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Schwarz P surfaces and a non local perturbation of the perimeter

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In the paper, we consider a small non local perturbation of the perimeter and we construct at least four critical points close to suitable translations of the Schwarz P surface with fixed volume.

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1. Introduction

A diblock copolymer is a complex molecule where chains of two different kinds of monomers, say A and B, are grafted together. Diblock copolymer melts are large collections of diblock copolymers. The experiments show that, above a certain temperature, these melts behave like fluids, that is the monomers are mixed in a disordered way, while below this critical temperature phase separation is observed. Some common periodic structures observed in experiments are spheres, cylinders, gyroids and lamellae (see Figure 1). These patterns can be found by minimising some energy. It looks reasonable to describe the phenomenon through an energy given by the sum of the perimeter, that forces the separation surfaces to be minimal, plus some non local term that keeps track of the long-range interactions between monomers. More explicitly, one can take the functional

$$\mathcal{E}(u) := \int_{T^3} |\nabla u| + \gamma \int_{T^3} \int_{T^3} G(x, y)\big(u(x) - m\big)\big(u(y) - m\big) dx dy \tag{1.1}$$

as an energy. Here T^3 is the three dimensional torus, that is the three dimensional unit cube with the identification of the opposite faces. This can be seen as the container where the diblock copolymer melt is confined, u is a bounded variation function in T^3 with values in $\{\pm 1\}$. For instance, we

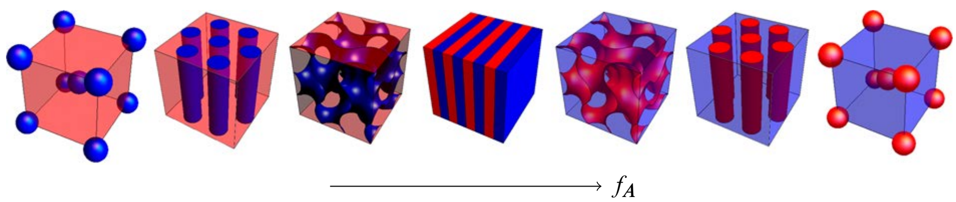

FIG. 1. The most commonly observed periodic structures are spheres, cylinders, gyroids and lamellae

can assume that $u(x) = 1$ if there are only monomers of type A at x, $u(x) = -1$ if there are only monomers of type B at x. In Figure 1, monomers of type A are represented in blue, while monomers of type B are represented in red (in a printed black-and-white version, blue and red just appear as different shades of grey, blue being the darker). $\int_\Omega |\nabla u|$ is its total variation, or equivalently the perimeter of the set $\{x \in \Omega : u(x) = 1\}$, G is the Green's function of $-\Delta$ on T^3, $\gamma \geqslant 0$ is a parameter depending on the material, that we will assume to be small.

This energy appears as the Γ-limit as $\varepsilon \to 0$ of the approximating functional

$$\mathcal{E}_\varepsilon(u) = \frac{\varepsilon}{2}\int_{T^3} |\nabla u|^2 dx + \frac{1}{\varepsilon}\int_{T^3} \frac{(1-u^2)^2}{4} dx + \frac{16\gamma}{3}\int_{T^3}\int_{T^3} G(x,y)\big(u(x)-m\big)\big(u(y)-m\big)dx\,dy,$$

introduced by Ohta and Kawasaki (see [2, 6–8]).

In a more geometric way our functional is given by

$$J_\gamma(E) := P_{T^3}(E) + \gamma\int_{T^3}\int_{T^3} G(x,y)\big(u_E(x)-m\big)\big(u_E(y)-m\big)dx\,dy \tag{1.2}$$

where

$$E := \{x \in \Omega : u(x) = 1\},$$

so that $u_E = \chi_E - \chi_{\Omega\setminus E}$. The first variation of J_γ is given by

$$J'_\gamma(E)[\varphi] = \int_\Sigma \big(H_\Sigma(x) + 4\gamma v_E(x)\big)\varphi(x)d\sigma(x), \tag{1.3}$$

while its second variation is given by

$$J''_\gamma(E)[\varphi,\psi] = \int_\Sigma L\varphi(x)\psi(x)d\sigma(x), \tag{1.4}$$

where

$$L\varphi = -\Delta_\Sigma\varphi - |A_\Sigma|^2\varphi + 8\gamma\int_\Sigma G(\cdot,y)\varphi(y)d\sigma(y) + 4\gamma\partial_\nu v\varphi. \tag{1.5}$$

Here A_Σ is the second fundamental form of the surface Σ, $|A_\Sigma|^2 := k_1^2 + k_2^2$ is the same of the squared principal curvatures and φ, ψ are in the space

$$W := \left\{w \in H^1(\Sigma) : \int_\Sigma w(x)\nu_i(x)d\sigma(x) = 0, \quad 1 \leqslant i \leqslant 3\right\}, \tag{1.6}$$

$\Sigma := \partial E$ and

$$v_E(x) := \int_{T^3} G(x,y)(u_E(y)-m)dy \tag{1.7}$$

is the unique solution to the problem

$$\begin{cases} -\Delta v_E = u_E - m & \text{in } T^3 \\ \int_{T^3} v_E\,dx = 0. \end{cases} \tag{1.8}$$

For an explicit computation of the first and the second variation, see for instance [9]. It is known that J_γ is translation invariant, that is $J_\gamma(E+\xi) = J_\gamma(E)$, for any $\xi \in T^3$ (see [2], [9]), thus, once we find a critical point of it, any translation in T^3 is still critical.

There are several results in the literature about critical points of this functional. For instance, an interesting problem is to understand whether all global minimisers are periodic, like the patterns described above (spheres, cylinders, gyroids and lamellae, see Figure 1). This is known to be true in dimension one (see [24]), but the problem is still open in higher dimension. We refer to [1, 33] for further results. Some other authors, such as Ren and Wei [28–32], constructed explicit examples of stable periodic local minimisers, that is with positive second variation. Moreover, Acerbi Fusco and Morini [2] showed that any stable critical point is actually a local minimiser with respect to small L^1 perturbations.

