Interfaces and Free Boundaries **20** (2018), 211–260
DOI 10.4171/IFB/401

Uniform ball property and existence of optimal shapes for a wide class of geometric functionals

JÉRÉMY DALPHIN

Institut Elie Cartan de Lorraine, UMR CNRS 7502, Université de Lorraine, BP 70239, 54506 Vandoeuvre-lès-Nancy Cedex, France

E-mail: *jeremy.dalphin@mines-nancy.org*

[Received 28 December 2016 and in revised form 18 January 2018]

In this article, we study shape optimization problems involving the geometry of surfaces (normal vector, principal curvatures). Given $\varepsilon > 0$ and a fixed non-empty large bounded open hold-all $B \subset \mathbb{R}^n$, $n \geqslant 2$, we consider a specific class $\mathcal{O}_\varepsilon(B)$ of open sets $\Omega \subset B$ satisfying a uniform ε-ball condition. First, we recall that this geometrical property $\Omega \in \mathcal{O}_\varepsilon(B)$ can be equivalently characterized in terms of $C^{1,1}$-regularity of the boundary $\partial\Omega \neq \emptyset$, and thus also in terms of positive reach and oriented distance function. Then, the main contribution of this paper is to prove the existence of a $C^{1,1}$-regular minimizer among $\Omega \in \mathcal{O}_\varepsilon(B)$ for a general range of geometric functionals and constraints defined on the boundary $\partial\Omega$, involving the first- and second-order properties of surfaces, such as problems of the form:

$$\inf_{\Omega \in \mathcal{O}_\varepsilon(B)} \int_{\partial\Omega} \Big(j_0\big[\mathbf{x}, \mathbf{n}(\mathbf{x})\big] + j_1\big[\mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\big] + j_2\big[\mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\big]\Big) dA(\mathbf{x}),$$

where $\mathbf{n}$, H, K respectively denote the unit outward normal vector, the scalar mean curvature and the Gaussian curvature. We only assume continuity of j_0, j_1, j_2 with respect to the set of variables and convexity of j_1, j_2 with respect to the last variable, but no growth condition on j_1, j_2 are imposed here regarding the last variable. Finally, we give various applications in the modelling of red blood cells such as the Canham–Helfrich energy and the Willmore functional.

2010 Mathematics Subject Classification: Primary 49Q10; Secondary 53A05, 49J45.

Keywords: Shape optimization, uniform ball condition, geometric functionals, Helfrich, Willmore, curvature depending energies.

1. Introduction

Many physical phenomena are governed by their surrounding geometry and are often modelled by energy minimization principles. Some problems like soap films involve the first-order properties of surfaces (normal vector, first fundamental form), while others such as the equilibrium shapes of red blood cells also concern the second-order ones (principal curvatures, second fundamental form).

In this article, we are interested in the existence of solutions to such shape optimization problems through the determination of a suitable class of admissible shapes. Indeed, a relevant framework of study is often provided by geometric measure theory [55], but the minimizer is usually less regular than expected, and it is hard to understand (and prove) in which sense singularities occur or not.

Using the viewpoint of shape optimization, the aim of this paper is to consider a more reasonable class of surfaces, in which there always exists an enough regular minimizer for general functionals

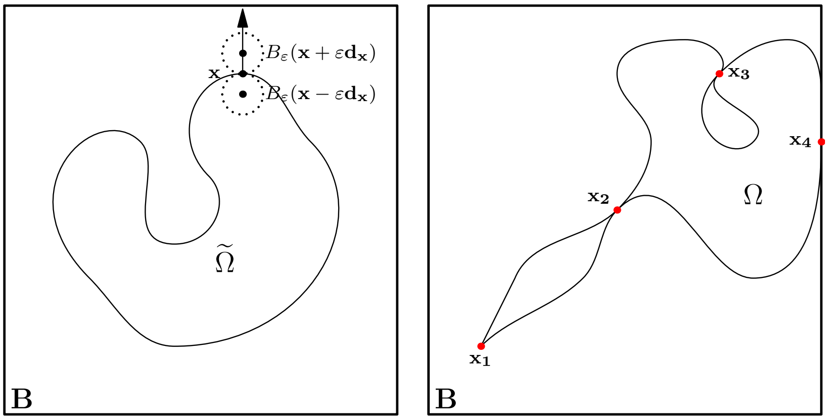

FIG. 1. Example of an open set $\widetilde{\Omega} \subset B$ satisfying the ε-ball condition whereas $\Omega \subset B$ does not. Indeed, there is no circle passing through the points $\mathbf{x}_1$ and $\mathbf{x}_2$ (respectively $\mathbf{x}_3$ and $\mathbf{x}_4$) whose inner domains are included in Ω (respectively in $B\backslash\overline{\Omega}$).

and constraints involving the first- and second-order geometric properties of surfaces. Inspired by the uniform cone property of Chenais [8], we define the uniform ball condition as follows.

DEFINITION 1.1 Let $\varepsilon > 0$ and $B \subseteq \mathbb{R}^n$ be open, $n \geqslant 2$. We say that an open set $\Omega \subset B$ with a non-empty boundary $\partial\Omega := \overline{\Omega}\backslash\Omega$ satisfies the ε-*ball condition* and we write $\Omega \in \mathcal{O}_\varepsilon(B)$ if for any point $\mathbf{x} \in \partial\Omega$, there exists a unit vector $\mathbf{d_x}$ of $\mathbb{R}^n$ such that $B_\varepsilon(\mathbf{x} - \varepsilon\mathbf{d_x}) \subseteq \Omega$ and $B_\varepsilon(\mathbf{x} + \varepsilon\mathbf{d_x}) \subseteq B\backslash\overline{\Omega}$, where $B_r(\mathbf{z}) := \{\mathbf{y} \in \mathbb{R}^n,\ \|\mathbf{y} - \mathbf{z}\| < r\}$ denotes the open ball of $\mathbb{R}^n$ centred at $\mathbf{z}$ and of radius r.

REMARK 1.2 The ε-ball condition, illustrated in Figure 1, only makes sense for sets having a non-empty boundary. Hence, we will always assume $\partial\Omega \neq \emptyset$ in the sequel, or equivalently $\Omega \notin \{\emptyset, \mathbb{R}^n\}$. Note also that Definition 2.1 imposes Ω to be the subset of a fixed set B. However, since we only require B to be open, one can take $B = \mathbb{R}^n$ and consider the class $\mathcal{O}_\varepsilon(\mathbb{R}^n)$ of open sets $\Omega \notin \{\emptyset, \mathbb{R}^n\}$ satisfying the ε-ball condition. This is what we have done for example in Theorems 2.5–2.7.

The uniform ball condition was already considered by Poincaré [50]. As illustrated in Figure 1, it avoids the formation of singularities such as corners, cracks, or self-intersections. In fact, it has been known to characterize the $C^{1,1}$-regularity of hypersurfaces for a long time by oral tradition. Consequently, it can also be linked to other well-known equivalent concepts, such as the notion of *positive reach* introduced by Federer in [30], and the local $C^{1,1}$-regularity of *oriented distance functions* introduced by Delfour and Zolésio in [20].

In [24, Chapter 7 Theorems 7.2–7.3 and 8.1–8.4], one can already find most of the material about these three last properties. However, as far as the uniform ball condition is here concerned, we believe it would be useful to unify the exposition and notation to the expense of a lengthy article. Hence, for completeness, three equivalent characterizations of Definition 1.1 are given in Section 2 with further references, namely Theorems 2.5–2.7 (but proofs are postponed to the Appendix).

Equipped with this class of admissible shapes, we can now state our main general existence result. For simplicity and to avoid here the introduction of too many hypothesis and notation, the result is stated in the three-dimensional Euclidean space $\mathbb{R}^3$ but of course, it can be generalized to $\mathbb{R}^n$ for any $n \geqslant 2$. We refer to Theorem 4.26 in Section 4.5 for its most general form.

MAIN THEOREM 1.3 Let $\varepsilon > 0$ and $B \subset \mathbb{R}^3$ be a non-empty bounded open set, large enough so that $\mathcal{O}_\varepsilon(B) \neq \emptyset$. We consider $(C, \widetilde{C}) \in \mathbb{R} \times \mathbb{R}$, five continuous maps $j_0, f_0, g_0, g_1, g_2 : \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$, and four maps $j_1, j_2, f_1, f_2 : \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ being continuous and convex in their last variable. Then, the following problem has at least one solution (see Remark 1.4):

$$\inf \int_{\partial\Omega} \Big(j_0\big[\mathbf{x}, \mathbf{n}(\mathbf{x})\big] + j_1\big[\mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\big] + j_2\big[\mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\big]\Big)\, dA(\mathbf{x}),$$

where the infimum is taken among $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying a finite number of constraints of the form:

$$\begin{cases} \displaystyle\int_{\partial\Omega} \Big(f_0\big[\mathbf{x}, \mathbf{n}(\mathbf{x})\big] + f_1\big[\mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\big] + f_2\big[\mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\big]\Big)\, dA(\mathbf{x}) \leqslant C \\ \displaystyle\int_{\partial\Omega} \Big(g_0\big[\mathbf{x}, \mathbf{n}(\mathbf{x})\big] + H(\mathbf{x})\, g_1\big[\mathbf{x}, \mathbf{n}(\mathbf{x})\big] + K(\mathbf{x})\, g_2\big[\mathbf{x}, \mathbf{n}(\mathbf{x})\big]\Big)\, dA(\mathbf{x}) = \widetilde{C}. \end{cases}$$

REMARK 1.4 We denote by $A(\bullet)$ (respectively $V(\bullet)$) the area (resp. the volume) i.e. the two(resp. three)-dimensional Hausdorff measure, and the integration on a surface is done with respect to A. The Gauss map $\mathbf{n} : \mathbf{x} \mapsto \mathbf{n}(\mathbf{x}) \in \mathbb{S}^2$ always refers to the unit outer normal field of the surface, while $H = \kappa_1 + \kappa_2$ is the scalar mean curvature and $K = \kappa_1\kappa_2$ is the Gaussian curvature. Note that the Gauss map is Lipschitz continuous since any $\Omega \in \mathcal{O}_\varepsilon(B)$ has a $C^{1,1}$-boundary (cf. Theorem 2.6 and Section 4.1). In particular, from Rademacher's Theorem, it is differentiable almost everywhere so in this context, any notion of curvatures is well defined almost everywhere and essentially bounded.

We mention that the particular case $j_0 \geqslant 0$, $j_1 = j_2 = 0$ without constraints is studied in [36]. The proof of Theorem 1.3 only relies on basic tools of analysis and does not use geometric measure theory. Following the usual direct method from Calculus of Variations, we establish:

(i) in Proposition 3.2 that the class $\mathcal{O}_\varepsilon(B)$ is sequentially compact for some various modes of convergence (for the Hausdorff distance of the complements in $\overline{B}$, of the adherences, of the boundaries, for the $L^1(B)$-norm of the characteristic functions, for the $W^{1,1}(B)$-norm of the oriented distance functions, and in the sense of compact sets, cf. Definition 3.1), allowing the extraction of a minimizing subsequence that converges to a candidate for being a minimizer;

(ii) in Section 4 that the functionals and inequality constraints considered in Theorem 1.3 are lower semi-continuous with respect to the convergence in the sense of compact sets provided the boundaries also converge for the Haudorff distance, while the equality constraints are really continuous, explaining why we only assume $(j_i, f_i)_{i=1,2}$ to be convex in their last variable to get lower semi-continuity (but note that no growth condition is imposed here) whereas the integrands containing $(g_i)_{i=1,2}$ have to be linear in H and K to get continuity.

Point (i) is a consequence of the fact that the ε-ball condition implies the uniform cone property (Theorem 2.6 (i)), for which we have the compactness result of Chenais [8], later refined by Delfour and Zolésio [20] [24, Chapter 7 Theorem 13.1]. Point (ii) is much harder to obtain. Our method is based on localization and the study of convergence for graphs of regular functions (Theorem 3.3). In Section 3, we show we can locally parametrize simultaneously by $C^{1,1}$-graphs in a fixed local frame the boundaries of a converging sequence in $\mathcal{O}_\varepsilon(B)$ (cf. Figure 2). Moreover, the local graphs converge strongly in $C^{1,1-\delta}$, $\delta \in]0, 1]$, and weakly-star in $W^{2,\infty}$. This allows to study and get (ii).

The author is aware of the important work of Delfour and Zolésio related to the *distance function* [21] [24, Chapter 6] and *oriented distance function* [20] [24, Chapter 7] with numerous applications in shape differential calculus [24, Chapter 9]. We refer to Section 3.1 for further references and an overview of the general background related to these concepts. Since the viewpoint of local graphs and oriented distance functions b_Ω are equivalent [24, Chapter 7 Theorem 8.2 (ii)], Theorem 3.3 and the continuity results of Section 4 can also be expressed and proved in terms of b_Ω.

However, we decide to consider here the graph approach for several reasons. First, the oriented distance functions do not remove the difficulty overcome by Theorem 3.3 i.e. the existence of a fixed set to properly study continuity. Indeed, let us assume the convergence of some $(\partial\Omega_i)_{i\in\mathbb{N}}$ to $\partial\Omega$ for the Hausdorff distance. We can find a common tubular neighbourhood $\mathcal{V}_r(\partial\Omega)$, $r > 0$, in which occurs the convergence of the associated oriented distance functions $(b_{\Omega_i})_{i\in\mathbb{N}}$ to b_Ω strongly in $C^{1,1-\delta}$ for any $\delta \in]0,1]$ and weakly-star in $W^{2,\infty}$. Nevertheless, the continuity of a functional $\omega \mapsto \int_{\partial\omega} j$ remains unclear because even if $\int_{\partial\Omega_i} j = \lim_{h\to 0} \frac{1}{2h} \int_{\mathcal{V}_h(\partial\Omega_i)} j \circ (Id - b_{\Omega_i}\nabla b_{\Omega_i})$ as in [15], the exchange of limits $i \to +\infty$ and $h \to 0$ requires some work, that we believe as technical as what we have done to get Theorem 3.3. If this issue is overcome, then the continuity results of Section 4 also follow from the various convergences of $(b_{\Omega_i})_{i\in\mathbb{N}}$ to b_Ω and the fact that ∇b_{Ω_i} is an extension of the unit outward normal field to $\partial\Omega_i$, $\mathrm{Hess}(b_{\Omega_i})$ of the second fundamental form, etc.

Moreover, the article aims to give general existence results for shape optimization problems involving a large range of geometric functionals and constraints. It is thus intended to a broad audience and the viewpoint of graph seems a rather usual approach, compared to further equivalent sophisticated tools that would certainly lighten the proofs. In addition, we only deal here with $C^{1,1}$-regularity and do not necessarily need very sharp tools for studying cracks or the fine geometric properties of shapes. Furthermore, this paper also intends to settle the framework for another future work that will soon be published [11] and study more complex problems of the form:

$$\inf_{\Omega\in\mathcal{O}_\varepsilon(B)} \int_{\partial\Omega} j\left[\mathbf{x}, \mathbf{n}_\Omega(\mathbf{x}), H_\Omega(\mathbf{x}), K_\Omega(\mathbf{x}), u_\Omega(\mathbf{x}), \nabla u_\Omega(\mathbf{x})\right] dA(\mathbf{x}),$$

where u_Ω is the solution of some second-order elliptic boundary-value problems posed on the inner domain enclosed by the shape $\partial\Omega$. In this direction, the convergence results of Theorem 3.3 are very useful to study the convergence of $(u_{\Omega_i} \circ X_i)_{i\in\mathbb{N}}$, where X_i is a local parametrization of $\partial\Omega_i$.

Finally, to our knowledge, the existence results presented here are new. Indeed, the functionals we consider are defined on the boundary of a domain, a case which is not covered by the usual existence theory in shape optimization. Moreover, we are able to extend and generalize the results given in [36] by using a similar framework [35]. If the compactness issue is quite straightforward, the continuity of quite general functionals defined on the boundary is not. In particular, we show in Section 4 how to use Theorem 3.3 in order to study the continuity and lower-semi-continuity properties for a wide range of geometric functionals. Although the statements of Sections 4.1–4.3 are rather expected consequences of Theorem 3.3 under the construction of a suitable partition of unity, the ones of Section 4.4 are not, especially the L^∞-weak-star convergence of the Gaussian curvature. In particular, in Section 4.4, we emphasize the fact that we manage to obtain the non-trivial continuity of non-linear functionals (such as the genus) by applying the Div-Curl Lemma to this geometric setting. To our knowledge, such a method is new. We now present three physical applications of Theorem 1.3 (further general examples in $\mathbb{R}^n$ are also detailed in Section 4.5).

1.1 *First application: minimizing the Canham–Helfrich energy with area and volume constraints*

In biology, when a sufficiently large amount of phospholipids is inserted in aqueous media, they immediately gather in pairs to form bilayers called vesicles. Devoid of nucleus among mammals, red blood cells are typical examples of vesicles on which is fixed a network of proteins playing the role of a skeleton inside the membrane [59]. In the 70s, Canham [7] then Helfrich [37] suggested a simple model to characterize vesicles. Imposing the area of the bilayer and the volume of fluid it contains, their shape is a minimizer for the following free-bending energy (see Remark 1.4):

$$\mathcal{E} = \frac{k_b}{2} \int_{\text{membrane}} (H - H_0)^2 \, dA + k_G \int_{\text{membrane}} K dA, \tag{1.1}$$

where $H_0 \in \mathbb{R}$ (called the spontaneous curvature) measures the asymmetry between the two layers, and where $k_b > 0$, $k_G < 0$ are two other physical constants. Note that if $k_G > 0$, for any $k_b, H_0 \in \mathbb{R}$, the Canham–Helfrich energy (1.1) with prescribed area A_0 and volume V_0 is not bounded from below. Indeed, in that case, from the Gauss–Bonnet Theorem, the second term tends to $-\infty$ as the genus $g \to +\infty$, while the first term remains bounded by $4|k_b|(12\pi + \frac{1}{4}H_0^2 A_0)$ (to see this last point, combine [41, Remark 1.7 (iii) (1.5)], [52, Theorem 1.1], and [56, Inequality (0.2)]).

The two-dimensional case of (1.1) is considered by Bellettini, Dal Maso, and Paolini in [3]. Some of their results is recovered by Delladio [25] in the framework of special generalized Gauss graphs from the theory of currents. Then, Choksi and Veneroni [9] solve the axisymmetric situation of (1.1) in $\mathbb{R}^3$ assuming $-2k_b < k_G < 0$. In the general case, this hypothesis gives a fundamental coercivity property [9, Lemma 2.1] and the integrand of (1.1) is standard in the sense of [40, Definition 4.1.2]. Hence, we get a minimizer for (1.1) in the class of rectifiable integer oriented 2-varifold in $\mathbb{R}^3$ with L^2-bounded generalized 2nd fundamental form [40, Theorem 5.3.2] [47, Section 2] [4, Appendix]. These compactness and lower semi-continuity properties were already noticed in [4, Section 9.3].

However, the regularity of minimizers remains an open problem and experiments show that singular behaviours can occur to vesicles such as the budding effect [53, 54]. This cannot happen to red blood cells because their skeleton prevents the membrane from bending too much locally [59, Section 2.1]. To take this aspect into account, the uniform ball condition of Definition 1.1 is also motivated by the modelization of the equilibrium shapes of red blood cells. We even have a clue for its physical value [59, Section 2.1.5]. Our result states as follows.

Theorem 1.5 *Let $H_0, k_G \in \mathbb{R}$ and $\varepsilon, k_b, A_0, V_0 > 0$ such that $A_0^3 > 36\pi V_0^2$. Then, the following problem has at least one solution (see Remark 1.4):*

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^3) \\ A(\partial\Omega) = A_0 \\ V(\Omega) = V_0}} \frac{k_b}{2} \int_{\partial\Omega} (H - H_0)^2 dA + k_G \int_{\partial\Omega} K dA.$$

REMARK 1.6 The hypothesis $A_0^3 > 36\pi V_0^2$ is rather natural. Indeed, any compact surface has to satisfy the isoperimetric inequality and equality only occurs for spheres. Hence, we have to assume $A_0^3 > 36\pi V_0^2$ otherwise the set of constraints in $\mathcal{O}_\varepsilon(\mathbb{R}^3)$ is either empty or reduced to a ball of area A_0 and volume V_0. Moreover, note that we only consider here the class $\mathcal{O}_\varepsilon(\mathbb{R}^3)$ and not $\mathcal{O}_\varepsilon(B)$ with B bounded as it is the case for Theorem 1.3. Indeed, a uniform bound on the diameter is implicitly given by the functional and the area constraint [56, Lemma 1.1]. Finally, the result of Theorem 1.5 also holds true if H_0 is a continuous function of the position and the normal vector.

1.2 *Second application: minimizing the Canham–Helfrich energy with genus, area, and volume prescribed*

The Gauss–Bonnet Theorem [30, Theorem 5.19] is valid for sets of positive reach (cf. Definition 2.1) thus we get from Theorem 2.5 that $\int_\Sigma K dA = 4\pi(1-g)$ for any compact connected $C^{1,1}$-surface Σ (without boundary embedded in $\mathbb{R}^3$) of genus $g \in \mathbb{N}$. Hence, instead of minimizing (1.1), people usually fix the topology and search for a minimizer of the following energy (see Remark 1.4):

$$\mathcal{H}(\Sigma) = \int_\Sigma (H - H_0)^2 \, dA, \tag{1.2}$$

with prescribed area and enclosed volume. The critical points of (1.2) are studied by Nagasawa and Yi in [49]. Like (1.1), such a functional depends on the surface but also on its orientation. However, in the case $H_0 \neq 0$, energy (1.2) is not even lower semi-continuous with respect to the varifold convergence [4, Section 9.3]: the counterexample is due to Große-Brauckmann [34]. In this case, we cannot directly use the tools of geometric measure theory but we can prove the following result.

Theorem 1.7 *Let $H_0 \in \mathbb{R}$, $g \in \mathbb{N}$, and $\varepsilon, A_0, V_0 > 0$ such that $A_0^3 > 36\pi V_0^2$. Then, the following problem has at least one solution (see Remark 1.4 and Remark 1.6):*

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^3) \\ \text{genus}(\partial\Omega) = g \\ A(\partial\Omega) = A_0 \\ V(\Omega) = V_0}} \int_{\partial\Omega} (H - H_0)^2 dA,$$

where $\text{genus}(\partial\Omega) = g$ *has to be understood as $\partial\Omega$ is a compact connected $C^{1,1}$-surface of genus g.*

1.3 *Third application: Minimizing the Willmore functional for various given constraints*

The particular case $H_0 = 0$ in (1.2) is known as the Willmore functional (see Remark 1.4):

$$\mathcal{W}(\Sigma) = \frac{1}{4} \int_\Sigma H^2 dA. \tag{1.3}$$

It has been widely studied by geometers. Without constraint, Willmore [60, Theorem 7.2.2] proved that spheres are the only global minimizers of (1.3). The existence was established by Simon [56] for genus-one surfaces, Bauer and Kuwert [2] for higher genus. Recently, Marques and Neves [45] solved the so-called Willmore conjecture: conformal transformations of the stereographic projection of the Clifford torus are the only global minimizers of (1.3) among smooth genus-one surfaces.

A main ingredient is the conformal invariance of (1.3), from which we can in particular deduce that minimizing (1.3) with prescribed isoperimetric ratio is equivalent to impose the area and the enclosed volume. In this direction, Schygulla [52] established the existence of a minimizer for (1.3) among analytic surfaces of zero genus and given isoperimetric ratio. For higher genus, Keller, Mondino, and Riviere [41] recently obtained similar results, using the point of view of immersions developed by Riviere [51] to characterize precisely the critical points of (1.3).

An existence result related to (1.3) is the particular case $H_0 = 0$ of Theorem 1.7. Again, the difficulty with these kind of functionals is not to obtain a minimizer (compactness and lower semi-continuity in the class of varifolds for example) but to show that it is regular in the usual sense (i.e. a smooth surface). We now give a last application of Theorem 1.3 which comes from the modelling of vesicles. It is known as the bilayer-couple model [53, Section 2.5.3] and it states as follows.

Theorem 1.8 *Let $M_0 \in \mathbb{R}$ and $\varepsilon, A_0, V_0 > 0$ such that $A_0^3 > 36\pi V_0^2$. Then, the following problem has at least one solution (see Remark 1.4 and Remark 1.6):*

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^3) \\ \text{genus}(\partial\Omega)=g \\ A(\partial\Omega)=A_0,\ V(\Omega)=V_0 \\ \int_{\partial\Omega} H dA = M_0}} \frac{1}{4}\int_{\partial\Omega} H^2 dA.$$

To conclude the introduction, we recall how the paper is organized. In Section 2, three equivalent characterizations of the uniform ball condition are stated, namely Theorem 2.5 in terms of positive reach, Theorem 2.6 in terms of $C^{1,1}$-hypersurface, and Theorem 2.7 in terms of $C^{1,1}$-regular oriented distance function. Proofs are postponed to the Appendix. Following the classical method from the Calculus of Variations, in Section 3.1, we first obtain the compactness of the class $\mathcal{O}_\varepsilon(B)$ for various modes of convergence. This essentially follows from the fact that the ε-ball condition implies a uniform cone property, for which we already have some good compactness results.

Then, in the remaining part of Section 3, we prove the key ingredient of Theorem 1.3 i.e. we manage to parametrize in a fixed local frame simultaneously all the graphs associated with the boundaries of a converging sequence in $\mathcal{O}_\varepsilon(B)$. We prove the $W^{2,\infty}$-weak-star and the $C^{1,1-\delta}$-strong convergence of these local graphs for any $\delta \in]0,1]$. Finally, in Section 4, we show how to use this local result on a suitable partition of unity to get the global continuity for a general range of geometric functionals. We conclude by giving some existence results in Section 4.5. In particular, we prove Theorem 1.3, its generalization to $\mathbb{R}^n$, and detail many applications such as Theorem 1.5 and Theorems 1.7–1.8, mainly coming from the modelling of vesicles and red blood cells.

2. Three characterizations of the uniform ball property

In this section, we recall three characterizations of the ε-ball condition, namely Theorems 2.5–2.7. First, it is equivalent to Federer's notion of positive reach [30]. Then, it is equivalent to a uniform $C^{1,1}$-regularity of hypersurfaces. Finally, it is equivalent to the local $C^{1,1}$-regularity of oriented distance functions introduced by Zolésio and Delfour [24, Chapter 7]. All this is known but for completeness and readability, the proofs are postponed to the Appendix, since we did not find references where these characterizations were gathered in the form given in Theorems 2.5–2.7.

Indeed, two equivalent characterizations in terms of positive reach, local graph, and oriented distance function can be found in [24, Chapter 7 Theorems 7.2-7.3 and 8.1-8.4], but they are not linked to the uniform ball condition studied in this paper. Moreover, many parts of Theorems 2.5–2.6 can be found in the literature as remarks [39, below Theorem 1.4] [46, (1.10)] [30, Remark 4.20], sometimes with proofs [31, Section 2.1] [35, Theorem 2.2] [42, §4 Theorem 1] [43, Proposition 1.4], or as consequences of results [32, Theorem 1.2] [1, Theorem 1.1 (1.2)]. Finally, we mention that the proofs of Theorems 2.5–2.6 were already given in [10] and are reproduced here for completeness.

Before stating the theorems, we recall some definitions and notation, used hereafter in the paper. Let $n \geqslant 2$ be an integer henceforth set. The space $\mathbb{R}^n$ whose points are marked $\mathbf{x} = (x_1, \ldots, x_n)$ is naturally provided with its usual Euclidean structure, $\langle \mathbf{x} \mid \mathbf{y} \rangle = \sum_{k=1}^n x_k y_k$ and $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x} \mid \mathbf{x} \rangle}$, but also with a direct orthonormal frame whose choice will be specified later. Inside this frame, every point $\mathbf{x}$ of $\mathbb{R}^n$ will be written into the form $(\mathbf{x}', x_n)$ such that $\mathbf{x}' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. In particular, the symbols $\mathbf{0}$ and $\mathbf{0}'$ respectively refer to the zero vector of $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$.

First, some of the notation introduced in [30, Section 4] by Federer are recalled. For every non-empty subset A of $\mathbb{R}^n$, the following map is well defined and 1-Lipschitz continuous:

$$d(\bullet, A): \begin{array}{ccl} \mathbb{R}^n & \longrightarrow & [0, +\infty[\\ \mathbf{x} & \longmapsto & d(\mathbf{x}, A) = \inf_{\mathbf{a}\in A} \|\mathbf{x}-\mathbf{a}\|. \end{array}$$

Furthermore, we set $\mathrm{Unp}(A) := \{\mathbf{x} \in \mathbb{R}^n \mid \exists! \mathbf{a} \in A, \quad \|\mathbf{x}-\mathbf{a}\| = d(\mathbf{x}, A)\}$. This is the set of points in $\mathbb{R}^n$ having a unique projection on A, i.e., the domain on which this map is well defined:

$$p_A : \mathbf{x} \in \mathrm{Unp}(A) \longmapsto p_A(\mathbf{x}) \in A,$$

where $p_A(\mathbf{x})$ is the unique point of A such that $\|p_A(\mathbf{x}) - \mathbf{x}\| = d(\mathbf{x}, A)$. We can also notice that $A \subseteq \mathrm{Unp}(A)$ thus in particular $\mathrm{Unp}(A) \neq \emptyset$. We can now express what is a set of positive reach.

DEFINITION 2.1 Consider any non-empty subset A of $\mathbb{R}^n$. First, we set for any point $\mathbf{a} \in A$:

$$\mathrm{Reach}(A, \mathbf{a}) = \sup\left\{r > 0, \quad B_r(\mathbf{a}) \subseteq \mathrm{Unp}(A)\right\},$$

with the convention $\sup \emptyset = 0$. Then, we define the *reach* of A as $\mathrm{Reach}(A) = \inf_{\mathbf{a}\in A} \mathrm{Reach}(A, \mathbf{a})$. Finally, we say that A has a *positive reach* if we have $\mathrm{Reach}(A) > 0$.

Definition 2.1 is the one given by Federer [30, Definition 4.1]. Note that if A is a non-empty open subset of $\mathbb{R}^n$, then $\mathrm{Unp}(A) = A$ so $\mathrm{Reach}(x, A) = d(x, \partial A)$ for any $x \in A$ and thus $\mathrm{Reach}(A) = 0$ [30, Remark 4.2]. Hence, the notion of reach is of little interest for open sets. This is reason why some authors often assume that A is closed in Definition 2.1, or equivalently, define $\mathrm{Reach}(A, \mathbf{a})$ for any $\mathbf{a} \in \overline{A}$ as in [24, Chapter 6 Definition 6.1]. Similarly, in order to ensure that any point of $\mathbb{R}^n$ has at least one projection on A [24, Chapter 6 Theorem 2.1 (ii)], some people often assume that A is closed, or equivalently, define the projection p_A as a map from $\mathrm{Unp}(A)$ into $\overline{A}$. In our case, we will always consider the reach of the boundary $\partial\Omega$, the closure $\overline{\Omega}$, or the complement $\mathbb{R}^n \backslash \Omega$ of an open set $\Omega \notin \{\emptyset, \mathbb{R}^n\}$ so Definition 2.1 and the one of p_A do not lead to any ambiguity here.

Then, we also recall the definition of a $C^{1,1}$-hypersurface in terms of local graph. Note that from the Jordan-Brouwer Separation Theorem, any compact topological hypersurface of $\mathbb{R}^n$ has a well-defined inner domain, and in particular a well-defined enclosed volume. If instead of being compact, it is connected and closed as a subset of $\mathbb{R}^n$, then it remains the boundary of an open set [48, Theorem 4.16] [27, Section 8.15], which is not unique and possibly unbounded in this case.

DEFINITION 2.2 Consider any non-empty subset $\mathcal{S}$ of $\mathbb{R}^n$. We say that $\mathcal{S}$ is a $C^{1,1}$*-hypersurface* if there exists an open subset Ω of $\mathbb{R}^n$ such that $\partial\Omega = \mathcal{S}$, and such that for any point $\mathbf{x_0} \in \partial\Omega$, there exists a direct orthonormal frame centred at $\mathbf{x_0}$ such that in this local frame, there exists a map $\varphi : D_r(\mathbf{0}') \to]-a, a[$ continuously differentiable with $a > 0$, such that φ and its gradient $\nabla\varphi$ are L-Lipschitz continuous with $L > 0$, satisfying $\varphi(\mathbf{0}') = 0$, $\nabla\varphi(\mathbf{0}') = \mathbf{0}'$, and also:

$$\begin{cases} \partial\Omega \cap (D_r(\mathbf{0}') \times]-a, a[) = \left\{(\mathbf{x}', \varphi(\mathbf{x}')), \quad \mathbf{x}' \in D_r(\mathbf{0}')\right\} \\ \Omega \cap (D_r(\mathbf{0}') \times]-a, a[) = \left\{(\mathbf{x}', x_n), \quad \mathbf{x}' \in D_r(\mathbf{0}') \text{ and } -a < x_n < \varphi(\mathbf{x}')\right\}, \end{cases}$$

with $D_r(\mathbf{0}') = \{\mathbf{x}' \in \mathbb{R}^{n-1}, \|\mathbf{x}'\| < r\}$ the open ball of $\mathbb{R}^{n-1}$ of radius $r > 0$ centred at the origin.

Finally, we recall the definition of the uniform cone property introduced by Chenais in [8], and from which the ε-ball condition is inspired. We also refer to [38, Definition 2.4.1].

DEFINITION 2.3 Let $\alpha \in]0, \frac{\pi}{2}[$ and $\Omega \subset \mathbb{R}^n$ be open with a non-empty boundary. We say that Ω satisfies the *α-cone condition* if for any $\mathbf{x} \in \partial\Omega$, there exists a unit vector $\mathbf{d}_{\mathbf{x}}$ of $\mathbb{R}^n$ such that:

$$\forall \mathbf{y} \in B_\alpha(\mathbf{x}) \cap \Omega, \qquad C_\alpha(\mathbf{y}, \mathbf{d}_{\mathbf{x}}) \subseteq \Omega,$$

where $C_\alpha(\mathbf{y}, \mathbf{d}_{\mathbf{x}}) = \{\mathbf{z} \in B_\alpha(\mathbf{y}),\ \|\mathbf{z} - \mathbf{y}\| \cos\alpha < \langle \mathbf{z} - \mathbf{y} \mid \mathbf{d}_{\mathbf{x}} \rangle\}$ refers to the open cone of vertex $\mathbf{y}$, direction $\mathbf{d}_{\mathbf{x}}$, and (half-)aperture α.

At last, we give the definition of the oriented distance function introduced by Delfour and Zolésio in [20], which provides a useful level-set description of a set.

