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Two-dimensional steady supersonic exothermically reacting Euler flows with strong contact discontinuity over a Lipschitz wall

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In this paper, we establish the global existence of supersonic entropy solutions with a strong contact discontinuity over a Lipschitz wall governed by the two-dimensional steady exothermically reacting Euler equations, when the total variation of both the initial data and slope of the Lipschitz wall is sufficiently small. Local and global estimates are developed and a modified Glimm-type functional is carefully designed. Next the validation of the quasi-one-dimensional approximation in the domain bounded by the wall and the strong contact discontinuity is rigorous justified by proving that the difference between the average of weak solution and the solution of quasi-one-dimensional system can be bounded by the square of the total variation of both the initial data and slope of the Lipschitz wall.

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1. Introduction

We are concerned with the global existence and the quasi-one-dimensional approximation of entropy solutions with strong contact discontinuity of two-dimensional steady supersonic exothermically reacting Euler flows, which are governed by

$$\begin{cases} (\rho u)_x + (\rho v)_y = 0, \\ (\rho u^2 + p)_x + (\rho uv)_y = 0, \\ (\rho uv)_x + (\rho v^2 + p)_y = 0, \\ \big((\rho E + p)u\big)_x + \big((\rho E + p)v\big)_y = 0, \\ (\rho u Z)_x + (\rho v Z)_y = -\rho\phi(T)Z. \end{cases} \tag{1.1}$$

Here (u, v) is the velocity. p, ρ, and T stand for the scalar pressure, the density, the temperature, respectively. Z represents the fraction of unburned gas, where $0 \leqslant Z \leqslant 1$. E denotes the specific total energy and is given by

$$E = e + \frac{1}{2}(u^2 + v^2) + q_0 Z, \tag{1.2}$$

where e is the specific internal energy, and $q_0 > 0$ is the specific binding energy of unburned gas. $\phi(T)$ is the reaction rate function, which is a C^1 function with respect to T on $(0, +\infty)$ and satisfies that

$$\lim_{T\to 0+} \phi(T) = 0 \qquad \text{and} \qquad \phi'(T) > 0.$$

As an example in [6], we call $\phi(T)$ has the Arrhenius form which vanishes only at absolute zero temperature, if

$$\phi(T) = T^{\mu} e^{-\mathcal{E}/RT},$$

where μ is a positive constant, and $\mathcal{E}$ is the action energy. For further information on this equation and related combustion theories, we refer the reader to [15, 23, 35, 38, 42].

The other thermodynamic variable is the entropy S, which is defined through the thermodynamical relation that:

$$TdS = de - \frac{p}{\rho^2} d\rho.$$

For the ideal polytropic gas, the constitutive relations are

$$p = R\rho T, \quad e = c_v T, \quad \gamma = 1 + \frac{R}{c_v} > 1,$$

where R, c_v, γ are all positive constants. Then the sonic speed is given by $c = \sqrt{\gamma p/\rho}$.

Obviously, by the thermodynamical relation and the constitutive relations, any two thermodynamic variables of e, p, T, ρ and S can be chosen as the independent variables. Here, we choose ρ and p as the independent variables, and let

$$U = (u, v, p, \rho, Z)^\top. \tag{1.3}$$

In this paper, we will study the two-dimensional steady supersonic exothermically reacting Euler flow with a strong contact discontinuity over a Lipschitz wall, which is small perturbation of the straight wall (see Fig. 1). Here a strong contact discontinuity is a free interface across which the flow direction and the pressure are continuous but the jump of the other quantities are not small, and the contact discontinuity propagates along the flow direction (see also Remark 1.5).

Mathematically, the problem we are concerned with is the following initial-boundary value problem of system (1.1) in Ω with the initial condition that

$$U(0, y) = U_0(y) = \begin{cases} U_2(y), & y^{(0)} < y < 0, \\ U_1(y), & y < y^{(0)}, \end{cases} \tag{1.4}$$

and the boundary condition that

$$(u, v) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{1.5}$$

Then we can define the global entropy solutions of problem (1.1) and (1.4)-(1.5).

DEFINITION 1.1 (Entropy Solutions) A vector-valued function $U(x, y) = (u, v, p, \rho, Z)^\top(x, y) \in L^\infty(\Omega)$ is called a global *entropy solution* of problem (1.1) and (1.4)–(1.5) if

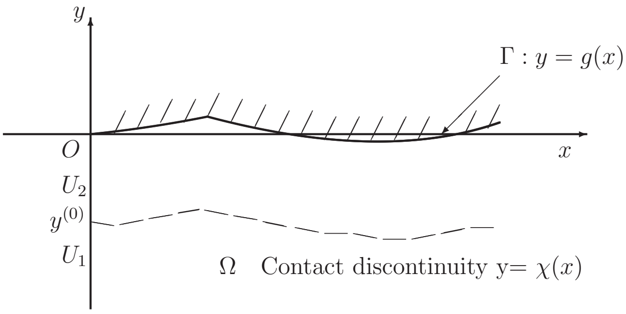

FIG. 1. Reacting Euler flow over a Lipschitz wall

(i) U is a weak solution of (1.1) in Ω in the distribution sense and satisfies (1.4)–(1.5) in the trace sense;
(ii) U satisfies the entropy inequality that

$$(\rho u S)_x + (\rho v S)_y \geqslant \frac{q_0 \rho \phi(T) Z}{T}$$

in the distribution sense in $\overline{\Omega}$.

REMARK 1.2 The entropy inequality in Definition 1.1 follows from Clausius–Duhem inequality in the time-dependent case, see, for instance, [19].

Before stating the main theorems of this paper, let us assume the following:

(H1) As shown in Fig. 1, there exists a Lipschitz function $g(x) \in \mathrm{Lip}(\mathbb{R}_+;\mathbb{R})$ with that $g(0) = 0$, $g'(0+) = 0$, and that $g'(x) \in BV(\mathbb{R}^+;\mathbb{R})$ such that

$$\Omega = \{(x, y) : y < g(x), x > 0\}, \quad \Gamma = \{(x, y) : y = g(x), x \geqslant 0\},$$

and $\mathbf{n}(x\pm) = \frac{(-g'(x\pm),1)}{\sqrt{(g'(x\pm))^2+1}}$ is the outer normal vectors to Γ at the points $x\pm$, respectively. Here and in sequel, $\mathrm{Lip}(\mathbb{R}_+;\mathbb{R})$ denotes the set of Lipschitz continuous functions, while $BV(\mathbb{R}^+;\mathbb{R})$ denotes the set of functions with bounded variations.

(H2) The upstream flow in $\{x = 0\}$ consists of two states $U_2(y) = (u_2, v_2, p_2, \rho_2, Z_2)^\top(y)$ when $y^{(0)} < y < 0$ and $U_1(y) = (u_1, v_1, p_1, \rho_1, Z_1)^\top(y)$ when $y < y^{(0)}$, separated by the interface $y = y_0$ such that

$$u_i > c_i > 0, \quad 0 \leqslant Z_i \leqslant 1, \quad \lim_{y\to-\infty} Z_1(y) = 0,$$

where $c_i = \sqrt{\gamma p_i/\rho_i}$ is the sonic speed of state U_i, for $i = 1, 2$.

(H3) There exists a positive constant $T^{(0)} > 0$, such that $T_i(y) > T^{(0)}$, for $i = 1, 2$.

REMARK 1.3 Assumption (H3) is to make sure that $\phi(T)$ admits a positive minimum value.

Our first result is to establish the nonlinear stability of strong contact discontinuity in the supersonic exothermically reacting Euler flows around a background solution, which is given by the case that $g(x) = 0$. In this case, the problem admits a solution consisting of one straight contact discontinuity $y = y^{(0)} < 0$ and two constant states:

$$U^{(0)} = \begin{cases} U_2^{(0)} = (u_2^{(0)}, 0, p_2^{(0)}, \rho_2^{(0)}, 0), & y^{(0)} < y < 0,\ x > 0, \\ U_1^{(0)} = (u_1^{(0)}, 0, p_1^{(0)}, \rho_1^{(0)}, 0), & y < y^{(0)},\ x > 0, \end{cases}$$

where $p_2^{(0)} = p_1^{(0)}$, $u_i^{(0)} > c_i^{(0)} > 0$, and the sonic speed $c_i^{(0)} = \sqrt{\gamma p_i^{(0)}/\rho_i^{(0)}}$, for $i = 1, 2$.

More precisely, we proved the following theorem.

Theorem 1.4 *Under assumptions* (H1)–(H3)*, there exist positive constants δ_0 and C depending only on $U^{(0)}$, such that if*

$$T.V.\{g'(\cdot) : [0, +\infty)\} < \delta_0, \tag{1.6}$$

and

$$\sup_{y<y^{(0)}} |U_1(y) - U_1^{(0)}| + \sup_{y^{(0)}<y<0} |U_2(y) - U_2^{(0)}| < \delta_0, \tag{1.7}$$

$$T.V.\{U_1(\cdot) : (-\infty, y^{(0)})\} + T.V.\{U_2(\cdot) : (y^{(0)}, 0)\} < \delta_0, \tag{1.8}$$

then the initial-boundary value problem (1.1) *and* (1.4)–(1.5) *admits a global entropy solution $U(x, y) \in L^\infty(\Omega)$ such that the following hold:*

(i) *for every $x \in [0, +\infty)$,*

$$T.V.\{U(x, \cdot) : (-\infty, g(x)]\} \leqslant C\delta_0. \tag{1.9}$$

(ii) *The curve $\{y = \chi(x)\}$ is a strong contact discontinuity emanating from the point $(0, y^{(0)})$ with $\chi(x) < g(x)$ for any $x > 0$, and is Lipschitz such that*

$$|\chi(x') - \chi(x'')| \leqslant C|x' - x''|, \tag{1.10}$$

for any $x', x'' \geqslant 0$. Furthermore, it holds that

$$\sup_{y<\chi(x)} |U(x, y) - U_1^{(0)}| \leqslant C\delta_0, \qquad \sup_{\chi(x)<y<g(x)} |U(x, y) - U_2^{(0)}| \leqslant C\delta_0. \tag{1.11}$$

REMARK 1.5 Indeed, the solution U given by Theorem 1.4 has the trace along the both sides of curve $y = \chi(x)$. Let $U_\pm = (u_\pm, v_\pm, p_\pm, \rho_\pm, Z_\pm)$ be the trace of U above or below the curve $y = \chi(x)$ respectively, and $\mathbf{n}_\emptyset = (\chi'(x), -1)$ is the normal to the curve $y = \chi(x)$. Due to the construction of the solution, we have that

$$p_+ = p_- \quad \text{and} \quad (u_+, v_+) \cdot \mathbf{n}_\emptyset = (u_-, v_-) \cdot \mathbf{n}_\emptyset, \qquad U_+ \neq U_- \tag{1.12}$$

almost everywhere along the curve $y = \chi(x)$. Then as in [19] and [17] such curve $y = \chi(x)$ is called the contact discontinuity here and in sequel (see also [2, 26, 41]). Moreover, compared to the strengths of the other waves, the strength $|U_+ - U_-|$ is relatively large, so the curve $y = \chi(x)$ is called the strong contact discontinuity.

Our second result is about the quasi-one-dimensional approximation. Physically, quasi-one-dimensional approach is based on the assumption that the motion of the nonuniform true flow is slowly various in some direction. Then by averaging the nonuniform flows over some direction or cross section, we can reduce the dimension to make the model simpler, see [14, 36, 42] for example. Therefore, the true nonuniform flow is replaced by a simpler flow, which is easy to be investigate and useful to study the basic properties of the true flow. In this case, it is possible to reduce the dimensionality of the problem. We will justify the quasi-one-dimensional approximation for our problem in this paper.

If the flow is slowly various in the y-direction compared to the x-direction, we can introduce the quasi-one-dimensional approximation in the domain $\{(x, y)|x > 0,\ \chi(x) < y < g(x)\}$ as follows. Neglect the changes of the solutions in y-direction, and let $A(x)$ be the distance between the wall and the strong contact discontinuity. Then the motion of the steady exothermically reacting Euler flows in the domain $\{(x, y)|x > 0,\ \chi(x) < y < g(x)\}$ can be described by the much simpler quasi-one-dimensional model:

$$\begin{cases} (\rho u A(x))_x = 0, \\ \big((\rho u^2 + p)A(x)\big)_x = A'(x)p, \\ \big((e + \frac{1}{2}u^2 + \frac{p}{\rho})\rho u A(x)\big)_x = q_0 A(x)\rho\phi(T)Z, \\ \big(\rho u Z A(x)\big)_x = -A(x)\rho\phi(T)Z. \end{cases} \tag{1.13}$$

Let $(\bar{\rho}_0, \bar{u}_0, \bar{p}_0, \bar{Z}_0)^\top$ be the integral average of the initial data $U_2(y)$ in the interval $y^{(0)} < y < 0$, that is

$$(\bar{\rho}_0, \bar{u}_0, \bar{p}_0, \bar{Z}_0)^\top = \frac{1}{|y^{(0)}|}\int_{y^{(0)}}^{0} U_2(y)dy.$$

Let $\bar{U}(x) = (\bar{\rho}, \bar{u}, \bar{p}, \bar{Z})^\top$ be the integral average of the solution of system (1.1) with respect to y between the wall and the strong contact discontinuity, that is,

$$\bar{U}(x) = \frac{1}{A(x)}\int_{\chi(x)}^{g(x)} U(x, y)dy,$$

and let $U_A(x) = (\rho_A, u_A, p_A, Z_A)^\top$ the solution of system (1.13) with the initial data $U_{A,0} = (\bar{\rho}_0, \bar{u}_0, \bar{p}_0, \bar{Z}_0)^\top$. Then our second result related to the quasi-one-dimensional approximation in the domain $\{(x, y)|x > 0,\ \chi(x) < y < g(x)\}$ is as follows.

Theorem 1.6 *Under assumptions* (H1)–(H3)*, there exist positive constants δ_0 and C depending only on $U^{(0)}$, such that if* (1.6)–(1.8) *hold, then for any $x \geqslant 0$, it holds that*

$$|\bar{U}(x) - U_A(x)| \leqslant C\delta_*^2,$$

where

$$\begin{aligned} \delta_* =& T.V.\{g'(\cdot) : [0, +\infty)\} + T.V.\{U_1(\cdot) : (-\infty, y^{(0)})\} \\ &+ T.V.\{U_2(\cdot) : (y^{(0)}, 0)\} + \sup_{y^{(0)} < y < 0} |U_2(y) - U_2^{(0)}|. \end{aligned}$$

Theorem 1.6 justifies the validation of the quasi-one-dimensional approximation of the supersonic exothermically reacting Euler flows if δ_* is sufficiently small, i.e., this theorem indicates that the difference between the integral average of the weak solution of (1.1) and the solution of

(1.13) can be bounded by the square of the total variation of both the initial data and slope of the Lipschitz wall.

We develop a fractional-step Glimm scheme to construct the approximate solutions to establish the global existence of the entropy solution of the initial boundary value problem (1.1) and (1.4)–(1.5). To make it, we have to design a Glimm-type functional based on local estimates obtained in Section 2. The key estimates are the reflection coefficient in front of the strength of the reflected 5-wave when the weak 1-wave hits the strong contact discontinuity from above governed by the corresponding homogeneous system (2.3) is strictly less than one, as well as the exponential decay estimate of the reactant Z in the reacting step. With the Glimm-type functional in hand, we can show that the total variation of the approximate solutions is uniformly bounded and actually small, and then by the standard argument developed in [26] to show Theorem 1.4. Another essential estimate is to trace the approximate strong contact discontinuity in order to establish the nonlinear stability of strong contact discontinuity in the supersonic exothermically reacting Euler flows under small perturbation.

We remark that although elegant results had been established for the existence of entropy solutions of hyperbolic balance laws in [16, 20, 29, 43], system (1.1) concerned in this paper does not satisfy the hypotheses there. In fact, the exothermic reaction can increase the total variation of the solutions. For example, the linearized stability analysis, as well as numerical and physical experiments, have shown that certain steady detonation waves are unstable [1, 21, 24, 32]. However, if assume that the reaction rate function $\phi(T)$ never vanishes, then the decay estimate of the reaction plays a key role in controlling the increasing of the total variation of solutions.

Next, in order to show Theorem 1.6, we need carefully to derive several estimates on error terms of different type to pass the limit $h \to 0$ such that we can get the equations that the integral average of weak solutions with respect to y satisfies. Then the validation of the quasi-one-dimensional approximation is rigorously justified by applying the decay estimates of the reactant Z of both system (1.1) and (1.13), and the smallness of the $B.V.$ bounds of solutions.

The importance of the problem of steady supersonic non-reacting Euler flow past a wedge has been introduced in Courant-Friedrichs' book [18]. When the flow behind the shock is smooth, the existence and asymptotic behaviour had been extensively studied by many authors (for instance, see [11–13, 22, 28, 33, 40]). See also [3, 37] for piecewise smooth shock free solutions. Next, for the non-piecewise smooth solutions, by developing a modified Glimm scheme or wave-front tracking scheme, global weak entropy solutions of the potential flow had been constructed in [44–46] when the wedge is a small perturbation of a straight wedge or a convex one. Later, global weak entropy solutions with a large shock or vortex sheet had been established for the full Euler equations in [9, 10]. Recently, global weak entropy solutions with transonic characteristic discontinuities had been obtained in [5, 30] when the steady supersonic non-reacting Euler flow past a convex corner surrounded by the static gas. Meanwhile, the quasi-one-dimensional approximation of isentropic or irrotational gas flow had been established in [14, 25] by applying the Riemann semigroup via the wave-front tracking scheme (see [2, 7] for more details of the techniques).

For the exothermically reacting Euler equations, the large-time existence of one-dimensional time-dependent entropy solutions of the Cauchy problem was established in [6]. Recently, the global existence of steady weak entropy solutions with a strong shock or strong rarefaction wave is established in [4, 8]. For further information on the reacting gas dynamic theory, we refer the reader to [35, 42].

The rest of this paper is organised as follows. In Section 2, several important local estimates including local interaction estimates and local estimates on the reacting step are established. In Section 3, we introduce the fractional-step Glimm scheme to construct approximate solutions and introduce a modified Glimm-type functional to prove the global estimates of the approximated solutions in the non-reacting step and the reacting step separately. Then we complete the proof of Theorem 1.4 in Section 3. Finally, section 4 is devoted to the proof of Theorem 1.6.

2. Local estimates of solutions of the steady exothermically reacting Euler equations

In this section, we will establish the local wave interaction estimates for the homogeneous system, and then the local estimates on the reacting step of the steady exothermically reacting Euler equations (1.1).

