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Approximation of sets of finite fractional perimeter by smooth sets and comparison of local and global s-minimal surfaces

LUCA LOMBARDINI
Università degli Studi di Milano, Via Cesare Saldini, 50, 20133 Milano, Italia
and
Université de Picardie Jules Verne, 33, rue Saint-Leu, 80039 Amiens CEDEX 1, France
E-mail: luca.lombardini@unimi.it

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This article is divided into two parts. In the first part we show that a set E has locally finite s-perimeter if and only if it can be approximated in an appropriate sense by smooth open sets. In the second part we prove some elementary properties of local and global s-minimal sets, such as existence and compactness. We also compare the two notions of minimizer (i.e., local and global), showing that in bounded open sets with Lipschitz boundary they coincide. Conversely, in general this is not true in unbounded open sets, where a global s-minimal set may fail to exist (we provide an example in the case of a cylinder $\Omega \times \mathbb{R}$).

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1. Introduction and main results

The aim of this paper consists in better understanding the behavior of the family of sets having (locally) finite fractional perimeter. In particular, we would like to show that this family is not "too different" from the family of Caccioppoli sets (which are the sets having locally finite classical perimeter).

This paper somehow continues the study started in [16]. In particular, we showed there (following an idea appeared in the seminal paper [20]) that sets having finite fractional perimeter can have a very rough boundary, which may indeed be a nowhere rectifiable fractal (like the von Koch snowflake).

This represents a dramatic difference between the fractional and the classical perimeter, since Caccioppoli sets have a "big" portion of the boundary, the so-called reduced boundary, which is $(n-1)$-rectifiable (by De Giorgi's structure Theorem).

Still, we prove in this paper that a set has (locally) finite fractional perimeter if and only if it can be approximated (in an appropriate way) by smooth open sets. To be more precise, we show that a set E has locally finite s-perimeter if and only if we can find a sequence of smooth open sets which converge in measure to E, whose boundaries converge to that of E in a uniform sense, and whose s-perimeters converge to that of E in every bounded open set.

Such a result is well known for Caccioppoli sets (see, e.g., [17]) and indeed this density property can be used to define the (classical) perimeter functional as the relaxation (with respect to L^1_{loc}

convergence) of the $\mathcal{H}^{n-1}$ measure of boundaries of smooth open sets, that is

$$P(E,\Omega)=\inf\Big\{\liminf_{k\to\infty}\mathcal{H}^{n-1}(\partial E_h\cap\Omega)\,\Big|\,E_h\subset\mathbb{R}^n \text{ open with smoothboundary, s.t. } E_h\xrightarrow{loc}E\Big\}. \tag{1.1}$$

The second part of this paper is concerned with sets minimizing the fractional perimeter. The boundaries of these minimizers are often referred to as nonlocal minimal surfaces and naturally arise as *limit interfaces* of long-range interaction phase transition models. In particular, in regimes where the long-range interaction is dominant, the nonlocal Allen-Cahn energy functional Γ-converges to the fractional perimeter (see [19]) and the minimal interfaces of the corresponding Allen-Cahn equation approach locally uniformly the nonlocal minimal surfaces (see [18]).

We remark that throughout the paper, given a set A and an open set Ω, we will write $A\subset\subset\Omega$ to mean that the closure $\overline{A}$ of A is compact and $\overline{A}\subset\Omega$. In particular, notice that if $A\subset\subset\Omega$, then A must be bounded.

We consider sets which are locally s-minimal in an open set $\Omega\subset\mathbb{R}^n$, namely sets which minimize the s-perimeter in every open subset $\Omega'\subset\subset\Omega$, and we prove existence and compactness results which extend those of [4].

We also compare this definition of local s-minimal set with the definition of s-minimal set introduced in [4], proving that they coincide when the domain Ω is a bounded open set with Lipschitz boundary (see Theorem 1.7).

In particular, the following existence results are proven:

- If Ω is an open set and E_0 is a fixed set, then there exists a set E which is locally s-minimal in Ω and such that $E\setminus\Omega=E_0\setminus\Omega$;
- there exist minimizers in the class of subgraphs, namely nonlocal nonparametric minimal surfaces (see Theorem 1.16 for a precise statement);
- if Ω is an open set which has finite s-perimeter, then for every fixed set E_0 there exists a set E which is s-minimal in Ω and such that $E\setminus\Omega=E_0\setminus\Omega$.

On the other hand, we show that when the domain Ω is unbounded the nonlocal part of the s-perimeter can be infinite, thus preventing the existence of competitors having finite s-perimeter in Ω and hence also of "global" s-minimal sets. In particular, we study this situation in a cylinder $\Omega^\infty:=\Omega\times\mathbb{R}\subset\mathbb{R}^{n+1}$, considering as exterior data the subgraph of a (locally) bounded function.

In the next sections we present the precise statements of the main results of this paper. We begin by recalling the definition of fractional perimeter.

1.1 *Sets of (locally) finite s-perimeter*

Let $s\in(0,1)$ and let $\Omega\subset\mathbb{R}^n$ be an open set. The s-fractional perimeter of a set $E\subset\mathbb{R}^n$ in Ω is defined as

$$P_s(E,\Omega):=\mathcal{L}_s(E\cap\Omega,\complement E\cap\Omega)+\mathcal{L}_s(E\cap\Omega,\complement E\setminus\Omega)+\mathcal{L}_s(E\setminus\Omega,\complement E\cap\Omega),$$

where

$$\mathcal{L}_s(A,B):=\int_A\int_B\frac{1}{|x-y|^{n+s}}\,dx\,dy,$$

for every couple of disjoint sets $A,\ B\subset\mathbb{R}^n$. We simply write $P_s(E)$ for $P_s(E,\mathbb{R}^n)$.

We say that a set $E \subset \mathbb{R}^n$ has locally finite s-perimeter in an open set $\Omega \subset \mathbb{R}^n$ if

$$P_s(E, \Omega') < \infty \qquad \text{for every open set } \Omega' \subset\subset \Omega. \tag{1.2}$$

We remark that the family of sets having finite s-perimeter in Ω need not coincide with the family of sets of locally finite s-perimeter in Ω, not even when Ω is "nice" (say bounded and with Lipschitz boundary). To be more precise, since

$$P_s(E, \Omega) = \sup_{\Omega' \subset\subset \Omega} P_s(E, \Omega'), \tag{1.3}$$

(see Proposition 2.9 and Remark 2.10), a set which has finite s-perimeter in Ω has also locally finite s-perimeter. However the converse, in general, is false.

When Ω is not bounded it is clear that also for sets of locally finite s-perimeter the sup in (1.3) may be infinite (consider, e.g., $\Omega = \mathbb{R}^n$ and $E = \{x_n \leqslant 0\}$).

Actually, as shown in Remark 2.11, this may happen even when Ω is bounded and has Lipschitz boundary. Roughly speaking, this is because the set E might oscillate more and more as it approaches the boundary $\partial\Omega$.

1.2 *Approximation by smooth open sets*

We denote by $N_\rho(\Gamma)$ the ρ-neighborhood of a set $\Gamma \subset \mathbb{R}^n$, that is

$$N_\rho(\Gamma) := \big\{x \in \mathbb{R}^n \,|\, d(x, \Gamma) < \rho\big\}.$$

The main approximation result is the following. In particular it shows that open sets with smooth boundary are dense in the family of sets of locally finite s-perimeter.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be an open set. A set $E \subset \mathbb{R}^n$ has locally finite s-perimeter in Ω if and only if there exists a sequence $E_h \subset \mathbb{R}^n$ of open sets with smooth boundary and $\varepsilon_h \longrightarrow 0^+$ such that*

(i) $E_h \xrightarrow{loc} E, \qquad \sup_{h \in \mathbb{N}} P_s(E_h, \Omega') < \infty$ *for every* $\Omega' \subset\subset \Omega$,

(ii) $\lim_{h\to\infty} P_s(E_h, \Omega') = P_s(E, \Omega')$ *for every* $\Omega' \subset\subset \Omega$,

(iii) $\partial E_h \subset N_{\varepsilon_h}(\partial E)$.

Moreover, if $\Omega = \mathbb{R}^n$ and the set E is such that $|E| < \infty$ and $P_s(E) < \infty$, then

$$E_h \longrightarrow E, \qquad \lim_{h\to\infty} P_s(E_h) = P_s(E), \tag{1.4}$$

and we can require each set E_h to be bounded (instead of asking (iii)).

The scheme of the proof is the following. First of all, in Section 3.1 we prove appropriate approximation results for the functional

$$\mathcal{F}(u, \Omega) = \frac{1}{2} \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x) - u(y)|}{|x - y|^{n+s}}\, dx\, dy,$$

which we believe might be interesting on their own.

Then we exploit the generalized coarea formula

$$\mathcal{F}(u,\Omega)=\int_{-\infty}^{\infty}P_s(\{u>t\},\Omega)\,dt,$$

and Sard's Theorem to obtain the approximation of the set E by superlevel sets of smooth functions which approximate χ_E.

Finally, a diagonal argument guarantees the convergence of the s-perimeters in every open set $\Omega'\subset\subset\Omega$.

REMARK 1.2 Let $\Omega\subset\mathbb{R}^n$ be a bounded open set with Lipschitz boundary and consider a set E which has finite s-perimeter in Ω. Notice that if we apply Theorem 1.1, in point (ii) we do not get the convergence of the s-perimeters in Ω, but only in every $\Omega'\subset\subset\Omega$. On the other hand, if we can find an open set $\mathcal{O}$ such that $\Omega\subset\subset\mathcal{O}$ and

$$P_s(E,\mathcal{O})<\infty,$$

then we can apply Theorem 1.1 in $\mathcal{O}$. In particular, since $\Omega\subset\subset\mathcal{O}$, by point (ii) we obtain

$$\lim_{h\to\infty}P_s(E_h,\Omega)=P_s(E,\Omega). \tag{1.5}$$

Still, when Ω is a bounded open set with Lipschitz boundary, we can always obtain the convergence (1.5) at the cost of weakening a little our request on the uniform convergence of the boundaries.

Theorem 1.3 *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set with Lipschitz boundary. A set $E\subset\mathbb{R}^n$ has finite s-perimeter in Ω if and only if there exists a sequence $\{E_h\}$ of open sets with smooth boundary and $\varepsilon_h\longrightarrow 0^+$ such that*

(i) $E_h\xrightarrow{loc}E,\qquad \sup_{h\in\mathbb{N}}P_s(E_h,\Omega)<\infty,$

(ii) $\lim_{h\to\infty}P_s(E_h,\Omega)=P_s(E,\Omega),$

(iii) $\partial E_h\setminus N_{\varepsilon_h}(\partial\Omega)\subset N_{\varepsilon_h}(\partial E).$

Notice that in point (iii) we do not ask the convergence of the boundaries in the whole of $\mathbb{R}^n$ but only in $\mathbb{R}^n\setminus N_\delta(\partial\Omega)$ (for any fixed $\delta>0$). Since $N_{\varepsilon_h}(\partial\Omega)\searrow\partial\Omega$, roughly speaking, the convergence holds in $\mathbb{R}^n$ "in the limit".

Moreover, we remark that point (ii) in Theorem 1.3 guarantees the convergence of the s-perimeters also in every $\Omega'\subset\subset\Omega$ (see Remark 3.6).

Finally, from the lower semicontinuity of the s-perimeter and Theorem 1.3, we obtain

Corollary 1.4 *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $E\subset\mathbb{R}^n$. Then*

$$P_s(E,\Omega)=\inf\Big\{\liminf_{h\to\infty}P_s(E_h,\Omega)\,\Big|\,E_h\subset\mathbb{R}^n \text{ open with smooth boundary, s.t. } E_h\xrightarrow{loc}E\Big\}. \tag{1.6}$$

For similar approximation results see also [5] and [6].

It is interesting to observe that in [13] the authors have proved, by exploiting the divergence Theorem, that if $E\subset\mathbb{R}^n$ is a bounded open set with smooth boundary, then

$$P_s(E)=c_{n,s}\int_{\partial E}\int_{\partial E}\frac{2-|\nu_E(x)-\nu_E(y)|^2}{|x-y|^{n+s-2}}\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y, \tag{1.7}$$

where ν_E denotes the external normal of E and

$$c_{n,s} := \frac{1}{2s(n+s-2)}.$$

Notice that in order to consider the right hand side of (1.7), we need the boundary of the set E to be at least locally $(n-1)$-rectifiable, so that the Hausdorff dimension of ∂E is $n-1$ and E has a well defined normal vector at $\mathcal{H}^{n-1}$-a.e. $x \in \partial E$. Therefore, the equality (1.7) cannot hold true for a generic set E having finite s-perimeter, since, as remarked in the beginning of the Introduction, such a set could have a nowhere rectifiable boundary.

Nevertheless, as a consequence of the equality (1.7), of the lower semicontinuity of the s-perimeter and of Theorem 1.1, we obtain the following Corollary, which can be thought of as an analogue of (1.1) in the fractional setting.

Corollary 1.5 *Let $E \subset \mathbb{R}^n$ be such that $|E| < \infty$. Then*

$$P_s(E) = \inf\left\{ \liminf_{h\to\infty} c_{n,s} \int_{\partial E_h} \int_{\partial E_h} \frac{2 - |\nu_{E_h}(x) - \nu_{E_h}(y)|^2}{|x-y|^{n+s-2}} \, d\mathcal{H}^{n-1}_x \, d\mathcal{H}^{n-1}_y \;\Big|\; E_h \subset \mathbb{R}^n \text{ bounded open set with smooth boundary, s.t. } E_h \xrightarrow{loc} E \right\}.$$

1.3 *Nonlocal minimal surfaces*

First of all we give the definition of (locally) s-minimal sets.

DEFINITION 1.6 Let $\Omega \subset \mathbb{R}^n$ be an open set and let $s \in (0,1)$. We say that a set $E \subset \mathbb{R}^n$ is *s-minimal* in Ω if $P_s(E,\Omega) < \infty$ and

$$F \setminus \Omega = E \setminus \Omega \quad \Longrightarrow \quad P_s(E,\Omega) \leqslant P_s(F,\Omega).$$

We say that a set $E \subset \mathbb{R}^n$ is *locally s-minimal* in Ω if it is s-minimal in every open subset $\Omega' \subset\subset \Omega$.

When the open set $\Omega \subset \mathbb{R}^n$ is bounded and has Lipschitz boundary, the notions of s-minimal set and locally s-minimal set coincide.