DEFINITION 2.4 Let $A \subseteq \mathbb{R}^n$ have a non-empty boundary. Then, the *oriented distance function* $b_A : \mathbb{R}^n \mapsto \mathbb{R}$ is defined as $b_A(\mathbf{x}) := d(\mathbf{x}, A) - d(\mathbf{x}, \mathbb{R}^n \backslash A)$ for any $\mathbf{x} \in \mathbb{R}^n$. In particular, we have:

$$\forall \mathbf{x} \in \mathbb{R}^n, \qquad b_A(\mathbf{x}) = \begin{cases} d(\mathbf{x}, \partial A) & \text{if } \mathbf{x} \in \mathbb{R}^n \backslash \overline{A} \\ 0 & \text{if } \mathbf{x} \in \partial A. \\ -d(\mathbf{x}, \partial A) & \text{if } \mathbf{x} \in \mathrm{Int}(A) \end{cases}$$

We are now in position to state three characterizations of the ε-ball condition. In Theorems 2.5–2.7, $V(\bullet)$ refers to the n-dimensional Lebesgue measure and the proofs are given in the Appendix.

Theorem 2.5 (A characterization in terms of positive reach) *Consider any open subset Ω of $\mathbb{R}^n$ with a non-empty boundary. Then, the following implications are true:*

(i) *if there exists $\varepsilon > 0$ such that $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$ as in Definition 1.1, then $\partial\Omega$ has a positive reach in the sense of Definition 2.1 with* $\mathrm{Reach}(\partial\Omega) \geqslant \varepsilon$ *and we have $V(\partial\Omega) = 0$;*
(ii) *if $\partial\Omega$ has a positive reach and $V(\partial\Omega) = 0$, then $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$ for any $\varepsilon \in]0, \mathrm{Reach}(\partial\Omega)[$, and moreover, if $\partial\Omega$ has a finite positive reach, then Ω also satisfies the* $\mathrm{Reach}(\partial\Omega)$*-ball condition.*

In particular, if $V(\partial\Omega) = 0$, then we have the following characterization:

$$\mathrm{Reach}(\partial\Omega) = \sup\left\{\varepsilon > 0, \quad \Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)\right\},$$

with the convention $\sup\emptyset = 0$. *Moreover, this supremum becomes a maximum if it is not zero and finite. Finally, we get* $\mathrm{Reach}(\partial\Omega) = +\infty$ *if and only if $\partial\Omega$ is an affine hyperplane of $\mathbb{R}^n$.*

Theorem 2.6 (A characterization in terms of $C^{1,1}$-regularity) *Let Ω be any open subset of $\mathbb{R}^n$ with a non-empty boundary. If there exists $\varepsilon > 0$ such that $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$, then its boundary $\partial\Omega$ is a $C^{1,1}$-hypersurface of $\mathbb{R}^n$ in the sense of Definition 2.2, where $a = \varepsilon$, the constants L, r depend only on ε, and where $\nabla\varphi$ is valued in $D_{\frac{32}{31}}(\mathbf{0}')$. Moreover, we have the following properties:*

(i) *Ω satisfies the $f^{-1}(\varepsilon)$-cone property as in Definition 2.3 with*
$f : \alpha \in]0, \frac{\pi}{2}[\mapsto \frac{2\alpha}{\cos\alpha} \in]0, +\infty[$;
(ii) *the $\mathbf{d}_{\mathbf{x}}$ of Definition 1.1 is the unit outer normal vector to the hypersurface at the point $\mathbf{x}$;*
(iii) *the Gauss map $\mathbf{d} : \mathbf{x} \in \partial\Omega \mapsto \mathbf{d}_{\mathbf{x}} \in \mathbb{S}^{n-1}$ is well defined and $\frac{1}{\varepsilon}$-Lipschitz continuous.*

Conversely, if $\mathcal{S}$ is a non-empty compact $C^{1,1}$-hypersurface of $\mathbb{R}^n$ in the sense of Definition 2.2, then there exists $\varepsilon > 0$ such that its inner domain $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$. In particular, it has a positive reach with $\mathrm{Reach}(\mathcal{S}) = \max\{\varepsilon > 0,\ \Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)\}$ *and we have $V(\mathcal{S}) = 0$.*

Theorem 2.7 (A characterization in terms of oriented distance function) *Let Ω be any open subset of $\mathbb{R}^n$ with a non-empty boundary. If there exists $\varepsilon > 0$ such that $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$ as in Definition 1.1, then the oriented distance function b_Ω introduced in Definition 2.4 is continuously differentiable on the open tubular neighbourhood $\mathcal{V}_\varepsilon(\partial\Omega) := \{\mathbf{x} \in \mathbb{R}^n,\ d(\mathbf{x}, \partial\Omega) < \varepsilon\}$. Moreover, we have $V(\partial\Omega) = 0$ and for any $r \in]0, \varepsilon[$, the map $\nabla b_\Omega : \mathcal{V}_r(\partial\Omega) \to \mathbb{S}^{n-1}$ is $\frac{2}{\varepsilon - r}$-Lipschitz continuous, having a unique $\frac{2}{\varepsilon - r}$-Lipschitz continuous extension to $\overline{\mathcal{V}_r(\partial\Omega)}$. Conversely, if there exists $\varepsilon > 0$ such that $b_\Omega \in C^{1,1}(\overline{B_\varepsilon(\mathbf{x})})$ and $V(B_\varepsilon(\mathbf{x}) \cap \partial\Omega) = 0$ for any $\mathbf{x} \in \partial\Omega$, then we have $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$.*

REMARK In Theorem 2.6, one can notice that a, L, and r only depend on ε for any point of the hypersurface. This uniform dependence of the $C^{1,1}$-regularity characterizes the class $\mathcal{O}_\varepsilon(\mathbb{R}^n)$. Indeed, the converse part of Theorem 2.6 also holds true if instead of being compact, the non-empty $C^{1,1}$-hypersurface $\mathcal{S}$ satisfies: $\exists \varepsilon > 0, \forall \mathbf{x_0} \in \mathcal{S}, \min(\frac{1}{L}, \frac{r}{3}, \frac{a}{3}) \geqslant \varepsilon$. In this case, we still have $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$ where Ω is the open set of Definition 2.2 such that $\partial\Omega = \mathcal{S}$.

REMARK 2.8 From Point (iii) of Theorem 2.6, the Gauss map $\mathbf{d}$ is $\frac{1}{\varepsilon}$-Lipschitz continuous. Hence, it is differentiable almost everywhere and its differential $D_\bullet \mathbf{d} : \mathbf{x} \in \partial\Omega \mapsto D_\mathbf{x}\mathbf{d} \in \mathcal{L}(T_\mathbf{x}\partial\Omega)$ is an L^∞-map satisfying $\|D_\bullet \mathbf{d}\|_{L^\infty(\partial\Omega)} \leqslant \frac{1}{\varepsilon}$ [38, Section 5.2.2]. In particular, the principal curvatures (see Section 4.1 for definitions and (4.19) for details) satisfy $\|\kappa_l\|_{L^\infty(\partial\Omega)} \leqslant \frac{1}{\varepsilon}$ for any $l \in \{1, \dots, n-1\}$.

3. Parametrization of a converging sequence from $\mathcal{O}_\varepsilon(B)$

In this section, we are interested in establishing some good compactness results. First, we recall the definitions of some various modes of convergence used thereafter.

DEFINITION 3.1 The *Hausdorff distance* $d_H(X, Y)$ between two compact sets $X, Y \subset \mathbb{R}^n$ is defined as $\max(\sup_{\mathbf{x}\in X} d(\mathbf{x}, Y), \sup_{\mathbf{y}\in Y} d(\mathbf{y}, X))$. We say that a sequence of compacts sets $(K_i)_{i\in\mathbb{N}}$ converges to a compact set K *for the Hausdorff distance* if $d_H(K_i, K) \to 0$ as $i \to +\infty$. Let $B \subset \mathbb{R}^n$ be non-empty bounded open. A sequence of open sets $(\Omega_i)_{i\in\mathbb{N}} \subset B$ converges to an open set $\Omega \subset B$:

(i) *in the Hausdorff sense* if $(\overline{B}\backslash\Omega_i)_{i\in\mathbb{N}}$ converges to $\overline{B}\backslash\Omega$ for the Hausdorff distance;
(ii) *in the sense of compact sets* if for any compact sets K and L such that $K \subset \Omega$ and $L \subset B\backslash\overline{\Omega}$, there exists $I \in \mathbb{N}$ such that for any integer $i \geqslant I$, we have $K \subset \Omega_i$ and $L \subset B\backslash\overline{\Omega_i}$;
(iii) *in the sense of characteristic functions* if we have $\int_B |\mathbf{1}_{\Omega_i}(\mathbf{x}) - \mathbf{1}_\Omega(\mathbf{x})| d\mathbf{x} \to 0$, where $\mathbf{1}_X$ is the characteristic function of X, valued one for the points of X, otherwise zero.

In Section 3.1, we recall some well-known compactness results about the uniform cone property. From Point (i) of Theorem 2.6, every set satisfying the ε-ball condition also satisfies the $f^{-1}(\varepsilon)$-cone property. Hence, we only have to check that $\mathcal{O}_\varepsilon(B)$ is closed under the convergence in the Hausdorff sense (cf. Definition 3.1 (i)) to get the following compactness result.

Proposition 3.2 *Let $\varepsilon > 0$ and $B \subset \mathbb{R}^n$ be a bounded open set, large enough to ensure that $\mathcal{O}_\varepsilon(B) \neq \emptyset$. If $(\Omega_i)_{i\in\mathbb{N}}$ is a sequence of elements from $\mathcal{O}_\varepsilon(B)$, then there exists $\Omega \in \mathcal{O}_\varepsilon(B)$ such that a subsequence $(\Omega_{i'})_{i\in\mathbb{N}}$ converges to Ω in the following senses (see Definition 3.1):*

- *$(\Omega_{i'})_{i\in\mathbb{N}}$ converges to Ω in the Hausdorff sense;*
- *$(\partial\Omega_{i'})_{i\in\mathbb{N}}$ converges to $\partial\Omega$ for the Hausdorff distance;*
- *$(\overline{\Omega_{i'}})_{i\in\mathbb{N}}$ converges to $\overline{\Omega}$ for the Hausdorff distance;*

- $(B\backslash\overline{\Omega_{i'}})_{i\in\mathbb{N}}$ *converges to* $B\backslash\Omega$ *in the Hausdorff sense;*
- $(\Omega_{i'})_{i\in\mathbb{N}}$ *converges to* Ω *in the sense of compact sets;*
- $(\Omega_{i'})_{i\in\mathbb{N}}$ *converges to* Ω *in the sense of characteristic functions.*

Moreover, considering the associated oriented distance functions introduced in Definition 2.4, we also have that $(b_{\Omega_{i'}})_{i\in\mathbb{N}}$ *strongly converges to* b_Ω *in* $W^{1,p}(B,\mathbb{R})$ *for any* $p\in[1,+\infty[$.

In Section 3.1, Proposition 3.2 is proved and for sake of completeness, further explanations and references are given with respect to this general compactness pattern. Then, in the remaining part of Section 3, we consider a sequence $(\Omega_i)_{i\in\mathbb{N}}$ of elements from $\mathcal{O}_\varepsilon(B)$ converging to $\Omega\in\mathcal{O}_\varepsilon(B)$ in the sense of compact sets (cf. Definition 3.1 (ii)). We prove that for any i sufficiently large, the boundary $\partial\Omega_i$ can be locally parametrized by a $C^{1,1}$-graph in a local frame associated with $\partial\Omega$. The key point here is that the local frame is fixed and does not depend in i. Moreover, we get the $C^{1,1-\delta}$-strong for any $\delta\in]0,1]$ and the $W^{2,\infty}$-weak-star convergence of a subsequence of these local graphs. The entire sequence converges under the additional assumption $\lim_{i\to+\infty} d_H(\partial\Omega_i,\partial\Omega)=0$. In this case, the limit graph is precisely the one associated with $\partial\Omega$. These results are illustrated in Figure 2 and will be fundamentally used in Section 4 to study the continuity of functionals.

MAIN THEOREM 3.3 Let $(\Omega_i)_{i\in\mathbb{N}}\subset\mathcal{O}_\varepsilon(B)$ converge to $\Omega\in\mathcal{O}_\varepsilon(B)$ as in Definition 3.1 (ii). Then, for any point $\mathbf{x_0}\in\partial\Omega$, there exists a direct orthonormal frame centred at $\mathbf{x_0}$, and also $I\in\mathbb{N}$ depending only on $\mathbf{x_0}$, ε, Ω, and $(\Omega_i)_{i\in\mathbb{N}}$, such that inside this frame, for any integer $i\geqslant I$, there exists a continuously differentiable map $\varphi_i : D_{\tilde r}(\mathbf{0'})\to]-\varepsilon,\varepsilon[$, whose gradient $\nabla\varphi_i$ is valued in $D_{\frac{32}{29}}(\mathbf{0'})$, where $\nabla\varphi_i$ and φ_i are L-Lipschitz continuous with $L>0$ and $\tilde r>0$ depending only on ε, and such that:

$$\begin{cases} \partial\Omega_i\cap(D_{\tilde r}(\mathbf{0'})\cap]-\varepsilon,\varepsilon[)=\big\{(\mathbf{x'},\varphi_i(\mathbf{x'})),\quad \mathbf{x'}\in D_{\tilde r}(\mathbf{0'})\big\}\\ \Omega_i\cap(D_{\tilde r}(\mathbf{0'})\cap]-\varepsilon,\varepsilon[)=\big\{(\mathbf{x'},x_n),\quad \mathbf{x'}\in D_{\tilde r}(\mathbf{0'}) \text{ and } -\varepsilon<x_n<\varphi_i(\mathbf{x'})\big\}.\end{cases}$$

Moreover, any of the $(\varphi_i)_{i\geqslant I}$ has a unique $C^{1,1}$-extension to the closure $\overline{D_{\tilde r}(\mathbf{0'})}$ and there exists $\varphi\in W^{2,\infty}(D_{\tilde r}(\mathbf{0'}))\cap C^1(\overline{D_{\tilde r}(\mathbf{0'})})$ such that a subsequence $(\varphi_{i'})_{i\geqslant I}$ satisfies:

$$\begin{cases} \varphi_{i'}\to\varphi & \text{strongly in } C^{1,1-\delta}(\overline{D_{\tilde r}(\mathbf{0'})}) \text{ for any } \delta\in]0,1],\\ \varphi_{i'}\rightharpoonup\varphi & \text{weakly star in } W^{2,\infty}\big(D_{\tilde r}(\mathbf{0'})\big).\end{cases}\tag{3.1}$$

If in addition, we assume that $(\partial\Omega_i)_{i\in\mathbb{N}}$ converges to $\partial\Omega$ for the Hausdorff distance, then the map φ is precisely the one of Definition 2.2 associated with the point $\mathbf{x_0}$ of $\partial\Omega$ and furthermore, the *whole* sequence $(\varphi_i)_{i\geqslant I}$ converge to φ in (3.1).

The proof of Theorem 3.3 is organized in the spirit of Sections A.1.2 and A.2.1–A.2.2. First, some geometric inequalities are given in Section 3.2. Then, the boundary $\partial\Omega_i$ is locally parametrized by a certain graph in Section 3.3. Finally, in Section 3.4, we obtain the $C^{1,1}$-regularity of this graph. We conclude Section 3 by proving Theorem 3.3 i.e. that (3.1) holds true for the graphs.

3.1 *Compactness of the class* $\mathcal{O}_\varepsilon(B)$

In this section, we recall the general background concerning the compactness results given by a uniform regularity. First, we consider the well-known case of the uniform cone property.

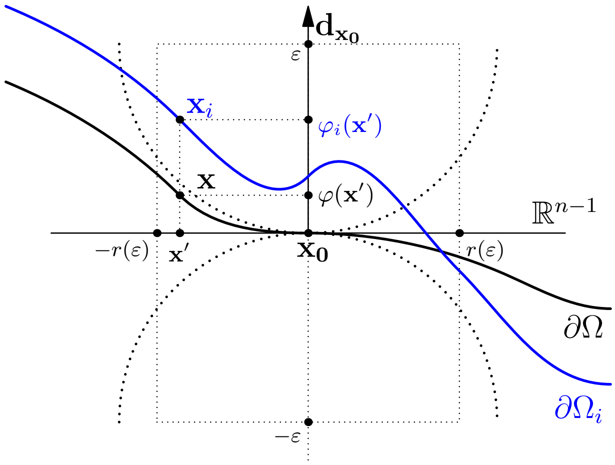

FIG. 2. Illustration of Theorem 3.3 stating that there exists a fixed common local frame in which a converging sequence of elements in $\mathcal{O}_\varepsilon(B)$ can be simultaneously parametrized by $C^{1,1}$-graphs

Theorem 3.4 *Let $\alpha \in]0, \frac{\pi}{2}[$ and B be a bounded open subset of $\mathbb{R}^n$. We set $\mathfrak{O}_\alpha(B)$ as the class of non-empty open sets $\Omega \subseteq B$ that satisfy the α-cone property of Definition 2.3. We assume that B is large enough to have $\mathfrak{O}_\alpha(B) \neq \emptyset$. If $(\Omega_i)_{i\in\mathbb{N}}$ is a sequence of elements from $\mathfrak{O}_\alpha(B)$, then there exists $\Omega \in \mathfrak{O}_\alpha(B)$ such that a subsequence $(\Omega_{i'})_{i\in\mathbb{N}}$ converges to Ω as in Proposition 3.2.*

Proof. First, for a proof of the convergence in the sense of characteristic functions, we refer to the original paper of Chenais [8, Theorem III.1]. Another proof is given in [24, Chapter 5 Theorem 6.11] but assume that the boundary ∂B is uniformly Lipschitz. Then, we refer to [38, Theorem 2.4.10] for further details concerning the proof of Theorem 3.4 that is not considering the convergence of the oriented distance functions. Finally, a complete proof of Theorem 3.4 can be found in Section [24, Chapter 7 Section 13 Theorem 13.1 and Corollary 2]. □

Working with a family of open sets Ω contained a bounded open hold-all $B \subset \mathbb{R}^n$ makes the boundaries compact and such a class becomes sequentially compact in the Hausdorff sense (cf. Definition 3.1 (i) and for a proof see e.g. [24, Chapter 6 Theorem 2.4 (ii)] or [38, Theorem 2.2.23]). As shown by Chenais in [8], adding a uniform Lipschitz condition on the local graph yields to a compactness result in terms of characteristic functions. In [58], Tiba obtain in a similar result by assuming only a uniform condition on the modulus of continuity of the local graph functions. In doing so, he generalized what Chenais did to domains with cusps.

However, there is a stronger and neater version which gives the convergence of the oriented distance functions b_Ω in $W^{1,p}(B)$ for any $p \in [1, +\infty[$. It was originally given in the first 2001 edition of the book of Delfour and Zolésio [24, Chapter 7 Section 13] in terms of a uniform fat segment condition that generalizes the uniform cone and cusp properties. In [22, 23], Delfour and Zolésio gave the equivalence between this condition and the one considered by Tiba [24, Chapter 7 Theorem 13.2]. Most compactness theorems (uniform cone property of Chenais [8], density perimeter and capacity condition of Bucur and Zolésio [5, 6], sets of bounded curvatures [24, Chapter 7, Section 11], and the graph version of the uniform cusp property of Tiba [58]) are not only true for the C^0-convergence of distance functions, or the L^p-

convergence of characteristic functions, but also for the finer $W^{1,p}$-convergence of oriented distance functions.

Moreover, this latter directly implies [24, Chapter 7 Theorem 4.1 (iv)-(v)] the other convergences, i.e., of the distance functions $d_{\overline{\Omega}}$, $d_{\mathbb{R}^n\backslash\Omega}$, and $d_{\partial\Omega}$ in $C^0(\overline{B})$ and in $W^{1,p}(B)$ for any $p \in [1,\infty[$, and of the characteristics functions $\mathbf{1}_\Omega$, $\mathbf{1}_{\mathbb{R}^n\backslash\Omega}$, and $\mathbf{1}_{\partial\Omega}$ in $L^p(B)$ for any $p \in [1,+\infty[$. Since we have $d_H(\overline{A},\overline{B}) = \|d_{\overline{A}} - d_{\overline{B}}\|_{C^0(\overline{B})}$ for any $A, B \subset \overline{B}$ (see, e.g., [38, Proposition 2.2.25] or [24, Chapter 6, Section 2.2]), this general pattern is exactly what we have stated in Proposition 3.2 for the uniform ball condition, which requires a uniform local Lipschitz condition on the local graph function and its gradient at each point of the boundary (cf. Theorem 2.6).

The oriented distance function of Definition 2.4 was initially introduced by Delfour and Zolésio in [20]. They were able to sharpen the local characterization of C^k-regular sets, $k \geqslant 2$, given by Gilbarg and Trudinger [33], and extended it to sets of class $C^{1,1}$. Therefore, for a set of class $C^{1,1}$ or with better regularity, the oriented distance function has the same regularity in the neighbourhood of each point of its boundary, and this is equivalent to a local graph representation with the same smoothness [24, Chapter 7 Theorem 8.2]. The sets of class $C^{1,1}$ have been extensively studied through the oriented distance functions b_Ω [15, 18, 19], and especially in the context of thin and asymptotics shells [12–14, 16, 17, 22]. In particular, the restriction of ∇b_Ω to $\partial\Omega$ is the unitary exterior normal vector to Ω, Hess(b_Ω) is the natural extension to $\mathbb{R}^n$ of the second fundamental form associated with $\partial\Omega$, $(\text{Hess}(b_\Omega))^2$ the third fundamental form, and so on. They exists almost everywhere with respect to the $(n-1)$-dimensional Hausdorff measure (see e.g. [17]). Under this point of view, the intrisic theory of Sobolev space on such $C^{1,1}$-hypersurfaces can be found in [15].

However, this article consider the more geometrical approach of the uniform ball condition in the context of shape optimization. Of course, the two concepts are equivalent as we have shown in Theorem 2.7, and both can be used to study these kind of problems. The reasons of this choice were already explained in Section 1, from below Theorem 1.3 until Section 1.1.

We are now in position to prove Proposition 3.2, mentioning that a proof can also be found in [35, Theorem 2.8]. More precisely, Guo and Yang prove that $\mathcal{O}_\varepsilon(B)$ is sequentially compact for the convergence in the Hausdorff sense (cf. Definition 3.1 (i)). Hence, combining this result with Theorem 3.4, we get that Proposition 3.2 holds true. The proof is short, see [35] for details.

***Proof of Proposition* 3.2.** Since $\mathcal{O}_\varepsilon(B) \subset \mathfrak{O}_{f^{-1}(\varepsilon)}(B)$ (Point (i) of Theorem 2.6), Theorem 3.4 holds true and we only have to check $\Omega \in \mathcal{O}_\varepsilon(B)$. Consider $\mathbf{x} \in \partial\Omega$. From [38, Proposition 2.2.14], there exists a sequence of points $\mathbf{x}_i \in \partial\Omega_i$ converging to $\mathbf{x}$. Then, we can apply the ε-ball condition on each point $\mathbf{x}_i$ so there exists a sequence of unit vector $\mathbf{d}_{\mathbf{x}_i}$ of $\mathbb{R}^n$ such that:

$$\forall i \in \mathbb{N}, \qquad B_\varepsilon(\mathbf{x}_i - \varepsilon\mathbf{d}_{\mathbf{x}_i}) \subseteq \Omega_i \quad \text{and} \quad B_\varepsilon(\mathbf{x}_i + \varepsilon\mathbf{d}_{\mathbf{x}_i}) \subseteq B\backslash\overline{\Omega_i}.$$

Since $\|\mathbf{d}_{\mathbf{x}_i}\| = 1$, there exists a unit vector $\mathbf{d}_\mathbf{x}$ of $\mathbb{R}^n$ such that, up to a subsequence, $(\mathbf{d}_{\mathbf{x}_i})_{i\in\mathbb{N}}$ converges to $\mathbf{d}_\mathbf{x}$. Finally, the inclusion is stable under the Hausdorff convergence [38, (2.16)] and we get the ε-ball condition of Definition 1.1 by letting $i \to +\infty$ in the above inclusions. □

3.2 *Some global and local geometric inequalities*

In the remaining part of Section 3, consider a sequence $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$ converging to $\Omega \in \mathcal{O}_\varepsilon(B)$ in the sense of compact sets (cf. Definition 3.1 (ii)). We also make the following hypothesis, which are only used throughout Section 3.2–3.4 to prove Theorem 3.3.

ASSUMPTION 3.5 Let $\mathbf{x}_0 \in \partial\Omega$ henceforth set. From the ε-ball condition, a unit vector $\mathbf{d}_{\mathbf{x}_0}$ is associated with the point $\mathbf{x}_0$ (which is unique from Proposition A.4). Moreover, we have:

$$B_\varepsilon(\mathbf{x}_0 - \varepsilon\mathbf{d}_{\mathbf{x}_0}) \subseteq \Omega \qquad \text{and} \qquad B_\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{d}_{\mathbf{x}_0}) \subseteq B\backslash\overline{\Omega}.$$

Then, we consider $\eta \in]0, \varepsilon[$. Since we assume that $(\Omega_i)_{i\in\mathbb{N}}$ converges to Ω as in Definition 3.1 (ii), there exists $I \in \mathbb{N}$ depending on $(\Omega_i)_{i\in\mathbb{N}}$, Ω, $\mathbf{x}_0$, ε and η, such that for any integer $i \geqslant I$, we have:

$$\overline{B_{\varepsilon-\eta}(\mathbf{x}_0 - \varepsilon\mathbf{d}_{\mathbf{x}_0})} \subseteq \Omega_i \qquad \text{and} \qquad \overline{B_{\varepsilon-\eta}(\mathbf{x}_0 + \varepsilon\mathbf{d}_{\mathbf{x}_0})} \subseteq B\backslash\overline{\Omega_i}. \tag{3.2}$$

Finally, we consider any integer $i \geqslant I$.

Proposition 3.6 *We assume that* (3.2) *holds true. Then, for any point* $\mathbf{x}_i \in \partial\Omega_i$, *we have*

$$\|\mathbf{d}_{\mathbf{x}_i} - \mathbf{d}_{\mathbf{x}_0}\|^2 \leqslant \frac{1}{\varepsilon^2}\|\mathbf{x}_i - \mathbf{x}_0\|^2 + \frac{(2\varepsilon)^2 - (2\varepsilon-\eta)^2}{\varepsilon^2}. \tag{3.3}$$

Proof. With (3.2) and the ε-ball condition at $\mathbf{x}_i \in \partial\Omega_i$, we get $\overline{B_{\varepsilon-\eta}(\mathbf{x}_0 \pm \varepsilon\mathbf{d}_{\mathbf{x}_0})} \cap B_\varepsilon(\mathbf{x}_i \mp \varepsilon\mathbf{d}_{\mathbf{x}_i}) = \emptyset$. We deduce $\|\mathbf{x}_i - \mathbf{x}_0 \mp \varepsilon(\mathbf{d}_{\mathbf{x}_i} + \mathbf{d}_{\mathbf{x}_0})\| \geqslant 2\varepsilon - \eta$. Squaring these two inequalities and summing them, we obtain the required one: $\|\mathbf{x}_i - \mathbf{x}_0\|^2 + 4\varepsilon^2 - (2\varepsilon-\eta)^2 \geqslant 2\varepsilon^2 - 2\varepsilon^2\langle\mathbf{d}_{\mathbf{x}_i} \mid \mathbf{d}_{\mathbf{x}_0}\rangle = \varepsilon^2\|\mathbf{d}_{\mathbf{x}_i} - \mathbf{d}_{\mathbf{x}_0}\|^2$. □

Proposition 3.7 *Under Assumption* 3.5, *for any* $\mathbf{x}_i \in \partial\Omega_i$, *we have the following global inequality:*

$$|\langle\mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| < \frac{1}{2\varepsilon}\|\mathbf{x}_i - \mathbf{x}_0\|^2 + \frac{\varepsilon^2 - (\varepsilon-\eta)^2}{2\varepsilon}. \tag{3.4}$$

Moreover, if we introduce the vector $(\mathbf{x}_i - \mathbf{x}_0)' = (\mathbf{x}_i - \mathbf{x}_0) - \langle\mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle\mathbf{d}_{\mathbf{x}_0}$ *and if we assume that* $\|(\mathbf{x}_i - \mathbf{x}_0)'\| < \varepsilon - \eta$ *and* $|\langle\mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| < \varepsilon$, *then we have the following local inequality:*

$$\frac{1}{2\varepsilon}\|\mathbf{x}_i - \mathbf{x}_0\|^2 + \frac{\varepsilon^2 - (\varepsilon-\eta)^2}{2\varepsilon} < \varepsilon - \sqrt{(\varepsilon-\eta)^2 - \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2}. \tag{3.5}$$

Proof. From (3.2), any point $\mathbf{x}_i \in \partial\Omega_i$ cannot belong to the sets $\overline{B_{\varepsilon-\eta}(\mathbf{x}_0 \pm \varepsilon\mathbf{d}_{\mathbf{x}_0})}$. Hence, we have: $\|\mathbf{x}_i - \mathbf{x}_0 \mp \varepsilon\mathbf{d}_{\mathbf{x}_0}\| > \varepsilon - \eta$. Squaring these two inequalities, we get the first required relation (3.4): $\|\mathbf{x}_i - \mathbf{x}_0\|^2 + \varepsilon^2 - (\varepsilon-\eta)^2 > 2\varepsilon|\langle\mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle|$. Then, by introducing the vector $(\mathbf{x}_i - \mathbf{x}_0)'$ of the statement, the previous inequality now takes the following form:

$$|\langle\mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle|^2 - 2\varepsilon|\langle\mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| + \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2 + \varepsilon^2 - (\varepsilon-\eta)^2 > 0.$$

We assume its left member is a second-order polynomial whose reduced discriminant is positive: $\Delta' := (\varepsilon-\eta)^2 - \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2 > 0$. Hence, the unknown satisfies either $|\langle\mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| < \varepsilon - \sqrt{\Delta'}$ or $|\langle\mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| > \varepsilon + \sqrt{\Delta'}$. We assume $|\langle\mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| < \varepsilon$ and the last case cannot hold true. Squaring the remaining inequality, we get: $|\langle\mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle|^2 + \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2 < \varepsilon^2 + (\varepsilon-\eta)^2 - 2\varepsilon\sqrt{\Delta'}$, which is the second required relation (3.5) since its left member is equal to $\|\mathbf{x}_i - \mathbf{x}_0\|^2$. □

Corollary 3.8 *Considering the assumptions and notation of Propositions* 3.6 *and* 3.7, *we have:*

$$\|\mathbf{x}_i - \mathbf{x}_0\| < 2\eta + 2\|(\mathbf{x}_i - \mathbf{x}_0)'\|, \tag{3.6}$$

$$\varepsilon\|\mathbf{d}_{\mathbf{x}_i} - \mathbf{d}_{\mathbf{x}_0}\| < 2\sqrt{2\varepsilon\eta} + \sqrt{2}\|(\mathbf{x}_i - \mathbf{x}_0)'\|. \tag{3.7}$$

Proof. Consider any $\mathbf{x}_i \in \partial\Omega_i$. We set $(\mathbf{x}_i - \mathbf{x}_0)' = (\mathbf{x}_i - \mathbf{x}_0) - \langle \mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle \mathbf{d}_{\mathbf{x}_0}$. We assume $\|(\mathbf{x}_i - \mathbf{x}_0)'\| < \varepsilon - \eta$ and $|\langle \mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| < \varepsilon$. The local estimation (3.5) of Proposition 3.7 gives:

$$\begin{aligned}\|\mathbf{x}_i - \mathbf{x}_0\|^2 &< \varepsilon^2 + (\varepsilon-\eta)^2 - 2\varepsilon\sqrt{(\varepsilon-\eta)^2 - \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2} \\ &= \frac{\left[\varepsilon^2 + (\varepsilon-\eta)^2\right]^2 - 4\varepsilon^2(\varepsilon-\eta)^2 + 4\varepsilon^2\|(\mathbf{x}_i - \mathbf{x}_0)'\|^2}{\varepsilon^2 + (\varepsilon-\eta)^2 + 2\varepsilon\sqrt{(\varepsilon-\eta)^2 - \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2}} \\ &< \left[\frac{\varepsilon^2 - (\varepsilon-\eta)^2}{\varepsilon}\right]^2 + 4\|(\mathbf{x}_i - \mathbf{x}_0)'\|^2 \quad < \quad 4\eta^2 + 4\|(\mathbf{x}_i - \mathbf{x}_0)'\|^2.\end{aligned}$$

Hence, we get: $\|\mathbf{x}_i - \mathbf{x}_0\| < 2\eta + 2\|(\mathbf{x}_i - \mathbf{x}_0)'\|$. Then, using (3.3), we also have:

$$\varepsilon\|\mathbf{d}_{\mathbf{x}_i} - \mathbf{d}_{\mathbf{x}_0}\| \leqslant \sqrt{4\varepsilon^2 - (2\varepsilon-\eta)^2 + \|\mathbf{x}_i - \mathbf{x}_0\|^2}.$$

Combining the above inequality with (3.5), we obtain:

$$\begin{aligned}\varepsilon\|\mathbf{d}_{\mathbf{x}_i} - \mathbf{d}_{\mathbf{x}_0}\| &< \sqrt{4\varepsilon\eta - \eta^2 + \varepsilon^2 + (\varepsilon-\eta)^2 - 2\varepsilon\sqrt{(\varepsilon-\eta)^2 - \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2}} \\ &= \sqrt{2\varepsilon\frac{4\varepsilon\eta + \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2}{\varepsilon + \eta + \sqrt{(\varepsilon-\eta)^2 - \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2}}} \quad < \quad 2\sqrt{2\varepsilon\eta} + \sqrt{2}\|(\mathbf{x}_i - \mathbf{x}_0)'\|.\end{aligned}$$

Consequently, the required inequalities (3.6)–(3.7) are established so Corollary 3.8 holds true. □

3.3 *A local parametrization of the boundary $\partial\Omega_i$*

Henceforth, we consider a basis $\mathcal{B}_{\mathbf{x}_0}$ of the hyperplane $\mathbf{d}_{\mathbf{x}_0}^{\perp}$ such that $(\mathbf{x}_0, \mathcal{B}_{\mathbf{x}_0}, \mathbf{d}_{\mathbf{x}_0})$ is a direct orthonormal frame. The position of any point is now determined in this local frame associated with $\mathbf{x}_0$. More precisely, for any point $\mathbf{x} \in \mathbb{R}^n$, we set $\mathbf{x}' = (x_1, \ldots, x_{n-1})$ such that $\mathbf{x} = (\mathbf{x}', x_n)$. In particular, the symbols $\mathbf{0}$ and $\mathbf{0}'$ respectively refer to the zero vector of $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$. Moreover, since $\mathbf{x}_0$ is identified with $\mathbf{0}$ in this new frame, Relations (3.2) of Assumption 3.5 take new forms:

$$\overline{B_{\varepsilon-\eta}(\mathbf{0}', -\varepsilon)} \subseteq \Omega_i \qquad \text{and} \qquad \overline{B_{\varepsilon-\eta}(\mathbf{0}', \varepsilon)} \subseteq B\backslash\overline{\Omega_i}. \tag{3.8}$$

We introduce two functions defined on $D_{\varepsilon-\eta}(\mathbf{0}') = \{\mathbf{x}' \in \mathbb{R}^{n-1},\ \|\mathbf{x}'\| < \varepsilon - \eta\}$. The first one determine around $\mathbf{x}_0$ the position of the boundary $\partial\Omega_i$ thanks to some exterior points, the other one with interior points. Then, we show these two maps coincide even if it means reducing η.