First, system (1.1) can be written in the following form:

$$W(U)_x + H(U)_y = G(U), \tag{2.1}$$

with $U = (u, v, p, \rho, Z)^\top$, where

$$\begin{aligned} W(U) &= \Big(\rho u, \rho u^2 + p, \rho u v, \rho u\Big(\frac{u^2+v^2}{2} + \frac{\gamma p}{(\gamma-1)\rho}\Big), \rho u Z\Big)^\top, \\ H(U) &= \Big(\rho v, \rho u v, \rho v^2 + p, \rho v\Big(\frac{u^2+v^2}{2} + \frac{\gamma p}{(\gamma-1)\rho}\Big), \rho v Z\Big)^\top, \\ G(U) &= \big(0, 0, 0, q_0 \rho Z \phi(T), -\rho\phi(T)Z\big)^\top. \end{aligned} \tag{2.2}$$

In the case when $G(U)$ is identically zero, (2.1) becomes the homogeneous system

$$W(U)_x + H(U)_y = 0. \tag{2.3}$$

2.1 *Elementary wave curves of the homogeneous system of (2.3)*

Before deriving the local estimates, we review certain basic properties of the homogeneous system of (2.3) and the solvability of several typical Riemann problems that appear in the process of the fractional-step Glimm scheme.

First, we remark that in this paper, M is a universal constant, depending only on the data and different at each occurrence, $O(1)$ is a quantity that is bounded by M, and $O_\epsilon(U)$ is a neighbourhood with radius $M\epsilon$ and center U.

If $u > c$, the homogeneous system (2.3) has five real eigenvalues in the x-direction, which are

$$\lambda_i = \frac{uv + (-1)^{\frac{i+3}{4}} c\sqrt{u^2+v^2-c^2}}{u^2-c^2}, \; i = 1, 5, \qquad \lambda_j = \frac{v}{u}, \; j = 2, 3, 4.$$

The associated linearly independent right eigenvectors are

$$r_i = \kappa_i\Big(-\lambda_i, 1, \rho(\lambda_i u - v), \frac{\rho(\lambda_i u - v)}{c^2}, 0\Big)^\top, \quad i = 1, 5; \tag{2.4}$$

$$r_2 = (u, v, 0, 0, 0)^\top, \quad r_3 = (0, 0, 0, \rho, 0)^\top, \quad r_4 = (0, 0, 0, 0, 1)^\top, \tag{2.5}$$

where κ_i are chosen so that $r_i \cdot \nabla\lambda_i = 1$ since the i-th characteristic fields are genuinely nonlinear for $i = 1, 5$. It is easy to see that $r_j \cdot \nabla\lambda_j = 0$, $j = 2, 3, 4$, which means these characteristic fields are linearly degenerate. By the straightforward calculation, we have the following lemma about the value of κ_i.

Lemma 2.1 *At the constant state* $U_k^{(0)} = (u_k^{(0)}, 0, p_k^{(0)}, \rho_k^{(0)}, 0)$ *with* $u_k^{(0)} > c_k^{(0)} > 0, k = 1, 2,$

$$\kappa_1\big(U_k^{(0)}\big) = \kappa_5\big(U_k^{(0)}\big) = 1/\big(\nabla_U \lambda_i \cdot (-\lambda_i, 1, \rho u\lambda_i, \rho u\lambda_i/c^2, 0)|_{U=U_k^{(0)}}\big) > 0, \quad i = 1, 5.$$

It implies that $\kappa_i(U) > 0$ *for any* $U \in O_\epsilon(U_k^{(0)})$ *since* $\kappa_i(U)$ *are continuous for* $i = 1, 5$.

Next, we will consider wave curves for $u > c$ in the phase space, especially in the neighborhood of $U_1^{(0)}$ and $U_2^{(0)}$. At each state $U_a = (u_a, v_a, p_a, \rho_a, Z_a)^\top$ with $u_a > c_a = \sqrt{\gamma p_a/\rho_a}$, there are five wave curves in the phase space through U_a.

The j-th contact discontinuity wave curve $C_j(U_a)$ for $j = 2, 3, 4$, are

$$C_j(U_a): \ dp = 0, \qquad vdu - udv = 0.$$

More precisely, by solving the following ODE problem

$$\begin{cases} \frac{dU}{d\sigma_j} = r_j(U), \ j = 2, 3, 4, \\ U|_{\sigma_j=0} = U_a, \end{cases}$$

we easily have that

$$C_2(U_a): \ U = (u_a e^{\sigma_2}, v_a e^{\sigma_2}, p_a, \rho_a, Z_a)^\top, \tag{2.6}$$

$$C_3(U_a): \ U = (u_a, v_a, p_a, \rho_a e^{\sigma_3}, Z_a)^\top, \tag{2.7}$$

$$C_4(U_a): \ U = (u_a, v_a, p_a, \rho_a, Z_a + \sigma_4)^\top. \tag{2.8}$$

The i-th rarefaction wave curve $R_i(U_a), i = 1, 5$,

$$R_i(U_a): \ dp = c^2 d\rho, \ du = -\lambda_i dv, \ \rho(\lambda_i u - v)dv = dp, \ dZ = 0. \tag{2.9}$$

The i-th shock wave curve $S_i(U_a), i = 1, 5$,

$$S_i(U_a): \ [p] = \frac{c_a^2}{\hat{\gamma}}[\rho], \ [u] = -s_i[v], \ \rho_a(s_i u_a - v_a)[v] = [p], \ [Z] = 0. \tag{2.10}$$

where $[\cdot]$ stands for the jump of a quantity across the shock, the slope of the discontinuity

$$s_i = \frac{u_a v_a + (-1)^{\frac{i+3}{4}} \hat{c}_a \sqrt{u_a^2 + v_a^2 - \hat{c}_a^2}}{u_a^2 - \hat{c}_a^2},$$

and $\hat{c}_a^2 = \frac{c_a^2}{\hat{\gamma}} \frac{\rho}{\rho_a}, \hat{\gamma} = \frac{\gamma+1}{2} - \frac{(\gamma-1)}{2} \frac{\rho}{\rho_a}$.

Following the ideas in [31], in a neighbourhood of $U_k^{(0)}, k = 1, 2$, we can parameterize any physically admissible wave curves above by

$$\alpha_i \mapsto \Phi_i(\alpha_i; U_a), \tag{2.11}$$

with $\Phi_i \in C^2$, $\Phi_i|_{\alpha_i=0} = U_a$, and $\frac{\partial \Phi_i}{\partial \alpha_i}|_{\alpha_i=0} = r_i(U_a)$. For $i = 1, 5$, the case $\alpha_i > 0$ corresponds to a rarefaction wave, while the case $\alpha_i < 0$ corresponds to a shock wave. Moreover, Φ_2, Φ_3, Φ_4 can be given with three independent parameters $(\sigma_2, \sigma_3, \alpha_4)$ as

$$\Phi_2(\sigma_2; U_a) = (u_a e^{\sigma_2}, v_a e^{\sigma_2}, p_a, \rho_a, Z_a), \tag{2.12}$$

$$\Phi_3(\sigma_3; U_a) = (u_a, v_a, p_a, \rho_a e^{\sigma_3}, Z_a), \tag{2.13}$$

$$\Phi_4(\alpha_4; U_a) = (u_a, v_a, p_a, \rho_a, Z_a + \alpha_4). \tag{2.14}$$

In particular, it holds that

$$U_2^{(0)} = \left(u_2^{(0)}, 0, p_2^{(0)}, \rho_2^{(0)}, 0\right)^\top = \left(u_1^{(0)} e^{\sigma_{20}}, 0, p_1^{(0)}, \rho_1^{(0)} e^{\sigma_{30}}, 0\right)^\top.$$

2.2 *Local interaction estimates*

Let us consider the local wave interaction estimates for the homogeneous system (2.3) first, which include the weak wave interactions, weak wave reflections on the boundary and the interaction between the strong contact discontinuity and weak waves.

First, let us consider the Riemann problem only involving weak waves for (2.3):

$$U|_{x=x_0} = \begin{cases} U_b = (u_b, v_b, p_b, \rho_b, Z_b)^\top, & y > y_0, \\ U_a = (u_a, v_a, p_a, \rho_a, Z_a)^\top, & y < y_0, \end{cases} \tag{2.15}$$

where the constant states U_a and U_b are the below state and above state with respect to the line $y = y_0$, respectively.

Let $\tilde{\Phi}_i, i = 1, 2, 3, 5$ be the vector which only contains the first four components of Φ_i, where Φ_i are defined in Section 2.1. For the simplicity, we set

$$\tilde{\Phi}\big(\alpha_5, \alpha_3, \alpha_2, \alpha_1; V_a\big) = \tilde{\Phi}_5\Big(\alpha_5; \tilde{\Phi}_3\Big(\alpha_3; \tilde{\Phi}_2\big(\alpha_2; \tilde{\Phi}_1(\alpha_1; V_a)\big)\Big)\Big),$$

with $V_a = (u_a, v_a, p_a, \rho_a)^\top$, and $\mathfrak{F}(\sigma_3, \sigma_2; V_a) = \tilde{\Phi}_3(\sigma_3; \tilde{\Phi}_2(\sigma_2; V_a)) = (u_a e^{\sigma_2}, v_a e^{\sigma_2}, p_a, \rho_a e^{\sigma_3})^\top$ for any $V_a \in O_\epsilon({V_1}^{(0)})$ with ${V_1}^{(0)} = (u_1^{(0)}, 0, p_1^{(0)}, \rho_1^{(0)})^\top$.

Following the argument in [31], we easily have the following lemma.

Lemma 2.2 *There exist positive constants ϵ and C, such that for any states $U_a, U_b \in O_\epsilon(U_k^{(0)}), k = 1, 2$, the Rieman problem (2.15) admits a unique admissible solution of five elementary waves. In addition, the state U_b can be represented by*

$$\begin{cases} V_b = \tilde{\Phi}(\alpha_5, \alpha_3, \alpha_2, \alpha_1; V_a), \\ Z_b = Z_a + \alpha_4, \end{cases} \tag{2.16}$$

with $V_b = (u_b, v_b, p_b, \rho_b)^\top$. Furthermore, it holds that $|U_b - U_a| \leqslant C \sum_{i=1}^{5} |\alpha_i|$.

Moreover, the Glimm interaction estimates theorem (see [19, 26, 41]) implies the following local weak wave interaction estimates.

PROPOSITION 2.3 Suppose that three states U_a, U_m, and $U_b \in O_\epsilon(U_k^{(0)}), k = 1, 2$, satisfy that

$$\begin{aligned} V_b &= \tilde{\Phi}(\gamma_5, \gamma_3, \gamma_2, \gamma_1; V_a), \quad Z_b = Z_a + \gamma_4, \\ V_b &= \tilde{\Phi}(\beta_5, \beta_3, \beta_2, \beta_1; V_m), \quad Z_b = Z_m + \beta_4, \\ V_m &= \tilde{\Phi}(\alpha_5, \alpha_3, \alpha_2, \alpha_1; V_a), \quad Z_m = Z_a + \alpha_4. \end{aligned}$$

(see Fig. 2). Then it holds that

$$\begin{cases} \gamma_i = \alpha_i + \beta_i + O(1)\Delta(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*), \ i = 1, 2, 3, 5, \\ \gamma_4 = \alpha_4 + \beta_4, \end{cases} \tag{2.17}$$

where $\Delta(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) = |\alpha_5|(|\beta_1| + |\beta_2| + |\beta_3|) + |\beta_1|(|\alpha_2| + |\alpha_3|) + \sum_{j=1,5} \Delta_j(\alpha_j, \beta_j)$ with

$$\Delta_j(\alpha_j, \beta_j) = \begin{cases} 0, & \alpha_j \geqslant 0 \, , \beta_j \geqslant 0, \\ |\alpha_j||\beta_j|, & \text{otherwise}. \end{cases}$$

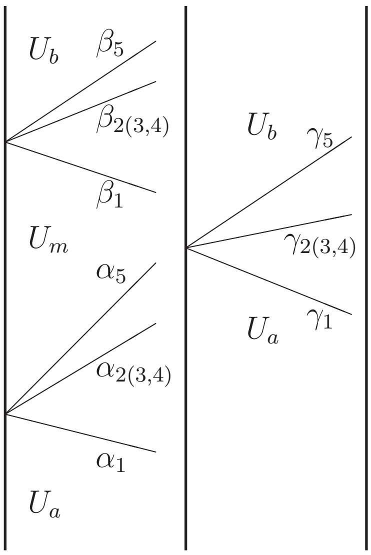

FIG. 2. Weak wave interactions

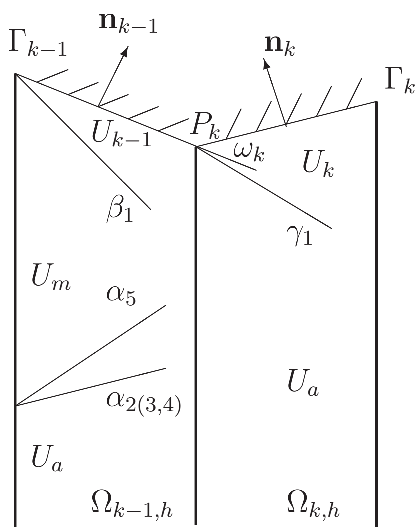

FIG. 3. Weak wave reflections on the boundary

Next, we consider the reflections and interactions of the waves near the boundary. Denote by $\{P_k\}_{k=0}^{\infty}$ the points $\{(x_k, y_k)\}_{k=0}^{\infty}$ in the x-y plane with $x_k := kh$ and $y_k := g(kh)$. Set

$$
\begin{aligned}
&\omega_{k,k+1} = \arctan\Big(\frac{y_{k+1}-y_k}{x_{k+1}-x_k}\Big), \quad \omega_k = \omega_{k,k+1} - \omega_{k-1,k}, \quad \omega_{-1,0} = 0,\\
&g_{k,h}(x) = y_k + (x - x_k)\tan(\omega_{k,k+1}), \quad x \in [x_k, x_{k+1}), \qquad (2.18)\\
&\Omega_{k,h} = \big\{(x, y) : x \in [x_k, x_{k+1}), y < g_{k,h}(x)\big\}, \quad \Omega_h = \bigcup_{k\geqslant 0} \Omega_{k,h},\\
&\Gamma_k = \big\{(x, y) : x \in [x_k, x_{k+1}), y = g_{k,h}(x)\big\}.
\end{aligned}
$$

Let $\mathbf{n}_k$ be the outer normal vector to Γ_k, *i.e.*,

$$
\mathbf{n}_k = \frac{(-y_{k+1} + y_k, x_{k+1} - x_k)}{\sqrt{(y_{k+1} - y_k)^2 + (x_{k+1} - x_k)^2}} = \big(-\sin(\omega_{k,k+1}), \cos(\omega_{k,k+1})\big). \qquad (2.19)
$$

Now, we consider the Riemann problem for (2.3) with boundary,

$$
\begin{cases}
W(U)_x + H(U)_y = 0, & \text{in } \Omega_{k,h},\\
U|_{\{x=kh\}} = U_a, & \\
(u, v)\cdot \mathbf{n}_k = 0 & \text{on } \Gamma_k,
\end{cases} \qquad (2.20)
$$

where U_a is a constant state (see Fig. 3).

For small angle $\omega_{k,k+1}$, we have the following for the solvability of the boundary Riemann problem (2.20).

Lemma 2.4 *There exists $\epsilon > 0$ such that, for $U_a \in O_\epsilon(U_2^{(0)})$ and $|\omega_{k,k+1}| < \epsilon$, there is only one admissible solution, consisting of a 1-wave with strength γ_1, that solves the boundary value problem (2.20). It also holds that*

$$
\gamma_1 = K_b \omega_{k,k+1} + O(1)\big(|\omega_{k,k+1}|^2 + |U_a - U_2^{(0)}|\big), \qquad (2.21)
$$

with the constant $K_b > 0$.

Proof. Let us consider the function

$$
\varphi_k(\gamma_1, \omega_{k,k+1}) = (u, v)\cdot \mathbf{n}_k = \big(\Phi_1^{(1)}(\gamma_1; U_a), \Phi_1^{(2)}(\gamma_1; U_a)\big)\cdot\big(-\sin(\omega_{k,k+1}), \cos(\omega_{k,k+1})\big),
$$

where $\Phi_1^{(i)} (i = 1, 2)$ is the i-th component of Φ_1 .

Note that $\varphi_k(0, 0)|_{\{U_a=U_2^{(0)}\}} = 0$, and

$$
\frac{\partial \varphi_k(\gamma_1, \omega_{k,k+1})}{\partial \gamma_1}\Big|_{\{\gamma_1=0,\omega_{k,k+1}=0,U_a=U_2^{(0)}\}} = \kappa_1\big(U_2^{(0)}\big) > 0,
$$

with $\kappa_1(U_2^{(0)})$ given by Lemma 2.1. It follows from the implicit function theorem that there exists $\epsilon > 0$, such that for $U_a \in O_\epsilon(U_2^{(0)})$ and $|\omega_{k,k+1}| < \epsilon$, equation $\varphi_k(\gamma_1, \omega_{k,k+1}) = 0$ admits a unique solution $\gamma_1(\omega_{k,k+1})$. Moreover, by the Taylor expansion formula, we have

$$
\gamma_1(\omega_{k,k+1}) = \gamma_1(0) + \frac{\partial \gamma_1}{\partial \omega_{k,k+1}}\Big|_{\{\omega_{k,k+1}=0\}}\omega_{k,k+1} + O(1)|\omega_{k,k+1}|^2.
$$

Differentiating $\varphi_k(\gamma_1(\omega_{k,k+1}), \omega_{k,k+1}) = 0$ with respect to $\omega_{k,k+1}$, and letting $\omega_{k,k+1} = 0$ and $U_a = U_2^{(0)}$, we have

$$\frac{\partial \gamma_1}{\partial \omega_{k,k+1}}\Big|_{\{\omega_{k,k+1}=0, U_a=U_2^{(0)}\}} = \frac{u_2^{(0)}}{\kappa_1\big(U_2^{(0)}\big)} > 0.$$

Thus, we have $K_b > 0$ for sufficiently small ϵ. □

Then, we can obtain the estimates of the weak wave reflection on the boundary.