Theorem 1.7 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $E \subset \mathbb{R}^n$. The following are equivalent*

(i) *E is s-minimal in Ω;*
(ii) *$P_s(E,\Omega) < \infty$ and*

$$P_s(E,\Omega) \leqslant P_s(F,\Omega) \qquad \text{for every } F \subset \mathbb{R}^n \quad \text{s.t.} \quad E\Delta F \subset\subset \Omega;$$

(iii) *E is locally s-minimal in Ω.*

We remark that a set as in (ii) is called a local minimizer for $P_s(-,\Omega)$ in [2] and a "nonlocal area minimizing surface" in Ω in [8].

REMARK 1.8 The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) actually hold in any open set $\Omega \subset \mathbb{R}^n$.

In [4] the authors proved that if Ω is a bounded open set with Lipschitz boundary, then given any fixed set $E_0 \subset \mathbb{R}^n$ we can find a set E which is s-minimal in Ω and such that $E \setminus \Omega = E_0 \setminus \Omega$.

This is because

$$P_s(E_0 \setminus \Omega, \Omega) \leqslant P_s(\Omega) < \infty,$$

so the exterior datum $E_0 \setminus \Omega$ is itself an admissible competitor with finite s-perimeter in Ω and we can use the direct method of the Calculus of Variations to obtain a minimizer.

In Section 2.3 we prove a compactness property which we use in Section 4.3 to prove the following existence results, which extend that of [4].

Theorem 1.9 *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $E_0 \subset \mathbb{R}^n$. Then there exists a set $E \subset \mathbb{R}^n$ s-minimal in Ω, with $E \setminus \Omega = E_0 \setminus \Omega$, if and only if there exists a set $F \subset \mathbb{R}^n$, with $F \setminus \Omega = E_0 \setminus \Omega$ and such that $P_s(F, \Omega) < \infty$.*

An immediate consequence of this Theorem is the existence of s-minimal sets in open sets having finite s-perimeter.

Corollary 1.10 *Let $s \in (0, 1)$ and let $\Omega \subset \mathbb{R}^n$ be an open set such that*

$$P_s(\Omega) < \infty.$$

Then for every $E_0 \subset \mathbb{R}^n$ there exists a set $E \subset \mathbb{R}^n$ s-minimal in Ω, with $E \setminus \Omega = E_0 \setminus \Omega$.

Even if we cannot find a competitor with finite s-perimeter, we can always find a locally s-minimal set.

Corollary 1.11 *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $E_0 \subset \mathbb{R}^n$. Then there exists a set $E \subset \mathbb{R}^n$ locally s-minimal in Ω, with $E \setminus \Omega = E_0 \setminus \Omega$.*

In Section 4.2 we also prove compactness results for (locally) s-minimal sets (by slightly modifying the proof of Theorem 3.3 of [4], which proved compactness for s-minimal sets in a ball). Namely, we prove that every limit set of a sequence of (locally) s-minimal sets is itself (locally) s-minimal.

Theorem 1.12 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $\{E_k\}$ be a sequence of s-minimal sets in Ω, with $E_k \xrightarrow{loc} E$. Then E is s-minimal in Ω and*

$$P_s(E, \Omega) = \lim_{k\to\infty} P_s(E_k, \Omega). \tag{1.8}$$

Corollary 1.13 *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $\{E_h\}$ be a sequence of sets locally s-minimal in Ω, with $E_h \xrightarrow{loc} E$. Then E is locally s-minimal in Ω and*

$$P_s(E, \Omega') = \lim_{h\to\infty} P_s(E_h, \Omega'), \qquad \textit{for every } \Omega' \subset\subset \Omega. \tag{1.9}$$

1.3.1 *Minimal sets in cylinders.* We have seen in Corollary 1.11 that a locally s-minimal set always exists, no matter what the domain Ω or the exterior data $E_0 \setminus \Omega$ are.

On the other hand, by Theorem 1.9 we know that the only requirement needed for the existence of an s-minimal set is the existence of a competitor with finite s-perimeter.

We show that even in the case of a regular domain, like the cylinder $\Omega^\infty := \Omega \times \mathbb{R}$, with $\Omega \subset \mathbb{R}^n$ bounded with $C^{1,1}$ boundary, such a competitor might not exist. Roughly speaking, this is a consequence of the unboundedness of the domain Ω^∞, which forces the nonlocal part of the s-perimeter to be infinite.

In Section 4.4 we study (locally) s-minimal sets in Ω^∞, with respect to the exterior data given by the subgraph of a function v, that is

$$\mathcal{S}g(v) = \big\{(x,t)\,|\,t < v(x)\big\}.$$

In particular, we consider sets which are s-minimal in the "truncated" cylinders $\Omega^k := \Omega \times (-k,k)$, showing that if the function v is locally bounded, then these s-minimal sets cannot "oscillate" too much. Namely their boundaries are constrained in a cylinder $\Omega \times (-M, M)$ independently on k.

As a consequence, we can find k_0 big enough such that a set E is locally s-minimal in Ω^∞ if and only if it is s-minimal in Ω^{k_0} (see Lemma 4.3 and Proposition 4.4 for the precise statements).

However, in general a set s-minimal in Ω^∞ does not exist. As an example we prove that there cannot exist an s-minimal set having as exterior data the subgraph of a bounded function.

Frst of all, we remark that we can write the fractional perimeter as the sum

$$P_s(E,\Omega) = P_s^L(E,\Omega) + P_s^{NL}(E,\Omega),$$

where

$$\begin{aligned} P_s^L(E,\Omega) &:= \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\cap\Omega) = \frac{1}{2}[\chi_E]_{W^{s,1}(\Omega)},\\ P_s^{NL}(E,\Omega) &:= \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\setminus\Omega) + \mathcal{L}_s(E\setminus\Omega, \mathcal{C}E\cap\Omega). \end{aligned}$$

We can think of $P_s^L(E,\Omega)$ as the local part of the fractional perimeter, in the sense that if $|(E\Delta F)\cap\Omega| = 0$, then $P_s^L(F,\Omega) = P_s^L(E,\Omega)$.

The main result of Section 4.4 is the following

Theorem 1.14 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $E \subset \mathbb{R}^{n+1}$ be such that*

$$\Omega\times(-\infty,-k] \subset E\cap\Omega^\infty \subset \Omega\times(-\infty,k], \tag{1.10}$$

for some $k\in\mathbb{N}$, and suppose that $P_s(E,\Omega^{k+1}) < \infty$. Then

$$P_s^L(E,\Omega^\infty) < \infty.$$

On the other hand, if

$$\{x_{n+1} \leqslant -k\} \subset E \subset \{x_{n+1} \leqslant k\}, \tag{1.11}$$

then

$$P_s^{NL}(E,\Omega^\infty) = \infty.$$

In particular, if Ω has $C^{1,1}$ boundary and $v \in L^\infty(\mathbb{R}^n)$, there cannot exist an s-minimal set in Ω^∞ with exterior data

$$\mathcal{S}g(v)\setminus\Omega^\infty = \big\{(x,t)\in\mathbb{R}^{n+1}\,|\,x\in\mathcal{C}\Omega, \quad t < v(x)\big\}.$$

REMARK 1.15 From Theorem 1.9 we see that if $v \in L^\infty(\mathbb{R}^n)$, there cannot exist a set $E \subset \mathbb{R}^{n+1}$ such that $E\setminus\Omega^\infty = \mathcal{S}g(v)\setminus\Omega^\infty$ and $P_s(E,\Omega^\infty) < \infty$.

As a consequence of the computations developed in the proof of Theorem 1.14, in the end of Section 4.4 we also show that we cannot define a "naive" fractional nonlocal version of the area functional as

$$\mathcal{A}_s(u,\Omega) := P_s\big(\mathcal{S}g(u),\Omega^\infty\big),$$

since this would be infinite even for very regular functions.

To conclude, we remark that as an immediate consequence of Corollary 1.11 and Theorem 1.1 in [11], we obtain an existence result for the Plateau's problem in the class of subgraphs.

Theorem 1.16 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^{1,1}$ boundary. For every function $v \in C(\mathbb{R}^n)$ there exists a function $u \in C(\overline{\Omega})$ such that, if*

$$\tilde{u} := \chi_\Omega u + (1-\chi_\Omega)v,$$

then $\mathcal{S}g(\tilde{u})$ is locally s-minimal in Ω^∞.

Notice that, as remarked in [11], the function $\tilde{u}$ need not be continuous. Indeed, because of boundary stickiness effects of s-minimal surfaces (see, e.g., [12]), in general we might have

$$u_{|\partial\Omega} \neq v_{|\partial\Omega}.$$

1.4 *Notation and assumptions*

- Unless otherwise stated, Ω and Ω' will always denote open sets.
- In $\mathbb{R}^n$ we will usually write $|E| = \mathcal{L}^n(E)$ for the n-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^n$.
- By $A_h \xrightarrow{loc} A$ we mean that $\chi_{A_h} \longrightarrow \chi_A$ in $L^1_{loc}(\mathbb{R}^n)$, i.e. for every bounded open set $\Omega \subset \mathbb{R}^n$ we have $|(A_h \Delta A) \cap \Omega| \longrightarrow 0$.
- We write $\mathcal{H}^d$ for the d-dimensional Hausdorff measure, for any $d \geqslant 0$.
- We define the dimensional constants

$$\omega_d := \frac{\pi^{\frac{d}{2}}}{\Gamma\big(\frac{d}{2}+1\big)}, \qquad d \geqslant 0.$$

  In particular, we remark that $\omega_k = \mathcal{L}^k(B_1)$ is the volume of the k-dimensional unit ball $B_1 \subset \mathbb{R}^k$ and $k\,\omega_k = \mathcal{H}^{k-1}(\mathbb{S}^{k-1})$ is the surface area of the $(k-1)$-dimensional sphere

$$\mathbb{S}^{k-1} = \partial B_1 = \big\{x \in \mathbb{R}^k \,|\, |x| = 1\big\}.$$

- Since

$$|E \Delta F| = 0 \quad \Longrightarrow \quad P_s(E,\Omega) = P_s(F,\Omega),$$

  we can and will implicitly identify sets up to sets of zero measure.
  In particular, equality and inclusions of sets will usually be considered in the measure sense, e.g., $E = F$ will usually mean $|E \Delta F| = 0$.
  Moreover, whenever needed we will implicitly choose a particular representative for the class of χ_E in $L^1_{loc}(\mathbb{R}^n)$, as in the Remark below.

REMARK 1.17 Let $E \subset \mathbb{R}^n$. Up to modifying E on a set of measure zero, we can assume (see, e.g., Appendix C of [16]) that E contains the measure theoretic interior

$$E_1 := \big\{x \in \mathbb{R}^n \,|\, \exists\, r > 0 \text{ s.t. } |E \cap B_r(x)| = \omega_n r^n\big\} \subset E,$$

the complementary $\complement E$ contains its measure theoretic interior

$$E_0 := \big\{x \in \mathbb{R}^n \,|\, \exists\, r > 0 \text{ s.t. } |E \cap B_r(x)| = 0\big\} \subset \complement E,$$

and the topological boundary of E coincides with its measure theoretic boundary, $\partial E = \partial^- E$, where

$$\partial^- E := \mathbb{R}^n \setminus (E_0 \cup E_1) = \big\{x \in \mathbb{R}^n \,|\, 0 < |E \cap B_r(x)| < \omega_n r^n \text{ for every } r > 0\big\}.$$

2. Tools

It is convenient to point out the following easy but useful result.

Proposition 2.1 *Let $\Omega' \subset \Omega \subset \mathbb{R}^n$ be open sets and let $E \subset \mathbb{R}^n$. Then*

$$P_s(E,\Omega) = P_s(E,\Omega') + \mathcal{L}_s(E \cap (\Omega \setminus \Omega'), \complement E \setminus \Omega') + \mathcal{L}_s\big(E \setminus \Omega, \complement E \cap (\Omega \setminus \Omega')\big). \tag{2.1}$$

As a consequence,

(i) *if $E \subset \Omega$, then*
$$P_s(E,\Omega) = P_s(E),$$

(ii) *if $E, F \subset \mathbb{R}^n$ have finite s-perimeter in Ω and $E \Delta F \subset \Omega' \subset \Omega$, then*

$$P_s(E,\Omega) - P_s(F,\Omega) = P_s(E,\Omega') - P_s(F,\Omega'). \tag{2.2}$$

REMARK 2.2 In particular, if E has finite s-perimeter in Ω, then it has finite s-perimeter also in every open set $\Omega' \subset \Omega$.

2.1 *Bounded open sets with Lipschitz boundary*

Given a set $E \subset \mathbb{R}^n$, with $E \neq \emptyset$, the distance function from E is defined as

$$d_E(x) = d(x,E) := \inf_{y \in E} |x-y|, \qquad \text{for } x \in \mathbb{R}^n.$$

The signed distance function from ∂E, negative inside E, is then defined as

$$\bar{d}_E(x) = \bar{d}(x,E) := d(x,E) - d(x,\complement E). \tag{2.3}$$

We also define for every $r \in \mathbb{R}$ the sets

$$E_r := \{x \in \mathbb{R}^n \,|\, \bar{d}_E(x) < r\}.$$

Notice that if $\rho > 0$, then

$$N_\rho(\partial\Omega) = \{|\bar{d}_\Omega| < \rho\} = \Omega_\rho \setminus \overline{\Omega_{-\rho}}$$

is the ρ-tubular neighborhood of $\partial\Omega$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. It is well known (see, e.g., Theorem 4.1 of [14]) that also the bounded open sets Ω_r have Lipschitz boundary, when r is small enough, say $|r| < r_0$.

Notice that

$$\partial\Omega_r = \{\bar{d}_\Omega = r\}.$$

Moreover the perimeter of Ω_r can be bounded uniformly in $r \in (-r_0, r_0)$ (see also Appendix B of [16] for a more detailed discussion).

Proposition 2.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then there exists $r_0 > 0$ such that Ω_r is a bounded open set with Lipschitz boundary for every $r \in (-r_0, r_0)$ and*

$$\sup_{|r|<r_0} \mathcal{H}^{n-1}\big(\{\bar{d}_\Omega = r\}\big) < \infty. \tag{2.4}$$

As a consequence, exploiting the embedding $BV(\mathbb{R}^n) \hookrightarrow W^{s,1}(\mathbb{R}^n)$ we obtain a uniform bound for the (global) s-perimeters of the sets Ω_r (see Corollary 1.2 of [16])

Corollary 2.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then there exists $r_0 > 0$ such that*

$$\sup_{|r|<r_0} P_s(\Omega_r) < \infty. \tag{2.5}$$

2.1.1 *Increasing sequences.* In particular, Proposition 2.3 shows that if Ω is a bounded open set with Lpschitz boundary, then we can approximate it strictly from the inside with a sequence of bounded open sets $\Omega_k := \Omega_{-1/k} \subset\subset \Omega$. Moreover, (2.4) gives a uniform bound on the measure of the boundaries of the approximating sets.

