\sigma v = 0 \tag{1.9}$$

$$-\Delta v + v = u \tag{1.10}$$

with the zero Neumann boundary condition for both u and v. The new parameters here are related to the earlier ones by

$$\epsilon^2 = \frac{\gamma D_1 B}{E D_2}, \ \epsilon\alpha = \frac{A + \frac{\delta C}{\gamma}}{E}, \ \epsilon\sigma = \frac{\delta B}{E\gamma}. \tag{1.11}$$

In particular both α and σ must be positive. The same type of scaling on ϵ also appears in other models; for instance, the article [23] and the references therein.

One solves (1.10) for v in terms of u so that $v = N_D u$, where $N_D u$ is the solution of (1.6) with χ_Ω replaced by u. Upon substitution (1.9) becomes

$$-\epsilon^2 \Delta u + u\Big(u - \frac{1}{2}\Big)(u-1) - \epsilon\alpha + \epsilon\sigma N_D u = 0 \ \text{ in } D; \ \ \partial_\nu u = 0 \ \text{ on } \partial D. \tag{1.12}$$

The equation (1.12) has a variational structure. It can be viewed as the equation for critical points of the functional

$$I_{D,\epsilon}(u) = \int_D \Big(\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{u^2(u-1)^2}{4} - \epsilon\alpha u + \frac{\epsilon\sigma u}{2} N_D u\Big)\, dx. \tag{1.13}$$

When D is bounded, this functional has a Γ-limit. More precisely, as $\epsilon \to 0$, $\epsilon^{-1} I_{D,\epsilon}$ Γ-converges to the functional

$$J_D(\Omega) = \tau P_D(\Omega) - \alpha|\Omega| + \frac{\sigma}{2}\int_\Omega N_D(\Omega)\, dx; \tag{1.14}$$

see [36, Proposition 2.1]. The functional in (1.14) is the same as the one in (1.3) except for an immaterial constant τ, which is given by

$$\tau = \int_0^1 \sqrt{\frac{u^2(u-1)^2}{2}}\, du = \frac{\sqrt{2}}{12}. \tag{1.15}$$

One can factor out τ in (1.14) by redefining α and σ. Hence J_D in (1.14) is equivalent to (1.3). Because of the Γ-convergence J_D is considered a singular limit of the FitzHugh–Nagumo system.

Several properties follow once the Γ-convergence of $\epsilon^{-1} I_{D,\epsilon}$ to J_D is established. A global minimizer of $I_{D,\epsilon}$ converges to a global minimizer of J_D, when $\epsilon \to 0$, at least along a subsequence. If J_D has a strict local minimizer, then nearby there is a local minimizer of $I_{D,\epsilon}$ if ϵ is sufficiently small [19, 27, 29, 36].

The paper [12] by Chmaj and Ren seems to be the first work on J_D, although it was motivated by a different problem. In one dimension ($D = (0,1)$), if $0 < \alpha < \sigma$, then the stationary sets of $J_{(0,1)}$ form an infinite but countable family [12, Theorem 2.6]. Every stationary set is a local minimizer. Later Dancer, Ren, and Yan studied J_D on a unit ball in $\mathbb{R}^n$, i.e., $D = B_1(0) \subset \mathbb{R}^n$, and found several radially symmetric stationary sets [14].

In this paper we first study J_D of (1.3) on the plane ($D = \mathbb{R}^2$) and find disc shaped stationary sets. Such stationary sets are termed bubbles.

Before discussing bubbles in $\mathbb{R}^2$, let us see what happens in $\mathbb{R}$. If $(-b, b)$ is a one dimensional bubble, then

$$N_{\mathbb{R}}(-b,b)(x) = \begin{cases} 1 - e^{-b}\cosh x & \text{if } \ x \in (-b,b) \\ e^{-|x|}\sinh b & \text{if } \ x \in \mathbb{R}\backslash(-b,b) \end{cases}. \tag{1.16}$$

The equation (1.8) becomes

$$-\alpha + \sigma e^{-b} \sinh b = 0 \tag{1.17}$$

and consequently

$$b = -\frac{1}{2} \log\Big(1 - \frac{2\alpha}{\sigma}\Big). \tag{1.18}$$

Since α and σ are positive, (1.18) implies that if and only if

$$0 < \alpha < \frac{\sigma}{2}, \tag{1.19}$$

there is a unique one dimensional bubble. One can also show that this bubble is stable.

The situation in $\mathbb{R}^2$ is far more complex. Depending on the values of α and σ, there may be zero, one, two, or even three bubbles. In Theorem 3.4 we give a complete answer regarding the existence and multiplicity of bubbles for all possible values of α and σ. Next in Section 4, one develops a framework that allows us to determine the stability of every bubble (see Theorems 4.3 and 4.4). The key idea is to use proper functions to describe deformations of a disc. These functions live in a Hilbert space, and the stability of a bubble is formulated as a nonlocal eigenvalue problem on the Hilbert space. The eigenvalue problem is solved explicitly with the help of the modified Bessel functions (see Lemma 4.1). For a stable bubble, except for the zero eigevalue due to translation, the remaining eigenvalues are all positive.

The modified Bessel functions of all orders show up naturally in the study of the existence and the stability of bubbles. Section 2 is a collection of some useful properties of these Bessel functions. More technical properties needed in this paper are given in appendices A and B.

Finally we present an application of Theorems 3.4 and 4.3 for J_D on a bounded D in $\mathbb{R}^2$. In one dimension (say $D = (0, 1)$), all stationary sets are known. It was shown in [12] that every stationary set Ω (not equal to the trivial sets $\emptyset$ or $(0, 1)$) is associated with a positive integer N such that the restriction of Ω to $(0, 1/N)$ is either $(0, \xi)$ or $(\xi, 1/N)$ for some $\xi \in (0, 1/N)$. The full set can be obtained from $(0, \xi)$ (or $(\xi, 1/N)$) via successive reflection from $(0, 1/N)$ to $(1/N, 2/N)$, then to $(2/N, 3/N)$, and so on. Indeed, in the $(0, \xi)$ case,

$$N_{(0,1)}(\Omega)(x) = \begin{cases} (\sinh\xi \coth\frac{1}{N} - \cosh\xi)\cosh x + 1 & \text{if} \quad x \in (0, \xi) \\ \sinh\xi \coth\frac{1}{N}(\cosh x - \tanh\frac{1}{N}\sinh x) & \text{if} \quad x \in (\xi, \frac{1}{N}) \end{cases} \tag{1.20}$$

and (1.8) becomes

$$-\alpha + \sigma\Big(\sinh\xi\cosh\xi\coth\frac{1}{N} - \sinh^2\xi\Big) = 0. \tag{1.21}$$

Since the quantity $\big(\sinh\xi\cosh\xi\coth\frac{1}{N} - \sinh^2\xi\big)$ increases from 0 to 1 when ξ varies from 0 to $1/N$, (1.21) is solvable if and only if

$$0 < \alpha < \sigma. \tag{1.22}$$

The same is true if the restriction of Ω to $(0, 1/N)$ is $(\xi, 1/N)$.

A complete description of stationary sets in a general bounded two dimensional domain D is a challenging problem. In Theorem 5.1 we identify a range for the parameters α and σ so that J_D admits stationary sets that are assemblies of perturbed discs. Such stationary sets are called droplet assemblies. The radii and the locations of the discs in stationary droplet assemblies are determined asymptotically.

2. Preliminary lemmas

Throughout this paper I_n and K_n, $n = 0, 1, 2, \ldots$, are the n-th order modified Bessel functions of the first and the second kind, respectively. These functions are positive on the positive real axis. In this section we collect some of their properties. When $D = \mathbb{R}^2$, one recalls (1.7) and simplifies the notation N_D to N.

Lemma 2.1 *Let $B_r(0) \subset \mathbb{R}^2$ be a ball of radius r centered at the origin. Then $N(B_r)$ is radially symmetric with*

$$N(B_r)(r) = rI_1(r)K_0(r) \tag{2.1}$$

and

$$\int_{B_r} N(B_r)\,dx = 2\pi r^2\Big(\frac{1}{2} - I_1(r)\,K_1(r)\Big)\,. \tag{2.2}$$

Proof. Let $r > 0$ and $v = N(B_r)$. As the solution $v = v(t)$ is radially symmetric, it satisfies

$$(tv')' - tv = \begin{cases} -t & \text{if} \quad 0 < t < r \\ 0 & \text{if} \quad r < t \end{cases},$$

together with the requirement that v and v' are continuous at r. Thus

$$v(t) = \begin{cases} 1 - \frac{K_0(r)}{I_0(r)K_0'(r) - I_0'(r)K_0(r)} I_0(t) & \text{if} \quad 0 < t < r \\ -\frac{I_0'(r)}{I_0(r)K_0'(r) - I_0'(r)K_0(r)} K_0(t) & \text{if} \quad r < t \end{cases}.$$

As $I_0' = I_1$ and $K_0' = -K_1$ (see [1, formula (9.6.27)]), it follows that $K_0(r)I_0'(r) - K_0'(r)I_0(r) = K_0(r)I_1(r) + K_1(r)I_0(r) = 1/r$ for all r (see [1, formula (9.6.15)]). Hence

$$v(t) = \begin{cases} 1 - rK_1(r)I_0(t) & \text{if} \quad 0 < t < r \\ rI_1(r)K_0(t) & \text{if} \quad r < t \end{cases}$$

from which (2.1) follows. Moreover,

$$\int_{B_r} N(B_r)\,dx = 2\pi \int_0^r v(t)\,t\,dt = 2\pi\Big(\frac{r^2}{2} - rK_1(r)\int_0^r tI_0\,dt\Big). \tag{2.3}$$

As $(tI_0')' - tI_0 = 0$, $\int_0^r tI_0\,dt = rI_0'(r) = rI_1(r)$ and (2.3) implies (2.2). □

Denote by J_n and Y_n the n-th order Bessel functions of the first and the second kind respectively.

Lemma 2.2 *Let $z \in \mathbb{R}$ and $n = 0, 1, 2, \ldots$. Then*

$$Y_n(\mathrm{i}z) = -\frac{2}{\pi \mathrm{i}^n}\Big(K_n(z) - (-1)^n \frac{\pi \mathrm{i}}{2} I_n(z)\Big).$$

Proof. The case $n = 0$ is documented in [1, (9.6.5)]. For other values of n, use (9.6.4), (9.1.3) and (9.6.3) of [1] in the following calculation:

$$K_n(z) = \frac{\pi \mathrm{i}}{2}\mathrm{i}^n H_n^{(1)}(\mathrm{i}z) = \frac{\pi \mathrm{i}}{2}\mathrm{i}^n\big(J_n(\mathrm{i}z) + \mathrm{i}Y_n(\mathrm{i}z)\big) = \frac{\pi \mathrm{i}}{2}\mathrm{i}^n\big(\mathrm{i}^n I_n(z) + \mathrm{i}Y_n(\mathrm{i}z)\big),$$

which gives the answer. Here $H_n^{(1)}$ is the Hankel function of the first kind. □

Lemma 2.3 *Let $z \in \mathbb{R}$ and $n = 0, 1, 2, \dots$. Then*

$$\int_0^{\pi/2} K_0(2z\sin t)\,\cos(2nt)\,dt = \frac{\pi}{2} I_n(z)\,K_n(z)\,.$$

Proof. By [1, formula (11.4.9)] one has that for any complex z

$$\int_0^{\pi/2} Y_0(2z\sin t)\,\cos(2nt)\,dt = \frac{\pi}{2} J_n(z)\,Y_n(z)\,.$$

Change z to iz in the above equation and use Lemma 2.2, so the above identity is equivalent to

$$\begin{aligned}\int_0^{\pi/2} \Big(\mathrm{i}I_0(2z\sin t) - \frac{2}{\pi}K_0(2z\sin t)\Big)\,\cos(2nt)\,dt &= \frac{\pi}{2} J_n(\mathrm{i}z)\,Y_n(\mathrm{i}z)\\ &= -I_n(z)\Big(K_n(z) - (-1)^n\frac{\pi\mathrm{i}}{2}I_n(z)\Big)\,.\end{aligned}$$

Taking the real parts on both sides, one completes the proof of the Lemma. □

Lemma 2.4 1. *Let $z > 0$ be fixed. Then*

$$I_0(z)\,K_0(z) > I_1(z)\,K_1(z) > I_2(z)\,K_2(z) > \cdots \tag{2.4}$$

2. *Let $n = 0, 1, 2, \dots$ and define $g_n : [0,\infty) \to [0,\infty)$ such that $g_n(z) \equiv zI_{n+1}(z)K_n(z)$. Then g_n is a strictly increasing function with $g_n'(z) = z(I_n\,K_n - I_{n+1}\,K_{n+1})$. Moreover $g_n(0) = 0$, $\lim_{z\to\infty} g_n(z) = 1/2$ and*

$$g_0(z) = \frac{1}{2}\left(1 - \frac{1}{2z} - \frac{3}{16z^3} + O\Big(\frac{1}{z^4}\Big)\right) \tag{2.5}$$

as $z \to \infty$.

Proof. Part 1 is proved in [4, Theorem 2]. For part 2 employing [1, (9.6.26)], namely $I_{n+1}' = I_n - \frac{n+1}{z}I_{n+1}$ and $K_n' = -K_{n+1} + \frac{n}{z}K_n$, one obtains $g_n'(z) = z(I_n\,K_n - I_{n+1}\,K_{n+1}) > 0$. Thus g_n is strictly increasing and its behavior as $z \to 0$ and $z \to \infty$ can be easily evaluated from formulas (9.6.7)–(9.6.9), (9.7.1) and (9.7.2) in [1]. From the cited formulas,

$$\begin{aligned} &zI_1(z)K_0(z)\\ &\quad= \frac{1}{2}\left(1 - \frac{3}{8z} - \frac{15}{128z^2} - \frac{105}{1024z^3} + O\Big(\frac{1}{z^4}\Big)\right)\left(1 - \frac{1}{8z} + \frac{9}{128z^2} - \frac{225}{3072z^3} + O\Big(\frac{1}{z^4}\Big)\right)\\ &\quad= \frac{1}{2}\left(1 - \frac{1}{2z} - \frac{3}{16z^3} + O\Big(\frac{1}{z^4}\Big)\right)\end{aligned}$$

as $z \to \infty$. The $O(1/z^2)$ terms on the right hand side happen to cancel. □

3. Bubbles in a plane

In this section we are interested in disc shaped stationary sets of J_D when $D = \mathbb{R}^2$. Such stationary sets are called bubbles in this paper. Write J for $J_{\mathbb{R}^2}$ for simplicity. Because of the translation invariance of J, only discs $B_r(0)$ centered at the origin are considered. Define

$$j(r, \sigma, \alpha) = J\big(B_r(0)\big), \tag{3.1}$$

namely, the energy of the disc $B_r(0)$ with a radius r, when the parameters of J are σ and α. By (1.3) and (2.2) one has

$$j(r, \sigma, \alpha) = 2\pi r - \alpha\pi r^2 + \sigma\pi r^2\Big(\frac{1}{2} - I_1(r)\, K_1(r)\Big). \tag{3.2}$$

It is known from [1, (9.6.26)] that $I_1' = I_0 - \frac{1}{r} I_1$ and $K_1' = -K_0 - \frac{1}{r} K_1$. These lead to

$$\frac{\partial j}{\partial r} = 2\pi - 2\pi\alpha r + \pi\sigma r - \pi\sigma r^2\big(I_0(r)K_1(r) - K_0(r)I_1(r)\big).$$

Since $I_0 K_1 + I_1 K_0 = 1/r$ [1, (9.6.15)], the above can be simplified to

$$\frac{\partial j}{\partial r} = 2\pi - 2\pi\alpha r + 2\pi\sigma r^2\, I_1(r)\, K_0(r). \tag{3.3}$$

A disc centered at the origin of radius b is a bubble if and only if $\frac{\partial j}{\partial r}\big|_{r=b} = 0$; namely that b is a solution of

$$1 - \alpha b + \sigma b^2 I_1(b) K_0(b) = 0. \tag{3.4}$$

With the Bessel functions involved, it is difficult to solve (3.4) for b directly. We take a different perspective, which will lead to a complete characterization of the solutions of (3.4). For each $\alpha > 0$, solve (3.4) for σ in terms of b to obtain a function s_α:

$$\sigma = s_\alpha(b) \equiv \frac{\alpha b - 1}{b^2 I_1 K_0}, \quad b \in \Big(\frac{1}{\alpha}, \infty\Big). \tag{3.5}$$

To make notation simpler, we have suppressed the dependence on b in (3.5) and written I_n for $I_n(b)$ and K_n for $K_n(b)$. This practice is continued throughout the paper.

The function $\sigma = s_\alpha(b)$ defines a curve $\{(s_\alpha(b), b) : b \in (\frac{1}{\alpha}, \infty)\}$ in the σ-b quadrant. With a slight abuse of notation, one again calls the curve s_α for each α. In fact every pair (σ, b) uniquely determines a positive α by (3.4):

$$\alpha = \frac{1 + \sigma b^2 I_1 K_0}{b}. \tag{3.6}$$

The curves s_α for distinct α are mutually disjoint and every point in the σ-b quadrant is passed by one of such curves.

Next study the monotonicity of the function s_α, or whether the curve s_α is turning left or right in the σ-b quadrant as b increases. Implicit differentiation of the equation $\frac{\partial j}{\partial r} = 0$ shows that

$$s_\alpha'(b) = -\frac{\frac{\partial^2 j}{\partial b^2}}{\frac{\partial^2 j}{\partial\sigma\partial b}} = \frac{\alpha - \sigma\big(bI_1K_0 + b^2(I_0K_0 - I_1K_1)\big)}{b^2 I_1 K_0} \tag{3.7}$$

because

$$\frac{\partial^2 j}{\partial b^2} = -2\pi\alpha + 2\pi\sigma\big(bI_1K_0 + b^2(I_0K_0 - I_1K_1)\big), \tag{3.8}$$

$$\frac{\partial^2 j}{\partial\sigma\partial b} = 2\pi b^2 I_1 K_0. \tag{3.9}$$

Note that (3.8) follows from Lemma 2.4. Use (3.5) to simplify (3.7) to

$$s'_\alpha(b) = \frac{1 - \sigma b^3(I_0K_0 - I_1K_1)}{b^3 I_1 K_0} \tag{3.10}$$

where $\sigma = s_\alpha(b)$. Since the denominator in (3.10) is positive, the sign of the numerator determines whether the curve s_α is going left or right. Thus motivated, define another curve

$$C_0 = \big\{(\sigma, b) \in \mathbb{R}_+ \times \mathbb{R}_+ :\ 1 - \sigma b^3(I_0K_0 - I_1K_1) = 0\big\}. \tag{3.11}$$

This curve divides the σ-b quadrant into two regions

$$R_i = \big\{(\sigma, b) \in \mathbb{R}_+ \times \mathbb{R}_+ :\ 1 - \sigma b^3(I_0K_0 - I_1K_1) > 0\big\} \tag{3.12}$$

$$R_d = \big\{(\sigma, b) \in \mathbb{R}_+ \times \mathbb{R}_+ :\ 1 - \sigma b^3(I_0K_0 - I_1K_1) < 0\big\} \tag{3.13}$$

so that $\mathbb{R}_+ \times \mathbb{R}_+ = R_i \cup C_0 \cup R_d$. In summary one has the following lemma.

Lemma 3.1 *For each $\alpha > 0$ let s_α be the curve given above.*

1. *The family $\{s_\alpha :\ \alpha > 0\}$ is a set of non-intersecting curves covering the σ-b quadrant.*
2. *Each curve s_α starts at the point $\big(0, \frac{1}{\alpha}\big)$ with $s'_\alpha(\frac{1}{\alpha}) > 0$ and ends with a vertical asymptote $\lim_{b\to\infty} s_\alpha(b) = 2\alpha$.*
3. *When it is in R_i, s_α goes right as b increases, i.e. $s'_\alpha(b) > 0$ if $(s_\alpha(b), b) \in R_i$.*
4. *When it is in R_d, s_α goes left as b increases, i.e. $s'_\alpha(b) < 0$ if $(s_\alpha(b), b) \in R_d$.*
5. *When it intersects C_0, s_α does so with a vertical slope, i.e. $s'_\alpha(b) = 0$ if $(s_\alpha(b), b) \in C_0$.*

Proof. It remains to prove Part 2. Since $s_\alpha(\frac{1}{\alpha}) = 0$ by (3.5), (3.7) implies

$$s'_\alpha\Big(\frac{1}{\alpha}\Big) = \frac{\alpha^3}{I_1(\frac{1}{\alpha})K_0(\frac{1}{\alpha})} > 0.$$

From Lemma 2.4, one deduces $\lim_{b\to\infty} s_\alpha(b) = 2\alpha$. □

To better understand the curves s_α, one studies the shape of the curve C_0 first. This curve can be viewed as the graph of the function

$$\sigma = C_0(b) \equiv \frac{1}{b^3(I_0K_0 - I_1K_1)}, \quad b \in (0, \infty). \tag{3.14}$$

Lemma 3.2 *The function C_0 has the following properties.*

1. $\lim_{b\to 0+} C_0(b) = \infty$ *and* $\lim_{b\to\infty} C_0(b) = 4$.