Proposition 3.9 *Under Assumption 3.5, the two following maps $\varphi_i^{\pm}$ are well defined:*

$$\begin{cases} \varphi_i^+ : & \mathbf{x}' \in D_{\varepsilon-\eta}(\mathbf{0}') & \longmapsto & \sup\left\{x_n \in [-\varepsilon, \varepsilon],\ (\mathbf{x}', x_n) \in \Omega_i\right\} \in\ ]-\varepsilon, \varepsilon[\\ \varphi_i^- : & \mathbf{x}' \in D_{\varepsilon-\eta}(\mathbf{0}') & \longmapsto & \inf\left\{x_n \in [-\varepsilon, \varepsilon],\ (\mathbf{x}', x_n) \in B\backslash\overline{\Omega_i}\right\} \in\ ]-\varepsilon, \varepsilon[, \end{cases}$$

Moreover, for any $\mathbf{x}' \in D_{\varepsilon-\eta}(\mathbf{0}')$, introducing the points $\mathbf{x}_i^{\pm} = (\mathbf{x}', \varphi_i^{\pm}(\mathbf{x}'))$, we have $\mathbf{x}_i^{\pm} \in \partial\Omega_i$ and also the following inequalities:

$$|\varphi_i^{\pm}(\mathbf{x}')| < \frac{1}{2\varepsilon}\|\mathbf{x}_i^{\pm} - \mathbf{x}_0\|^2 + \frac{\varepsilon^2 - (\varepsilon-\eta)^2}{2\varepsilon} < \varepsilon - \sqrt{(\varepsilon-\eta)^2 - \|\mathbf{x}'\|^2}. \tag{3.9}$$

Proof. Let $\mathbf{x}' \in D_{\varepsilon-\eta}(\mathbf{0}')$ and $g : t \in [-\varepsilon, \varepsilon] \mapsto (\mathbf{x}', t)$. Since $-\varepsilon \in g^{-1}(\Omega_i) \subseteq [-\varepsilon, \varepsilon]$, we can set $\varphi_i^+(\mathbf{x}') = \sup g^{-1}(\Omega_i)$. The map g is continuous so $g^{-1}(\Omega_i)$ is open and $\varphi_i^+(\mathbf{x}') \neq \varepsilon$ thus we get $\varphi_i^+(\mathbf{x}') \notin g^{-1}(\Omega_i)$ i.e. $\mathbf{x}_i^+ \in \overline{\Omega_i}\backslash\Omega_i$. Similarly, the map φ_i^- is well defined and $\mathbf{x}_i^- \in \partial\Omega_i$. Finally, we use (3.4) and (3.5) on the points $\mathbf{x}_0$ and $\mathbf{x}_i = \mathbf{x}_i^\pm$ in order to obtain (3.9). □

Lemma 3.10 *We make Assumption 3.5 and assume $\eta < \frac{\varepsilon}{3}$. We set $r = \frac{1}{2}\sqrt{4(\varepsilon-\eta)^2 - (\varepsilon+\eta)^2}$ and $\mathbf{x}' \in D_r(\mathbf{0}')$. Assume there exists $x_n \in]-\varepsilon, \varepsilon[$ such that $\mathbf{x}_i := (\mathbf{x}', x_n)$ belongs to $\partial\Omega_i$. We also consider $\tilde{x}_n \in \mathbb{R}$ satisfying the inequality $|\tilde{x}_n| < \varepsilon - \sqrt{(\varepsilon-\eta)^2 - \|\mathbf{x}'\|^2}$. Introducing $\tilde{\mathbf{x}}_i = (\mathbf{x}', \tilde{x}_n)$, then we have: $(\tilde{x}_n < x_n \Longrightarrow \tilde{\mathbf{x}}_i \in \Omega_i)$ and $\left(\tilde{x}_n > x_n \Longrightarrow \tilde{\mathbf{x}}_i \in B\backslash\overline{\Omega_i}\right)$.*

Proof. We assume $\eta < \frac{\varepsilon}{3}$ so we can set $r = \frac{1}{2}\sqrt{4(\varepsilon-\eta)^2 - (\varepsilon+\eta)^2}$. Consider any $\mathbf{x}' \in D_r(\mathbf{0}')$ and also $(x_n, \tilde{x}_n) \in]-\varepsilon, \varepsilon[^2$ such that $\mathbf{x}_i := (\mathbf{x}', x_n) \in \partial\Omega_i$ and $\tilde{\mathbf{x}}_i := (\mathbf{x}', \tilde{x}_n) \notin \overline{B_{\varepsilon-\eta}(\mathbf{0}', \pm\varepsilon)}$. We need to show that if $\tilde{x}_n \gtrless x_n$, then $\tilde{\mathbf{x}}_i \in B_\varepsilon(\mathbf{x}_i \pm \varepsilon\mathbf{d}_{\mathbf{x}_i})$. The ε-ball condition on Ω_i will give the result. Since $\mathbf{x}_i - \tilde{\mathbf{x}}_i = (x_n - \tilde{x}_n)\mathbf{d}_{\mathbf{x}_0}$, if we assume $\tilde{x}_n > x_n$, then we have:

$$\begin{aligned}
\|\tilde{\mathbf{x}}_i - \mathbf{x}_i - \varepsilon\mathbf{d}_{\mathbf{x}_i}\|^2 - \varepsilon^2 &= (\tilde{x}_n - x_n)^2 - 2\varepsilon(\tilde{x}_n - x_n)\langle \mathbf{d}_{\mathbf{x}_0} \mid \mathbf{d}_{\mathbf{x}_i}\rangle \\
&= |\tilde{x}_n - x_n| \left(|\tilde{x}_n - x_n| + \varepsilon\|\mathbf{d}_{\mathbf{x}_i} - \mathbf{d}_{\mathbf{x}_0}\|^2 - 2\varepsilon\right) \\
&\leqslant |\tilde{x}_n - x_n| \left(|\tilde{x}_n| + |x_n| + \frac{\|\mathbf{x}_i - \mathbf{x}_0\|^2 + (2\varepsilon)^2 - (2\varepsilon-\eta)^2}{\varepsilon} - 2\varepsilon\right),
\end{aligned}$$

where the last inequality comes from Proposition 3.6 (3.3) applied to $\mathbf{x}_i \in \partial\Omega_i$. Finally, we use the inequality involving $\tilde{x}_n$ and the ones (3.4)-(3.5) of Proposition 3.7 applied to $\mathbf{x}_i \in \partial\Omega_i$ to obtain:

$$\|\tilde{\mathbf{x}}_i - \mathbf{x}_i - \varepsilon\mathbf{d}_{\mathbf{x}_i}\|^2 - \varepsilon^2 < 4|x_n - \tilde{x}_n| \underbrace{\left(\frac{\varepsilon+\eta}{2} - \sqrt{(\varepsilon-\eta)^2 - \|\mathbf{x}'\|^2}\right)}_{\leqslant\left(\frac{\varepsilon+\eta}{2} - \sqrt{(\varepsilon-\eta)^2 - r^2}\right) = 0}.$$

Hence, if $\tilde{x}_n > x_n$, then we get $\tilde{\mathbf{x}}_i \in B_\varepsilon(\mathbf{x}_i + \varepsilon\mathbf{d}_{\mathbf{x}_i}) \subseteq B\backslash\overline{\Omega_i}$. Similarly, one can prove that if $\tilde{x}_n < x_n$, then we have $\tilde{\mathbf{x}}_i \in B_\varepsilon(\mathbf{x}_i - \varepsilon\mathbf{d}_{\mathbf{x}_i}) \subseteq \Omega_i$. □

Proposition 3.11 *Let η, r be as in Lemma 3.10. Then, the two functions $\varphi_i^\pm$ of Proposition 3.9 coincide on $D_r(\mathbf{0}')$. The map φ_i refers to their common restrictions and it satisfies:*

$$\begin{cases}
\partial\Omega_i \cap (D_r(\mathbf{0}') \cap]-\varepsilon, \varepsilon[) &= \left\{(\mathbf{x}', \varphi_i(\mathbf{x}')), \quad \mathbf{x}' \in D_r(\mathbf{0}')\right\} \\
\Omega_i \cap (D_r(\mathbf{0}') \cap]-\varepsilon, \varepsilon[) &= \left\{(\mathbf{x}', x_n), \quad \mathbf{x}' \in D_r(\mathbf{0}') \text{ and } -\varepsilon < x_n < \varphi_i(\mathbf{x}')\right\}.
\end{cases}$$

Proof. First, we assume by contradiction that there exists $\mathbf{x}' \in D_r(\mathbf{0}')$ such that $\varphi_i^-(\mathbf{x}') \neq \varphi_i^+(\mathbf{x}')$. The hypothesis of Lemma 3.10 are satisfied for the points $\mathbf{x}_i := (\mathbf{x}', \varphi_i^+(\mathbf{x}'))$ and $\tilde{\mathbf{x}}_i := (\mathbf{x}', \varphi_i^-(\mathbf{x}'))$ by using (3.9). Hence, either $(\varphi_i^-(\mathbf{x}') < \varphi_i^+(\mathbf{x}') \Rightarrow \tilde{\mathbf{x}}_i \in \Omega_i)$ or $(\varphi_i^-(\mathbf{x}') > \varphi_i^+(\mathbf{x}') \Rightarrow \tilde{\mathbf{x}}_i \in B\backslash\overline{\Omega}_i)$ whereas $\tilde{\mathbf{x}}_i \in \partial\Omega_i$. We deduce that $\varphi_i^-(\mathbf{x}') = \varphi_i^+(\mathbf{x}')$ for any $\mathbf{x}' \in D_r(\mathbf{0}')$. Then, we consider $\mathbf{x}' \in D_r(\mathbf{0}')$ and $x_n \in]-\varepsilon, \varepsilon[$. We set $\mathbf{x}_i = (\mathbf{x}', \varphi_i(\mathbf{x}'))$ and $\tilde{\mathbf{x}}_i = (\mathbf{x}', x_n)$. Proposition 3.9 ensures that if $x_n = \varphi_i(\mathbf{x}')$, then $\mathbf{x}_i \in \partial\Omega_i$. Moreover, if $-\varepsilon < x_n \leqslant -\varepsilon + \sqrt{(\varepsilon-\eta)^2 - \|\mathbf{x}'\|^2}$, then $\tilde{\mathbf{x}}_i \in \overline{B_{\varepsilon-\eta}(\mathbf{0}', -\varepsilon)} \subseteq \Omega_i$ and if $-\varepsilon + \sqrt{(\varepsilon-\eta)^2 - \|\mathbf{x}'\|^2} < x_n < \varphi(\mathbf{x}')$, then apply Lemma 3.10 to get $\tilde{\mathbf{x}}_i \in \Omega_i$. Consequently, we have proved: $\forall \mathbf{x}' \in D_r(\mathbf{0}'), \ -\varepsilon < x_n < \varphi_i(\mathbf{x}') \Longrightarrow (\mathbf{x}', x_n) \in \Omega_i$. To conclude, similar arguments hold true when $\varepsilon > x_n > \varphi_i(\mathbf{x}')$ and imply $(\mathbf{x}', x_n) \in B\backslash\overline{\Omega_i}$. □

3.4 *The $C^{1,1}$-regularity of the local graph φ_i*

We previously showed that the boundary $\partial\Omega_i$ is locally described by the graph of a well-defined map $\varphi_i : D_r(\mathbf{0}') \to]-\varepsilon,\varepsilon[$. Now we prove its $C^{1,1}$-regularity even if it means reducing η and r.

Lemma 3.12 *The following map is well defined, smooth, surjective and increasing:*

$$f_\eta : \quad]0,\frac{\pi}{2}[\longrightarrow]2\sqrt{2\varepsilon\eta},+\infty[$$
$$\alpha \longmapsto \frac{3\alpha + 2\sqrt{2\varepsilon\eta}}{\cos\alpha}.$$

In particular, it is an homeomorphism and its inverse f_η^{-1} satisfies the following inequality:

$$\forall\varepsilon > 0,\ \forall\eta \in \left]0,\frac{\varepsilon}{8}\right[,\quad f_\eta^{-1}(\varepsilon) < \frac{\varepsilon}{3}. \tag{3.10}$$

Proof. The proof is basic calculus. □

Proposition 3.13 *In Assumption 3.5, let $\eta < \frac{\varepsilon}{8}$ and consider $\alpha \in]0, f_\eta^{-1}(\varepsilon)]$, where f_η^{-1} has been introduced in Lemma 3.12. Then, we have:*

$$\forall \mathbf{x}_i \in B_\alpha(\mathbf{x}_0) \cap \overline{\Omega_i},\quad C_\alpha(\mathbf{x}_i, -\mathbf{d}_{\mathbf{x}_0}) \subseteq \Omega_i,$$

where $C_\alpha(\mathbf{x}_i, -\mathbf{d}_{\mathbf{x}_0})$ is defined in Definition 2.3.

Proof. Since we have $\eta < \frac{\varepsilon}{3}$, we can set $r = \frac{1}{2}\sqrt{4(\varepsilon-\eta)^2 - (\varepsilon+\eta)^2}$ and $\mathcal{C}_{r,\varepsilon} = D_r(\mathbf{0}')\times]-\varepsilon,\varepsilon[$. Moreover, we assume $\eta < \frac{\varepsilon}{8}$, i.e., $2\sqrt{2\varepsilon\eta} < \varepsilon$ so $f_\eta^{-1}(\varepsilon)$ is well defined. Choose $\alpha \in]0, f_\eta^{-1}(\varepsilon)]$ then consider $\mathbf{x}_i = (\mathbf{x}', x_n) \in B_\alpha(\mathbf{x}_0) \cap \overline{\Omega_i}$ and $\mathbf{y}_i = (\mathbf{y}', y_n) \in C_\alpha(\mathbf{x}_i, -\mathbf{d}_{\mathbf{x}_0})$. The proof of the assertion $\mathbf{y}_i \in \Omega_i$ is divided into the three following steps.

1. Check $\mathbf{x}_i \in \mathcal{C}_{r,\varepsilon}$ so as to introduce the point $\tilde{\mathbf{x}}_i = (\mathbf{x}', \varphi_i(\mathbf{x}'))$ of $\partial\Omega_i$ satisfying $x_n \leqslant \varphi_i(\mathbf{x}')$.
2. Consider $\tilde{\mathbf{y}}_i = (\mathbf{y}', y_n + \varphi_i(\mathbf{x}') - x_n)$ and prove $\tilde{\mathbf{y}}_i \in C_\alpha(\tilde{\mathbf{x}}_i, -\mathbf{d}_{\mathbf{x}_0}) \subseteq B_\varepsilon(\tilde{\mathbf{x}}_i - \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}_i}) \subseteq \Omega_i$.
3. Show $(\tilde{\mathbf{y}}_i, \mathbf{y}_i) \in \mathcal{C}_{r,\varepsilon} \times \mathcal{C}_{r,\varepsilon}$ in order to deduce $y_n + \varphi_i(\mathbf{x}') - x_n < \varphi_i(\mathbf{y}')$ and conclude $\mathbf{y}_i \in \Omega_i$.

First, from (3.10), we have: $\max(\|\mathbf{x}'\|, |x_n|) \leqslant \|\mathbf{x}_i - \mathbf{x}_0\| < \alpha \leqslant f_\eta^{-1}(\varepsilon) < \frac{\varepsilon}{3}$. Since $\eta < \frac{\varepsilon}{8}$, we get $r > \frac{1}{2}[4(\frac{7\varepsilon}{8})^2 - (\frac{9\varepsilon}{8})^2]^{\frac{1}{2}} > \frac{\varepsilon}{2}$ thus $\mathbf{x}_i \in \overline{\Omega_i} \cap \mathcal{C}_{r,\varepsilon}$. Hence, from Proposition 3.11, it comes $x_n \leqslant \varphi_i(\mathbf{x}')$. We set $\tilde{\mathbf{x}}_i = (\mathbf{x}', \varphi_i(\mathbf{x}')) \in \partial\Omega_i \cap \mathcal{C}_{r,\varepsilon}$. Then, we prove $C_\alpha(\tilde{\mathbf{x}}_i, -\mathbf{d}_{\mathbf{x}_0}) \subseteq B_\varepsilon(\tilde{\mathbf{x}}_i - \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}_i})$ so consider any $\mathbf{y} \in C_\alpha(\tilde{\mathbf{x}}_i, -\mathbf{d}_{\mathbf{x}_0})$. Combining the Cauchy–Schwartz inequality and $\mathbf{y} \in C_\alpha(\tilde{\mathbf{x}}_i, -\mathbf{d}_{\mathbf{x}_0})$, we get:

$$\begin{aligned}
\|\mathbf{y} - \tilde{\mathbf{x}}_i + \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}_i}\|^2 - \varepsilon^2 &\leqslant \|\mathbf{y} - \tilde{\mathbf{x}}_i\|^2 + 2\varepsilon\|\mathbf{y} - \tilde{\mathbf{x}}_i\|\|\mathbf{d}_{\tilde{\mathbf{x}}_i} - \mathbf{d}_{\mathbf{x}_0}\| - 2\varepsilon\|\mathbf{y} - \tilde{\mathbf{x}}_i\|\cos\alpha \\
&< 2\|\mathbf{y} - \tilde{\mathbf{x}}_i\|\left(\frac{\alpha}{2} + 2\sqrt{2\varepsilon\eta} + \sqrt{2}\|\mathbf{x}'\| - \varepsilon\cos\alpha\right) \\
&< 2\alpha\cos\alpha\left(f_\eta(\alpha) - \varepsilon\right) \leqslant 0,
\end{aligned}$$

where we used (3.7) on $\tilde{\mathbf{x}}_i \in \partial\Omega_i \cap \mathcal{C}_{r,\varepsilon}$ and $\|\mathbf{x}'\| \leqslant \|\mathbf{x}_i - \mathbf{x}_0\| < \alpha$. Hence, $\mathbf{y} \in B_\varepsilon(\tilde{\mathbf{x}}_i - \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}_i})$ so $C_\alpha(\tilde{\mathbf{x}}_i, -\mathbf{d}_{\mathbf{x}_0}) \subseteq B_\varepsilon(\tilde{\mathbf{x}}_i - \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}_i}) \subseteq \Omega_i$, using the ε-ball condition. Moreover, since $\tilde{\mathbf{y}}_i - \tilde{\mathbf{x}}_i = \mathbf{y}_i - \mathbf{x}_i$

and $\mathbf{y}_i \in C_\alpha(\mathbf{x}_i, -\mathbf{d}_{\mathbf{x}_0})$, we get $\tilde{\mathbf{y}}_i \in C_\alpha(\tilde{\mathbf{x}}_i, -\mathbf{d}_{\mathbf{x}_0})$, which ends the proof of $\tilde{\mathbf{y}}_i \in \Omega_i$. Finally, we check that $(\mathbf{y}_i, \tilde{\mathbf{y}}_i) \in \mathcal{C}_{r,\varepsilon} \times \mathcal{C}_{r,\varepsilon}$. We have successively:

$$\begin{cases} \|\mathbf{y}'\| \leqslant \|\mathbf{y}' - \mathbf{x}'\| + \|\mathbf{x}'\| < \sqrt{\alpha^2 - \alpha^2\cos^2\alpha} + \alpha = \frac{\alpha}{\cos\alpha}\left(\frac{1}{2}\sin 2\alpha + \cos\alpha\right) < \frac{f_\eta(\alpha)}{2} \leqslant \frac{\varepsilon}{2} < r \\ |y_n| \leqslant |y_n - x_n| + |x_n| \leqslant \|\mathbf{y}_i - \mathbf{x}_i\| + \|\mathbf{x}_i - \mathbf{x}_0\| < 2\alpha < f(\alpha) \leqslant \varepsilon \\ |\tilde{y}_n| = |y_n + \varphi_i(\mathbf{x}') - x_n| \leqslant \|\mathbf{y}_i - \mathbf{x}_i\| + \varepsilon - \sqrt{(\varepsilon-\eta)^2 - \|\mathbf{x}'\|^2} < \alpha + \frac{\eta(2\varepsilon-\eta)+\|\mathbf{x}'\|^2}{\varepsilon+\sqrt{(\varepsilon-\eta)^2-\|\mathbf{x}'\|^2}}. \end{cases}$$

Here, we used Relation (3.9), the fact that $\mathbf{y}_i \in C_\alpha(\mathbf{x}_i, -\mathbf{d}_{\mathbf{x}_0})$ and $\mathbf{x}_i \in B_\alpha(\mathbf{x}_0)$. Hence, we obtain: $|\tilde{y}_n| < 2\alpha + 2\eta < 2f_\eta^{-1}(\varepsilon) + 2\frac{\varepsilon}{8} \leqslant \frac{2\varepsilon}{3} + \frac{\varepsilon}{4} < \varepsilon$. To conclude, apply Proposition 3.11 to $\tilde{\mathbf{y}}_i \in \Omega_i \cap \mathcal{C}_{r,\varepsilon}$ in order to get $y_n + \varphi_i(\mathbf{x}') - x_n < \varphi(\mathbf{y}')$. Since we firstly proved $x_n \leqslant \varphi_i(\mathbf{x}')$, we have $y_n < \varphi_i(\mathbf{y}')$. Applying Proposition 3.11 to $\mathbf{y}_i \in \mathcal{C}_{r,\varepsilon}$, we get $\mathbf{y}_i \in \Omega_i$ as required. □

Lemma 3.14 *The following map is well defined, smooth, surjective and increasing:*

$$g: \quad]0, \frac{\pi}{8}[\longrightarrow]0, +\infty[\qquad \eta \longmapsto \frac{32\eta}{\cos^2(4\eta)}.$$

In particular, it is an homeomorphism and its inverse g^{-1} satisfies the following relations:

$$\forall \varepsilon > 0, \quad g^{-1}(\varepsilon) < \frac{\varepsilon}{32} \qquad \text{and} \qquad g^{-1}(\varepsilon) < \frac{1}{4} f^{-1}_{g^{-1}(\varepsilon)}(\varepsilon), \tag{3.11}$$

where f_η^{-1} is defined in Lemma 3.12.

Proof. We only prove the inequality $g^{-1}(\varepsilon) < \frac{1}{4} f^{-1}_{g^{-1}(\varepsilon)}(\varepsilon)$. The remaining part is basic calculus. Consider any $\varepsilon > 0$. There exists a unique $\eta \in]0, \frac{\pi}{8}[$ such that $g(\eta) = \varepsilon$ or equivalently $\eta = g^{-1}(\varepsilon)$. Hence, we have $4\eta \in]0, \frac{\pi}{2}[$ so we can compute, using the first inequality $\eta < \frac{\varepsilon}{32}$:

$$f_\eta(4\eta) = \frac{2\sqrt{2\eta\varepsilon}}{\cos(4\eta)}\left(3\sqrt{\frac{2\eta}{\varepsilon}} + 1\right) < \frac{2\sqrt{2\eta\varepsilon}}{\cos(4\eta)}\left(3\sqrt{\frac{2}{32}} + 1\right) < \frac{4\sqrt{2\varepsilon\eta}}{\cos(4\eta)} = \sqrt{g(\eta)\varepsilon} = \varepsilon.$$

As f_η is an increasing homeomorphism, so does f_η^{-1} and the inequality follows: $4\eta < f_\eta^{-1}(\varepsilon)$. □

Corollary 3.15 *In Assumption 3.5, we set $\eta = g^{-1}(\varepsilon)$, then consider $\alpha = f_\eta^{-1}(\varepsilon)$ and $\tilde{r} = \frac{1}{4}\alpha - \eta$. The restriction to $D_{\tilde{r}}(\mathbf{0}')$ of the map φ_i defined in Proposition 3.11 is $\frac{1}{\tan\alpha}$-Lipschitz continuous.*

Proof. Let $\eta = g^{-1}(\varepsilon)$ and using (3.11), we have $\eta < \frac{\varepsilon}{32}$ so we can set $r = \frac{1}{2}\sqrt{4(\varepsilon-\eta)^2 - (\varepsilon+\eta)^2}$ and $\alpha = f_\eta^{-1}(\varepsilon)$, but we also have $\tilde{r} := \frac{1}{4}\alpha - \eta > 0$. We consider any $(\mathbf{x}'_+, \mathbf{x}'_-) \in D_{\tilde{r}}(\mathbf{0}') \times D_{\tilde{r}}(\mathbf{0}')$. Using (3.10)-(3.11), we get $\tilde{r} < \frac{1}{4}f_\eta^{-1}(\varepsilon) < \frac{\varepsilon}{12} < \frac{1}{2}[4(\frac{31\varepsilon}{32})^2 - (\frac{33\varepsilon}{32})^2]^{\frac{1}{2}} < r$. From Proposition 3.11, we can define $\mathbf{x}_i^\pm := (\mathbf{x}'_\pm, \varphi_i(\mathbf{x}'_\pm)) \in \partial\Omega_i$. Then, we show that $\mathbf{x}_i^\pm \in \partial\Omega_i \cap B_\alpha(\mathbf{x}_0) \cap B_\alpha(\mathbf{x}_i^\mp)$. Relation (3.6) ensures that $\|\mathbf{x}_i^\pm - \mathbf{x}_0\| < 2\|\mathbf{x}'_\pm\| + 2\eta \leqslant 2\tilde{r} + 2\eta < \alpha$ and the triangle inequality gives $\|\mathbf{x}_i^+ - \mathbf{x}_i^-\| \leqslant \|\mathbf{x}_i^+ - \mathbf{x}_0\| + \|\mathbf{x}_0 - \mathbf{x}_i^-\| < 4\tilde{r} + 4\eta = \alpha$. Finally, we apply Proposition 3.13 to $\mathbf{x}_i^\pm \in \partial\Omega_i \cap B_\alpha(\mathbf{x}_0)$, which cannot belong to the cone $C_\alpha(\mathbf{x}_i^\mp, -\mathbf{d}_{\mathbf{x}_0}) \subseteq \Omega_i$. Hence, we obtain:

$$|\langle \mathbf{x}_i^+ - \mathbf{x}_i^- \mid \mathbf{d}_{\mathbf{x}_0}\rangle| \leqslant \cos\alpha\|\mathbf{x}_i^+ - \mathbf{x}_i^-\| = \cos\alpha\sqrt{\|\mathbf{x}'_+ - \mathbf{x}'_-\|^2 + |\langle \mathbf{x}_i^+ - \mathbf{x}_i^- \mid \mathbf{d}_{\mathbf{x}_0}\rangle|^2}.$$

Re-arranging the above inequality, we deduce that the map φ_i is L-Lipschitz continuous with $L > 0$ depending only on ε as required: $|\varphi_i(\mathbf{x}'_+) - \varphi_i(\mathbf{x}'_-)| = |\langle \mathbf{x}_i^+ - \mathbf{x}_i^- \mid \mathbf{d}_{\mathbf{x}_0}\rangle| \leqslant \frac{1}{\tan\alpha}\|\mathbf{x}'_+ - \mathbf{x}'_-\|$. □

Proposition 3.16 *We set $\tilde{r} = \frac{1}{4} f^{-1}_{g^{-1}(\varepsilon)}(\varepsilon) - g^{-1}(\varepsilon)$, where f and g are defined in Lemmas 3.12 and 3.14. Then, the restriction to $D_{\tilde{r}}(\mathbf{0}')$ of the map φ_i defined in Proposition 3.11 is differentiable:*

$$\forall \mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}'), \quad \nabla\varphi_i(\mathbf{a}') = \frac{-1}{\langle \mathbf{d}_{\mathbf{a}_i} \mid \mathbf{d}_{\mathbf{x}_0}\rangle}\mathbf{d}'_{\mathbf{a}_i} \qquad \text{where} \quad \mathbf{a}_i := (\mathbf{a}', \varphi_i(\mathbf{a}')).$$

Moreover, $\nabla\varphi_i : D_{\tilde{r}}(\mathbf{0}') \to \mathbb{R}^{n-1}$ is L-Lipschitz continuous with $L > 0$ depending only on ε, and the map is also uniformly bounded. More precisely, we have $\|\nabla\varphi_i(\mathbf{a}')\| < \frac{32}{29}$ for any $\mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}')$.

Proof. Let $\eta = g^{-1}(\varepsilon)$ and using (3.11), we have $\eta < \frac{\varepsilon}{32}$ so we can set $r = \frac{1}{2}\sqrt{4(\varepsilon-\eta)^2 - (\varepsilon+\eta)^2}$ and $\alpha = f_\eta^{-1}(\varepsilon)$, but we also have $\tilde{r} := \frac{1}{4}\alpha - \eta > 0$. Let $\mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}')$ and $\mathbf{x}' \in \overline{D_{\tilde{r}-\|\mathbf{a}'\|}(\mathbf{a}')}$. Hence, $(\mathbf{a}', \mathbf{x}') \in D_{\tilde{r}}(\mathbf{0}') \times D_{\tilde{r}}(\mathbf{0}')$. Using (3.10)-(3.11), $\tilde{r} < \frac{1}{4} f_\eta^{-1}(\varepsilon) < \frac{\varepsilon}{12} < \frac{1}{2}[4(\frac{31\varepsilon}{32})^2 - (\frac{33\varepsilon}{32})^2]^{\frac{1}{2}} < r$. From Proposition 3.11, we can define $\mathbf{x}_i^\pm := (\mathbf{x}'_\pm, \varphi_i(\mathbf{x}'_\pm)) \in \partial\Omega_i$. Then, we apply (A2) to Ω_i thus:

$$|\langle \mathbf{x}_i - \mathbf{a}_i \mid \mathbf{d}_{\mathbf{a}_i}\rangle| \leqslant \frac{1}{2\varepsilon}\left(\|\mathbf{x}' - \mathbf{a}'\|^2 + |\varphi_i(\mathbf{x}') - \varphi_i(\mathbf{a}')|^2\right) \leqslant \underbrace{\frac{1}{2\varepsilon}\left(1 + \frac{1}{\tan^2\alpha}\right)}_{:=C(\varepsilon)>0}\|\mathbf{x}' - \mathbf{a}'\|^2,$$

where we also used the Lipschitz continuity of φ_i on $D_{\tilde{r}}(\mathbf{0}')$ established in Corollary 3.15. We note that $\mathbf{d}_{\mathbf{a}_i} = (\mathbf{d}'_{\mathbf{a}_i}, (\mathbf{d}_{\mathbf{a}_i})_n)$ where $(\mathbf{d}_{\mathbf{a}_i})_n = \langle \mathbf{d}_{\mathbf{a}_i} \mid \mathbf{d}_{\mathbf{x}_0}\rangle$. Hence, the above inequality takes the form:

$$|\left(\varphi_i(\mathbf{x}') - \varphi_i(\mathbf{a}')\right)(\mathbf{d}_{\mathbf{a}_i})_n + \langle \mathbf{d}'_{\mathbf{a}_i} \mid \mathbf{x}' - \mathbf{a}'\rangle| \leqslant C(\varepsilon)\|\mathbf{x}' - \mathbf{a}'\|^2.$$

This last inequality is a first-order Taylor expansion of φ_i if it can be divided by a uniform positive constant smaller than $(\mathbf{d}_{\mathbf{a}_i})_n$. Let us justify this last assertion. From (3.3) and (3.5), we deduce:

$$(\mathbf{d}_{\mathbf{a}_i})_n = 1 - \frac{1}{2}\|\mathbf{d}_{\mathbf{a}_i} - \mathbf{d}_{\mathbf{x}_0}\|^2 \geqslant 1 - \frac{1}{2\varepsilon^2}\|\mathbf{a}_i - \mathbf{x}_0\|^2 - \frac{4\varepsilon\eta - \eta^2}{2\varepsilon^2} > \frac{1}{\varepsilon}\sqrt{(\varepsilon-\eta)^2 - \|\mathbf{a}'\|^2} - \frac{\eta}{\varepsilon}.$$

Then, inequality (3.11) gives $\frac{\eta}{\varepsilon} < \frac{1}{32}$ and from (3.10), it comes $\|\mathbf{a}'\| < \tilde{r} < \frac{\alpha}{4} < \frac{\varepsilon}{12}$. Consequently, we get $(\mathbf{d}_{\mathbf{a}_i})_n > [(\frac{31}{32})^2 - (\frac{1}{12})^2]^{\frac{1}{2}} - \frac{1}{32} > \frac{29}{32}$ and from the foregoing, we obtain:

$$\forall \mathbf{x}' \in \overline{D_{\tilde{r}-\|\mathbf{a}'\|}(\mathbf{a}')}, \qquad \left|\varphi_i(\mathbf{x}') - \varphi_i(\mathbf{a}') + \left\langle \frac{\mathbf{d}'_{\mathbf{a}_i}}{(\mathbf{d}_{\mathbf{a}_i})_n} \mid \mathbf{x}' - \mathbf{a}'\right\rangle\right| \leqslant \frac{32C(\varepsilon)}{29}\|\mathbf{x}' - \mathbf{a}'\|^2.$$

Therefore, φ_i is differentiable at any point $\mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}')$ with $\nabla\varphi_i(\mathbf{a}') = -\mathbf{d}'_{\mathbf{a}_i}/(\mathbf{d}_{\mathbf{a}_i})_n$. Moreover, the fact that $(\mathbf{d}_{\mathbf{a}_i})_n > \frac{29}{32}$ and $\|\mathbf{d}'_{\mathbf{a}_i}\| \leqslant \|\mathbf{d}_{\mathbf{a}_i}\| = 1$ also ensures that $\|\nabla\varphi_i(\mathbf{a}')\| < \frac{32}{29}$ for any $\mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}')$, i.e., the map $\nabla\varphi_i$ is uniformly bounded. Finally, we show that $\nabla\varphi_i : D_{\tilde{r}}(\mathbf{0}') \to \mathbb{R}^{n-1}$ is Lipschitz continuous. Let $(\mathbf{x}', \mathbf{a}') \in D_{\tilde{r}}(\mathbf{0}') \times D_{\tilde{r}}(\mathbf{0}')$. We have:

$$\begin{aligned} \|\nabla\varphi_i(\mathbf{x}') - \nabla\varphi_i(\mathbf{a}')\| &\leqslant \left|\frac{1}{(\mathbf{d}_{\mathbf{x}_i})_n} - \frac{1}{(\mathbf{d}_{\mathbf{a}_i})_n}\right| \|\mathbf{d}'_{\mathbf{x}_i}\| + \frac{1}{(d_{\mathbf{a}_i})_n}\|\mathbf{d}'_{\mathbf{a}_i} - \mathbf{d}'_{\mathbf{x}_i}\| \\ &\leqslant \frac{32}{29}\left(\frac{32}{29}|(\mathbf{d}_{\mathbf{a}_i})_n - (\mathbf{d}_{\mathbf{x}_i})_n| + \|\mathbf{d}_{\mathbf{a}_i} - \mathbf{d}_{\mathbf{x}_i}\|\right) \\ &\leqslant \frac{32}{29\varepsilon}\left(1 + \frac{32}{29}\right)\|\mathbf{x}_i - \mathbf{a}_i\| \leqslant \frac{32}{29\varepsilon}\left(1 + \frac{32}{29}\right)\sqrt{1 + \frac{1}{\tan^2\alpha}}\|\mathbf{x}' - \mathbf{a}'\|. \end{aligned}$$