Now we prove that any open set $\Omega \neq \emptyset$ can be approximated strictly from the inside with a sequence of bounded open sets with smooth boundaries.

Proposition 2.5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For every $\varepsilon > 0$ there exists a bounded open set $\mathcal{O}_\varepsilon \subset \mathbb{R}^n$ with smooth boundary, such that*

$$\mathcal{O}_\varepsilon \subset\subset \Omega \qquad \textit{and} \qquad \partial\mathcal{O}_\varepsilon \subset N_\varepsilon(\partial\Omega). \tag{2.6}$$

Proof. We show that we can approximate the set $\Omega_{-\varepsilon/2}$ with a bounded open set $\mathcal{O}_\varepsilon$ with smooth boundary such that $\partial\mathcal{O}_\varepsilon \subset N_{\varepsilon/4}(\partial\Omega_{-\varepsilon/2})$.

In general $\mathcal{O}_\varepsilon \not\subset \Omega_{-\varepsilon/2}$. However

$$\mathcal{O}_\varepsilon \subset N_{\varepsilon/4}(\Omega_{-\varepsilon/2}) \subset\subset \Omega \quad \text{and indeed} \quad \Omega_{-3\varepsilon/4} \subset \mathcal{O}_\varepsilon \subset \Omega_{-\varepsilon/4}, \tag{2.7}$$

proving the claim.

Let $u := \chi_{\Omega_{-\varepsilon/2}}$ and consider the regularized function

$$v := u_{\varepsilon/4} = u * \eta_{\varepsilon/4}$$

(see Section 3 for the details about the mollifier η_ε). Since $v \in C^\infty(\mathbb{R}^n)$, we know from Sard's Theorem that the superlevel set $\{v > t\}$ is an open set with smooth boundary for a.e. $t \in (0, 1)$. Moreover notice that $0 \leqslant v \leqslant 1$, with

$$\operatorname{supp} v \subset N_{\varepsilon/4}(\operatorname{supp} u) = N_{\varepsilon/4}(\Omega_{-\varepsilon/2}) \subset \Omega_{-\varepsilon/4},$$

and

$$v(x) = 1 \qquad \text{for every } x \in \Big\{y \in \Omega_{-\varepsilon/2} \,\Big|\, d(y, \partial\Omega_{-\varepsilon/2}) > \frac{\varepsilon}{4}\Big\} \supset \Omega_{-\frac{3}{4}\varepsilon}.$$

This shows that $\mathcal{O}_\varepsilon := \{v > t\}$ (for any "regular" t) satisfies (2.7). □

Corollary 2.6 *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then there exists a sequence $\{\Omega_k\}$ of bounded open sets with smooth boundary such that $\Omega_k \nearrow \Omega$ strictly, i.e.,*

$$\Omega_k \subset\subset \Omega_{k+1} \subset\subset \Omega \qquad \textit{and} \qquad \bigcup_{k\in\mathbb{N}} \Omega_k = \Omega. \tag{2.8}$$

In particular $\Omega_k \xrightarrow{loc} \Omega$.

Proof. It is enough to notice that we can approximate Ω strictly from the inside with bounded open sets $\mathcal{O}_k \subset \mathbb{R}^n$, that is

$$\mathcal{O}_k \subset\subset \mathcal{O}_{k+1} \subset\subset \Omega \qquad \text{and} \qquad \bigcup_{k\in\mathbb{N}} \mathcal{O}_k = \Omega.$$

Then we can exploit Proposition 2.5, and in particular (2.7), to find bounded open sets $\Omega_k \subset \mathbb{R}^n$ with smooth boundary such that

$$\mathcal{O}_k \subset\subset \Omega_k \subset\subset \mathcal{O}_{k+1}.$$

Indeed we can take as Ω_k a set $\mathcal{O}_\varepsilon$ corresponding to $\mathcal{O}_{k+1}$, with ε small enough to guarantee $\mathcal{O}_k \subset\subset \mathcal{O}_\varepsilon$.

As for the sets $\mathcal{O}_k$, if Ω is bounded we can simply take $\mathcal{O}_k := \Omega_{-2^{-k}}$. If Ω is not bounded, we can consider the sets $\Omega \cap B_{2^k}$ and define

$$\mathcal{O}_k := \Big\{x \in \Omega \cap B_{2^k} \mid d\big(x, \partial(\Omega \cap B_{2^k})\big) > 2^{-k}\Big\}.$$

To conclude, notice that we have $\chi_{\Omega_k} \longrightarrow \chi_\Omega$ pointwise everywhere in $\mathbb{R}^n$, which implies the convergence in $L^1_{loc}(\mathbb{R}^n)$. □

2.1.2 *Some uniform estimates for ρ-neighborhoods.* The uniform bound (2.4) on the perimeters of the sets Ω_δ allows us to obtain the following estimates, which will be used in the sequel.

Lemma 2.7 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $\delta \in (0, r_0)$. Then*

$$\begin{aligned} &\text{(i)} \quad \mathcal{L}_s(\Omega_{-\delta}, \Omega \setminus \Omega_{-\delta}) \leqslant C\,\delta^{1-s}, \\ &\text{(ii)} \quad \mathcal{L}_s(\Omega, \Omega_\delta \setminus \Omega) \leqslant C\,\delta^{1-s} \quad \textit{and} \quad \mathcal{L}_s(\Omega \setminus \Omega_{-\delta}, \mathcal{C}\Omega) \leqslant C\,\delta^{1-s}, \end{aligned} \tag{2.9}$$

where the constant C is

$$C := \frac{n\omega_n}{s(1-s)} \sup_{|r|<r_0} \mathcal{H}^{n-1}\big(\{\bar{d}_\Omega = r\}\big).$$

Proof. By using the coarea formula for $\bar{d}_\Omega$ and exploiting (2.4), we get

$$\begin{aligned}\mathcal{L}_s(\Omega_{-\delta}, \Omega \setminus \Omega_{-\delta}) &= \int_{-\delta}^{0} \Big(\int_{\{\bar{d}_\Omega = \rho\}} \Big(\int_{\Omega_{-\delta}} \frac{dx}{|x-y|^{n+s}} \Big) d\mathcal{H}^{n-1}_y \Big) d\rho \\ &\leqslant \int_{-\delta}^{0} \Big(\int_{\{\bar{d}_\Omega = \rho\}} \Big(\int_{\complement B_{\rho+\delta}(y)} \frac{dx}{|x-y|^{n+s}} \Big) d\mathcal{H}^{n-1}_y \Big) d\rho \\ &= \frac{n\omega_n}{s} \int_{-\delta}^{0} \frac{\mathcal{H}^{n-1}(\{\bar{d}_\Omega = \rho\})}{(\rho+\delta)^s} \, d\rho \\ &\leqslant M \frac{n\omega_n}{s(1-s)} \int_{-\delta}^{0} \frac{d}{d\rho}(\rho+\delta)^{1-s} \, d\rho = M \frac{n\omega_n}{s(1-s)} \, \delta^{1-s}.\end{aligned}$$

In the same way we obtain point (ii),

$$\begin{aligned}\mathcal{L}_s(\Omega_\delta \setminus \Omega, \Omega) &= \int_{0}^{\delta} \Big(\int_{\{\bar{d}_\Omega = \rho\}} \Big(\int_{\Omega} \frac{dx}{|x-y|^{n+s}} \Big) d\mathcal{H}^{n-1}_y \Big) d\rho \\ &\leqslant \int_{0}^{\delta} \Big(\int_{\{\bar{d}_\Omega = \rho\}} \Big(\int_{\complement B_{\rho}(y)} \frac{dx}{|x-y|^{n+s}} \Big) d\mathcal{H}^{n-1}_y \Big) d\rho \\ &= \frac{n\omega_n}{s} \int_{0}^{\delta} \frac{\mathcal{H}^{n-1}(\{\bar{d}_\Omega = \rho\})}{\rho^s} \, d\rho \\ &\leqslant M \frac{n\omega_n}{s(1-s)} \int_{0}^{\delta} \frac{d}{d\rho}\rho^{1-s} \, d\rho = M \frac{n\omega_n}{s(1-s)} \, \delta^{1-s},\end{aligned}$$

(the other estimate in point (ii) is analogous). □

2.2 *(Semi)continuity of the s-perimeter*

As shown in Theorem 3.1 of [4], Fatou's Lemma gives the lower semicontinuity of the functional $\mathcal{L}_s$.

Proposition 2.8 *Suppose*

$$A_k \xrightarrow{loc} A \qquad \text{and} \qquad B_k \xrightarrow{loc} B.$$

Then

$$\mathcal{L}_s(A, B) \leqslant \liminf_{k\to\infty} \mathcal{L}_s(A_k, B_k). \tag{2.10}$$

In particular, if

$$E_k \xrightarrow{loc} E \qquad \text{and} \qquad \Omega_k \xrightarrow{loc} \Omega,$$

then

$$P_s(E, \Omega) \leqslant \liminf_{k\to\infty} P_s(E_k, \Omega_k). \tag{2.11}$$

Proof. If the right hand side of (2.10) is infinite, we have nothing to prove, so we can suppose that it is finite. By definition of the liminf, we can find $k_i \nearrow \infty$ such that

$$\lim_{i\to\infty} \mathcal{L}_s(A_{k_i}, B_{k_i}) = \liminf_{k\to\infty} \mathcal{L}_s(A_k, B_k) =: I.$$

Since $\chi_{A_{k_i}} \to \chi_A$ and $\chi_{B_{k_i}} \to \chi_B$ in $L^1_{loc}(\mathbb{R}^n)$, up to passing to a subsequence we can suppose that

$$\chi_{A_{k_i}} \longrightarrow \chi_A \qquad \text{and} \qquad \chi_{B_{k_i}} \longrightarrow \chi_B \qquad \text{a.e. in } \mathbb{R}^n.$$

Then, since

$$\mathcal{L}_s(A_{k_i}, B_{k_i}) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n+s}} \chi_{A_{k_i}}(x)\chi_{B_{k_i}}(y)\,dx\,dy,$$

Fatou's Lemma gives

$$\mathcal{L}_s(A, B) \leqslant \liminf_{i\to\infty} \mathcal{L}_s(A_{k_i}, B_{k_i}) = I,$$

proving (2.10).

The second inequality follows just by summing the contributions defining the fractional perimeter. □

Keeping Ω fixed we obtain Theorem 3.1 of [4].

On the other hand, if we keep the set E fixed and approximate the open set Ω with a sequence of open subsets $\Omega_k \subset \Omega$, we get a continuity property.

Proposition 2.9 *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\{\Omega_k\}$ be any sequence of open sets such that $\Omega_k \xrightarrow{loc} \Omega$. Then for every set $E \subset \mathbb{R}^n$*

$$P_s(E, \Omega) \leqslant \liminf_{k\to\infty} P_s(E, \Omega_k).$$

Moreover, if $\Omega_k \subset \Omega$ for every k, then

$$P_s(E, \Omega) = \lim_{k\to\infty} P_s(E, \Omega_k), \tag{2.12}$$

(whether it is finite or not).

Proof. Since $\Omega_k \xrightarrow{loc} \Omega$, Proposition 2.8 gives the first statement. Now notice that if $\Omega_k \subset \Omega$, Proposition 2.1 implies

$$P_s(E, \Omega_k) \leqslant P_s(E, \Omega),$$

and hence

$$\limsup_{k\to\infty} P_s(E, \Omega_k) \leqslant P_s(E, \Omega),$$

concluding the proof. □

REMARK 2.10 As a consequence, exploiting Corollary 2.6, we get

$$P_s(E, \Omega) = \sup_{\Omega' \subsetneq \Omega} P_s(E, \Omega') = \sup_{\Omega' \subset\subset \Omega} P_s(E, \Omega'). \tag{2.13}$$

REMARK 2.11 Consider the set $E \subset \mathbb{R}$ constructed in the proof of Example 2.10 in [10]. That is, let $\beta_k > 0$ be a decreasing sequence such that

$$M := \sum_{k=1}^{\infty} \beta_k < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \beta_{2k}^{1-s} = \infty, \qquad \forall\, s \in (0, 1).$$

Then define

$$\sigma_m := \sum_{k=1}^{m} \beta_k, \qquad I_m := (\sigma_m, \sigma_{m+1}), \qquad E := \bigcup_{j=1}^{\infty} I_{2j},$$

and let $\Omega := (0, M)$. As shown in [10],

$$P_s(E, \Omega) = \infty, \qquad \forall\, s \in (0, 1).$$

On the other hand

$$P(E, \Omega') < \infty, \qquad \forall\, \Omega' \subset\subset \Omega,$$

hence E has locally finite s-perimeter in Ω, for every $s \in (0, 1)$.

Indeed, notice that the intervals I_{2j} accumulate near M. Thus, for every $\varepsilon > 0$, all but a finite number of the intervals I_{2j}'s fall outside of the open set $\mathcal{O}_\varepsilon := (\varepsilon, M-\varepsilon)$. Therefore $P(E, \mathcal{O}_\varepsilon) < \infty$ and hence

$$P_s(E, \mathcal{O}_\varepsilon) < \infty, \quad \forall\, s \in (0, 1).$$

Since $\mathcal{O}_\varepsilon \nearrow \Omega$ as $\varepsilon \to 0^+$, the set E has locally finite s-perimeter in Ω for every $s \in (0, 1)$.

Proposition 2.12 *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\{E_h\}$ be a sequence of sets such that*

$$E_h \xrightarrow{loc} E \qquad \textit{and} \qquad \lim_{h\to\infty} P_s(E_h, \Omega) = P_s(E, \Omega) < \infty.$$

Then

$$\lim_{h\to\infty} P_s(E_h, \Omega') = P_s(E, \Omega') \qquad \textit{for every open set } \Omega' \subset \Omega. \tag{2.14}$$

Proof. The claim follows from classical properties of limits of sequences.