2. *There exists* $\hat{b} > 0$ *such that* $C_0'(b) < 0$ *if* $b \in (0, \hat{b})$, $C_0'(b) = 0$ *if* $b = \hat{b}$, *and* $C_0'(b) > 0$ *if* $b \in (\hat{b}, \infty)$.

Proof. According to [1, (9.6.7)–(9.6.9)], for small b,

$$I_0(b) \sim 1,\ K_0(b) \sim \log\frac{1}{b},\ I_1(b) \sim \frac{b}{2},\ K_1(b) \sim \frac{1}{b}. \tag{3.15}$$

They imply $\lim_{b\to 0+} C_0(b) = \infty$. It is also known, [1, (9.7.5)], that as $b \to \infty$,

$$I_0(b)\,K_0(b) = \frac{1}{2b}\left(1 + \frac{1}{8b^2} + \frac{27}{128b^4} + O\Big(\frac{1}{b^6}\Big)\right),$$

$$I_1(b)\,K_1(b) = \frac{1}{2b}\left(1 - \frac{3}{8b^2} - \frac{45}{128b^4} + O\Big(\frac{1}{b^6}\Big)\right).$$

Thus

$$b^3\big(I_0(b)\,K_0(b) - I_1(b)\,K_1(b)\big) = \frac{1}{4} + \frac{9}{32b^2} + O\Big(\frac{1}{b^4}\Big), \tag{3.16}$$

and

$$C_0(b) = \frac{1}{b^3(I_0K_0 - I_1K_1)} = 4\left(1 - \frac{9}{8b^2} + O\Big(\frac{1}{b^4}\Big)\right). \tag{3.17}$$

Hence $\lim_{b\to\infty} C_0(b) = 4$. This proves the first part of the lemma. The proof of the second part is postponed to appendix A. □

Let $\hat{\sigma} = C_0(\hat{b})$ so that $(\hat{\sigma}, \hat{b})$ is the leftmost point on the curve C_0. Then

$$\hat{\sigma} < 4. \tag{3.18}$$

If $s_{\hat{\alpha}}$ is the curve from the family $\{s_\alpha : \alpha > 0\}$ that passes the point $(\hat{\sigma}, \hat{b})$, then according to (3.6)

$$\hat{\alpha} = \frac{1 + \hat{\sigma}\hat{b}^2 I_1(\hat{b})K_0(\hat{b})}{\hat{b}}. \tag{3.19}$$

It will be shown below that $\hat{\alpha} < 2$. Knowing the shape of C_0, one investigates how the curves s_α change as α varies.

Lemma 3.3 *There exist numbers* $\hat{\alpha}$ *and* $\bar{\alpha}$, $0 < \hat{\alpha} < \bar{\alpha} < 2$, *and real analytic functions* $\sigma_L : (\hat{\alpha}, 2) \to (\hat{\sigma}, 4)$, $\sigma_R : (\hat{\alpha}, \infty) \to (\hat{\sigma}, \infty)$, $b_L : (\hat{\alpha}, 2) \to (\hat{b}, \infty)$, *and* $b_R : (\hat{\alpha}, \infty) \to (0, \hat{b})$ *such that* σ_L, σ_R, *and* b_L *are increasing and* b_R *is decreasing,* $\hat{\sigma} < \sigma_L(\alpha) < \sigma_R(\alpha)$ *and* $b_L(\alpha) > \hat{b} > b_R(\alpha)$ *for all* $\alpha \in (\hat{\alpha}, 2)$. *The following properties hold for the curves* s_α.

1. *If* $\alpha \in (0, \hat{\alpha})$, *then the curve* s_α *stays in the region* R_i *and* $s_\alpha'(b) > 0$ *for all* $b \in (\frac{1}{\alpha}, \infty)$.
2. *If* $\alpha = \hat{\alpha}$, *then* $s_{\hat{\alpha}}$ *touches* C_0 *tangentially at one point* $(\hat{\sigma}, \hat{b})$ *where* $s_{\hat{\alpha}}'(\hat{b}) = C_0'(\hat{b}) = 0$. *Elsewhere* $s_{\hat{\alpha}}$ *stays in* R_i *and* $s_{\hat{\alpha}}'(b) > 0$ *if* $b \neq \hat{b}$.
3. *If* $\alpha \in (\hat{\alpha}, 2)$, *then* s_α *intersects* C_0 *twice transversally at* $(\sigma_R(\alpha), b_R(\alpha))$ *and* $(\sigma_L(\alpha), b_L(\alpha))$. *When* $b \in (\frac{1}{\alpha}, b_R(\alpha)) \cup (b_L(\alpha), \infty)$, $s_\alpha'(b) > 0$ *and* s_α *is in* R_i; *when* $b \in (b_R(\alpha), b_L(\alpha))$, $s_\alpha'(b) < 0$ *and* s_α *is in* R_d. *Furthermore,*

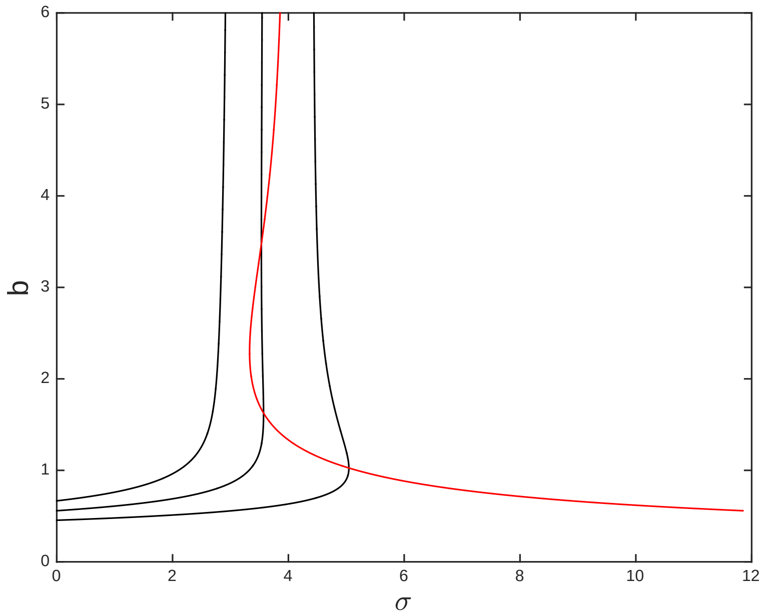

FIG. 1. The curve C_0 (in red) and the curves s_α for various α. As α increases (from 1.5 to 1.79 to 2.2 in this figure), the s_α curves shift to the right. (Only the ebook/online version of this article contains full color images.)

(a) *if* $\alpha \in (\hat{\alpha}, \bar{\alpha})$, *then*

$$\sup_{b \in (\frac{1}{\alpha}, \infty)} s_\alpha(b) = \lim_{b \to \infty} s_\alpha(b) = 2\alpha > \sigma_R(\alpha) = s_\alpha\big(b_R(\alpha)\big) > \sigma_L(\alpha) = s_\alpha\big(b_L(\alpha)\big);$$

(b) *if* $\alpha = \bar{\alpha}$, *then*

$$\sup_{b \in (\frac{1}{\bar{\alpha}}, \infty)} s_{\bar{\alpha}}(b) = \lim_{b \to \infty} s_{\bar{\alpha}}(b) = 2\bar{\alpha} = \sigma_R(\bar{\alpha}) = s_{\bar{\alpha}}\big(b_R(\bar{\alpha})\big) > \sigma_L(\bar{\alpha}) = s_{\bar{\alpha}}\big(b_L(\bar{\alpha})\big);$$

(c) *if* $\alpha \in (\bar{\alpha}, 2)$, *then*

$$\sup_{b \in (\frac{1}{\alpha}, \infty)} s_\alpha(b) = \sigma_R(\alpha) = s_\alpha\big(b_R(\alpha)\big) > \lim_{b \to \infty} s_\alpha(b) = 2\alpha > \sigma_L(\alpha) = s_\alpha\big(b_L(\alpha)\big).$$

4. *If* $\alpha \in [2, \infty)$, *then* s_α *intersects* C_0 *once transversally at* $(\sigma_R(\alpha), b_R(\alpha))$. *When* $b \in (\frac{1}{\alpha}, b_R(\alpha))$, $s'_\alpha(b) > 0$ *and* s_α *is in* R_i*; when* $b \in (b_R(\alpha), \infty))$, $s'_\alpha(b) < 0$ *and* s_α *is in* R_d.

Proof. Recall that $(\hat{\sigma}, \hat{b})$ is the leftmost point on C_0. Note that

$$s_\alpha(b) = \frac{\alpha b - 1}{b^2 I_1 K_0} < \frac{\alpha}{b I_1 K_0}$$

and $b I_1 K_0 \to \frac{1}{2}$ as $b \to \infty$ by Lemma 2.4. If α is sufficiently small, $s_\alpha(b) < \hat{\sigma}$ for all $b \in (\frac{1}{\alpha}, \infty)$. Then s_α stays in R_i and $s'_\alpha(b) > 0$ for every $b \in (\frac{1}{\alpha}, \infty)$.

Since $s_\alpha(b)$ is increasing with respect to α, the curve s_α sweeps to the right in the σ-b quadrant as α increases. Let us temporarily set

$$\hat{\alpha} = \sup\left\{\alpha > 0 : s_\alpha \text{ is in } R_i \text{ for all } b \in \left(\frac{1}{\alpha}, \infty\right)\right\}. \tag{3.20}$$

We claim that $\hat{\alpha} < 2$. By Lemma 2.4 for large b,

$$s_2(b) = \frac{2b-1}{\frac{b}{2}\left(1 - \frac{1}{2b} - \frac{3}{16b^3} + O(\frac{1}{b^4})\right)} = 4 + \frac{3}{4b^3} + O\left(\frac{1}{b^4}\right). \tag{3.21}$$

Comparing this to (3.17), one sees that the curve s_2 lies to the right of the vertical line $\sigma = 4$ and C_0 lies to the left of the vertical line when b is sufficiently large. Since $s_2(\frac{1}{2}) = 0 < C_0(\frac{1}{2})$, the curve s_2 must intersect C_0. As $s_\alpha(b)$ depends continuously on α, if $s_2(b) > 4$ at some large b where $C_0(b) < 4$, then $s_\alpha(b) > 4$ when α is slightly smaller than 2. Consequently s_α must also intersect C_0 if α is slightly smaller than 2. Hence $\hat{\alpha} < 2$.

In view of

$$\lim_{b\to\infty} s_{\hat{\alpha}}(b) = 2\hat{\alpha} < 4 = \lim_{b\to\infty} C_0(b), \tag{3.22}$$

$s_{\hat{\alpha}}$ must intersect C_0, for otherwise one can increase α slightly beyond $\hat{\alpha}$ and keep s_α in R_i. Since $s_{\hat{\alpha}}(b) \leqslant C_0(b)$ for all $b \in (\frac{1}{\hat{\alpha}}, \infty)$, the two curves can only intersect tangentially. At an intersection point, $s'_{\hat{\alpha}}$ is vanished by Lemma 3.1 (5). Then C'_0 also vanishes there. However C_0 has only one critical point $\hat{b}$ by Lemma 3.2 (2), so $s_{\hat{\alpha}}$ and C_0 intersect at only one point: $(\hat{\sigma}, \hat{b})$. Moreover $\hat{\alpha}$ defined in (3.20) is the same as the one given in (3.19). Parts 1 and 2 are proved.

Since $s_\alpha(b)$ is increasing with respect to α, for $\alpha > \hat{\alpha}$ the curve s_α must intersect C_0 at least once. Any intersection point in this case must be different from $(\hat{\sigma}, \hat{b})$. Hence s'_α vanishes but C'_0 does not at such an intersection point, so the two curves intersect transversally. For $\alpha > \hat{\alpha}$, as b increases from $\frac{1}{\alpha}$, set the first intersection point of s_α and C_0 to be $(\sigma_R(\alpha), b_R(\alpha))$. For b slightly greater than $b_R(\alpha)$, the point $(s_\alpha(b), b)$ is in R_d. Then $0 = s'_\alpha(b_R(\alpha)) \geqslant C'_0(b_R(\alpha))$. This forces $b_R(\alpha) < \hat{b}$ by Lemma 3.2 (2).

The curve s_α, $\alpha > \hat{\alpha}$, may stay in R_d for all $b > b_R(\alpha)$, or it may intersect C_0 again. When the latter happens, denote the second intersection point by $(\sigma_L(\alpha), b_L(\alpha))$, which is again a transversal intersection point and $b_L(\alpha) > \hat{b}$. Since $s_\alpha(b)$ is decreasing with respect to b when $b \in (b_R(\alpha), b_L(\alpha))$,

$$\hat{\sigma} < \sigma_L(\alpha) < \sigma_R(\alpha), \quad \text{and} \quad b_L(\alpha) > \hat{b} > b_R(\alpha). \tag{3.23}$$

After s_α exits R_d, it cannot intersect C_0 again. If there were a third intersection point (σ_*, b_*), then $b_* > \hat{b}$, and $C'_0(b_*) > 0 = s'_\alpha(b_*)$, which is a contradiction to $s_\alpha(b) < C_0(b)$ for $b \in (b_L(\alpha), b_*)$.

Now consider the case $\alpha \geqslant 2$. By (3.21) and the fact that $\lim_{b\to\infty} s_\alpha(b) = 2\alpha > 4$ for $\alpha > 2$, the curve s_α stays in R_d when b is sufficiently large. Such a curve s_α can only intersect C_0 once at $(\sigma_R(\alpha), b_R(\alpha))$. This proves part 4.

The last case is $\alpha \in (\hat{\alpha}, 2)$. Since $\lim_{b\to\infty} s_\alpha(b) = 2\alpha < 4$, the curve s_α is in R_i when b is large. As b increases, s_α must intersect C_0 twice, first at $(\sigma_R(\alpha), b_R(\alpha))$ and then at $(\sigma_L(\alpha), b_L(\alpha))$. Moreover

$$\begin{array}{lll} s'_\alpha(b) > 0 & \text{if} & b \in \left(\frac{1}{\alpha}, b_R(\alpha)\right), \\ s'_\alpha(b) < 0 & \text{if} & b \in \left(b_R(\alpha), b_L(\alpha)\right), \\ s'_\alpha(b) > 0 & \text{if} & b \in \left(b_L(\alpha), \infty\right). \end{array} \tag{3.24}$$

Note that $b_R(\alpha)$ is a local maximum point of s_α. To prove the three subcases in part 3, compare $\sigma_R(\alpha) = s_\alpha(b_R(\alpha))$ to $\lim_{b\to\infty} s_\alpha(b) = 2\alpha$. When $\alpha \in (\hat{\alpha}, 2)$ sufficiently close to $\hat{\alpha}$, the point $(\sigma_R(\alpha), b_R(\alpha))$ is close to $(\hat{\sigma}, \hat{b})$. Since $s'_{\hat{\alpha}}(b) > 0$ for all $b \in (\hat{b}, \infty)$, $\hat{\sigma} = s_{\hat{\alpha}}(\hat{b}) < \lim_{b\to\infty} s_{\hat{\alpha}}(b) = 2\hat{\alpha}$. Consequently $\sigma_R(\alpha) = s_\alpha(b_R(\alpha)) < \lim_{b\to\infty} s_\alpha(b) = 2\alpha$ when $\alpha \in (\hat{\alpha}, 2)$ sufficiently close to $\hat{\alpha}$. On the other hand, note that $s'_2(b) < 0$ for all $b \in (b_R(2), \infty)$, and $\sigma_R(2) = s_2(b_R(2)) > \lim_{b\to\infty} s_2(b) = 4$. Then for $\alpha \in (\hat{\alpha}, 2)$ sufficiently close to 2, $\sigma_R(\alpha) = s_\alpha(b_R(\alpha)) > \lim_{b\to\infty} s_\alpha(b) = 2\alpha$. Hence there exists $\bar{\alpha} \in (\hat{\alpha}, 2)$ such that

$$\sigma_R(\bar{\alpha}) = s_{\bar{\alpha}}\big(b_R(\bar{\alpha})\big) = \lim_{b\to\infty} s_{\bar{\alpha}}(b) = 2\bar{\alpha}. \tag{3.25}$$

Since $(\sigma_R(\bar{\alpha}), b_R(\bar{\alpha}))$ is an intersection point of $s_{\bar{\alpha}}$ and C_0, the triplet $(\sigma_R(\bar{\alpha}), b_R(\bar{\alpha}), \bar{\alpha})$ satisfies

$$1 - \alpha b + \sigma b^2 I_1 K_0 = 0, \tag{3.26}$$

$$1 - \sigma b^3 (I_0 K_0 - I_1 K_1) = 0, \tag{3.27}$$

$$\sigma = 2\alpha, \tag{3.28}$$

where (3.28) comes from (3.25). Eliminating σ and α in (3.26)-(3.28), one arrives at

$$b^2(I_0 K_0 - I_1 K_1) + b I_1 K_0 = \frac{1}{2}. \tag{3.29}$$

It is shown in appendix B that (3.29) admits a unique solution for b in $(0, \infty)$. This implies the uniqueness of $\bar{\alpha}$ and proves part 3.

As a point of intersection, $(\sigma_R(\alpha), b_R(\alpha))$ satisfies $\sigma = s_\alpha(b)$ and $\sigma = C_0(b)$. On eliminating σ one deduces that b_R is a function of α given implicitly by

$$s_\alpha(b) - C_0(b) = 0\,. \tag{3.30}$$

The left side of (3.30) is analytic with respect to α and b, and

$$\frac{\partial\big(s_\alpha(b) - C_0(b)\big)}{\partial b}\bigg|_{(\alpha,b)=(\alpha,b_R(\alpha))} = s'_\alpha\big(b_R(\alpha)\big) - C'_0\big(b_R(\alpha)\big) = -C'_0\big(b_R(\alpha)\big) \neq 0 \tag{3.31}$$

since $b_R(\alpha) < \hat{b}$ for all $\alpha > \hat{\alpha}$. By the implicit function theorem, $b_R(\alpha)$ is analytic with respect to α for $\alpha \in (\hat{\alpha}, \infty)$. Consequently $\sigma_R(\alpha) = s_\alpha(b_R(\alpha))$ is also analytic. The same argument works for b_L and σ_L as well. □

Numerical calculations show that $\hat{\alpha} \approx 1.71$ and $\bar{\alpha} \approx 1.80$, so the three critical values $\hat{\alpha}$, $\bar{\alpha}$ and 2 for α are fairly close to each other.

Recall that a bubble is a disc shaped solution of (1.8) in $\mathbb{R}^2$. The next theorem gives the exact number of bubbles up to translation for each parameter pair (σ, α).

Theorem 3.4 *Let $\hat{\alpha}$, $\bar{\alpha}$, σ_L, and σ_R be as in Lemma 3.3. The following statements hold.*

1. *If $\alpha \in (0, \hat{\alpha}]$, then*
   (a) *if $\sigma \in (0, 2\alpha)$, there is one bubble,*
   (b) *if $\sigma \in [2\alpha, \infty)$, there is no bubble.*

2. *If* $\alpha \in (\hat{\alpha}, \bar{\alpha})$*, then* $\sigma_R(\alpha) < 2\alpha$ *and*

   (a) *if* $\sigma \in (0, \sigma_L(\alpha))$*, there is one bubble,*
   (b) *if* $\sigma = \sigma_L(\alpha)$*, there are two bubbles,*
   (c) *if* $\sigma \in (\sigma_L(\alpha), \sigma_R(\alpha))$*, there are three bubbles,*
   (d) *if* $\sigma = \sigma_R(\alpha)$*, there are two bubbles,*
   (e) *if* $\sigma \in (\sigma_R(\alpha), 2\alpha)$*, there is one bubble,*
   (f) *if* $\sigma \in [2\alpha, \infty)$*, there is no bubble.*

3. *If* $\alpha = \bar{\alpha}$*, then* $\sigma_R(\alpha) = 2\alpha$ *and*

   (a) *if* $\sigma \in (0, \sigma_L(\alpha))$*, there is one bubble,*
   (b) *if* $\sigma = \sigma_L(\alpha)$*, there are two bubbles,*
   (c) *if* $\sigma \in (\sigma_L(\alpha), 2\alpha)$*, there are three bubbles,*
   (d) *if* $\sigma = 2\alpha$*, there is one bubble,*
   (e) *if* $\sigma \in (2\alpha, \infty)$*, there is no bubble.*

4. *If* $\alpha \in (\bar{\alpha}, 2)$*, then* $\sigma_L(\alpha) < 2\alpha < \sigma_R(\alpha)$ *and*

   (a) *if* $\sigma \in (0, \sigma_L(\alpha))$*, there is one bubble,*
   (b) *if* $\sigma = \sigma_L(\alpha)$*, there are two bubbles,*
   (c) *if* $\sigma \in (\sigma_L(\alpha), 2\alpha)$*, there are three bubbles,*
   (d) *if* $\sigma \in [2\alpha, \sigma_R(\alpha))$*, there are two bubbles,*
   (e) *if* $\sigma = \sigma_R(\alpha)$*, there is one bubble,*
   (f) *if* $\sigma \in (\sigma_R(\alpha), \infty)$*, there is no bubble.*

5. *If* $\alpha \in [2, \infty)$*, then* $2\alpha < \sigma_R(\alpha)$ *and*

   (a) *if* $\sigma \in (0, 2\alpha]$*, there is one bubble,*
   (b) *if* $\sigma \in (2\alpha, \sigma_R(\alpha))$*, there are two bubbles,*
   (c) *if* $\sigma = \sigma_R(\alpha)$*, there is one bubble,*
   (d) *if* $\sigma \in (\sigma_R(\alpha), \infty)$*, there is no bubble.*

Proof. To determine the number of bubbles for the parameters σ and α, consider the curve s_α and the vertical line σ in the σ-b quadrant. An intersection point of the σ-line and s_α corresponds to a bubble of J with σ and α being the parameters.