Other related results were proved, by variational techniques, by Cristoferi (see [12], Theorem 4.18), who constructed a family of isolated local minimizers of J_γ with respect to periodic perturbations, close to any given smooth periodic strictly stable constant mean curvature surface. However, our techniques are very different, since they rely on a Lyapunov–Schmidt reduction.

Here we add a small linear perturbation that corresponds to an external force f applied to the system, that can be taken to be $C^{0,1}(\mathbb{R}^3)$, periodic and satisfying the symmetries of the Schwarz P surface, that is

$$\begin{cases} f(x+e_i) = f(x), & \forall x \in \mathbb{R}^3, i = 1,2,3, \\ f(x) = f(T_j x), & j = 1,2,3, \end{cases} \tag{1.9}$$

where $\{e_i\}_{1\leqslant i\leqslant 3}$ is the standard basis of $\mathbb{R}^3$ and T_j are the reflections defined by

$$T_1(x_1,x_2,x_3) = (-x_1,x_2,x_3) \qquad T_2(x_1,x_2,x_3) = (x_1,-x_2,x_3) \qquad T_3(x_1,x_2,x_3) = (x_1,x_2,-x_3). \tag{1.10}$$

The energy becomes

$$I_\gamma(E) := J_\gamma(E) + \gamma \int_{T^3} f(x) u_E(x) dx. \tag{1.11}$$

The additional linear term breaks the translation invariance. We will construct at least four critical points F_j of I_γ, $1 \leqslant j \leqslant 4$, for γ small enough, that are close to suitable translations of the Schwarz P surface Σ (see figure 2), under the volume constraint

$$\mathfrak{L}_3(F_j) = \mathfrak{L}_3(E), \tag{1.12}$$

where E is the interior of Σ and $\mathfrak{L}_3$ is the Lebesgue measure in $\mathbb{R}^3$.

The critical points of this energy represent the equilibria of a physical system under the action of an external force, represented by f. Our main result, that is Theorem 1.2, shows that if, for instance, our model is referred to the behaviour of two fluids, the effect of the force is not so relevant as regards the pattern created by these fluids, which is reasonable since this force is multiplied by a small factor γ.

REMARK 1.1 The Schwartz P surface can be seen as a periodic surface in $\mathbb{R}^3$, with triple period 1. Moreover, it divides the unit cube into two components, an interior and an exterior. In the sequel, E will denote the interior part.

We will use a technique based on a finite dimensional Lyapunov–Schmidt reduction (see [4], Chapter 2.2), and on the Lusternik–Schnirelman theory (see [3], Chapter 9) for the multiplicity.

We point out that there exists a global parametrisation of Σ

$$\phi : Y \to \Sigma, \tag{1.13}$$

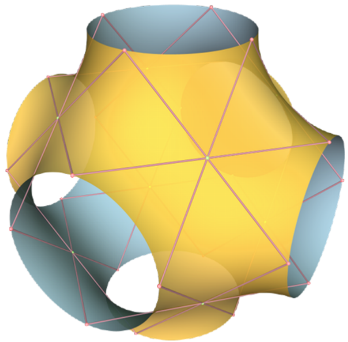

FIG. 2. Schwarz P surface

defined on an open set $Y \subset \mathbb{R}^2$ (see [15], section 3). The variables in Y will be denoted by $(\mathrm{y}_1, \mathrm{y}_2)$. For $0 < \alpha < 1$ and for any integer $k \geqslant 0$, we introduce the Hölder spaces

$$C_s^{k,\alpha}(\Sigma) := \left\{ w \in C^{k,\alpha}(\Sigma) : w(x) = w(T_j x), 1 \leqslant j \leqslant 3 \right\}, \tag{1.14}$$

where the T_j's are defined above (1.10). Here it is understood that we have put the origin in the centre of the cube (see Figure 2), in such a way that these spaces consist of functions that respect the symmetries of Σ, that is the symmetries with respect to the coordinate planes $\{x_j = 0\}$, $1 \leqslant j \leqslant 3$. We endow these spaces with the norm

$$||w||_{C^{k,\alpha}(\Sigma)} = \sum_{|\beta| \leqslant k} ||\partial^\beta (w \circ Y)||_{L^\infty(\Sigma)} + \sup_{|\beta|=k} \sup_{\mathrm{x},\mathrm{y} \in Y, \mathrm{x} \neq \mathrm{y}} \frac{|\partial^\beta (w \circ Y)(\mathrm{x}) - \partial^\beta (w \circ Y)(\mathrm{y})|}{d(\mathrm{x}, \mathrm{y})^\alpha}, \tag{1.15}$$

where d is the geodesics distance on Σ, $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$ is a multi-index, $|\beta| := \beta_1 + \beta_2$ and $\partial^\beta := \frac{\partial^{|\beta|}}{\partial_{\mathrm{y}_1}^{\beta_1} \partial_{\mathrm{y}_2}^{\beta_2}}$. We point out that we are allowed to define these norms, due to the regularity of Σ, which is actually analytic. For a proof of this fact, see for instance [17], Chapter 11, Theorem 11.8, or [11], Corollary 6.9.