We used the fact that $(\mathbf{d}_{\mathbf{a}_i})_n > \frac{29}{32}$, the Lipschitz continuity of φ_i proved in Corollary 3.15 and the one of the map $\mathbf{x}_i \in \partial\Omega_i \mapsto \mathbf{d}_{\mathbf{x}_i}$ coming from Proposition A.4 applied to $\Omega_i \in \mathcal{O}_\varepsilon(B)$. To conclude, $\nabla\varphi_i$ is an L-Lipschitz continuous map, where $L > 0$ depends only on ε. □

Proof of Theorem 3.3. Set $K := \overline{D_{\tilde{r}}(\mathbf{0}')}$ where $\tilde{r} := \frac{1}{4} f^{-1}_{g^{-1}(\varepsilon)}(\varepsilon) - g^{-1}(\varepsilon)$ is positive from (3.11). Using Propositions 3.11, 3.16 and Corollary 3.15, we have proved that each Ω_i is parametrized by a local graph $\varphi_i : D_{\tilde{r}}(\mathbf{0}') \to] - \varepsilon, \varepsilon[$ as stated in Theorem 3.3. Hence, it remains to prove the convergence of these graphs. First, any of the $(\varphi_i)_{i \geqslant I}$ is Lipschitz thus uniformly continuous on $D_{\tilde{r}}(\mathbf{0}')$ so it has a unique Lipschitz continuous extension to K. In addition, the sequence $(\varphi_i)_{i \geqslant I}$ is uniformly bounded and equi-Lipschitz continuous. Applying the Arzelà-Ascoli Theorem, it is uniformly converging, up to a subsequence, to a Lipschitz continuous function $\tilde{\varphi} : K \to [-\varepsilon, \varepsilon]$. Similarly, using Corollary 3.15, the sequence $(\nabla\varphi_i)_{i \geqslant I}$ is uniformly bounded and equi-L-Lipschitz continuous so up to a subsequence, it is uniformly converging on K to a Lipschitz continuous map, which has to be $\nabla\tilde{\varphi}$ (use the convergence in the sense of distributions and [30, Lemma 4.7]). Then, let $\delta \in]0, 1]$ and we have:

$$\sup_{\substack{(\mathbf{x},\mathbf{y}) \in K \times K \\ \mathbf{x} \neq \mathbf{y}}} \frac{\|\nabla(\varphi_i - \varphi)(\mathbf{x}) - \nabla(\varphi_i - \varphi)(\mathbf{y})\|}{\|\mathbf{x} - \mathbf{y}\|^{1-\delta}} \leqslant \left(L + \|\nabla\varphi\|_{C^{0,1}(K)}\right)^{1-\delta} \|\nabla\varphi_i - \nabla\varphi\|^{\delta}_{C^0(K)},$$

from which we deduce that up to a subsequence, $(\varphi_i)_{i \geqslant I}$ converges to φ in $C^{1,1-\delta}(K)$ for any $\delta \in]0, 1]$. Moreover, using [38, Section 5.2.2], each coefficient of the Hessian matrix of φ_i is uniformly bounded in $L^\infty(K)$ so up to a subsequence again [38, Lemma 2.2.27], each of them weakly-star converges in $L^\infty(K)$ to the ones of $\tilde{\varphi}$. Finally, we assume that $\lim_{i \to +\infty} d_H(\partial\Omega_i, \partial\Omega) = 0$. Even if it means reducing $\tilde{r}$ again, we can also assume that $\tilde{r} < r$, where $r > 0$ is the one of Theorem 2.6. Consequently, $K \subseteq D_r(\mathbf{0}')$ and we can consider the local map $\varphi : K \to] - \varepsilon, \varepsilon[$ associated with $\partial\Omega$. We now show that $\varphi \equiv \tilde{\varphi}$ on K. Let $\mathbf{x}' \in K$. We set $\mathbf{x} = (\mathbf{x}', \tilde{\varphi}(\mathbf{x}'))$ and $\mathbf{x}_i = (\mathbf{x}', \varphi_i(\mathbf{x}'))$. There exists $\mathbf{y} \in \partial\Omega$ such that $d(\mathbf{x}_i, \partial\Omega) = \|\mathbf{x}_i - \mathbf{y}\|$. We thus have:

$$\begin{aligned} d(\mathbf{x}, \partial\Omega) \leqslant \|\mathbf{x} - \mathbf{y}\| &\leqslant \|\mathbf{x} - \mathbf{x}_i\| + \|\mathbf{x}_i - \mathbf{y}\| = |\varphi_i(\mathbf{x}') - \tilde{\varphi}(\mathbf{x}')| + d(\mathbf{x_i}, \partial\Omega) \\ &\leqslant \|\varphi_i - \tilde{\varphi}\|_{C^0(K)} + d_H(\partial\Omega_i, \partial\Omega). \end{aligned}$$

As $i \to +\infty$, we obtain $\mathbf{x} \in \partial\Omega$. In particular, from the ε-ball condition, we deduce that $|\tilde{\varphi}(\mathbf{x}')| \neq \varepsilon$ otherwise $\mathbf{x} \in B_\varepsilon(\mathbf{0}', \varepsilon) \subseteq B\backslash\overline{\Omega}$ or $\mathbf{x} \in B_\varepsilon(\mathbf{0}', -\varepsilon) \subseteq \Omega$ which is not the case. Therefore, since $K \subset D_r(\mathbf{0}')$, we get $\mathbf{x} \in \partial\Omega \cap (D_r(\mathbf{0}') \times] - \varepsilon, \varepsilon[)$ and Theorem 2.6 yields to $\mathbf{x} = (\mathbf{x}', \varphi(\mathbf{x}'))$ i.e. $\varphi(\mathbf{x}') = \tilde{\varphi}(\mathbf{x}')$ for any $\mathbf{x}' \in K$. To conclude, we also have proved that φ is the unique limit of any converging subsequence of $(\varphi_i)_{i \geqslant I}$. Hence, the whole sequence $(\varphi_i)_{i \geqslant I}$ is converging to φ. □

4. Continuity of some geometric functionals in the class $\mathcal{O}_\varepsilon(B)$

In this section, we prove that the convergence properties and the uniform $C^{1,1}$-regularity of the class $\mathcal{O}_\varepsilon(B)$ ensure the continuity of a wide range of geometric functionals. More precisely, with a suitable partition of unity, we show how to use the local convergence results of Theorem 3.3 in order to get the global continuity of many functionals of the form $J : \Omega \in \mathcal{O}_\varepsilon(B) \mapsto \int_{\partial\Omega} j_\Omega(\mathbf{x})dA(\mathbf{x})$.

First, we study the case of integrands depending only on the position and the normal vector i.e. for any $j_\Omega : \mathbf{x} \in \partial\Omega \mapsto j[\mathbf{x}, \mathbf{n}(\mathbf{x})]$, where $j : \overline{B} \times \mathbb{S}^{n-1} \to \mathbb{R}$ is a continuous map. In Section 4.2, we explain how to build a partition of unity and the continuity of J will directly follow from the C^1-strong convergence of the local graphs given in Theorem 3.3.

Then, we aim to use the L^∞-weak star convergence of the Hessian-matrix coefficients associated with the local graphs. We thus consider integrands whose expressions in the local basis are linear in these coefficients. It is the case for the scalar mean curvature H and in Section 4.3, we obtain the continuity of J for any $j_\Omega : \mathbf{x} \in \partial\Omega \mapsto H(\mathbf{x}) j[\mathbf{x}, \mathbf{n}(\mathbf{x})]$, where $j : \overline{B} \times \mathbb{S}^{n-1} \to \mathbb{R}$ is continuous. Moreover, using classical arguments, we can relax the continuity results into lower semi-continuity ones by assuming only convexity with respect to H of integrands $j_\Omega : \mathbf{x} \in \partial\Omega \mapsto j[\mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})]$. In this case, note that we only have lower semi-continuity and **not** continuity (which requires the linearity of j in H). Note also that no growth condition on j is imposed here regarding the last variable. In particular, we are able to obtain the lower semi-continuity of $\Omega \in \mathcal{O}_\varepsilon(B) \mapsto \int_{\partial\Omega} |H| dA$, which is excluded from many statements of geometric measure theory (cf. Remark 4.12).

Furthermore, we only need to assume the continuity of j with respect to the set of variables in order to ensure that the functional J is well defined. Indeed, from Theorem 2.6, the Gauss map $\mathbf{n} : \mathbf{x} \in \partial\Omega \mapsto \mathbf{n}(\mathbf{x}) \in \mathbb{S}^{n-1}$ is $\frac{1}{\varepsilon}$-Lipschitz continuous. Rademacher's Theorem [29, Section 3.1.2] ensures it is differentiable almost everywhere and its differential $D_\bullet\mathbf{n} : \mathbf{x} \in \partial\Omega \mapsto D_\mathbf{x}\mathbf{n} \in \mathcal{L}(T_\mathbf{x}\partial\Omega)$ is an L^∞-map satisfying $\|D_\bullet\mathbf{n}\|_{L^\infty(\partial\Omega)} \leqslant \frac{1}{\varepsilon}$. We deduce that the map $\mathbf{x} \in \partial\Omega \mapsto (\mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x}))$ is valued in the compact set $\overline{B} \times \mathbb{S}^{n-1} \times [-\frac{n-1}{\varepsilon}, \frac{n-1}{\varepsilon}]$. In particular, the continuity of j and the compactness of $\partial\Omega$ ensure the existence of $\int_{\partial\Omega} j[\mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})]dA(\mathbf{x}) < +\infty$, i.e., $J : \mathcal{O}_\varepsilon(B) \to \mathbb{R}$ is well defined. These kind of arguments also work for any functional considered in Section 4.

Finally, we wonder if we can have the L^∞-weak star convergence of some non-linear functions of the Hessian-matrix coefficients associated with the local graphs. Considering the Gauss-Codazzi–Mainardi equations (4.14)–(4.15), we detail how to apply a version of the Div-Curl Lemma [57] to this geometrical setting. In Section 4.4, we obtain the L^∞-weak star convergence of the Gaussian curvature K, and more generally of (4.8) i.e. of the elementary symmetric polynomials $H^{(l)}$ of the principal curvatures. As before, we deduce continuity for integrands that are linear in K, $H^{(l)}$, and only lower semi-continuity for integrands that are convex in K, $H^{(l)}$, $l = 1 \ldots n-1$.

Note that for $C^{1,1}$-hypersurfaces, $(H^{(l)})_{0 \leqslant l \leqslant n-1}$ corresponds to the curvature measures defined more generally for sets of positive reach. Consequently, we have strengthened the results of Federer [30, Theorem 5.9] in the particular context of the ε-ball condition: the $(H^{(l)})_{0 \leqslant l \leqslant n-1}$ are not only converging in the sense of Radon measures but also L^∞-weakly star (cf. Remark 4.4). Throughout this section, we make the following hypothesis, that were exactly the one assumed in Theorem 3.3.

ASSUMPTION 4.1 Let $\varepsilon > 0$ and $B \subset \mathbb{R}^n$ be an open bounded set, large enough to ensure that $\mathcal{O}_\varepsilon(B) \neq \emptyset$. We assume $(\Omega_i)_{i\in\mathbb{N}}$ is a sequence of elements from $\mathcal{O}_\varepsilon(B)$ converging to $\Omega \in \mathcal{O}_\varepsilon(B)$ in the sense of compacts sets (cf. Definition 3.1 (ii)) and $\lim_{i\to+\infty} d_H(\partial\Omega_i, \partial\Omega) = 0$.

DEFINITION 4.2 Let f, $(f_i)_{i\in\mathbb{N}} : E \to F$ be continuous maps between two metric spaces. We say that $(f_i)_{i\in\mathbb{N}}$ *diagonally converges* to f if $\| f_i(t_i) - f(t) \|_F \to 0$ for any $(t_i)_{i\in\mathbb{N}}$ converging to t in E.

REMARK The uniform convergence implies the diagonal convergence implying the pointwise convergence. Conversely, any sequence of equi-continuous maps converging pointwise is diagonally convergent. If in addition, it is uniformly bounded, then we get the uniform convergence.

Section 4 is organized as follows. In Section 4.1, we recall some notions related to the geometry of $C^{1,1}$-hypersurfaces. In Section 4.2, we study the continuity of functionals depending on the position and the normal vector. In Section 4.3, we consider the dependence in the mean curvature. In Section 4.4, we treat the case of the Gaussian curvature in $\mathbb{R}^3$ and we prove its $\mathbb{R}^n$-version, namely Theorem 4.3 stated hereafter. We conclude by giving some existence results in Section 4.5. We prove Theorem 1.3, its generalization to $\mathbb{R}^n$, and detail many applications like Theorem 1.5 and Theorems 1.7–1.8, mainly coming from the modelling of vesicles and red blood cells.

Theorem 4.3 *Let ε, B, Ω, $(\Omega_i)_{i\in\mathbb{N}}$ be as in Assumption 4.1. We consider some continuous maps $j^l, j_i^l : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ such that each sequence $(j_i^l)_{i\in\mathbb{N}}$ is uniformly bounded on $\overline{B} \times \mathbb{S}^{n-1}$ and diagonally converges to j^l for any $l \in \{0, \dots, n-1\}$. Then, the following functional is continuous:*

$$J(\partial\Omega_i) := \sum_{l=0}^{n-1} \int_{\partial\Omega_i} \Bigg[\sum_{1\leqslant n_1 < \ldots < n_l \leqslant n-1} \kappa_{n_1}^{\partial\Omega_i}(\mathbf{x}) \ldots \kappa_{n_l}^{\partial\Omega_i}(\mathbf{x}) \Bigg] j_i^l \left[\mathbf{x}, \mathbf{n}^{\partial\Omega_i}(\mathbf{x}) \right] dA(\mathbf{x}) \underset{i\to+\infty}{\longrightarrow} J(\partial\Omega),$$

where $\kappa_1, \dots \kappa_{n-1}$ are the principal curvatures, $\mathbf{n}$ the unit outer normal field to the hypersurface, and where the integration is done with respect to the $(n-1)$-dimensional Hausdorff measure $A(\bullet)$.

REMARK 4.4 In the specific case of compact $C^{1,1}$-hypersurfaces, note that the above theorem is stronger than Federer's one on sets of positive reach [30, Theorem 5.9]. Indeed, in Theorem 4.3, taking $j_i^l(\mathbf{x}, \mathbf{n}(\mathbf{x})) = j^l(\mathbf{x})$ yields to the convergence of the curvature measures associated with $\partial\Omega_i$ to the ones of $\partial\Omega$ in the sense of Radon measures.

4.1 *On the geometry of hypersurfaces with $C^{1,1}$-regularity*

Let us consider a non-empty compact $C^{1,1}$-hypersurface $\mathcal{S} \subset \mathbb{R}^n$. Merely speaking, for any point $\mathbf{x_0} \in \mathcal{S}$, there exists $r_{\mathbf{x_0}} > 0$, $a_{\mathbf{x_0}} > 0$, and a unit vector $\mathbf{d_{x_0}}$ such that in the cylinder defined by:

$$\mathcal{C}_{r_{\mathbf{x_0}},a_{\mathbf{x_0}}}(\mathbf{x_0}) = \left\{ \mathbf{x} \in \mathbb{R}^n,\ |\langle \mathbf{x} - \mathbf{x_0} \mid \mathbf{d_{x_0}} \rangle| < a_{\mathbf{x_0}} \text{ and } \|(\mathbf{x} - \mathbf{x_0}) - \langle \mathbf{x} - \mathbf{x_0} \mid \mathbf{d_{x_0}} \rangle \mathbf{d_{x_0}} \| < r_{\mathbf{x_0}} \right\}, \quad (4.1)$$

the hypersurface $\mathcal{S}$ is the graph of a $C^{1,1}$-map. Introducing the orthogonal projection on the affine hyperplane $\mathbf{x_0} + \mathbf{d}_{\mathbf{x_0}}^{\perp}$:

$$\begin{aligned} \Pi_{\mathbf{x_0}} : \mathbb{R}^n &\longrightarrow \mathbf{x_0} + \mathbf{d}_{\mathbf{x_0}}^{\perp} \\ \mathbf{x} &\longmapsto \mathbf{x} - \langle \mathbf{x} - \mathbf{x_0} \mid \mathbf{d_{x_0}} \rangle \mathbf{d_{x_0}}, \end{aligned} \quad (4.2)$$

and considering the set $D_{r_{\mathbf{x_0}}}(\mathbf{x_0}) = \Pi_{\mathbf{x_0}}(\mathcal{C}_{r_{\mathbf{x_0}},a_{\mathbf{x_0}}}(\mathbf{x_0}))$, this means that there exists a continuously differentiable map $\varphi_{\mathbf{x_0}} : \mathbf{x}' \in D_{r_{\mathbf{x_0}}}(\mathbf{x_0}) \mapsto \varphi_{\mathbf{x_0}}(\mathbf{x}') \in]-a_{\mathbf{x_0}}, a_{\mathbf{x_0}}[$ such that its gradient $\nabla\varphi_{\mathbf{x_0}}$ and $\varphi_{\mathbf{x_0}}$

are $L_{\mathbf{x}_0}$-Lipschitz continuous maps, and such that:

$$\mathcal{S} \cap C_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0) = \{\mathbf{x}' + \varphi_{\mathbf{x}_0}(\mathbf{x}')\mathbf{d}_{\mathbf{x}_0}, \quad \mathbf{x}' \in D_{r_{\mathbf{x}_0}}(\mathbf{x}_0)\}.$$

Hence, we can introduce the local parametrization:

$$\begin{aligned} X_{\mathbf{x}_0} : D_{r_{\mathbf{x}_0}}(\mathbf{x}_0) &\longrightarrow \mathcal{S} \cap C_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0) \\ \mathbf{x}' &\longmapsto \mathbf{x}' + \varphi_{\mathbf{x}_0}(\mathbf{x}')\mathbf{d}_{\mathbf{x}_0} \end{aligned}$$

and $\mathcal{S}$ is a $C^{1,1}$-hypersurface in the sense of [48, Definition 2.2]. Indeed, $X_{\mathbf{x}_0}$ is an homeomorphism, its inverse map is the restriction of $\Pi_{\mathbf{x}_0}$ to $C_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0)$, and $X_{\mathbf{x}_0}$ is an immersion of class $C^{1,1}$.

DEFINITION 4.5 Let $n \geqslant 2$. We say that a non-empty subset $\mathcal{S}$ of $\mathbb{R}^n$ is a $C^{1,1}$*-hypersurface in the sense of [48, Definition 2.2]* if for any point $\mathbf{x} \in \mathcal{S}$, there exists an open set $U_{\mathbf{x}} \subseteq \mathbb{R}^{n-1}$, an open neighbourhood $V_{\mathbf{x}}$ of $\mathbf{x}$ in $\mathbb{R}^n$, and a $C^{1,1}$-map $X_{\mathbf{x}} : U_{\mathbf{x}} \to V_{\mathbf{x}} \cap \mathcal{S}$, which is an homeomorphism and such that its differential $D_{\mathbf{y}}X_{\mathbf{x}} : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is injective for any $\mathbf{y} \in U_{\mathbf{x}}$.

We usually drop the dependence in $\mathbf{x}_0$ to lighten notation, and consider a direct orthonormal frame $(\mathbf{x}_0, \mathcal{B}_{\mathbf{x}_0}, \mathbf{d}_{\mathbf{x}_0})$ where $\mathcal{B}_{\mathbf{x}_0}$ is a basis of $\mathbf{d}_{\mathbf{x}_0}^{\perp}$. In this local frame, $\mathbf{x}_0$ is identified with $\mathbf{0} \in \mathbb{R}^n$, the affine hyperplane $\mathbf{x}_0 + \mathbf{d}_{\mathbf{x}_0}^{\perp}$ with $\mathbb{R}^{n-1}$ and $\mathbf{x}_0 + \mathbb{R}\mathbf{d}_{\mathbf{x}_0}$ with $\mathbb{R}$. The cylinder $\mathcal{C}_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0)$ becomes $D_r(\mathbf{0}')\times]-a,a[$, $\varphi_{\mathbf{x}_0}$ is the $C^{1,1}$-map $\varphi : D_r(\mathbf{0}') \to]-a,a[$, projection $\Pi_{\mathbf{x}_0}$ is $X^{-1} : (\mathbf{x}', x_n) \mapsto \mathbf{x}'$, and parametrization $X_{\mathbf{x}_0}$ is the $C^{1,1}$-map $X : \mathbf{x}' \in D_r(\mathbf{0}') \mapsto (\mathbf{x}', \varphi(\mathbf{x}')) \in \mathcal{S} \cap (D_r(\mathbf{0}')\times]-a,a[)$. In this setting, $\mathcal{S}$ is a $C^{1,1}$-hypersurface in the sense of Definition 2.2.

Since $\mathbf{x}' \in D_r(\mathbf{0}') \mapsto D_{\mathbf{x}'}X$ is injective, the vectors $\partial_1 X, \ldots, \partial_{n-1}X$ are linearly independent. For any $\mathbf{x} \in \mathcal{S} \cap (D_r(\mathbf{0}')\times]-a,a[)$, we define the tangent hyperplane $T_{\mathbf{x}}\mathcal{S}$ by $D_{X^{-1}(\mathbf{x})}X(\mathbb{R}^{n-1})$. It is an $(n-1)$-dimensional vector space so $(\partial_1 X, \ldots, \partial_{n-1}X)$ forms a basis of $T_{\mathbf{x}}\mathcal{S}$. However, this basis is not necessarily orthonormal. Consequently, the first fundamental form of $\mathcal{S}$ at $\mathbf{x}$ is defined as the restriction of the usual scalar product in $\mathbb{R}^n$ to the tangent hyperplane $T_{\mathbf{x}}\mathcal{S}$, i.e. as $\mathbf{I}(\mathbf{x}) : (\mathbf{v}, \mathbf{w}) \in T_{\mathbf{x}}\mathcal{S} \times T_{\mathbf{x}}\mathcal{S} \mapsto \langle \mathbf{v} \mid \mathbf{w}\rangle$. In the basis $(\partial_1 X, \ldots, \partial_{n-1}X)$, it is represented by a positive-definite symmetric matrix usually referred to as $(g_{ij})_{1\leqslant i,j\leqslant n-1}$ and its inverse denoted by $(g^{ij})_{1\leqslant i,j\leqslant n-1}$ is also explicitly given in this case:

$$g_{ij} = \langle \partial_i X \mid \partial_j X\rangle = \delta_{ij} + \partial_i\varphi\partial_j\varphi, \tag{4.3}$$

$$g^{ij} = \delta_{ij} - \frac{\partial_i\varphi\partial_j\varphi}{1 + \|\nabla\varphi\|^2}. \tag{4.4}$$

As a function of $\mathbf{x}'$, note that each coefficient of these two matrices is Lipschitz continuous so it is a $W^{1,\infty}$-map [29, Section 4.2.3], and from Rademacher's Theorem [29, Section 3.1.2], its differential exists almost everywhere. Moreover, any $\mathbf{v} \in T_{\mathbf{x}}\mathcal{S}$ can be decomposed in the basis $(\partial_1 X, \ldots, \partial_{n-1}X)$. Denoting by V_i the component of $\partial_i X$ and $v_i = \langle \mathbf{v} \mid \partial_i X\rangle$, we have:

$$\mathbf{v} = \sum_{i=1}^{n-1} V_i \partial_i X \implies v_j = \sum_{i=1}^{n} V_i g_{ij} \implies V_i = \sum_{j=1}^{n-1} g^{ij} v_j \implies \mathbf{v} = \sum_{i=1}^{n-1}\left(\sum_{j=1}^{n-1} g^{ij} v_j\right)\partial_i X. \tag{4.5}$$

In particular, we deduce $\mathbf{I}(\mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{n-1} g^{ij} v_i w_j$. Then, the orthogonal of the tangent hyperplane is one dimensional. Hence, there exists a unique unit vector $\mathbf{n}$ orthogonal to the $(n-1)$ vectors $\partial_1 X$,

..., $\partial_{n-1}X$ and pointing outwards the inner domain of $\mathcal{S}$ i.e. $\det(\partial_1 X, \ldots, \partial_{n-1}X, \mathbf{n}) > 0$. It is called the unit outer normal vector to the hypersurface and we have its explicit expression:

$$\forall \mathbf{x}' \in D_r(\mathbf{0}'), \quad \mathbf{n} \circ X(\mathbf{x}') = \frac{1}{\sqrt{1 + \|\nabla\varphi(\mathbf{x}')\|^2}} \begin{pmatrix} -\nabla\varphi(\mathbf{x}') \\ 1 \end{pmatrix}. \tag{4.6}$$

It is a Lipschitz continuous map, like the coefficients of the first fundamental form. In particular, it is differentiable almost everywhere and introducing the Gauss map $\mathbf{n} : \mathbf{x} \in \mathcal{S} \mapsto \mathbf{n}(\mathbf{x}) \in \mathbb{S}^{n-1}$, we can compute its differential almost everywhere called the Weingarten map:

$$\begin{aligned} D_{\mathbf{x}}\mathbf{n} : T_{\mathbf{x}}\mathcal{S} = D_{X^{-1}(\mathbf{x})}X(\mathbb{R}^2) &\longrightarrow T_{\mathbf{n}(\mathbf{x})}\mathbb{S}^{n-1} = D_{X^{-1}(\mathbf{x})}(\mathbf{n} \circ X)(\mathbb{R}^2) \\ \mathbf{v} = D_{X^{-1}(\mathbf{x})}X(\mathbf{w}) &\longmapsto D_{\mathbf{x}}\mathbf{n}(\mathbf{v}) = D_{X^{-1}(\mathbf{x})}(\mathbf{n} \circ X)(\mathbf{w}). \end{aligned} \tag{4.7}$$

Note that $T_{\mathbf{n}(\mathbf{x})}\mathbb{S}^{n-1} = D_{X^{-1}(\mathbf{x})}(\mathbf{n} \circ X)(\mathbb{R}^2)$ because $\mathbf{n} \circ X$ is a Lipschitz parametrization of $\mathbb{S}^{n-1}$. Since $T_{\mathbf{n}(\mathbf{x})}\mathbb{S}^{n-1} \sim \mathbf{n}(\mathbf{x})^{\perp}$ can be identified with $T_{\mathbf{x}}\mathcal{S}$, the map $D_{\mathbf{x}}\mathbf{n}$ is an endomorphism of $T_{\mathbf{x}}\mathcal{S}$. Moreover, one can prove it is self-adjoint so it can be diagonalized to obtain $n-1$ eigenvalues denoted by $\kappa_1(\mathbf{x}), \ldots, \kappa_{n-1}(\mathbf{x})$ and called the principal curvatures. Recall that the eigenvalues of an endomorphism do not depend on the chosen basis and thus are really properties of the operator. This assertion also holds true for the coefficients of the characteristic polynomial associated with $D_{\mathbf{x}}\mathbf{n}$ so we can introduce them:

$$\forall l \in \{0, \ldots, n-1\}, \quad H^{(l)}(\mathbf{x}) = \sum_{1 \leqslant n_1 < \ldots < n_l \leqslant n-1} \kappa_{n_1}(\mathbf{x}) \ldots \kappa_{n_l}(\mathbf{x}). \tag{4.8}$$

In particular, $H^{(0)} = 1$, $H^{(1)} = H$ is called the scalar mean curvature, and $H^{(n-1)} = K$ refers to the Gaussian curvature:

$$H(\mathbf{x}) = \kappa_1(\mathbf{x}) + \ldots + \kappa_{n-1}(\mathbf{x}) \qquad \text{and} \qquad K(\mathbf{x}) = \kappa_1(\mathbf{x})\kappa_2(\mathbf{x}) \ldots \kappa_{n-1}(\mathbf{x}). \tag{4.9}$$

Moreover, introducing the symmetric matrix $(b_{ij})_{1 \leqslant i,j \leqslant n-1}$ defined by:

$$b_{ij} = -\langle D\mathbf{n}(\partial_i X) \mid \partial_j X\rangle = -\left\langle \partial_i(\mathbf{n} \circ X) \mid \partial_j X\right\rangle = \frac{\text{Hess}\,\varphi}{\sqrt{1 + \|\nabla\varphi\|^2}} = \left\langle \mathbf{n} \circ X \mid \partial_{ij} X\right\rangle, \tag{4.10}$$

we get from (4.5) that the Weingarten map $D\mathbf{n}$ is represented in the local basis $(\partial_1 X, \ldots, \partial_{n-1}X)$ by the following symmetric matrix:

$$(h_{ij})_{1 \leqslant i,j \leqslant n-1} = \left(-\sum_{k=1}^{n-1} g^{ik} b_{kj}\right) = \left(-\sum_{k=1}^{n-1} \left(\delta_{ik} - \frac{\partial_i\varphi\partial_j\varphi}{1 + \|\nabla\varphi\|^2}\right) \frac{\partial_{kj}\varphi}{\sqrt{1 + \|\nabla\varphi\|^2}}\right). \tag{4.11}$$

Finally, we introduce the symmetric bilinear form whose representation in the local basis is (b_{ij}). It is called the second fundamental form of the hypersurface and it is defined by:

$$\begin{aligned} \mathbf{II}(\mathbf{x}) : T_{\mathbf{x}}(\mathcal{S}) \times T_{\mathbf{x}}(\mathcal{S}) &\to \mathbb{R} \\ (\mathbf{v}, \mathbf{w}) &\mapsto \langle -D_{\mathbf{x}}\mathbf{n}(\mathbf{v}) | \mathbf{w}\rangle = \sum_{i,j,k,l=1}^{n-1} g^{ij} v_j g^{kl} w_l b_{il} = \sum_{i,j,k=1}^{n-1} g^{ij} v_j v_k h_{ki}. \end{aligned} \tag{4.12}$$

We can also decompose $\partial_{ij} X$ in the basis $(\partial_1 X, \dots, \partial_{n-1} X, \mathbf{n})$ and its coefficients in the tangent space are the Christoffel symbols:

$$\partial_{ij} X = \sum_{k=1}^{n-1} \Gamma_{ij}^k \partial_k X + b_{ij} \mathbf{n}.$$

Note that the Christoffel symbols are symmetric with respect to the lower indices: $\Gamma_{ij}^k = \Gamma_{ji}^k$. They can be expressed only in terms of coefficients of the first fundamental form:

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{n-1} g^{kl} \left(\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij} \right). \tag{4.13}$$

Like the first fundamental form, it is an intrinsic notion, which in particular do not depend on the orientation chosen for the hypersurface, while the Gauss map, the Weingarten map, and the second fundamental form does. Note that in local coordinates, the coefficients of the first fundamental form and the Gauss map are Lipschitz continuous functions, i.e., $\mathbf{n} \circ X, g_{ij}, g^{ij} \in W^{1,\infty}(D_r(\mathbf{0}'))$. Hence, the Christoffel symbols, the Weingarten map and the coefficients of the second fundamental form exist almost everywhere and $\Gamma_{ij}^k, b_{ij}, h_{ij} \in L^\infty(D_r(\mathbf{0}'))$. Furthermore, one can prove that a $C^{1,1}$-hypersurface satisfies the so-called Gauss and Codazzi–Mainardi equations in distributional sense:

$$\partial_l \Gamma_{ij}^k - \partial_j \Gamma_{il}^k + \sum_{m=1}^{n-1} \left(\Gamma_{ij}^m \Gamma_{ml}^k - \Gamma_{il}^m \Gamma_{mj}^k \right) = \sum_{m=1}^{n-1} g^{km} \left(b_{ij} b_{ml} - b_{il} b_{mj} \right) \tag{4.14}$$

$$\partial_k b_{ij} - \partial_j b_{ik} = \sum_{l=1}^{n-1} \left(\Gamma_{ik}^l b_{lj} - \Gamma_{ij}^l b_{lk} \right). \tag{4.15}$$

In fact, the converse statement also holds in $\mathbb{R}^3$: these equations characterize uniquely a surface and it is referred as the Fundamental Theorem of Surface Theory, valid with $C^{1,1}$-regularity [44]. Given a simply-connected open subset $\omega \subseteq \mathbb{R}^2$, a symmetric positive-definite (2×2)-matrix $(g_{ij})_{1\leqslant i,j\leqslant 2} \in W^{1,\infty}(\omega)$ and a symmetric matrix $(b_{ij})_{1\leqslant i,j\leqslant 2} \in L^\infty(\omega)$ satisfying (4.14) and (4.15) in the sense of distributions, then there exists an injective $C^{1,1}$-immersion $X : \omega \to \mathbb{R}^3$, unique up to proper isometries of $\mathbb{R}^3$, such that the surface $\mathcal{S} := X(\omega)$ has (g_{ij}) and (b_{ij}) as coefficients of the first and second fundamental forms. To conclude, we recall that $A(\bullet)$ (respectively $V(\bullet)$) refers to the $n-1$(resp. n)-dimensional Hausdorff measure. The integration is always be done with respect to A and we have $(dA \circ X)(\mathbf{x}') = \sqrt{\det(g_{ij})} d\mathbf{x}' = \sqrt{1 + \|\nabla\varphi(\mathbf{x}')\|^2} d\mathbf{x}'$. We refer to [26, 48] for a more detailed exposition on all the notions quickly introduced here.

4.2 *Geometric functionals involving the position and the normal vector*

Proposition 4.6 *Consider Assumption 4.1. Then, for any continuous map $j : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$:*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \int_{\partial\Omega} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}).$$

In particular, the area and the volume are continuous: $A(\partial\Omega_i) \longrightarrow A(\partial\Omega)$ and $V(\Omega_i) \longrightarrow V(\Omega)$.

REMARK Note that the above result states the convergence of $(\partial\Omega_i)_{i\in\mathbb{N}}$ to $\partial\Omega$ in the sense of oriented varifolds [4, Appendix B] [55]. Similar results were obtained in [36]. Moreover, the continuity of volume and the lower semi-continuity of area are already implied by the convergence of characteristic functions (cf. Definition 3.1 (iii) and Proposition 3.2) [38, Proposition 2.3.6].