When $\alpha \in (0, \hat{\sigma}]$, Lemma 3.1 (2) and Lemma 3.3 (1), (2) imply that if $\sigma \in (0, 2\alpha)$, the σ-line intersects s_α once; if $\sigma \in [2\alpha, \infty)$, there is no intersection. This proves Part 1 of the theorem.

When $\alpha \in (\hat{\alpha}, \bar{\alpha})$, Lemma 3.3 (3), particularly subcase (a), describes the shape of the curve s_α in the σ-b quadrant. It implies that the σ- line intersects s_α once if $\sigma \in (0, \sigma_L(\alpha))$, twice if $\sigma = \sigma_L(\alpha)$, three times if $\sigma \in (\sigma_L(\alpha), \sigma_R(\alpha))$, twice if $\sigma = \sigma_R(\alpha)$, and once if $\sigma \in (\sigma_R(\alpha), 2\alpha)$. There is no intersection if $\sigma \in [2\alpha, \infty)$.

Other parts of the theorem follow from the other cases of Lemma 3.3 similarly. □

The regions stated in Theorem 3.4 are numerically calculated and presented in Figure 2. The borderline cases (2(b), 2(d), 3(b), 3(d), 4(b), 4(e), and 5(c)) are not depicted.

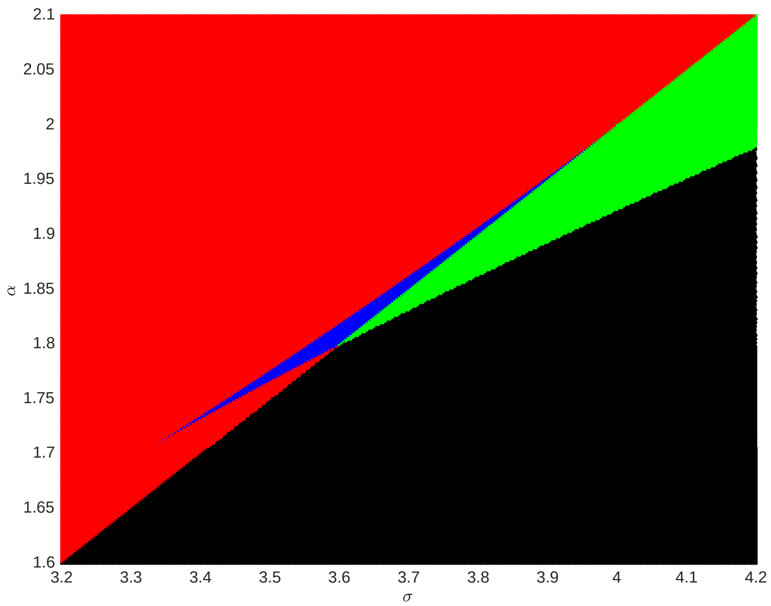

FIG. 2. When (σ,α) is in the red region, there is one radially unstable bubble; in the green region there are two bubbles of which one is radially stable and the other is radially unstable; in the blue region there are three bubbles of which one is radially stable and two are radially unstables; finally in the black region there is no bubble at all. For the blue region, its lowest pointed apex has $\alpha = \hat{\alpha}$ while its highest pointed apex has $\alpha = 2$. The point surrounded by 4 colors has $\alpha = \bar{\alpha}$. From left the blue region is bounded by the curve $\sigma = \sigma_L(\alpha)$, $\alpha \in (\hat{\alpha}, 2)$. From right the union of the blue and green regions is bounded by $\sigma = \sigma_R(\alpha)$, $\alpha \in (\hat{\alpha}, \infty)$. The union of the red and blue regions is separated from the union of the green and black regions by $\sigma = 2\alpha$, $\alpha \in (0,\infty)$. (Only the ebook/online version of this article contains full color images.)

4. Bubble stability

We now set up a framework to facilitate a discussion of the stability of the bubbles found in Theorem 3.4. Let $L^2(\mathbb{S}^1)$ be the L^2 space of the unit circle, or the interval $[0, 2\pi]$ with end points identified, under the Lebesgue measure. It is a Hilbert space with the inner product

$$\langle \phi, \psi \rangle = \int_0^{2\pi} \phi(\theta)\psi(\theta)\, d\theta. \tag{4.1}$$

Denote by $H^1(\mathbb{S}^1)$ and $H^2(\mathbb{S}^1)$ the standard Sobolev spaces whose norms are given by

$$\|\phi\|_{H^1}^2 = \int_0^{2\pi} \left((\phi')^2 + \phi^2\right) d\theta, \quad \|\phi\|_{H^2}^2 = \int_0^{2\pi} \left((\phi'')^2 + (\phi')^2 + \phi^2\right) d\theta \tag{4.2}$$

respectively. The three spaces are nested: $H^2(\mathbb{S}^1) \subset H^1(\mathbb{S}^1) \subset L^2(\mathbb{S}^1)$.

Consider a bubble $B_b(0)$, i.e. b is a solution of (3.4). Given a 2π-periodic function ϕ, $1 + 2\phi(\theta) > 0$ for all $\theta \in \mathbb{S}^1$, let $\Omega = P_\phi$ be a perturbed disc:

$$P_\phi = \{re^{\mathrm{i}\theta} : \ r \in \left[0, b\sqrt{1 + 2\phi(\theta)}\right], \ \theta \in \mathbb{S}^1\}. \tag{4.3}$$

As a functional of ϕ, the domain of J is

$$Dom(J) = \{\phi \in H^1(\mathbb{S}^1) : \|\phi\|_{H^1} < \eta_0\} \tag{4.4}$$

with $\eta_0 > 0$ small enough so that

$$\sqrt{1 + 2\phi(\theta)} \in (0, 2), \text{ for all } \theta \in \mathbb{S}^1; \tag{4.5}$$

in particular $J(P_\phi)$ is written more explicitly as

$$J(\phi) = \int_0^{2\pi} b\sqrt{\frac{(\phi')^2}{1+2\phi} + 1 + 2\phi}\, d\theta - \alpha b^2 \int_0^{2\pi} \frac{1+2\phi}{2}\, d\theta + \frac{\sigma}{2}\int_{P_\phi}\int_{P_\phi} \frac{1}{2\pi} K_0(|x-y|)\, dx dy\ . \tag{4.6}$$

For a general $\Omega \subset \mathbb{R}^2$ and a deformation Ω_ε of Ω, there is a first variation formula:

$$\frac{dJ(\Omega_\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = -\int_{\partial\Omega} \big(K(\partial\Omega) - \alpha + \sigma N(\Omega)\big)\mathbf{N}\cdot\mathbf{X}\, ds, \tag{4.7}$$

which holds for sufficiently smooth deformation Ω_ε. The vector $\mathbf{N}$ is the inward pointing unit normal on $\partial\Omega$ and ds is the arc length element of $\partial\Omega$. The vector $\mathbf{X}$ is the infinitesimal element of the deformation of $\partial\Omega$. If $\partial\Omega_\varepsilon$ is parametrized by $\mathbf{R}_\varepsilon(\theta)$, then

$$\mathbf{X}(\theta) = \frac{\partial \mathbf{R}_\varepsilon(\theta)}{\partial\varepsilon}\Big|_{\varepsilon=0}. \tag{4.8}$$

Note that the sign convention of the curvature $K(\partial\Omega)$ in (4.7) must be consistent with the direction of the normal vector $\mathbf{N}$. In other words, $K(\partial\Omega)\mathbf{N}$ is the curvature vector of $\partial\Omega$. If Ω were convex, $K(\partial\Omega)$ would be non-negative.

In the present setting, one can easily define a deformation of P_ϕ by deforming ϕ to

$$\phi + \varepsilon\psi,\ \phi \in Dom(J) \text{ and } \psi \in H^1(\mathbb{S}^1). \tag{4.9}$$

This gives rise to $P_{\phi+\varepsilon\psi}$, which implies

$$\mathbf{R}_\varepsilon(\theta) = b\sqrt{1 + 2\big(\phi(\theta) + \varepsilon\psi(\theta)\big)}\, e^{\mathrm{i}\theta},\ \theta \in \mathbb{S}^1. \tag{4.10}$$

Denote the inward normal vector of $\mathbf{R}_0$ by $\mathbf{N}$ and the infinitesimal element of the deformation $\mathbf{R}_\varepsilon$ by $\mathbf{X}$. Then, if $\phi \in H^2(\mathbb{S}^1)$,

$$\begin{aligned}
\mathbf{N}\cdot\mathbf{X}\, ds &= \mathrm{i}\frac{\partial\mathbf{R}_\varepsilon}{\partial\theta}\Big|_{\varepsilon=0} \cdot \frac{\partial\mathbf{R}_\varepsilon}{\partial\varepsilon}\Big|_{\varepsilon=0} d\theta \\
&= \Big(\frac{b\frac{\partial\phi}{\partial\theta}}{\sqrt{1+2\phi}} e^{\mathrm{i}\theta}\mathrm{i} - b\sqrt{1+2\phi}e^{\mathrm{i}\theta}\Big)\cdot\Big(\frac{b\psi}{\sqrt{1+2\phi}}e^{\mathrm{i}\theta}\Big)\, d\theta \\
&= -b^2\psi\, d\theta,
\end{aligned} \tag{4.11}$$

where complex numbers are used to denote vectors for simpler notation. The first variation of J now becomes

$$\frac{dJ(\phi+\varepsilon\psi)}{d\varepsilon}\Big|_{\varepsilon=0} = \int_0^{2\pi} \big(K(\partial P_\phi) - \alpha + \sigma N(P_\phi)\big)b^2\psi\, d\theta\ . \tag{4.12}$$

Introduce a nonlinear operator $S: Dom(S) \to L^2(\mathbb{S}^1)$ with the domain

$$Dom(S) = \{\phi \in H^2(\mathbb{S}^1) : \|\phi\|_{H^2} < \eta_0\}, \tag{4.13}$$

where η_0 is the same as the one in (4.4). Let

$$S(\phi) = b^2\big(K(\partial P_\phi) - \alpha + \sigma N(P_\phi)\big) . \tag{4.14}$$

If one computes the curvature of ∂P_ϕ in terms of ϕ, then an explicit formula is obtained:

$$S(\phi)(\theta) = \frac{b\Big(1 + 2\phi + \frac{3(\phi')^2}{1+2\phi} - \phi''\Big)}{\Big(1 + 2\phi + \frac{(\phi')^2}{1+2\phi}\Big)^{3/2}} - \alpha b^2 + \sigma b^2 \int_{P_\phi} \frac{1}{2\pi} K_0\big(\big|b\sqrt{1+2\phi(\theta)}\, e^{i\theta} - y\big|\big)\, dy. \tag{4.15}$$

The Euler–Lagrange equation (1.8) now becomes the equation $S(\phi) = 0$.

With the operator S one can write the first variation (4.12) more concisely as

$$\frac{dJ(\phi + \varepsilon\psi)}{d\varepsilon}\Big|_{\varepsilon=0} = \langle S(\phi), \psi\rangle. \tag{4.16}$$

While the right side of (4.16) is obviously meaningful if $\phi \in Dom(S)$ and $\psi \in L^2(\mathbb{S}^1)$, the left side of (4.16) is still defined if $\phi \in Dom(J)$ and $\psi \in H^1(\mathbb{S}^1)$. Henceforth $\langle S(\phi), \psi\rangle$ is interpreted this way in the latter case. One also finds the second variation of J:

$$\frac{\partial^2 J(\phi + \varepsilon_1\psi + \varepsilon_2 \upsilon)}{\partial\varepsilon_1\partial\varepsilon_2}\Big|_{\varepsilon_1=\varepsilon_2=0} = \langle S'(\phi)\psi, \upsilon\rangle. \tag{4.17}$$

In (4.17), S' is the Fréchet derivative of S. For each $\phi \in Dom(S)$, $S'(\phi)$ is a linear operator from $H^2(\mathbb{S}^1)$ to $L^2(\mathbb{S}^1)$.

Note that the bubble $B_b(0)$ corresponds to $\phi = 0$. Hence $\phi = 0$ is a critical point of J, so $S(0) = 0$. A more involved calculation shows

$$\begin{aligned} S'(0)\psi &= b(-\psi'' - \psi) + \sigma \int_0^{2\pi} \frac{b^4}{2\pi} K_0\big(\big|be^{i\theta} - be^{i\omega}\big|\big)\psi(\omega)\, d\omega \\ &\quad + \sigma \int_{|y|<b} \frac{b^3}{2\pi} K_0'\big(\big|be^{i\theta} - y\big|\big) \frac{be^{i\theta} - y}{|be^{i\theta} - y|}\, dy \cdot e^{i\theta}\psi(\theta) . \end{aligned} \tag{4.18}$$

Let $H_n = \{\psi : \psi = A\cos n\theta + B\sin n\theta,\ A, B \in \mathbb{R}\}$ and decompose

$$L^2(\mathbb{S}^1) = \oplus_{n=0}^{\infty} H_n . \tag{4.19}$$

Lemma 4.1 *Each of the spaces H_n, $n = 0, 1, 2, \ldots$, is an eigenspace of $S'(0)$ for the eigenvalue*

$$\lambda_n = b(n^2 - 1) + \sigma b^4\big(I_n(b)K_n(b) - I_1(b)K_1(b)\big). \tag{4.20}$$

Proof. It suffices to study the integral operator

$$\psi \to \int_0^{2\pi} \frac{b^4}{2\pi} K_0\big(\big|be^{i\theta} - be^{i\omega}\big|\big)\psi(\omega)\, d\omega \tag{4.21}$$

and the multiplication operator

$$\psi \to \int_{|y|<b} \frac{b^3}{2\pi} K_0'\big(|be^{i\theta} - y|\big) \frac{be^{i\theta} - y}{|be^{i\theta} - y|}\, dy \cdot e^{i\theta} \psi(\theta). \tag{4.22}$$

For the integral operator (4.21) take ψ to be $\cos n\theta$, with $n = 0, 1, 2, \dots$, to deduce

$$\begin{aligned}
&\int_0^{2\pi} \frac{1}{2\pi} K_0\big(|be^{i\theta} - be^{i\omega}|\big) \cos n\omega \, d\omega \\
&\quad = \int_0^{2\pi} \frac{1}{2\pi} K_0\big(b\big|1 - e^{i(\omega-\theta)}\big|\big)\big(\cos n(\omega - \theta)\cos n\theta - \sin n(\omega-\theta)\sin n\theta\big)\, d\omega \\
&\quad = \cos n\theta \int_0^{2\pi} \frac{1}{2\pi} K_0\big(b\big|1 - e^{i\omega}\big|\big) \cos n\omega\, d\omega - \sin n\theta \int_0^{2\pi} \frac{1}{2\pi} K_0\big(b\big|1-e^{i\omega}\big|\big) \sin n\omega \, d\omega \\
&\quad = \cos n\theta \int_0^{\pi} \frac{1}{\pi} K_0\Big(2b \sin\frac{\omega}{2}\Big) \cos n\omega \, d\omega. \qquad (4.23)
\end{aligned}$$

Similarly, take ψ to be $\sin n\theta$ for $n = 1, 2, \dots$, to find

$$\int_0^{2\pi} \frac{1}{2\pi} K_0\Big(\big|be^{i\theta} - be^{i\omega}\big|\Big) \sin n\omega \, d\omega = \sin n\theta \int_0^{\pi} \frac{1}{\pi} K_0\Big(2b\sin\frac{\omega}{2}\Big) \cos n\omega \, d\omega. \tag{4.24}$$

By Lemma 2.3 one concludes from (4.23) and (4.24) that on $\cos n\theta$ and $\sin n\theta$ the integral operator (4.21) acts like

$$\cos n\theta \to b^4 I_n(b) K_n(b) \cos n\theta, \quad \sin n\theta \to b^4 I_n(b) K_n(b) \sin n\theta. \tag{4.25}$$

For the multiplication operator (4.22), direct calculation gives

$$\begin{aligned}
&\frac{1}{b}\int_{|y|<b} \frac{1}{2\pi} K_0'\big(|be^{i\theta} - y|\big) \frac{be^{i\theta} - y}{|be^{i\theta} - y|}\, dy \cdot e^{i\theta} \\
&\qquad = \int_{|z-1|<1} \frac{b}{2\pi} K_0'(b|z|) \frac{e^{i\theta} z}{|z|}\, dz \cdot e^{i\theta}, \quad \big(y = be^{i\theta}(1-z)\big) \\
&\qquad = \int_0^2 \int_{-\arccos(r/2)}^{\arccos(r/2)} \frac{b}{2\pi} K_0'(b\, r)(\cos a) r\, da dr, \quad (z = re^{ia}) \\
&\qquad = \int_0^2 \frac{b}{\pi} K_0'(b\, r) \sqrt{1 - \Big(\frac{r}{2}\Big)^2}\; r\, dr \\
&\qquad = \int_{-1}^{1} \frac{b}{2\pi} K_0'\big(b\sqrt{2-2u}\big)\sqrt{2+2u}\, du, \quad \big(r = \sqrt{2-2u}\big) \\
&\qquad = -\int_0^{\pi} \frac{1}{\pi} K_0\Big(2b\sin\frac{\omega}{2}\Big) \cos\omega\, d\omega, \quad (\text{integration by parts and } u = \cos\omega) \\
&\qquad = -I_1(b) K_1(b), \qquad (4.26)
\end{aligned}$$

where the last line follows from Lemma 2.3 with $n = 1$. Hence the multiplication operator (4.22) is simply

$$\psi \to -b^4 I_1(b) K_1(b) \psi. \tag{4.27}$$

Then (4.20) follows from (4.25) and (4.27). □

Because of the translation invariance of the problem, $S'(0)$ always has a nontrivial kernel; namely when $n = 1$, $\lambda_1 = 0$ and the corresponding eigenspace H_1 is always contained in the kernel. The stability of the bubble depends on whether the other λ_n's, $n = 0, 2, 3, 4,..$, are all positive. To study this issue, define the curves

$$C_n = \{(\sigma, b) : n^2 - 1 + \sigma b^3 \big(I_n(b)K_n(b) - I_1(b)K_1(b)\big) = 0\} \tag{4.28}$$

in the σ-b quadrant. Any (σ, b)-point on the curve C_n corresponds to a bubble whose eigenvalue λ_n vanishes. Note that when $n = 0$, one has the same curve C_0 previously given in (3.11). As in the C_0 case one may regard C_n as the graph of the function

$$\sigma = C_n(b) \equiv \frac{1 - n^2}{b^3\big(I_n(b)K_n(b) - I_1(b)K_1(b)\big)}, \quad b \in (0, \infty). \tag{4.29}$$

Each C_n divides the σ-b quadrant into two regions:

$$R_{n,s} = \{(\sigma, b) : n^2 - 1 + \sigma b^3\big(I_n(b)K_n(b) - I_1(b)K_1(b)\big) > 0\}, \tag{4.30}$$

$$R_{n,u} = \{(\sigma, b) : n^2 - 1 + \sigma b^3\big(I_n(b)K_n(b) - I_1(b)K_1(b)\big) < 0\}. \tag{4.31}$$

Note that

$$R_{0,s} = R_d, \quad R_{0,u} = R_i \tag{4.32}$$

where R_d and R_i are given in (3.13) and (3.12), respectively. Note that $R_{0,s}$ lies on the right of the curve C_0, while $R_{n,s}$ is on the left of C_n for $n = 2, 3, \ldots$

If (σ, b) is in $R_{n,s}$, then the bubble $B_b(0)$ is stable with respect to the n-th mode; if (σ, b) is in $R_{n,u}$, then the bubble is unstable with respect to this mode. In the case $n = 0$, $(\sigma, b) \in R_{0,s} = R_d$ means that the bubble is stable with respect to radial perturbations, i.e. radially stable; if $(\sigma, b) \in R_{0,u} = R_i$, then the bubble is radially unstable. A stable bubble is stable with respect to all modes, and hence (σ, b) must be in the intersection

$$\bigcap_{n=0,2,3,\ldots} R_{n,s}. \tag{4.33}$$

The following lemma gives a simple description of this set.

Lemma 4.2 *For every $b > 0$,*

1. $C_{n+1}(b) > C_n(b)$ *when* $n = 2, 3, \ldots,$
2. $C_2(b) > C_0(b)$.