Main Theorem 1.2 *Let I_γ be defined as in (1.11) and $\nu(x)$ be the outward-pointing unit normal to the Schwarz P surface Σ. Then there exists $\gamma_0 > 0$ such that, for any $0 < \gamma < \gamma_0$, there exist $\xi_j \in T^3$, $1 \leqslant j \leqslant 4$, and $w_{\gamma,j} \in C_s^{2,\alpha}(\Sigma)$, with*

$$||w_{\gamma,j}||_{C^{2,\alpha}(\Sigma)} \leqslant c\gamma, \tag{1.16}$$

such that the sets F_j defined as the interior of

$$\Gamma_j := \left\{ x + \xi_j + \nu(x) w_{\gamma,j}(x) : x \in \Sigma \right\} \tag{1.17}$$

are critical points of I_γ under the volume constraint

$$\mathfrak{L}_3(F_j) = \mathfrak{L}_3(E). \tag{1.18}$$

A crucial tool in the proof is non degeneracy up to translations of the Jacobi operator of the Schwarz P surface. In [27], Ross showed that the Schwarz P surface is a critical point of the area and it is *volume preserving stable*, that is the second variation of the area is non-negative on any normal variation with zero average. More precisely, setting $I_0 := P_{T^3}$ and $|A_\Sigma|^2 := k_1^2 + k_2^2$, where A_Σ is the second fundamental form of Σ and k_1, k_2 are the principal curvatures, we have

$$I_0''(E)[\varphi,\varphi] = \int_\Sigma |\nabla_\Sigma \varphi|^2 - |A_\Sigma|^2\varphi^2 d\sigma \geqslant 0 \tag{1.19}$$

for any $\varphi \in H^1(\Sigma)$ satisfying

$$\int_\Sigma \varphi d\sigma = 0, \tag{1.20}$$

(see Theorem 1 of [27]). Let $\nu(x)$ denote the exterior unit normal to Σ at x. Since I_0 is translation invariant, then $\nu_i(x) := (\nu(x), e_i)$ are Jacobi fields of Σ, that is they satisfy

$$-\Delta_\Sigma \nu_i - |A_\Sigma|^2 \nu_i = 0 \quad \text{in } \Sigma, \tag{1.21}$$

(see [2], [9]). Moreover, Grosse-Brauckmann and Wohlgemuth showed in ([20]) that Σ is non degenerate up to translations, that is there are no other nontrivial Jacobi fields. In other words

$$Ker\big(I_0''(E)\big) = span\{\nu_i\}_{1\leqslant i\leqslant 3}. \tag{1.22}$$

REMARK 1.3 Let us observe that the ν_i's are linearly independent. In fact, if not, there would exist a constant vector $b = (b_1, b_2, b_3) \neq 0$ such that $0 = (b, \nu(x))$ for any $x \in \Sigma$, but this contradicts the geometry of Σ.

We note that the ν_i's have zero average, since

$$\int_\Sigma \nu_i(x) d\sigma(x) = \int_{T^3} \mathrm{div} e_i = 0. \tag{1.23}$$

In addition, we decompose $H^1(\Sigma)$ into the orthogonal sum

$$H^1(\Sigma) = span\{\nu_i\}_{1\leqslant i\leqslant 3} \oplus W, \tag{1.24}$$

(see (1.6) for the definition of W), and we define

$$W^0 := \Big\{ w \in W : \int_\Sigma w(x) d\sigma(x) = 0 \Big\}. \tag{1.25}$$

The above discussion can be rephrased by saying that

$$\int_\Sigma |\nabla_\Sigma w|^2 - |A_\Sigma|^2 w^2 d\sigma \geqslant c\|w\|^2_{H^1(\Sigma)} \quad \text{for any } w \in W^0. \tag{1.26}$$

2. The proof of Theorem 1.2: Lyapunov–Schmidt reduction

We need to find at least four sets F of the form (1.16), (1.17) and a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$H_{\partial F}(y) + 4\gamma v_F(y) + \gamma f(y) = \lambda \quad \forall y \in \partial F, \tag{2.1}$$

where $H_{\partial F} := k_{1,\partial F} + k_{2,\partial F}$ is the mean curvature of ∂F, that is the sum of the principal curvatures $k_{i,\partial F}$ of ∂F, or equivalently

$$I'_\gamma(F) = \lambda. \tag{2.2}$$

Exploiting the variational nature of the problem and the fact that Σ has zero mean curvature, namely $H_\Sigma = 0$, equation (2.1) is equivalent to

$$\lambda = 4\gamma v_E(x) + Lw(x) + Q(w)(x) + \gamma f(y), \quad \forall x \in \Sigma, \tag{2.3}$$

where $y = x + \xi + w(x)\nu(x)$. Here we have set

$$Q(w)(x) := H_{\partial F}(y) + 4\gamma\big(v_F(y) - v_E(x)\big) - Lw(x). \tag{2.4}$$

and

$$Lw = -\Delta_\Sigma w - |A_\Sigma|^2 w + \gamma \tilde{L} w, \tag{2.5}$$

where

$$\tilde{L}w = 8\int_\Sigma G(\cdot\,,\zeta)w(\zeta)d\sigma(\zeta) + 4\partial_\nu v_E w. \tag{2.6}$$

We can see that (2.3) is equivalent to

$$-\Delta_\Sigma w - |A_\Sigma|^2 w = \lambda + \mathfrak{F}(\gamma,\xi,w), \tag{2.7}$$

where the nonlinear functional $\mathfrak{F}$ is given by

$$\mathfrak{F}(\gamma,\xi,w)(x) := -4\gamma v_E(x) - \gamma\tilde{L}w(x) - Q(w)(x) - \gamma f(y), \quad \forall x \in \Sigma. \tag{2.8}$$

The unknowns are the function w, $\xi \in T^3$ and $\lambda \in \mathbb{R}$. We note that, since f satisfies (1.9), if $w \in C_s^{2,\alpha}(\Sigma)$, then $\mathfrak{F}(\gamma,\xi,w)(x) \in C_s^{0,\alpha}(\Sigma)$.