Proof. Consider Assumption 4.1. Hence, from Theorem 3.3, the boundaries $(\partial\Omega_i)_{i\in\mathbb{N}}$ are locally parametrized by graphs of $C^{1,1}$-maps φ_i that converge strongly in C^1 and weakly-star in $W^{2,\infty}$ to the map φ associated with $\partial\Omega$. We now detail the procedure which allows to pass from this local result to the global one thanks to a suitable partition of unity. For any $\mathbf{x} \in \partial\Omega$, we introduce the cylinder $\mathcal{C}_{\tilde{r},\varepsilon}(\mathbf{x})$ defined by (4.1) and we assume that $\tilde{r} > 0$ is the one given in Theorem 3.3. In particular, it only depends on ε. Since $\partial\Omega$ is compact, there exists a finite number $K \geqslant 1$ of points written $\mathbf{x}_1, \ldots, \mathbf{x}_K$, such that $\partial\Omega \subseteq \bigcup_{k=1}^K \mathcal{C}_{\frac{\tilde{r}}{2},\frac{\varepsilon}{2}}(\mathbf{x}_k)$. We set $\delta = \min(\frac{\tilde{r}}{2}, \frac{\varepsilon}{2}) > 0$. From the triangle inequality, the tubular neighbourhood $\mathcal{V}_\delta(\partial\Omega) = \{\mathbf{y} \in \mathbb{R}^n, \quad d(\mathbf{y}, \partial\Omega) < \delta\}$ has its closure embedded in $\bigcup_{k=1}^K \mathcal{C}_{\tilde{r},\varepsilon}(\mathbf{x}_k)$. Then, we can introduce a partition of unity on this set. There exists K non-negative C^∞-maps ξ^k with compact support in $\mathcal{C}_{\tilde{r},\varepsilon}(\mathbf{x}_k)$ and such that $\sum_{k=1}^K \xi^k(\mathbf{x}) = 1$ for any point $\mathbf{x} \in \mathcal{V}_\delta(\partial\Omega)$. Now, we can apply Theorem 3.3 to the K points $\mathbf{x}_k$. There exists K integers $I_k \in \mathbb{N}$ and some maps $\varphi_i^k : D_{\tilde{r}}(\mathbf{x}_k) \mapsto]-\varepsilon, \varepsilon[$, with $i \geqslant I_k$ and $K \geqslant k \geqslant 1$, such that:

$$\begin{cases} \partial\Omega_i \cap \mathcal{C}_{\tilde{r},\varepsilon}(\mathbf{x}_k) = \left\{(\mathbf{x}', \varphi_i^k(\mathbf{x}')), \quad \mathbf{x}' \in D_{\tilde{r}}(\mathbf{x}_k)\right\} \\ \Omega_i \cap \mathcal{C}_{\tilde{r},\varepsilon}(\mathbf{x}_k) = \left\{(\mathbf{x}', x_n), \quad \mathbf{x}' \in D_{\tilde{r}}(\mathbf{x}_k) \quad \text{and} \quad -\varepsilon < x_n < \varphi_i^k(\mathbf{x}')\right\}. \end{cases}$$

Moreover, the K sequences of functions $(\varphi_i^k)_{i\geqslant I_k}$ and $(\nabla\varphi_i^k)_{i\geqslant I_k}$ converge uniformly on $\overline{D_{\tilde{r}}(\mathbf{x}_k)}$ respectively to the maps φ^k and $\nabla\varphi^k$ associated with $\partial\Omega$ at each point $\mathbf{x}_k$. From the Hausdorff convergence of the boundaries given in Assumption 4.1, there also exists $I_0 \in \mathbb{N}$ such that for any integer $i \geqslant I_0$, we have $\partial\Omega_i \in \mathcal{V}_\delta(\partial\Omega)$. Hence, we set $I = \max_{0\leqslant k\leqslant K} I_k$, which thus only depends on $(\Omega_i)_{i\in\mathbb{N}}$, Ω and ε. Then, we deduce that for any integer $i \geqslant I$, we have:

$$\begin{aligned} &J(\partial\Omega_i) \\ &:= \int_{\partial\Omega_i} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \int_{\partial\Omega_i \cap \mathcal{V}_\delta(\partial\Omega)} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \\ &= \int_{\partial\Omega_i} \left(\sum_{k=1}^K \xi^k(\mathbf{x})\right) j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \sum_{k=1}^K \int_{\partial\Omega_i \cap \mathcal{C}_{r,\varepsilon}(\mathbf{x}_k)} \xi^k(\mathbf{x})\, j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \\ &= \sum_{k=1}^K \int_{D_{\tilde{r}}(\mathbf{x}_k)} \xi^k \begin{pmatrix} \mathbf{x}' \\ \varphi_i^k(\mathbf{x}') \end{pmatrix} j\left[\begin{pmatrix} \mathbf{x}' \\ \varphi_i^k(\mathbf{x}') \end{pmatrix}, \begin{pmatrix} \frac{-\nabla\varphi_i^k(\mathbf{x}')}{\sqrt{1+\|\nabla\varphi_i^k(\mathbf{x}')\|^2}} \\ \frac{1}{\sqrt{1+\|\nabla\varphi_i^k(\mathbf{x}')\|^2}} \end{pmatrix}\right] \sqrt{1 + \|\nabla\varphi_i^k(\mathbf{x}')\|^2} d\mathbf{x}'. \end{aligned}$$

The last equality comes from [48, Proposition 5.13] and Relation (4.6). The uniform convergence of the K sequences $(\varphi_i^k)_{i\geqslant I}$ and $(\nabla\varphi_i^k)_{i\geqslant I}$ on the compact set $\overline{D_{\tilde{r}}(\mathbf{x}_k)}$ combined with the continuity of j and $(\xi^k)_{1\leqslant k\leqslant K}$ allows one to let $i \to \infty$ in the above expression. Observing that the limit expression obtained is equal to $J(\partial\Omega)$, we proved that the functional J is continuous. Finally, for the area, take $j \equiv 1$ and for the volume, applying the Divergence Theorem, take $j[\mathbf{x}, \mathbf{n}(\mathbf{x})] = \frac{1}{n}\langle \mathbf{x} \mid \mathbf{n}(\mathbf{x})\rangle$. □

Proposition 4.7 *Consider Assumption 4.1 and some continuous maps* $j, j_i : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ *such that* $(j_i)_{i\in\mathbb{N}}$ *is uniformly bounded on* $\overline{B} \times \mathbb{S}^{n-1}$ *and diagonally converges to* j *in the sense of Definition 4.2. Then, we have:*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} j_i\,[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}) = \int_{\partial\Omega} j\,[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}).$$

Proof. The proof is identical to the one of Proposition 4.6. Using the same partition of unity and the same notation, we get that $\int_{\partial\Omega_i} j_i[\mathbf{x}, \mathbf{n}(\mathbf{x})]dA(\mathbf{x})$ is equal to:

$$\sum_{k=1}^{K} \int_{D_{\tilde{r}}(\mathbf{x}_k)} \xi^k \begin{pmatrix} \mathbf{x}' \\ \varphi_i^k(\mathbf{x}') \end{pmatrix} j_i \left[\begin{pmatrix} \mathbf{x}' \\ \varphi_i^k(\mathbf{x}') \end{pmatrix}, \begin{pmatrix} \frac{-\nabla\varphi_i^k(\mathbf{x}')}{\sqrt{1+\|\nabla\varphi_i^k(\mathbf{x}')\|^2}} \\ \frac{1}{\sqrt{1+\|\nabla\varphi_i^k(\mathbf{x}')\|^2}} \end{pmatrix} \right] \sqrt{1+\|\nabla\varphi_i^k(\mathbf{x}')\|^2}\, d\mathbf{x}'.$$

Then, instead of using the uniform convergence of each integrand on a compact set as it is the case in Proposition 4.6, we apply instead Lebesgue's Dominated Convergence Theorem. Indeed, the diagonal convergence ensures the pointwise convergence of each integrand, which are also, using the other hypothesis, uniformly bounded. Hence, we can let $i \to +\infty$ in the above expression. □

DEFINITION 4.8 Let $\mathcal{S}, \mathcal{S}_i$ be some non-empty compact C^1-hypersurfaces of $\mathbb{R}^n$ such that $(\mathcal{S}_i)_{i\in\mathbb{N}}$ converges to $\mathcal{S}$ for the Hausdorff distance: $d_H(\mathcal{S}_i, \mathcal{S}) \longrightarrow_{i\to+\infty} 0$. On each hypersurface $\mathcal{S}_i$, we also consider a continuous vector field $\mathbf{V}_i : \mathbf{x} \in \mathcal{S}_i \mapsto \mathbf{V}_i(\mathbf{x}) \in T_{\mathbf{x}}\mathcal{S}_i$. We say that $(\mathbf{V}_i)_{i\in\mathbb{N}}$ is *diagonally converging to a vector field* on $\mathcal{S}$ denoted by $\mathbf{V} : \mathbf{x} \in \mathcal{S} \mapsto \mathbf{V}(\mathbf{x}) \in T_{\mathbf{x}}\mathcal{S}$ if for any point $\mathbf{x} \in \mathcal{S}$ and for any sequence of points $\mathbf{x}_i \in \mathcal{S}_i$ that converges to $\mathbf{x}$, we have $\|\mathbf{V}_i(\mathbf{x}_i) - \mathbf{V}(\mathbf{x})\| \longrightarrow_{i\to+\infty} 0$.

REMARK In Definition 4.8, $(\mathbf{V}_i(\mathbf{x}_i))_{i\in\mathbb{N}}$ is assumed to converge to $\mathbf{V}(\mathbf{x})$ as a sequence of points in $\mathbb{R}^n$, although $\mathbf{V}_i(\mathbf{x}_i)$ and $\mathbf{V}(\mathbf{x})$ belong to different linear spaces $T_{\mathbf{x}_i}\mathcal{S}_i$ and $T_{\mathbf{x}}\mathcal{S}$.

Corollary 4.9 *Let* $\varepsilon, B, \Omega, (\Omega_i)_{i\in\mathbb{N}}$ *be as in Assumption 4.1 and consider some continuous vector fields* $\mathbf{V}_i$ *on* $\partial\Omega_i$ *converging to a continuous vector field* $\mathbf{V}$ *on* $\partial\Omega$ *as in Definition 4.8. We also assume that* $(\mathbf{V}_i)_{i\in\mathbb{N}}$ *is uniformly bounded. If* $j : \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R}^n \to \mathbb{R}$ *is a continuous map, then:*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} j\,[\mathbf{x}, \mathbf{n}(\mathbf{x}), \mathbf{V}_i(\mathbf{x})]\, dA(\mathbf{x}) = \int_{\partial\Omega} j\,[\mathbf{x}, \mathbf{n}(\mathbf{x}), \mathbf{V}(\mathbf{x})]\, dA(\mathbf{x}).$$

Of course, this continuity result can be extended to a finite number of vector fields.

Proof. We only have to check that the maps $j_i : (\mathbf{x}, \mathbf{u}) \in \partial\Omega_i \times \mathbb{S}^{n-1} \to j[\mathbf{x}, \mathbf{u}, \mathbf{V}_i(\mathbf{x})]$ can be extended to $\mathbb{R}^n \times \mathbb{S}^{n-1}$ such that their extension satisfy the hypothesis of Proposition 4.7. This is a standard procedure [38, Section 5.4.1]. Using the partition of unity given in Proposition 4.6 and introducing the $C^{1,1}$-diffeomorphisms $\Psi_i^k : (\mathbf{x}', x_n) \in \mathcal{C}_{r,\varepsilon}(\mathbf{x_k}) \mapsto (\mathbf{x}', \varphi_i^k(\mathbf{x}') - x_n)$, we can set:

$$\forall(\mathbf{x},\mathbf{u}) \in \mathbb{R}^n \times \mathbb{S}^{n-1}, j_i(\mathbf{x},\mathbf{u}) = \sum_{k=1}^{K} \xi^k(\mathbf{x}) j[(\Psi_i^k)^{-1} \circ \Pi_{\mathbf{x}_k} \circ \Psi_i^k(\mathbf{x}), \mathbf{u}, \mathbf{V}_i \circ (\Psi_i^k)^{-1} \circ \Pi_{\mathbf{x}_k} \circ \Psi_i^k(\mathbf{x})].$$

We recall that $\Pi_{\mathbf{x}_k}$ is defined by (4.2). Finally, $(j_i)_{i\in\mathbb{N}}$ diagonally converges to the extension of $(\mathbf{x}, \mathbf{u}) \mapsto j[\mathbf{x}, \mathbf{u}, \mathbf{V}(\mathbf{x})]$, since $(V_i)_{i\in\mathbb{N}}$ is diagonally converging to V. Moreover, $(\Omega_i)_{i\in\mathbb{N}} \subset B$, the Gauss map is always valued in $\mathbb{S}^{n-1}$, and $(V_i)_{i\in\mathbb{N}}$ is uniformly bounded. Hence, $(\mathbf{x}, \mathbf{n}_{\partial\Omega_i}(\mathbf{x}), \mathbf{V}_i(\mathbf{x}))$ is valued in a compact set. Since j is continuous on this compact set, it is bounded and $(j_i)_{i\in\mathbb{N}}$ is thus uniformly bounded on $\overline{B} \times \mathbb{S}^{n-1}$. Finally, we can apply Proposition 4.7 to let $i \to +\infty$. □

4.3 *Some linear functionals involving the second fundamental form*

From Theorem 3.3, we only have the L^∞-weak-star convergence of the coefficients associated with the Hessian of the local maps φ_i^k so we consider here the case of functionals whose expressions in the parametrization are linear in $\partial_{pq}\varphi_i^k$. This is the case for the scalar mean curvature and the second fundamental form of two vector fields.

Proposition 4.10 *Consider Assumption 4.1 and a continuous map $j : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$. Then, the functional $\Omega \in \mathcal{O}_\varepsilon(B) \mapsto \int_{\partial\Omega} H(\mathbf{x}) j[\mathbf{x}, \mathbf{n}(\mathbf{x})] dA(\mathbf{x})$ is continuous:*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} H(\mathbf{x})\, j\,[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}) = \int_{\partial\Omega} H(\mathbf{x})\, j\,[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}).$$

Proof. The proof is identical to the one of Proposition 4.6. Using the same notation and the same partition of unity, we have to check that in the parametrization $X_i^k : \mathbf{x}' \in D_{\tilde{r}}(\mathbf{x}_k) \mapsto (\mathbf{x}', \varphi_i^k(\mathbf{x}'))$, the scalar mean curvature L^∞-weakly-star converges. It is the trace (4.9) of the Weingarten map defined by (4.7) so relation (4.11) gives:

$$(H \circ X_i^k) = -\sum_{p,q=1}^{n-1} g^{pq} b_{qp} = -\sum_{p,q=1}^{n-1} \left(\delta_{pq} - \frac{\partial_p\varphi_i^k \partial_q\varphi_i^k}{1 + \|\nabla\varphi_i^k\|^2}\right)\left(\frac{\partial_{pq}\varphi_i^k}{\sqrt{1 + \|\nabla\varphi_i^k\|^2}}\right). \tag{4.16}$$

Using Theorem 3.3, the K sequences $(H \circ X_i^k)_{i\in\mathbb{N}}$ weakly-star converge in $L^\infty(D_{\tilde{r}}(\mathbf{x}_k))$ respectively to $H \circ X^k$. The remaining part of each integrand below uniformly converges to the one of $\partial\Omega$ so we can let $i \to +\infty$ inside:

$$\sum_{k=1}^{K} \int_{D_{\tilde{r}}(\mathbf{x}_k)} (H \circ X_i^k)(\mathbf{x}')(\xi^k \circ X_i^k)(\mathbf{x}') j[X_i^k(\mathbf{x}'), (\mathbf{n} \circ X_i^k)(\mathbf{x}')](dA \circ X_i^k)(\mathbf{x}'),$$

to get the limit asserted in Proposition 4.10. □

Corollary 4.11 *Consider Assumption 4.1 and a continuous map $j : \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{R}$ which is convex in its last variable. Then, we have:*

$$\int_{\partial\Omega} j\,[\mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})]\, dA(\mathbf{x}) \leqslant \liminf_{i\to+\infty} \int_{\partial\Omega_i} j\,[\mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})]\, dA(\mathbf{x}).$$

REMARK 4.12 In particular, this result implies that the Helfrich (1.2) and the Willmore functional (1.3) are lower semi-continuous, and so does the p-th power norm of the mean curvature $\int |H|^p dA$, $p \geqslant 1$. Note that we are able to treat the critical case $p = 1$, while it is often excluded from many statements of geometric measure theory [25, Example 4.1] [47, Definition 2.2] [40, Definition 4.1.2]. We emphasize the fact that we have here lower semi-continuity and *not* continuity.

Proof. The arguments are standard [57, §2 Theorem 4]. We only sketch the proof. First, assume that j is the maximum of finitely many affine functions according to its last variable:

$$\forall t \in \mathbb{R}, \quad j(\mathbf{x}, \mathbf{n}(\mathbf{x}), t) = \max_{0\leqslant l\leqslant L} j_l\,[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, t + \tilde{j}_l\,[\mathbf{x}, \mathbf{n}(\mathbf{x})]. \tag{4.17}$$

For simplicity, let us assume that j only depends on the position. Using a partition of unity as in Proposition 4.6, we introduce the local parametrizations $X^k : \mathbf{x}' \in D_{\tilde{r}}(\mathbf{x}_k) \mapsto (\mathbf{x}', \varphi^k(\mathbf{x}'))$ and we make a partition of the set $D_{\tilde{r}}(\mathbf{x}_k)$ into L disjoints sets. We define recursively for any $l \in \{1, \dots L\}$:

$$D_l^k = \{\mathbf{x}' \in D_{\tilde{r}}(\mathbf{x}_k) \setminus \bigcup_{\tilde{l}=1}^{l} D_{\tilde{l}}^k, j\left[X^k(\mathbf{x}'), (H \circ X^k)(\mathbf{x}')\right] \\ = j_l\left[X^k(\mathbf{x}')\right] H\left[X^k(\mathbf{x}')\right] + \tilde{j}_l\left[X^k(\mathbf{x}')\right]\}.$$

Then, applying Proposition 4.10, we have successively:

$$\begin{aligned}\int_{\partial\Omega} j[\mathbf{x}, H(\mathbf{x})] dA(\mathbf{x}) &= \sum_{k=1}^{K} \int_{D_{\tilde{r}}(\mathbf{x}_k)} (\xi^k \circ X^k) j[X^k, (H \circ X^k)](dA \circ X^k) \\ &= \sum_{k=1}^{K} \sum_{l=1}^{L} \int_{D_l^k} (\xi^k \circ X^k) \left(j_l[X^k] H[X^k] + \tilde{j}_l[X^k]\right) (dA \circ X^k) \\ &= \sum_{k=1}^{K} \sum_{l=1}^{L} \lim_{i \to +\infty} \int_{D_l^k} (\xi^k \circ X_i^k) \left(j_l[X_i^k] H[X_i^k] + \tilde{j}_l[X_i^k]\right) (dA \circ X_i^k) \\ &\leqslant \sum_{k=1}^{K} \sum_{l=1}^{L} \liminf_{i \to +\infty} \int_{D_l^k} (\xi^k \circ X_i^k) j[X_i^k, (H \circ X_i^k)](dA \circ X_i^k) \\ &\leqslant \liminf_{i \to +\infty} \int_{\partial\Omega_i} j[\mathbf{x}, H(\mathbf{x})] dA(\mathbf{x}).\end{aligned}$$

The result holds true for maps j that are maximum of finitely many planes. In general, we write $j = \lim_{L \to +\infty} j_L$ where j_L is defined by (4.17) and apply the Monotone Convergence Theorem. □

Proposition 4.13 *Consider Assumption 4.1 and some continuous maps* $j, j_i : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ *such that* $(j_i)_{i \in \mathbb{N}}$ *is uniformly bounded on* $\overline{B} \times \mathbb{S}^{n-1}$ *and diagonally converges to* j *in the sense of Definition 4.2. Then, we have:*

$$\lim_{i \to +\infty} \int_{\partial\Omega_i} H(\mathbf{x})\, j_i[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}) = \int_{\partial\Omega} H(\mathbf{x})\, j[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}).$$

REMARK 4.14 As in Corollary 4.9, we can consider here that j_i is a continuous map of the position, the normal vector, and a finite number of uniformly bounded vector fields diagonally converging in the sense of Definition 4.8.

Proof. The proof is identical to the one of Proposition 4.10. Writing the functional in terms of local parametrizations, it remains to check that we can let $i \to +\infty$ in each integral. From (4.16), $(H \circ X_i^k)_{i \in \mathbb{N}}$ weakly-star converges in $L^\infty(D_{\tilde{r}}(\mathbf{0}'))$ to $H \circ X^k$, while the remaining part of the integrand is strongly converging in $L^1(D_{\tilde{r}}(\mathbf{0}'))$, since the hypothesis allows one to apply Lebesgue's Dominated Convergence Theorem. Hence, Proposition 4.13 holds true. □

Proposition 4.15 *Consider Assumption 4.1 and some uniformly bounded continuous vector fields $\mathbf{V}_i$ and $\mathbf{W}_i$ on $\partial\Omega_i$ that are diagonally converging to continuous vector fields $\mathbf{V}$ and $\mathbf{W}$ on $\partial\Omega$ in the sense of Definition 4.8. Let $j, j_i : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ be continuous maps such that $(j_i)_{i\in\mathbb{N}}$ is uniformly bounded on $\overline{B} \times \mathbb{S}^{n-1}$ and diagonally converges to j as in Definition 4.2. Then, we have:*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} \mathbf{II}\left[\mathbf{V}_i(\mathbf{x}), \mathbf{W}_i(\mathbf{x})\right] j_i\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \int_{\partial\Omega} \mathbf{II}\left[\mathbf{V}(\mathbf{x}), \mathbf{W}(\mathbf{x})\right] j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}).$$

REMARK 4.16 Note that if $j_i = j$ for any $i \in \mathbb{N}$, then the above assertion states that a functional which is linear in the second fundamental form is continuous. Hence, adapting the arguments of Corollary 4.11, any functional whose integrand is a continuous map of the position, the normal vector, and the second fundamental form, convex in its last variable, is lower semi-continuous.

Proof. We write the integral in terms of local parametrizations and check that we can let $i \to +\infty$. In the local basis $(\partial_1 X_i^k, \ldots, \partial_{n-1} X_i^k)$, using (4.12), the second fundamental form takes the form:

$$\left(\mathbf{II} \circ X_i^k\right)\left(\mathbf{V}_i \circ X_i^k, \mathbf{W}_i \circ X_i^k\right) = \sum_{p,q,r,s=1}^{n-1} \left\langle \mathbf{V}_i \circ X_i^k \mid \partial_p X_i^k \right\rangle g^{pq} b_{qr} g^{rs} \left\langle \mathbf{W}_i \circ X_i^k \mid \partial_s X_i^k \right\rangle.$$

Hence, each integrand is the product of $g^{pq} b_{qr} g^{rs}$ with a remaining term. Using the assumptions, the convergence results of Theorem 3.3, and Lebesgue's Dominated Convergence Theorem, we get that $g^{pq} b_{qr} g^{rs}$ weakly-star converges in L^∞, while the remaining term L^1-strongly converges. □

4.4 *Some non-linear functionals involving the second fundamental form*

All the previous continuity results were obtained by expressing the integrals in the parametrizations associated with a suitable partition of unity, and by observing that each integrand is the product of b_{pq} converging L^∞-weakly-star with a remaining term converging L^1-strongly. We are wondering here if a non-linear function such as the determinant of the (b_{pq}) can also L^∞-weakly-star converge. Note that the convergence is in L^∞ and *not* in $W^{1,p}$ so we cannot use e.g. [28, Section 8.2.4.b].

However, the coefficients of the first and second fundamental forms are not random. They characterize the hypersurfaces through the Gauss-Codazzi–Mainardi equations (4.14) and (4.15). Hence, using the differential structure of these equations, we want to obtain the L^∞-weak-star convergence of non-linear functions of the b_{pq}. This is done by considering a generalization of the Div-Curl Lemma due to Tartar. We refer to [57, Section 6] for details and it states as follows.

Proposition 4.17 (Tartar [57, Section 6, Corollary 13]) *Let $n \geqslant 3$ and $U \subset \mathbb{R}^{n-1}$ be open and bounded with smooth boundary. We consider a sequence of maps $(u_i)_{i\in\mathbb{N}}$ weakly-star converging to u in $L^\infty(U, \mathbb{R}^M)$, $M \geqslant 1$, and a continuous functional $F : \mathbb{R}^M \to \mathbb{R}$ such that $(F(u_i))_{i\in\mathbb{N}}$ is weakly-star converging in $L^\infty(U, \mathbb{R})$. Let us suppose we are given P first-order constant coefficient differential operators $A^p v := \sum_{q=1}^{n-1} \sum_{m=1}^{M} a_{mq}^p \partial_q v_m$ such that the sequences $(A^p u_i)_{i\in\mathbb{N}}$ lies in a compact subset of $H^{-1}(U)$. We also assume that $(u_i)_{i\in\mathbb{N}}$ is almost everywhere valued in K for some given compact set $K \subset \mathbb{R}^M$. We introduce the following wave cone:*

$$\Lambda = \left\{ \lambda \in \mathbb{R}^M \mid \exists \mu \in \mathbb{R}^{n-1} \backslash \{\mathbf{0}'\}, \forall p \in \{1, \ldots P\}, \quad \sum_{q=1}^{n-1} \sum_{m=1}^{M} a_{mq}^p \lambda_m \mu_q = 0 \right\}.$$

If F is a quadratic form and $F = 0$ on Λ, then the weak-star limit of $(F(u_i))_{i\in\mathbb{N}}$ is $F(u)$.

We now treat the case of $\mathbb{R}^3$ to get familiar with the notation and observe how Proposition 4.17 can be used here to obtain the L^∞-weak-star convergence of the Gaussian curvature $K = \kappa_1\kappa_2$. Let $n = 3$, $U = D_{\tilde{r}}(\mathbf{x}_k)$, and $u_i : \mathbf{x}' \mapsto (b_{pq}) \in \mathbb{R}^{2^2}$ defined by (4.10) with $X_i^k : \mathbf{x}' \mapsto (\mathbf{x}', \varphi_i^k(\mathbf{x}')) \in \partial\Omega_i$. First, we show that the assumptions of Proposition 4.17 are satisfied. From Theorem 3.3, $(u_i)_{i\in\mathbb{N}}$ $L^\infty(U)$-weakly-star converges to u and it is uniformly bounded so it is valued in a compact set. Moreover, in the case $n = 3$, there are only two Codazzi–Mainardi equations (4.15):

$$\begin{cases} \partial_1 b_{12} - \partial_2 b_{11} = \left(\Gamma_{11}^1 b_{12} - \Gamma_{12}^1 b_{11}\right) + \left(\Gamma_{11}^2 b_{22} - \Gamma_{12}^2 b_{21}\right) \\ \partial_1 b_{22} - \partial_2 b_{21} = \left(\Gamma_{21}^1 b_{12} - \Gamma_{22}^1 b_{11}\right) + \left(\Gamma_{21}^2 b_{22} - \Gamma_{22}^2 b_{21}\right). \end{cases}$$

Hence, the two differential operators $A^1 u_i := \partial_1 b_{12} - \partial_2 b_{11}$ and $A^2 u_i := \partial_1 b_{22} - \partial_2 b_{21}$ are valued and uniformly bounded in $L^\infty(U)$, which is compactly embedded in $H^{-1}(U)$ (Rellich–Kondrachov Embedding Theorem), so we deduce that up to a subsequence, $(A^1 u_i)_{i\in\mathbb{N}}$ and $(A^2 u_i)_{i\in\mathbb{N}}$ lies in a compact subset of $H^{-1}(U)$. Let us now have a look at the wave cone:

$$\Lambda = \left\{ \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \in \mathbb{R}^{2^2} \mid \exists \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \right.$$
$$\left. \mu_1\lambda_{12} - \mu_2\lambda_{11} = 0 \text{ and } \mu_1\lambda_{22} - \mu_2\lambda_{21} = 0 \right\}.$$

REMARK 4.18 The wave cone Λ is the set of (2×2)-matrices with zero determinant.

Consequently, if we want to apply Proposition 4.17 on a quadratic form in the b_{pq}, we get from Remark 4.18 that the determinant is one possibility. Indeed, if we set $F(u_i) = \det(u_i)$, then F is quadratic and $F(\lambda) = 0$ for any $\lambda \in \Lambda$. Since $(F(u_i))_{i\in\mathbb{N}}$ is uniformly bounded in $L^\infty(U)$, up to a subsequence, it is converging and applying Proposition 4.17, the limit is $F(u)$. This also proves that $F(u)$ is the unique limit of any converging subsequence. Hence, the whole sequence is converging to $F(u)$ and we are now in position to prove the following result.

Proposition 4.19 *Consider Assumption 4.1 and some continuous maps $j, j_i : \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ such that $(j_i)_{i\in\mathbb{N}}$ is uniformly bounded on $\overline{B} \times \mathbb{S}^2$ and diagonally converges to j as in Definition 4.2. Then, we have (note that Remarks 4.14 and 4.16 also hold true here):*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} K(\mathbf{x})\, j_i\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \int_{\partial\Omega} K(\mathbf{x})\, j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}).$$

In particular, the genus is continuous: $\text{genus}(\partial\Omega_i) \longrightarrow_{i\to+\infty} \text{genus}(\partial\Omega)$.

Proof. As in the proof of Proposition 4.6, we can express the functional in the parametrizations associated with the partition of unity. Then, we have to check we can let $i \to +\infty$ in each integral. Note that K is the determinant (4.9) of the Weingarten map (4.7) so we get from (4.11):

$$K \circ X_i^k = \det(h) = \det(-g^{-1}b) = -\frac{\det(b_{pq})}{\det(g_{rs})}.$$

From the foregoing and the uniform convergence of (g_{rs}), we get that the sequences $(K \circ X_i^k)_{i\in\mathbb{N}}$ converge L^∞-weakly-star respectively to $K \circ X^k$, whereas the remaining term in the integrand is L^1-strongly converging using the hypothesis and Lebesgue's Dominated Convergence Theorem. Hence, we can let $i \to +\infty$ and Proposition 4.19 holds true. Finally, concerning the genus, we apply the Gauss–Bonnet Theorem $\int_{\partial\Omega_i} K dA = 4\pi(1-g_i) \longrightarrow_{i\to+\infty} \int_{\partial\Omega} K dA = 4\pi(1-g)$. □

We now establish the equivalent of Proposition 4.19 in $\mathbb{R}^n$. First, instead of working with the coefficients (b_{pq}) of the second fundamental form (4.10), we prefer to work with the ones (h_{pq}) representing the Weingarten map. We set $n > 3$, $U = D_{\tilde{r}}(\mathbf{x}_k)$, and $u_i : \mathbf{x}' \in U \mapsto (h_{pq}) \in \mathbb{R}^{(n-1)^2}$ defined by (4.11) in the local parametrizations $X_i^k : \mathbf{x}' \in U \mapsto (\mathbf{x}', \varphi_i^k(\mathbf{x}')) \in \partial\Omega_i$ introduced in the proof of Proposition 4.6. Then, we check that the hypothesis of Proposition 4.17 are satisfied. From Theorem 3.3, $(u_i)_{i\in\mathbb{N}}$ weakly-star converges to u in $L^\infty(U)$ and it is uniformly bounded so it is valued in a compact set. Using the Codazzi–Mainardi equations (4.15), the differential operators:

$$\partial_{q'} h_{pq} - \partial_q h_{pq'} = \sum_{m=1}^{n-1} \left((\partial_{q'} g^{pm}) b_{mq} - (\partial_q g^{pm}) b_{mq'} \right) + \sum_{m=1}^{n-1} g^{pm} \left(\partial_{q'} b_{mq} - \partial_q b_{mq'} \right),$$

are valued and uniformly bounded in $L^\infty(U)$, which is compactly embedded in $H^{-1}(U)$ (Rellich-Kondrachov Embedding Theorem), so up to a subsequence, they lies in a compact set of $H^{-1}(U)$. Finally, we introduce the wave cone of Proposition 4.17:

$$\Lambda = \left\{ \lambda \in \mathbb{R}^{(n-1)^2} \mid \exists \mu \neq 0_{(n-1)\times 1}, \forall (p,q,m) \in \{1,\dots,n-1\}^3, \quad \mu_m \lambda_{pq} - \mu_q \lambda_{pm} = 0 \right\}.$$

DEFINITION 4.20 A *pth-order minor* of a square $(n-1)^2$-matrix M is the determinant of any $(p \times p)$-matrix $M[I,J]$ formed by the coefficients of M corresponding to rows with index in I and columns with index in J, where $I, J \subset \{1,\dots,n-1\}$ have p elements, i.e., $\sharp I = \sharp J = p$.

REMARK 4.21 The wave cone Λ is the set of square $(n-1)^2$-matrices of rank zero or one. In particular, any minor of order two is zero for such matrices.

Consequently, Remark 4.21 combined with Proposition 4.17 tells us that continuous functionals are given by the ones whose expressions in the local parametrizations (cf. proof of Proposition 4.6) are linear in terms of the form $h_{pq}h_{p'q'} - h_{pq'}h_{p'q}$. However, such terms depend on the partition of unity and on the parametrizations i.e. on the chosen basis $(\partial_1 X_i^k, \dots, \partial_{n-1} X_i^k)$ whereas the integrand of the functional cannot. We now give three applications for which it is the case.

Proposition 4.22 *Consider Assumption 4.1 and some continuous maps $j, j_i : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ so that $(j_i)_{i\in\mathbb{N}}$ is uniformly bounded on $\overline{B} \times \mathbb{S}^{n-1}$ and diagonally converges to j in the sense of Definition 4.2. Then, introducing $H^{(2)} = \sum_{1\leqslant p<q\leqslant n-1} \kappa_p\kappa_q$ defined in (4.8), we have:*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} H^{(2)}(\mathbf{x})\, j_i\,[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}) = \int_{\partial\Omega} H^{(2)}(\mathbf{x})\, j\,[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}).$$

Note that Remarks 4.14 and 4.16 also hold true here.

Proof. First, using the notation of Definition 4.20, note that the characteristic polynomial of (h_{pq}), which is the matrix (4.11) representing the Weingarten map (4.7) in the basis $(\partial_1 X_i^k, \dots \partial_{n-1} X_i^k)$, can be expressed as:

$$P(t) = \det(h - tI_{n-1}) = (-1)^n t^n + \sum_{m=1}^{n-1} (-1)^{n-m} \Big(\sum_{\sharp I = m} \det(h[I,I]) \Big) t^{n-m},$$

but we can also represent the Weingarten map in the basis associated with the principal curvatures:

$$P(t) = \prod_{m=1}^{n-1}\left(\left(\kappa_m \circ X_i^k\right) - t\right) = \sum_{m=0}^{n-1}(-1)^{n-m}\left(H^{(m)} \circ X_i^k\right)t^{n-m}.$$

Since each coefficients of the characteristic polynomial do not depend on the chosen basis, we get:

$$\forall m \in \{0,\dots,n-1\}, \quad H^{(m)} \circ X_i^k = \sum_{\sharp I = m} \det\big(h[I,I]\big). \tag{4.18}$$

If we set $F(\lambda) = \sum_{\sharp I=2} \det(\lambda[I,I])$, then F is quadratic and from Remark 4.21 we get $F(\lambda) = 0$ for any $\lambda \in \Lambda$. Since $(F(u_i))_{i\in\mathbb{N}}$ is uniformly bounded in $L^\infty(U)$, up to a subsequence, it is converging and applying Proposition 4.17, the limit is $F(u)$, unique limit of any converging subsequence so the whole sequence is converging to $F(u)$. Using (4.18), we get that the sequences $(H^{(2)} \circ X_i^k)_{i\in\mathbb{N}}$ converge L^∞-weakly-star respectively to $H^{(2)} \circ X^k$, whereas the remaining term in the integrand is L^1-strongly converging using the hypothesis and Lebesgue's Dominated Convergence Theorem. Hence, we can let $i \to +\infty$ and the functional is continuous. □

Corollary 4.23 *Considering Assumption 4.1, a continuous map $j : \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{R}$ convex in its last variable, and the Frobenius L^2-norm $\|D_{\mathbf{x}}\mathbf{n}\|_2 = \sqrt{\mathrm{trace}(D_{\mathbf{x}}\mathbf{n} \circ D_{\mathbf{x}}\mathbf{n}^T)} = (\sum_{m=1}^{n-1}\kappa_m^2)^{\frac{1}{2}}$ of the Weingarten map (4.7), we have:*

$$\int_{\partial\Omega} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x}), \|D_{\mathbf{x}}\mathbf{n}\|_2^2\right] dA(\mathbf{x}) \leqslant \liminf_{i\to+\infty} \int_{\partial\Omega_i} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x}), \|D_{\mathbf{x}}\mathbf{n}\|_2^2\right] dA(\mathbf{x}).$$

The pth-power of the 2nd-fundamental-form L^2-norm $\int \|\mathbf{II}\|_2^p dA$, $p \geqslant 2$ is lower semi-continuous.