PROPOSITION 2.5 Suppose that the three constant states U_a, U_m and $U_{k-1} \in O_\epsilon(U_2^{(0)})$ satisfy that (see Fig. 3)

$$V_m = \tilde{\Phi}(\alpha_5, \alpha_3, \alpha_2; V_a), \quad Z_m = Z_a + \alpha_4, \tag{2.22}$$

$$U_{k-1} = \Phi_1(\beta_1; U_m), \quad (u_{k-1}, v_{k-1}) \cdot \mathbf{n}_{k-1} = 0. \tag{2.23}$$

Then, for constant state $U_k \in O_\epsilon(U_2^{(0)})$ which satisfies that

$$U_k = \Phi_1(\gamma_1; U_a), \quad (u_k, v_k) \cdot \mathbf{n}_k = 0,$$

it holds that

$$\gamma_1 = \beta_1 + K_{b0}\omega_k + K_{b2}\alpha_2 + K_{b3}\alpha_3 + K_{b5}\alpha_5, \tag{2.24}$$

where $K_{b0}, K_{b2}, K_{b3}, K_{b5}$ are C^2-functions of $\beta_1, \omega_k, \alpha_2, \alpha_3, \alpha_5, \omega_{k-1,k}$ and U_a. Furthermore, K_{b0} is bounded, and when $\beta_1 = \omega_k = \alpha_2 = \alpha_3 = \alpha_5 = \omega_{k-1,k} = 0, U_a = U_2^{(0)}$, it holds that

$$K_{b5} = 1, \quad K_{bi} = 0, \quad i = 2, 3. \tag{2.25}$$

Proof. Let us consider the function:

$$\begin{aligned}\varphi_{k,k-1}(\gamma_1, \beta_1, \omega_k, \alpha_2, \alpha_3, \alpha_5) := \big(&\Phi_1^{(1)}(\gamma_1; V_a), \Phi_1^{(2)}(\gamma_1; V_a)\big) \cdot \mathbf{n}_k \\ &- \Big(\Phi_1^{(1)}\big(\beta_1; \tilde{\Phi}(\alpha_5, \alpha_3, \alpha_2; V_a)\big), \Phi_1^{(2)}\big(\beta_1; \tilde{\Phi}(\alpha_5, \alpha_3, \alpha_2; V_a)\big)\Big) \cdot \mathbf{n}_{k-1}.\end{aligned}$$

Note that $\varphi_{k,k-1}(0,0,0,0,0,0,0) = 0$ and $\frac{\partial \varphi_{k,k-1}}{\partial \gamma_1}\big|_{\{\gamma_1=0, U_a=U_2^{(0)}, \omega_{k,k+1}=0\}} = \kappa_1(U_2^{(0)}) > 0$, it follows from the implicit function theorem that γ_1 can be solved as a C^2 function of $\beta_1, \omega_k, \alpha_2, \alpha_3, \alpha_5, \omega_{k-1,k}$, and V_a. Next, by the Taylor expansion formula, we have

$$\begin{aligned}\gamma_1 &= \gamma_1(\beta_1, 0, 0, 0, 0) + \gamma_1(\beta_1, \omega_k, 0, 0, 0) - \gamma_1(\beta_1, 0, 0, 0, 0) + \gamma_1(\beta_1, \omega_k, \alpha_2, 0, 0) \\ &\quad - \gamma_1(\beta_1, \omega_k, 0, 0, 0) + \gamma_1(\beta_1, \omega_k, \alpha_2, \alpha_3, 0) - \gamma_1(\beta_1, \omega_k, \alpha_2, 0, 0) \\ &\quad + \gamma_1(\beta_1, \omega_k, \alpha_2, \alpha_3, \alpha_5) - \gamma_1(\beta_1, \omega_k, \alpha_2, \alpha_3, 0) \\ &= \beta_1 + K_{b0}\omega_k + K_{b2}\alpha_2 + K_{b3}\alpha_3 + K_{b5}\alpha_5.\end{aligned}$$

Differentiating $\varphi_{k,k-1}(\gamma_1, \beta_1, \omega_k, \alpha_2, \alpha_3, \alpha_5) = 0$ with respect to $\omega_k, \alpha_2, \alpha_3, \alpha_5$, and letting $\beta_1 = \omega_k = \alpha_2 = \alpha_3 = \alpha_5 = \omega_{k-1,k} = 0$, and letting $U_a = U_2^{(0)}$, we have

$$\frac{\partial \gamma_1}{\partial \omega_k} = \frac{u_2^{(0)}}{\kappa_1\big(U_2^{(0)}\big)}, \quad \frac{\partial \gamma_1}{\partial \alpha_i} = \frac{r_i^{(2)}\big(U_2^{(0)}\big)}{\kappa_1\big(U_2^{(0)}\big)}, \quad i = 2, 3, 5,$$

where $r_5^{(2)}(U_2^{(0)}) = \kappa_5(U_2^{(0)}), r_2^{(2)}(U_2^{(0)}) = r_3^{(2)}(U_2^{(0)}) = 0$. Then by Lemma 2.1, we have (2.25). □

Finally, let us consider the wave interaction estimates involving the strong contact discontinuity for (2.3). First we have the following lemma.

Lemma 2.6 *For the constant states* $V_1^{(0)} = (u_1^{(0)}, 0, p_1^{(0)}, \rho_1^{(0)})^\top$ *and* $V_2^{(0)} = (u_2^{(0)}, 0, p_2^{(0)}, \rho_2^{(0)})^\top$, *it holds that*

(i)
$$\begin{aligned}&\det\big(\tilde r_5(V_2^{(0)}), \partial_{\sigma_3}\mathfrak{F}(\sigma_{30},\sigma_{20};V_1^{(0)}), \partial_{\sigma_2}\mathfrak{F}(\sigma_{30},\sigma_{20};V_1^{(0)}), \nabla_V\mathfrak{F}(\sigma_{30},\sigma_{20};V_1^{(0)})\cdot\tilde r_1(V_1^{(0)})\big)\\ &= \kappa_1(V_1^{(0)})\kappa_5(V_2^{(0)})(\rho_1^{(0)})^2(u_1^{(0)})^2 e^{\sigma_{20}+\sigma_{30}}\big(\lambda_5(V_2^{(0)})e^{2\sigma_{20}+\sigma_{30}} + \lambda_5(V_1^{(0)})\big) > 0. \quad (2.26)\end{aligned}$$

where $\tilde r_i\,(i=1,5)$ *is the vector which only contains the first four components of* r_i.

(ii) *For any* $V_a \in O_\epsilon(V_1^{(0)})$ *and* $\sigma_j \in O_{\hat\epsilon}(\sigma_{j0})$ *which satisfies that* $\mathfrak{F}(\sigma_3,\sigma_2;V_a) \in O_\epsilon(V_2^{(0)})$ *with some* $\hat\epsilon = \hat\epsilon(\epsilon) \to 0$ *as* $\epsilon \to 0$, *it holds that*

$$|\mathfrak{F}(\sigma_3,\sigma_2;V_a) - \mathfrak{F}(\sigma_{30},\sigma_{20};V_a)| \leqslant C(|\sigma_3-\sigma_{30}| + |\sigma_2-\sigma_{20}|), \quad (2.27)$$

for some constant C.

Proof. Since $\mathfrak{F}(\sigma_3,\sigma_2;V_a) = (u_a e^{\sigma_2}, v_a e^{\sigma_2}, p_a, \rho_a e^{\sigma_3})^\top$ for any $V_a \in O_\epsilon(V_1{}^{(0)})$, $u_2^{(0)} = u_1^{(0)}e^{\sigma_{20}}$, and $\rho_2^{(0)} = \rho_1^{(0)}e^{\sigma_{30}}$, direct calculations gives that,

$$\begin{aligned}&\det\Big(\tilde r_5\big(V_2^{(0)}\big), \partial_{\sigma_3}\mathfrak{F}(\sigma_{30},\sigma_{20};V_1^{(0)}), \partial_{\sigma_2}\mathfrak{F}\big(\sigma_{30},\sigma_{20};V_1^{(0)}\big), \nabla_V\mathfrak{F}\big(\sigma_{30},\sigma_{20};V_1^{(0)}\big)\cdot\tilde r_1\big(V_1^{(0)}\big)\Big)\\
&= \kappa_1\big(V_1^{(0)}\big)\kappa_5\big(V_2^{(0)}\big)\begin{vmatrix} -\lambda_5\big(V_2^{(0)}\big) & 0 & u_1^{(0)}e^{\sigma_{20}} & -\lambda_1\big(V_1^{(0)}\big)e^{\sigma_{20}}\\ 1 & 0 & 0 & e^{\sigma_{20}}\\ \lambda_5\big(V_2^{(0)}\big)\rho_2^{(0)}u_2^{(0)} & 0 & 0 & \lambda_1\big(V_1^{(0)}\big)\rho_1^{(0)}u_1^{(0)}\\ \lambda_5\big(V_2^{(0)}\big)\rho_2^{(0)}u_2^{(0)}/\big(c_2^{(0)}\big)^2 & \rho_1^{(0)}e^{\sigma_{30}} & 0 & \lambda_1\big(V_2^{(0)}\big)\rho_1^{(0)}u_1^{(0)}e^{\sigma_{30}}/\big(c_1^{(0)}\big)^2\end{vmatrix}\\
&= \kappa_1\big(V_1^{(0)}\big)\kappa_5\big(V_2^{(0)}\big)(\rho_1^{(0)})^2(u_1^{(0)})^2e^{\sigma_{20}+\sigma_{30}}\big(\lambda_5\big(V_2^{(0)}\big)e^{2\sigma_{20}+\sigma_{30}} + \lambda_5\big(V_1^{(0)}\big)\big) > 0.\end{aligned}$$

Moreover, note that

$$\mathfrak{F}(\sigma_3,\sigma_2;V_a) - \mathfrak{F}(\sigma_{30},\sigma_{20};V_a) = \big(u_a(e^{\sigma_2}-e^{\sigma_{20}}), v_a(e^{\sigma_2}-e^{\sigma_{20}}), 0, \rho_a(e^{\sigma_3}-e^{\sigma_{30}})\big)^\top,$$

then by the Taylor expansion formula, we can obtain (2.27). □

We remark that (2.26) is essential to estimate the strengths of reflected weak waves in the wave interaction of the strong contact discontinuity and weak waves governed by (2.3). Now, we can establish the solvability of the Riemann problem involving the strong contact discontinuity.

Lemma 2.7 *There exists* $\epsilon > 0$ *such that, for any given constant states* $U_a \in O_\epsilon(U_1^{(0)})$ *and* $U_b \in O_\epsilon(U_2^{(0)})$, *the Riemann problem* (2.15) *admits a unique admissible solution that consists of a weak 1-wave, a strong contact discontinuity, and a weak 5-wave. In addition,* U_b *can be represented by*

$$\begin{cases} V_b = \tilde\Phi_5\Big(\alpha_5;\mathfrak{F}\big(\sigma_3,\sigma_2;\tilde\Phi_1(\alpha_1;V_a)\big)\Big),\\ Z_b = Z_a + \alpha_4,\end{cases} \quad (2.28)$$

with $V_b = (u_b, v_b, p_b, \rho_b)^\top$.

Proof. It is clear from (2.14) that $Z_b = Z_a + \alpha_4$.

Next, let us consider the function:

$$\varphi_c(\alpha_5, \sigma_3, \sigma_2, \alpha_1, V_a, V_b) = \tilde{\Phi}_5\Big(\alpha_5; \mathcal{T}\big(\sigma_3, \sigma_2; \tilde{\Phi}_1(\alpha_1; V_a)\big)\Big) - V_b.$$

Obviously, we have $\varphi_c(0, \sigma_{30}, \sigma_{20}, 0, V_1^{(0)}, V_2^{(0)}) = 0$, and

$$\begin{aligned}
&\det\Big(\frac{\partial\varphi_c(\alpha_5, \sigma_3, \sigma_2, \alpha_1, V_a, V_b)}{\partial(\alpha_5, \sigma_3, \sigma_2, \alpha_1)}\Big)\Big|_{\{\alpha_5=\alpha_1=0, \sigma_3=\sigma_{30}, \sigma_2=\sigma_{20}, V_a=V_1^{(0)}, V_b=V_2^{(0)}\}}\\
&= \det\Big(\tilde{r}_5\big(V_2^{(0)}\big), \partial_{\sigma_3}\mathcal{T}\big(\sigma_{30}, \sigma_{20}; V_1^{(0)}\big), \partial_{\sigma_2}\mathcal{T}\big(\sigma_{30}, \sigma_{20}; V_1^{(0)}\big), \tilde{r}_1\big(V_1^{(0)}\big)\Big)\\
&= \kappa_1(V_1^{(0)})\kappa_5\big(V_2^{(0)}\big)\begin{vmatrix} -\lambda_5\big(V_2^{(0)}\big) & 0 & u_1^{(0)}e^{\sigma_{20}} & -\lambda_1\big(V_1^{(0)}\big)\\ 1 & 0 & 0 & 1\\ \lambda_5\big(V_2^{(0)}\big)\rho_2^{(0)}u_2^{(0)} & 0 & 0 & \lambda_1\big(V_1^{(0)}\big)\rho_1^{(0)}u_1^{(0)}\\ \lambda_5\big(V_2^{(0)}\big)\rho_2^{(0)}u_2^{(0)}/\big(c_2^{(0)}\big)^2 & \rho_1^{(0)}e^{\sigma_{30}} & 0 & \lambda_1\big(V_2^{(0)}\big)\rho_1^{(0)}u_1^{(0)}/\big(c_1^{(0)}\big)^2 \end{vmatrix}\\
&= \kappa_1\big(V_1^{(0)}\big)\kappa_5\big(V_2^{(0)}\big)\big(\rho_1^{(0)}\big)^2\big(u_1^{(0)}\big)^2 e^{\sigma_{20}+\sigma_{30}}\Big(\lambda_5\big(V_2^{(0)}\big)e^{\sigma_{20}+\sigma_{30}} + \lambda_5\big(V_1^{(0)}\big)\Big) > 0.
\end{aligned}$$

Then it follows from the implicit function theorem that there exists $\epsilon > 0$, such that for any given constant states $U_a \in O_\epsilon(U_1^{(0)})$ and $U_b \in O_\epsilon(U_2^{(0)})$, the equation

$$\varphi_c(\alpha_5, \sigma_3, \sigma_2, \alpha_1, V_a, V_b) = 0$$

admits a unique solution $\alpha_5, \sigma_3, \sigma_2, \alpha_1$. □

Now we shall derive the wave interaction estimates between the strong contact discontinuity and weak waves. There are two cases depending on how the strong contact discontinuity and weak waves interact. The first case is that, as shown in Fig. 4, the weak waves approach the strong contact discontinuity from the above. For this case, we have the following lemma.

PROPOSITION 2.8 For any given three constant states $U_a \in O_\epsilon(U_1^{(0)})$, and $U_m, U_b \in O_\epsilon(U_2^{(0)})$, (see Fig. 4), with the assumptions that

$$\begin{aligned}
&V_m = \tilde{\Phi}_5\Big(\alpha_5; \mathcal{T}\big(\sigma_3, \sigma_2; \tilde{\Phi}_1(\alpha_1; V_a)\big)\Big), \quad Z_m = Z_a + \alpha_4,\\
&V_b = \tilde{\Phi}(\beta_5, \beta_3, \beta_2, \beta_1; V_m), \quad Z_b = Z_m + \beta_4,\\
&V_b = \tilde{\Phi}_5\Big(\gamma_5; \mathcal{T}\big(\sigma_3', \sigma_2'; \tilde{\Phi}_1(\gamma_1; V_a)\big)\Big), \quad Z_b = Z_a + \gamma_4,
\end{aligned}$$

it holds that

$$\begin{cases} \gamma_1 = K_{21}\beta_1 + \alpha_1 + O(1)\Delta'(\alpha_5, \boldsymbol{\beta}^*),\\ \sigma_i' = K_{2i}\beta_1 + \beta_i + \sigma_i + O(1)\Delta'(\alpha_5, \boldsymbol{\beta}^*), i = 2, 3,\\ \gamma_4 = \alpha_4 + \beta_4,\\ \gamma_5 = K_{25}\beta_1 + \alpha_5 + \beta_5 + O(1)\Delta'(\alpha_5, \boldsymbol{\beta}^*), \end{cases} \tag{2.29}$$

where $\Delta'(\alpha_5, \boldsymbol{\beta}^*) = |\alpha_5|(|\beta_1| + |\beta_2| + |\beta_3|) + \Delta_5(\alpha_5, \beta_5)$. Furthermore, $\sum_{i=1}^{3}|K_{2i}|$ is bounded, and when $\beta = \alpha_1 = \alpha_4 = \alpha_5 = 0, \sigma_2 = \sigma_{20}, \sigma_3 = \sigma_{30}$, it holds that $|K_{25}| < 1$.

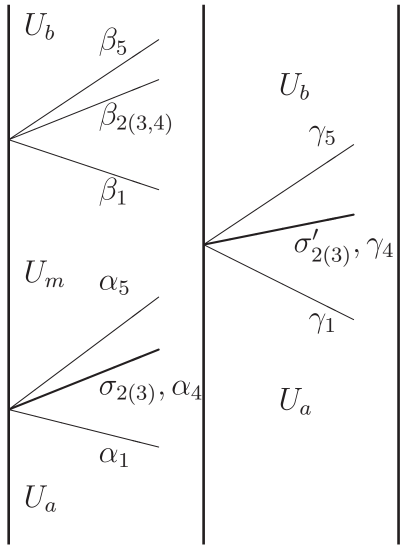

FIG. 4. Weak waves approach the strong contact discontinuity from above

REMARK 2.9 The essential feature of homogeneous system (2.3) is that the reflection coefficient K_{25} is less than one, which is the stability condition in [17, 39].

Proof. First, it is obvious that $\gamma_4 = \alpha_4 + \beta_4$.

Then, for any state $V_{\widetilde{m}} \in O_\epsilon(V_2^{(0)})$, we define

$$\tilde{\Phi}\big(\delta_5, \delta_3, \delta_2, \delta_1; V_{\widetilde{m}}\big) = \tilde{\Phi}\big(\beta_5, \beta_3, \beta_2, \beta_1; \tilde{\Phi}_5(\alpha_5; V_{\widetilde{m}})\big). \tag{2.30}$$

By applying Proposition 2.3, we have

$$\begin{aligned} \delta_i &= \beta_i + O(1)\Delta'(\alpha_5, \boldsymbol{\beta}^*), \quad i = 1, 2, 3, \\ \delta_5 &= \alpha_5 + \beta_5 + O(1)\Delta'(\alpha_5, \boldsymbol{\beta}^*), \end{aligned} \tag{2.31}$$

where $\Delta'(\alpha_5, \boldsymbol{\beta}^*) = |\alpha_5|(|\beta_1| + |\beta_2| + |\beta_3|) + \Delta_5(\alpha_5, \beta_5)$.