2.1 *The volume constraint*

Now we will consider the relation between the volume of F and w. In order to do so, we observe that the parametrisation ϕ of Σ defined in (1.13) induces a change of coordinates on a neighbourhood of Σ given by

$$X(\mathrm{y}_1,\mathrm{y}_2,z) := \exp_{\phi(\mathrm{y}_1,\mathrm{y}_2)}\big(z\nu(\mathrm{y}_1,\mathrm{y}_2)\big), \tag{2.9}$$

where, with an abuse of notation, $\nu(\mathrm{y}_1,\mathrm{y}_2)$ is the outward-pointing unit normal to Σ at $\phi(\mathrm{y}_1,\mathrm{y}_2)$ and $\exp_{\phi(\mathrm{y}_1,\mathrm{y}_2)}$ is the exponential map of T^3 at $\phi(\mathrm{y}_1,\mathrm{y}_2)$. The volume of F is given by

$$\mathfrak{L}_3(F) = \mathfrak{L}_3(E) + \int_Y d\mathrm{y}\int_0^{w(\mathrm{y})} \det JX(\mathrm{y},z)dz,$$

where JX is the Jacobian of X. We expand

$$\det JX(\mathrm{y},z) = \det JX(\mathrm{y},0) + O(z),$$

thus we get

$$\mathfrak{L}_3(F) = \mathfrak{L}_3(E) + \int_Y \det JX(\mathrm{y},0)w(\mathrm{y})d\mathrm{y} + \int_Y B(\mathrm{y},w(\mathrm{y}))d\mathrm{y}.$$

where B is quadratic in w, that is it satisfies

$$\left|B\big(y, w_1(y)\big) - B\big(y, w_2(y)\big)\right| \leqslant c\big(\|w_1\|_{C^{2,\alpha}(\Sigma)} + \|w_2\|_{C^{2,\alpha}(\Sigma)}\big)\|w_1 - w_2\|_{C^{2,\alpha}(\Sigma)}, \quad \forall w_1, w_2 \in C^{2,\alpha}(\Sigma). \tag{2.10}$$

Since $\det JX(y,0) = (\nu(y), \partial_{y_1}\phi \times \partial_{y_2}\phi) \neq 0$ for any $y \in Y$,

$$\mathfrak{L}_3(F) = \mathfrak{L}_3(E) + \int_\Sigma w(x)d\sigma(x) + \int_\Sigma \tilde{Q}\big(x, w(x)\big)d\sigma(x), \tag{2.11}$$

where

$$\tilde{Q}(x, w) = \frac{1}{\det JX(x)} B\big(x, w(x)\big) \tag{2.12}$$

satisfies (2.10). Therefore the volume constraint is equivalent to an equation of the form

$$\int_\Sigma w(x)dx = -\int_\Sigma \tilde{Q}\big(x, w(x)\big)d\sigma(x). \tag{2.13}$$

2.2 *The auxiliary equation*

The aim is to solve (2.7) under the volume constraint (2.13). However, since, by (1.22) and (1.26), the Jacobi operator $-\Delta_\Sigma - |A_\Sigma|^2$ is non-degenerate up to translations, we can actually solve the system

$$\begin{cases} -\Delta_\Sigma w - |A_\Sigma|^2 w = \lambda + P\mathfrak{F}(\gamma, \xi, w) & \text{in } \Sigma \\ \int_\Sigma w d\sigma = -\int_\Sigma \tilde{Q}\big(x, w(x)\big)d\sigma(x), \end{cases} \tag{2.14}$$

where $P : L^2(\Sigma) \to X$ is the projection onto the space

$$X := \left\{\varphi \in L^2(\Sigma) : \int_\Sigma \varphi(x)\nu_i(x)d\sigma(x) = 0, \quad 1 \leqslant i \leqslant 3\right\}. \tag{2.15}$$

This will be done by a fixed point argument in the following Lemma, proved in Section 3.

Lemma 2.1 *For any $\xi \in T^3$ and for any γ sufficiently small, there exists a unique solution $(w_{\gamma,\xi}, \lambda_{\gamma,\xi}) \in C_s^{2,\alpha}(\Sigma) \times \mathbb{R}$ to problem (2.14) satisfying*

$$\|w_{\gamma,\xi}\|_{C^{2,\alpha}(\Sigma)} + |\lambda_{\gamma,\xi}| \leqslant C\gamma, \tag{2.16}$$

$$\int_\Sigma w(x)\nu_i(x)d\sigma(x) = 0, \quad 1 \leqslant i \leqslant 3, \tag{2.17}$$

for some constant $C > 0$. Moreover, the solution is Lipschitz continuous with respect to the parameter ξ, that is

$$\|w_{\gamma,\xi_1} - w_{\gamma,\xi_2}\|_{C^{2,\alpha}(\Sigma)} + |\lambda_{\gamma,\xi_1} - \lambda_{\gamma,\xi_1}| \leqslant C\gamma|\xi_1 - \xi_2|, \quad \forall \xi_1, \xi_2 \in T^3. \tag{2.18}$$

REMARK 2.2 If we take $f \equiv 0$, in order to get the right correction w, we just solve (2.14) for $\xi = 0$, due to the translation invariance of J_γ. We do not need the Lyapunov–Schmidt reduction.

2.3 *The bifurcation equation*

In order to conclude the proof of Theorem 1.2, we have to find at least four points $\xi \in T^3$ such that $(Id - P)\mathfrak{F}(\gamma, \xi, w_{\gamma,\xi})(x) = 0$, or equivalently

$$\int_{\Sigma} \mathfrak{F}(\gamma, \xi, w_{\gamma,\xi})(x)\nu_i(x)d\sigma(x) = 0, \tag{2.19}$$

for $i = 1, 2, 3$.