Proof. First, assume that j is linear in its last argument. Note that the Frobenius norm $\|.\|_2$ does not depend on the chosen basis so we can consider the one associated with the principal curvatures, and we get $\|D\mathbf{n}\|_2^2 = \sum_{m=1}^{n-1}\kappa_m^2 = (\sum_{m=1}^{n-1}\kappa_m)^2 - \sum_{p\neq q}\kappa_p\kappa_q = H^2 - 2H^{(2)}$. Hence, there exists a continuous map $\tilde{j} : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ such that $\int_{\partial\Omega_i} j[\mathbf{x}, \mathbf{n}(\mathbf{x}), \|D_{\mathbf{x}}\mathbf{n}\|_2^2] dA(\mathbf{x})$ is equal to:

$$\int_{\partial\Omega_i} H^2(\mathbf{x})\, \tilde{j}\,[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}) - 2\int_{\partial\Omega_i} H^{(2)}(\mathbf{x})\, \tilde{j}\,[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}).$$

In the left term, the integrand is convex in H so Corollary 4.11 furnishes its lower semi-continuity. Concerning the right one, apply Proposition 4.22 to get its continuity. Therefore, the functional is lower semi-continuous if j is linear in its last variable. Then, we can apply the standard procedure [57, §2 Theorem 4] described in Corollary 4.11 to get the same result in the general case. Finally, $\|\mathbf{II}(\mathbf{x})\|_2^2 = \|D_{\mathbf{x}}\mathbf{n}\|_2^2$ and if $p \geqslant 2$, $t \mapsto t^{\frac{p}{2}}$ is convex thus $\int \|\mathbf{II}\|_2^p dA$ is lower semi-continuous. □

Proposition 4.24 *Consider Assumption 4.1, some continuous maps $j, j_i : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ such that $(j_i)_{i\in\mathbb{N}}$ is uniformly bounded on $\overline{B} \times \mathbb{S}^{n-1}$ and diagonally converges to j as in Definition 4.2, and some vector fields $\mathbf{V}_i$ and $\mathbf{W}_i$ on $\partial\Omega_i$ uniformly bounded and diagonally converging to vector fields $\mathbf{V}$ and $\mathbf{W}$ on $\partial\Omega$ in the sense of Definition 4.8. Then, the following functional is continuous (note that Remarks 4.14 and 4.16 also hold true here):*

$$J(\partial\Omega_i) := \int_{\partial\Omega_i} \langle D_{\mathbf{x}}\mathbf{n}[\mathbf{V}_i(\mathbf{x})] \mid D_{\mathbf{x}}\mathbf{n}[\mathbf{W}_i(\mathbf{x})] - H(\mathbf{x})\mathbf{W}_i(\mathbf{x})\rangle\, j_i[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}) \underset{i\to+\infty}{\longrightarrow} J(\partial\Omega).$$

Proof. Again, the idea is to check that the expression of the functional in the parametrization is linear in a term of the form $b_{pq}b_{p'q'} - b_{pq'}b_{p'q'}$. First, the linear term can be expressed as:

$$\sum_{p,p',p"=1}^{n-1} \sum_{q,q',q"=1}^{n-1} \left\langle \mathbf{V}_i \circ X_i^k \mid \partial_q X_i^k \right\rangle g^{pq} g^{p'q'} \left(b_{q'p} b_{p"p'} - b_{q'p'} b_{pp"} \right) g^{p"q"} \left\langle \mathbf{W}_i \circ X_i^k \mid \partial_{q"} X_i^k \right\rangle$$

Note that until now, in Section 4, we never used the fact that (g_{pq}), (g^{pq}), (b_{pq}) or (h_{pq}) are symmetric matrices. Here, let us invert the two indices $b_{pp"} = b_{p"p}$ in the above expression. Then, $b_{q'p}b_{p"p'} - b_{q'p'}b_{p"p}$ is L^{∞}-weakly-star converging. Indeed, as we did for (h_{pq}), we can use the Codazzi–Mainardi equations (4.15) and Remark 4.21 to apply Proposition 4.17 on (b_{pq}). Finally, the hypothesis and the convergence results of Theorem 3.3 gives the L^1-strong convergence of the remaining term so we can let $i \to +\infty$ in each integral and the functional is continuous. □

Note also that, in Section 4.4, we only used the Codazzi–Mainardi equations (4.15). We want here to use the Gauss equations (4.14) because from the foregoing, its right member is L^{∞}-weakly-star converging. For this purpose, we need to introduce some concepts of Riemannian geometry which are beyond the scope of the article. Hence, we refer to [60] for precise definitions. Merely speaking, the Riemann curvature tensor R of a Riemannian manifold measures the extend to which the first fundamental form is not locally isometric to an Euclidean space, i.e. the non-commutativity of the covariant derivative. In the basis $(\partial_1 X, \dots, \partial_{n-1} X)$, we have [60, Section 2.6]:

$$R_{jli}^{k} = \sum_{m=1}^{n-1} g^{km} R_{mjli} = \partial_l \Gamma_{ij}^{k} - \partial_j \Gamma_{il}^{k} + \sum_{m=1}^{n-1} \left(\Gamma_{ij}^{m} \Gamma_{ml}^{k} - \Gamma_{il}^{m} \Gamma_{mj}^{k} \right),$$

where the Christoffels symbols Γ_{ij}^{k} were defined in (4.13). Hence, the Gauss equations (4.14) state that in the local parametrization, the Riemann curvature tensor is given by:

$$R_{jli}^{k} = \sum_{m=1}^{n-1} g^{km} \left(b_{ij} b_{ml} - b_{il} b_{mj} \right),$$

which is thus L^{∞}-weakly-star converging, and so does the Ricci curvature tensor [60, Section 3.3] $Ric_{ij} = \sum_{k=1}^{n-1} R_{ikj}^{k}$ and the scalar curvature $\mathfrak{R} = \sum_{i,j=1}^{n-1} g^{ij} R_{ij}$. Hence, we get the following result.

Proposition 4.25 *Consider Assumption 4.1, some continuous maps $j, j_i : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ such that $(j_i)_{i\in\mathbb{N}}$ is uniformly bounded on $\overline{B} \times \mathbb{S}^{n-1}$ and diagonally converges to j as in Definition 4.2, and some vector fields $\mathbf{T}_i, \mathbf{U}_i, \mathbf{V}_i, \mathbf{W}_i$ on $\partial\Omega_i$ uniformly bounded and diagonally converging to vector fields $\mathbf{T}, \mathbf{U}, \mathbf{V}, \mathbf{W}$ on $\partial\Omega$ in the sense of Definition 4.8. Then, the three following functionals are continuous (note that Remarks 4.14 and 4.16 also hold true here):*

$$\begin{cases} J(\partial\Omega_i) := \int_{\partial\Omega_i} \langle R_{\mathbf{x}} [\mathbf{T}_i(\mathbf{x}), \mathbf{U}_i(\mathbf{x})] \mathbf{V}_i(\mathbf{x}) \mid \mathbf{W}_i(\mathbf{x}) \rangle \, j_i [\mathbf{x}, \mathbf{n}(\mathbf{x})] \, dA(\mathbf{x}) \underset{i\to+\infty}{\longrightarrow} J(\partial\Omega) \\ J'(\partial\Omega_i) := \int_{\partial\Omega_i} Ric_{\mathbf{x}} [\mathbf{V}_i(\mathbf{x}), \mathbf{W}_i(\mathbf{x})] \, j_i [\mathbf{x}, \mathbf{n}(\mathbf{x})] \, dA(\mathbf{x}) \underset{i\to+\infty}{\longrightarrow} J'(\partial\Omega) \\ J''(\partial\Omega_i) := \int_{\partial\Omega_i} \mathfrak{R}(\mathbf{x}) \; j_i [\mathbf{x}, \mathbf{n}(\mathbf{x})] \, dA(\mathbf{x}) \underset{i\to+\infty}{\longrightarrow} J"(\partial\Omega). \end{cases}$$

Proof. The proof is same than the previous ones. Write the functional in the local parametrizations, and observe that it is a finite sum of integrals whose integrand is the product of a L^{∞}-weakly-star converging term, while the other one is converging L^1-strongly so we can let $i \to +\infty$. □

Note that in the case $n = 3$, the scalar curvature $\mathfrak{R}$ is twice the Gaussian curvature $K = \kappa_1\kappa_2$. Hence, the continuity of the last functional above is the generalization of Proposition 4.19 to $\mathbb{R}^n$, $n > 3$, which was the task of the subsection. We conclude by proving Theorem 4.3.

Proof of Theorem 4.3. Using Proposition 4.17 and (4.15), we showed how to get the L^∞-weakly-star convergence of any $h[pp', qq'] := h_{pq}h_{p'q'} - h_{pq'}h_{p'q}$ from the one of (h_{pq}) defined in (4.11). Now, we want to apply Proposition 4.17 to $(h[pp', qq'])$. For this purpose, we need to find differential operators which are valued and uniformly bounded in L^∞. Using (4.15), this is the case for:

$$\begin{aligned}\begin{vmatrix} \partial_q & h_{pq} & h_{p'q} \\ \partial_{q'} & h_{pq'} & h_{p'q'} \\ \partial_{q''} & h_{pq''} & h_{p'q''} \end{vmatrix} &= \partial_q h[pp', q'q''] - \partial_{q'} h[pp', qq''] + \partial_{q''} h[pp', qq'] \\ &= \left(\partial_q h_{pq'} - \partial_{q'} h_{pq}\right) h_{p'q''} + \left(\partial_{q'} h_{p'q} - \partial_q h_{p'q'}\right) h_{pq''} \\ &\quad + \left(\partial_q h_{p'q''} - \partial_{q''} h_{p'q}\right) h_{pq'} + \left(\partial_{q''} h_{p'q'} - \partial_{q'} h_{p'q''}\right) h_{pq} \\ &\quad + \left(\partial_{q''} h_{pq} - \partial_q h_{pq''}\right) h_{p'q'} + \left(\partial_{q'} h_{p'q''} - \partial_{q''} h_{pq'}\right) h_{p'q}.\end{aligned}$$

Then, the wave cone associated with these differential operators is thus given by:

$$\Lambda = \left\{ \lambda \in \mathbb{R}^{(n-1)^4} \mid \exists \mu \neq 0_{(n-1)\times 1}, \forall (p, p', q, q', q'') \in \{0, \dots, n-1\}, \begin{vmatrix} \mu_q & \lambda_{pq} & \lambda_{p'q} \\ \mu_{q'} & \lambda_{pq'} & \lambda_{p'q'} \\ \mu_{q''} & \lambda_{pq''} & \lambda_{p'q''} \end{vmatrix} = 0 \right\}.$$

As in Remark 4.18, one can check that the wave cone is given by all $(n-1)^2$-matrices for which any minor of order three are zero in the sense of Definition 4.20. Finally, combining (4.18) and Proposition 4.17, we get that functionals linear in $H^{(3)}$ are continuous. This procedure can be done recursively similarly to $H^{(l)}$ for any $l \geqslant 3$ so Theorem 4.3 holds true. □

4.5 *Existence of a minimizer for various geometric functionals*

We are now in position to establish general existence results in the class $\mathcal{O}_\varepsilon(B)$. More precisely, we can minimize any functional (and constraints) built from those given before in Section 4. Indeed, considering a minimizing sequence in $\mathcal{O}_\varepsilon(B)$, there exists a converging subsequence as stated in Proposition 3.2 and Assumption 4.1 holds true. Then, applying the appropriate continuity results, we can pass to the limit in the functional and the constraints to get the existence of a minimizer.

In this section, we first give a proof of Theorem 1.3 and state/prove its generalization to $\mathbb{R}^n$. Then, we establish the existence for a very general model of vesicles. In particular, we prove that Theorems 1.5, 1.7, and 1.8 hold true. We refer to Sections 1.1, 1.2, and 1.3 of the introduction for a detailed exposition on these three models. Finally, we present two more applications that show how to use other continuity results to get the existence of a minimizer in the class $\mathcal{O}_\varepsilon(B)$.

Proof of Theorem 1.3. Consider any minimizing sequence $(\Omega_i)_{i\in\mathbb{N}}$ of $\mathcal{O}_\varepsilon(B)$. Proposition 3.2 ensures that up to a subsequence, $(\Omega_i)_{i\in\mathbb{N}}$ is converging to an open set $\Omega \in \mathcal{O}_\varepsilon(B)$ as stated in

Assumption 4.1. We can thus combine Propositions 4.6, 4.10, and 4.19 to let $i \to +\infty$ in:

$$\int_{\partial\Omega_i} g_0\left[\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega_i} H(\mathbf{x})\, g_1\left[\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega_i} K(\mathbf{x})\, g_2\left[\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \widetilde{C}.$$

Then, apply Proposition 4.6, Corollary 4.11 and Remark 4.16 on Proposition 4.19, to obtain the lower semi-continuity of the functional and that the inequality constraints remain true as $i \to +\infty$. Therefore, Ω is a minimizer of the functional satisfying the constraints in the class $\mathcal{O}_\varepsilon(B)$. □

MAIN THEOREM 4.26 Let $\varepsilon > 0$ and $B \subset \mathbb{R}^n$ be a bounded open set, large enough to contain an open ball of radius 3ε. Consider $(C, \widetilde{C}) \in \mathbb{R}\times\mathbb{R}$, some continuous maps $j_0, f_0, g_0, g_l : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$, and some maps $j_l, f_l : \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{R}$ which are continuous and convex in their last variable for any $l \in \{1,\ldots,n-1\}$. Then, the following problem has at least one solution (for the notation, we refer to Section 4.1):

$$\inf \int_{\partial\Omega} j_0\left[\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \sum_{l=1}^{n-1}\int_{\partial\Omega} j_l\left[\mathbf{x},\mathbf{n}(\mathbf{x}), H^{(l)}(\mathbf{x})\right] dA(\mathbf{x}),$$

where the infimum is taken among $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying a finite number of constraints of the form:

$$\begin{cases} \displaystyle\int_{\partial\Omega} f_0\left[\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \sum_{l=1}^{n-1}\int_{\partial\Omega} f_l\left[\mathbf{x},\mathbf{n}(\mathbf{x}), H^{(l)}(\mathbf{x})\right] dA(\mathbf{x}) \leqslant C \\ \displaystyle\int_{\partial\Omega} g_0\left[\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \sum_{l=1}^{n-1}\int_{\partial\Omega} H^{(l)}(\mathbf{x})\, g_l\left[\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \widetilde{C}. \end{cases}$$

Proof. Consider a minimizing sequence $(\Omega_i)_{i\in\mathbb{N}}$ of $\mathcal{O}_\varepsilon(B)$. Proposition 3.2 ensures that up to a subsequence, $(\Omega_i)_{i\in\mathbb{N}}$ is converging to an open set $\Omega \in \mathcal{O}_\varepsilon(B)$ as stated in Assumption 4.1. We can thus apply Theorem 4.3 to let $i \to +\infty$ in the following equality:

$$\int_{\partial\Omega_i} g_0\left[\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \sum_{l=1}^{n-1}\int_{\partial\Omega_i} H^{(l)}(\mathbf{x})\, g_l\left[\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \widetilde{C}.$$

Then, we can use again Theorem 4.3 for any $l_0 \in \{1,\ldots,n-1\}$ by setting $j_{l_0} = g_{l_0}$ and $j_l = 0$ for any $l \neq l_0$ to obtain the continuity of any $\int H^{(l_0)}(\bullet) g_{l_0}[\bullet,\mathbf{n}(\bullet)]$ and Remark 4.16 gives the lower semi-continuity of any $\int f_{l_0}[\bullet,\mathbf{n}(\bullet), H^{(l_0)}(\bullet)]$ and $\int j_{l_0}[\bullet,\mathbf{n}(\bullet), H^{(l_0)}(\bullet)]$. Hence, the functional is lower-semi-continuous and the inequality constraint remains true as $i \to +\infty$. Therefore, Ω is a minimizer of the functional satisfying the constraints. □

Proposition 4.27 *Let $H_0, M_0, k_G, k_m \in \mathbb{R}$ and $\varepsilon, k_b, A_0, V_0 > 0$ such that $A_0^3 > 36\pi V_0^2$. Then, the following problem modelling the equilibrium shapes of vesicles [53, Section 2.5] has at least one solution (see Remark 1.4):*

$$\inf_{\substack{\Omega\in\mathcal{O}_\varepsilon(\mathbb{R}^3)\\ A(\partial\Omega)=A_0\\ V(\Omega)=V_0}} \frac{k_b}{2}\int_{\partial\Omega}(H-H_0)^2 dA + k_G\int_{\partial\Omega} K\,dA + k_m\left(\int_{\partial\Omega} H\,dA - M_0\right)^2.$$

Proof. Let us consider a minimizing sequence $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(\mathbb{R}^3)$ of the functional satisfying the area and volume constraints. First, we need to find an open ball B such that $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$. This can be done if we can bound the diameter thanks to the functional and the area constraint. The first step is to control the Willmore energy (1.3). Introducing the functional of the statement $J : \Omega \in \mathcal{O}_\varepsilon(B) \mapsto \frac{k_b}{2}\int_{\partial\Omega}(H-H_0)^2 dA + k_G\int_{\partial\Omega} K dA + k_m(\int_{\partial\Omega} H dA - M_0)^2$, we have:

$$\begin{aligned}\frac{k_b}{4}\int_{\partial\Omega} H^2 dA &= \frac{k_b}{4}\int_{\partial\Omega}(H-H_0+H_0)^2 dA \leqslant \frac{k_b}{2}\int_{\partial\Omega}(H-H_0)^2 dA + \frac{k_b H_0^2}{2}A(\partial\Omega)\\ &\leqslant J(\partial\Omega) + \frac{k_b H_0^2}{2}A(\partial\Omega) + |k_G|\left|\int_{\partial\Omega} K dA\right| + |k_m|\left(\int_{\partial\Omega} H dA - M_0\right)^2\\ &\leqslant J(\partial\Omega) + \frac{k_b H_0^2}{2}A(\partial\Omega) + |k_G|\int_{\partial\Omega}|K| dA + 2|k_m|\left(\int_{\partial\Omega} H dA\right)^2 + 2|k_m|M_0^2.\end{aligned}$$

The second step is to use point (iii) in Theorem 2.6 and Remark 2.8. Considering a point $\mathbf{x} \in \partial\Omega$ in which the Gauss map $\mathbf{n}$ is differentiable, and a unit eigenvector $\mathbf{e}_l$ associated with the eigenvalue κ_l of the Weingarten map $D_{\mathbf{x}}\mathbf{n}$, we have:

$$|\kappa_l(\mathbf{x})| = \|\kappa_l(\mathbf{x})\mathbf{e}_l\| = \|D_{\mathbf{x}}\mathbf{n}(\mathbf{e}_l)\| \leqslant \|D_{\mathbf{x}}\mathbf{n}\|_{\mathcal{L}(T_{\mathbf{x}}\partial\Omega)}\|\mathbf{e}_l\| \leqslant \frac{1}{\varepsilon}, \tag{4.19}$$

from which we deduce that $\max_{1\leqslant l\leqslant n-1}\|\kappa_l\|_{L^\infty(\partial\Omega)} \leqslant \frac{1}{\varepsilon}$. Hence, we obtain:

$$\frac{k_b}{4}\int_{\partial\Omega} H^2 dA \leqslant J(\partial\Omega) + \frac{k_b H_0^2}{2}A(\partial\Omega) + \frac{|k_G|}{\varepsilon^2}A(\partial\Omega) + \frac{8|k_m|}{\varepsilon^2}A(\partial\Omega)^2 + 2|k_m|M_0^2.$$

The final step is to apply [56, Lemma 1.1] to get four positive constants C_0, C_1, C_2, C_3 such that:

$$\operatorname{diam}(\Omega) \leqslant C_0 J(\partial\Omega)A(\partial\Omega) + C_1 A(\partial\Omega) + C_2 A(\partial\Omega)^2 + C_3 A(\partial\Omega)^3.$$

Hence, we can bound uniformly the diameter of the Ω_i and there exists a ball $B \subset \mathbb{R}^n$ sufficiently large such that $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$. From Proposition 3.2, up to a subsequence, it is converging to an $\Omega \in \mathcal{O}_\varepsilon(B)$ as stated in Assumption 4.1. Then, we can apply:

- Cor. 4.11 with $j(x,y,z) = \frac{k_b}{2}(z-H_0)^2$ to get the lower semi-continuity of $\frac{k_b}{2}\int(H-H_0)^2$;
- Prop. 4.19 with $j_i \equiv 1$ to obtain the continuity of $\kappa_G\int K$;
- Prop. 4.10 with $j \equiv 1$ to have the continuity of $\int H dA$ thus the one of $k_m(\int H dA - M_0)^2$.

The functional is lower semi-continuous and from Proposition 4.6 with $j \equiv 1$ and $j(x,y) = \langle x \mid y\rangle$, the area and volume constraints are also continuous so let $i \to +\infty$ and Ω is a minimizer. □

Proof of Theorem 1.5. It is the particular case $k_m = 0$ in Proposition 4.27. This can be also deduced from Theorem 1.3, it suffices to follow the method described in the next proof. □

Proof of Theorem 1.7. First, as in the proof of Proposition 4.27 , one can show that minimizing in $\mathcal{O}_\varepsilon(\mathbb{R}^n)$ or in $\mathcal{O}_\varepsilon(B)$ is equivalent here. Then, apply Theorem 1.3 by setting $j_0 = j_2 \equiv 0$ and $j_1(x,y,z) = (z-H_0)^2$ which is continuous and convex in z. The area and volume constraints can be expressed as in Proposition 4.6 by setting $g_1 = g_2 \equiv 0$ and successively $g_0 \equiv 1$, $g_0(x,y) =$

$\langle x \mid y \rangle$. Using the Gauss–Bonnet Theorem, the genus constraint is written as $\int K dA = 4\pi(1-g) := K_0$. Hence, Theorem 1.3 gives the existence of a minimizer satisfying the three constraints. Finally, we can apply [38, Proposition 2.2.17] to ensure that the compact minimizer is connected since it is the case for any minimizing sequence of compact sets. Hence, using again the Gauss–Bonnet Theorem, the minimizer has the right genus so Theorem 1.7 holds true. □

Proof of Theorem 1.8. The proof is identical to the previous one. We just need to set $H_0 = 0$ and add a fourth equality constraint of the form $g_0 = g_2 \equiv 0$, $g_1 \equiv 1$. □

Proposition 4.28 *Let $\varepsilon > 0$ and $B \subset \mathbb{R}^4$ be a bounded open set, large enough to contain an open ball of radius 3ε. Consider two bounded continuous vector fields of $\mathbb{R}^4$ denoted by $\mathbf{V}, \mathbf{W} : \mathbb{R}^4 \to \mathbb{R}^4$ and a continuous map $j : \mathbb{R}^4 \times \mathbb{S}^3 \times \mathbb{R}$, which is convex in its last variable. Then, the following problem has at least one solution (for the notation, see Section 4.1 and above Proposition 4.25):*

$$\inf \int_{\partial\Omega} j\big[\mathbf{x}, \mathbf{n}(\mathbf{x}), Ric_{\mathbf{x}}\big(\mathbf{V}(\mathbf{x}) \wedge \mathbf{n}(\mathbf{x}), \mathbf{W}(\mathbf{x}) - \langle \mathbf{W}(\mathbf{x}) \mid \mathbf{n}(\mathbf{x})\rangle \mathbf{n}(\mathbf{x})\big)\big] dA(\mathbf{x}),$$

where the infimum is taken among all $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying the following constraint:

$$\int_{\partial\Omega} \mathfrak{R}(\mathbf{x}) \langle \mathbf{V}(\mathbf{x}) \mid \mathbf{n}(\mathbf{x})\rangle \, dA(\mathbf{x}) = \int_{\partial\Omega} H^{(2)}(\mathbf{x}) \langle \mathbf{W}(\mathbf{x}) \mid \mathbf{n}(\mathbf{x})\rangle \, dA(\mathbf{x}).$$

Proof. Consider a minimizing sequence $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$ of the functional satisfying the constraint. From Proposition 3.2, up to a subsequence, it is converging to a set $\Omega \in \mathcal{O}_\varepsilon(B)$. We define $V_i := \mathbf{V} \wedge \mathbf{n}_{\partial\Omega_i}$ and $W_i := \mathbf{W} - \langle \mathbf{W} \mid \mathbf{n}_{\partial\Omega_i}\rangle \mathbf{n}_{\partial\Omega_i}$ which are two continuous vector fields on $\partial\Omega_i$, uniformly bounded since $\mathbf{V}$ and $\mathbf{W}$ are. We now check the diagonal convergence. Choose any sequence of points $\mathbf{x}_i \in \partial\Omega_i$ converging to $\mathbf{x} \in \partial\Omega$. Using the partition of unity introduced in Proposition 4.6, we get that $\mathbf{x} \in \partial\Omega \cap \mathcal{C}_{\tilde{r},\varepsilon}(\mathbf{x}_k)$ for some $k \in \{1, \ldots K\}$. Hence, there exists $\mathbf{x}' \in D_{\tilde{r}}(\mathbf{x}_k)$ such that $\mathbf{x} = (\mathbf{x}', \varphi^k(\mathbf{x}'))$. Since $(\mathbf{x}_i)_{i\in\mathbb{N}}$ is converging to $\mathbf{x}$, for i sufficiently large, we can write $\mathbf{x}_i = (\mathbf{x}'_i, \varphi_i^k(\mathbf{x}'_i))$ with $\mathbf{x}'_i \in D_{\tilde{r}}(\mathbf{x}_k)$. Hence, $\mathbf{x}'_i \to \mathbf{x}'$ and $\varphi_i^k(\mathbf{x}'_i) \to \varphi^k(\mathbf{x}')$, but we also have from the triangle inequality:

$$\|\nabla\varphi_i^k(\mathbf{x}'_i) - \nabla\varphi^k(\mathbf{x}')\| \leqslant \|\nabla\varphi_i^k - \nabla\varphi^k\|_{C^0(\overline{D_{\tilde{r}}(\mathbf{x}_k)})} + \|\nabla\varphi^k(\mathbf{x}'_i) - \nabla\varphi^k(\mathbf{x}')\|.$$

From (3.1) and the continuity of $\nabla\varphi^k$, we can let $i \to +\infty$ and the diagonal convergence of $(\nabla\varphi_i^k)_{i\in\mathbb{N}}$ to $\nabla\varphi^k$ holds true. Then, using (4.6), $n_{\partial\Omega_i}$ is also diagonally converging to $n_{\partial\Omega}$, and so does V_i and W_i. If j is linear in its last variable, we can apply Proposition 4.25 to obtain the continuity of the functional, otherwise we can use Remark 4.16 on the previous case to get the lower semi-continuity of the functional. Finally, apply Theorem 4.3 with $j_i^l \equiv 0$ if $l \neq 2$ and $j_i^2 = \langle \mathbf{V} \mid \mathbf{n}\rangle$ to have the continuity of the left member of the constraint. The continuity of the right one comes from Proposition 4.25 on J'' with $j_i = \langle \mathbf{W} \mid \mathbf{n}\rangle$. Hence, we can let $i \to +\infty$ in the constraint.. □

Proposition 4.29 *Let $\varepsilon, A_0, V_0 > 0$ be such that $A_0^3 > 36\pi V_0^2$, and let $B \subset \mathbb{R}^3$ be a bounded open set, large enough to contain an open ball of radius 3ε. We consider a bounded vector field in $\mathbb{R}^3$ denoted by $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ and a continuous map $j : \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ which is convex in its last variable. Then, the following problem has at least one solution:*

$$\inf_{\substack{\Omega\in\mathcal{O}_\varepsilon(B)\\ A(\partial\Omega)=A_0\\ V(\Omega)=V_0}} \int_{\partial\Omega} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x}), \kappa_{\mathbf{v}}(\mathbf{x})\right] dA(\mathbf{x}),$$

where $\kappa_{\mathbf{v}}$ is the normal curvature at $\mathbf{x}$ i.e. the curvature at $\mathbf{x}$ of the curve formed by the intersection of the surface $\partial\Omega$ with the plane spanned by $\mathbf{n}(\mathbf{x})$ and the vector $\mathbf{v} := \mathbf{V}(\mathbf{x}) - \langle \mathbf{V}(\mathbf{x}) \mid \mathbf{n}(\mathbf{x})\rangle \mathbf{n}(\mathbf{x})$.

Proof. First, [48, Proposition 3.26, Remark 3.27] gives $\kappa_{\mathbf{v}} = \kappa_1 |\langle \mathbf{v}|\mathbf{e}_1\rangle|^2 + \kappa_2 |\langle \mathbf{v}|\mathbf{e}_2\rangle|^2 = \mathbf{II}(\mathbf{v}, \mathbf{v})$. Then, as in the previous proof, we can show that $\mathbf{v}_{\partial\Omega_i}$ is diagonally converging to $\mathbf{v}_{\partial\Omega}$. Finally, if j is linear in its last variable, we can apply Proposition 4.15 to get its continuity, otherwise use Remark 4.16 to get its lower semi-continuity. The area and volume constraints are continuous from Proposition 4.6. Hence, from Proposition 3.2, a minimizing sequence has a converging subsequence to a certain Ω and from the foregoing we can let $i \to +\infty$ in the functional and constraints. □

A. The proofs of Theorems 2.5–2.7

A.1 *The sets of positive reach and the uniform ball condition*

Throughout this section, Ω refers to any non-empty open subset of $\mathbb{R}^n$ different from $\mathbb{R}^n$. Hence, its boundary $\partial\Omega$ is not empty and $\mathrm{Reach}(\partial\Omega)$ is well defined (cf. Remark 1.2). First, we establish some properties that were mentioned in Federer's paper [30] and we show Theorem 2.5 holds true.

A.1.1 *Positive reach implies uniform ball condition.* The point of view adopted here is slightly different from the usual one [30, Theorem 4.8]. Indeed, in order to get the ε-ball condition at a given point $\mathbf{x} \in \partial\Omega$, we need to exhibit points outside the boundary whose projections are precisely $\mathbf{x}$, whereas people usually assume that they exist [24, Chapter 6 Theorem 6.2 (ii) and Chapter 7 Theorem 7.2 (ii)] or only consider the projection of points outside the boundary [24, Chapter 6 Theorems 6.2 (iii) and Chapter 7 Theorem 7.2 (iii)]. However, in order to do so, we have to prevent the open sets to have a thick boundary i.e. having a non-empty interior and thus a non-zero Lebesgue measure [24, Chapter 5 Example 6.2].

Lemma A.1 *For any* $\mathbf{x} \in \partial\Omega$*, we have:* $\mathrm{Reach}(\partial\Omega, \mathbf{x}) = \min\big(\mathrm{Reach}(\overline{\Omega}, \mathbf{x}), \mathrm{Reach}(\mathbb{R}^n\backslash\Omega, \mathbf{x})\big)$.