Indeed, let

$$a_h := P_s(E_h, \Omega'),$$
$$b_h := \mathcal{L}_s\big(E_h \cap (\Omega \setminus \Omega'), \mathcal{C}E_h \setminus \Omega'\big) + \mathcal{L}_s\big(E_h \setminus \Omega, \mathcal{C}E_h \cap (\Omega \setminus \Omega')\big),$$

and let a and b be the corresponding terms for E.
Notice that, by Proposition 2.1, we have

$$P_s(E_h, \Omega) = a_h + b_h \qquad \text{and} \qquad P_s(E, \Omega) = a + b.$$

From Proposition 2.8 we have

$$a \leqslant \liminf_{h\to\infty} a_h \qquad \text{and} \qquad b \leqslant \liminf_{h\to\infty} b_h,$$

and by hypothesis we know that

$$\lim_{h\to\infty} (a_h + b_h) = a + b.$$

Therefore

$$a + b \leqslant \liminf_{h\to\infty} a_h + \liminf_{h\to\infty} b_h \leqslant \liminf_{h\to\infty} (a_h + b_h) = a + b,$$

and hence

$$0 \leqslant \liminf_{h\to\infty} b_h - b = a - \liminf_{h\to\infty} a_h \leqslant 0,$$

so that

$$a = \liminf_{h\to\infty} a_h \qquad \text{and} \qquad b = \liminf_{h\to\infty} b_h.$$

Then, since

$$\limsup_{h\to\infty} a_h + \liminf_{h\to\infty} b_h \leqslant \limsup_{h\to\infty}(a_h + b_h) = a + b,$$

we obtain

$$a = \liminf_{h\to\infty} a_h \leqslant \limsup_{h\to\infty} a_h \leqslant a,$$

concluding the proof. □

2.3 *Compactness*

Proposition 2.13 (Compactness) *Let $\Omega \subset \mathbb{R}^n$ be an open set. If $\{E_h\}$ is a sequence of sets such that*

$$\limsup_{h\to\infty} P_s^L(E_h, \Omega') \leqslant c(\Omega') < \infty, \qquad \forall\, \Omega' \subset\subset \Omega, \tag{2.15}$$

then there exists a subsequence $\{E_{h_i}\}$ and $E \subset \mathbb{R}^n$ such that

$$E_{h_i} \cap \Omega \xrightarrow{loc} E \cap \Omega.$$

Proof. We want to use a compact Sobolev embedding (Corollary 7.2 of [9]) to construct a limit set via a diagonal argument.

Thanks to Corollary 2.6 we know that we can find an increasing sequence of bounded open sets $\{\Omega_k\}$ with smooth boundary such that

$$\Omega_k \subset\subset \Omega_{k+1} \subset\subset \Omega \qquad \text{and} \qquad \bigcup_{k\in\mathbb{N}} \Omega_k = \Omega.$$

Moreover, Hypothesis (2.15) guarantees that

$$\forall k \quad \exists h(k) \text{ s.t.} \quad P_s^L(E_h, \Omega_k) \leqslant c_k < \infty, \quad \forall h \geqslant h(k). \tag{2.16}$$

Clearly

$$\|\chi_{E_h}\|_{L^1(\Omega_k)} \leqslant |\Omega_k| < \infty,$$

and hence, since $[\chi_{E_h}]_{W^{s,1}(\Omega_k)} = 2P_s^L(E_h, \Omega_k)$, we have

$$\|\chi_{E_h}\|_{W^{s,1}(\Omega_k)} \leqslant c'_k, \quad \forall h \geqslant h(k).$$

Therefore Corollary 7.2 of [9] (notice that each Ω_k is an extension domain) guarantees for every fixed k the existence of a subsequence $h_i \nearrow \infty$ (with $h_1 \geqslant h(k)$) such that

$$E_{h_i} \cap \Omega_k \xrightarrow{i\to\infty} E^k$$

in measure, for some set $E^k \subset \Omega_k$.

Applying this argument for $k = 1$ we get a subsequence $\{h_i^1\}$ with

$$E_{h_i^1} \cap \Omega_1 \xrightarrow{i\to\infty} E^1.$$

Applying again this argument in Ω_2, with $\{E_{h_i^1}\}$ in place of $\{E_h\}$, we get a subsequence $\{h_i^2\}$ of $\{h_i^1\}$ with

$$E_{h_i^2} \cap \Omega_2 \xrightarrow{i\to\infty} E^2.$$

Notice that, since $\Omega_1 \subset \Omega_2$, we must have $E^2 \cap \Omega_1 = E^1$ in measure (by the uniqueness of the limit in Ω_1). We can also suppose that $h_1^2 > h_1^1$.
Proceeding inductively in this way we get an increasing subsequence $\{h_1^k\}$ such that

$$E_{h_1^i} \cap \Omega_k \xrightarrow{i\to\infty} E^k, \qquad \text{for every } k \in \mathbb{N},$$

with $E^{k+1} \cap \Omega_k = E^k$. Therefore if we define $E := \bigcup_k E^k$, since $\bigcup_k \Omega_k = \Omega$, we get

$$E_{h_1^i} \cap \Omega \xrightarrow{loc} E,$$

concluding the proof. □

REMARK 2.14 If E_h is s-minimal in Ω_k for every $h \geqslant h(k)$, then by minimality we get

$$P_s^L(E_h, \Omega_k) \leqslant P_s(E_h, \Omega_k) \leqslant P_s(E_h \setminus \Omega_k, \Omega_k) \leqslant P_s(\Omega_k) =: c_k < \infty,$$

since Ω_k is bounded and has Lipschitz boundary. Therefore $\{E_h\}$ satisfies the hypothesis of Proposition 2.13 and we can find a convergent subsequence.

3. Generalized coarea and approximation by smooth sets

We begin by showing that the s-perimeter satisfies a generalized coarea formula (see also [20] and Lemma 10 in [2]). In the end of this section we will exploit this formula to prove that a set E of locally finite s-perimeter can be approximated by smooth sets whose s-perimeter converges to that of E.

Let $\Omega \subset \mathbb{R}^n$ be an open set. Given a function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$, we define the functional

$$\mathfrak{F}(u, \Omega) := \frac{1}{2}\int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x - y|^{n+s}} dx\, dy + \int_\Omega \int_{\complement\Omega} \frac{|u(x) - u(y)|}{|x - y|^{n+s}} dx\, dy, \tag{3.1}$$

that is, half the "Ω-contribution" to the $W^{s,1}$-seminorm of u.

Notice that

$$\mathfrak{F}(\chi_E, \Omega) = P_s(E, \Omega)$$

and, clearly,

$$\mathfrak{F}(u, \mathbb{R}^n) = \frac{1}{2}[u]_{W^{s,1}(\mathbb{R}^n)}.$$

Proposition 3.1 (Coarea) *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$. Then*

$$\mathfrak{F}(u, \Omega) = \int_{-\infty}^{\infty} P_s\big(\{u > t\}, \Omega\big)\, dt. \tag{3.2}$$

In particular

$$\frac{1}{2}[u]_{W^{s,1}(\Omega)} = \int_{-\infty}^{\infty} P_s^L\big(\{u > t\}, \Omega\big)\, dt.$$

Proof. Notice that for every $x,\ y \in \mathbb{R}^n$ we have

$$|u(x) - u(y)| = \int_{-\infty}^{\infty} |\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|\, dt. \tag{3.3}$$

Indeed, the function $t \longmapsto |\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|$ takes only the values $\{0, 1\}$ and it is different from 0 precisely in the interval having $u(x)$ and $u(y)$ as extremes. Therefore, if we plug (3.3) into (3.1) and use Fubini's Theorem, we get

$$\mathfrak{F}(u, \Omega) = \int_{-\infty}^{\infty} \mathfrak{F}\big(\chi_{\{u>t\}}, \Omega\big)\, dt = \int_{-\infty}^{\infty} P_s\big(\{u > t\}, \Omega\big)\, dt,$$

as wanted. □

3.1 *Approximation results for the functional $\mathfrak{F}$*

In this section we prove the approximation properties for the functional $\mathfrak{F}$ which we need for the proofs of Theorem 1.1 and Theorem 1.3. To this end we consider a (symmetric) smooth function η such that

$$\eta \in C_c^\infty(\mathbb{R}^n), \quad \operatorname{supp} \eta \subset B_1, \quad \eta \geqslant 0, \quad \eta(-x) = \eta(x), \quad \int_{\mathbb{R}^n} \eta\, dx = 1,$$

and we define the mollifier

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta\Big(\frac{x}{\varepsilon}\Big),$$

for every $\varepsilon \in (0, 1)$. Notice that $\operatorname{supp} \eta_\varepsilon \subset B_\varepsilon$ and $\int_{\mathbb{R}^n} \eta_\varepsilon = 1$.

Given $u \in L^1_{loc}(\mathbb{R}^n)$, we define the ε-regularization of u as the convolution

$$u_\varepsilon(x) := (u * \eta_\varepsilon)(x) = \int_{\mathbb{R}^n} u(x - \xi)\eta_\varepsilon(\xi)\, d\xi, \quad \text{for every } x \in \mathbb{R}^n.$$

It is well known that $u_\varepsilon \in C^\infty(\mathbb{R}^n)$ and

$$u_\varepsilon \longrightarrow u \qquad \text{in } L^1_{loc}(\mathbb{R}^n).$$

Moreover, if $u = \chi_E$, then

$$0 \leqslant u_\varepsilon \leqslant 1 \qquad \text{and} \quad u_\varepsilon(x) = \begin{cases} 1, & \text{if } |B_\varepsilon(x) \setminus E| = 0 \\ 0, & \text{if } |B_\varepsilon(x) \cap E| = 0 \end{cases}, \tag{3.4}$$

(see, e.g., Section 12.3 of [17]).

Lemma 3.2 (i) *Let $u \in L^1_{loc}(\mathbb{R}^n)$ and let $\Omega \subset \mathbb{R}^n$ be an open set. Then*

$$\mathcal{F}(u,\Omega) < \infty \quad \Longrightarrow \quad \lim_{\varepsilon\to 0^+} \mathcal{F}(u_\varepsilon,\Omega') = \mathcal{F}(u,\Omega') \qquad \forall\, \Omega' \subset\subset \Omega. \tag{3.5}$$

(ii) *Let $u \in W^{s,1}(\mathbb{R}^n)$. Then*

$$\lim_{\varepsilon\to 0}[u_\varepsilon]_{W^{s,1}(\mathbb{R}^n)} = [u]_{W^{s,1}(\mathbb{R}^n)}.$$

(iii) *Let $u \in W^{s,1}(\mathbb{R}^n)$. Then there exists $\{u_k\} \subset C^\infty_c(\mathbb{R}^n)$ such that*

$$\|u - u_k\|_{L^1(\mathbb{R}^n)} \longrightarrow 0 \quad \text{and} \quad \lim_{k\to\infty}[u_k]_{W^{s,1}(\mathbb{R}^n)} = [u]_{W^{s,1}(\mathbb{R}^n)}.$$

Moreover, if $u = \chi_E$, then $0 \leqslant u_k \leqslant 1$.

Proof. (i) Given $\mathcal{O} \subset \mathbb{R}^n$, let $Q(\mathcal{O}) := \mathbb{R}^{2n} \setminus (\mathcal{C}\mathcal{O})^2$, so that

$$\mathcal{F}(u,\mathcal{O}) = \frac{1}{2}\int_{Q(\mathcal{O})} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy.$$

Notice that if $\mathcal{O} \subset \Omega$, then $Q(\mathcal{O}) \subset Q(\Omega)$ and hence

$$\mathcal{F}(u,\mathcal{O}) \leqslant \mathcal{F}(u,\Omega). \tag{3.6}$$

Now let $\Omega' \subset\subset \Omega$ and notice that for ε small enough we have

$$Q(\Omega' - \varepsilon\xi) \subset Q(\Omega) \qquad \text{for every } \xi \in B_1. \tag{3.7}$$

As a consequence

$$\mathcal{F}(u_\varepsilon,\Omega') \leqslant \int_{B_1} \mathcal{F}(u,\Omega'-\varepsilon\xi)\eta(\xi)\,d\xi \leqslant \mathcal{F}(u,\Omega). \tag{3.8}$$

The second inequality follows from (3.7), (3.6) and $\int_{B_1}\eta = 1$.
As for the first inequality, we have

$$\begin{aligned}
\int_{Q(\Omega')} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|}{|x-y|^{n+s}}dx\,dy &= \int_{Q(\Omega')}\Big|\int_{\mathbb{R}^n}\big(u(x-\xi)-u(y-\xi)\big)\frac{1}{\varepsilon^n}\eta\Big(\frac{\xi}{\varepsilon}\Big)\,d\xi\Big|\frac{dx\,dy}{|x-y|^{n+s}}\\
&= \int_{Q(\Omega')}\Big|\int_{B_1}\big(u(x-\varepsilon\xi)-u(y-\varepsilon\xi)\big)\eta(\xi)\,d\xi\Big|\frac{dx\,dy}{|x-y|^{n+s}}\\
&\leqslant \int_{B_1}\Big(\int_{Q(\Omega')}\frac{|u(x-\varepsilon\xi)-u(y-\varepsilon\xi)|}{|x-y|^{n+s}}dx\,dy\Big)\eta(\xi)\,d\xi\\
&= \int_{B_1}\Big(\int_{Q(\Omega'-\varepsilon\xi)}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}dx\,dy\Big)\eta(\xi)\,d\xi.
\end{aligned}$$

We prove something stronger than the claim, that is

$$\lim_{\varepsilon\to 0^+} \mathcal{F}(u_\varepsilon - u,\Omega') = 0. \tag{3.9}$$

Indeed, notice that

$$|\mathcal{F}(u_\varepsilon, \Omega') - \mathcal{F}(u, \Omega')| \leqslant \mathcal{F}(u_\varepsilon - u, \Omega').$$

Let $\psi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be defined as

$$\psi(x, y) := \frac{u(x) - u(y)}{|x - y|^{n+s}}.$$

Moreover, for every $\varepsilon > 0$ and $\xi \in B_1$, we consider the left translation by $\varepsilon(\xi, \xi)$ in $\mathbb{R}^{2n}$, that is

$$(L_{\varepsilon\xi} f)(x, y) := f(x - \varepsilon\xi, y - \varepsilon\xi),$$

for every $f : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$.