Let $\boldsymbol{\delta}^* = (\delta_5, \delta_3, \delta_2, \delta_1)$. By (2.30), let us consider the following function:

$$\begin{aligned} &\varphi_d(\gamma_5, \sigma'_3, \sigma'_2, \gamma_1, \boldsymbol{\delta}^*, \sigma_3, \sigma_2, \alpha_1) \\ &\quad = \tilde{\Phi}_5\Big(\gamma_5; \mathfrak{T}\big(\sigma'_3, \sigma'_2; \tilde{\Phi}_1(\gamma_1; V_a)\big)\Big) - \tilde{\Phi}\bigg(\beta_5, \beta_3, \beta_2, \beta_1; \tilde{\Phi}_5\Big(\alpha_5; \mathfrak{T}\big(\sigma_3, \sigma_2; \tilde{\Phi}_1(\alpha_1; V_a)\big)\Big)\bigg) \\ &\quad = \tilde{\Phi}_5\Big(\gamma_5; \mathfrak{T}\big(\sigma'_3, \sigma'_2; \tilde{\Phi}_1(\gamma_1; V_a)\big)\Big) - \tilde{\Phi}\Big(\delta_5, \delta_3, \delta_2, \delta_1; \mathfrak{T}\big(\sigma_3, \sigma_2; \tilde{\Phi}_1(\alpha_1; V_a)\big)\Big). \end{aligned}$$

It is clear that $\varphi_d(0, \sigma_{30}, \sigma_{20}, 0, \mathbf{0}, \sigma_{30}, \sigma_{20}, 0) = 0$. By (2.26), we have

$$\begin{aligned}
&\det\left(\frac{\partial \varphi_d(\gamma_5, \sigma_3', \sigma_2', \gamma_1, \boldsymbol{\delta}^*, \sigma_3, \sigma_2, \alpha_1)}{\partial(\gamma_5, \sigma_3', \sigma_2', \gamma_1)}\right)\Big|_{\{\gamma_1=\gamma_5=0, \sigma_3'=\sigma_{30}, \sigma_2'=\sigma_{20}, V_a=V_1^{(0)}\}} \\
&= \det\Big(\tilde{r}_5(V_2^{(0)}), \partial_{\sigma_3}\mathfrak{F}(\sigma_{30}, \sigma_{20}; V_1^{(0)}), \partial_{\sigma_2}\mathfrak{F}(\sigma_{30}, \sigma_{20}; V_1^{(0)}), \\
&\qquad\qquad \nabla_V \mathfrak{F}(\sigma_{30}, \sigma_{20}; V_1^{(0)}) \cdot \tilde{r}_1(V_1^{(0)})\Big) > 0.
\end{aligned}$$

Then it follows from the implicit function theorem that $\gamma_i, i = 1, 5$, and $\sigma_j', j = 2, 3$, can be solved as a C^2 function of $\gamma_5, \sigma_3', \sigma_2', \gamma_1, \boldsymbol{\delta}^*, \sigma_3, \sigma_2, \alpha_1$, and V_a. Thus, we have

$$\begin{aligned}
\sigma_j' &= \sigma_j'(\delta_5, \delta_3, \delta_2, \delta_1, \sigma_3, \sigma_2, \alpha_1) - \sigma_j'(\delta_5, \delta_3, \delta_2, 0, \sigma_3, \sigma_2, \alpha_1) + \sigma_j'(\delta_5, \delta_3, \delta_2, 0, \sigma_3, \sigma_2, \alpha_1) \\
&= K_{2j}\delta_1 + \delta_j + \sigma_j, \quad j = 2, 3,
\end{aligned}$$

Similarly, it holds that

$$\gamma_1 = K_{21}\delta_1 + \alpha_1, \qquad \text{and} \qquad \gamma_5 = K_{25}\delta_1 + \delta_5.$$

Then by (2.31), we can obtain (2.29).

Differentiating the equation $\varphi_d = 0$ with respect to δ_1, and letting $\boldsymbol{\delta}^* = \alpha_1 = 0, \sigma_3 = \sigma_{30}$, $\sigma_2 = \sigma_{20}$, and $U_a = U_1^{(0)}$, we have

$$\begin{aligned}
&\partial_{\delta_1}\gamma_5 \tilde{r}_5(V_2^{(0)}) + \partial_{\delta_1}\sigma_3'\partial_{\sigma_3}\mathfrak{F}(\sigma_{30}, \sigma_{20}; V_1^{(0)}) + \partial_{\delta_1}\sigma_2'\partial_{\sigma_2}\mathfrak{F}(\sigma_{30}, \sigma_{20}; V_1^{(0)}) \\
&\qquad + \partial_{\delta_1}\gamma_1 \nabla_V \mathfrak{F}(\sigma_{30}, \sigma_{20}; V_1^{(0)}) \cdot \tilde{r}_1(V_1^{(0)}) = \tilde{r}_1(V_2^{(0)}).
\end{aligned}$$

It is clear that $K_{2i}, i = 1, 2, 3$ are bounded. By (2.26) and Lemma 2.1, it holds that

$$\begin{aligned}
&|\partial_{\delta_1}\gamma_5| \\
&= \left|\frac{\det\Big(\tilde{r}_1(V_2^{(0)}), \partial_{\sigma_3}\mathfrak{F}(\sigma_{30}, \sigma_{20}; V_1^{(0)}), \partial_{\sigma_2}\mathfrak{F}(\sigma_{30}, \sigma_{20}; V_1^{(0)}), \nabla_V \mathfrak{F}(\sigma_{30}, \sigma_{20}; V_1^{(0)}) \cdot \tilde{r}_1(V_1^{(0)})\Big)}{\det\Big(\tilde{r}_5(V_2^{(0)}), \partial_{\sigma_3}\mathfrak{F}(\sigma_{30}, \sigma_{20}; V_1^{(0)}), \partial_{\sigma_2}\mathfrak{F}(\sigma_{30}, \sigma_{20}; V_1^{(0)}), \nabla_V \mathfrak{F}(\sigma_{30}, \sigma_{20}; V_1^{(0)}) \cdot \tilde{r}_1(V_1^{(0)})\Big)}\right| \\
&= \left|\frac{\kappa_1(V_1^{(0)})\kappa_1(V_2^{(0)})(\rho_1^{(0)})^2(u_1^{(0)})^2 e^{\sigma_{20}+\sigma_{30}}(\lambda_5(V_1^{(0)}) - \lambda_5(V_2^{(0)})e^{2\sigma_{20}+\sigma_{30}})}{\kappa_1(V_1^{(0)})\kappa_5(V_2^{(0)})(\rho_1^{(0)})^2(u_1^{(0)})^2 e^{\sigma_{20}+\sigma_{30}}\Big(\lambda_5(V_1^{(0)}) + \lambda_5(V_2^{(0)})e^{2\sigma_{20}+\sigma_{30}}\Big)}\right| \\
&= \left|\frac{\lambda_5(V_1^{(0)}) - \lambda_5(V_2^{(0)})e^{2\sigma_{20}+\sigma_{30}}}{\lambda_5(V_1^{(0)}) + \lambda_5(V_2^{(0)})e^{2\sigma_{20}+\sigma_{30}}}\right| < 1.
\end{aligned}$$

This completes the proof. □

The second case is that the weak waves approach the strong contact discontinuity from the below (Fig. 5). By the symmetry, we can easily obtain the following proposition.

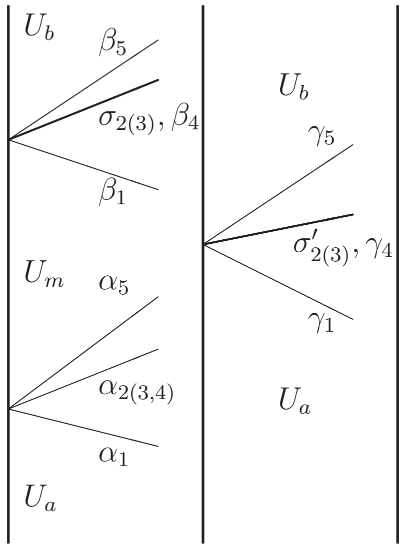

FIG. 5. Weak waves approach the strong contact discontinuity from below

PROPOSITION 2.10 For any given three constant states $U_a, U_m \in O_\epsilon(U_1^{(0)})$, and $U_b \in O_\epsilon(U_2^{(0)})$ with the assumptions that

$$V_m = \tilde{\Phi}(\alpha_5, \alpha_3, \alpha_2, \alpha_1; V_a), \quad Z_m = Z_a + \alpha_4,$$
$$V_b = \tilde{\Phi}_5\Big(\beta_5; \mathfrak{T}\big(\sigma_3, \sigma_2; \tilde{\Phi}_1(\beta_1; V_m)\big)\Big), \quad Z_b = Z_m + \beta_4,$$
$$V_b = \tilde{\Phi}_5\Big(\gamma_5; \mathfrak{T}\big(\sigma'_3, \sigma'_2; \tilde{\Phi}_1(\gamma_1; V_a)\big)\Big), \quad Z_b = Z_a + \gamma_4,$$

it holds that

$$\begin{cases} \gamma_1 = K_{11}\alpha_5 + \alpha_1 + \beta_1 + O(1)\Delta''(\boldsymbol{\alpha}^*, \beta_1), \\ \sigma'_i = K_{1i}\alpha_5 + \alpha_i + \sigma_i + O(1)\Delta''(\boldsymbol{\alpha}^*, \beta_1), i = 2, 3, \\ \gamma_4 = \alpha_4 + \beta_4, \\ \gamma_5 = K_{15}\alpha_5 + \beta_5 + O(1)\Delta''(\boldsymbol{\alpha}^*, \beta_1), \end{cases} \tag{2.32}$$

where $\Delta''(\boldsymbol{\alpha}^*, \beta_1) = |\beta_1|(|\alpha_5| + |\alpha_3| + |\alpha_2|) + \Delta_1(\alpha_1, \beta_1)$.

2.3 *Local estimates on the reacting step*

Let $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{p}, \tilde{\rho}, \tilde{Z})^\top$ be the value of U after the reaction. It means that $\tilde{U}$ satisfies the equation

$$W(\tilde{U}) = W(U) + G(U)h,$$

which is precisely of the following form

$$\begin{cases} \tilde{\rho}\tilde{u} = \rho u, \\ \tilde{\rho}\tilde{u}^2 + \tilde{p} = \rho u^2 + p, \\ \tilde{\rho}\tilde{u}\tilde{v} = \rho u v, \\ (\tilde{\rho}\tilde{E} + \tilde{p})\tilde{u} = (\rho E + p)u + q_0\rho\phi(T)Zh, \\ \tilde{\rho}\tilde{u}\tilde{Z} = \rho u Z - \rho\phi(T)Zh. \end{cases} \tag{2.33}$$

Then we have the following property that indicates the change of the solutions $\tilde{U}$ with respect to h.

Lemma 2.11 *Suppose that* $0 \leqslant Z \leqslant 1$ *and* $T \geqslant T_0$ *for some positive constant* T_0*, then there exists a constant* $l > 0$*, such that for sufficiently small* $h > 0$*, it holds that*

$$\tilde{T} \geqslant T \geqslant T_0 > 0, \quad \tilde{V} - V = O(1)Zh, \quad 0 \leqslant \tilde{Z} \leqslant e^{-lh}Z \leqslant 1,$$

where $\tilde{V} = (\tilde{u}, \tilde{v}, \tilde{p}, \tilde{\rho})^\top$*, and* $V = (u, v, p, \rho)^\top$.

Proof. By $(2.33)_1$ and $(2.33)_2$, we have that

$$\tilde{u} - u = -\frac{1}{\rho u}(\tilde{p} - p). \tag{2.34}$$

By $(2.33)_1$ and $(2.33)_3$, we have that

$$\tilde{v} = v. \tag{2.35}$$

By $(2.33)_1$ and $(2.33)_5$, we know that

$$\tilde{Z} = \Big(1 - \frac{\phi(T)}{u}h\Big)Z. \tag{2.36}$$

Moreover, $(2.33)_1$ also means that

$$\tilde{\rho} - \rho = -\frac{\rho}{u}(\tilde{u} - u) + O(h^2). \tag{2.37}$$

Note that by the thermodynamical relation, we know that $T = \frac{\gamma-1}{R}e = \frac{p}{R\rho}$. Then by the assumption $u^2 > c^2 = \gamma R T$ and from all the above identities and $(2.33)_4$, we have that

$$\tilde{T} - T = \frac{(\gamma - 1)(u^2 - RT)}{R\rho u(u^2 - \gamma R T)} q_0\rho\phi(T)Zh + O(h^2) \geqslant 0,$$

which shows that the temperature T does not decrease due to the reaction.

Next, (2.36) also means that $0 \leqslant \tilde{Z} \leqslant 1$. Since $\phi(T)$ is assumed to be Lipschitz continuous, nonnegative, and increasing, there exists a constant $l > 0$, such that $\tilde{Z} \leqslant e^{-lh}Z$, which implies the decay property of the reactant Z.

Finally, we will consider the change of $V = (u, v, p, \rho)^\top$. It follows from the implicit function theorem that $\tilde{V} = (\tilde{u}, \tilde{v}, \tilde{p}, \tilde{\rho})^\top$ can be solved as a C^2 function of V and Zh by the first four equations of (2.33). By the Taylor expansion, one can easily see that there exists a function $\widetilde{\mho}$ such that

$$\tilde{V} = V + \widetilde{\mho}(V, Zh)Zh. \tag{2.38}$$

This completes the proof. □

In sequel, if $\tilde{V}$ and V satisfy (2.38), and if $\tilde{Z}$ and Z satisfy (2.36), then we say $\tilde{U} = (\tilde{V}, \tilde{Z})$ is the value of $U = (V, Z)$ after the reaction step.

Now, we are going to consider the change of the wave strength after the reaction step. The analysis is divided into the following three cases.

Case 1. U_a and U_b are connected only by the weak waves.

PROPOSITION 2.12 Let $U_a, U_b \in O_\epsilon(U_k^{(0)}), k = 1, 2$ with

$$V_b = \tilde{\Phi}(\gamma_5, \gamma_3, \gamma_2, \gamma_1; V_a), \quad Z_b = Z_a + \gamma_4.$$

Let $\tilde{U}_a = (\tilde{V}_a, \tilde{Z}_a)$ and $\tilde{U}_b = (\tilde{V}_b, \tilde{Z}_b)$ be the value of U_a and U_b after the reaction step respectively. Assume that

$$\tilde{V}_b = \tilde{\Phi}(\tilde{\gamma}_5, \tilde{\gamma}_3, \tilde{\gamma}_2, \tilde{\gamma}_1; \tilde{V}_a), \quad \tilde{Z}_b = \tilde{Z}_a + \tilde{\gamma}_4.$$

Then it holds that

$$\begin{cases} \tilde{\gamma}_i = \gamma_i + O(1)|\boldsymbol{\gamma}^*|Z_a h + O(1)|\gamma_4|h, \quad i = 1, 2, 3, 5, \\ \tilde{\gamma}_4 = (1 - \phi(T_b)h/u_b)\gamma_4 + O(1)|\boldsymbol{\gamma}^*|Z_a h. \end{cases} \tag{2.39}$$

where $|\boldsymbol{\gamma}^*| = |\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_5|$.

Proof. By (2.36), it is obvious that $\tilde{Z}_b = (1 - \phi(T_b)h/u_b)Z_b$, and $\tilde{Z}_a = (1 - \phi(T_a)h/u_a)Z_a$. Hence we have $\tilde{\gamma}_4 = (1 - \phi(T_b)h/u_b)\gamma_4 + (\phi(T_a)/u_a - \phi(T_b)/u_b)Z_a h$, which implies $(2.39)_2$.

Next, by (2.38), we need to find the solution $\tilde{\boldsymbol{\gamma}}^*$ as a function of $\boldsymbol{\gamma}^*$, $Z_a h$, $Z_b h$, and V_a such that

$$\tilde{\Phi}\big(\tilde{\boldsymbol{\gamma}}^*; V_a + \widetilde{\mho}(V_a, Z_a h)Z_a h\big) = \tilde{\Phi}(\boldsymbol{\gamma}^*; V_a) + \widetilde{\mho}(V_b, Z_b h)Z_b h,$$

where $\tilde{\boldsymbol{\gamma}}^* = (\tilde{\gamma}_5, \tilde{\gamma}_3, \tilde{\gamma}_2, \tilde{\gamma}_1)$, and $\boldsymbol{\gamma}^* = (\gamma_5, \gamma_3, \gamma_2, \gamma_1)$.

First, it follows from the implicit function theorem that $\tilde{\gamma}_i, i = 1, 2, 3, 5$ can be solved as a C^2-function of $(\boldsymbol{\gamma}^*, Z_a h, Z_b h, V_a)$ uniquely. Then, we have

$$\begin{aligned} \tilde{\gamma}_i(\boldsymbol{\gamma}^*, Z_a h, Z_b h, V_a) &= O(1)|Z_a - Z_b|h + \tilde{\gamma}_i(\boldsymbol{\gamma}^*, Z_a h, Z_a h, V_a) \\ &= O(1)|Z_a - Z_b|h + O(1)|\boldsymbol{\gamma}^*|Z_a h + \tilde{\gamma}_i(\boldsymbol{\gamma}^*, 0, 0, V_a) \\ &\quad + \tilde{\gamma}_i(0, Z_a h, Z_a h, V_a) - \tilde{\gamma}_i(0, 0, 0, V_a) \\ &= \gamma_i + O(1)|\boldsymbol{\gamma}^*|Z_a h + O(1)|\gamma_4|h. \end{aligned}$$

It completes the proof of this proposition. □

Case 2. U_a and U_k are connected by a weak 1-wave near the boundary Γ_k.

PROPOSITION 2.13 Let $U_a, U_k \in O_\epsilon(U_2^{(0)})$ with

$$V_k = \Phi_1(\gamma_1; V_a), \quad (u_k, v_k) \cdot \mathbf{n}_k = 0.$$

Let $\tilde{U}_a = (\tilde{V}_a, \tilde{Z}_a)$ and $\tilde{U}_k = (\tilde{V}_k, \tilde{Z}_k)$ be the value of U_a, U_k after the reaction step respectively. Assume that

$$\tilde{V}_k = \Phi_1(\tilde{\gamma}_1; \tilde{V}_a), \quad (\tilde{u}_k, \tilde{v}_k) \cdot \mathbf{n}_k = 0.$$

Then, it holds that

$$\tilde{\gamma}_1 = \gamma_1 + O(1)Z_a h. \tag{2.40}$$

Proof. By (2.38), we need to find the solution $\tilde{\gamma}_1$ as a function of $\gamma_1, Z_a h$, and V_a such that

$$\Big(\Phi_1^{(1)}(\tilde{\gamma}_1; V_a + \widetilde{\mho}(V_a, Z_a h)Z_a h), \Phi_1^{(2)}(\tilde{\gamma}_1; V_a + \widetilde{\mho}(V_a, Z_a h)Z_a h)\Big) \cdot \mathbf{n}_k = \big(\Phi_1^{(1)}(\gamma_1; V_a), \Phi_1^{(2)}(\gamma_1; V_a)\big) \cdot \mathbf{n}_k.$$

Obviously, it follows from the implicit function theorem that $\tilde{\gamma}_1$ can be solved as a C^2-function of $(\gamma_1, Z_a h, V_a)$ uniquely. Moreover, by the Taylor expansion formula, we have that

$$\tilde{\gamma}_1(\gamma_1, Z_a h, V_a) = \tilde{\gamma}_1(\gamma_1, 0, V_a) + \tilde{\gamma}_1(\gamma_1, Z_a h, V_a) - \tilde{\gamma}_1(\gamma_1, 0, V_a) = \gamma_1 + O(1)Z_a h. \qquad \square$$

Case 3. U_a and U_b are connected by a weak 1-wave, a strong contact discontinuity, and a weak 5-wave.

PROPOSITION 2.14 Let $U_a \in O_\epsilon(U_1^{(0)}), U_b \in O_\epsilon(U_2^{(0)})$ with

$$V_b = \tilde{\Phi}_5\Big(\gamma_5, \mathfrak{F}\big(\sigma_3, \sigma_2; \tilde{\Phi}_1(\gamma_1; V_a)\big)\Big), \quad Z_b = Z_a + \gamma_4.$$

Let $\tilde{U}_a$ and $\tilde{U}_b$ be the value of U_a and U_b after the reaction step respectively. Assume that

$$\tilde{V}_b = \tilde{\Phi}_5\Big(\tilde{\gamma}_5, \mathfrak{F}\big(\tilde{\sigma}_3, \tilde{\sigma}_2; \tilde{\Phi}_1(\tilde{\gamma}_1; \tilde{V}_a)\big)\Big), \quad \tilde{Z}_b = \tilde{Z}_a + \tilde{\gamma}_4.$$

Then, it holds that

$$\begin{cases} \tilde{\gamma}_i = \gamma_i + O(1)Z_a h + O(1)|\gamma_4|h, \ i = 1, 5, \\ \tilde{\sigma}_j = \sigma_j + O(1)Z_a h + O(1)|\gamma_4|h, \ j = 2, 3, \\ \tilde{\gamma}_4 = (1 - \phi(T_b)h/u_b)\gamma_4 + O(1)Z_a h. \end{cases} \tag{2.41}$$

Proof. By (2.36), it is clear that $\tilde{Z}_b = (1-\phi(T_b)h/u_b)Z_b$, and $\tilde{Z}_a = (1-\phi(T_a)h/u_a)Z_a$. Hence we have $\tilde{\gamma}_4 = (1 - \phi(T_b)h/u_b)\gamma_4 + (\phi(T_a)/u_a - \phi(T_b)/u_b)Z_a h$, which implies $(2.41)_3$.