Since $\partial\Sigma = \emptyset$, an integration by parts yields

$$\int_{\Sigma} (-\Delta_{\Sigma} w_{\gamma,\xi} - |A_{\Sigma}|^2 w_{\gamma,\xi})(x)\nu_i(x)d\sigma(x) = 0,$$

for $i = 1, 2, 3$, thus by (2.14) we can see that w solves

$$P(4\gamma v_E + Lw + Q(w) + \gamma f(y) - \lambda) = 0, \tag{2.20}$$

or equivalently, for any $\gamma > 0$ small enough and $\xi \in T^3$, there exist coefficients $A_{i,\gamma,\xi} \in \mathbb{R}$ such that

$$4\gamma v_E + Lw + Q(w) + \gamma f(y) - \lambda = \sum_{i=1}^{3} A_{i,\gamma,\xi}\nu_i. \tag{2.21}$$

Since, by construction,

$$\begin{aligned}
\mathfrak{F}(\gamma, \xi, w_{\gamma,\xi}) &= -\big(4\gamma v_E + Lw + Q(w) + \gamma f(y)\big) - \Delta_{\Sigma} w_{\gamma,\xi} - |A_{\Sigma}|^2 w_{\gamma,\xi} \\
&= -\big(4\gamma v_E + Lw + Q(w) + \gamma f(y)\big) + \lambda + P\mathfrak{F}\big(\gamma, \xi, w_{\gamma,\xi}\big) \\
&= -\sum_{i=1}^{3} A_{i,\gamma,\xi}\nu_i + P\mathfrak{F}(\gamma, \xi, w_{\gamma,\xi}),
\end{aligned}$$

and (2.21) holds, we can see that (2.19) is equivalent to

$$A_{i,\gamma,\xi} = 0 \quad \text{for} \quad i = 1, 2, 3. \tag{2.22}$$

Lemma 2.3 *Equation (2.22) is satisfied if ξ is a critical point of the function $\Phi_\gamma : T^3 \to \mathbb{R}$ defined by*

$$\Phi_\gamma(\xi) := I_\gamma(F), \tag{2.23}$$

where F is the interior of

$$\Gamma := \big\{x + \xi + w_{\gamma,\xi}(x)\nu(x) : x \in \Sigma\big\}.$$

The proof of Lemma 2.3 will be carried out in Section 4. In order to find critical points on the torus T^3 of Φ_γ, we will use the Lusternik–Schnirelmann theory and the compactness of the Torus. To make it clear, let us give some definitions and recall a Theorem proved in [3]. We say that a subset $\mathfrak{A}$ of a topological set M is *contractible* if it is homotopic to a point, that is if there exist a point $p \in M$ and a function $H \in C^0([0, 1] \times \mathfrak{A}, M)$ such that $H(0, u) = u$ and $H(1, u) = p$, for any $u \in \mathfrak{A}$.

DEFINITION 2.4 ([3], Definition 9.2) The category of $\mathfrak{A}$ in M, denoted $cat(\mathfrak{A}, M)$, is the least integer k such that $\mathfrak{A} \subset \mathfrak{A}_1 \cup \cdots \cup \mathfrak{A}_k$, with $\mathfrak{A}_i$ closed and contractible. Moreover, we set $cat(\emptyset, M) := 0$ and $cat(\mathfrak{A}, M) = +\infty$ if there are no integers with the above property. We will write $cat(M)$ for $cat(M, M)$.

Sometimes, the *category* is also referred to as *Lusternik–Schnirelmann category*. Moreover, we set

$$cat_k(M) := \sup \{cat(\mathfrak{A}, M) : \mathfrak{A} \subset M, \mathfrak{A} \text{ compact}\}.$$

Theorem 2.5 ([3], Theorem 9.10) *Let M be a Banach space or manifold and $\Phi : M \to \mathbb{R}$ a $C^{1,1}$ functional. If Φ is bounded from below on M and the Palais-Smale condition is satisfied, then Φ has at least $cat_k(M)$ critical points on M.*

The statement by Ambrosetti and Malchiodi in [3] is actually more general, however Theorem 2.5 is enough for our purposes. Therefore it is possible to see that Φ_γ actually admits at least 4 critical points, due to Theorem 2.5 applied to I_γ, with $M = T^3$, which has category 4. In general, the k-dimensional Torus T^k has category $k+1$ (see [3], example 9.4, *(iii)*). The compactness of the torus T^3 is crucial, since it guarantees that I_γ is bounded from below on M and the Palais-Smale condition is satisfied.

3. Solving the auxiliary equation

The aim of this section is to prove Lemma 2.1. First, in Section 3.1, we will treat the corresponding linear problem, then, in Section 3.2, we will solve problem (2.14) by a fixed point argument.

3.1 *The linear problem*

Lemma 3.1 *Let $a \in \mathbb{R}$ and $\varphi \in C_s^{0,\alpha}(\Sigma)$ be such that*

$$\int_\Sigma \varphi \nu_i \, d\sigma = 0 \quad \text{for } i = 1, 2, 3. \tag{3.1}$$

Then there exists a unique solution $(w, \lambda) = \Psi(\varphi, a) \in C_s^{2,\alpha}(\Sigma) \times \mathbb{R}$ to the problem

$$\begin{cases} -\Delta_\Sigma w - |A_\Sigma|^2 w = \lambda + \varphi & \text{in } \Sigma \\ \int_\Sigma w \nu_i \, d\sigma = 0 & \text{for } 1 \leqslant i \leqslant 3, \\ \int_\Sigma w \, d\sigma = a. \end{cases} \tag{3.2}$$

Moreover, we have the estimate

$$||w||_{C^{2,\alpha}(\Sigma)} + |\lambda| \leqslant c(||\varphi||_{C^{0,\alpha}(\Sigma)} + |a|). \tag{3.3}$$

REMARK 3.2 Since the ν_i's are linearly independent (see Remark 1.3), then the matrix

$$L_{ki} := \int_\Sigma \nu_k \nu_i \, d\sigma \tag{3.4}$$

is invertible. Indeed, we can show that if $< Lc, c >= 0$, then $c = 0$. By definition of L,

$$\langle Lc, c\rangle = \sum_{i,j=1}^{3} \left(\int_\Sigma \nu_i(x)\nu_j(x)d\sigma(x) \right) c_i c_j = 0.$$

This is equivalent to

$$\int_\Sigma \langle c, \nu(x)\rangle^2 d\sigma(x) = 0,$$

hence $\langle c, \nu(x)\rangle \equiv 0$, that yields that $c = 0$, by the geometry of Σ.