Since $\psi \in L^1(Q(\Omega))$, for every $\delta > 0$ there exists $\Psi \in C_c^1(Q(\Omega))$ such that

$$\|\psi - \Psi\|_{L^1(Q(\Omega))} \leqslant \frac{\delta}{2}.$$

We have

$$\begin{aligned}
\mathcal{F}(u_\varepsilon - u, \Omega') &= \int_{Q(\Omega')} \frac{|u_\varepsilon(x) - u_\varepsilon(y) - u(x) + u(y)|}{|x - y|^{n+s}} dx\, dy \\
&\leqslant \int_{B_1} \Big(\int_{Q(\Omega')} \frac{|u(x - \varepsilon\xi) - u(y - \varepsilon\xi) - u(x) + u(y)|}{|x - y|^{n+s}} dx\, dy \Big) \eta(\xi)\, d\xi \\
&= \int_{B_1} \|L_{\varepsilon\xi}\psi - \psi\|_{L^1(Q(\Omega'))} \eta(\xi)\, d\xi \\
&\leqslant \int_{B_1} \Big(\|L_{\varepsilon\xi}\psi - L_{\varepsilon\xi}\Psi\|_{L^1(Q(\Omega'))} + \|L_{\varepsilon\xi}\Psi - \Psi\|_{L^1(Q(\Omega'))} \\
&\qquad\qquad + \|\Psi - \psi\|_{L^1(Q(\Omega'))} \Big) \eta(\xi)\, d\xi.
\end{aligned}$$

Notice that

$$\|L_{\varepsilon\xi}\psi - L_{\varepsilon\xi}\Psi\|_{L^1(Q(\Omega'))} = \|\psi - \Psi\|_{L^1(Q(\Omega' - \varepsilon\xi))} \leqslant \|\psi - \Psi\|_{L^1(Q(\Omega))}$$

and hence

$$\mathcal{F}(u_\varepsilon - u, \Omega') \leqslant \delta + \int_{B_1} \|L_{\varepsilon\xi}\Psi - \Psi\|_{L^1(Q(\Omega'))} \eta(\xi)\, d\xi.$$

For $\varepsilon > 0$ small enough we have

$$\operatorname{supp}(L_{\varepsilon\xi}\Psi - \Psi) \subset N_1(\operatorname{supp} \Psi) =: K \subset\subset \mathbb{R}^{2n},$$

and

$$|\Psi(x - \varepsilon\xi, y - \varepsilon\xi) - \Psi(x, y)| \leqslant 2 \max_{\operatorname{supp} \Psi} |\nabla \Psi|\, \varepsilon.$$

Thus

$$\int_{B_1} \|L_{\varepsilon\xi}\Psi - \Psi\|_{L^1(Q(\Omega'))} \eta(\xi)\, d\xi \leqslant 2|K| \max_{\operatorname{supp} \Psi} |\nabla \Psi|\, \varepsilon.$$

Passing to the limit as $\varepsilon \to 0^+$ then gives

$$\limsup_{\varepsilon \to 0^+} \mathcal{F}(u_\varepsilon - u, \Omega') \leqslant \delta.$$

Since δ is arbitrary, we get (3.9).

(ii) Reasoning as above we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|}{|x-y|^{n+s}} dx\,dy &\leqslant \int_{B_1} \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x-\varepsilon\xi) - u(y-\varepsilon\xi)|}{|x-y|^{n+s}} dx\,dy \Big) \eta(\xi)\,d\xi \\
&= \int_{B_1} \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x-y|^{n+s}} dx\,dy \Big) \eta(\xi)\,d\xi \\
&= [u]_{W^{s,1}(\mathbb{R}^n)} \int_{B_1} \eta(\xi)\,d\xi,
\end{aligned}$$

that is

$$[u_\varepsilon]_{W^{s,1}(\mathbb{R}^n)} \leqslant [u]_{W^{s,1}(\mathbb{R}^n)}. \tag{3.10}$$

This and Fatou's Lemma give

$$[u]_{W^{s,1}(\mathbb{R}^n)} \leqslant \liminf_{\varepsilon \to 0} [u_\varepsilon]_{W^{s,1}(\mathbb{R}^n)} \leqslant \limsup_{\varepsilon \to 0} [u_\varepsilon]_{W^{s,1}(\mathbb{R}^n)} \leqslant [u]_{W^{s,1}(\mathbb{R}^n)},$$

concluding the proof.

(iii) The proof is a classical cut-off argument. We consider a sequence of cut-off functions $\psi_k \in C_c^\infty(\mathbb{R}^n)$ such that

$$0 \leqslant \psi_k \leqslant 1, \quad \operatorname{supp} \psi_k \subset B_{k+1} \quad \text{and} \quad \psi_k \equiv 1 \quad \text{in } B_k.$$

We can also assume that

$$\sup_{k \in \mathbb{N}} |\nabla \psi_k| \leqslant M_0 < \infty.$$

It is enough to show that

$$\lim_{k\to\infty} \|u - \psi_k u\|_{L^1(\mathbb{R}^n)} = 0 \quad \text{and} \quad \lim_{k\to\infty} [\psi_k u]_{W^{s,1}(\mathbb{R}^n)} = [u]_{W^{s,1}(\mathbb{R}^n)}. \tag{3.11}$$

Indeed then we can use (ii) to approximate each $\psi_k u$ with a smooth function $u_k := (u\psi_k) * \eta_{\varepsilon_k}$, for ε_k small enough to have

$$\|\psi_k u - u_k\|_{L^1(\mathbb{R}^n)} < 2^{-k} \quad \text{and} \quad |[\psi_k u]_{W^{s,1}(\mathbb{R}^n)} - [u_k]_{W^{s,1}(\mathbb{R}^n)}| < 2^{-k}.$$

Therefore

$$\|u - u_k\|_{L^1(\mathbb{R}^n)} \leqslant \|u - \psi_k u\|_{L^1(\mathbb{R}^n)} + 2^{-k} \longrightarrow 0$$

and

$$|[u]_{W^{s,1}(\mathbb{R}^n)} - [u_k]_{W^{s,1}(\mathbb{R}^n)}| \leqslant |[u]_{W^{s,1}(\mathbb{R}^n)} - [\psi_k u]_{W^{s,1}(\mathbb{R}^n)}| + 2^{-k} \longrightarrow 0.$$

Also notice that

$$\operatorname{supp} u_k \subset N_{\varepsilon_k}(\operatorname{supp} \psi_k u) \subset B_{k+2}$$

so that $u_k \in C_c^\infty(\mathbb{R}^n)$ for every k. Moreover, from the definition of u_k it follows that if $u = \chi_E$, then $0 \leqslant u_k \leqslant 1$.

For a proof of (3.11) see, e.g., Lemma 12 in [15]. □

Now we show that if Ω is a bounded open set with Lipschitz boundary and if $u = \chi_E$, then we can find smooth functions u_h such that

$$\mathcal{F}(u_h, \Omega) \longrightarrow \mathcal{F}(u, \Omega).$$

We first need the following two results.

Lemma 3.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $u \in L^\infty(\mathbb{R}^n)$ be such that $\mathcal{F}(u, \Omega) < \infty$. For every $\delta \in (0, r_0)$ let*

$$\varphi_\delta := 1 - \chi_{\{|\bar{d}_\Omega| < \delta\}}.$$

Then

$$u\varphi_\delta \xrightarrow{\delta \to 0} u \quad \text{in } L^1(\mathbb{R}^n), \tag{3.12}$$

and

$$\lim_{\delta \searrow 0^+} \mathcal{F}(u\varphi_\delta, \Omega) = \mathcal{F}(u, \Omega). \tag{3.13}$$

Proof. First of all, notice that

$$\int_{\mathbb{R}^n} |u\varphi_\delta - u|\, dx = \int_{\{|\bar{d}_\Omega| < \delta\}} |u|\, dx \leqslant \|u\|_{L^\infty(\mathbb{R}^n)}\, |\{|\bar{d}_\Omega| < \delta\}| \xrightarrow{\delta \to 0} 0.$$

Now

$$\begin{aligned} &\int_\Omega \int_\Omega \frac{|(u\varphi_\delta)(x) - (u\varphi_\delta)(y)|}{|x - y|^{n+s}}\, dx\, dy \\ &\quad = \int_{\Omega_{-\delta}} \int_{\Omega_{-\delta}} \frac{|u(x) - u(y)|}{|x - y|^{n+s}}\, dx\, dy + 2\int_{\Omega_{-\delta}} \Big(\int_{\Omega \setminus \Omega_{-\delta}} \frac{|u(x)|}{|x - y|^{n+s}}\, dy \Big) dx. \end{aligned}$$

Since $\Omega_{-\delta} \subset \Omega$, we have

$$\int_{\Omega_{-\delta}} \int_{\Omega_{-\delta}} \frac{|u(x) - u(y)|}{|x - y|^{n+s}}\, dx\, dy \leqslant \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x - y|^{n+s}}\, dx\, dy.$$

On the other hand, since $|\Omega \setminus \Omega_{-\delta}| \longrightarrow 0$, we get

$$\frac{|u(x) - u(y)|}{|x - y|^{n+s}} \chi_{\Omega_{-\delta}}(x) \chi_{\Omega_{-\delta}}(y) \xrightarrow{\delta \to 0} \frac{|u(x) - u(y)|}{|x - y|^{n+s}} \chi_\Omega(x) \chi_\Omega(y),$$

for a.e. $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Therefore, by Fatou's Lemma we obtain

$$[u]_{W^{s,1}(\Omega)} \leqslant \liminf_{\delta \searrow 0} [u]_{W^{s,1}(\Omega_{-\delta})} \leqslant \limsup_{\delta \searrow 0} [u]_{W^{s,1}(\Omega_{-\delta})} \leqslant [u]_{W^{s,1}(\Omega)}. \tag{3.14}$$

Moreover, by point (i) of (2.9) we get

$$\begin{aligned} 2\int_{\Omega_{-\delta}} \Big(\int_{\Omega \setminus \Omega_{-\delta}} \frac{|u(x)|}{|x - y|^{n+s}}\, dy \Big) dx &\leqslant 2\|u\|_{L^\infty(\mathbb{R}^n)} \mathcal{L}_s(\Omega_{-\delta}, \Omega \setminus \Omega_{-\delta}) \\ &\leqslant 2C\, \|u\|_{L^\infty(\mathbb{R}^n)}\, \delta^{1-s}. \end{aligned}$$

Therefore we find

$$\lim_{\delta\searrow 0}[u\varphi_\delta]_{W^{s,1}(\Omega)} = [u]_{W^{s,1}(\Omega)}. \tag{3.15}$$

Now

$$\begin{aligned}\int_\Omega\int_{\complement\Omega} &\frac{|(u\varphi_\delta)(x)-(u\varphi_\delta)(y)|}{|x-y|^{n+s}}\,dx\,dy\\ &= \int_{\Omega_{-\delta}}\int_{\complement\Omega_\delta}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy + \int_{\Omega_{-\delta}}\Big(\int_{\Omega_\delta\setminus\Omega}\frac{|u(x)|}{|x-y|^{n+s}}\,dy\Big)dx\\ &\qquad + \int_{\Omega\setminus\Omega_{-\delta}}\Big(\int_{\complement\Omega_\delta}\frac{|u(x)|}{|x-y|^{n+s}}\,dy\Big)dx.\end{aligned}$$

Since $\Omega_{-\delta}\subset\Omega$ and $\complement\Omega_\delta\subset\complement\Omega$, we have

$$\int_{\Omega_{-\delta}}\int_{\complement\Omega_\delta}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy \leqslant \int_\Omega\int_{\complement\Omega}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy.$$

Moreover, since both $|\Omega\setminus\Omega_{-\delta}|\longrightarrow 0$ and $|\complement\Omega\setminus\complement\Omega_\delta|\longrightarrow 0$, we have

$$\frac{|u(x)-u(y)|}{|x-y|^{n+s}}\chi_{\Omega_{-\delta}}(x)\chi_{\complement\Omega_\delta}(y)\xrightarrow{\delta\to 0}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}\chi_\Omega(x)\chi_{\complement\Omega}(y),$$

for a.e. $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$.

Therefore, again by Fatou's Lemma we obtain

$$\lim_{\delta\searrow 0}\int_{\Omega_{-\delta}}\int_{\complement\Omega_\delta}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy = \int_\Omega\int_{\complement\Omega}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy. \tag{3.16}$$

Furthermore, by point (ii) of (2.9) we get

$$\begin{aligned}\int_{\Omega_{-\delta}}\Big(\int_{\Omega_\delta\setminus\Omega}\frac{|u(x)|}{|x-y|^{n+s}}\,dy\Big)dx &\leqslant \|u\|_{L^\infty(\mathbb{R}^n)}\mathfrak{L}_s(\Omega_{-\delta},\Omega_\delta\setminus\Omega)\\ &\leqslant \|u\|_{L^\infty(\mathbb{R}^n)}\mathfrak{L}_s(\Omega,\Omega_\delta\setminus\Omega)\leqslant C\|u\|_{L^\infty(\mathbb{R}^n)}\delta^{1-s}\end{aligned}$$

and also

$$\int_{\Omega\setminus\Omega_{-\delta}}\Big(\int_{\complement\Omega_\delta}\frac{|u(x)|}{|x-y|^{n+s}}\,dy\Big)dx \leqslant C\|u\|_{L^\infty(\mathbb{R}^n)}\delta^{1-s}.$$

Thus

$$\lim_{\delta\searrow 0}\int_\Omega\int_{\complement\Omega}\frac{|(u\varphi_\delta)(x)-(u\varphi_\delta)(y)|}{|x-y|^{n+s}}\,dx\,dy = \int_\Omega\int_{\complement\Omega}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy, \tag{3.17}$$

concluding the proof. □

Lemma 3.4 *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $v\in L^\infty(\mathbb{R}^n)$ be such that $\mathcal{F}(v,\Omega)<\infty$ and*

$$v\equiv 0 \quad \textit{in } \{|\bar{d}_\Omega|<\delta/2\},$$

for some $\delta\in(0,r_0)$. Then

$$\big|\mathcal{F}(v,\Omega)-\mathcal{F}(v,\Omega_{-\delta/2})\big|\leqslant C\|v\|_{L^\infty(\mathbb{R}^n)}\delta^{1-s}, \tag{3.18}$$

where $C=C(n,s,\Omega)>0$ does not depend on v.