Proof. Denote $I_\nu K_\nu$ by P_ν where $\nu \in \mathbb{R}_+$. Since

$$b^3\big(C_{n+1}(b) - C_n(b)\big) = \frac{(2n+1)P_1 - (n^2 + 2n)P_n + (n^2 - 1)P_{n+1}}{(P_1 - P_n)(P_1 - P_{n+1})}, \tag{4.34}$$

it suffices to show the numerator on the right is positive. It is known [4, Theorem 1 (4)] that the function $t \to P_{\sqrt{t}}(b)$ is log-convex for $t > 0$. Since a log-convex function is convex, for any $0 < \mu < 1$

$$P_{\sqrt{(1-\mu)\nu_1^2 + \mu\nu_2^2}}(b) < (1-\mu)P_{\nu_1}(b) + \mu P_{\nu_2}(b)\ .$$

Part 1 follows if $\nu_1 = 1$, $\nu_2 = n+1$ and $\mu = \frac{n^2-1}{n^2+2n}$.

Next observe that by [4, Theorem 2, Eqn. (19)] and the fact that P_n is deceasing in n,

$$P_1 \leqslant \frac{1}{2}P_0 + \frac{1}{2}P_2 < \frac{3}{4}P_0 + \frac{1}{4}P_2, \tag{4.35}$$

from which Part 2 follows. □

Lemma 4.2 implies that the curves C_n in the σ-b quadrant do not intersect. If $n < m$, C_n is to the left of C_m. The set $\cap_{n=0,2,3,\dots} R_{n,s}$ is simply the region between C_0 and C_2, i.e.,

$$\bigcap_{n=0,2,3,\dots} R_{n,s} = R_d \cap R_{2,s}. \tag{4.36}$$

The stability of bubbles is summarized in the following theorem.

Theorem 4.3 *Let α and σ be given positive numbers.*

1. *The functional J with α and σ being its parameters can have at most one radially stable bubble. Consequently there can be at most one stable bubble.*
2. *A bubble $B_b(0)$ is stable if $C_0(b) < \sigma < C_2(b)$.*

Proof. Part 2 follows from the discussion above and (4.29) that defines C_n. For part 1, note that a radially stable bubble corresponds to a point $(\sigma, b) \in R_{0,s} = R_d$ in the σ-b quadrant. In R_d, (recall the proof of Theorem 3.4), a s_α curve and a verticle σ line can intersect at most once, since $s_\alpha(b)$ is decreasing with respect to b in R_d. □

We have learned that for given α and σ there may be zero, one, two, or even three bubbles. So the bubble radius b is not uniquely determined by (σ, α). On the other hand for given σ and b, there is a unique α by (3.6). Define

$$T : (\sigma, b) \to (\sigma, \alpha) \equiv \Big(\sigma, \frac{1 + \sigma b^2 I_1 K_0}{b}\Big). \tag{4.37}$$

If T is restricted to the region R_d (also termed $R_{0,s}$), then the restriction, denoted by T_{R_d}, is one to one. Denote the inverse of T_{R_d} by $T_{R_d}^{-1}$, which maps from $T(R_d)$ back to R_d. Note that $T(\cap_{n=0,2,3,\dots} R_{n,s}) \subset T(R_d)$ since $\cap_{n=0,2,3,\dots} R_{n,s} \subset R_d$. With the help of T one can precisely identify regions for (σ, α) that give rise to radially stable bubbles and stable bubbles, respectively.

Theorem 4.4 *If $(\sigma, \alpha) \in T(R_d)$, then with $(\sigma, b) = T_{R_d}^{-1}(\sigma, \alpha)$, $B_b(0)$ is a radially stable bubble. If $(\sigma, \alpha) \in T(\cap_{n=0,2,3,\dots} R_{n,s})$, then with $(\sigma, b) = T_{R_d}^{-1}(\sigma, \alpha)$, $B_b(0)$ is a stable bubble.*

Figure 3 shows $T(\cap_{n=0,2,3,\dots} R_{n,s})$ in purple and $T(R_d)\backslash T(\cap_{n=0,2,3,\dots} R_{n,s})$ in yellow. Note that the union of the yellow and the purple regions in Figure 3 equals the union of the green and the blue regions in Figure 2.

5. Droplet assemblies on a bounded domain

Beginning in this section we study the singular limit J_D on a bounded domain D and show that when α and σ are in a proper range, the functional J_D admits stable stationary sets that are assemblies of perturbed small discs.

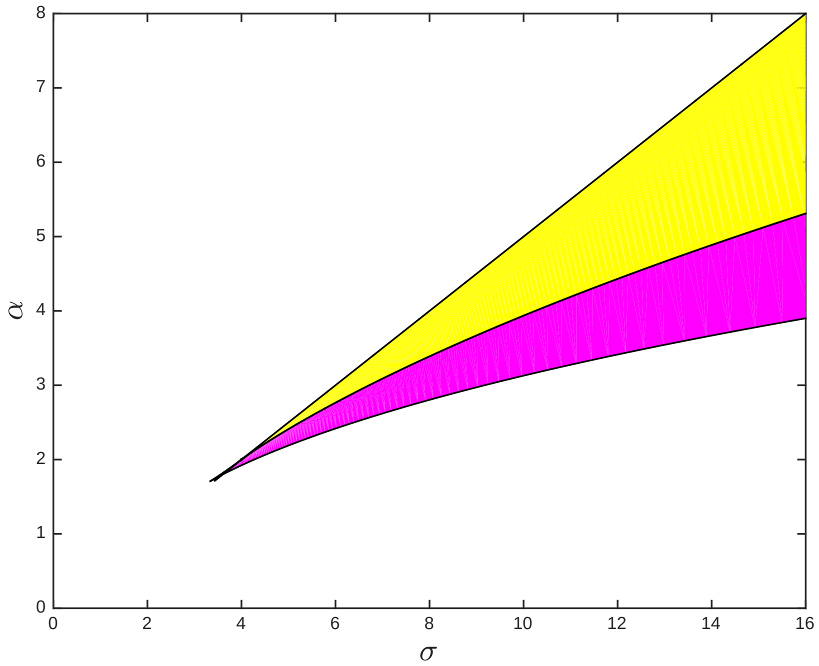

FIG. 3. If (σ, α) is in the purple region, there is a stable bubble; if (σ, α) is in the yellow region, there is a bubble that is stable with respect to radial perturbations, but not stable with respect to some other perturbations. The union of the yellow and the purple regions is the same as the union of the blue and green regions in Figure 2. The yellow region is separated from the purple region by the image of the curve C_2 under T. (Only the ebook/online version of this article contains full color images.)

Since

$$I_0(b)K_0(b) = \Big(\log\frac{1}{b}\Big)\big(1+o(1)\big), \quad I_n(b)K_n(b) = \frac{1}{2n}\big(1+o(1)\big), \quad n = 1,2,3,\dots$$

when b is small [1, (9.6.7)–(9.6.9)], one deduces that

$$C_0(b) = \frac{1}{b^3(I_0K_0 - I_1K_1)} = \frac{1}{b^3\log\frac{1}{b}}\big(1+o(1)\big), \tag{5.1}$$

$$C_2(b) = \frac{3}{b^3(I_1K_1 - I_2K_2)} = \frac{12}{b^3}\big(1+o(1)\big). \tag{5.2}$$

Consequently for every $\delta > 0$ there exists $b_0 > 0$ such that

$$\left\{(\sigma, b) : 0 < b < b_0,\ \frac{1+\delta}{b^3\log\frac{1}{b}} < \sigma < \frac{12-\delta}{b^3}\right\} \subset \bigcap_{n=0,2,3,\dots} R_{n,s}\ . \tag{5.3}$$

The set on the left side of (5.3) will play a critical role in the next theorem. First assume that D is sufficiently smooth and there is a Green's function of the equation (1.6), denoted by $G = G(x, y)$, so that

$$N_D(\Omega)(x) = \int_D G(x, y)\chi_\Omega(y)\, dy. \tag{5.4}$$

One can decompose G into two parts:

$$G(x, y) = \frac{1}{2\pi} K_0\big(|x - y|\big) + R(x, y), \tag{5.5}$$

where the modified Bessel function $K_0(|x - y|)$ is singular when $x = y$ while the regular part R is smooth. Second introduce a key quantity in this section, a function F defined in terms of G and R, for K distinct points $\xi^1, \xi^2, \dots, \xi^K$ in D:

$$F(\xi^1, \xi^2, \dots, \xi^K) = \sum_{k=1}^{K} R(\xi^k, \xi^k) + \sum_{k=1}^{K} \sum_{l=1, l\neq k}^{K} G(\xi^k, \xi^l). \tag{5.6}$$

Third recall the mapping T from (4.37) and its restriction T_{R_d} to R_d.

Theorem 5.1 *Let D be a bounded and sufficiently smooth domain in $\mathbb{R}^2$, $K \in \mathbb{N}$ and $\delta > 0$. There exists $b_0 > 0$ depending on D, K, and δ, such that if*

1. $(\sigma, \alpha) \in T(R_d)$,
2. $b < b_0$,
3. $\frac{1+\delta}{b^3 \log \frac{1}{b}} < \sigma < \frac{12-\delta}{b^3}$,

where $(\sigma, b) = T_{R_d}^{-1}(\sigma, \alpha)$, then J_D, with α and σ being its parameters, admits a stationary set that is an assembly of K perturbed discs. Moreover,

1. *the radii of all the perturbed discs are approximately the same;*
2. *if the centers of the perturbed discs are $\xi^1_*, \xi^2_*, \dots, \xi^K_*$, then as $b \to 0$, $(\xi^1_*, \xi^2_*, \dots, \xi^K_*)$ converges to a minimum of F, possibly along a subsequence;*
3. *this stationary assembly is stable in some sense.*

A few remarks are in order. This strategy of the proof of Theorem 5.1 was first developed by Ren and Wei in [38, 39] for the Ohta-Kawasaki diblock copolymer problem [31]. It was later successfully applied to more sophisticated nonlocal geometric variational problems [35, 40]. The main difference between the diblock copolymer problem studied in [38] and the present problem lies in the the bubble profile used in the construction. In [38] there always exists a unique bubble simply because of an area constraint on the permissible sets. That bubble is stable if the nonlocal interaction term in [38] is not too large. Here we have no area constraint. The very subtle existence and stability properties of the bubble used in the proof of Theorem 5.1 come from Theorems 3.4 and 4.3.

For the diblock copolymer problem, Choski and Sternberg [13] studied the stability issue by a second variation formula, and Acerbi, Fusco and Morini [2] showed that positivity of the second variation implies local minimality for stationary sets.

The smoothness requirement on D is only needed to ensure that the operator N_D is well defined. This is the case if D is of the class $C^{2,\alpha}$ [22, Section 6.7]. By (5.3) the range of (σ, α) specified by the three conditions stated in the theorem is not empty. The corresponding (σ, b) is in a tail of $\cap_{n=0,2,3,\dots} R_{n,s}$ with small b. In this tail, $b \to 0$ is equivalent to $\sigma \to \infty$.

To get a rough idea what α and σ look like when $b \to 0$ in this theorem, note

$$\alpha = \frac{1 + \sigma b^2 I_1 K_0}{b} = \frac{1 + \frac{\sigma b^3}{2} \log \frac{1}{b}\big(1 + o(1)\big)}{b} \tag{5.7}$$

by [1, (9.6.7) and (9.6.8)]. In the two borderline cases of the third assumption of the theorem,

1. if $\sigma \sim \frac{1}{b^3 \log \frac{1}{b}}$, then $\alpha \sim \frac{3}{2b}$;
2. if $\sigma \sim \frac{12}{b^3}$, then $\alpha \sim \frac{6}{b} \log \frac{1}{b}$.

Hence as $b \to 0$, both α and σ approach ∞, although σ does so much faster than α does.

We start with a construction of assemblies of exact discs whose radii are close to b. First let

$$\Xi = \Big\{\xi = \big(\xi^1, \xi^2, \ldots, \xi^K\big) : \ \xi^k \in D, \ \xi^k \neq \xi^l \text{ if } k \neq l\Big\}, \tag{5.8}$$

which is the domain of the function F given in (5.6). Then, for $\eta > 0$, define

$$\Xi_\eta = \Big\{\xi = \big(\xi^1, \xi^2, \ldots, \xi^K\big) : \ \xi^k \in D, \ \operatorname{dist}\big(\xi^k, \partial D\big) > \eta, \ \operatorname{dist}\big(\xi^k, \xi^l\big) > 2\eta \text{ for } k \neq l\Big\}. \tag{5.9}$$

Regarding F defined in (5.6), note that $G(x, y) \to \infty$ if $y \to x$. The techniques in [35] may also be used to study the behavior of $R(x, y)$ when x and y are close to ∂D. In particular based on the property that $R(z, z) \to \infty$ if $z \to \partial D$, one can find a sufficiently small η such that

$$\inf_{\xi \in \Xi \setminus \Xi_\eta} F(\xi) > \inf_{\xi \in \Xi} F(\xi). \tag{5.10}$$

This implies that the minimum of F on Ξ is achieved inside Ξ_η. The number η is fixed throughout the paper.

Let $\xi = (\xi^1, \xi^2, \ldots, \xi^K) \in \overline{\Xi_\eta}$, the closure of Ξ_η in $\mathbb{R}^{2K}$, and make an approximate solution that is an assembly of small discs centered at the ξ^k's. The radii of the discs are $b\beta^k$ where the β^k's are not yet determined. Collectively write $\beta = (\beta^1, \beta^2, \ldots, \beta^K) \in \overline{B_\tau}$, where $\overline{B_\tau}$ is the closure of a box B_τ in $\mathbb{R}^K$ defined by

$$B_\tau = \big\{\beta : \ \beta^k \in (1 - \tau, 1 + \tau)\big\}. \tag{5.11}$$

The number τ will be specified and made small later, so the radii $b\beta^k$ become close to b. For now we only require that $\tau \in (0, 1)$. Take b_0 to be sufficiently small so that when $b < b_0$,

$$b, \ b\beta^k < \frac{\eta}{2} \ \text{ for all } \beta \in B_\tau. \tag{5.12}$$

Let P_0 be the union of discs P_0^k centered at ξ^k of radii $b\beta^k$:

$$P_0 = \cup_{k=1}^K P_0^k \ \text{ where } \ P_0^k = \big\{x \in \mathbb{R}^2 : \ |x - \xi^k| \leqslant b\beta^k\big\}. \tag{5.13}$$

Because of (5.12) the assembly P_0 consists of non-overlapping discs, and P_0 is contained in D.

Lemma 5.2 *Let $\xi \in \overline{\Xi_\eta}$ and $\beta \in \overline{B_\tau}$. Then*

$$\begin{aligned} J_D(P_0) = &\sum_{k=1}^K \Big(2\pi b\beta^k - \alpha\pi(b\beta^k)^2 + \frac{\sigma\pi(b\beta^k)^2}{2}\big(1 - 2I_1(b\beta^k)K_1(b\beta^k)\big)\Big) \\ &+ \sum_{k=1}^K \frac{\sigma\pi^2(b\beta^k)^4}{2} R(\xi^k, \xi^k) + \sum_{k=1}^K \sum_{l=1, l\neq k}^K \frac{\sigma\pi^2(b\beta^k)^2(b\beta^l)^2}{2} G(\xi^k, \xi^l) + O(\sigma b^6). \end{aligned}$$

Proof. Since, by Lemma 2.1,

$$\int_{|x|<t}\int_{|y|<t}\frac{1}{2\pi}K_0(|x-y|)\,dy\,dx=\pi t^2\big(1-2I_1(t)K_1(t)\big),$$

one obtains (with $t=b\beta^k$)

$$\begin{aligned}J_D(P_0)&=\sum_{k=1}^K\Big(2\pi b\beta^k-\alpha\pi(b\beta^k)^2+\frac{\sigma}{2}\int_{P_0^k}\int_{P_0^k}\frac{1}{2\pi}K_0(|x-y|)\,dy\,dx\Big)\\&\quad+\sum_{k=1}^K\frac{\sigma}{2}\int_{P_0^k}\int_{P_0^k}R(x,y)\,dy\,dx+\sum_{k=1}^K\sum_{l=1,l\neq k}^K\frac{\sigma}{2}\int_{P_0^k}\int_{P_0^l}G(x,y)\,dy\,dx\\&=\sum_{k=1}^K\Big(2\pi b\beta^k-\alpha\pi(b\beta^k)^2+\frac{\sigma\pi(b\beta^k)^2}{2}\big(1-2I_1(b\beta^k)K_1(b\beta^k)\big)\Big)\\&\quad+\sum_{k=1}^K\frac{\sigma}{2}\int_{P_0^k}\int_{P_0^k}R(x,y)\,dy\,dx+\sum_{k=1}^K\sum_{l=1,l\neq k}^K\frac{\sigma}{2}\int_{P_0^k}\int_{P_0^l}G(x,y)\,dy\,dx.\end{aligned}$$

Denote the gradient of $R(x,y)$ ($G(x,y)$ resp.) with respect to x by $\nabla R(x,y)$ ($\nabla G(x,y)$ resp.), and the gradient with respect to y by $\widetilde{\nabla}R(x,y)$ ($\widetilde{\nabla}G(x,y)$ resp.). Then

$$\begin{aligned}\int_{P_0^k}R(x,y)\,dy&=\int_{|y-\xi^k|<b\beta^k}\big(R(x,\xi^k)+\widetilde{\nabla}R(x,\xi^k)\cdot(y-\xi^k)+O(b^2)\big)\,dy\\&=\pi(b\beta^k)^2R(x,\xi^k)+O(b^4)\end{aligned}$$

and

$$\begin{aligned}&\int_{P_0^k}\int_{P_0^k}R(x,y)\,dy\,dx\\&\quad=\int_{|x-\xi^k|<b\beta^k}\big(\pi(b\beta^k)^2R(x,\xi^k)+O(b^4)\big)\,dx\\&\quad=\int_{|x-\xi^k|<b\beta^k}\big(\pi(b\beta^k)^2R(\xi^k,\xi^k)+(b\beta^k)^2\nabla R(\xi^k,\xi^k)\cdot(x-\xi^k)+O(b^4)\big)\,dx\\&\quad=\pi^2(b\beta^k)^4R(\xi^k,\xi^k)+O(b^6).\end{aligned}$$

Similarly

$$\int_{P_0^k}\int_{P_0^l}G(x,y)\,dy\,dx=\pi^2(b\beta^k)^2(b\beta^l)^2G(\xi^k,\xi^l)+O(b^6).$$

The lemma then follows from these estimates. □

To introduce a perturbed assembly, let ϕ^k, $k=1,2,\dots,K$, be 2π-periodic functions. Let

$$P_{\phi^k}^k=\{\xi^k+re^{\mathrm{i}\theta}:\ r\in\Big[0,b\beta^k\sqrt{1+2\phi^k(\theta)}\Big],\ \theta\in\mathbb{S}^1\}\tag{5.14}$$

be a perturbed disc, and

$$P_\phi = \cup_{k=1}^K P^k_{\phi^k} \tag{5.15}$$

be a perturbed assembly. Here one writes $\phi = (\phi^1, \phi^2, \ldots, \phi^K)$ and P_ϕ to emphasize that this assembly depends on ϕ. In fact, P_ϕ also depends on $\beta = (\beta^1, \beta^2, \ldots, \beta^K)$ and $\xi = (\xi^1, \xi^2, \ldots, \xi^K)$, and these dependencies will be exploited later.

Next specify the domain for ϕ so that J_D can be viewed as a functional of ϕ. The first space is $L^2(\mathbb{S}^1, \mathbb{R}^K)$, a Hilbert space with the inner product

$$\langle \phi, \psi \rangle = \sum_{k=1}^K \int_0^{2\pi} \phi^k(\theta)\psi^k(\theta)\, d\theta. \tag{5.16}$$

The second space is $H^1(\mathbb{S}^1, \mathbb{R}^K)$; the third space is $H^2(\mathbb{S}^1, \mathbb{R}^K)$. They are standard Sobolev spaces, with range in $\mathbb{R}^K$. The norms of the latter two spaces are defined by

$$\|\phi\|^2_{H^1} = \int_0^{2\pi} \sum_{k=1}^K \Big((\phi^{k,\prime})^2 + (\phi^k)^2\Big)\, d\theta,\ \|\phi\|^2_{H^2} = \int_0^{2\pi} \sum_{k=1}^K \Big((\phi^{k,\prime\prime})^2 + (\phi^{k,\prime})^2 + (\phi^k)^2\Big)\, d\theta, \tag{5.17}$$

respectively. The three spaces are nested: $H^2(\mathbb{S}^1, \mathbb{R}^K) \subset H^1(\mathbb{S}^1, \mathbb{R}^K) \subset L^2(\mathbb{S}^1, \mathbb{R}^K)$.