Proof. We only sketch the proof. Observe $d(\mathbf{x}, \partial\Omega) = \max(d(\mathbf{x}, \overline{\Omega}), d(\mathbf{x}, \mathbb{R}^n\backslash\Omega))$ for any $\mathbf{x} \in \mathbb{R}^n$ to get $\mathrm{Unp}(\partial\Omega) = \mathrm{Unp}(\overline{\Omega}) \cap \mathrm{Unp}(\mathbb{R}^n\backslash\Omega)$ and the equality of Lemma A.1 follows from definitions. □

Proposition A.2 (Federer [30, Theorem 4.8 (6)]) *Let $A \neq \emptyset$ be closed in $\mathbb{R}^n$, $\mathbf{x} \in A$, and $\mathbf{v} \in \mathbb{R}^n$. If the set $\{t > 0, \mathbf{x} + t\mathbf{v} \in \mathrm{Unp}(A)$ and $p_A(\mathbf{x} + t\mathbf{v}) = \mathbf{x}\}$ is not empty and bounded from above, then its supremum τ is well defined and $\mathbf{x} + \tau\mathbf{v}$ cannot belong to the interior of $\mathrm{Unp}(A)$.*

Proof. We refer to [30] for a proof using Peano's Existence Theorem on differential equations. □

Corollary A.3 *If $V(\partial\Omega) = 0$, then for any point $\mathbf{x} \in \partial\Omega$ satisfying $\mathrm{Reach}(\partial\Omega, \mathbf{x}) > 0$, there exists two different points $\mathbf{y} \in \mathrm{Unp}(\overline{\Omega})\backslash\{\mathbf{x}\}$ and $\tilde{\mathbf{y}} \in \mathrm{Unp}(\mathbb{R}^n\backslash\Omega)\backslash\{\mathbf{x}\}$ such that $p_{\overline{\Omega}}(\mathbf{y}) = p_{\mathbb{R}^n\backslash\Omega}(\tilde{\mathbf{y}}) = \mathbf{x}$.*

Proof. Consider $\mathbf{x} \in \partial\Omega$ satisfying $\mathrm{Reach}(\partial\Omega, \mathbf{x}) > 0$. From Lemma A.1, there exists $r > 0$ such that $\overline{B_r(\mathbf{x})} \subseteq \mathrm{Unp}(\overline{\Omega})$. Let $(\mathbf{x}_i)_{i\in\mathbb{N}}$ be a sequence of elements in $B_{\frac{r}{2}}(\mathbf{x})\backslash\overline{\Omega}$ converging to $\mathbf{x}$. Such a sequence exists otherwise $B_{\frac{r}{2}}(\mathbf{x}) \subset \overline{\Omega}$ and $\mathrm{Reach}(\mathbf{x}, \mathbb{R}^n\backslash\Omega) > 0$ would imply $V(\partial\Omega) > 0$. We set:

$$\forall i \in \mathbb{N},\ \forall t \in \mathbb{R}, \quad \mathbf{z}_i(t) = p_{\overline{\Omega}}(\mathbf{x}_i) + t\frac{\mathbf{x}_i - p_{\overline{\Omega}}(\mathbf{x}_i)}{\|\mathbf{x}_i - p_{\overline{\Omega}}(\mathbf{x}_i)\|} \qquad \text{and} \qquad t_i = \frac{r}{2} + d(\mathbf{x}_i, \overline{\Omega}),$$

which is well defined as $\mathbf{x}_i \in \mathrm{Unp}(\overline{\Omega})$. First, $\mathbf{z}_i(t) \in \overline{B_{\frac{r}{2}}(\mathbf{x}_i)} \subseteq B_r(\mathbf{x}) \subseteq \mathrm{Unp}(\overline{\Omega})$ for any $t \in [0, t_i]$. Then, using Federer's result recalled in Proposition A.2, one can prove by contradiction that:

$$\forall t \in [0, t_i], \quad p_{\overline{\Omega}}(\mathbf{z}_i(t)) = p_{\overline{\Omega}}(\mathbf{x}_i).$$

Finally, the sequence $\mathbf{y}_i := \mathbf{z}_i(t_i)$ satisfies $\|\mathbf{y}_i - \mathbf{x}_i\| = \frac{r}{2}$ and also $p_{\overline{\Omega}}(\mathbf{y}_i) = p_{\overline{\Omega}}(\mathbf{x}_i)$. Moreover, since it is bounded, $(\mathbf{y}_i)_{i\in\mathbb{N}}$ is converging, up to a subsequence, to a point denoted $\mathbf{y} \in \overline{B_r(\mathbf{x})} \subseteq \mathrm{Unp}(\overline{\Omega})$. Using the continuity of $p_{\overline{\Omega}}$ [30, Theorem 4.8 (4)], we get $\mathbf{y} \in \mathrm{Unp}(\overline{\Omega})\backslash\{\mathbf{x}\}$ and $p_{\overline{\Omega}}(\mathbf{y}) = p_{\overline{\Omega}}(\mathbf{x}) = \mathbf{x}$. To conclude, similar arguments work by replacing $\overline{\Omega}$ with $\mathbb{R}^n\backslash\Omega$ so Corollary A.3 holds true. □

Proof of Point (ii) in Theorem 2.5. The hypothesis $\partial\Omega \neq \emptyset$ ensures that its reach is well defined. Assume $\mathrm{Reach}(\partial\Omega) > 0$ and $V(\partial\Omega) = 0$. We choose $\varepsilon \in]0, \mathrm{Reach}(\partial\Omega)[$ and let $\mathbf{x} \in \partial\Omega$. From Corollary A.3, there exists $\mathbf{y} \in \mathrm{Unp}(\overline{\Omega})\backslash\{\mathbf{x}\}$ such that $p_{\overline{\Omega}}(\mathbf{y}) = \mathbf{x}$ so we can set $\mathbf{d}_\mathbf{x} = \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|}$. From Lemma A.1, we get $\mathbf{x} + [0, \varepsilon]\mathbf{d}_\mathbf{x} \subseteq \mathrm{Unp}(\overline{\Omega})$. Then, we use Proposition A.2 again to prove by contradiction that $p_{\overline{\Omega}}(\mathbf{x} + t\mathbf{d}_\mathbf{x}) = \mathbf{x}$ for any $t \in [0, \varepsilon]$. In particular, we have $\|\mathbf{z} - (\mathbf{x} + \varepsilon\mathbf{d}_\mathbf{x})\| > \varepsilon$ for any point $\mathbf{z} \in \overline{\Omega}\backslash\{\mathbf{x}\}$ from which we deduce that:

$$\overline{\Omega} \subseteq \{\mathbf{x}\} \cup \left(\mathbb{R}^n\backslash\overline{B_\varepsilon(\mathbf{x} + \varepsilon\mathbf{d}_\mathbf{x})}\right) \iff \overline{B_\varepsilon(\mathbf{x} + \varepsilon\mathbf{d}_\mathbf{x})}\backslash\{\mathbf{x}\} \subseteq \mathbb{R}^n\backslash\overline{\Omega}.$$

Similarly, there exists a unit vector ${}_{\mathbf{x}}$ of $\mathbb{R}^n$ such that we get $\overline{B_\varepsilon(\mathbf{x} + \varepsilon\,{}_{\mathbf{x}})}\backslash\{\mathbf{x}\} \subseteq \Omega$. Since we have $\overline{B_\varepsilon(\mathbf{x} + \varepsilon\,{}_{\mathbf{x}})} \cap \overline{B_\varepsilon(\mathbf{x} + \varepsilon\mathbf{d}_\mathbf{x})} = \{\mathbf{x}\}$, we obtain $\mathbf{d}_\mathbf{x} = -{}_{\mathbf{x}}$. To conclude, if $\mathrm{Reach}(\partial\Omega) < +\infty$, then observe that $B_{\mathrm{Reach}(\partial\Omega)}(\mathbf{x} \pm \mathrm{Reach}(\partial\Omega)\mathbf{d}_\mathbf{x}) = \bigcup_{0<\varepsilon<\mathrm{Reach}(\partial\Omega)} \overline{B_\varepsilon(\mathbf{x} \pm \varepsilon\mathbf{d}_\mathbf{x})}\backslash\{\mathbf{x}\}$ in order to check that Ω also satisfies the $\mathrm{Reach}(\partial\Omega)$-ball condition. □

A.1.2 *Uniform ball condition implies positive reach.*

Proposition A.4 *Assume that there exists $\varepsilon > 0$ such that $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$. Then, we have:*

$$\forall(\mathbf{x}, \mathbf{y}) \in \partial\Omega \times \partial\Omega, \quad \|\mathbf{d}_\mathbf{x} - \mathbf{d}_\mathbf{y}\| \leqslant \frac{1}{\varepsilon}\|\mathbf{x} - \mathbf{y}\|. \tag{A1}$$

In particular, if $\mathbf{x} = \mathbf{y}$, then $\mathbf{d}_\mathbf{x} = \mathbf{d}_\mathbf{y}$ which ensures the unit vector $\mathbf{d}_\mathbf{x}$ of Definition 1.1 is unique. In other words, the map $\mathbf{d} : \mathbf{x} \in \partial\Omega \mapsto \mathbf{d}_\mathbf{x} \in \mathbb{S}^{n-1}$ is well defined and $\frac{1}{\varepsilon}$-Lipschitz continuous.

Proof. Let $\varepsilon > 0$ and $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$. Since $\partial\Omega \neq \emptyset$, we can consider $(\mathbf{x}, \mathbf{y}) \in \partial\Omega \times \partial\Omega$. First, from the ε-ball condition on $\mathbf{x}$ and $\mathbf{y}$, we have $B_\varepsilon(\mathbf{x} \pm \varepsilon\mathbf{d}_\mathbf{x}) \cap B_\varepsilon(\mathbf{y} \mp \varepsilon\mathbf{d}_\mathbf{y}) = \emptyset$, from which we deduce $\|\mathbf{x} - \mathbf{y} \pm \varepsilon(\mathbf{d}_\mathbf{x} + \mathbf{d}_\mathbf{y})\| \geqslant 2\varepsilon$. Then, squaring these two inequalities and summing them, one obtains the result (A1) of the statement: $\|\mathbf{x} - \mathbf{y}\|^2 \geqslant 2\varepsilon^2 - 2\varepsilon^2\langle\mathbf{d}_\mathbf{x} \mid \mathbf{d}_\mathbf{y}\rangle = \varepsilon^2\|\mathbf{d}_\mathbf{x} - \mathbf{d}_\mathbf{y}\|^2$. □

Proof of Point (i) in Theorem 2.5. Let $\varepsilon > 0$ and assume that Ω satisfies the ε-ball condition. Since $\partial\Omega \neq \emptyset$ we can choose any $\mathbf{x} \in \partial\Omega$ and let us prove $B_\varepsilon(\mathbf{x}) \subseteq \mathrm{Unp}(\partial\Omega)$. We first assume that $\mathbf{y} \in B_\varepsilon(\mathbf{x}) \cap \Omega$. Since $\partial\Omega$ is closed, there exists $\mathbf{z} \in \partial\Omega$ such that $d(\mathbf{y}, \partial\Omega) = \|\mathbf{z} - \mathbf{y}\|$. Moreover, we obtain from the ε-ball condition and $\mathbf{y} \in \Omega$:

$$\left(B_\varepsilon(\mathbf{z} + \varepsilon\mathbf{d}_\mathbf{z}) \subseteq \mathbb{R}^n\backslash\overline{\Omega} \quad \text{and} \quad B_{d(\mathbf{y},\partial\Omega)}(\mathbf{y}) \subseteq \Omega\right) \implies B_\varepsilon(\mathbf{z} + \varepsilon\mathbf{d}_\mathbf{z}) \cap B_{d(\mathbf{y},\partial\Omega)}(\mathbf{y}) = \emptyset.$$

Therefore, we deduce that $\mathbf{y} = \mathbf{z} - d(\mathbf{y}, \partial\Omega)\mathbf{d}_\mathbf{z}$. Then, we show that such a $\mathbf{z}$ is unique. Considering another projection $\tilde{\mathbf{z}}$ of $\mathbf{y}$ on $\partial\Omega$, we get from the foregoing: $\mathbf{y} = \mathbf{z} - d(\mathbf{y}, \partial\Omega)\mathbf{d}_\mathbf{z} = \tilde{\mathbf{z}} - d(\mathbf{y}, \partial\Omega)\mathbf{d}_{\tilde{\mathbf{z}}}$.

Using (A1), we have:

$$\|\mathbf{d_z} - \mathbf{d_{\tilde{z}}}\| \leqslant \frac{1}{\varepsilon}\|\mathbf{z} - \tilde{\mathbf{z}}\| = \frac{d(\mathbf{y}, \partial\Omega)}{\varepsilon}\|\mathbf{d_z} - \mathbf{d_{\tilde{z}}}\|.$$

Since $d(\mathbf{y}, \partial\Omega) \leqslant \|\mathbf{x} - \mathbf{y}\| < \varepsilon$, the above inequality can only hold true if $\|\mathbf{d_z} - \mathbf{d_{\tilde{z}}}\| = 0$ i.e. $\mathbf{z} = \tilde{\mathbf{z}}$. Hence, we obtain $B_\varepsilon(\mathbf{x}) \cap \Omega \subseteq \mathrm{Unp}(\partial\Omega)$ and similarly, one can prove $B_\varepsilon(\mathbf{x}) \cap (\mathbb{R}^n \backslash \overline{\Omega}) \subseteq \mathrm{Unp}(\partial\Omega)$. Since $\partial\Omega \subseteq \mathrm{Unp}(\partial\Omega)$, we finally get $B_\varepsilon(\mathbf{x}) \subseteq \mathrm{Unp}(\partial\Omega)$. We thus have $\mathrm{Reach}(\partial\Omega, \mathbf{x}) \geqslant \varepsilon$ for every $\mathbf{x} \in \partial\Omega$ i.e. $\mathrm{Reach}(\partial\Omega) \geqslant \varepsilon$ as required. To conclude the proof of Theorem 2.5, we can simply get $V(\partial\Omega) = 0$ from Theorem 2.6 proved in Section A.2. Indeed, $\partial\Omega$ can be written as a countable union of Lipschitz graphs which have zero Lebesgue measure [29, combine Sections 2.2 and 2.4.1]. □

Proposition A.5 *Assume that there exists $\varepsilon > 0$ such that $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$. Then, we have:*

$$\forall (\mathbf{a}, \mathbf{x}) \in \partial\Omega \times \partial\Omega, \quad |\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a} \rangle| \leqslant \frac{1}{2\varepsilon}\|\mathbf{x} - \mathbf{a}\|^2. \tag{A2}$$

Moreover, introducing the vector $(\mathbf{x} - \mathbf{a})' = (\mathbf{x} - \mathbf{a}) - \langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a} \rangle \mathbf{d_a}$, if we assume $\|(\mathbf{x} - \mathbf{a})'\| < \varepsilon$ and $|\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a} \rangle| < \varepsilon$, then the following local inequality holds true:

$$\frac{1}{2\varepsilon}\|\mathbf{x} - \mathbf{a}\|^2 \leqslant \varepsilon - \sqrt{\varepsilon^2 - \|(\mathbf{x} - \mathbf{a})'\|^2}. \tag{A3}$$

Proof. Let $\varepsilon > 0$ and $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$. Since $\partial\Omega \neq \emptyset$, we can consider $(\mathbf{a}, \mathbf{x}) \in \partial\Omega \times \partial\Omega$. Observe that the point $\mathbf{x}$ cannot belong neither to $B_\varepsilon(\mathbf{a} - \varepsilon\mathbf{d_a}) \subseteq \Omega$ nor to $B_\varepsilon(\mathbf{a} + \varepsilon\mathbf{d_a}) \subseteq \mathbb{R}^n \backslash \overline{\Omega}$. Hence, we have $\|\mathbf{x} - \mathbf{a} \mp \varepsilon\mathbf{d_a}\| \geqslant \varepsilon$. Squaring these two inequalities, we obtain that (A2) holds true:

$$\|\mathbf{x} - \mathbf{a}\|^2 \geqslant 2\varepsilon |\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a} \rangle| \Longleftrightarrow |\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a} \rangle|^2 - 2\varepsilon |\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a} \rangle| + \|(\mathbf{x} - \mathbf{a})'\|^2 \geqslant 0.$$

It is a second-order polynomial inequality and we assume that its reduced discriminant is positive: $\Delta' = \varepsilon^2 - \|(\mathbf{x} - \mathbf{a})'\|^2 > 0$. Hence, the unknown cannot be located between the two roots: either $|\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a} \rangle| \leqslant \varepsilon - \sqrt{\Delta'}$ or $|\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a} \rangle| \geqslant \varepsilon + \sqrt{\Delta'}$. We assume $|\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a} \rangle| < \varepsilon$ and the last case cannot hold true. Squaring the remaining relation, we get the local inequality (A3) of the statement: $\|\mathbf{x} - \mathbf{a}\|^2 = |\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a} \rangle|^2 + \|(\mathbf{x} - \mathbf{a})'\|^2 \leqslant 2\varepsilon^2 - 2\varepsilon\sqrt{\varepsilon^2 - \|(\mathbf{x} - \mathbf{a})'\|^2}$. □

A.2 *The uniform ball condition and the compact $C^{1,1}$-hypersurfaces*

In this section, Theorem 2.6 is proved. First, we show $\partial\Omega$ can be considered locally as the graph of a function whose $C^{1,1}$-regularity is then established. Finally, we prove that the converse statement holds true in the compact case. Hence, it is the optimal regularity we can expect from the uniform ball property. The proofs in Sections A.1.2 and A.2.1–A.2.2 inspire those of Sections 3.2–3.4.

A.2.1 *A local parametrization of the boundary $\partial\Omega$.* We now set $\varepsilon > 0$ and assume that the open set Ω satisfies the ε-ball condition. Since $\Omega \notin \{\emptyset, \mathbb{R}^n\}$, $\partial\Omega$ is not empty so we consider any point $\mathbf{x_0} \in \partial\Omega$ and its unique vector $\mathbf{d_{x_0}}$ from Proposition A.4. We choose a basis $\mathcal{B}_{\mathbf{x_0}}$ of the hyperplane $\mathbf{d}_{\mathbf{x_0}}^{\perp}$ so that $(\mathbf{x_0}, \mathcal{B}_{\mathbf{x_0}}, \mathbf{d_{x_0}})$ is a direct orthonormal frame. Inside this frame, any point $\mathbf{x} \in \mathbb{R}^n$ is of the form $(\mathbf{x}', x_n)$ such that $\mathbf{x}' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. The zero vector $\mathbf{0}$ of $\mathbb{R}^n$ is now identified with $\mathbf{x_0}$ so we have $B_\varepsilon(\mathbf{0}', -\varepsilon) \subseteq \Omega$ and $B_\varepsilon(\mathbf{0}', \varepsilon) \subseteq \mathbb{R}^n \backslash \overline{\Omega}$.

Proposition A.6 *The following maps $\varphi^{\pm}$ are well defined on $D_\varepsilon(\mathbf{0}') = \{\mathbf{x}' \in \mathbb{R}^{n-1},\ \|\mathbf{x}'\| < \varepsilon\}$:*

$$\begin{cases} \varphi^{+}: & \mathbf{x}' \in D_\varepsilon(\mathbf{0}') \longmapsto \sup\{x_n \in [-\varepsilon,\varepsilon],\ (\mathbf{x}',x_n) \in \Omega\} \in\ ]-\varepsilon,\varepsilon[, \\ \varphi^{-}: & \mathbf{x}' \in D_\varepsilon(\mathbf{0}') \longmapsto \inf\{x_n \in [-\varepsilon,\varepsilon],\ (\mathbf{x}',x_n) \in \mathbb{R}^n\backslash\overline{\Omega}\} \in\ ]-\varepsilon,\varepsilon[. \end{cases}$$

Moreover, for any $\mathbf{x}' \in D_\varepsilon(\mathbf{0}')$, introducing the points $\mathbf{x}^{\pm} = (\mathbf{x}', \varphi^{\pm}(\mathbf{x}'))$, we have $\mathbf{x}^{\pm} \in \partial\Omega$ and:

$$|\varphi^{\pm}(\mathbf{x}')| \leqslant \frac{1}{2\varepsilon}\|\mathbf{x}^{\pm} - \mathbf{x}_0\|^2 \leqslant \varepsilon - \sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2}. \tag{A4}$$

Proof. Let $\mathbf{x}' \in D_\varepsilon(\mathbf{0}')$ and $g : t \in [-\varepsilon,\varepsilon] \mapsto (\mathbf{x}', t)$. Since $-\varepsilon \in g^{-1}(\Omega) \subseteq [-\varepsilon,\varepsilon]$, we can set $\varphi^{+}(\mathbf{x}') = \sup g^{-1}(\Omega)$. The map g is continuous so $g^{-1}(\Omega)$ is open and $\varphi^{+}(\mathbf{x}') \neq \varepsilon$ thus we get $\varphi(\mathbf{x}') \notin g^{-1}(\Omega)$ i.e. $\mathbf{x}^{+} \in \overline{\Omega}\backslash\Omega$. Similarly, the map φ^{-} is well defined and $\mathbf{x}^{-} \in \partial\Omega$. Finally, we use (A2) and (A3) on the points $\mathbf{x}_0$ and $\mathbf{x} = \mathbf{x}^{\pm}$ in order to obtain (A4). □

Lemma A.7 *Let $r = \frac{\sqrt{3}}{2}\varepsilon$ and $\mathbf{x}' \in D_r(\mathbf{0}')$. We assume that there exists $x_n \in]-\varepsilon,\varepsilon[$ such that $\mathbf{x} = (\mathbf{x}', x_n) \in \partial\Omega$ and $\tilde{x}_n \in \mathbb{R}$ such that $|\tilde{x}_n| \leqslant \varepsilon - \sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2}$. Then, we introduce $\tilde{\mathbf{x}} = (\mathbf{x}', \tilde{x}_n)$ and the following implications hold true: $(\tilde{x}_n < x_n \Longrightarrow \tilde{\mathbf{x}} \in \Omega)$* and *$(\tilde{x}_n > x_n \Longrightarrow \tilde{\mathbf{x}} \in \mathbb{R}^n\backslash\overline{\Omega})$.*

Proof. Let $\mathbf{x}' \in D_r(\mathbf{0}')$. Since $\tilde{\mathbf{x}} - \mathbf{x} = (\tilde{x}_n - x_n)\mathbf{d}_{\mathbf{x}_0}$, if we assume $\tilde{x}_n > x_n$, then we have:

$$\begin{aligned} \|\tilde{\mathbf{x}} - \mathbf{x} - \varepsilon\mathbf{d}_{\mathbf{x}}\|^2 - \varepsilon^2 &= |\tilde{x}_n - x_n|\left(|\tilde{x}_n - x_n| + \varepsilon\|\mathbf{d}_{\mathbf{x}} - \mathbf{d}_{\mathbf{x}_0}\|^2 - 2\varepsilon\right) \\ &\leqslant |\tilde{x}_n - x_n|\left(|\tilde{x}_n| + |x_n| + \frac{1}{\varepsilon}\|\mathbf{x} - \mathbf{x}_0\|^2 - 2\varepsilon\right) \\ &\leqslant |\tilde{x}_n - x_n|\left(2\varepsilon - 4\sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2}\right) < |\tilde{x}_n - x_n|(2\varepsilon - 4\sqrt{\varepsilon^2 - r^2}) = 0. \end{aligned}$$

Indeed, we used (A1) with $\mathbf{x} \in \partial\Omega$ and $\mathbf{y} = \mathbf{x}_0$, (A2) and (A3) applied to $\mathbf{x} \in \partial\Omega$ and $\mathbf{a} = \mathbf{x}_0$, and also the hypothesis made on $\tilde{x}_n$. Hence, we proved that if $\tilde{x}_n > x_n$, then $\tilde{\mathbf{x}} \in B_\varepsilon(\mathbf{x} + \varepsilon\mathbf{d}_{\mathbf{x}}) \subseteq \mathbb{R}^n\backslash\overline{\Omega}$. Similarly, one can prove that if $\tilde{x}_n < x_n$, then we have $\tilde{\mathbf{x}} \in B_\varepsilon(\mathbf{x} - \varepsilon\mathbf{d}_{\mathbf{x}}) \subseteq \Omega$. □

Proposition A.8 *Set $r = \frac{\sqrt{3}}{2}\varepsilon$. Then, the two maps $\varphi^{\pm}$ of Proposition A.6 coincide on $D_r(\mathbf{0}')$. We denote by φ their common restriction. Moreover, we have $\varphi(\mathbf{0}') = 0$ and also:*

$$\begin{cases} \partial\Omega \cap (D_r(\mathbf{0}')\times]-\varepsilon,\varepsilon[) = \left\{(\mathbf{x}', \varphi(\mathbf{x}')),\quad \mathbf{x}' \in D_r(\mathbf{0}')\right\} \\ \Omega \cap (D_r(\mathbf{0}')\times]-\varepsilon,\varepsilon[) = \left\{(\mathbf{x}', x_n),\quad \mathbf{x}' \in D_r(\mathbf{0}') \text{ and } -\varepsilon < x_n < \varphi(\mathbf{x}')\right\}. \end{cases}$$

Proof. Assume by contradiction that there exists $\mathbf{x}' \in D_r(\mathbf{0}')$ such that $\varphi^{-}(\mathbf{x}') \neq \varphi^{+}(\mathbf{x}')$. We set $\mathbf{x} = (\mathbf{x}', \varphi^{+}(\mathbf{x}'))$ and $\tilde{\mathbf{x}} = (\mathbf{x}', \varphi^{-}(\mathbf{x}'))$. By using (A4), the hypothesis of Lemma A.7 are satisfied for $\mathbf{x}$ and $\tilde{\mathbf{x}}$. Hence, either $(\varphi^{-}(\mathbf{x}') < \varphi^{+}(\mathbf{x}') \Rightarrow \tilde{\mathbf{x}} \in \Omega)$ or $(\varphi^{-}(\mathbf{x}') > \varphi^{+}(\mathbf{x}') \Rightarrow \tilde{\mathbf{x}} \in \mathbb{R}^n\backslash\overline{\Omega})$ whereas $\tilde{\mathbf{x}} \in \partial\Omega$. We deduce $\varphi^{-}(\mathbf{x}') = \varphi^{+}(\mathbf{x}')$ for any $\mathbf{x}' \in D_r(\mathbf{0}')$. Now consider $\mathbf{x}' \in D_r(\mathbf{0}')$ and $x_n \in]-\varepsilon,\varepsilon[$. We set $\mathbf{x} = (\mathbf{x}', \varphi(\mathbf{x}'))$ and $\tilde{\mathbf{x}} = (\mathbf{x}', x_n)$. If $x_n = \varphi(\mathbf{x}')$, then Proposition A.6 ensures that $\mathbf{x} \in \partial\Omega$. Moreover, if $-\varepsilon < x_n < -\varepsilon + \sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2}$, then $\tilde{\mathbf{x}} \in B_\varepsilon(\mathbf{0}', -\varepsilon) \subseteq \Omega$, and if $-\varepsilon + \sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2} \leqslant x_n < \varphi(\mathbf{x}')$, then apply Lemma A.7 to get $\tilde{\mathbf{x}} \in \Omega$. Consequently, we proved $(-\varepsilon < x_n < \varphi(\mathbf{x}') \Longrightarrow (\mathbf{x}', x_n) \in \Omega)$ for any $\mathbf{x}' \in D_r(\mathbf{0}')$. Similar arguments hold true when $\varepsilon > x_n > \varphi(\mathbf{x}')$ and imply $(\mathbf{x}', x_n) \in \mathbb{R}^n\backslash\overline{\Omega}$. To conclude, note that $\mathbf{x}_0 = \mathbf{0} = (\mathbf{0}', \varphi(\mathbf{0}'))$. □

A.2.2 *The $C^{1,1}$-regularity of the local graph.*

Lemma A.9 *The map $f : \alpha \in]0, \frac{\pi}{2}[\mapsto \frac{2\alpha}{\cos\alpha} \in]0, +\infty[$ is well defined, continuous, surjective and increasing. In particular, it is an homeomorphism and its inverse f^{-1} satisfies:*

$$\forall \varepsilon > 0, \quad f^{-1}(\varepsilon) < \frac{\varepsilon}{2}. \tag{A5}$$

Proof. The proof is basic calculus. □

Proposition A.10 (Point (i) of Theorem 2.6) *Consider any $\alpha \in]0, f^{-1}(\varepsilon)]$ where f is defined in Lemma A.9. Then, we have $C_\alpha(\mathbf{x}, -\mathbf{d}_{\mathbf{x}_0}) \subseteq \Omega$ for any $\mathbf{x} \in B_\alpha(\mathbf{x}_0) \cap \overline{\Omega}$. In particular, the set Ω satisfies the $f^{-1}(\varepsilon)$-cone property in the sense of Definition 2.3.*

Proof. We set $r = \frac{\sqrt{3}}{2}\varepsilon$ and $\mathfrak{C}_{r,\varepsilon} = D_r(\mathbf{0}')\times]-\varepsilon, \varepsilon[$. We choose any $\alpha \in]0, f^{-1}(\varepsilon)]$ then consider $\mathbf{x} = (\mathbf{x}', x_n) \in B_\alpha(\mathbf{x}_0) \cap \overline{\Omega}$ and $\mathbf{y} = (\mathbf{y}', y_n) \in C_\alpha(\mathbf{x}, -\mathbf{d}_{\mathbf{x}_0})$. The proof of the assertion $\mathbf{y} \in \Omega$ is divided into three steps:

- check that $\mathbf{x} \in \mathfrak{C}_{r,\varepsilon}$ so as to introduce the point $\tilde{\mathbf{x}} = (\mathbf{x}', \varphi(\mathbf{x}'))$ of $\partial\Omega$ satisfying $x_n \leqslant \varphi(\mathbf{x}')$;
- consider $\tilde{\mathbf{y}} = (\mathbf{y}', y_n + \varphi(\mathbf{x}') - x_n)$ and prove that $\tilde{\mathbf{y}} \in C_\alpha(\tilde{\mathbf{x}}, -\mathbf{d}_{\mathbf{x}_0}) \subseteq B_\varepsilon(\tilde{\mathbf{x}} - \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}}) \subseteq \Omega$;
- show that $(\tilde{\mathbf{y}}, \mathbf{y}) \in \mathfrak{C}_{r,\varepsilon} \times \mathfrak{C}_{r,\varepsilon}$ in order to deduce $y_n + \varphi(\mathbf{x}') - x_n < \varphi(\mathbf{y}')$ and conclude $\mathbf{y} \in \Omega$.

First, from (A5), we have: $\max(\|\mathbf{x}'\|, |x_n|) \leqslant \|\mathbf{x} - \mathbf{x}_0\| < \alpha \leqslant f^{-1}(\varepsilon) < \frac{\varepsilon}{2}$. Hence, we get $\mathbf{x} \in \overline{\Omega} \cap \mathfrak{C}_{r,\varepsilon}$ and applying Proposition A.8, it comes $x_n \leqslant \varphi(\mathbf{x}')$. We set $\tilde{\mathbf{x}} = (\mathbf{x}', \varphi(\mathbf{x}')) \in \partial\Omega \cap \mathfrak{C}_{r,\varepsilon}$. Note that $\tilde{\mathbf{x}} \in B_{\alpha\sqrt{2}}(\mathbf{x}_0)$ because Relation (A4) applied to $\tilde{\mathbf{x}} = (\mathbf{x}', \varphi(\mathbf{x}'))$ gives:

$$\|\tilde{\mathbf{x}} - \mathbf{x}_0\|^2 \leqslant 2\varepsilon^2 - 2\varepsilon\sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2} = \frac{4\varepsilon^2\|\mathbf{x}'\|^2}{2\varepsilon^2 + 2\varepsilon\sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2}} \leqslant 2\|\mathbf{x}'\|^2 \leqslant 2\|\mathbf{x} - \mathbf{x}_0\|^2 < 2\alpha^2.$$

Then, we prove $C_\alpha(\tilde{\mathbf{x}}, -\mathbf{d}_{\mathbf{x}_0}) \subseteq B_\varepsilon(\tilde{\mathbf{x}} - \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}})$ so consider any point $\mathbf{z} \in C_\alpha(\tilde{\mathbf{x}}, -\mathbf{d}_{\mathbf{x}_0})$. Using the Cauchy–Schwartz inequality, (A1) applied to $\tilde{\mathbf{x}} \in \partial\Omega$ and $\mathbf{y} = \mathbf{x}_0$, the fact that $\mathbf{z} \in C_\alpha(\tilde{\mathbf{x}}, -\mathbf{d}_{\mathbf{x}_0})$, and the foregoing observation $\tilde{\mathbf{x}} \in B_{\alpha\sqrt{2}}(\mathbf{x}_0)$, we have successively:

$$\begin{aligned}
\|\mathbf{z} - \tilde{\mathbf{x}} + \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}}\|^2 - \varepsilon^2 &\leqslant \|\mathbf{z} - \tilde{\mathbf{x}}\|^2 + 2\varepsilon\|\mathbf{z} - \tilde{\mathbf{x}}\|\|\mathbf{d}_{\tilde{\mathbf{x}}} - \mathbf{d}_{\mathbf{x}_0}\| + 2\varepsilon\left\langle \mathbf{z} - \tilde{\mathbf{x}} \mid \mathbf{d}_{\mathbf{x}_0}\right\rangle \\
&< \|\mathbf{z} - \tilde{\mathbf{x}}\|^2 + 2\|\mathbf{z} - \tilde{\mathbf{x}}\|\|\tilde{\mathbf{x}} - \mathbf{x}_0\| - 2\varepsilon\|\mathbf{z} - \tilde{\mathbf{x}}\|\cos\alpha \\
&< \|\mathbf{z} - \tilde{\mathbf{x}}\| \left[(1 + 2\sqrt{2})\alpha - 2\varepsilon\cos\alpha\right] < 2\|\mathbf{z} - \tilde{\mathbf{x}}\|\cos\alpha\,(f(\alpha) - \varepsilon) \leqslant 0.
\end{aligned}$$

Hence, we get $\mathbf{z} \in B_\varepsilon(\tilde{\mathbf{x}} - \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}})$, i.e., $C_\alpha(\tilde{\mathbf{x}}, -\mathbf{d}_{\mathbf{x}_0}) \subseteq B_\varepsilon(\tilde{\mathbf{x}} - \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}}) \subseteq \Omega$ using the ε-ball condition. Moreover, since $\tilde{\mathbf{y}} - \tilde{\mathbf{x}} = \mathbf{y} - \mathbf{x}$ and $\mathbf{y} \in C_\alpha(\mathbf{x}, -\mathbf{d}_{\mathbf{x}_0})$, we obtain $\tilde{\mathbf{y}} \in C_\alpha(\tilde{\mathbf{x}}, -\mathbf{d}_{\mathbf{x}_0})$ and thus $\tilde{\mathbf{y}} \in \Omega$. Finally, we show that $(\mathbf{y}, \tilde{\mathbf{y}}) \in \mathfrak{C}_{r,\varepsilon} \times \mathfrak{C}_{r,\varepsilon}$. We have successively:

$$\begin{cases}
\|\mathbf{y}'\| \leqslant \|\mathbf{y}' - \mathbf{x}'\| + \|\mathbf{x}'\| < \sqrt{\alpha^2 - \alpha^2\cos^2\alpha} + \alpha = \frac{\alpha}{\cos\alpha}\left(\frac{\sin 2\alpha}{2} + \cos\alpha\right) \leqslant \frac{3f(\alpha)}{4} \leqslant \frac{3\varepsilon}{4} < r \\
|y_n| \leqslant |y_n - x_n| + |x_n| \leqslant \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{x}_0\| < 2\alpha < f(\alpha) \leqslant \varepsilon \\
|y_n + \varphi(\mathbf{x}') - x_n| \leqslant \|\mathbf{y} - \mathbf{x}\| + \varepsilon - \sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2} < \alpha + \frac{\|\mathbf{x}'\|^2}{\varepsilon + \sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2}} \leqslant \alpha + \frac{\alpha^2}{\varepsilon} < \frac{3}{2}\alpha \leqslant \varepsilon.
\end{cases}$$

We used (A4)–(A5), the fact that $\mathbf{y} \in C_\alpha(\mathbf{x}, -\mathbf{d}_{\mathbf{x}_0})$, and $\mathbf{x} \in B_\alpha(\mathbf{x}_0)$. To conclude, Proposition A.8 applied to $\tilde{\mathbf{y}} \in \Omega \cap \mathfrak{C}_{r,\varepsilon}$ yields to $y_n + \varphi(\mathbf{x}') - x_n < \varphi(\mathbf{y}')$. Since we firstly proved $x_n \leqslant \varphi(\mathbf{x}')$, we deduce that $y_n < \varphi(\mathbf{y}')$. Applying Proposition A.8 to $\mathbf{y} \in \mathfrak{C}_{r,\varepsilon}$, we get $\mathbf{y} \in \Omega$ as required. □

Corollary A.11 *The map φ restricted to $D_{\frac{\sqrt{2}}{4}f^{-1}(\varepsilon)}(\mathbf{0}')$ is $\frac{1}{\tan[f^{-1}(\varepsilon)]}$-Lipschitz continuous.*

Proof. We set $\alpha = f^{-1}(\varepsilon)$, $r = \frac{\sqrt{3}}{2}\varepsilon$, and $\tilde{r} = \frac{\sqrt{2}}{4}f^{-1}(\varepsilon)$. We choose $(\mathbf{x}'_+, \mathbf{x}'_-) \in D_{\tilde{r}}(\mathbf{0}') \times D_{\tilde{r}}(\mathbf{0}')$. From (A5), we get $\tilde{r} < r$ so we can consider $\mathbf{x}_\pm = (\mathbf{x}'_\pm, \varphi(\mathbf{x}'_\pm))$ and Proposition A.6 gives:

$$\|\mathbf{x}_\pm - \mathbf{x_0}\|^2 \leqslant 2\varepsilon^2 - 2\varepsilon\sqrt{\varepsilon^2 - \|\mathbf{x}'_\pm\|^2} = \frac{4\varepsilon^2\|\mathbf{x}'_\pm\|^2}{2\varepsilon^2 + 2\varepsilon\sqrt{\varepsilon^2 - \|\mathbf{x}'_\pm\|^2}} \leqslant 2\|\mathbf{x}'_\pm\|^2 < 2\tilde{r}^2 < \alpha^2.$$

Hence, we get $\mathbf{x}_\pm \in B_\alpha(\mathbf{x_0}) \cap \partial\Omega$. We also have: $\|\mathbf{x}_+ - \mathbf{x}_-\| \leqslant \|\mathbf{x}_+ - \mathbf{x_0}\| + \|\mathbf{x_0} - \mathbf{x}_-\| < 2\tilde{r}\sqrt{2} = \alpha$. Finally, applying Proposition A.10, the points $\mathbf{x}_\pm$ cannot belong to the cones $C_\alpha(\mathbf{x}_\mp, -\mathbf{d}_{\mathbf{x_0}}) \subseteq \Omega$ thus we get: $|\langle \mathbf{x}_+ - \mathbf{x}_- \mid \mathbf{d}_{\mathbf{x_0}}\rangle| \leqslant \cos\alpha\|\mathbf{x}_+ - \mathbf{x}_-\| = \cos\alpha\sqrt{\|\mathbf{x}'_+ - \mathbf{x}'_-\|^2 + |\langle \mathbf{x}_+ - \mathbf{x}_- \mid \mathbf{d}_{\mathbf{x_0}}\rangle|^2}$. Consequently, one can re-arrange these terms in order to obtain the expected result of the statement: $|\varphi(\mathbf{x}'_+) - \varphi(\mathbf{x}'_-)| = |\langle \mathbf{x}_+ - \mathbf{x}_- \mid \mathbf{d}_{\mathbf{x_0}}\rangle| \leqslant \frac{1}{\tan\alpha}\|\mathbf{x}'_+ - \mathbf{x}'_-\|$. ☐

Proposition A.12 *Set $\tilde{r} = \frac{\sqrt{2}}{4}f^{-1}(\varepsilon)$. The map φ of Proposition A.8 restricted to $D_{\tilde{r}}(\mathbf{0}')$ is differentiable and its gradient $\nabla\varphi : D_{\tilde{r}}(\mathbf{0}') \to \mathbb{R}^{n-1}$ is L-Lipschitz continuous where $L > 0$ depends only on ε. Moreover, we have $\nabla\varphi(\mathbf{0}') = \mathbf{0}'$ and also:*

$$\forall \mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}'), \quad \nabla\varphi(\mathbf{a}') = \frac{-1}{\langle \mathbf{d_a} \mid \mathbf{d}_{\mathbf{x_0}}\rangle}\mathbf{d}'_{\mathbf{a}}, \qquad \text{where} \quad \mathbf{a} = (\mathbf{a}', \varphi(\mathbf{a}')).$$

Furthermore, the gradient map $\nabla\varphi : D_{\tilde{r}}(\mathbf{0}') \to \mathbb{R}^{n-1}$ is bounded and valued in the set $D_{\frac{32}{31}}(\mathbf{0}')$.