Proof. Since

$$v \equiv 0 \quad \text{in } \{|\bar{d}_\Omega| < \delta/2\},$$

we have

$$\mathfrak{F}(v, \Omega) = \mathfrak{F}(v, \Omega_{-\delta/2}) + 2 \int_{\Omega \setminus \Omega_{-\delta/2}} \Big(\int_{\complement\Omega_{\delta/2}} \frac{|v(y)|}{|x-y|^{n+s}}\, dy \Big) dx.$$

Now, by point (ii) of (2.9) we have

$$\begin{aligned}\int_{\Omega \setminus \Omega_{-\delta/2}} \Big(\int_{\complement\Omega_{\delta/2}} \frac{|v(y)|}{|x-y|^{n+s}}\, dy \Big) &\leqslant \|v\|_{L^\infty(\mathbb{R}^n)} \mathfrak{L}_s(\Omega \setminus \Omega_{-\delta/2}, \complement\Omega) \\ &\leqslant 2^{s-1} C \|v\|_{L^\infty(\mathbb{R}^n)}\, \delta^{1-s}.\end{aligned}$$

□

Proposition 3.5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $u \in L^\infty(\mathbb{R}^n)$ be such that $\mathfrak{F}(u, \Omega) < \infty$. Then there exists a sequence $\{u_h\} \subset C^\infty(\mathbb{R}^n)$ such that*

$$\begin{aligned}&\text{(i)} \quad \|u_h\|_{L^\infty(\mathbb{R}^n)} \leqslant \|u\|_{L^\infty(\mathbb{R}^n)}, \quad \textit{and} \quad 0 \leqslant u_h \leqslant 1 \quad \textit{if} \quad 0 \leqslant u \leqslant 1, \\ &\text{(ii)} \quad u_h \xrightarrow{h\to\infty} u \quad \textit{in } L^1_{loc}(\mathbb{R}^n), \\ &\text{(iii)} \quad \lim_{h\to\infty} \mathfrak{F}(u_h, \Omega) = \mathfrak{F}(u, \Omega).\end{aligned} \tag{3.19}$$

Proof. By Lemma 3.3 we know that for every $h \in \mathbb{N}$ we can find δ_h small enough such that

$$\|u - u\varphi_{\delta_h}\|_{L^1(\mathbb{R}^n)} < 2^{-h} \quad \text{and} \quad \big|\mathfrak{F}(u, \Omega) - \mathfrak{F}(u\varphi_{\delta_h}, \Omega)\big| < 2^{-h}. \tag{3.20}$$

We can assume that $\delta_h \searrow 0$.

By point (i) of Lemma 3.2 we know that for every h we can find ε_h small enough such that

$$\|(u\varphi_{\delta_h}) * \eta_{\varepsilon_h} - u\varphi_{\delta_h}\|_{L^1(B_h)} < 2^{-h} \tag{3.21}$$

and

$$\big|\mathfrak{F}(u\varphi_{\delta_h}, \Omega_{-\delta_h/2}) - \mathfrak{F}((u\varphi_{\delta_h}) * \eta_{\varepsilon_h}, \Omega_{-\delta_h/2})\big| < 2^{-h}. \tag{3.22}$$

Taking ε_h small enough, we can also assume that

$$(u\varphi_{\delta_h}) * \eta_{\varepsilon_h} \equiv 0 \quad \text{in } \{|\bar{d}_\Omega| < \delta_h/2\}, \tag{3.23}$$

since the ε-convolution enlarges the support at most to an ε-neighborhood of the original support.

Let $u_h := (u\varphi_{\delta_h}) * \eta_{\varepsilon_h}$. Since we are taking the ε_h-regularization of the function $u\varphi_{\delta_h}$, which is just the product of u with a characteristic function, point (i) of our claim is immediate.

By (3.21) and the first part of (3.20) we get point (ii).

As for point (iii), exploiting (3.23) and Lemma 3.4, we obtain

$$\begin{aligned}\big|\mathfrak{F}(u, \Omega) - \mathfrak{F}(u_h, \Omega)\big| &\leqslant \big|\mathfrak{F}(u, \Omega) - \mathfrak{F}(u\varphi_{\delta_h}, \Omega)\big| + \big|\mathfrak{F}(u\varphi_{\delta_h}, \Omega) - \mathfrak{F}(u\varphi_{\delta_h}, \Omega_{-\delta_h/2})\big| \\ &\quad + \big|\mathfrak{F}(u\varphi_{\delta_h}, \Omega_{-\delta_h/2}) - \mathfrak{F}(u_h, \Omega_{-\delta_h/2})\big| \\ &\quad + \big|\mathfrak{F}(u_h, \Omega_{-\delta_h/2}) - \mathfrak{F}(u_h, \Omega)\big| \\ &\leqslant 2^{-h} + 2^s C \|u\|_{L^\infty(\mathbb{R}^n)} \delta_h^{1-s} + 2^{-h},\end{aligned}$$

which goes to 0 as $h \longrightarrow \infty$. □

3.2 *Proof of Theorem* 1.1 *and Theorem* 1.3

Exploiting Lemma 3.2 and the coarea formula, we can now prove Theorem 1.1.

Proof of Theorem 1.1. The "if part" is trivial. Indeed, just from point (i) and the lower semicontinuity of the s-perimeter we get

$$P_s(E,\Omega') \leqslant \liminf_{h\to\infty} P_s(E_h,\Omega') < \infty,$$

for every $\Omega' \subset\subset \Omega$.

Now suppose that E has locally finite s-perimeter in Ω.

The scheme of the proof is similar to that of the classical case (see, e.g., the proof of Theorem 13.8 of [17]).

Given a sequence $\varepsilon_h \searrow 0^+$ we consider the ε_h-regularization of $u := \chi_E$ and define the sets

$$E_h^t := \{u_{\varepsilon_h} > t\} \quad \text{with } t \in (0,1).$$

Sard's Theorem guarantees that for a.e. $t \in (0,1)$ the sequence $\{E_h^t\}_h$ is made of open sets with smooth boundary. We will get our sets E_h by opportunely choosing t.

Since $u_{\varepsilon_h} \longrightarrow \chi_E$ in $L^1_{loc}(\mathbb{R}^n)$, it is readily seen that for a.e. $t \in (0,1)$

$$E_h^t \xrightarrow{loc} E,$$

and hence the lower semicontinuity of the s-perimeter gives

$$P_s(E,\mathcal{O}) \leqslant \liminf_{h\to\infty} P_s(E_h^t,\mathcal{O}), \tag{3.24}$$

for every open set $\mathcal{O} \subset \mathbb{R}^n$.

Moreover from (3.4) we have

$$\{0 < u_\varepsilon < 1\} \subset N_\varepsilon(\partial E) \qquad \forall\, \varepsilon > 0,$$

and hence, since $\partial E_h^t \subset \{u_{\varepsilon_h} = t\}$, we obtain

$$\partial E_h^t \subset N_{\varepsilon_h}(\partial E), \tag{3.25}$$

which will give (iii) once we choose our t.

We improve (3.24) by showing that, if $\Omega' \subset\subset \Omega$ is a fixed bounded open set, then for a.e. $t \in (0,1)$ (with the set of exceptional values of t possibly depending on Ω'),

$$P_s(E,\Omega') = \liminf_{h\to\infty} P_s(E_h^t,\Omega'). \tag{3.26}$$

By (3.24) and Fatou's Lemma, we have

$$P_s(E,\Omega') \leqslant \int_0^1 \liminf_{h\to\infty} P_s(E_h^t,\Omega')\,dt \leqslant \liminf_{h\to\infty} \int_0^1 P_s(E_h^t,\Omega')\,dt. \tag{3.27}$$

Let $\mathcal{O}$ be a bounded open set such that $\Omega' \subset\subset \mathcal{O} \subset\subset \Omega$. Since E has locally finite s-perimeter in Ω, we have $P_s(E,\mathcal{O}) < \infty$. Then, since $\Omega' \subset\subset \mathcal{O}$, point (i) of Lemma 3.2 (with $\mathcal{O}$ in the place of Ω) implies

$$\lim_{h\to\infty} \mathcal{F}(u_{\varepsilon_h},\Omega') = \mathcal{F}(\chi_E,\Omega') = P_s(E,\Omega'). \tag{3.28}$$

Since $0 \leqslant u_{\varepsilon_h} \leqslant 1$, we have $E_h^t = \mathbb{R}^n$ if $t < 0$ and $E_h^t = \emptyset$ if $t > 1$, and hence rewriting (3.28) exploiting the coarea formula,

$$\lim_{h\to\infty} \int_0^1 P_s(E_h^t, \Omega')\,dt = P_s(E, \Omega').$$

This and (3.27) give

$$\int_0^1 \liminf_{h\to\infty} P_s(E_h^t, \Omega')\,dt = P_s(E, \Omega') = \int_0^1 P_s(E, \Omega')\,dt,$$

which implies

$$P_s(E, \Omega') = \liminf_{h\to\infty} P_s(E_h^t, \Omega'), \quad \text{for a.e. } t \in (0, 1), \tag{3.29}$$

as claimed.

Now let the sets $\Omega_k \subset\subset \Omega$ be as in Corollary 2.6. From (3.29) we deduce that for a.e. $t \in (0, 1)$ we have

$$P_s(E, \Omega_k) = \liminf_{h\to\infty} P_s(E_h^t, \Omega_k), \qquad \forall\, k \in \mathbb{N}. \tag{3.30}$$

Therefore, combining all we wrote so far, we find that for a.e. $t \in (0, 1)$ the sequence $\{E_h^t\}_h$ is made of open sets with smooth boundary such that $E_h^t \xrightarrow{loc} E$ and both (3.25) and (3.30) hold true.

To conclude, by a diagonal argument we can find $t_0 \in (0, 1)$ and $h_i \nearrow \infty$ such that, if we define $E_i := E_{h_i}^{t_0}$, then $\{E_i\}$ is a sequence of open sets with smooth boundary such that $E_i \xrightarrow{loc} E$, with $\partial E_i \subset N_{\varepsilon_{h_i}}(\partial E)$, and

$$P_s(E, \Omega_k) = \lim_{i\to\infty} P_s(E_i, \Omega_k), \qquad \forall\, k \in \mathbb{N}. \tag{3.31}$$

Now notice that if $\Omega' \subset\subset \Omega$, then there exists a k such that $\Omega' \subset\subset \Omega_k$. Therefore by (3.31) and Proposition 2.12 we get (ii).

This concludes the proof of the first part of the claim.

Now suppose that $\Omega = \mathbb{R}^n$ and $|E|,\ P_s(E) < \infty$.

Since $|E| < \infty$, we know that $u_\varepsilon \longrightarrow \chi_E$ in $L^1(\mathbb{R}^n)$. Therefore we obtain $E_h^t \longrightarrow E$ for a.e. $t \in (0, 1)$.

Moreover, from point (ii) of Lemma 3.2 we know that

$$\mathcal{F}(u, \mathbb{R}^n) < \infty \qquad \Longrightarrow \qquad \lim_{\varepsilon\to 0} \mathcal{F}(u_\varepsilon, \mathbb{R}^n) = \mathcal{F}(u, \mathbb{R}^n).$$

We can thus repeat the proof above and obtain

$$P_s(E) = \liminf_{h\to\infty} P_s(E_h^t),$$

for a.e. $t \in (0, 1)$. For any fixed "good" $t_0 \in (0, 1)$ this directly implies, with no need of a diagonal argument, the existence of a subsequence $h_i \nearrow \infty$ such that

$$P_s(E) = \lim_{i\to\infty} P_s(E_{h_i}^{t_0}).$$

We are left to show that in this case we can take the sets E_h to be bounded.

To this end, it is enough to replace the functions u_{ε_k} with the functions u_k obtained in point (iii) of Lemma 3.2.

Indeed, since u_k has compact support, for each $t \in (0,1)$ the set

$$E_k^t := \{u_k > t\}$$

is bounded. Since $u_k \longrightarrow u$ in $L^1(\mathbb{R}^n)$ we still find

$$E_k^t \xrightarrow{loc} E \quad \text{for a.e. } t \in (0,1),$$

and, since $0 \leqslant u_k \leqslant 1$ and

$$\lim_{k\to\infty} \mathcal{F}(u_k, \mathbb{R}^n) = P_s(E),$$

we can use again the coarea formula to conclude as above. □

Proof of Theorem 1.3. Exploiting the approximating sequence obtained in Proposition 3.5, we can now prove Theorem 1.3 exactly as above.

As for point (iii), recall that the functions u_h of Proposition 3.5 are defined as

$$u_h = (\chi_E \varphi_{\delta_h}) * \eta_{\varepsilon_h}.$$

Notice that, since we can suppose that $\varepsilon_h < \delta_h/2$, we have

$$u_h = \chi_E * \eta_{\varepsilon_h}, \qquad \text{in } \mathbb{R}^n \setminus N_{2\delta_h}(\partial\Omega).$$

Therefore, for every $t \in (0,1)$ we find

$$\partial\{u_h > t\} \subset N_{\varepsilon_h}(\partial E) \subset N_{2\delta_h}(\partial E), \qquad \text{in } \mathbb{R}^n \setminus N_{2\delta_h}(\partial\Omega).$$

This gives point (iii) once we choose an appropriate t, as in the proof of Theorem 1.1. □

REMARK 3.6 We remark that by Proposition 2.12 we have also

$$\lim_{h\to\infty} P_s(E_h, \Omega') = P_s(E, \Omega'), \qquad \text{for every } \Omega' \subset\subset \Omega.$$

4. Existence and compactness of s-minimal sets

4.1 *Proof of Theorem* 1.7

Proof of Theorem 1.7. (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (iii) Let $\Omega' \subset\subset \Omega$ and let $F \subset \mathbb{R}^n$ be such that $F \setminus \Omega' = E \setminus \Omega'$.

Since $E \Delta F \subset \Omega' \subset\subset \Omega$, we have

$$P_s(E, \Omega) \leqslant P_s(F, \Omega).$$

Then, since $F \setminus \Omega' = E \setminus \Omega'$, by Proposition 2.1 we get

$$P_s(E, \Omega') \leqslant P_s(F, \Omega').$$

(iii) $\Longrightarrow$ (i) Let E be locally s-minimal in Ω.

First of all we prove that $P_s(E,\Omega) < \infty$.

Indeed, since E is locally s-minimal in Ω, in particular it is s-minimal in every Ω_r, with $r \in (-r_0, 0)$. Thus, by minimality and (2.5), we get

$$P_s(E,\Omega_r) \leqslant P_s(E \setminus \Omega_r, \Omega_r) \leqslant P_s(\Omega_r) \leqslant M < \infty,$$

for every $r \in (-r_0, 0)$. Therefore by (2.12) we obtain $P_s(E,\Omega) \leqslant M$.

Now let $F \subset \mathbb{R}^n$ be such that $F \setminus \Omega = E \setminus \Omega$. Take a sequence $\{r_k\} \subset (-r_0, 0)$ such that $r_k \nearrow 0$, let $\Omega_k := \Omega_{r_k}$, and define

$$F_k := (F \cap \Omega_k) \cup (E \setminus \Omega_k).$$

The local minimality of E gives

$$P_s(E,\Omega_k) \leqslant P_s(F_k,\Omega_k), \qquad \text{for every } k \in \mathbb{N},$$

and by (2.12) we know that

$$P_s(E,\Omega) = \lim_{k\to\infty} P_s(E,\Omega_k). \tag{4.1}$$

Since $F_k = F$ outside $\Omega \setminus \Omega_k$, and $F_k = E$ in $\Omega \setminus \Omega_k$, we obtain

$$\begin{aligned} P_s(F,\Omega_k) - P_s(F_k,\Omega_k) = {} & \mathcal{L}_s\big(F \cap \Omega_k, \complement F \cap (\Omega \setminus \Omega_k)\big) \\ & + \mathcal{L}_s\big(\complement F \cap \Omega_k, F \cap (\Omega \setminus \Omega_k)\big) - \mathcal{L}_s\big(F \cap \Omega_k, \complement E \cap (\Omega \setminus \Omega_k)\big) \\ & - \mathcal{L}_s\big(\complement F \cap \Omega_k, E \cap (\Omega \setminus \Omega_k)\big). \end{aligned}$$

Notice that each of the four terms in the right hand side is less or equal than $\mathcal{L}_s(\Omega_k, \Omega \setminus \Omega_k)$. Thus

$$a_k := \big|P_s(F,\Omega_k) - P_s(F_k,\Omega_k)\big| \leqslant 4\,\mathcal{L}_s(\Omega_k, \Omega \setminus \Omega_k).$$

Notice that from point (i) of (2.9) we have $a_k \longrightarrow 0$.