Proof. Step (i)*: Existence and uniqueness.* First we look for a weak solution $w \in W$. We write any $w \in W$ as

$$w = w_0 + \frac{1}{|\Sigma|}\int_\Sigma w d\sigma,$$

with $w_0 \in W^0$. The linear problem can be rephrased as follows

$$\begin{cases} -\Delta_\Sigma w_0 - |A_\Sigma|^2 w_0 = \lambda + \varphi + |A_\Sigma|^2 \frac{a}{|\Sigma|} & \text{in } \Sigma \\ \int_\Sigma w_0 = 0. \end{cases} \tag{3.5}$$

We note that the right-hand side of (3.5) is orthogonal to ν_i, for $i = 1, 2, 3$, due to the fact that

$$\int_\Sigma |A_\Sigma|^2 \frac{a}{|\Sigma|}\nu_i(x)d\sigma = \int_\Sigma \left(\Delta_\Sigma \nu_i + |A_\Sigma|^2\nu_i\right)(x)\frac{a}{|\Sigma|}d\sigma = 0, \tag{3.6}$$

since $\partial\Sigma = \emptyset$, and

$$\int_\Sigma \nu_i(x)d\sigma(x) = \int_E \operatorname{div}(e_1)dx = 0. \tag{3.7}$$

In addition, the norm defined by

$$||w|| := \left\{ \int_\Sigma \left(|\nabla_\Sigma w|^2 - |A_\Sigma|^2 w^2\right)d\sigma \right\}^{1/2} \tag{3.8}$$

is equivalent to the $H^1(\Sigma)$-norm on W^0, thus the functional

$$\mathcal{G}(w) = \int_\Sigma |\nabla_\Sigma w|^2 - |A_\Sigma|^2 w^2 d\sigma - \int_\Sigma \left(\varphi + |A_\Sigma|^2\frac{a}{|\Sigma|}\right) w d\sigma$$

is bounded from below by

$$\mathcal{G}(w) \geqslant c||w||^2_{H^1(\Sigma)} - ||\varphi||_{L^2(\Sigma)}||w||_{H^1(\Sigma)}, \tag{3.9}$$

on W^0, hence it is coercive on it. Moreover, this functional is also weakly lower semi-continuous and strictly convex on W^0, therefore any minimising sequence $w_k \in W^0$ weakly converges, up to subsequence, to the unique minimiser $w_0 \in W^0$, which satisfies the Euler–Lagrange equation

$$\int_\Sigma \left((\nabla_\Sigma w_0, \nabla_\Sigma v) - |A_\Sigma|^2 w_0 v\right) d\sigma$$
$$= \lambda\int_\Sigma v d\sigma + \sum_{i=1}^{3}\beta_i \int_\Sigma \nu_i v d\sigma + \int_\Sigma \varphi v d\sigma + \int_\Sigma |A_\Sigma|^2 \frac{a}{|\Sigma|} v d\sigma, \tag{3.10}$$

for any $v \in H^1(\Sigma)$, for some Lagrange multipliers $\lambda, \beta_i \in \mathbb{R}$. Since $\varphi \in C^{0,\alpha}(\Sigma)$, then $w_0 \in C^{2,\alpha}(\Sigma)$, hence w_0 satisfies

$$-\Delta_\Sigma w_0 - |A_\Sigma|^2 w_0 = \lambda + \sum_{i=1}^{3} \beta_i \nu_i + \varphi + |A_\Sigma|^2 \frac{a}{|\Sigma|} \quad \text{in } \Sigma,$$

in the classical sense. Moreover, w_0 respects the symmetries of the Schwarz P surface, that is $w_0(x) = w_0(T_j x)$, $1 \leqslant j \leqslant 3$, where the T_j are defined above (1.10), because of the symmetries of the Laplace–Beltrami operator and uniqueness. Taking ν_j as a test function in (3.10), using (3.7), (3.1), (3.6) and integrating by parts, we get

$$\sum_{i=1}^{3} \beta_i \int_\Sigma \nu_i \nu_j \, d\sigma = 0,$$

therefore by Remark 3.2, $\beta_i = 0$.

Step (ii)*: Regularity estimates.* Multiplying (3.5) by w_0, integrating by parts, using (1.26) and Hölder's inequality, we can see that

$$\begin{aligned} c\|w_0\|^2_{H^1(\Sigma)} &\leqslant \int_\Sigma |\nabla_\Sigma w_0|^2 - |A_\Sigma|^2 w_0^2 d\sigma = \int_\Sigma \varphi w_0 d\sigma + \frac{a}{|\Sigma|} \int_\Sigma |A_\Sigma|^2 w_0 d\sigma \\ &\leqslant \|w_0\|_{L^2(\Sigma)} \big(\|\varphi\|_{L^2(\Sigma)} + \tilde{c}|a|\big) \leqslant \|w_0\|_{H^1(\Sigma)} \big(\|\varphi\|_{L^2(\Sigma)} + \tilde{c}|a|\big). \end{aligned}$$

Since $\|w\|^2_{H^1(\Sigma)} = \|w_0\|^2_{H^1(\Sigma)} + a^2$, then

$$\|w\|_{H^1(\Sigma)} \leqslant c\big(\|\varphi\|_{L^2(\Sigma)} + |a|\big).$$

In order to estimate λ, we integrate (3.2) and we get

$$\lambda|\Sigma| + \int_\Sigma \varphi \, d\sigma = -\int_\Sigma |A_\Sigma|^2 w \, d\sigma,$$

thus

$$|\lambda| \leqslant c\big(\|\varphi\|_{L^2(\Sigma)} + \|w\|_{L^2(\Sigma)}\big).$$