Proof. Let $\mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}')$ and $\mathbf{x}' \in \overline{D_{\tilde{r}-\|\mathbf{a}'\|}(\mathbf{a}')}$. Consequently, we have $(\mathbf{a}', \mathbf{x}') \in D_{\tilde{r}}(\mathbf{0}') \times D_{\tilde{r}}(\mathbf{0}')$ and from (A5), we get $\tilde{r} < \frac{\sqrt{3}}{2}\varepsilon$. Hence, using Proposition A.8, we can introduce $\mathbf{x} := (\mathbf{x}', \varphi(\mathbf{x}'))$ and $\mathbf{a} := (\mathbf{a}', \varphi(\mathbf{a}'))$. Applying (A2) to $(\mathbf{a}, \mathbf{x}) \in \partial\Omega \times \partial\Omega$ and using the Lipschitz continuity of φ on $D_{\tilde{r}}(\mathbf{0}')$ proved in Corollary A.11, we deduce that:

$$\left| \big(\varphi(\mathbf{x}') - \varphi(\mathbf{a}')\big)\,\mathbf{d}_{\mathbf{a}n} + \langle \mathbf{d}'_{\mathbf{a}} \mid \mathbf{x}' - \mathbf{a}'\rangle \right| \leqslant \frac{1}{2\varepsilon}\|\mathbf{x} - \mathbf{a}\|^2 \leqslant \underbrace{\frac{1}{2\varepsilon}\left(1 + \frac{1}{\tan^2[f^{-1}(\varepsilon)]}\right)}_{:=C(\varepsilon)>0}\|\mathbf{x}' - \mathbf{a}'\|^2,$$

where we set $\mathbf{d_a} = (\mathbf{d}'_{\mathbf{a}}, \mathbf{d}_{\mathbf{a}n})$ with $\mathbf{d}_{\mathbf{a}n} = \langle \mathbf{d_a} \mid \mathbf{d}_{\mathbf{x_0}}\rangle$. It represents a first-order Taylor expansion of the map φ if we can divide the above inequality by a uniform positive constant smaller than $\mathbf{d}_{\mathbf{a}n}$. Let us justify this assertion. Apply (A1) to $\mathbf{x} = \mathbf{a}$ and $\mathbf{y} = \mathbf{x_0}$, then use (A4) to get:

$$\mathbf{d}_{\mathbf{a}n} = 1 - \frac{1}{2}\|\mathbf{d_a} - \mathbf{d}_{\mathbf{x_0}}\|^2 \geqslant 1 - \frac{1}{2\varepsilon^2}\|\mathbf{a} - \mathbf{x_0}\|^2 \geqslant 1 - \frac{\varepsilon - \sqrt{\varepsilon^2 - \|\mathbf{a}'\|^2}}{\varepsilon} = 1 - \frac{\|\mathbf{a}'\|^2}{\varepsilon(\varepsilon + \sqrt{\varepsilon^2 - \|\mathbf{a}'\|^2})}.$$

Hence, using (A5), we obtain $\mathbf{d}_{\mathbf{a}n} > 1 - \frac{\tilde{r}^2}{\varepsilon^2} > \frac{31}{32} > 0$. Therefore, φ is a differentiable map at any point $\mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}')$ and its gradient is the one given in the statement:

$$\forall \mathbf{x}' \in \overline{D_{\tilde{r}-\|\mathbf{a}'\|}(\mathbf{a}')}, \quad \left| \varphi(\mathbf{x}') - \varphi(\mathbf{a}') + \left\langle \frac{\mathbf{d}'_{\mathbf{a}}}{\mathbf{d}_{\mathbf{a}n}} \mid \mathbf{x}' - \mathbf{a}' \right\rangle \right| \leqslant \frac{32}{31}C(\varepsilon)\|\mathbf{x}' - \mathbf{a}'\|^2.$$

Moreover, for any $(\mathbf{a}', \mathbf{x}') \in D_{\tilde{r}}(\mathbf{0}') \times D_{\tilde{r}}(\mathbf{0}')$, we have successively:

$$\|\nabla\varphi(\mathbf{x}') - \nabla\varphi(\mathbf{a}')\| \leqslant \left| \frac{1}{\mathbf{d}_{\mathbf{a}n}} - \frac{1}{\mathbf{d}_{\mathbf{x}n}} \right| \|\mathbf{d}'_{\mathbf{x}}\| + \left| \frac{1}{\mathbf{d}_{\mathbf{a}n}} \right| \|\mathbf{d}'_{\mathbf{a}} - \mathbf{d}'_{\mathbf{x}}\| \leqslant \frac{32}{31}\left(1 + \frac{32}{31}\right) \|\mathbf{d}_{\mathbf{a}} - \mathbf{d}_{\mathbf{x}}\|$$
$$\leqslant \frac{32}{31\varepsilon}\left(1 + \frac{32}{31}\right) \|\mathbf{x} - \mathbf{a}\| \leqslant \frac{32}{31\varepsilon}\left(1 + \frac{32}{31}\right) \sqrt{1 + \frac{1}{\tan^2[f^{-1}(\varepsilon)]}} \|\mathbf{x}' - \mathbf{a}'\|.$$

We applied (A1) to $\mathbf{x}$ and $\mathbf{y} = \mathbf{a}$, then used the Lipschitz continuity of φ proved in Corollary A.11. Hence, $\nabla\varphi : \mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}') \mapsto \nabla\varphi(\mathbf{a}')$ is L-Lipschitz continuous with $L > 0$ depending only on ε. To conclude, from the foregoing, we deduce $\|\nabla\varphi(\mathbf{x}')\| = |(\mathbf{d}_{\mathbf{a}n})^{-1}| \, \|\mathbf{d}'_{\mathbf{a}}\| < \frac{32}{31}\|\mathbf{d}_{\mathbf{a}}\| = \frac{32}{31}$ for any $\mathbf{x}' \in D_{\tilde{r}}(\mathbf{0}')$ and the map $\nabla\varphi : D_{\tilde{r}}(\mathbf{0}') \to \mathbb{R}^{n-1}$ is thus well valued in $D_{\frac{32}{31}}(\mathbf{0}')$. □

Corollary A.13 (Points (ii) and (iii) of Theorem 2.6) *The unit vector $\mathbf{d}_{\mathbf{x}_0}$ of Definition 1.1 is the outer normal vector to $\partial\Omega$ at the point $\mathbf{x}_0$. In particular, the $\frac{1}{\varepsilon}$-Lipschitz continuous map $\mathbf{d} : \mathbf{x} \mapsto \mathbf{d}_{\mathbf{x}}$ of Proposition A.4 is the Gauss map associated with the $C^{1,1}$-hypersurface $\partial\Omega$.*

Proof. Consider the map $\varphi : D_{\tilde{r}}(\mathbf{0}') \to]-\varepsilon, \varepsilon[$ whose $C^{1,1}$-regularity comes from Proposition A.12. We define the $C^{1,1}$-map $X : D_{\tilde{r}}(\mathbf{0}') \to \partial\Omega$ by $X(\mathbf{x}') = (\mathbf{x}', \varphi(\mathbf{x}'))$ then we consider $\mathbf{x}' \in D_{\tilde{r}}(\mathbf{0}')$. We denote by $(e_k)_{1 \leqslant k \leqslant n-1}$ the first vectors of our local basis. The tangent plane of $\partial\Omega$ at $X(\mathbf{x}')$ is spanned by the vectors $\partial_k X(\mathbf{x}') = e_k + (\mathbf{0}', \partial_k\varphi(\mathbf{x}'))$. Since any normal vector $\mathbf{u} = (u_1, \ldots, u_n)$ to this hyperplane is orthogonal to this $(n-1)$ vectors, we have: $\langle \mathbf{u} \mid \partial_k X(\mathbf{x}')\rangle = 0 \Leftrightarrow u_k = \frac{u_n}{\mathbf{d}_{\mathbf{x}n}} \mathbf{d}_{\mathbf{x}k}$. Hence, we obtain $\mathbf{u} = \frac{u_n}{\mathbf{d}_{\mathbf{x}n}} \mathbf{d}_{\mathbf{x}}$ so $\mathbf{u}$ is collinear to $\mathbf{d}_{\mathbf{x}}$. Now, if we impose that $\mathbf{u}$ points outwards Ω and if we assume $\|\mathbf{u}\| = 1$, then we get $\mathbf{u} = \mathbf{d}_{\mathbf{x}}$. □

A.2.3 *The compact case: When $C^{1,1}$-regularity implies the uniform ball condition.*

Proof of Theorem 2.6. Combining Proposition A.10 and Corollary A.13, it remains to prove the converse part of Theorem 2.6. Consider any non-empty compact $C^{1,1}$-hypersurface $\mathcal{S}$ of $\mathbb{R}^n$ and its associated inner domain Ω. Choose any $\mathbf{x}_0 \in \partial\Omega$ and its local frame as in Definition 2.2. First, we have for any $(\mathbf{x}', \mathbf{y}') \in D_r(\mathbf{0}') \times D_r(\mathbf{0}')$:

$$|\varphi(\mathbf{y}') - \varphi(\mathbf{x}') - \langle \nabla\varphi(\mathbf{x}') \mid \mathbf{y}' - \mathbf{x}'\rangle| \leqslant \int_0^1 \|\nabla\varphi\left(\mathbf{x}' + t(\mathbf{y}' - \mathbf{x}')\right) - \nabla\varphi(\mathbf{x}')\| \|\mathbf{y}' - \mathbf{x}'\| dt$$
$$\leqslant \frac{L}{2}\|\mathbf{y}' - \mathbf{x}'\|^2.$$

Then, we set $\varepsilon_0 = \min(\frac{1}{L}, \frac{r}{3}, \frac{a}{3})$ and consider any $\mathbf{x} \in B_{\varepsilon_0}(\mathbf{x}_0) \cap \partial\Omega$. Since $\varepsilon_0 \leqslant \min(r, a)$, there exists $\mathbf{x}' \in D_r(\mathbf{0}')$ such that $\mathbf{x} = (\mathbf{x}', \varphi(\mathbf{x}'))$. We introduce the notation $\mathbf{d}_{\mathbf{x}n} = (1 + \|\nabla\varphi(\mathbf{x}')\|^2)^{-\frac{1}{2}}$ and $\mathbf{d}'_{\mathbf{x}} = -\mathbf{d}_{\mathbf{x}n}\nabla\varphi(\mathbf{x}')$ so that $\mathbf{d}_{\mathbf{x}} := (\mathbf{d}'_{\mathbf{x}}, \mathbf{d}_{\mathbf{x}n})$ is a unit vector. Now, let us show that Ω satisfy the ε_0-ball condition at the point $\mathbf{x}$ so choose any $\mathbf{y} \in B_{\varepsilon_0}(\mathbf{x} + \varepsilon_0\mathbf{d}_{\mathbf{x}}) \subseteq B_{2\varepsilon_0}(\mathbf{x}) \subseteq B_{3\varepsilon_0}(\mathbf{x}_0)$. Since $3\varepsilon_0 \leqslant \min(r, a)$, there exists $\mathbf{y}' \in D_r(\mathbf{0}')$ and $y_n \in]-a, a[$ such that $\mathbf{y} = (\mathbf{y}', y_n)$. Moreover, we have $\mathbf{y} \in \mathbb{R}^n \backslash \overline{\Omega}$ iff $y_n > \varphi(\mathbf{y}')$. Observing that $\|\mathbf{y} - \mathbf{x} - \varepsilon_0\mathbf{d}_{\mathbf{x}}\| < \varepsilon_0 \Leftrightarrow \frac{1}{2\varepsilon_0}\|\mathbf{y} - \mathbf{x}\|^2 < \langle \mathbf{y} - \mathbf{x} \mid \mathbf{d}_{\mathbf{x}}\rangle$, we obtain successively:

$$
\begin{aligned}
y_n - \varphi(\mathbf{y}') &= \frac{1}{\mathbf{d}_{\mathbf{x}n}}\Big[\mathbf{d}_{\mathbf{x}n}(y_n - \varphi(\mathbf{x}')) + \langle \mathbf{d}'_{\mathbf{x}} \mid \mathbf{y}' - \mathbf{x}'\rangle - \langle \mathbf{d}'_{\mathbf{x}} \mid \mathbf{y}' - \mathbf{x}'\rangle + \mathbf{d}_{\mathbf{x}n}\left(\varphi(\mathbf{x}') - \varphi(\mathbf{y}')\right)\Big] \\
&= \frac{1}{\mathbf{d}_{\mathbf{x}n}}\langle \mathbf{y} - \mathbf{x} \mid \mathbf{d}_{\mathbf{x}}\rangle - \varphi(\mathbf{y}') + \varphi(\mathbf{x}') + \langle \nabla\varphi(\mathbf{x}') \mid \mathbf{y}' - \mathbf{x}'\rangle \\
&> \frac{\|\mathbf{y} - \mathbf{x}\|^2}{2\varepsilon_0 \mathbf{d}_{\mathbf{x}n}} - \frac{L}{2}\|\mathbf{y}' - \mathbf{x}'\|^2 > \frac{1}{2}\|\mathbf{y}' - \mathbf{x}'\|^2\left(\frac{1}{\varepsilon_0} - L\right) \geqslant 0.
\end{aligned}
$$

Consequently, we get $\mathbf{y} \notin \overline{\Omega}$ and we proved $B_{\varepsilon_0}(\mathbf{x} + \varepsilon_0\mathbf{d}_{\mathbf{x}}) \subseteq \mathbb{R}^n \backslash \overline{\Omega}$. Similarly, we can obtain $B_{\varepsilon_0}(\mathbf{x} - \varepsilon_0\mathbf{d}_{\mathbf{x}}) \subseteq \Omega$. Hence, for any $\mathbf{x}_0 \in \partial\Omega$, there exists $\varepsilon_0 > 0$ such that $\Omega \cap B_{\varepsilon_0}(\mathbf{x}_0)$ satisfies the ε_0-ball condition. Finally, as $\partial\Omega$ is compact, it is included in a finite reunion of such balls $B_{\varepsilon_0}(\mathbf{x}_0)$. Define $\varepsilon > 0$ as the minimum of this finite number of ε_0 and Ω will satisfy the ε-ball property. □

A.3 *The uniform ball property and the oriented distance functions*

Proof of Theorem 2.7. Let $\Omega \subset \mathbb{R}^n$ be open with $\partial\Omega \neq \emptyset$. First, from [20, Theorem 5.1 (i)], the oriented distance function $b_\Omega : \mathbb{R}^n \to \mathbb{R}$ is 1-Lipschitz continuous. Using Rademacher's Theorem [29, Section 3.1.2], b_Ω is differentiable almost everywhere with $\|\nabla b_\Omega\|_{L^\infty(\mathbb{R}^n)} \leqslant 1$. Then, we assume that there exists $\varepsilon > 0$ such that $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$. Let $\mathbf{x} \in \partial\Omega$. Following the arguments used in the proof of the point (i) of Theorem 2.5 (cf. Section A.1.2), we get for any $\mathbf{y} \in B_\varepsilon(\mathbf{x})$, that there exists a unique projection $p_{\partial\Omega}(\mathbf{y})$ on $\partial\Omega$ satisfying:

$$
\forall \mathbf{y} \in B_\varepsilon(\mathbf{x}), \quad p_{\partial\Omega}(\mathbf{y}) = \mathbf{y} - b_\Omega(\mathbf{y})\mathbf{d}_{p_{\partial\Omega}(\mathbf{y})}, \tag{A6}
$$

where $\mathbf{d}_{p_{\partial\Omega}(\mathbf{y})}$ is the unique vector of Proposition A.4. Consequently, combining Proposition A.4 with [20, Theorem 5.1 (i)] and $d(\mathbf{y}, \partial\Omega) \leqslant \|\mathbf{x} - \mathbf{y}\| < \varepsilon$, we deduce from (A6):

$$
\forall (\mathbf{y}, \tilde{\mathbf{y}}) \in B_\varepsilon(\mathbf{x}) \times B_\varepsilon(\mathbf{x}), \ \|p_{\partial\Omega}(\mathbf{y}) - p_{\partial\Omega}(\tilde{\mathbf{y}})\| \leqslant \frac{2\varepsilon}{\varepsilon - d(\mathbf{y}, \partial\Omega)}\|\mathbf{y} - \tilde{\mathbf{y}}\|. \tag{A7}
$$

Hence, $p_{\partial\Omega}(\mathbf{y}_i) \to p_{\partial\Omega}(\mathbf{y})$ for any $\mathbf{y}_i \to \mathbf{y} \in B_\varepsilon(\mathbf{x})$ and $p_{\partial\Omega} \in C^0(B_\varepsilon(\mathbf{x}))$. As $B_\varepsilon(\mathbf{x}) \subseteq \mathrm{Unp}(\partial\Omega)$, we apply [20, Theorem 5.1 (iv)]. We get $b_\Omega^2 \in C^1(B_\varepsilon(\mathbf{x}))$ and [20, Theorem 5.1 (ii))] ensures that $\nabla b_\Omega : B_\varepsilon(\mathbf{x})\backslash\partial\Omega \to B_1(\mathbf{0})$ is a well-defined map. From (A6) and [20, Theorem 5.1 (iii))], we obtain $\nabla b_\Omega(\mathbf{y}) = \mathbf{d}_{p_{\partial\Omega}(\mathbf{y})}$ for any $\mathbf{y} \in B_\varepsilon(\mathbf{x})\backslash\partial\Omega$. Since $V(\partial\Omega) = 0$ from Theorem 2.5 (i), we have $\nabla b_\Omega = \mathbf{d} \circ p_{\partial\Omega}$ almost everywhere on $B_\varepsilon(\mathbf{x})$. From (A7) and Proposition A.4, $\mathbf{d} \circ p_{\partial\Omega}$ is continuous on $B_\varepsilon(\mathbf{x})$ so [30, Lemma 4.7] yields $\nabla b_\Omega = \mathbf{d} \circ p_{\partial\Omega}$ everywhere on $B_\varepsilon(\mathbf{x})$ and $b_\Omega \in C^1(B_\varepsilon(\mathbf{x}))$ for any $\mathbf{x} \in \partial\Omega$ i.e. $b_\Omega \in C^1(\mathcal{V}_\varepsilon(\partial\Omega))$. Finally, if $\mathbf{y} \in B_r(\mathbf{x})$ for some $r \in]0, \varepsilon[$, we deduce from (A7) the $\frac{2\varepsilon}{\varepsilon - r}$-Lipschitz continuity of $p_{\partial\Omega} : B_r(\mathbf{x}) \to \partial\Omega$. By Proposition A.4, the map $\nabla b_\Omega : B_r(\mathbf{x}) \to \mathbb{S}^{n-1}$ is $\frac{2}{\varepsilon - r}$-Lipschitz continuous. In particular, ∇b_Ω is bounded and uniformly continuous: it has a unique Lipschitz continuous extension to $\overline{B_r(\mathbf{x})}$. Moreover, the Lipschitz constant $\frac{2}{\varepsilon - r}$ does not depend on $\mathbf{x} \in \partial\Omega$ thus $b_\Omega \in C^{1,1}(\overline{V_r(\partial\Omega)})$ for any $r \in]0, \varepsilon[$. Conversely, if there now exists $\varepsilon > 0$ such that $b_\Omega \in C^{1,1}(\overline{B_\varepsilon(\mathbf{x})})$ and $V(B_\varepsilon(\mathbf{x}) \cap \partial\Omega) = 0$ for any $\mathbf{x} \in \partial\Omega$, then we apply [24, Chapter 7 Theorem 8.3 (ii)] to obtain $B_\varepsilon(\mathbf{x}) \subseteq \mathrm{Unp}(\partial\Omega)$ i.e. $\mathrm{Reach}(\mathbf{x}, \partial\Omega) \geqslant \varepsilon$ for any $\mathbf{x} \in \partial\Omega$, from which we deduce $\mathrm{Reach}(\partial\Omega) \geqslant \varepsilon$. We can now use Theorem 2.5 (i). For this purpose, we check that $V(\partial\Omega) \leqslant \sum_{\mathbf{x} \in \partial\Omega \cap \mathbb{Q}^n} V(B_\varepsilon(\mathbf{x}) \cap \partial\Omega) = 0$ and we get $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$, concluding the proof. □

Acknowledgements. First, the author wants to warmly thank the referees who gave several important suggestions to wisely improve the paper. This work started in September 2011 as a part of a thesis directed by Antoine Henrot and Takéo Takahashi at the IECL (Institut Elie Cartan de Lorraine, Université de Lorraine BP70239 54506 Vandoeuvre-lès-Nancy Cedex, France). The author is very grateful for having given him this opportunity that now sounds as old good times. This work is also supported by the project ANR-12-BS01-0007-01-OPTIFORM *Optimisation de formes* financed by the French ANR. The final form of the paper grew while the author was under a postdoctoral position of the CMM (Centro de Modelamiento Matemático, Facultad de Ciencas Físicas y Matemáticas (FCFM), UMR 2071 CNRS-Universidad de Chile, Beauchef 851, Santiago, Chile).

References

[1] Alvarado, R., Brigham, D., Maz'ya, V., Mitrea, M., & Ziadé, E., On the regularity of domains satisfying a uniform hour-glass condition and a sharp version of the Hopf–Oleinik boundary point principle. *Journal of Mathematical Sciences* **176** (2011), 281–360. Zbl1290.35046 MR2839047

[2] Bauer, M., & Kuwert, E., Existence of minimizing Willmore surfaces of prescribed genus. *International Mathematics Research Notices* **2003** (2003), 553–576. Zbl1029.53073 MR1941840

[3] Bellettini, G., Dal Maso, G., & Paolini, M., Semicontinuity and relaxation properties of curvature depending functional in 2D. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série* **20** (1993), 247–297. Zbl0797.49013 MR1233638

[4] Bellettini, G., & Mugnai, L., Approximation of the Helfrich's functional via diffuse interfaces. *SIAM Journal on Mathematical Analysis* **42** (2010), 2402–2433. Zbl1230.49006 MR2733254

[5] Bucur, D., & Zolésio, J.-P., N-dimensional shape optimization under capacitary constraint. *Journal of Differential Equations* **123** (1995), 504–522. Zbl0847.49029 MR1362884

[6] Bucur, D., & Zolésio, J.-P., Free boundary problems and density perimeter. *Journal of Differential Equations* **126** (1996), 224–243. Zbl0856.35137 MR1383977

[7] Canham, P. B., The minimum energy of bending as a possible explanation of the biconcave shape of the human red blood cell. *Journal of Theoretical Biology* **26** (1970), 61–76.

[8] Chenais, D., On the existence of a solution in a domain identification problem. *Journal of Mathematical Analysis and Applications* **52** (1975), 189–219. Zbl0317.49005 MR0385666

[9] Choksi, R., & Veneroni, M., Global minimizers for the doubly-constrained Helfrich energy: The axisymmetric case. *Calculus of Variations and Part. Diff. Equ.* **48** (2013), 337–366. Zbl1278.49049 MR3116014

[10] Dalphin, J., Some characterizations of a uniform ball property. *ESAIM: proceedings and surveys* **45** (2014), 437–446. Zbl1338.49087 MR3451856

[11] Dalphin, J., Uniform ball condition and existence of optimal shapes for geometric functionals involving boundary-value problems. To appear.

[12] Delfour, M. C., Intrisic differential geometric methods in the asymptotic analysis of linear thin shells. In: *Boundaries, Interfaces, and Transitions* (Banff, 1995), CRM Proceedings and Lecture Notes, American Mathematical Society, Providence, (1998), 19–90. Zbl0916.73023 MR1619112

[13] Delfour, M. C., Intrinsic P(2,1) thin shell model and Naghdi's models without a priori assumption on the stress tensor. In: *Optimal Control and Partial Diff. Equations* (Chemnitz, 1998), International Series of Numerical Mathematics, Birkhäuser, Basel, (1999), 99–113. Zbl0948.74031 MR1725030

[14] Delfour, M. C., Membrane shell equations: characterization of the space of solutions. In: *Control of Distributed Parameter and Stochastic Systems* (Hangzhou, 1998), IFIP Advances in Information and Communication Technology, Chapman & Hall, New-York, (1999), 21–29. Zbl1006.74063 MR1777391

[15] Delfour, M. C., Tangential differential calculus and functional analysis on a $C^{1,1}$ submanifold. In: *Differential-Geometric Methods in the Control of Part. Diff. Equ.* (Boulder, 1999), Contemporary Mathematics, American Mathematical Society, Providence, (2000), 83–115. Zbl0977.58004 MR1804791

[16] Delfour, M. C., Characterization of the space of the membrane shell equation for arbitrary $C^{1,1}$ midsurfaces. *Control and Cybernetics* **28** (2008), 481–501. Zbl0981.74035 MR1782013

[17] Delfour, M. C., Representation of hypersurfaces and minimal smoothness of the midsurface in the theory of shells. *Control and Cybernetics* **37** (2008), 879–911. Zbl1260.53018 MR2536480

[18] Delfour, M. C., Representations, composition, and decomposition of $C^{1,1}$-hypersurfaces. In: *Optimal Control of Coupled Systems of Partial Differential Equations* (Oberwolfach, 2008), International Series of Numerical Mathematics, Birkhäuser, Basel, (2009), 85–104. Zbl 05785777 MR2588550

[19] Delfour, M. C., Metric Spaces of shapes and geometries from set parametrized functions. In: *New Trends in Shape Optimization* (Erlangen, 2015), International Series of Numerical Mathematics, Springer, Boston, (2015), 57–101. Zbl1329.49084 MR3467377

[20] Delfour, M. C., & Zolésio, J.-P., Shape analysis via oriented distance functions. *Journal of Functional Analysis* **123** (1994), 129–201. Zbl0814.49032 MR1279299

[21] Delfour, M. C., & Zolésio, J.-P., Shape analysis via distance functions: Local theory. In: *Boundaries, Interfaces, and Transitions*, CRM Proceedings and Lecture Notes, American Mathematical Society, Providence, (1998), 91–124. Zbl0955.49023 MR1619113

[22] Delfour, M. C., & Zolésio, J.-P., Oriented distance function and its evolution equation for initial sets with thin boundary. *SIAM J. on Control and Optim.* **42** (2004), 2286–2304. Zbl1058.49034 MR2081420

[23] Delfour, M. C., & Zolésio, J.-P., Uniform fat segment and cusp properties for compactness in shape optimization. *Applied Mathematics and Optimization* **55** (2007), 385–419. Zbl1119.49034 MR2314779

[24] Delfour, M. C., & Zolésio, J.-P., *Shapes and Geometries: Metrics, Analysis, Differential Calculus, and Optimization.* 2nd ed., Society for Ind. and App. Math., Philadelphia (2011). Zbl1251.49001 MR2731611

[25] Delladio, S., Special generalized Gauss graphs and their application to minimization of functional involving curvatures. *J. für die Reine und Angewandte Math.* **486** (1997), 17–43. Zbl0871.49034 MR1450749

[26] Do Carmo, M. P., *Differential Geometry of Curves and Surfaces.* Prentice-Hall (1976). Zbl0326.53001 MR0394451

[27] Dold, A., *Lectures on Algebraic Topology.* Springer Verlag, Berlin (1972). Zbl0234.55001 MR0415602

[28] Evans, L. C., *Partial Differential Equations.* American Math. Society, Providence (1998). Zbl0902.35002 MR1625845

[29] Evans, L. C., & Gariepy, R. F., *Measure Theory and Fine Properties of Functions.* CRC-Press, New-York (1992). Zbl0804.28001 MR1158660

[30] Federer, H., Curvature measures. *Trans. of the American Math. Society* **93** (1959), 418–491. Zbl0089.38402 MR0110078

[31] Fu, J. H. G., Tubular neighborhoods in Euclidean spaces. *Duke Mathematical Journal* **52** (1985), 1025–1046. Zbl0592.52002 MR0816398

[32] Ghomi, M., & Howard, R., Tangent cones and the regularity of real hypersurfaces. *J. Reine Angew. Math.* **697** (2014), 221–247. Zbl1305.53010 MR3281655

[33] Gilbarg, D., & Trudinger, N. S., *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin (2001). Zbl1042.35002 MR1814364

[34] Große-Brauckmann, K., New surfaces of constant mean curvature. *Mathematische Zeitschrift* **214** (1993), 527–565. Zbl0806.53005 MR1248112

[35] Guo, B. Z., & Yang, D. H., Some compact classes of open sets under Hausdorff distance and application to shape optimization. *SIAM J. of Control Optimization* **50** (2012), 222–242. Zbl1243.49055 MR2888263

[36] Guo, B. Z., & Yang, D. H., On convergence of boundary Hausdorff measure and application to a boundary shape optimization problem. *SIAM J. of Control Optim.* **51** (2013), 253–272. Zbl1263.49052 MR3032875

[37] Helfrich, W., The elastic properties of lipid bilayers: Theory and possible experiments. *Zeitschrift für Naturforschung C* **28** (1973), 693–703.

[38] Henrot, A. & Pierre, M., *Variation et optimisation de formes: une analyse géométrique.* Springer-Verlag, Berlin (2005). Zbl1098.49001 MR2512810

[39] Hörmander, L., Weak linear convexity and a related notion of concavity. *Mathematica Scandinavica* **102** (2008), 73–100. Zbl1158.32005 MR2420680

[40] Hutchinson, J. E., Second fundamental form for varifolds and the existence of surfaces minimizing curvature. *Indiana University Mathematical Journal* **35** (1986), 45–71. Zbl0561.53008 MR0825628

[41] Keller, L. G. A., Mondino, A., & Rivière, T., Embedded surfaces of arbitrary genus minimizing the Willmore energy under isoperimetric constraint. *Archive for Rational Mechanics and Analysis* **212** (2014), 645–682. Zbl1292.49039 MR3176354

[42] Lucas, K. R., Submanifolds of dimension $n-1$ in $\mathcal{E}^n$ with normals satisfying a Lipschitz condition. Technical Report, University of Kansas (1957). MR2938884

[43] Lytchak, A., Almost convex subsets. *Geometricae Dedicata* **115** (2005), 201–218. Zbl1088.53025 MR2180048

[44] Mardare, S., The fundamental theorem of surface theory for surfaces with little regularity. *Journal of Elasticity* **73** (2003), 251–290. Zbl1076.53003 MR2057747

[45] Marques, F. C., & Neves, A., Min-max theory and the Willmore conjecture. *Annals of Mathematics* **179** (2014), 683–782. Zbl1297.49079 MR3152944

[46] Mitrea, D., & Mitrea, M., On the well-posedness of the Dirichlet problem in certain classes of nontangentially accessible domains. In: *Analysis, Partial Differential Equations and Applications: The Vladimir Maz'ya Anniversary Volume* (Rome, 2008), Operator Theory: Advances and Applications, Birkhäuser, Basel, (2009), 171–195. Zbl1185.35061 MR2766073

[47] Mondino, A., Existence of integral m-varifolds minimizing $\int |A|^p$ and $\int |H|^p$, $p > m$, in Riemannian manifolds. *Calculus of Variations and Part. Diff. Equ.* **49** (2014), 431–470. Zbl1282.49041 MR3148123

[48] Montiel, S., & Ros, A., *Curves and Surfaces.* American Math. Society, Providence (2005). Zbl1075.53001 MR2165790

[49] Nagasawa, T., & Yi, T., Local existence and uniqueness for the n-dimensional Helfrich flow as a projected gradient flow. *Hokkaido Mathematical Journal* **41** (2012), 209–226. Zbl1247.53083 MR2977045

[50] Poincaré, H., Sur les équations aux dérivées partielles de la physique mathématique. *American Journal of Mathematics* **12** (1890), 211–294. JFM 22.0977.03 MR1505534

[51] Rivière, T., Analysis aspects of Willmore surfaces. *Inventiones Math.* **174** (2008), 1–45. Zbl1155.53031 MR2430975

[52] Schygulla, J., Willmore minimizers with prescribed isoperimetric ratio. *Archive for Rational Mechanics and Analysis* **203** (2012), 901–941. Zbl1288.74027 MR2928137

[53] Seifert, U., Configurations of fluid membranes and vesicles. *Adv. Phys.* **46** (1997), 13–137.

[54] Seifert, U., Berndl, K., & Lipowsky, R., Shape transformations of vesicles: Diagram for spontaneous-curvature and bilayer-coupling models. *Phys. Rev. A* **44** (1991), 1182–1202.

[55] Simon, L., *Lectures On Geometric Measure Theory.* Australian National University (1984). Zbl0546.49019 MR0756417

[56] Simon, L., Existence of surfaces minimizing the Willmore functional. *Communications in Analysis and Geometry* **1** (1993), 281–326. Zbl0848.58012 MR1243525

[57] Tartar, L., Compensated compactness and applications to partial differential equations. In: *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium*, volume IV, Research Notes in Mathematics, Pitman Press, London (1980). Zbl0437.35004 MR0584398

[58] Tiba, D., A property of Sobolev spaces and existence in optimal design. *Applied Mathematics and Optimization* **47** (2002), 45–58. Zbl1012.49030 MR1941911

[59] Lim, G. H. W., Wortis, M., & Mukhopadhyay, R., *Lipid Bilayers and Red Blood Cells.* Wiley-VCH, Weinheim (2008).

[60] Willmore, T. J., *Total Curvature in Riemannian Geometry.* Ellis Horwood (1982). Zbl0501.53038 MR0686105