Now

$$P_s(F,\Omega) + a_k \geqslant P_s(F,\Omega_k) + a_k \geqslant P_s(F_k,\Omega_k) \geqslant P_s(E,\Omega_k),$$

and hence, passing to the limit $k \to \infty$, we get

$$P_s(F,\Omega) \geqslant P_s(E,\Omega).$$

Since F was an arbitrary competitor for E, we see that E is s-minimal in Ω. □

4.2 *Compactness*

Proof of Theorem 1.12. Assume $F = E$ outside Ω and let

$$F_k := (F \cap \Omega) \cup (E_k \setminus \Omega).$$

Since $F_k = E_k$ outside Ω and E_k is s-minimal in Ω, we have

$$P_s(F_k,\Omega) \geqslant P_s(E_k,\Omega).$$

On the other hand, since $F_k = F$ inside Ω, we have

$$\big|P_s(F_k,\Omega) - P_s(F,\Omega)\big| \leqslant \mathfrak{L}_s(\Omega,(F_k\Delta F)\setminus\Omega) = \mathfrak{L}_s(\Omega,(E_k\Delta E)\setminus\Omega) =: b_k.$$

Thus

$$P_s(F,\Omega) + b_k \geqslant P_s(F_k,\Omega) \geqslant P_s(E_k,\Omega).$$

If we prove that $b_k \longrightarrow 0$, then by lower semicontinuty of the fractional perimeter

$$P_s(F,\Omega) \geqslant \limsup_{k\to\infty} P_s(E_k,\Omega) \geqslant \liminf_{k\to\infty} P_s(E_k,\Omega) \geqslant P_s(E,\Omega). \tag{4.2}$$

This shows that E is s-minimal in Ω. Moreover, (1.8) follows from (4.2) by taking $F = E$.

We are left to show $b_k \longrightarrow 0$.

Let r_0 be as in Proposition 2.3 and let $R > r_0$. In the end we will let $R \longrightarrow \infty$. Define

$$a_k(r) := \mathcal{H}^{n-1}\big((E_k\Delta E)\cap\{\bar{d}_\Omega = r\}\big)$$

for every $r \in [0, r_0)$.

We split b_k as the sum

$$b_k = \mathfrak{L}_s\big(\Omega,(E_k\Delta E)\cap(\Omega_{r_0}\setminus\Omega)\big) + \mathfrak{L}_s\big(\Omega,(E_k\Delta E)\cap(\Omega_R\setminus\Omega_{r_0})\big) + \mathfrak{L}_s\big(\Omega,(E_k\Delta E)\setminus\Omega_R\big).$$

Notice that if $x \in \Omega$ and $y \in (\Omega_R\setminus\Omega_{r_0})$, then $|x-y| \geqslant r_0$, and hence

$$\begin{aligned}\mathfrak{L}_s\big(\Omega,(E_k\Delta E)\cap(\Omega_R\setminus\Omega_{r_0})\big) &= \int_{\Omega_R\setminus\Omega_{r_0}} \chi_{E_k\Delta E}(y)\,dy\int_\Omega \frac{1}{|x-y|^{n+s}}dx\\ &\leqslant \frac{|\Omega|}{r_0^{n+s}}|(E_k\Delta E)\cap(\Omega_R\setminus\Omega_{r_0})|.\end{aligned}$$

Since $E_k \xrightarrow{loc} E$ and $\Omega_R\setminus\Omega_{r_0}$ is bounded, for every fixed R we find

$$\lim_{k\to\infty}\mathfrak{L}_s\big(\Omega,(E_k\Delta E)\cap(\Omega_R\setminus\Omega_{r_0})\big) = 0.$$

As for the last term, we have

$$\mathfrak{L}_s\big(\Omega,(E_k\Delta E)\setminus\Omega_R\big) \leqslant \mathfrak{L}_s(\Omega,\complement\Omega_R) \leqslant \int_\Omega dx\int_{\complement B_R(x)}\frac{dy}{|x-y|^{n+s}} = \frac{n\omega_n}{s\,R^s}|\Omega|.$$

We are left to estimate the first term. By using the coarea formula, we obtain

$$\begin{aligned}&\mathfrak{L}_s\big(\Omega,(E_k\Delta E)\cap(\Omega_{r_0}\setminus\Omega)\big)\\ &\qquad= \int_0^{r_0}\Big(\int_{\{\bar{d}_\Omega=r\}}\chi_{E_k\Delta E}(y)\Big(\int_\Omega\frac{dx}{|x-y|^{n+s}}\Big)d\mathcal{H}_y^{n-1}\Big)dr\\ &\qquad\leqslant \int_0^{r_0}\Big(\int_{\{\bar{d}_\Omega=r\}}\chi_{E_k\Delta E}(y)\Big(\int_{\complement B_r(y)}\frac{dx}{|x-y|^{n+s}}\Big)d\mathcal{H}_y^{n-1}\Big)dr\\ &\qquad= \frac{n\omega_n}{s}\int_0^{r_0}\frac{a_k(r)}{r^s}\,dr.\end{aligned}$$

Notice that

$$\int_0^{r_0} a_k(r)\,dr = \big|(E_k \Delta E) \cap (\Omega_{r_0} \setminus \Omega)\big| \xrightarrow{k\to\infty} 0,$$

so that

$$a_k(r) \xrightarrow{k\to\infty} 0 \qquad \text{for a.e. } r \in [0, r_0).$$

Moreover, exploiting (2.4) we get

$$\int_0^{r_0} \frac{a_k(r)}{r^s}\,dr \leqslant M \int_0^{r_0} \frac{1}{r^s}\,dr = \frac{M}{1-s} r_0^{1-s},$$

and hence, by dominated convergence, we obtain

$$\lim_{k\to\infty} \int_0^{r_0} \frac{a_k(r)}{r^s}\,dr = 0.$$

Therefore

$$\limsup_{k\to\infty} b_k \leqslant \frac{n\omega_n}{s} |\Omega|\, R^{-s}.$$

Letting $R \longrightarrow \infty$, we obtain $b_k \longrightarrow 0$, concluding the proof. □

Proof of Corollary 1.13. Let the sets $\Omega_k \subset\subset \Omega$ be as in Corollary 2.6. By Theorem 1.12 we see that E is s-minimal in each Ω_k. Moreover (1.8) gives

$$P_s(E, \Omega_k) = \lim_{h\to\infty} P_s(E_h, \Omega_k),$$

for every k. Now if $\Omega' \subset\subset \Omega$, then $\Omega' \subset \Omega_k$ for some k. Thus E is s-minimal in Ω' and we obtain (1.9) by Proposition 2.12. □

4.3 *Existence of (locally) s-minimal sets*

Proof of Theorem 1.9. The "only if" part is trivial. Now suppose there exists a competitor for E_0 with finite s-perimeter in Ω. Then

$$\inf\big\{P_s(E, \Omega) \,|\, E \setminus \Omega = E_0 \setminus \Omega\big\} < \infty$$

and we can find a minimizing sequence, that is $\{E_h\}$ with $E_h \setminus \Omega = E_0 \setminus \Omega$ and

$$\lim_{h\to\infty} P_s(E_h, \Omega) = \inf\big\{P_s(E, \Omega) \,|\, E \setminus \Omega = E_0 \setminus \Omega\big\}.$$

Let $\Omega' \subset\subset \Omega$. Since, for every $h \in \mathbb{N}$ we have

$$P_s(E_h, \Omega') \leqslant P_s(E_h, \Omega) \leqslant M < \infty,$$

we can use Proposition 2.13 to find a set $E' \subset \Omega$ such that

$$E_h \cap \Omega \xrightarrow{loc} E'$$

(up to subsequence). Since $E_h \setminus \Omega = E_0 \setminus \Omega$ for every h, if we set $E := E' \cup (E_0 \setminus \Omega)$, then

$$E_h \xrightarrow{loc} E.$$

The semicontinuity of the fractional perimeter concludes the proof. □

REMARK 4.1 In particular, if Ω is a bounded open set with Lipschitz boundary, then (as already proved in [4]) we can always find an s-minimal set for every $s \in (0,1)$, no matter what the external data $E_0 \setminus \Omega$ is. Indeed in this case

$$P_s(E_0 \setminus \Omega, \Omega) \leqslant P_s(\Omega) < \infty.$$

Actually, in order to have the existence of s-minimal sets for some fixed $s \in (0,1)$, the open set Ω need not be bounded nor have a regular boundary. It is enough to have

$$P_s(\Omega) < \infty.$$

Then $E_0 \setminus \Omega$ has finite s-perimeter in Ω and we can apply Theorem 1.9.

Proof of Corollary 1.11. Let the sets Ω_k be as in Corollary 2.6.

From Theorem 1.9 and Remark 4.1 we know that for every k we can find a set E_k which is s-minimal in Ω_k and such that $E_k \setminus \Omega_k = E_0 \setminus \Omega_k$.

Notice that, since the sequence Ω_k is increasing, the set E_h is s-minimal in Ω_k for every $h \geqslant k$.

This gives us a sequence $\{E_h\}$ satisfying the hypothesis of Proposition 2.13 (see Remark 2.14), and hence (up to a subsequence)

$$E_h \cap \Omega \xrightarrow{loc} F,$$

for some $F \subset \Omega$. Since $E_h \setminus \Omega = E_0 \setminus \Omega$ for every h, if we set $E := F \cup (E_0 \setminus \Omega)$, we obtain

$$E_h \xrightarrow{loc} E.$$

Theorem 1.12 guarantees that E is s-minimal in every Ω_k and hence also locally s-minimal in Ω. Indeed, if $\Omega' \subset\subset \Omega$, then for some k big enough we have $\Omega' \subset \Omega_k$. Now, since E is s-minimal in Ω_k, it is s-minimal also in Ω'. □

4.4 *Locally s-minimal sets in cylinders*

Given a bounded open set $\Omega \subset \mathbb{R}^n$, we consider the cylinders

$$\Omega^k := \Omega \times (-k, k), \qquad \Omega^\infty := \Omega \times \mathbb{R}.$$

We recall that, given any set $E_0 \subset \mathbb{R}^{n+1}$, by Corollary 1.11 we can find a set $E \subset \mathbb{R}^{n+1}$ which is locally s-minimal in Ω^∞ and such that $E \setminus \Omega^\infty = E_0 \setminus \Omega^\infty$.

REMARK 4.2 Actually, if Ω has Lipschitz boundary then E is s-minimal in every cylinder $\mathcal{O} = \Omega \times (a,b)$ of finite height (notice that $\mathcal{O}$ is not compactly contained in Ω^∞). Indeed, $\mathcal{O}$ is a bounded open set with Lipschitz boundary and E is locally s-minimal in $\mathcal{O}$. Thus, by Theorem 1.7, E is s-minimal in $\mathcal{O}$.
As a consequence, E is s-minimal in every bounded open subset $\Omega' \subset \Omega^\infty$.

We are going to consider as exterior data the subgraph

$$E_0 = \mathcal{S}g(v) := \big\{(x,t) \in \mathbb{R}^{n+1} \,|\, t < v(x)\big\},$$

of a function $v : \mathbb{R}^n \longrightarrow \mathbb{R}$, which is locally bounded, i.e.

$$M_r := \sup_{|x| \leqslant r} |v(x)| < \infty, \qquad \text{for every } r > 0. \tag{4.3}$$

The following result is an immediate consequence of (the proof of) Lemma 3.3 of [11].

Lemma 4.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^{1,1}$ boundary and let $v : \mathbb{R}^n \longrightarrow \mathbb{R}$ be locally bounded. There exists a constant $M = M(n, s, \Omega, v) > 0$ such that if $E \subset \mathbb{R}^{n+1}$ is locally s-minimal in Ω^∞, with $E \setminus \Omega^\infty = \mathcal{S}g(v) \setminus \Omega^\infty$, then*

$$\Omega \times (-\infty, -M] \subset E \cap \Omega^\infty \subset \Omega \times (-\infty, M].$$

As a consequence

$$E \setminus \big(\Omega \times [-M, M]\big) = \mathcal{S}g(v) \setminus \big(\Omega \times [-M, M]\big). \tag{4.4}$$

Proof. By Remark 4.2, the set E is s-minimal in Ω^∞ in the sense considered in [11]. Lemma 3.3 of [11] then guarantees that

$$E \cap \Omega^\infty \subset \Omega \times (-\infty, M].$$

Moreover, the same argument used in the proof shows also that

$$\mathcal{C}E \cap \Omega^\infty \subset \Omega \times [-M, \infty),$$

(up to considering a bigger M).

Since $M > M_{R_0}$, where R_0 is such that $\Omega \subset\subset B_{R_0}$, we get (4.4), concluding the proof. □

Roughly speaking, Lemma 4.3 gives an a priori bound on the variation of ∂E in the "vertical" direction. In particular, from (4.4) we see that it is enough to look for a locally s-minimal set among sets which coincide with $\mathcal{S}g(v)$ out of $\Omega \times [-M, M]$.

As a consequence, we can prove that a set is locally s-minimal in Ω^∞ if and only if it is s-minimal in $\Omega \times [-M, M]$.

Proposition 4.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^{1,1}$ boundary and let $v : \mathbb{R}^n \longrightarrow \mathbb{R}$ be locally bounded. Let M be as in Lemma 4.3 and let k_0 be the smallest integer $k_0 > M$. Let $F \subset \mathbb{R}^{n+1}$ be s-minimal in Ω^{k_0}, with respect to the exterior data*

$$F \setminus \Omega^{k_0} = \mathcal{S}g(v) \setminus \Omega^{k_0}. \tag{4.5}$$

Then F is s-minimal in Ω^k for every $k \geqslant k_0$, hence is locally s-minimal in Ω^∞.