To sum up, we have the estimate

$$|\lambda| + \|w\|_{H^1(\Sigma)} \leqslant c\big(\|\varphi\|_{L^2(\Sigma)} + |a|\big), \tag{3.11}$$

In order to get the estimate with respect to the norms we are interested in, we point out that, by the Sobolev embeddings

$$\begin{aligned} \|w\|_{L^\infty(B_\delta(x))} \leqslant c\|w\|_{W^{2,2}(B_\delta(x))} &\leqslant c\big(\|w\|_{L^2(B_{2\delta}(x))} + \|\varphi + \lambda\|_{L^2(B_{2\delta}(x))} + |a|\big) \\ &\leqslant c\big(\|\varphi\|_{L^2(\Sigma)} + |a|\big), \end{aligned}$$

for any $\delta > 0$ small but fixed and $x \in \Sigma$ (here, $B_\delta(x)$ is the geodesic ball of radius δ centred at x in Σ). In particular,

$$\|w\|_{L^\infty(\Sigma)} \leqslant c\big(\|\varphi\|_{L^\infty(\Sigma)} + |a|\big).$$

By the Schauder regularity estimates, we conclude that,

$$\|w\|_{C^{2,\alpha}(\Sigma)} \leqslant c\big(\|w\|_{L^\infty(\Sigma)} + \|\varphi + \lambda\|_{C^{0,\alpha}(\Sigma)}\big) \leqslant c\big(\|\varphi\|_{C^{0,\alpha}(\Sigma)} + |a|\big),$$

(see [16], Chapter 6, Theorem 6.30). Since the same is true for $|\lambda|$, the proof is over. □

3.2 *The proof of Lemma 2.1: A fixed point argument*

Now we are ready to show existence, uniqueness and Lipschitz continuity with respect to ξ of the solution $(w_{\gamma,\xi}, \lambda_{\gamma,\xi})$ to (2.14).

Step (i)*: Existence and uniqueness.* We solve our problem by a fixed point argument. In fact the map

$$T(w,\lambda) = \Psi\Big(P\mathfrak{F}(\gamma,\xi,w), -\int_\Sigma \tilde{Q}(w)\Big)$$

is a contraction on the product $B \times \Lambda$, where $\Lambda = (-C\gamma, C\gamma)$ and

$$B := \big\{w \in W \cap C_s^{2,\alpha}(\Sigma) : ||w||_{C^{2,\alpha}(\Sigma)} < C\gamma\big\}, \tag{3.12}$$

provided C is large enough. In fact

$$\begin{aligned} ||\mathfrak{F}(\gamma,\xi,w)||_{C^{0,\alpha}(\Sigma)} &\leqslant \gamma\big(4||v_E||_{C^{2,\alpha}(\Sigma)} + ||f||_{C^{0,\alpha}(\Sigma)}\big) + c\gamma||w||_{C^{2,\alpha}(\Sigma)} \\ &\leqslant \gamma\big(4||v_E||_{C^{2,\alpha}(\Sigma)} + ||f||_{C^{0,\alpha}(\Sigma)}\big) + cC\gamma^2 < C\gamma \end{aligned}$$

provided $C > 2(4||v_E||_{C^{2,\alpha}(\Sigma)} + ||f||_{C^{0,\alpha}(\Sigma)})$ and γ is small enough. Similarly, we can see that $\mathfrak{F}(\gamma,\xi,w)$ is Lipschitz continuous in w with Lipschitz constant of order γ.

In addition, the second component fulfills

$$\left|\int_\Sigma \tilde{Q}(w)\right| \leqslant c||w||^2_{C^{2,\alpha}(\Sigma)} \leqslant cC^2\gamma^2 < C\gamma$$

if γ is small enough, and the same is true for the Lipschitz constant.

Lipschitz continuity with respect to ξ. In order to prove (2.18), we point out that, if we set $w_i := w_{\gamma,\xi_i}$ and $y_i := x + \xi_i + w_i(x)\nu(x)$, for $i = 1, 2$,

$$||f(y_1) - f(y_2)||_{C^{0,\alpha}(\Sigma)} \leqslant c\big(|\xi_1 - \xi_2| + ||w_1 - w_2||_{C^{2,\alpha}(\Sigma)}\big)$$

and

$$||\tilde{L}w_1 - \tilde{L}w_2||_{C^{0,\alpha}(\Sigma)} \leqslant c||w_1 - w_2||_{C^{2,\alpha}(\Sigma)},$$

where $\tilde{L}$ is defined in (2.6), and

$$\begin{aligned} ||Q(w_1) - Q(w_2)||_{C^{0,\alpha}(\Sigma)} &\leqslant c\big(||w_1||_{C^{2,\alpha}(\Sigma)} + ||w_2||_{C^{2,\alpha}(\Sigma)}\big)||w_1 - w_2||_{C^{2,\alpha}(\Sigma)} \\ &\leqslant cC\gamma||w_1 - w_2||_{C^{2,\alpha}(\Sigma)}. \end{aligned}$$

Similarly, we can show that

$$\left|\int_\Sigma (\tilde{Q}(w_1) - \tilde{Q}(w_2))d\sigma\right| \leqslant c\gamma||w_1 - w_2||_{C^{2,\alpha}(\Sigma)},$$

thus, applying Ψ,

$$|\lambda_1 - \lambda_2| + ||w_1 - w_2||_{C^{2,\alpha}(\Sigma)} \leqslant c\gamma\big(||w_1 - w_2||_{C^{2,\alpha}(\Sigma)} + |\xi_1 - \xi_2|\big).$$

In conclusion, for γ small enough,

$$|\lambda_1 - \lambda_2| + \frac{1}{2}||w_1 - w_2||_{C^{2,\alpha}(\Sigma)} \leqslant c\gamma|\xi_1 - \xi_2|.$$