As a functional of ϕ, the domain of J_D is

$$Dom(J_D) = \big\{\phi \in H^1(\mathbb{S}^1, \mathbb{R}^K) : \|\phi\|_{H^1} < \eta_1\big\} \tag{5.18}$$

with $\eta_1 > 0$ small enough so that

$$\sqrt{1 + 2\phi^k(\theta)} \in (0, 2), \text{ for all } \theta \in \mathbb{S}^1. \tag{5.19}$$

This and (5.12) imply that

$$b\beta^k \sqrt{1 + 2\phi^k(\theta)} \in (0, \eta) \tag{5.20}$$

where η is given in (5.10). Hence P_ϕ comprises of non-overlapping perturbed discs and $P_\phi \subset D$. As a functional of ϕ, $J_D(P_\phi)$ is written more explicitly as

$$J_D(\phi) = \sum_{k=1}^K \int_0^{2\pi} b\beta^k \sqrt{\frac{(\phi^{k,\prime})^2}{1 + 2\phi^k} + 1 + 2\phi^k}\, d\theta$$
$$- \alpha \sum_{k=1}^K (b\beta^k)^2 \int_0^{2\pi} \frac{1 + 2\phi^k}{2}\, d\theta + \frac{\sigma}{2} \int_{P_\phi} \int_{P_\phi} G(x, y)\, dx dy. \tag{5.21}$$

Again for a general $\Omega \subset D$ and a deformation Ω_ε of Ω, there is a first variation formula:

$$\frac{d J_D(\Omega_\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = -\int_{\partial\Omega} \big(K(\partial\Omega) - \alpha + \sigma N_D(\Omega)\big)\mathbf{N} \cdot \mathbf{X}\, ds, \tag{5.22}$$

which holds for sufficiently smooth Ω_ε that does not meet the boundary of D. The vector $\mathbf{N}$ is the inward pointing unit normal on $\partial\Omega$ and ds is the arc length element of $\partial\Omega$. The vector $\mathbf{X}$ is the infinitesimal element of the deformation of $\partial\Omega$. The Euler–Lagrange equation (1.8) also follows from formula (5.22).

Define a deformation of P_ϕ by deforming ϕ to

$$\phi + \varepsilon\psi,\ \phi \in Dom(J_D) \text{ and } \psi \in H^1(\mathbb{S}^1, \mathbb{R}^K), \tag{5.23}$$

which gives rise to $P_{\phi+\varepsilon\psi}$. Denote the boundary of the k-th component of this assembly by

$$\mathbf{R}^k_\varepsilon(\theta) = \xi^k + b\beta^k \sqrt{1 + 2\big(\phi^k(\theta) + \varepsilon\psi^k(\theta)\big)}\, e^{\mathrm{i}\theta},\ \theta \in \mathbb{S}^1. \tag{5.24}$$

Denote the inward normal vector of $\mathbf{R}^k_0$ by $\mathbf{N}^k$ and the infinitesimal element of the deformation $\mathbf{R}^k_\varepsilon$ by $\mathbf{X}^k$. Then, if $\phi \in H^2(\mathbb{S}^1, \mathbb{R}^K)$,

$$\begin{aligned}
\mathbf{N}^k \cdot \mathbf{X}^k\, ds &= \mathrm{i}\frac{\partial \mathbf{R}^k_\varepsilon}{\partial\theta}\Big|_{\varepsilon=0} \cdot \frac{\partial \mathbf{R}^k_\varepsilon}{\partial\varepsilon}\Big|_{\varepsilon=0} d\theta \\
&= \Big(\frac{b\beta^k \frac{\partial\phi^k}{\partial\theta}}{\sqrt{1+2\phi^k}} e^{\mathrm{i}\theta}\mathrm{i} - b\beta^k\sqrt{1+2\phi^k}\, e^{\mathrm{i}\theta}\Big) \cdot \Big(\frac{b\beta^k\psi^k}{\sqrt{1+2\phi^k}} e^{\mathrm{i}\theta}\Big)\, d\theta \\
&= -(b\beta^k)^2\psi^k\, d\theta.
\end{aligned} \tag{5.25}$$

The first variation of J_D now becomes

$$\frac{dJ_D(\phi+\varepsilon\psi)}{d\varepsilon}\Big|_{\varepsilon=0} = \sum_{k=1}^{K} \int_0^{2\pi} \big(K(\partial P^k_{\phi^k}) - \alpha + \sigma N_D(P_\phi)\big)(b\beta^k)^2\psi^k\, d\theta. \tag{5.26}$$

Introduce a nonlinear operator $S_D : Dom(S_D) \to L^2(\mathbb{S}^1, \mathbb{R}^K)$ with the domain

$$Dom(S_D) = \big\{\phi \in H^2(\mathbb{S}^1, \mathbb{R}^K) :\ \|\phi\|_{H^2} < \eta_1\big\} \tag{5.27}$$

where η_1 is the same as the one in (5.18). Let $S_D = (S^1_D, S^2_D, \ldots, S^K_D)$ and define its k-th component by

$$S^k_D(\phi) = (b\beta^k)^2\big(K(\partial P^k_{\phi^k}) - \alpha + \sigma N_D(P_\phi)\big) \tag{5.28}$$

where the right side of (5.28) is evaluated at $\partial P^k_{\phi^k}$. More specifically this quantity is evaluated at $\xi^k + b\beta^k\sqrt{1+2\phi^k(\theta)}\, e^{\mathrm{i}\theta}$ and viewed as a function with θ being the input variable. If one computes the curvature of $\partial P^k_{\phi^k}$ in terms of ϕ^k, then an explicit formula is obtained:

$$\begin{aligned}
S^k_D(\phi)(\theta) = {} & \frac{b\beta^k\Big(1 + 2\phi^k + \frac{3(\phi^{k,\prime})^2}{1+2\phi^k} - \phi^{k,\prime\prime}\Big)}{\Big(1 + 2\phi^k + \frac{(\phi^{k,\prime})^2}{1+2\phi^k}\Big)^{3/2}} - \alpha(b\beta^k)^2 \\
& + \sigma(b\beta^k)^2 \int_{P_\phi} G(\xi^k + b\beta^k\sqrt{1+2\phi^k(\theta)}\, e^{\mathrm{i}\theta}, y)\, dy.
\end{aligned} \tag{5.29}$$

The Euler–Lagrange equation (1.8) now becomes the equation $S_D(\phi) = 0$.

With the operator S_D one can write the first variation (5.26) more concisely as

$$\frac{dJ_D(\phi+\varepsilon\psi)}{d\varepsilon}\Big|_{\varepsilon=0} = \langle S_D(\phi), \psi\rangle. \tag{5.30}$$

The right side of (5.30) is meaningful if either $\phi \in Dom(S_D)$ and $\psi \in L^2(\mathbb{S}^1, \mathbb{R}^K)$, or $\phi \in Dom(J_D)$ and $\psi \in H^1(\mathbb{S}^1, \mathbb{R}^K)$.

Next consider the second variation of J_D, which is also the Fréchet derivative of S_D. Let $\phi + \varepsilon_1\psi + \varepsilon_2\upsilon$ be a two parameter deformation of ϕ. Then

$$\frac{\partial^2 J_D(\phi+\varepsilon_1\psi+\varepsilon_2\upsilon)}{\partial\varepsilon_1\partial\varepsilon_2}\Big|_{\varepsilon_1=\varepsilon_2=0} = \langle S_D'(\phi)\psi, \upsilon\rangle \tag{5.31}$$

Here $S_D'(\phi)$ is the Fréchet derivative of S_D at ϕ and

$$\psi \to S_D'(\phi)\psi \tag{5.32}$$

is a self-adjoint linear operator from $H^2(\mathbb{S}^1, \mathbb{R}^K) \subset L^2(\mathbb{S}^1, \mathbb{R}^K)$ to $L^2(\mathbb{S}^1, \mathbb{R}^K)$. The quadratic form $\langle S_D'(\phi)\psi, \upsilon\rangle$ is defined if $\phi \in Dom(S_D)$, $\psi \in H^2(\mathbb{S}^1, \mathbb{R}^K)$, and $\upsilon \in L^2(\mathbb{S}^1, \mathbb{R}^K)$. Moreover, one can also use the left side of (5.31) to interpret the quadratic form if $\phi \in Dom(J_D)$, and $\psi, \upsilon \in H^1(\mathbb{S}^1, \mathbb{R}^K)$.

Introduce some subspaces

$$L^2_\flat(\mathbb{S}^1, \mathbb{R}^K) = \Big\{\phi \in L^2(\mathbb{S}^1, \mathbb{R}^K) : \int_0^{2\pi} \phi^k(\theta)\, d\theta = \int_0^{2\pi} \phi^k(\theta)\cos\theta\, d\theta = \int_0^{2\pi} \phi^k(\theta)\sin\theta\, d\theta = 0,\ \forall k\Big\} \tag{5.33}$$

$$H^1_\flat(\mathbb{S}^1, \mathbb{R}^K) = \Big\{\phi \in H^1(\mathbb{S}^1, \mathbb{R}^K) : \int_0^{2\pi} \phi^k(\theta)\, d\theta = \int_0^{2\pi} \phi^k(\theta)\cos\theta\, d\theta = \int_0^{2\pi} \phi^k(\theta)\sin\theta\, d\theta = 0,\ \forall k\Big\} \tag{5.34}$$

$$H^2_\flat(\mathbb{S}^1, \mathbb{R}^K) = \Big\{\phi \in H^2(\mathbb{S}^1, \mathbb{R}^K) : \int_0^{2\pi} \phi^k(\theta)\, d\theta = \int_0^{2\pi} \phi^k(\theta)\cos\theta\, d\theta = \int_0^{2\pi} \phi^k(\theta)\sin\theta\, d\theta = 0,\ \forall k\Big\} \tag{5.35}$$

and denote the orthogonal projection from $L^2(\mathbb{S}^1, \mathbb{R}^K)$ to $L^2_\flat(\mathbb{S}^1, \mathbb{R}^K)$ by Π. Geometrically one can interpret an element in $L^2_\flat(\mathbb{S}^1, \mathbb{R}^K)$ (or $H^1_\flat(\mathbb{S}^1, \mathbb{R}^K)$ or $H^2_\flat(\mathbb{S}^1, \mathbb{R}^K)$) as an assembly whose perturbed discs have well defined centers ξ^k and well defined radii $b\beta^k$. More specifically, the condition $\int_0^{2\pi} \phi^k(\theta)\, d\theta = 0$ implies that $b\beta^k$ can be interpreted as the radius of the perturbed disc $P^k_{\phi^k}$; the condition $\int_0^{2\pi} \phi^k(\theta)\cos\theta\, d\theta = \int_0^{2\pi} \phi^k(\theta)\sin\theta\, d\theta = 0$ defines ξ^k as the center of $P^k_{\phi^k}$.

6. Solving $\Pi S_D(\phi) = 0$

It is not realistic to solve $S_D(\phi) = 0$ for arbitrarily given β and ξ. Instead in this section we solve a weaker equation, $\Pi S_D(\phi) = 0$, in $H^2_\flat(\mathbb{S}^1, \mathbb{R}^K)$.

Lemma 6.1 *There exists $C_1 > 0$ such that $\|\Pi S_D(0)\|_{L^2} \leqslant C_1 \sigma b^6$.*

Proof. Insert P_0 into the left side of the Euler–Lagrange equation (1.8). Since

$$K(\partial P_0^k) = \frac{1}{b\beta^k}, \tag{6.1}$$

it remains to estimate

$$\begin{aligned} N_D(P_0)(\xi^k + b\beta^k e^{i\theta}) &= \int_{P_0} G\big(\xi^k + b\beta^k e^{i\theta}, y\big)\, dy \\ &= \int_{P_0^k} \frac{1}{2\pi} K_0\big(\big|\xi^k + b\beta^k e^{i\theta} - y\big|\big)\, dy + \int_{P_0^k} R\big(\xi^k + b\beta^k e^{i\theta}, y\big)\, dy \\ &\quad + \sum_{l=1, l\neq k}^{K} \int_{P_0^l} G\big(\xi^k + b\beta^k e^{i\theta}, y\big)\, dy. \end{aligned}$$

The first integral of the last line was calculated in (2.1):

$$\int_{P_0^k} \frac{1}{2\pi} K_0\big(\big|\xi^k + b\beta^k e^{i\theta} - y\big|\big)\, dy = \int_{|y|<b\beta^k} \frac{1}{2\pi} K_0\big(\big|b\beta^k e^{i\theta} - y\big|\big)\, dy = b\beta^k I_1(b\beta^k) K_0(b\beta^k). \tag{6.2}$$

As in the proof of Lemma 5.2, the second and the third integrals become

$$\begin{aligned} &\int_{P_0^k} R(\xi^k + b\beta^k e^{i\theta}, y)\, dy \\ &\quad = \int_{P_0^k} \big(R(\xi^k + b\beta^k e^{i\theta}, \xi^k) + \widetilde{\nabla} R(\xi^k + b\beta^k e^{i\theta}, \xi^k) \cdot (y - \xi^k) + O(b^2)\big)\, dy \\ &\quad = \pi (b\beta^k)^2 R(\xi^k + b\beta^k e^{i\theta}, \xi^k) + O(b^4) \\ &\quad = \pi (b\beta^k)^2 R(\xi^k, \xi^k) + \pi (b\beta^k)^2 \nabla R(\xi^k, \xi^k) \cdot b\beta^k e^{i\theta} + O(b^4) \end{aligned} \tag{6.3}$$

$$\begin{aligned} &\int_{P_0^l} G(\xi^k + b\beta^k e^{i\theta}, y)\, dy \\ &\quad = \int_{P_0^l} \big(G(\xi^k + b\beta^k e^{i\theta}, \xi^l) + \widetilde{\nabla} G(\xi^k + b\beta^k e^{i\theta}, \xi^l) \cdot (y - \xi^l) + O(b^2)\big)\, dy \\ &\quad = \pi (b\beta^l)^2 G(\xi^k + b\beta^k e^{i\theta}, \xi^l) + O(b^4) \\ &\quad = \pi (b\beta^l)^2 G(\xi^k, \xi^l) + \pi (b\beta^l)^2 \nabla G(\xi^k, \xi^l) \cdot b\beta^k e^{i\theta} + O(b^4), \end{aligned} \tag{6.4}$$

uniformly with respect to θ. By (6.1), (6.2), (6.3), and (6.4), one derives

$$\begin{aligned} &K(\partial P_0) - \alpha + \sigma N_D(P_0) \\ &= \frac{1}{b\beta^k} - \alpha + \sigma b\beta^k I_1(b\beta^k) K_0(b\beta^k) + \sigma\pi (b\beta^k)^2 R(\xi^k, \xi^k) + \sigma\pi (b\beta^k)^3 \nabla R(\xi^k, \xi^k) \cdot e^{i\theta} \\ &\quad + \sum_{l=1, l\neq k}^{K} \sigma\pi (b\beta^l)^2 G(\xi^k, \xi^l) + \sum_{l=1, l\neq k}^{K} \sigma\pi (b\beta^l)^2 (b\beta^k) \nabla G(\xi^k, \xi^l) \cdot e^{i\theta} + O(\sigma b^4) \end{aligned} \tag{6.5}$$

on ∂P_0^k. The lemma follows from (6.5), (5.28), and the definition of the Π operator. □

In Lemma 6.1 the upper case C_1 is used in an upper bound estimate and the subscript 1 reminds us that the estimate is for the first variation of J_D at 0. In the next lemma we will use a lower case c_2 in a lower bound estimate on the second variation of J_D at 0.

Lemma 6.2 *There exists $c_2 > 0$ such that the following two statements hold when σ is sufficiently large.*

1. *For all $\psi \in H^2_\flat(\mathbb{S}^1, \mathbb{R}^K)$,*
$$\|\Pi S'_D(0)\psi\|_{L^2} \geqslant c_2 b \|\psi\|_{H^2}.$$
The linear map $\Pi S'_D(0)$ is one-to-one and onto from $H^2_\flat(\mathbb{S}^1, \mathbb{R}^K)$ to $L^2_\flat(\mathbb{S}^1, \mathbb{R}^K)$ whose inverse is bounded by $\|(\Pi S'_D(0))^{-1}\| \leqslant \frac{1}{c_2 b}$.
2. *For all $\psi \in H^1_\flat(\mathbb{S}^1, \mathbb{R}^K)$,*
$$\langle \Pi S'_D(0)\psi, \psi\rangle \geqslant c_2 b \|\psi\|^2_{H^1}.$$

Proof. One computes S'_D from (5.29) and finds the k-th component of the linear operator $S'_D(0)$ to be

$$S'^{,k}_D(0)\psi = b\beta^k(-\psi^{k,\prime\prime} - \psi^k) + \sigma \sum_{l=1}^{K} (b\beta^k)^2 (b\beta^l)^2 \int_0^{2\pi} G(\xi^k + b\beta^k e^{i\theta}, \xi^l + b\beta^l e^{i\omega}) \psi^l(\omega)\, d\omega$$
$$+ \sigma \sum_{l=1}^{K} (b\beta^k)^2 (b\beta^l) \int_{P^l_0} \nabla G(\xi^k + b\beta^k e^{i\theta}, y)\, dy \cdot e^{i\theta} \psi^l(\theta). \quad (6.6)$$

One separates $S'_D(0)$ into a major part L and a minor part M: $S'_D(0) = L + M$. Here

$$\begin{aligned} L^k\psi &= b\beta^k(-\psi^{k,\prime\prime} - \psi^k) + \sigma \int_0^{2\pi} \frac{(b\beta^k)^4}{2\pi} K_0(|b\beta^k e^{i\theta} - b\beta^k e^{i\omega}|)\psi^k(\omega)\, d\omega \\ &+ \sigma \int_{|y|<b\beta^k} \frac{(b\beta^k)^3}{2\pi} K'_0(|b\beta^k e^{i\theta} - y|) \frac{b\beta^k e^{i\theta} - y}{|b\beta^k e^{i\theta} - y|}\, dy \cdot e^{i\theta}\psi^k(\theta) \\ &+ \sigma \int_0^{2\pi} (b\beta^k)^4 R(\xi^k, \xi^k)\psi^k(\omega)\, d\omega \\ &+ \sigma \sum_{l=1, l\neq k}^{K} \int_0^{2\pi} (b\beta^k)^2 (b\beta^l)^2 G(\xi^k, \xi^l)\psi^l(\omega)\, d\omega, \end{aligned} \quad (6.7)$$

$$\begin{aligned} M^k\psi &= \sigma \int_0^{2\pi} (b\beta^k)^4 \big(R(\xi^k + b\beta^k e^{i\theta}, \xi^k + b\beta^k e^{i\omega}) - R(\xi^k, \xi^k)\big)\psi^k(\omega)\, d\omega \\ &+ \sigma \int_{P^k_0} (b\beta^k)^3 \nabla R(\xi^k + b\beta^k e^{i\theta}, y)\, dy \cdot e^{i\theta}\psi^k(\theta) \\ &+ \sigma \sum_{l=1, l\neq k}^{K} \int_0^{2\pi} (b\beta^k)^2 (b\beta^l)^2 \big(G(\xi^k + b\beta^k e^{i\theta}, \xi^l + b\beta^l e^{i\omega}) - G(\xi^k, \xi^l)\big)\psi^l(\omega)\, d\omega \\ &+ \sigma \sum_{l=1, l\neq k}^{K} \int_{P^l_0} (b\beta^k)^2 (b\beta^l) \nabla G(\xi^k + b\beta^k e^{i\theta}, y)\, dy \cdot e^{i\theta}\psi^l(\theta). \end{aligned} \quad (6.8)$$

The minor part is estimated easily:

$$\|M\psi\|_{L^2} \leqslant C\sigma b^5 \|\psi\|_{L^2}, \ \forall \psi \in L^2(\mathbb{S}^1, \mathbb{R}^K). \tag{6.9}$$

For the major part L, decompose the space $L^2(\mathbb{S}^1, \mathbb{R}^K)$ into a Hilbert sum:

$$L^2(\mathbb{S}^1, \mathbb{R}^K) = H_0 \oplus \big(\oplus_{n=1}^{\infty} \oplus_{k=1}^{K} H_n^k \big) \tag{6.10}$$

where

$$H_0 = \{\psi = (c^1, c^2, \ldots, c^K) : \ c^k \in \mathbb{R}\} \tag{6.11}$$

$$H_n^k = \{\psi = (0, 0, \ldots, 0, \psi^k, 0, \ldots, 0) : \ \psi^k = A\cos n\theta + B \sin n\theta, \ A, B \in \mathbb{R}\}. \tag{6.12}$$

Each H_n^k is a two dimensional eigenspace of L, whose corresponding eigenvalue is

$$\lambda_n^k = b\beta^k(n^2 - 1) + \sigma(b\beta^k)^4 \big(I_n(b\beta^k)K_n(b\beta^k) - I_1(b\beta^k)K_1(b\beta^k)\big). \tag{6.13}$$

To prove (6.13), since the last two terms of $L^k\psi$ in (6.7) vanish on H_n^k whenever $n \geqslant 1$, it suffices to study the integral operator

$$\psi^k \to \int_0^{2\pi} \frac{(b\beta^k)^4}{2\pi} K_0\big(|b\beta^k e^{\mathrm{i}\theta} - b\beta^k e^{\mathrm{i}\omega}|\big)\psi^k(\omega)\, d\omega \tag{6.14}$$

and the multiplication operator

$$\psi^k \to \int_{|y|<b\beta^k} \frac{(b\beta^k)^3}{2\pi} K_0'\big(|b\beta^k e^{\mathrm{i}\theta} - y|\big) \frac{b\beta^k e^{\mathrm{i}\theta} - y}{|b\beta^k e^{\mathrm{i}\theta} - y|}\, dy \cdot e^{\mathrm{i}\theta}\psi^k(\theta). \tag{6.15}$$

As in (4.25), taking ψ^k to be $\cos n\theta$ or $\sin n\theta$, one finds that the integral operator (6.14) acts like

$$\cos n\theta \to (b\beta^k)^4 I_n(b\beta^k)K_n(b\beta^k)\cos n\theta, \quad \sin n\theta \to (b\beta^k)^4 I_n(b\beta^k)K_n(b\beta^k)\sin n\theta. \tag{6.16}$$

For the multiplication operator (6.15), arguing like (4.27) yields

$$\psi^k \to -(b\beta^k)^4 I_1(b\beta^k)K_1(b\beta^k)\psi^k. \tag{6.17}$$

Then (6.13) follows from (6.16) and (6.17).