Next, by (2.38), we need to find the solution $\tilde{\gamma}_i, i = 1, 5$ and $\tilde{\sigma}_j, j = 2, 3$ as a function of $\gamma_5, \sigma_3, \sigma_2, \gamma_1, Z_a h, Z_b h$ and V_a such that

$$\tilde{\Phi}_5\Big(\tilde{\gamma}_5, \mathfrak{F}\big(\tilde{\sigma}_3, \tilde{\sigma}_2; \tilde{\Phi}_1(\tilde{\gamma}_1; V_a + \widetilde{\mho}(V_a, Z_a h)Z_a h)\big)\Big) = \tilde{\Phi}_5\Big(\gamma_5, \mathfrak{F}\big(\sigma_3, \sigma_2; \tilde{\Phi}_1(\gamma_1; V_a)\big)\Big) + \widetilde{\mho}(V_b, Z_b h)Z_b h. \tag{2.42}$$

It follows from the implicit function theorem that $\tilde{\gamma}_i, i = 1, 5$ and $\tilde{\sigma}_j, j = 2, 3$ can be solved as a C^2-function of $(\gamma_5, \sigma_3, \sigma_2, \gamma_1, Z_a h, Z_b h, V_a)$ uniquely. Moreover, we can obtain

$$\begin{aligned} \tilde{\gamma}_i(\gamma_5, \sigma_3, \sigma_2, \gamma_1, Z_a h, Z_b h, V_a) &= O(1)|Z_a - Z_b|h + \tilde{\gamma}_i(\gamma_5, \sigma_3, \sigma_2, \gamma_1, Z_a h, Z_a h, V_a) \\ &= O(1)|Z_a - Z_b|h + O(1)Z_a h + \tilde{\gamma}_i(\gamma_5, \sigma_3, \sigma_2, \gamma_1, 0, 0, V_a) \\ &= \gamma_i + O(1)Z_a h + O(1)|\gamma_4|h. \end{aligned}$$

The proof of $(2.41)_2$ can be derived in the same way. It completes the proof. $\square$

3. Global entropy solutions of the steady exothermically reacting Euler equations

Thanks to the local estimates obtained in Section 2, in this section, we will introduce the fractional-step Glimm scheme and a Glimm-type functional to construct the approximate solutions for the initial boundary value problem (2.1) and (1.4)–(1.5), by deriving global estimates on the non-reacting step and the reacting step. With these in hand, the global existence of entropy solutions with a strong contact discontinuity is obtained.

3.1 *The Glimm fractional-step scheme*

As shown in Fig. 6, we use the notations in (2.18)–(2.19), and let $h > 0$ and $s > 0$ be the step-length in the x and y directions respectively.

The construction of the fractional-step scheme for the inhomogeneous system (2.1) is as follows.

By (1.6), the boundary $y = g(x)$ is a perturbation of the straight wall. It means that for sufficiently small $\delta_0 > 0$, we have

$$\sup_{x\geqslant 0} |g'(x)| < \delta_0.$$

Therefore,

$$m := \sup_{k\geqslant 0} \left\{ \frac{|y_{k+1} - y_k|}{h} \right\} < \delta_0. \tag{3.1}$$

Let $y^{(0)}$ be given by (1.4). Choose s such that $y^{(0)}/s = 2N$ is an even number, and the following Courant–Friedrichs–Lewy condition holds:

$$\frac{s}{h} > \max_{j=1,5} \left(\sup_{U\in O_\epsilon(U_1^{(0)})\cup O_\epsilon(U_2^{(0)})} |\lambda_j(U)| \right) + m.$$

For any positive integer k and negative integer n, i.e., $k \geqslant 1$ and $n \leqslant -1$, define

$$y_{k,n} = y_k + (2n + 1 + \theta_k)s,$$

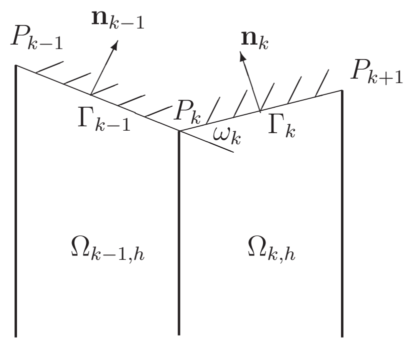

FIG. 6. The Glimm fractional-step scheme

where θ_k is randomly chosen in $(-1, 1)$. Define

$$P_{k,n} = (kh, y_{k,n}),$$

to be the mesh points.

Now we can define the approximate solutions $U_{h,\theta}$ in Ω_h, where $\theta = (\theta_1, \theta_2, \cdots)$, inductively as follows.

First, for initial data $U_0(y)$ and for $y \in (2ns, (2n+2)s)$, let

$$U_{h,0}(y) = \frac{1}{2s}\int_{2ns}^{(2n+2)s} U_0(y)dy.$$

Second, assume that $U_{h,\theta}$ has been constructed in $\{0 \leqslant x < kh\} \cap \Omega_h$, then for $y \in (y_k + 2ns, y_k + 2(n+1)s)$, define $U^0_{k,n}$ and $\widetilde{U}^0_{k,n}$ such that

$$\begin{cases} U^0_{k,n} := U_h(kh-, y_{k,n}), \\ W(\widetilde{U}^0_{k,n}) := W(U^0_{k,n}) + G(U^0_{k,n})h. \end{cases} \tag{3.2}$$

Now we are going to define $U_{h,\theta}$ in $\Omega_{k,h}$.

The first case is the Riemann problem with the boundary. Let $T_{k,0}$ be the diamond with the vertices that (kh, y_k), $(kh, y_k - s)$, $((k+1)h, y_{k+1} - s)$, and $((k+1)h, y_{k+1})$. Then $U_{h,\theta} = U_{k,0}$ in $T_{k,0}$ is the solution of the following Riemann problem:

$$\begin{cases} W(U_{k,0})_x + H(U_{k,0})_y = 0, & \text{in } T_{k,0}, \\ U_{k,0}|_{x=kh} = \widetilde{U}^0_{k,-1}, & y_k - s < y < y_k, \\ (u_{k,0}, v_{k,0}) \cdot \mathbf{n}_k = 0, & \text{on } \Gamma_k. \end{cases} \tag{3.3}$$

The second case is the Riemann problem without the boundary. For $n \leqslant -1$, let $T_{k,n}$ be the diamond with the vertices that $(kh, y_k + (2n+1)s)$, $(kh, y_k + (2n-1)s)$, $((k+1)h, y_{k+1} + (2n-1)s)$, and $((k+1)h, y_{k+1} + (2n+1)s)$. Then $U_{h,\theta} = U_{k,n}$ in $T_{k,n}$ is the solution of the following Riemann problem:

$$\begin{cases} W(U_{k,n})_x + H(U_{k,n})_y = 0 & \text{in } T_{k,n}, \\ U_{k,n}|_{x=kh} = \begin{cases} \widetilde{U}^0_{k,n}, & y_k + 2ns < y < y_k + (2n+1)s, \\ \widetilde{U}^0_{k,n-1}, & y_k + (2n-1)s < y < y_k + 2ns. \end{cases} \end{cases} \tag{3.4}$$

Therefore, we constructed an approximate solution $U_{h,\theta}$ globally in Ω_h provided that we can obtain a uniform bound of $U_{h,\theta}$, which is the main objective in the remaining part of this section.

3.2 *Glimm-type functional*

In order to obtain a uniform bound of $U_{h,\theta}$, let us introduce the Glimm-type functional in this section. Assume that $U_{h,\theta}$ has been defined in $\{0 \leqslant x < kh\} \cap \Omega_h$ and the following conditions are satisfied:

$A_1(k-1)$: In each $\Omega_{h,i}$, $0 \leqslant i \leqslant k-1$, there is a strong contact discontinuity $y = \chi^{(i)}$ with strength $(\sigma_2^{(i)}, \sigma_3^{(i)}, \gamma_4^{(i)})$ so that $\sigma_j^{(i)} \in O_{\hat{\epsilon}}(\sigma_{j0})$, $j = 2, 3$, $\gamma_4^{(i)} \in O_{\hat{\epsilon}}(0)$. $y = \chi^{(i)}$ divides $\Omega_{h,i}$ into two subregions: $\Omega_{h,i}^{(1)}$ and $\Omega_{h,i}^{(2)}$, where $\Omega_{h,i}^{(2)}$ is the region bounded by $y = \chi^{(i)}$ and Γ_i.

$A_2(k-1)$: $U_{h,\theta}|_{\Omega_{h,i}^{(1)}} \in O_\epsilon(U_1^{(0)})$, and $U_{h,\theta}|_{\Omega_{h,i}^{(2)}} \in O_\epsilon(U_2^{(0)})$, $0 \leqslant i \leqslant k-1$.

$A_3(k-1)$: $\{\chi^{(i)}\}_{i=0}^{k-1}$ together forms $y = \chi_{h,\theta}(x)$, which is the strong contact discontinuity in $\{0 \leqslant x < kh\} \cap \Omega_h$ and emanating from the point $(0, y^{(0)})$.

Then we shall prove that $U_{h,\theta}$ defined in $\Omega_{h,k}$ by Section 3.1 satisfies $A_1(k)$, $A_2(k)$ and $A_3(k)$. From the construction in Section 3.1, there exists a strong contact discontinuity $y = \chi^{(k)}$ in a diamond $T_{k,n}$. We extend $\chi_{h,\theta}$ to $\Omega_{h,k}$ such that $\chi_{h,\theta}=\chi^{(k)}$ in $\Omega_{h,k}$ and define $\Omega_{h,k}^{(1)}$ and $\Omega_{h,k}^{(2)}$ in the same way as in $A_1(k-1)$. So it is sufficient to show that $A_2(k)$ holds such that

$$U_{h,\theta}|_{\Omega_{h,k}^{(i)}} \in O_\epsilon\big(U_i^{(0)}\big), \quad i = 1, 2, \qquad \sigma_j^{(k)} \in O_{\hat{\epsilon}}(\sigma_{j0}), \quad j = 2, 3, \qquad \gamma_4^{(k)} \in O_{\hat{\epsilon}}(0).$$

To achieve this, as in [26], we introduce the mesh curves to establish the bound on the total variation of $U_{h,\theta}$.

DEFINITION 3.1 A *k-mesh curve* J is a piecewise unbounded linear curve lying in the strip $\{(k-1)h \leqslant x \leqslant (k+1)h\}$ and consists of the diamond boundaries of the form $P_{k,n-1}N(\theta_{k+1}, n)$, $P_{k,n-1}S(\theta_k, n)$, $S(\theta_k, n)P_{k,n}$, and $N(\theta_{k+1}, n)P_{k,n}$, where

$$N(\theta_{k+1}, n) = \begin{cases} P_{k+1,n} & \theta_{k+1} \leqslant 0, \\ P_{k+1,n-1} & \theta_{k+1} > 0, \end{cases} \qquad S(\theta_k, n) = \begin{cases} P_{k-1,n-1} & \theta_k \leqslant 0, \\ P_{k+1,n} & \theta_k > 0. \end{cases}$$

DEFINITION 3.2 We call mesh curve I is an *immediate successor* to mesh curve J, if all but one mesh points of I are on J and I lies on the right hand side of J.

Then, we define the Glimm-type functional $F(J)$ on J.

DEFINITION 3.3 Let

$$F(J) = L(J) + KQ(J),$$

with

$$\begin{aligned}
L(J) &= L_c(J) + L^1(J) + L^2(J), \\
L_c(J) &= C_1^*\big(|\sigma_2^J - \sigma_{20}| + |\sigma_3^J - \sigma_{30}|\big) + C_2^*|\gamma_4^J|, \\
L^1(J) &= K_{11}^* L_1^1(J) + K_{12}^* L_2^1(J) + K_{13}^* L_3^1(J) + K_{14}^* L_4^1(J) + K_{15}^* L_5^1(J), \\
L^2(J) &= K_{20}^* L_0(J) + L_1^2(J) + K_{22}^* L_2^2(J) + K_{23}^* L_3^2(J) + K_{24}^* L_4^2(J) + K_{25}^* L_5^2(J), \\
Q(J) &= \sum \big\{|\alpha_j||\beta_i| : \text{both weak waves } \alpha_j \text{ and } \beta_i \text{ across } J \text{ and approach}, i, j \neq 4.\big\}, \\
L_0(J) &= \sum \big\{|\omega_k(P_k)| : P_k \in \Gamma_J\big\}, \quad \Gamma_J = \big\{P_k = (kh, y_k) : P_k \in J^+ \cap \partial\Omega_h, k \geqslant 0\big\}, \\
L_j^i(J) &= \sum \big\{|\alpha_j| : \alpha_j \text{ across } J \text{ in region } \Omega_{h,k-1}^{(i)} \cup \Omega_{h,k}^{(i)}, \ i = 1, 2, \ j = 1, 2, 3, 4, 5.\big\},
\end{aligned}$$

where σ_2^J, σ_3^J and γ_4^J stand for the strength of the strong contact discontinuity across J, and J^+ denotes the subregion of Ω_h such that all the points in J^+ lie at the right hand side of J.

The positive constants C_1^*, C_2^* and K in Definition 3.3 will be defined later. The other constants are given in the following lemma.

Lemma 3.4 *There exist positive constants $K_{1i}^*, i = 1,2,3,4,5$, and $K_{2i}^*, i = 0,2,3,4,5$, such that*

$$K_{20}^* > |K_{b0}|, \quad K_{2i}^* > |K_{bi}|, i = 2,3,5, \quad K_{24}^* > C_2^*, \quad K_{14}^* > C_2^*,$$
$$K_{11}^* < \frac{1 - K_{25}^*|K_{25}|}{|K_{21}|}, \quad K_{15}^* > K_{25}^*|K_{15}| + K_{11}^*|K_{11}|,$$

and $K_{1i}^*, i = 2,3$, are arbitrarily large positive constants.

Proof. By Proposition 2.5 and Proposition 2.8, we know that $K_{b5} = 1$ and $|K_{25}| < 1$. Hence there exists a constant K_{25}^* such that $K_{b5} < K_{25}^* < 1/|K_{25}|$. Then we can choose a positive constant K_{11}^* satisfying

$$0 < K_{11}^* < \frac{1 - K_{25}^*|K_{25}|}{|K_{21}|}.$$

This completes the proof. □

3.3 *Global estimates of the approximate solutions*

In this section, we will show that the functional $F(J)$ is decreasing to establish the global estimates of the approximate solutions. First let us consider the estimates on the non-reacting step.

PROPOSITION 3.5 Suppose that $g(x)$ satisfies (3.1), and suppose that I and J are two k-mesh curves such that J is an immediate successor of I. If

$$U_h|_{I\cap(\Omega_{h,k-1}^{(i)}\cup\Omega_{h,k}^{(i)})} \in O_\epsilon\big(U_i^{(0)}\big), \quad i = 1,2; \quad |\sigma_j^I - \sigma_{j0}| < \hat{\epsilon}, \quad j = 2,3; \quad |\gamma_4^I| < \hat{\epsilon},$$

for some $\epsilon, \hat{\epsilon} > 0$, then there exists $\tilde{\epsilon} > 0$ such that if $F(I) \leqslant \tilde{\epsilon}$, then it holds that

$$F(J) \leqslant F(I). \tag{3.5}$$

Proof. Let Λ be the diamond that is formed by I and J. Then assume that $I = I_0 \cup I'$ and $J = I_0 \cup J'$ such that $\partial\Lambda = I' \cup J'$. We will show this proposition case by case depending on the location of Λ.

Case 1 (Fig. 7): Λ lies in the interior of Ω_h and only weak waves enter Λ. Without loss of the generality, we assume that Λ lies in region (1). Denote $Q(\Lambda) = \Delta(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$, where $\Delta(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ is defined in (2.17). Then by Proposition 2.3, we have

$$L^1(J) - L^1(I) \leqslant M(K_{11}^* + K_{12}^* + K_{13}^* + K_{15}^*)Q(\Lambda),$$
$$Q(J) - Q(I) \leqslant (ML(I_0) - 1)Q(\Lambda).$$

Note that $F(I) \leqslant \tilde{\epsilon}$ for sufficiently small $\tilde{\epsilon}$, then it holds that

$$F(J) - F(I) \leqslant \big(M(K_{11}^* + K_{12}^* + K_{13}^* + K_{15}^*) + K(ML(I_0) - 1)\big)Q(\Lambda)$$
$$\leqslant -\frac{1}{2}Q(\Lambda),$$

provided that K is suitably large.

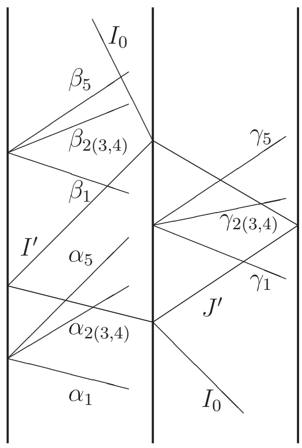

FIG. 7. Case 1: In the interior of Ω_h

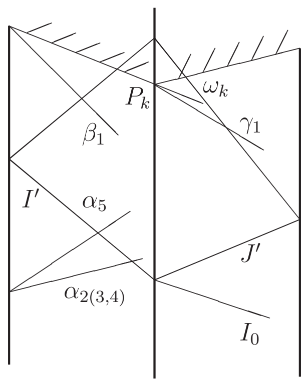

FIG. 8. Case 2: Near the boundary

Case 2 (Fig. 8): Λ touches the approximate boundary $\partial\Omega_h$, and $\Gamma_I = \Gamma_J \cup \{P_k\}$ for certain k. Using Proposition 2.5, we can obtain

$$\begin{aligned}
&L_0(J) - L_0(I) = -|\omega_k|,\\
&L_1^2(J) - L_1^2(I) \leqslant |K_{b0}||\omega_k| + \sum_{i=2,3,5} |K_{bi}||\alpha_i|,\\
&L_i^2(J) - L_i^2(I) = -|\alpha_i|, \quad i = 2,3,4,5.\\
&Q(J) - Q(I) \leqslant (|K_{b0}||\omega_k| + \sum_{i=2,3,5} |K_{bi}||\alpha_i|)L(I_0).
\end{aligned}$$

It implies that

$$L^2(J) - L^2(I) \leqslant (|K_{b0}| - K_{20}^*)|\omega_k| + \sum_{i=2,3,5} (|K_{bi}| - K_{2i}^*)|\alpha_i|.$$

Therefore, if $F(I) \leqslant \tilde{\epsilon}$ for sufficiently small $\tilde{\epsilon}$, then it holds that $F(J) \leqslant F(I)$ by the choice of K_{20}^* and K_{2i}^* in Lemma 3.4.