4. Solving the bifurcation equation

The parametrisation $\phi : Y \to \Sigma$ of Σ introduced in (2.9) induces a parametrisation $\beta : Y \to \Gamma := \partial F$ given by

$$\beta(y_1, y_2) := \phi(y_1, y_2) + \xi + w_{\gamma,\xi}(y_1, y_2)\nu(y_1, y_2). \tag{4.1}$$

The volume element can be expressed in terms of ϕ in this way

$$|\beta_{y_1} \times \beta_{y_2}| = |\phi_{y_1} \times \phi_{y_2}| + L^1_\xi w_{\gamma,\xi} + Q^1_\xi w_{\gamma,\xi}, \tag{4.2}$$

where L^1_ξ depends linearly on $w_{\gamma,\xi}$ and on its gradient and Q^1_ξ is quadratic in the same quantities. More precisely, they satisfy the estimates

$$\begin{cases} |L^1_\xi w| \leqslant c||w||_{C^{2,\alpha}(\Sigma)} \\ |Q^1_\xi (w)| \leqslant c||w||^2_{C^{2,\alpha}(\Sigma)}. \end{cases} \tag{4.3}$$

Using the Taylor expansion of the function $\frac{1}{1+s}$, we can show that the outward-pointing unit normal to Γ is

$$\nu_\Gamma = \frac{\beta_{y_1} \times \beta_{y_2}}{|\beta_{y_1} \times \beta_{y_2}|} = \frac{\phi_{y_1} \times \phi_{y_2}}{|\phi_{y_1} \times \phi_{y_2}|} + \tilde{L}^1_\xi w_{\gamma,\xi} + \tilde{Q}^1_\xi w_{\gamma,\xi} = \nu + \tilde{L}^1_\xi w_{\gamma,\xi} + \tilde{Q}^1_{\gamma,\xi} w_\xi, \tag{4.4}$$

with $\tilde{L}^1_\xi$ and $\tilde{Q}^1_\xi$ satisfying (4.3).

Now we point out that, if ξ is a critical point of Φ_γ, then

$$\partial_{\xi_i} \Phi_\gamma(\xi) = 0. \tag{4.5}$$

We will rephrase this fact in a more convenient way, that will be more suitable for the forthcoming computations. We define the one-parameter family of diffeomorphisms

$$y_t : Y \to \mathbb{R}^3$$

by

$$y_t(y_1, y_2) := \phi(y_1, y_2) + \xi + te_i + w_{\gamma,\xi+te_i}(y_1, y_2)\nu(y_1, y_2), \tag{4.6}$$

for $i = 1, 2, 3$; $\Gamma_t := y_t(Y)$ is the image of y_t. By construction, Γ_t is actually a submanifold of T^3 and $\Gamma_0 = \Gamma$. In terms of Γ_t, condition (4.5) is equivalent to

$$\frac{d}{dt} I_\gamma(\Gamma_t)|_{t=0} = 0. \tag{4.7}$$

By a result of Fall and Mahmoudi (see [13]),

$$0 = \frac{d}{dt} I_\gamma(\Gamma_t)|_{t=0} = \int_\Gamma (H_\Gamma + 4\gamma v_F + f)(\zeta, \nu_\Gamma) d\sigma_\Gamma + \frac{1}{|\partial\Gamma|} \int_{\partial\Gamma} (\zeta, \nu^\Gamma_{\partial\Gamma}) ds, \tag{4.8}$$

where

$$\zeta = \frac{d}{dt} y_t(x)|_{t=0} = e_i + \partial_{\xi_i} w_{\gamma,\xi} \nu. \tag{4.9}$$

and $\nu_{\partial\Gamma}^{\Gamma}$ is the unit normal to $\partial\Gamma$ in Γ. The boundary term vanishes since, by periodicity, $\partial\Gamma = \emptyset$. Using the parametrisation β of Γ and expansions (4.4) and (4.2), the latter relation becomes

$$\int_Y \Big\{(H_\Gamma + 4\gamma v_F + f)(\beta(y_1, y_2))\big(e_i + \partial_{\xi_i} w_{\gamma,\xi}\nu, \nu + \tilde{L}^1_\xi w_{\gamma,\xi} + \tilde{Q}^1_\xi w_{\gamma,\xi}\big) \\ \big(|\phi_x \times \phi_y| + L^1_\xi w_{\gamma,\xi} + Q^1_\xi w_{\gamma,\xi}\big)\Big\} dy_1 dy_2 = 0.$$

By the auxiliary equation, we know that

$$(H_\Gamma + 4\gamma v_F + f)(\beta(y_1, y_2)) = \sum_{k=1}^{3} A_{k,\gamma,\xi}\nu_k(y_1, y_2) + \lambda, \tag{4.10}$$

thus

$$\sum_{k=1}^{3} A_{k,\gamma,\xi}\left(\int_\Sigma \nu_k\nu_i d\sigma + b_{ki}\right) + \lambda\int_\Gamma (\zeta, \nu_\Gamma)d\sigma_\Gamma = 0, \quad \text{for} \quad i = 1, 2, 3, \tag{4.11}$$

with $b_{ki} = O(\gamma)$. Moreover, once again by [13], we know that

$$\frac{d}{dt}\mathfrak{L}_3(F_t) = \int_\Gamma (\zeta, \nu_F)d\sigma_\Gamma,$$

hence, by the volume constraint,

$$\int_\Gamma (\zeta, \nu_F)d\sigma_\Gamma = 0,$$

thus we get

$$\sum_{k=1}^{3} A_{k,\gamma,\xi}\left(\int_\Sigma \nu_k\nu_i d\sigma + b_{ki}\right) = 0, \quad \text{for} \quad i = 1, 2, 3. \tag{4.12}$$

Since the matrix L_{ki} is invertible (see Remark 3.2) and the coefficients b_{ki} are small, the matrix $L_{ki} + b_{ki}$ is invertible too, therefore $A_{k,\gamma,\xi} = 0$ for $k = 1, 2, 3$.

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