The space H_0 is not an eigenspace, but an invariant subspace under L. However H_0 and H_1^k are not of concern at this point, since only the restriction of L to $H_\flat^2(\mathbb{S}^1, \mathbb{R}^K)$ needs to be analyzed. Note that

$$L_\flat^2(\mathbb{S}^1, \mathbb{R}^K) = \oplus_{n=2}^{\infty} \oplus_{k=1}^{K} H_n^k, \tag{6.18}$$

so L maps $H_\flat^2(\mathbb{S}^1, \mathbb{R}^K)$ into $L_\flat^2(\mathbb{S}^1, \mathbb{R}^K)$.

To further study (6.13), let $t = b\beta^k$ and set

$$\Lambda_n(t) = t(n^2 - 1) + \sigma t^4 \big(I_n(t)K_n(t) - I_1(t)K_1(t)\big). \tag{6.19}$$

Then

$$\lambda_n^k = \Lambda_n(b\beta^k) \tag{6.20}$$

At this point we use the important upper bound condition $\sigma b^3 < 12 - \delta$, part of condition 3, in Theorem 5.1. Choose τ in (5.11) small enough so that

$$\sigma t^3 < 12 - \frac{\delta}{2} \tag{6.21}$$

for all $\beta \in \overline{B_\tau}$. It is known that

$$\lim_{t\to 0} I_n(t)K_n(t) = \frac{1}{2n}, \text{ for } n = 1, 2, \dots \tag{6.22}$$

by [1, (9.6.7)-(9.6.9)]. For $n = 2$, by (6.21), (6.22) and $I_2(t)K_2(t) - I_1(t)K_1(t) < 0$,

$$\liminf_{t\to 0} \frac{\Lambda_2(t)}{t} \geqslant (2^2 - 1) + \Big(12 - \frac{\delta}{2}\Big)\Big(\frac{1}{4} - \frac{1}{2}\Big) > 0. \tag{6.23}$$

For $n \geqslant 3$, since $I_n(t)K_n(t) - I_1(t)K_1(t) > -I_1(t)K_1(t)$, and $n^2 - 1 \geqslant 3^2 - 1 = 8$ for $n = 3, 4, 5, \dots$, one has

$$\frac{\Lambda_n(t)}{t} > 8 - \Big(12 - \frac{\delta}{2}\Big)I_1(t)K_1(t) \to 8 - \Big(12 - \frac{\delta}{2}\Big)\Big(\frac{1}{2}\Big) > 0, \text{ as } t \to 0. \tag{6.24}$$

Moreover, since $|I_n(t)K_n(t) - I_1(t)K_1(t)| < I_1(t)K_1(t)$ and $I_1(t)K_1(t)$ is bounded on $(0, 1)$ by (6.22),

$$\lim_{n\to\infty} \frac{\Lambda_n(t)}{n^2 t} = \lim_{n\to\infty} \frac{n^2 - 1 + \sigma t^3\big(I_n(t)K_n(t) - I_1(t)K_1(t)\big)}{n^2} = 1 \tag{6.25}$$

uniformly with respect to $t \in (0, 1)$. By (6.23), (6.24) and (6.25), there exist $c_2 > 0$ and $t_0 > 0$ such that

$$\frac{\Lambda_n(t)}{n^2 t} \geqslant 3c_2, \quad \text{for all } t \in (0, t_0) \text{ and all } n = 2, 3, 4, \dots \tag{6.26}$$

Consequently for sufficiently small b and τ

$$\frac{\lambda_n^k}{n^2} \geqslant 2c_2 b, \text{ for all } n = 2, 3, \dots \tag{6.27}$$

For any $\psi \in H^2_\flat(\mathbb{S}^1, \mathbb{R}^K)$, if one expands it by a trigonometric series

$$\psi = \sum_{n=2}^{\infty} \big(A_n \cos n\theta + B_n \sin n\theta\big), \; A_n, B_n \in \mathbb{R}^K, \tag{6.28}$$

then the H^2 norm of ψ can be equivalently given by

$$\|\psi\|^2_{H^2} = \sum_{n=2}^{\infty} \pi\big(|A_n|^2 n^4 + |B_n|^2 n^4\big). \tag{6.29}$$

On the series (6.28), L acts like

$$L\psi = \sum_{n=2}^{\infty} \left(\begin{pmatrix} \lambda_n^1 A_n^1 \\ \lambda_n^2 A_n^2 \\ \dots \\ \lambda_n^K A_n^K \end{pmatrix} \cos n\theta + \begin{pmatrix} \lambda_n^1 B_n^1 \\ \lambda_n^2 B_n^2 \\ \dots \\ \lambda_n^K B_n^K \end{pmatrix} \sin n\theta \right). \tag{6.30}$$

The L^2 norm of $L\psi$ is

$$\|L\psi\|_{L^2}^2 = \sum_{n=2}^{\infty}\sum_{k=1}^{K} \pi\big((\lambda_n^k A_n^k)^2 + (\lambda_n^k B_n^k)^2\big). \tag{6.31}$$

It follows from (6.27), (6.29), and (6.31) that

$$\|L\psi\|_{L^2} \geqslant 2c_2 b\|\psi\|_{H^2} \tag{6.32}$$

for all $\psi \in H_\flat^2(\mathbb{S}^1, \mathbb{R}^K)$. Combining (6.32) with (6.9), one obtains

$$\begin{aligned}\|\Pi S'_D(0)\psi\|_{L^2} &\geqslant \|L\psi\|_{L^2} - \|M\psi\|_{L^2}\\ &\geqslant 2c_2 b\|\psi\|_{H^2} - C\sigma b^5\|\psi\|_{L^2}\\ &\geqslant c_2 b\|\psi\|_{H^2}\end{aligned} \tag{6.33}$$

for all $\psi \in H_\flat^2(\mathbb{S}^1, \mathbb{R}^K)$, if b is sufficiently small.

Similarly, for any $\psi \in H_\flat^1(\mathbb{S}^1, \mathbb{R}^K)$, if one expands it by a trigonometric series as in (6.28), then the H^1 norm of ψ can be equivalently given by

$$\|\psi\|_{H^1}^2 = \sum_{n=2}^{\infty} \pi\big(|A_n|^2 n^2 + |B_n|^2 n^2\big). \tag{6.34}$$

The inner product $\langle L\psi, \psi\rangle$ is

$$\langle L\psi, \psi\rangle = \sum_{n=2}^{\infty}\sum_{k=1}^{K} \pi\big(\lambda_n^k (A_n^k)^2 + \lambda_n^k (B_n^k)^2\big), \ \forall\psi \in H_\flat^1(\mathbb{S}^1, \mathbb{R}^K). \tag{6.35}$$

Then by (6.27), (6.34), and (6.35),

$$\langle L\psi, \psi\rangle \geqslant 2c_2 b\|\psi\|_{H^1}^2, \ \forall\psi \in H_\flat^1(\mathbb{S}^1, \mathbb{R}^K); \tag{6.36}$$

by (6.9) and (6.36)

$$\begin{aligned}\langle \Pi S'_D(0)\psi, \psi\rangle &= \langle L\psi, \psi\rangle + \langle M\psi, \psi\rangle\\ &\geqslant 2c_2 b\|\psi\|_{H^1}^2 - \|M\psi\|_{L^2}\|\psi\|_{L^2}\\ &\geqslant 2c_2 b\|\psi\|_{H^1}^2 - C\sigma b^5\|\psi\|_{L^2}^2\\ &\geqslant c_2 b\|\psi\|_{H^1}^2, \ \forall\psi \in H_\flat^1(\mathbb{S}^1, \mathbb{R}^K),\end{aligned} \tag{6.37}$$

if b is small.

The standard theory of second order linear differential equations asserts that $\Pi S'_D(0)$ is an unbounded self-adjoint operator on $L_\flat^2(\mathbb{S}^1, \mathbb{R}^K)$ with the domain $H_\flat^2(\mathbb{S}^1, \mathbb{R}^K) \subset L_\flat^2(\mathbb{S}^1, \mathbb{R}^K)$. If $\tilde{\psi} \in L_\flat^2(\mathbb{S}^1, \mathbb{R}^K)$ is perpendicular to the range of $\Pi S'_D(0)$, i.e. $\langle \Pi S'_D(0)\psi, \tilde{\psi}\rangle = 0$ for all $\psi \in H_\flat^2(\mathbb{S}^1, \mathbb{R}^K)$, then the self-adjointness of $\Pi S'_D(0)$ implies that $\tilde{\psi} \in H_\flat^2(\mathbb{S}^1, \mathbb{R}^K)$ and $\Pi S'_D(0)\tilde{\psi} = 0$. This together with (6.33) shows $\tilde{\psi} = 0$. Hence, the range of $\Pi S'_D(0)$ is dense in $L_\flat^2(\mathbb{S}^1, \mathbb{R}^K)$. Finally (6.33) implies that the range of $\Pi S'_D(0)$ is a closed subset of $L_\flat^2(\mathbb{S}^1, \mathbb{R}^K)$. Therefore $\Pi S'_D(0)$ is onto. □

An upper bound for the third variation of J_D, i.e. the second Fréchet derivative of S_D, is needed in later work. Following our convention, we use an upper case C_3 in the next lemma for this bound.

Lemma 6.3 1. *There exists $C_3 > 0$ such that for all $\phi \in Dom(S_D)$, and all $\psi, \upsilon \in H^2(\mathbb{S}^1, \mathbb{R}^K)$,*

$$\|S''_D(\phi)(\psi, \upsilon)\|_{L^2(\mathbb{S}^1,\mathbb{R}^K)} \leqslant C_3(b + \sigma b^4)\|\psi\|_{H^2(\mathbb{S}^1,\mathbb{R}^K)}\|\upsilon\|_{H^2(\mathbb{S}^1,\mathbb{R}^K)}.$$

2. *There exists $C_3 > 0$ such that for all $\phi \in Dom(S_D)$, and all $\psi \in H^2(\mathbb{S}^1, \mathbb{R}^K)$ and $\upsilon \in H^1(\mathbb{S}^1, \mathbb{R}^K)$,*

$$|\langle S''_D(\phi)(\psi, \upsilon), \upsilon\rangle| \leqslant C_3(b + \sigma b^4)\|\psi\|_{H^2(\mathbb{S}^1,\mathbb{R}^K)}\|\upsilon\|^2_{H^1(\mathbb{S}^1,\mathbb{R}^K)}.$$

The proof, which is skipped, follows from straightforward calculation. Part 1 is similar to [39, Lemma 3.2] or [38, Lemma 6.1], and Part 2 is similar to [39, Lemma 4.1] or [38, Lemma 7.2]. The next lemma is the key result of this section.

Lemma 6.4 *There exists $\phi_* \in H^2_\flat(\mathbb{S}^1, \mathbb{R}^K) \cap Dom(S_D)$ such that $\|\phi_*\|_{H^2} = O(\sigma b^5)$ and $\Pi S_D(\phi_*) = 0$.*

Proof. For $\phi \in H^2_\flat(\mathbb{S}^1, \mathbb{R}^K) \cap Dom(S_D)$, write

$$\Pi S_D(\phi) = \Pi S_D(0) + \Pi S'_D(0)\phi + \Pi R(\phi), \tag{6.38}$$

where $R(\phi)$ is a higher order term defined by (6.38). Define an operator T from $H^2_\flat(\mathbb{S}^1, \mathbb{R}^K) \cap Dom(S_D)$ into $H^2_\flat(\mathbb{S}^1, \mathbb{R}^K)$ by

$$T(\phi) = -\big(\Pi S'_D(0)\big)^{-1}\big(\Pi S_D(0) + \Pi R(\phi)\big), \tag{6.39}$$

and rewrite the equation $\Pi S_D(\phi) = 0$ as a fixed point problem $T(\phi) = \phi$.

Let $c \in (0, \eta_1)$, where η_1 is given in (5.18) and (5.27), and define a closed ball

$$W = \big\{\phi \in H^2(\mathbb{S}^1, \mathbb{R}^K) : \|\phi\|_{H^2} \leqslant c\big\}.$$

For $\phi \in W$,

$$\|R(\phi)\|_{L^2} \leqslant \frac{1}{2}\sup_{\tau\in(0,1)}\|S''_D(\tau\phi)(\phi, \phi)\|_{L^2} \leqslant \frac{C_3(b + \sigma b^4)}{2}\|\phi\|^2_{H^2} \tag{6.40}$$

by Lemma 6.3 (1). Then by Lemma 6.1 and Lemma 6.2 (1),

$$\begin{aligned}\|T(\phi)\|_{H^2} &\leqslant \big\|\big(\Pi S'_D(0)\big)^{-1}\big\|\big(\|\Pi S_D(0)\|_{L^2} + \|\Pi R(\phi)\|_{L^2}\big)\\ &\leqslant \frac{1}{c_2 b}\Big(C_1\sigma b^6 + \frac{C_3(b + \sigma b^4)}{2}c^2\Big)\\ &\leqslant \frac{C_1\sigma b^5}{c_2} + \frac{C_3(1 + \sigma b^3)}{2c_2}c^2.\end{aligned} \tag{6.41}$$

Hence T maps W into itself if one first chooses c small and then b small.

Let $\phi, \tilde{\phi} \in W$ and use Lemma 6.3 (1) to derive

$$\begin{aligned}
\|T(\phi) - T(\tilde{\phi})\|_{H^2} &\leqslant \|(\Pi S_D'(0))^{-1}\| \, \|\Pi R(\phi) - \Pi R(\tilde{\phi})\|_{L^2} \\
&\leqslant \frac{1}{c_2 b}\|\Pi S_D(\phi) - \Pi S_D(\tilde{\phi}) - \Pi S_D'(0)(\phi - \tilde{\phi})\|_{L^2} \\
&\leqslant \frac{1}{c_2 b}\|\Pi S_D(\phi) - \Pi S_D(\tilde{\phi}) - \Pi S_D'(\tilde{\phi})(\phi - \tilde{\phi})\|_{L^2} \\
&\qquad + \frac{1}{c_2 b}\|(\Pi S_D'(\tilde{\phi}) - \Pi S_D'(0))(\phi - \tilde{\phi})\|_{L^2} \\
&\leqslant \frac{1}{2c_2 b} \sup_{\tau \in (0,1)} \|\Pi S_D''((1-\tau)\tilde{\phi} + \tau\phi)(\phi - \tilde{\phi}, \phi - \tilde{\phi})\|_{L^2} \\
&\qquad + \frac{1}{c_2 b} \sup_{\tau \in (0,1)} \|\Pi S_D''(\tau\tilde{\phi})(\tilde{\phi}, \phi - \tilde{\phi})\|_{L^2} \\
&\leqslant \frac{C_3(b + \sigma b^4)}{2c_2 b}\|\phi - \tilde{\phi}\|_{H^2}^2 + \frac{C_3(b + \sigma b^4)}{c_2 b}\|\tilde{\phi}\|_{H^2}\|\phi - \tilde{\phi}\|_{H^2} \\
&\leqslant \frac{C_3(b + \sigma b^4)}{2c_2 b}(2c + 2c)\|\phi - \tilde{\phi}\|_{H^2} \\
&= \frac{2C_3(1 + \sigma b^3)c}{c_2}\|\phi - \tilde{\phi}\|_{H^2}.
\end{aligned} \tag{6.42}$$

Hence T is a contraction if c is small, and the contraction mapping principle asserts that T has a fixed point in W. This fixed point is denoted ϕ_*, as it solves $\Pi S_D(\phi) = 0$.

To prove the estimate of ϕ_*, revisit the equation $\phi = T(\phi)$, satisfied by ϕ_*, and derive from (6.39) and (6.40) that

$$\begin{aligned}
\|\phi_*\|_{H^2} &\leqslant \left\|\left(\Pi S_D'(0)\right)^{-1}\right\|(\|\Pi S_D(0)\|_{L^2} + \|\Pi R(\phi_*)\|_{L^2}) \\
&\leqslant \frac{1}{c_2 b}\left(C_1\sigma b^6 + \frac{C_3(b + \sigma b^4)}{2}\|\phi_*\|_{H^2}^2\right).
\end{aligned}$$

Rewrite the above as

$$\left(1 - \frac{C_3(1 + \sigma b^3)}{2c_2}\|\phi_*\|_{H^2}\right)\|\phi_*\|_{H^2} \leqslant \frac{C_1\sigma b^5}{c_2}. \tag{6.43}$$

In (6.43)

$$\frac{C_3(1 + \sigma b^3)}{2c_2}\|\phi_*\|_{H^2} \leqslant \frac{C_3(1 + \sigma b^3)}{2c_2}c \leqslant \frac{1}{2} \tag{6.44}$$

if c is sufficiently small. The estimate of ϕ_* follows from (6.43) and (6.44). □

Note that P_{ϕ_*} is not yet a stationary assembly of J_D. The equation $\Pi S_D(\phi_*) = 0$ means that there exist $A_0^k, A_1^k, B_1^k \in \mathbb{R}$ such that

$$S_D^k(\phi_*) = A_0^k + A_1^k \cos\theta + B_1^k \sin\theta, \ k = 1, 2, \ldots, K. \tag{6.45}$$

Later we will find particular β and ξ such that A_0^k, A_1^k, B_1^k in (6.45) vanish.

Some properties of ϕ_* are shown in the next lemma. The first part gives a measurement of the non-degeneracy of the assembly ϕ_* within the class of assemblies with well defined centers and radii; the second part shows that ϕ_* is locally energy minimizing in this class.

Lemma 6.5 *Let c_2 be the same positive constant as in Lemma 6.2.*

1. *When σ is large,*

$$\|\Pi S'_D(\phi_*)\psi\|_{L^2} \geqslant \frac{c_2 b}{2}\|\psi\|_{H^2}$$

for all $\psi \in H^2_\flat(\mathbb{S}^1, \mathbb{R}^K)$.

2. *When σ is large,*

$$\langle \Pi S'_D(\phi_*)\psi, \psi\rangle \geqslant \frac{c_2 b}{2}\|\psi\|^2_{H^1}$$

for all $\psi \in H^1_\flat(\mathbb{S}^1, \mathbb{R}^K)$.

Proof. By Lemmas 6.2 (1), 6.3 (1), and 6.4,

$$\begin{aligned}
\|\Pi S'_D(\phi_*)\psi\|_{L^2} &\geqslant \|\Pi S'_D(0)\psi\|_{L^2} - \sup_{\tau\in(0,1)} \|\Pi S''_D(\tau\phi^*)(\phi^*, \psi)\|_{L^2} \\
&\geqslant c_2 b\|\psi\|_{H^2} - C_3(b + \sigma b^4)\|\phi_*\|_{H^2}\|\psi\|_{H^2} \\
&\geqslant \big(c_2 b - C C_3(b + \sigma b^4)\sigma b^5\big)\|\psi\|_{H^2} \\
&\geqslant \frac{c_2 b}{2}\|\psi\|_{H^2}.
\end{aligned}$$

if b is sufficiently small. This proves Part 1.