Case 3.1 (Fig. 9): The diamond Λ covers $\chi^{(k-1)}$ and the weak waves lying in region (2) interact with $\chi^{(k-1)}$ from the above. By applying Proposition 2.8, we have

$$\begin{aligned}
&L_1^1(J) - L_1^1(I) \leqslant |K_{21}||\beta_1| + M\Delta'(\alpha_5, \boldsymbol{\beta}^*),\\
&L_i^2(J) - L_i^2(I) = -|\beta_i|, \quad i = 1,2,3,4,\\
&L_5^2(J) - L_5^2(I) \leqslant |K_{25}||\beta_1| + M\Delta'(\alpha_5, \boldsymbol{\beta}^*),\\
&|\sigma_j^J - \sigma_j^I| \leqslant |K_{2i}||\beta_1| + |\beta_j| + M\Delta'(\alpha_5, \boldsymbol{\beta}^*), \quad j = 2,3\\
&|\gamma_4^J - \gamma_4^I| = |\beta_4|,\\
&Q(J) - Q(I) \leqslant \big(|K_{21}| + |K_{25}|\big)|\beta_1|L(I_0) + \big(ML(I_0) - 1\big)\Delta'(\alpha_5, \boldsymbol{\beta}^*).
\end{aligned}$$

It implies that

$$L^1(J) + L^2(J) - L^1(I) - L^2(I) \leqslant \big(K_{11}^*|K_{21}| + K_{25}^*|K_{25}| - 1\big)|\beta_1| - \sum_{i=2,3,4} K_{2i}^*|\beta_i| + M\Delta'(\alpha_5, \boldsymbol{\beta}^*).$$

Therefore, if $F(I) \leqslant \tilde{\epsilon}$ for sufficiently small $\tilde{\epsilon}$, then from the facts that $K_{11}^*|K_{21}| + K_{25}^*|K_{25}| < 1$ and that $K_{24}^* > C_2^*$ by Lemma 3.4, it holds that $F(J) \leqslant F(I)$ by choosing suitably small C_1^* and suitably large K.

Case 3.2 (Fig. 9): The diamond Λ covers $\chi^{(k-1)}$ and the weak waves lying in region (1) interact with $\chi^{(k-1)}$ from the below. By Proposition 2.10, we can obtain

$$\begin{aligned}
&L_1^1(J) - L_1^1(I) \leqslant |K_{11}||\alpha_5| + M\Delta''(\boldsymbol{\alpha}^*, \beta_1),\\
&L_i^1(J) - L_i^1(I) = -|\alpha_i|, \quad i = 2, 3, 4, 5,\\
&L_5^2(J) - L_5^2(I) \leqslant |K_{15}||\alpha_5| + M\Delta''(\boldsymbol{\alpha}^*, \beta_1),\\
&|\sigma_j^J - \sigma_j^I| \leqslant |K_{1i}||\alpha_5| + |\alpha_j| + M\Delta''(\boldsymbol{\alpha}^*, \beta_1), \quad j = 2, 3\\
&|\gamma_4^J - \gamma_4^I| = |\alpha_4|,\\
&Q(J) - Q(I) \leqslant \big(|K_{11}| + |K_{15}|\big)|\alpha_5|L(I_0) + \big(ML(I_0) - 1\big)\Delta''(\boldsymbol{\alpha}^*, \beta_1).
\end{aligned}$$

It implies that

$$L^1(J) + L^2(J) - L^1(I) - L^2(I) \leqslant (K_{11}^*|K_{11}| + K_{25}^*|K_{15}| - K_{15}^*)|\alpha_5| - \sum_{i=2,3,4} K_{1i}^*|\alpha_i| + M\Delta''(\boldsymbol{\alpha}^*, \beta_1).$$

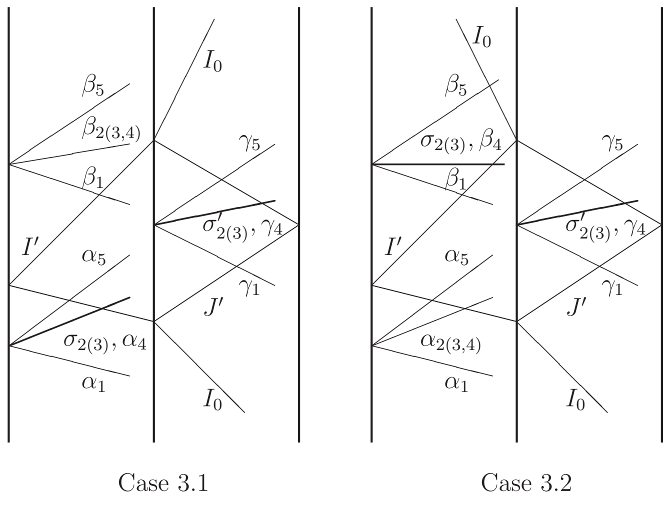

FIG. 9. Near the strong contact discontinuity

So if $F(I) \leqslant \tilde{\epsilon}$ for sufficiently small $\tilde{\epsilon}$, then from the facts that $K_{11}^*|K_{11}| + K_{25}^*|K_{15}| < K_{15}^*$ and $K_{14}^* > C_2^*$ by Lemma 3.4, it holds that $F(J) \leqslant F(I)$ by choosing suitably small C_1^* and suitably large K. □

In order to analyze the effect of the exothermic reaction on the functionals L and Q, as in [6], we introduce a new mesh curve $\tilde{J}$, which, as a curve, is the same as the mesh curve J, but upon which the states $\tilde{U}$ are the values of the states U on J after a single reaction step along J.

Let J_k and $\tilde{J}_k$ be the k-mesh curve lying in $\{kh \leqslant x \leqslant (k+1)h\}$. By Proposition 3.5, we have

Corollary 3.6 *Suppose that $g(x)$ satisfies (3.1). Let $\epsilon, \hat{\epsilon}, \tilde{\epsilon}$ be the constants given in Proposition 3.5 such that the induction hypotheses $A_1(k-1)$-$A_3(k-1)$ hold. If $F(\tilde{J}_{k-1}) \leqslant \tilde{\epsilon}$, then it holds that*

$$F(J_k) \leqslant F(\tilde{J}_{k-1}). \tag{3.6}$$

Next, let us consider the estimates on the reacting step.

PROPOSITION 3.7 There exists a positive constant M such that

$$\begin{aligned} L(\tilde{J}_k) &\leqslant L(J_k) + Me^{-lkh}h\|Z_0\|_\infty\big(L(J_k)+1\big), \\ Q(\tilde{J}_k) &\leqslant Q(J_k) + Me^{-lkh}h\|Z_0\|_\infty\big(L(J_k)+1\big)^2. \end{aligned} \tag{3.7}$$

It implies that

$$F(\tilde{J}_k) \leqslant F(J_k) + Me^{-lkh}h\|Z_0\|_\infty\big(F(J_k)+2\big)^2. \tag{3.8}$$

Here $\|\cdot\|_\infty$ stands for L^∞ norm.

Proof. By Lemma 2.11 and by the induction method, we can easily obtain that

$$\|Z_{h,\theta}(kh+,\cdot)\|_\infty \leqslant e^{-lkh}\|Z_0\|_\infty. \tag{3.9}$$

Then we will consider the change of L on the reaction step, which is the first inequality of (3.7). The analysis is divided into three cases depending on the location of Λ.

(1) Λ lies in the interior of Ω_h so that only weak waves γ go out of Λ through J_k. Without loss of the generality, we assume that Λ lies in region (1). With notations in Proposition 2.12, we use (2.39) to deduce the following,

$$\begin{aligned} L_i^1(\tilde{J}_k) &\leqslant L_i^1(J_k) + Me^{-lkh}h\|Z_0\|_\infty|\boldsymbol{\gamma}^*| + M|\gamma_4|h, \\ L_4^1(\tilde{J}_k) &\leqslant L_4^1(J_k) + Me^{-lkh}h\|Z_0\|_\infty|\boldsymbol{\gamma}^*| - l|\gamma_4|h. \end{aligned}$$

Then it holds that $L(\tilde{J}_k) \leqslant L(J_k) + Me^{-lkh}h\|Z_0\|_\infty|\boldsymbol{\gamma}^*|$ by choosing suitably large K_{14}^*.

(2) Λ covers a part of $\partial\Omega_h$. It follows from (2.40) that $L(\tilde{J}_k) \leqslant L(J_k) + Me^{-lkh}h\|Z_0\|_\infty$.

(3) Λ covers the strong contact discontinuity $\chi^{(k)}$ so that $\chi^{(k)}$ with strength $(\sigma_2^{(k)}, \sigma_3^{(k)}, \gamma_4^{(k)})$ goes out of Λ through J_k. By (2.41), we can obtain

$$\begin{aligned} L_1^1(\tilde{J}_k) &\leqslant L_1^1(J_k) + Me^{-lkh}h\|Z_0\|_\infty + M|\gamma_4^{(k)}|h, \\ L_5^2(\tilde{J}_k) &\leqslant L_5^2(J_k) + Me^{-lkh}h\|Z_0\|_\infty + M|\gamma_4^{(k)}|h, \\ |\tilde{\sigma}_j^{(k)} - \sigma_j^{(k)}| &\leqslant Me^{-lkh}h\|Z_0\|_\infty + M|\gamma_4^{(k)}|h, \quad j = 2, 3, \\ |\tilde{\gamma}_4^{(k)} - \gamma_4^{(k)}| &\leqslant Me^{-lkh}h\|Z_0\|_\infty - l|\gamma_4^{(k)}|h. \end{aligned}$$

Then it holds that $L(\tilde{J}_k) \leqslant L(J_k) + Me^{-lkh}h\|Z_0\|_\infty$ by choosing suitably large C_2^*.

Thus, combining these three cases, we proved the first inequality of (3.7).

The second estimate in (3.7) and the estimate in (3.8) can be derived in the same way. The proof is complete. □

Now in order to obtain the uniform bound of the total variation of $U_{h,\theta}$, we introduce the following functional:

$$F_c(J_k) = F(J_k) + K_z \sum_{j=k+1}^{\infty} e^{-ljh} h \|Z_0\|_\infty,$$

where constant K_z will be defined later.

Lemma 3.8 *There exist positive constants K_z and $\tilde{\epsilon}$, such that if $F_c(\tilde{J}_{k-1}) \leqslant \tilde{\epsilon}$, then it holds that*

$$F_c(\tilde{J}_k) \leqslant F_c(\tilde{J}_{k-1}), \tag{3.10}$$

and

$$U_h|_{\Omega^{(i)}_{h,k}} \in O_\epsilon(U_i^{(0)}),\ i = 1, 2, \quad |\sigma_j^{(k)} - \sigma_{j0}| < \hat{\epsilon},\ j = 2, 3, \quad |\gamma_4^{(k)}| < \hat{\epsilon}.$$

Proof. From the estimates (3.6) and (3.8), we have

$$F(\tilde{J}_k) \leqslant F(\tilde{J}_{k-1}) + Me^{-lkh} h \|Z_0\|_\infty \big(F(J_k) + 2\big)^2.$$

It implies

$$\begin{aligned} F_c(\tilde{J}_k) - F_c(\tilde{J}_{k-1}) &= F(\tilde{J}_k) - F(\tilde{J}_{k-1}) - K_z e^{-lkh} h \|Z_0\|_\infty \\ &\leqslant (M\big(F(J_k) + 2\big)^2 - K_z) e^{-lkh} h \|Z_0\|_\infty. \end{aligned}$$

Note that $F(J_k) < \tilde{\epsilon}$. So we can choose suitably large K_z such that $F_c(\tilde{J}_k) \leqslant F_c(\tilde{J}_{k-1})$ and $|\sigma_j^{(k)} - \sigma_{j0}| < \hat{\epsilon}$, $j = 2, 3$, and $|\gamma_4^{(k)}| < \hat{\epsilon}$.

Next, for any $k \geqslant 0$, define $U_{h,\theta}(kh+, -\infty) = \lim_{y \to -\infty} U_{h,\theta}(kh+, y)$. Then by the fact that $\lim_{y \to -\infty} Z_1(y) = 0$ and from the construction of the approximate solutions, we have that

$$U_h(kh+, -\infty) = \lim_{y \to -\infty} U_0(y).$$

Then by Lemma 2.2 and (2.27), for sufficiently small $\tilde{\epsilon}$, it holds that $U_h|_{\Omega^{(i)}_{h,k}} \in O_\epsilon(U_i^{(0)})$, $i = 1, 2$. □

Based on Proposition 3.5, Proposition 3.7, and Lemma 3.8, we have the following theorems on the uniform B.V. bound of the approximate solution $U_{h,\theta}$.

Theorem 3.9 *Under assumptions* (H1)–(H3), *there exist positive constants δ_0 and C depending only on $U^{(0)}$, such that if* (1.6)–(1.8) *hold, then for any $\theta \in \prod_{k=1}^{\infty}(-1, 1)$ and h, the modified Glimm scheme defines global approximate solutions $U_{h,\theta}$ in Ω_h, which satisfy $A_1(k)$–$A_3(k)$ given in Section 3.2 for $k \geqslant 0$. In addition,*

$$T.V.\{U_{h,\theta}(kh-, \cdot) : (-\infty, y_k]\} \leqslant C\delta_0, \tag{3.11}$$

for any $k \geqslant 0$ and

$$|\chi_{h,\theta}(x') - \chi_{h,\theta}(x'')| \leqslant C(|x' - x''| + h), \tag{3.12}$$

for any $x', x'' \geqslant 0$.

Based on Theorem 3.9, now we can show the global existence of entropy solutions of (1.1) as follows.

Proof of Theorem 1.4. The convergence of the approximate solutions to a global entropy solution can be carried out in the standard way as the one in [6, 26, 44] by using the structure of the approximate solutions. By (3.12), $\chi_{h,\theta}(x)$ converges to $\chi(x)$ uniformly in any bounded x-interval such that (1.10) and (1.11) hold. Therefore, we can establish the global existence of entropy solutions of (1.1), i.e., Theorem 1.4. □

4. Error estimate of the quasi-one-dimensional approximation

In this section, we shall study the quasi-one-dimensional approximation of two-dimensional steady supersonic exothermically reacting Euler flows between the Lipschitz wall $g(x)$ and strong contact discontinuity $\chi(x)$. To do that, we first solve the quasi-one-dimensional model, and then introduce several integral identities of the approximate solutions to show that the distance between the wall and the strong contact discontinuity has positive lower and upper bounds. Then we introduce the integral average of the approximate solutions with respect to y, and find the equations which the integral average satisfies as $h \to 0$. Based on them, the difference between the integral average of the weak solution and the solution of the quasi-one-dimensional system can be estimated by analyzing the error terms.

4.1 *Quasi-one-dimensional model*

In this section, we shall establish the global existence of solution to quasi-one-dimensional model (1.13).

First system (1.13) with initial data $U_{A,0} = (\rho_{A,0}, u_{A,0}, p_{A,0}, Z_{A,0})^\top$ can be written equivalently as

$$\begin{cases} \rho u A(x) = \rho_{A,0} u_{A,0} A(0), \\ u + \frac{A(x)}{\rho_{A,0}u_{A,0}A(0)} p = u_{A,0} + \frac{A(0)}{\rho_{A,0}u_{A,0}A(0)} p_{A,0} + \frac{1}{\rho_{A,0}u_{A,0}A(0)} \int_0^x A'(\tau) p \, d\tau, \\ \frac{\gamma p}{(\gamma-1)\rho} + \frac{1}{2}u^2 = \frac{\gamma p_{A,0}}{(\gamma-1)\rho_{A,0}} + \frac{1}{2}u_{A,0}^2 + \frac{q_0}{\rho_{A,0}u_{A,0}A(0)} \int_0^x A(\tau)\rho\phi(T) Z \, d\tau, \\ Z = Z_{A,0} - \frac{1}{\rho_{A,0}u_{A,0}A(0)} \int_0^x A(\tau)\rho\phi(T) Z \, d\tau. \end{cases} \tag{4.1}$$

Then we have the following lemma.

Lemma 4.1 *There exist positive constants δ_0, C, C_*, and C^*, such that if $|U_{A,0} - \hat{U}_2^{(0)}| \leqslant \delta_0$, with $\hat{U}_2^{(0)} = (\rho_2^{(0)}, u_2^{(0)}, p_2^{(0)}, 0)^\top$ and $\int_0^\infty |A'(\tau)| d\tau \leqslant \delta_0$, then the system (4.1) admits a unique global solution $U_A(x)$ satisfying that*

$$\max_{x \geqslant 0} \left| U_A(x) - \hat{U}_2^{(0)} \right| \leqslant C\delta_0, \quad Z_{A,0} e^{-C^* x} \leqslant Z_A \leqslant Z_{A,0} e^{-C_* x}. \tag{4.2}$$

Proof. We use the following the iteration scheme to establish a sequence of functions convergent to a solution. Let

$$\left(\rho_A^{(0)}, u_A^{(0)}, p_A^{(0)}\right) = \left(\rho_{A,0}, u_{A,0}, p_{A,0}\right),$$

and $Z_A^{(0)}$ is given by the last equation of (4.3) for $n=0$. Precisely, we have

$$Z_A^{(0)} = Z_{A,0}\exp(-\frac{1}{\rho_{A,0}u_{A,0}A(0)}\int_0^x A(\tau)\rho_{A,0}\phi(T_{A,0})d\tau).$$

Then for any $n \geqslant 1$, the functions $U_A^{(n)}(x) = (\rho_A^{(n)}, u_A^{(n)}, p_A^{(n)}, Z_A^{(n)})$ are determined inductively by

$$\begin{cases} \rho_A^{(n)}u_A^{(n)}A(x) = \rho_{A,0}u_{A,0}A(0),\\ u_A^{(n)} + \frac{A(x)}{\rho_{A,0}u_{A,0}A(0)}p_A^{(n)} = u_{A,0} + \frac{A(0)}{\rho_{A,0}u_{A,0}A(0)}p_{A,0} + \frac{1}{\rho_{A,0}u_{A,0}A(0)}\int_0^x A'(\tau)p_A^{(n-1)}d\tau,\\ \frac{\gamma p_A^{(n)}}{(\gamma-1)\rho_A^{(n)}} + \frac{1}{2}(u_A^{(n)})^2 = \frac{\gamma p_{A,0}}{(\gamma-1)\rho_{A,0}} + \frac{1}{2}u_{A,0}^2 + \frac{q_0}{\rho_{A,0}u_{A,0}A(0)}\int_0^x A(\tau)\rho_A^{(n-1)}\phi(T_A^{(n-1)})Z_A^{(n-1)}d\tau,\\ Z_A^{(n)} = Z_{A,0} - \frac{1}{\rho_{A,0}u_{A,0}A(0)}\int_0^x A(\tau)\rho_A^{(n)}\phi(T_A^{(n)})Z_A^{(n)}d\tau. \end{cases} \tag{4.3}$$

First, let us prove inductively that for any $n \geqslant 0$, $U_A^{(n)}(x)$ are well defined and that there exist positive constants δ_0 and C, such that the following inequality holds

$$\max_{x\geqslant 0}\left|U_A^{(n)}(x) - \hat{U}_2^{(0)}\right| \leqslant C\delta_0. \tag{4.4}$$

Obviously, it is true for $n=0$. Now assume that the estimate (4.4) holds for $n=k-1, k\geqslant 1$, then we have

$$C_* \leqslant \frac{1}{\rho_{A,0}u_{A,0}A(0)}A(x)\rho_A^{(k-1)}\phi\big(T_A^{(k-1)}\big) \leqslant C^*,$$

for some constants C_* and C^*. The last equation of (4.3) yields that

$$Z_A^{(k-1)} = Z_{A,0}\exp(-\frac{1}{\rho_{A,0}u_{A,0}A(0)}\int_0^x A(\tau)\rho_A^{(k-1)}\phi\big(T_A^{(k-1)}\big)d\tau).$$

It implies that

$$Z_{A,0}e^{-C^*x} \leqslant Z_A^{(k-1)} \leqslant Z_{A,0}e^{-C_*x}. \tag{4.5}$$

Let $\mathcal{H}(V_A^{(k)}, A(x)) = (\rho_A^{(k)}u_A^{(k)}A(x),\ u_A^{(k)} + \frac{A(x)}{\rho_{A,0}u_{A,0}A(0)}p_A^{(k)},\ \frac{\gamma p_A^{(k)}}{(\gamma-1)\rho_A^{(k)}} + \frac{1}{2}(u_A^{(k)})^2)^\top$ and $V_A^{(k)} = (\rho_A^{(k)},\ u_A^{(k)},\ p_A^{(k)})^\top$, then the first three equations of (4.3) can be written as

$$\mathcal{H}\big(V_A^{(k)}, A(x)\big) = \mathcal{H}\big(V_{A,0}, A(x)\big) + \mathcal{H}_e\big(V_A^{(k-1)}, V_{A,0}, A(x), A(0), Z_A^{(k-1)}\big). \tag{4.6}$$

where $V_{A,0} = (\rho_{A,0},\ u_{A,0},\ p_{A,0})^\top$, and the term $\mathcal{H}_e$ can be defined without confusion.