Proof. Let $E \subset \mathbb{R}^{n+1}$ be locally s-minimal in Ω^∞, with respect to the exterior data

$$E \setminus \Omega^\infty = \mathcal{S}g(v) \setminus \Omega^\infty.$$

Recall that by Remark 4.2 the set E is s-minimal in Ω^k for every k. In particular

$$P_s(E, \Omega^k) < \infty \qquad \forall\, k \in \mathbb{N}.$$

To prove the Proposition, it is enough to show that

$$P_s(F, \Omega^k) = P_s(E, \Omega^k), \qquad \text{for every } k \geqslant k_0. \tag{4.6}$$

Indeed, notice that by (4.5) and (4.4) we have

$$F \setminus \Omega^{k_0} = \mathcal{S}g(v) \setminus \Omega^{k_0} = E \setminus \Omega^{k_0}, \tag{4.7}$$

hence, clearly,

$$F \setminus \Omega^k = E \setminus \Omega^k, \qquad \forall\, k \geqslant k_0.$$

Then, since E is s-minimal in Ω^k, from (4.6) we conclude that also F is s-minimal in Ω^k, for every $k \geqslant k_0$. In turn, this implies that F is locally s-minimal in Ω^∞.

Exploiting Proposition 2.1, by (4.7) we obtain that for every $k \geqslant k_0$

$$P_s(F,\Omega^k) = P_s(F,\Omega^{k_0}) + c_k, \qquad P_s(E,\Omega^k) = P_s(E,\Omega^{k_0}) + c_k, \tag{4.8}$$

where

$$c_k = \mathcal{L}_s(\mathcal{S}g(v) \cap (\Omega^k \setminus \Omega^{k_0}), \mathcal{C}\mathcal{S}g(v) \setminus \Omega^{k_0}) + \mathcal{L}_s\big(\mathcal{S}g(v) \setminus \Omega^k, \mathcal{C}\mathcal{S}g(v) \cap (\Omega^k \setminus \Omega^{k_0})\big),$$

which is finite and does not depend on E nor F. To see that c_k is finite, simply notice that

$$c_k \leqslant P_s(E,\Omega^k) < \infty.$$

Now, by (4.7) and the minimality of F we have

$$P_s(F,\Omega^{k_0}) \leqslant P_s(E,\Omega^{k_0}).$$

On the other hand, since also the set E is s-minimal in Ω^{k_0}, again by (4.7) we get

$$P_s(E,\Omega^{k_0}) \leqslant P_s(F,\Omega^{k_0}).$$

This and (4.8) give

$$P_s(F,\Omega^k) = P_s(F,\Omega^{k_0}) + c_k = P_s(E,\Omega^k),$$

proving (4.6) and concluding the proof. □

It is now natural to wonder whether the set F is actually s-minimal in Ω^∞. The answer, in general, is no. Indeed, Theorem 1.14 shows that in general we cannot hope to find an s-minimal set in Ω^∞.

Proof of Theorem 1.14. Notice that by (1.10) we have

$$E \cap (\Omega^\infty \setminus \Omega^{k+1}) = \Omega \times (-\infty, -k-1),$$
$$\mathcal{C}E \cap (\Omega^\infty \setminus \Omega^{k+1}) = \Omega \times (k+1, \infty),$$

and

$$E \cap \Omega^{k+1} \subset \Omega \times (-k-1, k), \qquad \mathcal{C}E \cap \Omega^{k+1} \subset \Omega \times (-k, k+1).$$

Thus

$$\begin{aligned} P_s^L(E,\Omega^\infty) = P_s^L(E,\Omega^{k+1}) &+ \mathcal{L}_s(E \cap (\Omega^\infty \setminus \Omega^{k+1}), \mathcal{C}E \cap \Omega^{k+1}) \\ &+ \mathcal{L}_s(\mathcal{C}E \cap (\Omega^\infty \setminus \Omega^{k+1}), E \cap \Omega^{k+1}) + P_s^L(E, \Omega^\infty \setminus \Omega^{k+1}) \\ \leqslant P_s^L(E,\Omega^{k+1}) &+ 2\mathcal{L}_s\big(\Omega \times (-\infty,-k-1), \Omega \times (-k,k+1)\big) \\ &+ \mathcal{L}_s\big(\Omega \times (-\infty,-k-1), \Omega \times (k+1,\infty)\big). \end{aligned}$$

Since $d\big(\Omega \times (-\infty,-k-1), \Omega \times (-k,k+1)\big) = 1$, we get

$$\begin{aligned} \mathcal{L}_s\big(\Omega \times (-\infty,-k-1), \Omega \times (-k,k+1)\big) &\leqslant \int_{\Omega \times (-k,k+1)} \Big(\int_{\mathcal{C}B_1(X)} \frac{dY}{|X-Y|^{n+1+s}} \Big) dX \\ &= \frac{(n+1)\omega_{n+1}}{s} (2k+1)|\Omega|. \end{aligned}$$

As for the last term, since $n+1 \geqslant 2$, we have

$$\begin{aligned}
\mathcal{L}_s\big(\Omega \times (-\infty,-k-1), \Omega\times(k+1,\infty)\big)
&= \int_\Omega \int_\Omega \Big(\int_{-\infty}^{-k-1} \int_{k+1}^{\infty} \frac{dt\, d\tau}{(|x-y|^2+(t-\tau)^2)^{\frac{n+1+s}{2}}} \Big) dx\, dy \\
&\leqslant |\Omega|^2 \int_{-\infty}^{-k-1} \Big(\int_{k+1}^{\infty} \frac{dt}{(t-\tau)^{n+1+s}} \Big) d\tau \\
&= \frac{|\Omega|^2}{n+s} \int_{-\infty}^{-k-1} \frac{d\tau}{(k+1-\tau)^{n+s}} \\
&= \frac{|\Omega|^2}{(n+s)(n-1+s)} \frac{1}{(2k+2)^{n-1+s}}.
\end{aligned}$$

This shows that $P_s^L(E,\Omega^\infty) < \infty$.

Now suppose that $E \subset \mathbb{R}^{n+1}$ satisfies (1.11). Then

$$P_s^{NL}(E,\Omega^\infty) \geqslant 2\mathcal{L}_s\big(\Omega \times (-\infty,-k), \complement\Omega \times (k,\infty)\big).$$

Since Ω is bounded, we can take $R > 0$ big enough such that $\Omega \subset\subset B_R$. For every $T > T_0 := \max\{k, R\}$ we have

$$\Omega \times (-\infty,-T) \subset \Omega \times (-\infty,-k) \quad \text{and} \quad (B_T \setminus B_R) \times (T,\infty) \subset \complement\Omega \times (k,\infty).$$

Thus for every $T > T_0$

$$\begin{aligned}
\mathcal{L}_s\big(\Omega \times (-\infty,-k), \complement\Omega \times (k,\infty)\big) &\geqslant \mathcal{L}_s\big(\Omega \times (-\infty,-T), (B_T \setminus B_R) \times (T,\infty)\big) \\
&= \int_\Omega dx \int_{B_T\setminus B_R} dy \int_{-\infty}^{-T} dt \int_T^\infty \frac{d\tau}{(|x-y|^2+(\tau-t)^2)^{\frac{n+1+s}{2}}} =: a_T.
\end{aligned}$$

Notice that for every $x \in \Omega,\ y \in B_T \setminus B_R,\ t \in (-\infty,-T)$ and $\tau \in (T,\infty)$, we have

$$|x-y| \leqslant |x| + |y| \leqslant R + T \leqslant 2T \leqslant \tau - t,$$

and hence

$$\begin{aligned}
a_T &\geqslant \frac{1}{2^{\frac{n+1+s}{2}}} \int_\Omega dx \int_{B_T\setminus B_R} dy \int_{-\infty}^{-T} dt \int_T^\infty \frac{d\tau}{(\tau-t)^{n+1+s}} \\
&= \frac{|\Omega|}{2^{\frac{n+1+s}{2}}(n+s)(n-1+s)} \frac{|B_T \setminus B_R|}{(2T)^{n-1+s}}.
\end{aligned}$$

Since $|B_T \setminus B_R| \sim T^n$ as $T \to \infty$, we get $a_T \longrightarrow \infty$. Therefore, since

$$P_s^{NL}(E,\Omega^\infty) \geqslant 2a_T \quad \text{for every } T > T_0,$$

we obtain $P_s^{NL}(E,\Omega^\infty) = \infty$.

To conclude, let Ω be bounded, with $C^{1,1}$ boundary, and let $v \in L^\infty(\mathbb{R}^n)$.

Suppose that there exists a set $E \subset \mathbb{R}^{n+1}$ which is s-minimal in Ω^∞ with respect to the exterior data $E \setminus \Omega^\infty = \mathcal{S}g(v) \setminus \Omega^\infty$.

Then, thanks to Lemma 4.3, we can find k big enough such that E satisfies (1.11). Since this implies $P_s(E, \Omega^\infty) = \infty$, we reach a contradiction concluding the proof. □

Corollary 4.5 *In particular*

$$u \in BV_{loc}(\mathbb{R}^n) \cap L^\infty_{loc}(\mathbb{R}^n) \quad \Longrightarrow \quad P^L_s(\mathcal{S}g(u), \Omega^\infty) < \infty, \tag{4.9}$$

and

$$u \in L^\infty(\mathbb{R}^n) \quad \Longrightarrow \quad P^{NL}_s(\mathcal{S}g(u), \Omega^\infty) = \infty, \tag{4.10}$$

for every bounded open set $\Omega \subset \mathbb{R}^n$.

Furthermore, if $|u| \leqslant M$ in Ω and there exists $\Sigma \subset \mathbb{S}^{n-1}$ with $\mathcal{H}^{n-1}(\Sigma) > 0$ such that either

$$u(r\omega) \leqslant M \quad \text{or} \quad u(r\omega) \geqslant -M \qquad \text{for every } \omega \in \Sigma \quad \text{and} \quad r \geqslant r_0,$$

then $P^{NL}_s(\mathcal{S}g(u), \Omega^\infty) = \infty$.

Proof. Both (4.9) and (4.10) are immediate from Theorem 1.14, so we only need to prove the last claim.

Since Ω is bounded, we can find $R > 0$ such that $\Omega \subset\subset B_R$.

For every $T > T_0 := \max\{M, R, r_0\}$ define

$$\mathcal{S}(T) := \big\{x = r\omega \in \mathbb{R}^n \,|\, r \in (T_0, T),\, \omega \in \Sigma\big\}.$$

Notice that $\mathcal{S}(T) \subset B_T$ and

$$\begin{aligned} |\mathcal{S}(T)| &= \int_{T_0}^T \Big(\int_{\partial B_r} \chi_{\mathcal{S}(T)}\, d\mathcal{H}^{n-1}\Big) dr = \int_{T_0}^T \mathcal{H}^{n-1}(r\Sigma)\, dr \\ &= \frac{\mathcal{H}^{n-1}(\Sigma)}{n}(T^n - T_0^n). \end{aligned}$$

Suppose that $u(r\omega) \leqslant M$ for every $r \geqslant r_0$ and $\omega \in \Sigma$. Then, arguing as in the second part of the proof of Theorem 1.14, we obtain

$$\begin{aligned} P^{NL}_s(\mathcal{S}g(u), \Omega^\infty) &\geqslant \mathcal{L}_s(\mathcal{S}g(u) \cap \Omega^\infty, \mathcal{C}\mathcal{S}g(u) \setminus \Omega^\infty) \\ &\geqslant \mathcal{L}_s\big(\Omega \times (-\infty, -T), \mathcal{S}(T) \times (T, \infty)\big) \\ &\geqslant \frac{|\Omega|}{2^{\frac{n+1+s}{2}}(n+s)(n-1+s)} \frac{|\mathcal{S}(T)|}{(2T)^{n-1+s}}, \end{aligned}$$

for every $T > T_0$. Since

$$\frac{|\mathcal{S}(T)|}{(2T)^{n-1+s}} \sim T^{1-s},$$

which tends to ∞ as $T \to \infty$, we get our claim. □

In the classical framework, the area functional of a function $u \in C^{0,1}(\mathbb{R}^n)$ is defined as

$$\mathcal{A}(u,\Omega) := \int_\Omega \sqrt{1+|\nabla u|^2}\,dx = \mathcal{H}^n\Big(\big\{\big(x,u(x)\big) \in \mathbb{R}^{n+1} \,|\, x \in \Omega\big\}\Big),$$

for any bounded open set $\Omega \subset \mathbb{R}^n$. Exploiting the subgraph of u one then defines the relaxed area functional of a function $u \in BV_{loc}(\mathbb{R}^n)$ as

$$\mathcal{A}(u,\Omega) := P\big(\mathcal{S}g(u),\Omega^\infty\big). \tag{4.11}$$

Notice that when u is Lipschitz the two definitions coincide.

One might then be tempted to define a nonlocal fractional version of the area functional by replacing the classical perimeter in (4.11) with the s-perimeter, that is

$$\mathcal{A}_s(u,\Omega) := P_s\big(\mathcal{S}g(u),\Omega^\infty\big).$$

However Corollary 4.5 shows that this definition is ill-posed even for regular functions u.

On the other hand, it is worth remarking that one could use just the local part of the s-perimeter, but then the resulting functional

$$\mathcal{A}_s^L(u,\Omega) := P_s^L\big(\mathcal{S}g(u),\Omega^\infty\big) = \frac{1}{2}[\chi_{\mathcal{S}g(u)}]_{W^{s,1}(\Omega^\infty)}$$

has a local nature.

Exploiting Theorem 1 of [7], we obtain the following

Lemma 4.6 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $u \in BV(\Omega) \cap L^\infty(\Omega)$. Then*

$$\lim_{s\to 1^-}(1-s)\mathcal{A}_s^L(u,\Omega) = \omega_n \mathcal{A}(u,\Omega). \tag{4.12}$$

Proof. Let k be such that $|u| \leqslant k$. Then $E = \mathcal{S}g(u)$ satisfies (1.10) and hence, arguing as in the beginning of the proof of Theorem 1.14, we get

$$\mathcal{A}_s^L(u,\Omega) = P_s^L\big(\mathcal{S}g(u),\Omega^{k+1}\big) + O(1),$$

as $s \to 1$. Since $\mathcal{S}g(u)$ has finite perimeter in Ω^{k+1}, which is a bounded open set with Lipschitz boundary, we conclude using Theorem 1 of [7] (see also, e.g., [16] for the asymptotics as $s \to 1$ of the s-perimeter).

Indeed, notice that since $|u| \leqslant k$, we have

$$P(\mathcal{S}g(u),\Omega^{k+1}) = P\big(\mathcal{S}g(u),\Omega^\infty\big) = \mathcal{A}(u,\Omega). \qquad \square$$

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