There exists $\tilde\tau \in (0, 1)$ such that

$$\langle \Pi S'(\phi_*)\psi, \psi\rangle = \langle \Pi S'_D(0)\psi, \psi, \rangle + \langle \Pi S''(\tilde\tau\phi_*)(\phi_*, \psi), \psi\rangle.$$

By Lemma 6.3 (2),

$$|\langle \Pi S''_D(\tilde\tau\phi_*)(\phi_*, \psi), \psi\rangle| \leqslant C_3(b + \sigma b^4)\|\phi_*\|_{H^2}\|\psi\|^2_{H^1}. \tag{6.46}$$

The second part follows from Lemmas 6.2 (2) and 6.4, since

$$\langle \Pi S'_D(\phi_*)\psi, \psi\rangle \geqslant c_2 b\|\psi\|^2_{H^1} - C C_3(b + \sigma b^4)\sigma b^5\|\psi\|^2_{H^1} \geqslant \frac{c_2 b}{2}\|\psi\|^2_{H^1}$$

if b is sufficiently small. □

7. Finding the correct β and ξ

Now we investigate the impacts of $\beta = (\beta^1, \beta^2, \ldots, \beta^K)$ and $\xi = (\xi^1, \xi^2, \ldots, \xi^K)$. Denote the ϕ_* in Lemma 6.4 by $\phi_*(\cdot, \beta, \xi)$ to emphasize its dependence on β and ξ. Define

$$J(\beta, \xi) = J_D\big(\phi_*(\cdot, \beta, \xi)\big), \quad (\beta, \xi) \in \overline{B_\tau} \times \overline{\Xi_\eta}. \tag{7.1}$$

Lemma 7.1 *Any critical point $(\beta_c, \xi_c) \in B_\tau \times \Xi_\eta$ of J corresponds to a stationary assembly $P_{\phi_*(\cdot,\beta_c,\xi_c)}$ of J_D, i.e. $S_D(\phi_*(\cdot, \beta_c, \xi_c)) = 0$.*

Proof. Let us denote the parametrization of the boundary of $P_{\phi_*(\cdot,\beta,\xi)}$ by $\mathbf{R} = (\mathbf{R}^1, \mathbf{R}^2, \ldots, \mathbf{R}^K)$ where

$$\mathbf{R}^k(\theta,\beta,\xi) = \xi^k + b\beta^k \sqrt{1 + 2\phi_*^k(\theta,\beta,\xi)}\, e^{\mathrm{i}\theta},\ k = 1, 2, \ldots, K. \tag{7.2}$$

The unit tangent and normal vectors of $\mathbf{R}^k$ are

$$\mathbf{T}^k(\theta,\beta,\xi) = \frac{\frac{\partial \mathbf{R}^k(\theta,\beta,\xi)}{\partial\theta}}{\left|\frac{\partial \mathbf{R}^k(\theta,\beta,\xi)}{\partial\theta}\right|},\quad \mathbf{N}^k(\theta,\beta,\xi) = \mathrm{i}\,\mathbf{T}^k(\theta,\xi), \tag{7.3}$$

respectively. Note that $\mathbf{N}^k(\theta,\beta,\xi)$ is inward pointing with respect to $P^k_{\phi^k_*}$

Let one of $\beta^1, \beta^2, \ldots, \beta^K$, say β^k, vary, and keep the others fixed; all of $\xi^1, \xi^2, \ldots, \xi^K$ are also fixed. One treats β^k as a deformation parameter and $\mathbf{R}$ as a deformation. Unlike the deformation (5.23), this deformation does not preserve radii. By (5.22) and (5.28),

$$\frac{\partial J_D\big(\phi_*(\cdot,\beta,\xi)\big)}{\partial\beta^k} = -\sum_{l=1}^{K} \frac{1}{(b\beta^l)^2} \int_{\partial P^l_{\phi^l_*}} S^l_D(\phi_*)\mathbf{N}^l \cdot \mathbf{X}^l\, ds. \tag{7.4}$$

Here $\mathbf{N}^l$ is the normal vector defined in (7.3) and $\mathbf{X}^l$ is the infinitesimal element of the deformation:

$$\mathbf{X}^l(\theta,\beta,\xi) = \frac{\partial \mathbf{R}^l(\theta,\beta,\xi)}{\partial\beta^k}. \tag{7.5}$$

Note that

$$\begin{aligned}
\mathbf{N}^l \cdot \mathbf{X}^l \frac{ds}{d\theta} &= \mathrm{i}\frac{\partial \mathbf{R}^l(\theta,\beta,\xi)}{\partial\theta} \cdot \mathbf{X}^l(\theta,\beta,\xi) \\
&= \Big(\frac{b\beta^l \frac{\partial\phi^l_*}{\partial\theta}}{\sqrt{1+2\phi^l_*}} e^{\mathrm{i}\theta}\mathrm{i} - b\beta^l\sqrt{1+2\phi^l_*}\, e^{\mathrm{i}\theta}\Big) \cdot \begin{cases} \Big(b\sqrt{1+2\phi^k_*} + \frac{b\beta^k \frac{\partial\phi^k_*}{\partial\beta^k}}{\sqrt{1+2\phi^k_*}}\Big) e^{\mathrm{i}\theta} & \text{if } l = k \\ \frac{b\beta^l \frac{\partial\phi^l_*}{\partial\beta^k}}{\sqrt{1+2\phi^l_*}} e^{\mathrm{i}\theta} & \text{if } l \neq k \end{cases} \\
&= \begin{cases} -\Big(b^2\beta^k(1+2\phi^k_*) + b^2(\beta^k)^2\frac{\partial\phi^k_*}{\partial\beta^k}\Big) & \text{if } l = k \\ -(b\beta^l)^2\frac{\partial\phi^l_*}{\partial\beta^k} & \text{if } l \neq k \end{cases}.
\end{aligned} \tag{7.6}$$

Since $\Pi S_D(\phi_*) = 0$, there exist $A^l_0(\beta,\xi), A^l_1(\beta,\xi), B^l_1(\beta,\xi) \in \mathbb{R}$, $l = 1, 2, \ldots, K$, such that

$$S^l_D\big(\phi_*(\cdot,\beta,\xi)\big) = A^l_0(\beta,\xi) + A^l_1(\beta,\xi)\cos\theta + B^l_1(\beta,\xi)\sin\theta. \tag{7.7}$$

Using the facts that $\int_0^{2\pi} \phi^l_*\, d\theta = \int_0^{2\pi} \phi^l_* \cos\theta\, d\theta = \int_0^{2\pi} \phi^l_* \sin\theta\, d\theta = 0$, one obtains

$$\int_0^{2\pi} \frac{\partial\phi^l_*}{\partial\beta^k}\, d\theta = \int_0^{2\pi} \frac{\partial\phi^l_*}{\partial\beta^k}\cos\theta\, d\theta = \int_0^{2\pi} \frac{\partial\phi^l_*}{\partial\beta^k}\sin\theta\, d\theta = 0.$$

It follows from (7.4), (7.6), and (7.7) that

$$\frac{\partial J_D\big(\phi_*(\cdot,\beta,\xi)\big)}{\partial\beta^k} = 2\pi A^k_0(\beta,\xi)\Big(\frac{1}{\beta^k}\Big). \tag{7.8}$$

At the critical point (β_c, ξ_c) of J,

$$\frac{\partial J_D\big(\phi_*(\cdot,\beta,\xi)\big)}{\partial \beta^k}\Big|_{(\beta,\xi)=(\beta_c,\xi_c)} = 0. \tag{7.9}$$

Therefore (7.8) implies

$$A_0^k(\beta_c, \xi_c) = 0 \tag{7.10}$$

which holds for every $k = 1, 2, \ldots, K$.

Next let the first component of ξ^k, denoted ξ_1^k, vary and keep the other components of ξ and all components of β fixed. Again treat ξ_1^k as a deformation parameter and $\mathbf{R}$ as a deformation. Unlike (5.23) this deformation does not preserve centers. Now let $\mathbf{X}$ be the infinitesimal element of this deformation:

$$\mathbf{X}^l(\theta,\beta,\xi) = \frac{\partial \mathbf{R}^l(\theta,\beta,\xi)}{\partial \xi_1^k}. \tag{7.11}$$

Again compute

$$\begin{aligned}
\mathbf{N}^l \cdot \mathbf{X}^l \frac{ds}{d\theta} &= \mathrm{i}\frac{\partial \mathbf{R}^l(\theta,\beta,\xi)}{\partial \theta} \cdot \mathbf{X}^l(\theta,\beta,\xi) \\
&= \Big(\frac{b\beta^l \frac{\partial \phi_*^l}{\partial \theta}}{\sqrt{1+2\phi_*^l}} e^{\mathrm{i}\theta}\mathrm{i} - b\beta^l \sqrt{1+2\phi_*^l}\, e^{\mathrm{i}\theta}\Big) \cdot \begin{cases} \Big((1,0) + \frac{b\beta^k \frac{\partial \phi_*^k}{\partial \xi_1^k}}{\sqrt{1+2\phi_*^k}} e^{\mathrm{i}\theta}\Big) & \text{if } l = k \\ \frac{b\beta^l \frac{\partial \phi_*^l}{\partial \xi_1^k}}{\sqrt{1+2\phi_*^l}} e^{\mathrm{i}\theta} & \text{if } l \neq k \end{cases} \\
&= \begin{cases} \Big(-\frac{b\beta^k \frac{\partial \phi_*^k}{\partial \theta}}{\sqrt{1+2\phi_*^k}} \sin\theta - b\beta^k \sqrt{1+2\phi_*^k} \cos\theta - (b\beta^k)^2 \frac{\partial \phi_*^k}{\partial \xi_1^k}\Big) & \text{if } l = k \\ -(b\beta^l)^2 \frac{\partial \phi_*^l}{\partial \xi_1^k} & \text{if } l \neq k \end{cases},
\end{aligned} \tag{7.12}$$

and consequently deduce

$$\begin{aligned}
\frac{\partial J_D\big(\phi_*(\cdot,\beta,\xi)\big)}{\partial \xi_1^k} &= \frac{1}{(b\beta^k)^2} \int_0^{2\pi} \Big(A_0^k(\beta,\xi) + A_1^k(\beta,\xi)\cos\theta + B_1^k(\beta,\xi)\sin\theta\Big) \\
&\qquad \Big(\frac{b\beta^k \frac{\partial \phi_*^k}{\partial \theta}}{\sqrt{1+2\phi_*^k}} \sin\theta + b\beta^k \sqrt{1+2\phi_*^k}\cos\theta\Big)\, d\theta. \\
&= A_0^k(\beta,\xi) \int_0^{2\pi} \frac{1}{b\beta^k} \Big(\frac{\frac{\partial \phi_*^k}{\partial \theta}}{\sqrt{1+2\phi_*^k}} \sin\theta + \sqrt{1+2\phi_*^k}\cos\theta\Big)\, d\theta \\
&\quad + A_1^k(\beta,\xi) \int_0^{2\pi} \frac{1}{b\beta^k} \Big(\frac{\frac{\partial \phi_*^k}{\partial \theta}}{\sqrt{1+2\phi_*^k}} \sin\theta + \sqrt{1+2\phi_*^k}\cos\theta\Big) \cos\theta\, d\theta \\
&\quad + B_1^k(\beta,\xi) \int_0^{2\pi} \frac{1}{b\beta^k} \Big(\frac{\frac{\partial \phi_*^k}{\partial \theta}}{\sqrt{1+2\phi_*^k}} \sin\theta + \sqrt{1+2\phi_*^k}\cos\theta\Big) \sin\theta\, d\theta.
\end{aligned} \tag{7.13}$$

Since $\|\phi_*\|_{H^2} = O(\sigma b^5)$ according to Lemma 6.4, the last two integrals are estimated as follows:

$$\int_0^{2\pi} \frac{1}{b\beta^k} \left(\frac{\frac{\partial \phi_*^k}{\partial\theta}}{\sqrt{1+2\phi_*^k}} \sin\theta + \sqrt{1+2\phi_*^k}\cos\theta \right) \cos\theta \, d\theta = \frac{\pi}{b\beta^k} + O(\sigma b^4) \tag{7.14}$$

$$\int_0^{2\pi} \frac{1}{b\beta^k} \left(\frac{\frac{\partial \phi_*^k}{\partial\theta}}{\sqrt{1+2\phi_*^k}} \sin\theta + \sqrt{1+2\phi_*^k}\cos\theta \right) \sin\theta \, d\theta = O(\sigma b^4). \tag{7.15}$$

At (β_c, ξ_c), since

$$\frac{\partial J_D(\phi_*(\cdot,\beta,\xi))}{\partial \xi_1^k}\Big|_{(\beta,\xi)=(\beta_c,\xi_c)} = 0 \tag{7.16}$$

and $A_0(\beta_c, \xi_c) = 0$ by (7.10), (7.13) becomes

$$A_1^k(\beta_c,\xi_c)\Big(\frac{\pi}{b\beta^k} + O(\sigma b^4)\Big) + B_1^k(\beta_c,\xi_c)\big(0 + O(\sigma b^4)\big) = 0. \tag{7.17}$$

Finally if ξ_2^k, the second component of ξ^k, is taken as the deformation parameter, then in an analogous way one deduces

$$A_1^k(\beta_c,\xi_c)\big(0 + O(\sigma b^4)\big) + B_1^k(\beta_c,\xi_c)\Big(\frac{\pi}{b\beta^k} + O\big(\sigma b^4\big)\Big) = 0. \tag{7.18}$$

The equations (7.17) and (7.18) follow form a linear homogeneous system for $A_1^k(\beta_c,\xi_c)$ and $B_1^k(\beta_c,\xi_c)$. This system is non-singular if b is small. Hence

$$A_1^k(\beta_c,\xi_c) = B_1^k(\beta_c,\xi_c) = 0, \tag{7.19}$$

for every $k = 1, 2, \ldots, K$. Combining (7.19) with (7.10) one deduces that $S_D(\phi_*(\cdot,\beta_c,\xi_c)) = 0$. □

For $J(\beta,\xi)$ there is the following estimate.

Lemma 7.2 *It holds uniformly with respect to* $(\beta,\xi) \in \overline{B_\tau} \times \overline{\Xi_\eta}$ *that*

$$\begin{aligned} J(\beta,\xi) = {} & \sum_{k=1}^K \left(2\pi b\beta^k - \alpha\pi(b\beta^k)^2 + \frac{\sigma\pi(b\beta^k)^2}{2}\big(1 - 2I_1(b\beta^k)K_1(b\beta^k)\big) \right) \\ & + \sum_{k=1}^K \frac{\sigma\pi^2(b\beta^k)^4}{2} R(\xi^k,\xi^k) + \sum_{k=1}^K \sum_{l=1, l\neq k}^K \frac{\sigma\pi^2(b\beta^k)^2(b\beta^l)^2}{2} G(\xi^k,\xi^l) + O(\sigma b^6). \end{aligned}$$

Proof. Expanding $J_D(P_{\phi_*})$ yields

$$J_D(P_{\phi_*}) = J_D(P_0) + \sum_{k=1}^K \int_0^{2\pi} S_D^k(0)\phi_*^k \, d\theta + \frac{1}{2} \sum_{k=1}^K \int_0^{2\pi} (S_D'^{,k}(0)\phi_*)\phi_*^k \, d\theta + O(\sigma^3 b^{16}). \tag{7.20}$$

The error term in (7.20) is obtained by Lemmas 6.3 (2) and 6.4.

On the other hand $\Pi S_D(\phi_*) = 0$ implies that

$$\Pi\big(S_D(0) + S'_D(0)\phi_* + R(\phi_*)\big) = 0.$$

Take the inner product of the last equation with ϕ_* and integrate to derive, with the help of Lemma 6.3 (1),

$$\sum_{k=1}^{K}\int_0^{2\pi} S_D^k(0)\phi_*^k\,d\theta + \sum_{k=1}^{K}\int_0^{2\pi}\big(S_D^{\prime,k}(0)\phi_*\big)\phi_*^k\,d\theta = O(\sigma^3 b^{16}).$$

Rewrite (7.20) as

$$J_D(P_{\phi_*}) = J_D(P_0) + \frac{1}{2}\sum_{k=1}^{K}\int_0^{2\pi} S_D^k(0)\phi_*^k\,d\theta + O(\sigma^3 b^{16}).$$

Lemmas 6.1, 6.4, and the fact that $\phi_* \in H_\flat^2(\mathbb{S}^1, \mathbb{R}^K)$ imply that

$$J_D(P_{\phi_*}) = J_D(P_0) + O(\sigma^2 b^{11}) + O(\sigma^3 b^{16}).$$

Then apply Lemma 5.2 and treat σb^3 as $O(1)$ to complete the proof. □

Proof of Theorem 5.1. Let (β_*, ξ_*) be a minimum of J on $\overline{B_\tau} \times \overline{\Xi_\eta}$. We need to show that if b is small, then (β_*, ξ_*) is in $B_\tau \times \Xi_\eta$, the interior of $\overline{B_\tau} \times \overline{\Xi_\eta}$, so that (β_*, ξ_*) is a critical point of J and Lemma 7.1 applies. Let $b \to 0$ and (β_*, ξ_*) converge, possibly along a subsequence, to $(\beta_\circ, \xi_\circ) \in \overline{B_\tau} \times \overline{\Xi_\eta}$.

First we claim that $\beta_\circ = (1, 1, \ldots, 1)$, which is in B_τ. One needs the estimate

$$I_1(t)K_1(t) = \frac{1}{2} + \Big(\frac{t}{2}\Big)^2 \log\frac{t}{2} + O(t^2) \tag{7.21}$$

for $I_1(b\beta)K_1(b\beta^k)$, and the estimate

$$1 - \alpha b + \sigma b^2\Big(\frac{b}{2} + O(b^2)\Big)\Big(\log\frac{2}{b} + O(1)\Big) = 0 \tag{7.22}$$

for α. Here (7.21) follows from [1, (9.6.10) and (9.6.11)]; (7.22) follows from (3.4) and [1, (9.6.10) and (9.6.13)].

Assume that $\sigma b^3 \log\frac{1}{b} \to \gamma$. Because of the lower bound condition $1 + \delta < \sigma b^3 \log\frac{1}{b}$, part of condition 3, in Theorem 5.1, γ must fall in the range $[1+\delta, \infty]$, a crucial fact in this proof. Note that γ may be ∞. To find a uniform limit of $\frac{1}{\sigma b^4 \log\frac{1}{b}} J(\beta, \xi)$ as $b \to 0$, we appeal to Lemma 7.2 and derive the following limits:

$$\frac{1}{\sigma b^4 \log\frac{1}{b}}(2\pi b\beta^k) \to \frac{2\pi}{\gamma}\beta^k, \tag{7.23}$$

$$\frac{1}{\sigma b^4 \log\frac{1}{b}}\big(-\alpha\pi(b\beta^k)^2\big) \to -\pi\Big(\frac{1}{\gamma} + \frac{1}{2}\Big)(\beta^k)^2 \tag{7.24}$$

by (7.22),

$$\frac{1}{\sigma b^4 \log \frac{1}{b}} \Big(\frac{\sigma\pi (b\beta^k)^2}{2} \big(1 - 2I_1(b\beta^k)K_1(b\beta^k)\big)\Big) \to \frac{\pi}{4}(\beta^k)^4 \tag{7.25}$$

by (7.21), and

$$\frac{1}{\sigma b^4 \log \frac{1}{b}} \Big(\sum_{k=1}^{K} \frac{\sigma\pi^2 (b\beta^k)^4}{2} R(\xi^k, \xi^k) + \sum_{k=1}^{K} \sum_{l=1, l\neq k}^{K} \frac{\sigma\pi^2 (b\beta^k)^2 (b\beta^l)^2}{2} G(\xi^k, \xi^l) + O(\sigma b^6)\Big) \to 0. \tag{7.26}$$

Then Lemma 7.2 implies that

$$\frac{1}{\sigma b^4 \log \frac{1}{b}} J(\beta, \xi) \to \sum_{k=1}^{K} \Big(\frac{2\pi}{\gamma}\beta^k - \pi\big(\frac{1}{\gamma} + \frac{1}{2}\big)(\beta^k)^2 + \frac{\pi}{4}(\beta^k)^4 \Big) \tag{7.27}$$

uniformly on $\overline{B_\tau} \times \overline{\Xi_\eta}$ as $b \to 0$. It is interesting to note that the above right hand side is independent of ξ. Consequently

$$\begin{aligned}
&\frac{1}{\sigma b^4 \log \frac{1}{b}} \big(J(\beta_*, \xi_*) - J((1, 1, \dots, 1), \xi_*)\big) \\
&\to \sum_{k=1}^{K} \Big(\frac{2\pi}{\gamma}\beta_\circ^k - \pi\big(\frac{1}{\gamma} + \frac{1}{2}\big)(\beta_\circ^k)^2 + \frac{\pi}{4}(\beta_\circ^k)^4 \Big) - \sum_{k=1}^{K} \Big(\frac{2\pi}{\gamma}1 - \pi\big(\frac{1}{\gamma} + \frac{1}{2}\big)1^2 + \frac{\pi}{4}1^4 \Big).
\end{aligned} \tag{7.28}$$

The function

$$f(w) = \frac{2\pi}{\gamma} w - \pi\big(\frac{1}{\gamma} + \frac{1}{2}\big) w^2 + \frac{\pi}{4} w^4 \tag{7.29}$$

has $w = 1$ as a strict local minimum, since $\gamma > 1$ and so $1/\gamma \in [0, 1)$. Fix τ in (5.11) to be small enough so that in the interval $(1 - 2\tau, 1 + 2\tau)$, $w = 1$ is the only critical point of the function f. If $\beta_\circ$ were not $(1, 1, \dots, 1)$, then the right side of (7.28) is positive since each $\beta_\circ^k$ is in $[1 - \tau, 1 + \tau]$. Consequently when b is sufficiently small $J(\beta_*, \xi_*) > J((1, 1, \dots, 1), \xi_*)$, a contradiction to the choice of (β_*, ξ_*).

Next show that $\xi_\circ$ is a minimum of F on Ξ. Let ξ_m be a minimum of F. Then $\xi_m \in \Xi_\eta$ by (5.10). By Lemma 7.2 and the just proved fact that $\beta_* \to (1, 1, \dots, 1)$,

$$\begin{aligned}
&\frac{1}{\sigma b^4}\big(J(\beta_*, \xi_*) - J(\beta_*, \xi_m)\big) \\
&\quad = \sum_{k=1}^{K} \frac{\pi^2 (\beta_*^k)^4}{2} R(\xi_*^k, \xi_*^k) + \sum_{k=1}^{K} \sum_{l=1, l\neq k}^{K} \frac{\pi^2 (\beta_*^k)^2 (\beta_*^l)^2}{2} G(\xi_*^k, \xi_*^l) \\
&\quad\quad - \sum_{k=1}^{K} \frac{\pi^2 (\beta_*^k)^4}{2} R(\xi_m^k, \xi_m^k) - \sum_{k=1}^{K} \sum_{l=1, l\neq k}^{K} \frac{\pi^2 (\beta_*^k)^2 (\beta_*^l)^2}{2} G(\xi_m^k, \xi_m^l) + O(b^2) \\
&\quad \to \frac{\pi^2}{2}\big(F(\xi_\circ) - F(\xi_m)\big).
\end{aligned} \tag{7.30}$$

If $\xi_\circ$ were not a minimum of F, then the last line of (7.30) would be positive, a contradiction to the choice of (β_*, ξ_*). Now that $\xi_\circ$ is a minimum of F, $\xi_\circ$ is necessarily in Ξ_η by (5.10). Therefore $\xi_* \in \Xi_\eta$ when b is sufficiently small.