From the fact that $\int_0^\infty |A'(\tau)|d\tau \leqslant \delta_0$, and the estimate (4.5), we have

$$\begin{aligned} &|A(0) - A(x)| \leqslant \delta_0,\\ &\left|\int_0^x A'(\tau)p_A^{(k-1)}d\tau\right| \leqslant \int_0^x |A'(\tau)||p_A^{(k-1)} - p_2^{(0)}|d\tau + \int_0^x |A'(\tau)|p_2^{(0)}d\tau\\ &\qquad \leqslant C\delta_0^2 + p_2^{(0)}\delta_0, \end{aligned}$$

and

$$\begin{aligned}\Big|\int_0^x A(\tau)\rho_A^{(k-1)}\phi(T_A^{(k-1)})Z_A^{(k-1)}d\tau\Big| &\leqslant \int_0^x |A(\tau)|\rho_A^{(k-1)}\phi(T_A^{(k-1)})Z_A^{(k-1)}d\tau\\ &\leqslant C^*\rho_{A,0}u_{A,0}A(0)Z_{A,0}\int_0^x e^{-C_*\tau}d\tau\\ &\leqslant C^*\rho_{A,0}u_{A,0}A(0)\delta_0/C_*.\end{aligned}$$

Therefore, $\mathcal{H}_e$ is bounded by $O(1)\delta_0$. Then it follows from the implicit function theorem that $\max_{x\geqslant 0}|V_A^{(k)}(x)-V_{A,0}|\leqslant C'\delta_0$, by choosing suitably large C' and suitably small δ_0.

Again, from the last equation of (4.3), we have

$$Z_A^{(k)} = Z_{A,0}\exp(-\frac{1}{\rho_{A,0}u_{A,0}A(0)}\int_0^x A(\tau)\rho_A^{(k)}\phi\big(T_A^{(k)}\big)d\tau), \tag{4.7}$$

which implies $Z_{A,0}e^{-C^*x}\leqslant Z_A^{(k)}\leqslant Z_{A,0}e^{-C_*x}$. So we obtain the estimate (4.4) for $n=k$.

Second, we will show the convergence of the sequence $\{U_A^{(n)}(x)\}_{n=0}^{\infty}$.

Define

$$w^{(n)} = -\frac{1}{\rho_{A,0}u_{A,0}A(0)}\int_0^x A(\tau)\rho_A^{(n)}\phi\big(T_A^{(n)}\big)d\tau.$$

Then by (4.7), we can obtain

$$\begin{aligned}\big|Z_A^{(n)}-Z_A^{(n-1)}\big| &= \Big|Z_{A,0}\big(w^{(n)}-w^{(n-1)}\big)\int_0^1 \exp\big(sw^{(n)}+(1-s)w^{(n-1)}\big)ds\Big|\\ &\leqslant \frac{Z_{A,0}e^{-C_*x}}{\rho_{A,0}u_{A,0}A(0)}\int_0^x A(\tau)\Big|\rho_A^{(n)}\phi\big(T_A^{(n)}\big)-\rho_A^{(n-1)}\phi\big(T_A^{(n-1)}\big)\Big|d\tau\\ &\leqslant O(1)\delta_0\max_{0\leqslant\tau\leqslant x}\Big(\big|\rho_A^{(n)}-\rho_A^{(n-1)}\big|+\big|T_A^{(n)}-T_A^{(n-1)}\big|\Big).\end{aligned} \tag{4.8}$$

Next, by (4.6), it holds that

$$\begin{aligned}\mathcal{H}\big(V_A^{(n)},A(x)\big)-\mathcal{H}\big(V_A^{(n-1)},A(x)\big) &= \mathcal{H}_e\big(V_A^{(n-1)},V_{A,0},A(x),A(0),Z_A^{(n-1)}\big)\\ &\quad -\mathcal{H}_e\big(V_A^{(n-2)},V_{A,0},A(x),A(0),Z_A^{(n-2)}\big).\end{aligned} \tag{4.9}$$

Noticing the fact that $\int_0^\infty |A'(\tau)|d\tau\leqslant\delta_0$, and the estimate (4.8), we have

$$\Big|\frac{1}{\rho_{A,0}u_{A,0}A(0)}\int_0^x A'(\tau)\big(p_A^{(n-1)}-p_A^{(n-2)}\big)d\tau\Big|\leqslant O(1)\delta_0\max_{0\leqslant\tau\leqslant x}\Big|p_A^{(n-1)}-p_A^{(n-2)}\Big|,$$

and

$$\begin{aligned}
&\left|\frac{q_0}{\rho_{A,0}u_{A,0}A(0)}\int_0^x A(\tau)\Big(\rho_A^{(n-1)}\phi\big(T_A^{(n-1)}\big)Z_A^{(n-1)}-\rho_A^{(n-2)}\phi\big(T_A^{(n-2)}\big)Z_A^{(n-2)}\Big)d\tau\right|\\
&\quad\leqslant \frac{q_0}{\rho_{A,0}u_{A,0}A(0)}\int_0^x A(\tau)\big|\rho_A^{(n-1)}\phi\big(T_A^{(n-1)}\big)-\rho_A^{(n-2)}\phi\big(T_A^{(n-2)}\big)\big|Z_A^{(n-1)}d\tau\\
&\qquad+\frac{q_0}{\rho_{A,0}u_{A,0}A(0)}\int_0^x A(\tau)\rho_A^{(n-2)}\phi\big(T_A^{(n-2)}\big)\big|Z_A^{(n-1)}-Z_A^{(n-2)}\big|d\tau\\
&\quad\leqslant O(1)Z_{A,0}\int_0^x e^{-C_*\tau}d\tau\max_{0\leqslant\tau\leqslant x}\Big(\big|\rho_A^{(n-1)}-\rho_A^{(n-2)}\big|+\big|T_A^{(n-1)}-T_A^{(n-2)}\big|\Big)\\
&\qquad+O(1)Z_{A,0}\int_0^x e^{-C_*\tau}\tau d\tau\max_{0\leqslant\tau\leqslant x}\Big(\big|\rho_A^{(n-1)}-\rho_A^{(n-2)}\big|+\big|T_A^{(n-1)}-T_A^{(n-2)}\big|\Big)\\
&\quad\leqslant O(1)\delta_0\max_{0\leqslant\tau\leqslant x}\Big(\big|\rho_A^{(n-1)}-\rho_A^{(n-2)}\big|+\big|T_A^{(n-1)}-T_A^{(n-2)}\big|\Big).
\end{aligned}$$

Therefore, the right-hand side of (4.9) is bounded by $O(1)\delta_0\max_{0\leqslant\tau\leqslant x}|V_A^{(n-1)}-V_A^{(n-2)}|$. Then it follows from the implicit function theorem that

$$\max_{x\geqslant 0}\big|V_A^{(n)}(x)-V_A^{(n-1)}(x)\big|\leqslant\frac{1}{2}\max_{x\geqslant 0}\big|V_A^{(n-1)}(x)-V_A^{(n-2)}(x)\big|. \tag{4.10}$$

by choosing suitably small δ_0. Combining (4.8) and (4.10), we know that the limit $U_A(x)$ is an unique solution of (4.1), which belongs to $C([0,\infty),\mathbb{R}^4)$ and satisfies

$$\max_{x\geqslant 0}\big|U_A(x)-\hat{U}_2^{(0)}\big|\leqslant C\delta_0,\quad Z_{A,0}e^{-C^*x}\leqslant Z_A\leqslant Z_{A,0}e^{-C_*x}. \qquad\square$$

4.2 *Integral identities of the approximate solutions*

Let $U_{h,\theta}$ be the solution obtained by Theorem 3.9. Let $\Omega_{i,h}$ be the domain with the boundaries that $x=(i-1)h$, $x=ih$, $y=g_{i-1,h}(x)$, and $y=\chi^{(i-1)}(x)$. Let b_{i-1} be the slope of $y=g_{i-1,h}(x)$. And let $s^{(i-1)}$ be the slope of $y=\chi^{(i-1)}(x)$ emanating from point $((i-1)h, y_{i-1,s})$, where $y_{i-1,s}=y_{i-1}+2n_{i-1}s$ with a negative integer n_{i-1}. By applying the divergence theorem in domain $\Omega_{i,h}$ and using the Rankine–Hugoniot conditions, we have the following integral identities.

$$\int_{y_{i-1,s}+s^{(i-1)}h}^{y_i}(\rho_{h,\theta}u_{h,\theta})(ih-,y)dy-\int_{y_{i-1,s}}^{y_{i-1}}(\rho_{h,\theta}u_{h,\theta})\big((i-1)h+,y\big)dy=0, \tag{4.11}$$

$$\begin{aligned}
&\int_{y_{i-1,s}+s^{(i-1)}h}^{y_i}(\rho_{h,\theta}u_{h,\theta}^2+p_{h,\theta})(ih-,y)dy-\int_{y_{i-1,s}}^{y_{i-1}}(\rho_{h,\theta}u_{h,\theta}^2+p_{h,\theta})\big((i-1)h+,y\big)dy\\
&\quad+\int_{(i-1)h}^{ih}\big(-b_{i-1}p_{h,\theta}(\tau,y)|_{y=g_{i-1,h}(\tau)}+s^{(i-1)}p_{h,\theta}(\tau,y)|_{y=\chi^{(i-1)}(\tau)}\big)d\tau=0,
\end{aligned} \tag{4.12}$$

$$\int_{y_{i-1,s}+s^{(i-1)}h}^{y_i} \Big(\rho_{h,\theta}u_{h,\theta}\Big(\frac{\gamma p_{h,\theta}}{(\gamma-1)\rho_{h,\theta}}+\frac{1}{2}u_{h,\theta}^2+\frac{1}{2}v_{h,\theta}^2\Big)\Big)(ih-,y)dy$$
$$-\int_{y_{i-1,s}}^{y_{i-1}} \Big(\rho_{h,\theta}u_{h,\theta}\Big(\frac{\gamma p_{h,\theta}}{(\gamma-1)\rho_{h,\theta}}+\frac{1}{2}u_{h,\theta}^2+\frac{1}{2}v_{h,\theta}^2\Big)\Big)\big((i-1)h+,y\big)dy=0, \tag{4.13}$$

$$\int_{y_{i-1,s}+s^{(i-1)}h}^{y_i} (\rho_{h,\theta}u_{h,\theta}Z_{h,\theta})(ih-,y)dy-\int_{y_{i-1,s}}^{y_{i-1}} (\rho_{h,\theta}u_{h,\theta}Z_{h,\theta})\big((i-1)h+,y\big)dy=0. \tag{4.14}$$

Therefore, for any $x\in((k-1)h,kh)$, summing over (4.11)–(4.14) with respect to $1\leqslant i\leqslant k-1$ respectively, we have that

$$\int_{\chi^{(k-1)}(x)}^{g_{k-1,h}(x)} (\rho_{h,\theta}u_{h,\theta})(x-,y)dy+\sum_{i=1}^{k-1}E_{1,i}(h,\theta)=\int_{y^{(0)}}^{0}(\rho_{h,\theta}u_{h,\theta})(0+,y)dy, \tag{4.15}$$

$$\int_{\chi^{(k-1)}(x)}^{g_{k-1,h}(x)} (\rho_{h,\theta}u_{h,\theta}^2+p_{h,\theta})(x-,y)dy+\sum_{i=1}^{k-1}E_{2,i}(h,\theta)=\int_{y^{(0)}}^{0}(\rho_{h,\theta}u_{h,\theta}^2+p_{h,\theta})(0+,y)dy$$
$$+\sum_{i=1}^{k-1}\int_{(i-1)h}^{ih}\big(b_{i-1}p_{h,\theta}(\tau,y)|_{y=g_{i-1}(\tau)}-s^{(i-1)}p_{h,\theta}(\tau,y)|_{y=\chi^{(i-1)}(\tau)}\big)d\tau$$
$$+\int_{(k-1)h}^{x}\big(b_{k-1}p_{h,\theta}(\tau,y)|_{y=g_{k-1}(\tau)}-s^{(k-1)}p_{h,\theta}(\tau,y)|_{y=\chi^{(k-1)}(\tau)}\big)d\tau, \tag{4.16}$$

$$\int_{\chi^{(k-1)}(x)}^{g_{k-1,h}(x)} \Big(\rho_{h,\theta}u_{h,\theta}\Big(\frac{\gamma p_{h,\theta}}{(\gamma-1)\rho_{h,\theta}}+\frac{1}{2}u_{h,\theta}^2+\frac{1}{2}v_{h,\theta}^2\Big)\Big)(x-,y)dy+\sum_{i=1}^{k-1}E_{3,i}(h,\theta)$$
$$=\int_{y^{(0)}}^{0}\Big(\rho_{h,\theta}u_{h,\theta}\Big(\frac{\gamma p_{h,\theta}}{(\gamma-1)\rho_{h,\theta}}+\frac{1}{2}u_{h,\theta}^2+\frac{1}{2}v_{h,\theta}^2\Big)\Big)(0+,y)dy, \tag{4.17}$$

$$\int_{\chi^{(k-1)}(x)}^{g_{k-1,h}(x)} (\rho_{h,\theta}u_{h,\theta}Z_{h,\theta})(x-,y)dy+\sum_{i=1}^{k-1}E_{4,i}(h,\theta)=\int_{y^{(0)}}^{0}(\rho_{h,\theta}u_{h,\theta}Z_{h,\theta})(0+,y)dy, \tag{4.18}$$

where $E_{l,i}(h,\theta)(l=1,2,3,4)$ is the l-th component of the error term $E_i(h,\theta)$, and

$$E_i(h,\theta)=\int_{y_{i-1,s}+s^{(i-1)}h}^{y_i}\big(W(U_{h,\theta})\big)(ih-,y)dy-\int_{y_{i,s}}^{y_i}\big(W(U_{h,\theta})\big)(ih+,y)dy. \tag{4.19}$$

Now we are going to analyze the error terms $E_i(h,\theta)$ across the line $x=kh$. Note the fact that

$$W\big(U_{h,\theta}(ih+,y)\big)=W\big(U_{h,\theta}(ih-,y_{i,n})\big)+G\big(U_{h,\theta}(ih-,y_{i,n})\big)h. \tag{4.20}$$

in the interval $y_i + 2ns < y < y_i + 2(n+1)s, n \leqslant -1$, and the fact that

$$\sum_{i=1}^{k-1} \int_{y_{i,s}}^{y_i} G\big(U_{h,\theta}(ih-, y_{i,n})\big) h\,dy \to \int_0^x \int_{\chi(\tau)}^{g(\tau)} G\big(U(\tau, y)\big)\,dy\,d\tau,$$

when $h \to 0$ by the convergence of the approximate solutions. Let

$$\tilde{E}_i(h,\theta) = \int_{y_{i-1,s}+s^{(i-1)}h}^{y_i} W\big(U_{h,\theta}(ih-, y)\big)dy - \int_{y_{i,s}}^{y_i} W\big(U_{h,\theta_i}(ih-, y)\big)dy,$$

where $W(U_{h,\theta_i}(ih-, y)) = W(U_{h,\theta}(ih-, y_{i,n}))$ in the interval $y_i + 2ns < y < y_i + 2(n+1)s$, for $n \leqslant -1$. Obviously

$$E_i(h,\theta) = \tilde{E}_i(h,\theta) - \int_{y_{i,s}}^{y_i} G\big(U_{h,\theta}(ih-, y_{i,n})\big) h\,dy. \tag{4.21}$$

Therefore, in order to estimate $E_i(h,\theta)$, we only need to estimate $\tilde{E}_i(h,\theta)$.

To get the more specific expression of $\tilde{E}_i(h,\theta)$, let

$$d_i = \frac{s^{(i-1)}h - (y_i - y_{i-1})}{s}.$$

Obviously, $d_i \in (-1, 1)$, and is independent of θ_i.

Now we will divide our analysis into two cases based on d_i.