We have shown that (β_*, ξ_*) is a critical point of J. Lemma 7.1 implies that $P_{\phi_*(\cdot,\beta_*,\xi_*)}$ is a stationary assembly of J_D. The first additional assertion of the theorem that all the perturbed discs in $P_{\phi_*(\cdot,\beta_*,\xi_*)}$ have approximately the same radius comes from the fact that $\beta_* \to (1, 1, \ldots, 1)$; the second assertion follows from the facts that $\xi_* \to \xi_\circ$ and $\xi_\circ$ is a minimum of F.

A thorough study of the stability of the stationary assembly found in Theorem 5.1 is beyond the scope of this paper. To get a sense of stability for the stationary assemby $P_{\phi_*(\cdot,\beta_*,\xi_*)}$, recall that $P_{\phi_*(\cdot,\beta_*,\xi_*)}$ is found in two steps. First for each $(\beta, \xi) \in \overline{B_\tau} \times \overline{\Xi_\eta}$, a fixed point $P_{\phi_*(\cdot,\beta,\xi)}$ is constructed in a class of assemblies with well defined centers ξ^k and radii $b\beta^k$. The fixed point is shown to be locally minimizing J_D within this class of assemblies by Lemma 6.5 (2). In the second step J_D is minimized among the $P_{\phi_*(\cdot,\beta,\xi)}$'s where (β, ξ) ranges in $\overline{B_\tau} \times \overline{\Xi_\eta}$, and $P_{\phi_*(\cdot,\beta_*,\xi_*)}$ emerges as a minimum. Therefore $P_{\phi_*(\cdot,\beta_*,\xi_*)}$ is a mimimum of minima, on which we base our assertion that $P_{\phi_*(\cdot,\beta_*,\xi_*)}$ is stable. We feel that the second variation of J_D should be positive definite based on the properties of $P_{\phi_*(\cdot,\beta_*,\xi_*)}$ proved in this paper, but we do not have a proof. □

Appendix A. The shape of C_0

Let i be a nonnegative integer and $P_i \equiv I_i K_i$ be the product of modified Bessel functions. Using $I_0' = I_1$, $K_0' = -K_1$, $I_1' = I_0 - \frac{1}{z} I_1$, $K_1' = -K_0 - \frac{1}{z} K_1$ and $I_1 K_0 + I_0 K_1 = \frac{1}{z}$, a direct computation yields

$$P_0' = 2 I_1 K_0 - \frac{1}{z}\ , \tag{A.1}$$

$$P_1' = \frac{1}{z} - 2 I_1 K_0 - \frac{2}{z} P_1\ . \tag{A.2}$$

Define $h : (0, \infty) \to \mathbb{R}$ such that $h(z) \equiv z^3 (P_0(z) - P_1(z))$. Since h is analytic, at every critical point ζ, one can write h as $h(z) = h(\zeta) + (z - \zeta)^\nu g(z)$ where $\nu \geqslant 2$ is an integer and g is analytic at ζ and $g(\zeta) \neq 0$. If ν is even, then ζ is called a type-1 critical point. Moreover it is a strict local minimum, termed minimizing type-1 critical point, if $g(\zeta) > 0$; it is a strict local maximum, termed maximizing type-1 critical point, if $g(\zeta) < 0$. If ν is odd, then h is strictly monotone in a neighborhood of ζ and ζ is called a type-2 critical point.

The asymptotic behavior of I_n and K_n for small and large b [1, (9.6.10)-(9.6.13) and (9.7.5)] implies that $h(0) = 0$, $h(\infty) = 1/4$, $h'(z) > 0$ if z is small, and $h'(z) < 0$ if z is large. Therefore a global maximum $\hat{b}$ exists. It can only be a maximizing type-1 critical point. Also, since h is analytic, there are only finitely many critical points on $(0, \infty)$.

In the first step of the proof one shows that $\hat{b}$ is the only type-1 critical point that h has. Suppose on the contrary that there were more than one type-1 critical points. Let the smallest type-1 critical point be z_1. Since $h'(z) > 0$ for small z, z_1 must be maximizing. Otherwise, if z_1 were minimizing, there would be another maximizing type-1 critical point smaller than z_1. Let z_2 be the second smallest type-1 critical point. Since h is strictly decreasing on (z_1, z_2), z_2 must be minimizing. One claims that

$$z_2 > \frac{3}{2}. \tag{A.3}$$

By (A.1) and (A.2),

$$h' = z^2(3P_0 - P_1 - 2 + 4zI_1K_0), \tag{A.4}$$

and

$$h'' - \left(\frac{4}{z}\right)h' = 2z\big(-3P_0 + 2P_1 + 2z^2(P_0 - P_1)\big) \tag{A.5}$$

Denote

$$W = -3P_0 + 2P_1 + 2z^2(P_0 - P_1). \tag{A.6}$$

One shows that $W(z) < 0$ when $z \in (0, 3/2 + \delta)$ for some $\delta > 0$.

When $z \in (0, 1]$,

$$W = -P_0 + 2(1 - z^2)(-P_0 + P_1) < 0 \tag{A.7}$$

since $P_0 > P_1 > 0$. When $z \in (1, \sqrt{3/2}]$,

$$W = (-3 + 2z^2)P_0 + 2(1 - z^2)P_1 \leqslant 2(1 - z^2)P_1 < 0\,. \tag{A.8}$$

When $z \in (\sqrt{3/2}, 3/2]$, one writes

$$W = 2(z^2 - 1)\Big(\frac{2z^2 - 3}{2(z^2 - 1)}P_0 - P_1\Big). \tag{A.9}$$

Observe that $\frac{2z^2-3}{2(z^2-1)}$ is increasing on $(1, \infty)$, so $\frac{2z^2-3}{2(z^2-1)} \leqslant \frac{2(3/2)^2-3}{2((3/2)^2-1)} = 3/5$. Then (A.9) implies

$$W \leqslant 2(z^2 - 1)\Big(\frac{3P_0}{5} - P_1\Big). \tag{A.10}$$

It is known that both P_0 and P_1 are decreasing functions [32, Theorem 3.1]. Hence

$$\frac{3P_0(z)}{5} - P_1(z) \leqslant \left(\frac{3}{5}\right)P_0\Big(\sqrt{\frac{3}{2}}\Big) - P_1\left(\frac{3}{2}\right) = -0.0115\ldots < 0. \tag{A.11}$$

Therefore $W < 0$ on $(\sqrt{3/2}, 3/2 + \delta)$ for some $\delta > 0$. Combining this with (A.7) and (A.8) one has $W < 0$ on $(0, 3/2 + \delta)$.

Then (A.5) implies

$$h'' - \left(\frac{4}{z}\right)h' < 0 \tag{A.12}$$

on $(0, 3/2 + \delta)$. By the maximum principle, the local minimum z_2 cannot be in $(0, 3/2 + \delta)$. This proves (A.3).

Since z_2 is a minimizing type-1 critical point and $h'(z) < 0$ when z is large, there must be at least another type-1 critical point z_3 after z_2 and z_3 must be maximizing. Between z_2 and z_3, h is strictly increasing. Then $h'(z_2) = h'(z_3) = 0$, and $h'(z) \geqslant 0$ on (z_2, z_3). Let $y \in (z_2, z_3)$ be a maximum of h' on $[z_2, z_3]$.

Write (A.5) as

$$h'' - \frac{4}{z}h' - 4h = 2z(-3P_0 + 2P_1).$$

Differentiating yet one more time yields

$$h''' - \frac{4}{z}h'' + \Big(\frac{9}{z^2} - 4\Big)h' = 9(P_0 - P_1) > 0. \tag{A.13}$$

By the maximum principle, the non-negative local maximum y of h' cannot be in $(3/2, \infty)$ where $\frac{9}{z^2} - 4 < 0$. Hence

$$y \leqslant \frac{3}{2}. \tag{A.14}$$

With (A.3) and (A.14) one has a contradiction: $3/2 < z_2 < y \leqslant 3/2$. Therefore h has only one type-1 critical point $\hat{b}$ which is maximizing.

In the second step one shows that h has no type-2 critical point. Indeed, if h has a type-2 critical point ζ, then h takes the form $h(z) = h(\zeta) + (z-\zeta)^\nu g(z)$, where $\nu \geqslant 3$ is odd and $g(\zeta) \neq 0$. Since $h'(\zeta) = h''(\zeta) = 0$, (A.12) indicates no such ζ can exist in $(0, 3/2 + \delta)$.

Finally suppose there is a type-2 critical point $\zeta \in (3/2, \infty)$, then (A.13) implies that

$$h'''(\zeta) > 0, \tag{A.15}$$

and hence h has the form $h(z) = h(\zeta) + (z - \zeta)^3 g(z)$ with $g(\zeta) > 0$. This gives $h'(\zeta) = 0$ and $h'(z) > 0$ when $z \in (\zeta, \zeta + \epsilon)$ for some $\epsilon > 0$. Since $h'(z) < 0$ when z is sufficiently large, h' has a positive local maximum in $(\zeta, \infty) \subset (3/2, \infty)$. This violates the maximum principle applied to (A.13).

In conclusion h has a unique critical point $\hat{b}$, the global maximum of h; $h'(z) > 0$ if $z \in (0, \hat{b})$ and $h'(z) < 0$ if $z \in (\hat{b}, \infty)$.

Appendix B. Uniqueness of $\bar{\alpha}$

We show that the equation

$$b^2(I_0K_0 - I_1K_1) + bI_1K_0 = \frac{1}{2} \tag{B.1}$$

admits a unique positive solution.

Define $Q : (0, \infty) \to \mathbf{R}$ such that $Q(z) \equiv \frac{z}{2} - z^2 I_1(z)\, K_0(z)$. Then

$$Q'(z) = \frac{1}{2} - zI_1K_0 - z^2(I_0K_0 - I_1K_1) = \frac{Q(z) - h(z)}{z}, \tag{B.2}$$

where h is the same function as defined in Appendix A. It is clear that critical points of Q are roots of (B.1). Our task amounts to showing Q has exactly one critical point.

First extract some asymptotic information about Q for later use. For small positive z,

$$Q(z) \sim \frac{z}{2}, \quad h(z) \sim -z^3 \log z, \quad Q'(z) \sim 1/2. \tag{B.3}$$

When z is large, one has

$$Q(z) = \frac{1}{4} + O(\frac{1}{z^2}), \quad h(z) = \frac{1}{4} + O(\frac{1}{z^2}) \tag{B.4}$$

by (2.5) and 3.16) respectively.

Now solve the equation

$$Q'(z) = \frac{Q(z) - h(z)}{z}$$

for Q with a general initial condition at $(z_0, Q(z_0))$ to get

$$Q(z) = z\Big(\frac{Q(z_0)}{z_0} - \int_{z_0}^{z} \frac{h(t)}{t^2}\,dt\Big) = z\Big(\frac{Q(z_0)}{z_0} + \frac{h(t)}{t}\Big|_{z_0}^{z} - \int_{z_0}^{z} \frac{h'(t)}{t}\,dt\Big).$$

Let $z_0 \to \infty$ and use (B.4) to deduce

$$Q(z) = h(z) + z\int_{z}^{\infty} \frac{h'(t)}{t}\,dt. \tag{B.5}$$

Consequently

$$Q'(z) = \int_{z}^{\infty} \frac{h'(t)}{t}\,dt, \tag{B.6}$$

and

$$Q''(z) = -\frac{h'(z)}{z}. \tag{B.7}$$

Recall that $h'(t) > 0$ when $t \in (0, \hat{b})$ and $h'(t) < 0$ when $t > \hat{b}$. Then $Q'(z) < 0$ if $z \geqslant \hat{b}$ and Q' is strictly increasing on $(\hat{b}, \infty)$. On $(0, \hat{b})$, Q' is strictly decreasing. Since $Q'(z) \sim \frac{1}{2} > 0$ for small z by (B.3) and $Q'(\hat{b}) < 0$, Q' has a unique zero in $(0, \hat{b})$.

Acknowledgments. Part of the work was done when Choi and Ren were visiting the National Center for Theoretical Sciences, ROC, and Chen was visiting the University of Connecticut. Chen was supported by MOST grants 102-2115-M-018-002-MY3 and 105-2115-M-007-009-MY3; Ren was supported by NSF grants DMS-1311856 and DMS-1714371.

References

1. Abramowitz, M., & Stegun, I. A., *Handbook of Mathematical Functions*, revised ed. Dover Publications, 1965.
2. Acerbi, E., Fusco, N., & Morini, M., Minimality via second variation for a nonlocal isoperimetric problem. *Comm. Math. Phys.* **322** (2013), 515–557. Zbl1270.49043 MR3077924
3. Arfken, G. B., & Weber, H. J., *Mathematical Methods for Physicists*, sixth ed. Elsevier Academic Press, Amsterdam, 2005. Zbl1066.00001
4. Baricz, A. & Ponnusamy, S., On Turán type inequalities for modified Bessel functions. *Proc. Ameri. Math. Soc.* **141** (2013), 523–532. Zbl1272.33005 MR2996956
5. Chen, C.-N., & Choi, Y. S., Standing pulse solutions to FitzHugh–Nagumo equations. *Arch. Rational Mech. Anal.* **206** (2012), 741–777. Zbl1264.35119 MR2989442
6. Chen, C.-N., & Choi, Y. S., Traveling pulse solutions to FitzHugh–Nagumo equations. *Calc. Var. Partial Differential Equations* **54** (2015), 1–45. Zbl1328.34037 MR3385151
7. Chen, C.-N., Ei, I.-I., Lin, Y.-P., & Kung, S.-Y., Standing waves joining with Turing patterns in FitzHugh–Nagumo type systems. *Comm. Partial Differential Equations* **36** (2011), 998–1015. Zbl1233.35114 MR2765427
8. Chen, C.-N., & Hu, X., Maslov index for homoclinic orbits of Hamiltonian systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **24** (2007), 589–603. Zbl1202.37081 MR2334994
9. Chen, C.-N., & Hu, X., Stability analysis for standing pulse solutions to FitzHugh–Nagumo equations. *Calc. Var. Partial Differential Equations* **49** (2014), 827–845. Zbl1315.35030 MR3148136

10. Chen, C.-N., Kung, S.-Y., & Morita, Y., Planar standing wavefronts in the FitzHugh-Nagumo equations. *SIAM J. Math. Anal.* **46** (2014), 657–690. Zbl1301.34060 MR3163242
11. Chen, C.-N., & Tanaka, K., A variational approach for standing waves of FitzHugh-Nagumo type systems. *J. Differential Equations* **257** (2014), 109–144. Zbl1292.35117 MR3197242
12. Chmaj, A., & Ren, X., Multiple layered solutions of the nonlocal bistable equation. *Physica D* **147** (2000), 135–154. Zbl1005.74008 MR1793237
13. Choksi, R., & Sternberg, P., On the first and second variations of a nonlocal isoperimetric problem. *J. Reine Angew. Math.* **611** (2007), 75–108. Zbl1132.35029 MR2360604
14. Dancer, E. N., Ren, X., & Yan, S., On multiple radial solutions of a singularly perturbed nonlinear elliptic system. *SIAM J. Math. Anal.* **38** (2007), 2005–2041. Zbl1135.35028 MR2299439
15. Dancer, E. N., & Yan, S., Multipeak solutions for the Neumann problem of an elliptic system of FitzHugh–Nagumo type. *Proc. London Math. Soc.* **90** (2005), 209–244. Zbl1172.35402 MR2107042
16. Dancer, E. N., & Yan, S., Multipeak solutions for an elliptic system of FitzHugh–Nagumo type. *Math. Ann.* **335** (2006), 527–569. Zbl1260.35042 MR2221124
17. Dancer, E. N., & Yan, S., Solutions with interior and boundary peaks for the Neumann problem of an elliptic system of FitzHugh–Nagumo type. *Indiana Univ. Math. J.* **55** (2006), 217–258. Zbl1186.35053 MR2207551
18. de Figueiredo, D. G., & Mitidieri, E., A maximum principle for an elliptic system and applications to semilinear problems. *SIAM J. Math. Anal.* **17** 4 (1986), 836–849. Zbl0608.35022 MR0846392
19. De Giorgi, E., Sulla convergenza di alcune successioni d'integrali del tipo dell'area. *Rend. Mat. (6)* **8** (1975), 277–294. Zbl0316.35036 MR0375037
20. Fife, P. C., *Mathematical Aspects of Reacting and Diffusing Systems*. Springer, Berlin, 1979. Zbl0403.92004 MR0527914
21. FitzHugh, R. A., Impulses and physiological states in theoretical models of nerve membrane. *Biophys. J.* **1** (1961), 445–466.
22. Gilbarg, D., & Trudinger, S. N., *Elliptic Partial Differential Equations of Second Order*, second ed. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983. Zbl0562.35001 MR0737190
23. van Heijster, P., Chen, C.-N., Nishiura, Y., & Teramoto, T., Localized patterns in a three-component FitzHugh–Nagumo model revisited via an action functional. *J. Dyn. Differ. Equ.* Doi 10.1007/s10884-016-9557-z (2016).
24. Keener, J., & Syned, J., *Mathematical Physiology I: Cellular Physiology*, second ed. Springer, 2009. Zbl1273.92017 MR2447178
25. Klaasen, G., & Mitidieri, E., Standing wave solutions for a system derived from the FitzHugh–Nagumo equation for nerve conduction. *SIAM J. Math. Anal.* **17** 1 (1986), 74–83. Zbl0593.35043 MR0819214
26. Klaasen, G., & Troy, W., Stationary wave solutions of a system of reaction-diffusion equations derived from the FitzHugh–Nagumo equations. *SIAM J. Appl. Math.* **44** (1984), 96–110. Zbl0543.35051 MR0730003
27. Kohn, R., & Sternberg, P., Local minimisers and singular perturbations. *Proc. Roy. Soc. Edinburgh Sect. A* **111** (1989), 69–84. Zbl0676.49011 MR0985990
28. Liehr, A. W., *Dissipative Solitons in Reaction-Diffusion Systems*. Springer-Verlag, Berlin, 2013. Zbl1270.92001 MR3052731
29. Modica, L., The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rat. Mech. Anal.* **98** (1987), 123–142. Zbl0616.76004 MR0866718
30. Nagumo, J., Arimoto, S., & Yoshikawa, Y. An active pulse transmission line simulating nerve axton. *Proc. IRE* **50** (1962), 2061.
31. Ohta, T., & Kawasaki, K., Equilibrium morphology of block copolymer melts. *Macromolecules* **19** 10 (1986), 2621–2632.

32. Penfold, R., Vanden-Broeck, J. M., & Grandison, S., Monotonicity of some modified Bessel function products. *Integral Transforms Spec. Funct.* **18** (2007), 139–144. Zbl1109.33006 MR2290352
33. Reinecke, C., & Sweers, G., A positive solution on r^n to a equations of FitzHugh–Nagumo type. *J. Differential Equations* **153** (1999), 292–312. Zbl0929.35042 MR1683624
34. Reinecke, C., & Sweers, G., Solutions with internal jump for an autonomous elliptic system of FitzHugh–Nagumo type. *Math. Nachr.* **251** (2003), 64–87. Zbl1118.35009 MR1960805
35. Ren, X., & Shoup, D., The impact of the domain boundary on an inhibitory system: existence and location of a stationary half disc. *Comm. Math. Phys.* **340** (2015), 355–412. Zbl1332.82095 MR3395156
36. Ren, X., & Wei, J., On the multiplicity of solutions of two nonlocal variational problems. *SIAM J. Math. Anal.* **31** (2000), 909–924. Zbl0973.49007 MR1752422
37. Ren, X., & Wei, J., Nucleation in the FitzHugh-Nagumo system: Interface-spike solutions. *J. Differential Equations* **209** (2005), 266–301. Zbl1072.35037 MR2110206
38. Ren, X., & Wei, J., Many droplet pattern in the cylindrical phase of diblock copolymer morphology. *Rev. Math. Phys.* **19** (2007), 879–921. Zbl1145.82007 MR2349026
39. Ren, X., & Wei, J., Single droplet pattern in the cylindrical phase of diblock copolymer morphology. *J. Nonlinear Sci.* **17** (2007), 471–503. Zbl1134.82048 MR2361417
40. Ren, X., & Wei, J., A double bubble assembly as a new phase of a ternary inhibitory system. *Arch. Rat. Mech. Anal.* **215** (2015), 967–1034. Zbl1309.35172 MR3302114
41. Turing, A. M., The chemical basis of morphogenesis. *Phil. Transact. Royal Soc. B* **237** (1952), 37–72. Zbl06853054 MR3363444
42. Wei, J., & Winter, M., Clustered spots in the FitzHugh–Nagumo system. *Math. Ann.* **213** (2005), 121–145. Zbl1330.35022 MR2139340