The first case is that $d_i < 0$. In this case, if $\theta_i \in (-1, d_i + 1)$, then we have $y_{i,s} = y_i + 2n_{i-1}s$, and

$$\begin{aligned}\tilde{E}_i(h,\theta) = &\int_{y_i+2n_{i-1}s}^{y_i} \Big(W\big(U_{h,\theta}(ih-, y)\big) - W\big(U_{h,\theta_i}(ih-, y)\big)\Big)dy \\ &+ \int_{y_i+(2n_{i-1}+d_i)s}^{y_i+2n_{i-1}s} W\big(U_{h,\theta}(ih-, y)\big)dy.\end{aligned} \tag{4.22}$$

If $\theta_i \in (d_i + 1, 1)$, then we have $y_{i,s} = y_i + 2(n_{i-1} - 1)s$, and

$$\begin{aligned}\tilde{E}_i(h,\theta) = &\int_{y_i+2n_{i-1}s}^{y_i} \Big(W\big(U_{h,\theta}(ih-, y)\big) - W\big(U_{h,\theta_i}(ih-, y)\big)\Big)dy \\ &+ \int_{y_i+(2n_{i-1}+d_i)s}^{y_i+2n_{i-1}s} W\big(U_{h,\theta}(ih-, y)\big)dy - \int_{y_i+2(n_{i-1}-1)s}^{y_i+2n_{i-1}s} W\big(U_{h,\theta_i}(ih-, y)\big)dy.\end{aligned} \tag{4.23}$$

The second case is that $d_i > 0$. In this case, if $\theta_i \in (-1, d_i - 1)$, then we have $y_{i,s} = y_i + 2(n_{i-1} + 1)s$, and

$$\tilde{E}_i(h,\theta) = \int_{y_i+2n_{i-1}s}^{y_i} \big(W(U_{h,\theta}(ih-,y)) - W(U_{h,\theta_i}(ih-,y))\big)dy - \int_{y_i+2n_{i-1}s}^{y_i+(2n_{i-1}+d_i)s} W\big(U_{h,\theta}(ih-,y)\big)dy + \int_{y_i+2n_{i-1}s}^{y_i+2(n_{i-1}+1)s} W\big(U_{h,\theta_i}(ih+,y)\big)dy. \tag{4.24}$$

If $\theta_i \in (d_i - 1, 1)$, then we have $y_{i,s} = y_i + 2n_{i-1}s$, and

$$\tilde{E}_i(h,\theta) = \int_{y_i+2n_{i-1}s}^{y_i} \Big(W\big(U_{h,\theta}(ih-,y)\big) - W(U_{h,\theta_i}(ih-,y))\Big)dy - \int_{y_i+2n_{i-1}s}^{y_i+(2n_{i-1}+d_i)s} W\big(U_{h,\theta}(ih-,y)\big)dy. \tag{4.25}$$

Let $\mathbf{1}_B$ be the characteristic function of set B. Then, combining (4.22)–(4.25) together, we have

$$\begin{aligned}\tilde{E}_i(h,\theta) = {}& \mathbf{1}_{(-1,0)}(d_i)\Bigg(\int_{y_i+2n_{i-1}s}^{y_i} \Big(W\big(U_{h,\theta}(ih-,y)\big) - W\big(U_{h,\theta_i}(ih-,y)\big)\Big)dy \\ &+ \int_{y_i+(2n_{i-1}+d_i)s}^{y_i+2n_{i-1}s} W(U_{h,\theta}(ih-,y))dy - \mathbf{1}_{(d_i+1,1)}(\theta_i)\int_{y_i+2(n_{i-1}-1)s}^{y_i+2n_{i-1}s} W\big(U_{h,\theta_i}(ih-,y)\big)dy\Bigg) \\ &+ \mathbf{1}_{(0,1)}(d_i)\Bigg(\int_{y_i+2n_{i-1}s}^{y_i} \Big(W\big(U_{h,\theta}(ih-,y)\big) - W\big(U_{h,\theta_i}(ih-,y)\big)\Big)dy \\ &- \int_{y_i+2n_{i-1}s}^{y_i+(2n_{i-1}+d_i)s} W\big(U_{h,\theta}(ih-,y)\big)dy + \mathbf{1}_{(-1,d_i-1)}(\theta_i)\int_{y_i+2n_{i-1}s}^{y_i+2(n_{i-1}+1)s} W\big(U_{h,\theta_i}(ih-,y)\big)dy\Bigg).\end{aligned} \tag{4.26}$$

For the error term $\tilde{E}_i(h,\theta)$, we have the following lemma.

Lemma 4.2 *For any $x \geqslant 0$, there exist a null set $\mathscr{N}_1 \subset \prod_{k=1}^{\infty}(-1,1)$ and a subsequence $\{h_j\}_{j=1}^{\infty}$, which tends to 0, such that when $h_j \to 0$, it holds that*

$$\sum_{i=1}^{k-1} \tilde{E}_i(h_j,\theta) \to 0, \tag{4.27}$$

for any $\theta \in \prod_{k=1}^{\infty}(-1,1)\backslash\mathscr{N}_1$.

Proof. Note that n_{i-1} is independent of θ_i, then we have

$$
\begin{aligned}
&\frac{1}{2}\int_{-1}^{1}\int_{y_i+2n_{i-1}s}^{y_i}\Big(W\big(U_{h,\theta}(ih-,y)\big)-W\big(U_{h,\theta_i}(ih-,y)\big)\Big)dyd\theta_i\\
&=\frac{1}{2}\int_{-1}^{1}\sum_{n=n_{i-1}}^{-1}\int_{y_i+2ns}^{y_i+2(n+1)s}\Big(W\big(U_{h,\theta}(ih-,y)\big)-W\Big(U_{h,\theta}\big(ih-,y_i+(2n+1+\theta_i)\big)s\Big)\Big)dyd\theta_i\\
&=\sum_{n=n_{i-1}}^{-1}\Big(\int_{y_i+2ns}^{y_i+2(n+1)s}W\big(U_{h,\theta}(ih-,y)\big)dy-s\int_{-1}^{1}W\big(U_{h,\theta}(ih-,y_i+(2n+1+\theta_i)s)\big)d\theta_i\Big)\\
&=0.
\end{aligned}
$$

Next, note that if $d_i<0$, then $W(U_{h,\theta}(ih-,y))$ is a constant state independent of θ_i in the interval $(y_i+(2n_{i-1}+d_i)s, y_i+2n_{i-1}s)$; while if $d_i>0$, then $W(U_{h,\theta}(ih-,y))$ is a constant state independent of θ_i in the interval $(y_i+2n_{i-1}s, y_i+(2n_{i-1}+d_i)s)$. Hence it follows from (4.26) that

$$\frac{1}{2}\int_{-1}^{1}\tilde{E}_i(h,\theta)d\theta_i=0.$$

Therefore, we have

$$\int\Big|\sum_{i=1}^{k-1}\tilde{E}_i(h,\theta)\Big|^2d\theta=\sum_{i=1}^{[x/h]}\int\Big|\tilde{E}_i(h,\theta)\Big|^2d\theta\leqslant Cx(\frac{s}{h})^2h.$$

for some constant $C>0$. Then, we can show (4.27) by choosing a subsequence $\{h_j\}_{j=1}^{\infty}$ with $\sum_{j=1}^{\infty}h_j<\infty$. □

Moreover, by (4.15) and (4.27), we also have the following lemma.

Lemma 4.3 *There exist positive constants A_1 and A_2, such that for any $x\geqslant 0$,*

$$A_1\leqslant g(x)-\chi(x)\leqslant A_2.$$

4.3 *Integral average of the approximate solutions*

If $\tau\in((i-1)h, ih)$, we define the integral average of the approximate solutions as

$$\bar{U}_h(\tau-):=\frac{1}{g_{i-1,h}(\tau)-\chi^{(i-1)}(\tau)}\int_{\chi^{(i-1)}(\tau)}^{g_{i-1,h}(\tau)}U_{h,\theta}(\tau-,y)dy,$$

and the integral average of the approximate initial data as

$$\bar{U}_{h,0}:=\frac{1}{|y^{(0)}|}\int_{y^{(0)}}^{0}U_{h,0}(y)dy.$$

Now, we will derive the equation satisfied by the integral average of the weak solution. Replacing the approximate solutions in equations (4.15)–(4.18) by the integral average of the approximate solutions, (4.15)–(4.18) can be rewritten as

$$
\begin{aligned}
&\big(g_{k-1,h}(x)-\chi^{(k-1)}(x)\big)\bar\rho_h\bar u_h+\sum_{i=1}^{k-1}E_{1,i}(h,\theta)\\
&=-y^{(0)}\bar\rho_{h,0}\bar u_{h,0}-\int_{\chi^{(k-1)}(x)}^{g_{k-1,h}(x)}(\rho_{h,\theta}-\bar\rho_h)(u_{h,\theta}-\bar u_h)dy+\int_{y^{(0)}}^{0}(\rho_{h,0}-\bar\rho_{h,0})(u_{h,0}-\bar u_{h,0})dy,
\end{aligned}
\tag{4.28}
$$

$$
\begin{aligned}
&\big(g_{k-1,h}(x)-\chi^{(k-1)}(x)\big)(\bar\rho_h\bar u_h^2+\bar p_h)+\sum_{i=1}^{k-1}E_{2,i}(h,\theta)\\
&=-y^{(0)}(\bar\rho_{h,0}\bar u_{h,0}^2+\bar p_{h,0})+\sum_{i=1}^{k-1}(b_{i-1}-s^{(i-1)})\int_{(i-1)h}^{ih}\bar p_h d\tau+(b_{k-1}-s^{(k-1)})\int_{(k-1)h}^{x}\bar p_h d\tau\\
&\quad-\int_{\chi^{(k-1)}(x)}^{g_{k-1,h}(x)}(\rho_{h,\theta}u_{h,\theta}-\overline{\rho_h u_h})(u_{h,\theta}-\bar u_h)dy-\bar u_h\int_{\chi^{(k-1)}(x)}^{g_{k-1,h}(x)}(\rho_{h,\theta}-\bar\rho_h)(u_{h,\theta}-\bar u_h)dy\\
&\quad+\int_{y^{(0)}}^{0}(\rho_{h,0}u_{h,0}-\overline{\rho_{h,0}u_{h,0}})(u_{h,0}-\bar u_{h,0})dy+\bar u_{h,0}\int_{y^{(0)}}^{0}(\rho_{h,0}-\bar\rho_{h,0})(u_{h,0}-\bar u_{h,0})dy\\
&\quad+\sum_{i=1}^{k-1}\int_{(i-1)h}^{ih}\Big((b_{i-1}-s^{(i-1)})(p_{h,\theta}|_{y=\chi^{(i-1)}}-\bar p_h)+b_{i-1}(p_{h,\theta}|_{y=g_{i-1}}-p_{h,\theta}|_{y=\chi^{(i-1)}})\Big)d\tau\\
&\quad+\int_{(k-1)h}^{x}\Big((b_{k-1}-s^{(k-1)})(p_{h,\theta}|_{y=\chi^{(k-1)}}-\bar p_h)+b_{k-1}(p_{h,\theta}|_{y=g_{k-1}}-p_{h,\theta}|_{y=\chi^{(k-1)}})\Big)d\tau,
\end{aligned}
\tag{4.29}
$$

$$
\begin{aligned}
&\big(g_{k-1,h}(x)-\chi^{(k-1)}(x)\big)\bar\rho_h\bar u_h\Big(\frac{\gamma\bar p_h}{(\gamma-1)\bar\rho_h}+\frac{1}{2}\bar u_h^2\Big)+\sum_{i=1}^{k-1}E_{3,i}(h,\theta)\\
&=-y^{(0)}\bar\rho_{h,0}\bar u_{h,0}\Big(\frac{\gamma\bar p_{h,0}}{(\gamma-1)\bar\rho_{h,0}}+\frac{1}{2}\bar u_{h,0}^2\Big)-\frac{\bar u_h^2}{2}\int_{\chi^{(k-1)}(x)}^{g_{k-1,h}(x)}(\rho_{h,\theta}-\bar\rho_h)(u_{h,\theta}-\bar u_h)dy\\
&\quad-\frac{1}{2}\overline{(u_h-\bar u_h)^2+v_h^2}\int_{\chi^{(k-1)}(x)}^{g_{k-1,h}(x)}\rho_{h,\theta}u_{h,\theta}dy-\frac{\gamma}{\gamma-1}\int_{\chi^{(k-1)}(x)}^{g_{k-1,h}(x)}(p_{h,\theta}-\bar p_h)(u_{h,\theta}-\bar u_h)dy
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\int_{\chi^{(k-1)}(x)}^{g_{k-1,h}(x)}(\rho_{h,\theta}u_{h,\theta}-\overline{\rho_h u_h})(u_{h,\theta}^2+v_{h,\theta}^2-\overline{u_h^2+v_h^2})dy\\
&+\frac{\bar{u}_{h,0}^2}{2}\int_{y^{(0)}}^{0}(\rho_{h,0}-\bar{\rho}_{h,0})(u_{h,0}-\bar{u}_{h,0})dy\\
&+\frac{1}{2}\overline{(u_{h,0}-\bar{u}_{h,0})^2+v_{h,0}^2}\int_{y^{(0)}}^{0}\rho_{h,0}u_{h,0}dy+\frac{\gamma}{\gamma-1}\int_{y^{(0)}}^{0}(p_{h,0}-\bar{p}_{h,0})(u_{h,0}-\bar{u}_{h,0})dy\\
&+\frac{1}{2}\int_{y^{(0)}}^{0}(\rho_{h,0}u_{h,0}-\overline{\rho_{h,0}u_{h,0}})(u_{h,0}^2+v_{h,0}^2-\overline{u_{h,0}^2+v_{h,0}^2})dy,
\end{aligned}
\tag{4.30}
$$

and

$$
\begin{aligned}
&\big(g_{k-1,h}(x)-\chi^{(k-1)}(x)\big)(\bar{\rho}_h\bar{u}_h\bar{Z}_h)+\sum_{i=1}^{k-1}E_{4,i}(h,\theta)\\
&=-y^{(0)}\bar{\rho}_{h,0}\bar{u}_{h,0}\bar{Z}_{h,0}\\
&\quad-\int_{\chi^{(k-1)}(x)}^{g_{k-1,h}(x)}(\rho_{h,\theta}u_{h,\theta}-\overline{\rho_h u_h})(Z_{h,\theta}-\bar{Z}_h)dy-\bar{Z}_h\int_{\chi^{(k-1)}(x)}^{g_{k-1,h}(x)}(\rho_{h,\theta}-\bar{\rho}_h)(u_{h,\theta}-\bar{u}_h)dy\\
&\quad+\int_{y^{(0)}}^{0}(\rho_{h,0}u_{h,0}-\overline{\rho_{h,0}u_{h,0}})(Z_{h,0}-\bar{Z}_{h,0})dy+\bar{Z}_{h,0}\int_{y^{(0)}}^{0}(\rho_{h,0}-\bar{\rho}_{h,0})(u_{h,0}-\bar{u}_{h,0})dy.
\end{aligned}
\tag{4.31}
$$

Therefore, in order to derive the equations that the integral average of the solution of (1.1) satisfies, we need to analyze the error terms at the right hand side of (4.28)–(4.31) as $h \to 0$ first.

By Theorem 1.4, it holds that the terms like $\int_{\chi(\tau)}^{g(\tau)}(\rho-\bar{\rho})(u-\bar{u})dy$ can be bounded by the square of the total variation of the weak solution, i.e.,

$$
\int_{\chi(\tau)}^{g(\tau)}(\rho-\bar{\rho})(u-\bar{u})dy = O(1)\delta_*^2,
$$

with δ_* in Theorem 1.6. Next, from the decay property of the reactant Z, i.e., Lemma 2.11 and

(3.9), we know that

$$\int_0^x \int_{\chi(\tau)}^{g(\tau)} \rho\phi(T)Z\,dy\,d\tau - \int_0^x \big(g(\tau)-\chi(\tau)\big)\bar{\rho}\phi(\bar{T})\bar{Z}\,d\tau$$
$$= \int_0^x \int_{\chi(\tau)}^{g(\tau)} \rho\big(\phi(T)-\phi(\bar{T})\big)Z\,dy\,d\tau + \int_0^x \int_{\chi(\tau)}^{g(\tau)} (\rho-\bar{\rho})\phi(\bar{T})(Z-\bar{Z})\,dy\,d\tau$$
$$= O(1)\delta_*^2.$$

Therefore, we only need to estimate the last two terms in the right hand side of (4.29). To do that, we will carefully derive several estimates on the approximate strong contact discontinuity. Using the notations in the proof of Proposition 3.5, we define $Q_{h,\theta}(\Lambda)$ based on the location of Λ,

$$Q_{h,\theta}(\Lambda) = \begin{cases} Q(\Lambda) & \text{for Case 1,} \\ |\omega_k| + \sum_{i=2}^5 |\alpha_i| & \text{for Case 2,} \\ \sum_{i=1}^4 |\beta_i| & \text{for Case 3.1,} \\ \sum_{i=2}^5 |\alpha_i| & \text{for Case 3.2.} \end{cases} \tag{4.32}$$

Let $\Lambda_b = \cup_{k=1}^{+\infty} \Lambda_{k,0}$, where $\Lambda_{k,0}$ is the diamond centered at P_k. Let $L_{h,\theta}^b(\Lambda_b)$ be the summation of the strengths of the 1-waves leaving Λ_b.

Similarly, let $\Lambda_c = \cup_{k=1}^{+\infty} \Lambda_{k,n_k}$, where Λ_{k,n_k} is the diamond covering the strong contact discontinuity. Let $L_{h,\theta}^c(\Lambda_c)$ be the summation of the strengths of the 5-waves leaving Λ_c. Then, by (2.29), (2.32), and (3.10), we have

Lemma 4.4 *There exists a constant M, independent of $U_{h,\theta}$, θ, and h, such that*

$$\sum_{\Lambda} Q_{h,\theta}(\Lambda) \leqslant M, \quad L_{h,\theta}^b(\Lambda_b) \leqslant M, \quad L_{h,\theta}^c(\Lambda_c) \leqslant M. \tag{4.33}$$

where the summation is over all the diamonds Λ.

Next, let $\theta \in \prod_{k=1}^{\infty}(-1,1)\backslash\mathscr{N}$ be equidistributed, then we will prove the following lemma.

Lemma 4.5 *There exists a positive constant C, such that*

$$\int_0^{+\infty} T.V.\left\{\Big(\frac{v(\tau,\cdot)}{u(\tau,\cdot)},\ p(\tau,\cdot)\Big)\Big|_{(\chi(\tau),g(\tau))}\right\} d\tau \leqslant C\delta_*.$$

Proof. Since the velocity ratio $v_{h,\theta}/u_{h,\theta}$ and the pressure $p_{h,\theta}$ are invariant across the contact discontinuity, we only need to estimate the strengths of the weak 1-wave and the weak 5-wave. As in [27], we denote by $dQ_{h,\theta}$ the measure assigning to $Q_{h,\theta}(\Lambda)$, and by $dL_{h,\theta}^b$ and $dL_{h,\theta}^c$ the measure assigning to $L_{h,\theta}^b(\Lambda_b)$ and $L_{h,\theta}^c(\Lambda_c)$, respectively.

As shown in Fig. 10, let the line $x = X_{k-1}^*$ intersect $\partial\Omega$ and $y = \chi(x)$ at (X_{k-1}^*, Y_{k-1}^*) and $(X_{k-1}^*, \hat{Y}_{k-1}^*)$ respectively.

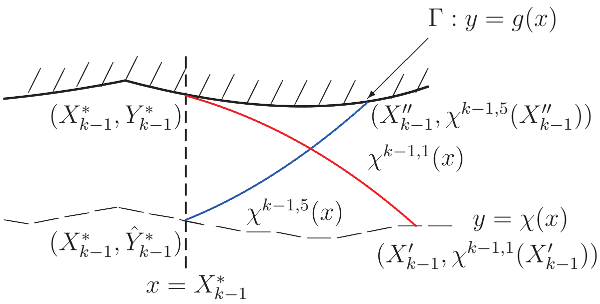

FIG. 10. Generalized characteristics in $U_{h,\theta}$

Let $y = \chi^{k-1,1}(x)$ be the maximal 1-generalized characteristics in $U_{h,\theta}$ emanating from the point (X^*_{k-1}, Y^*_{k-1}), and let $y = \chi^{k-1,5}(x)$ be the minimum 5-generalized characteristics in U_h emanating from the point $(X^*_{k-1}, \hat{Y}^*_{k-1})$. Moreover, let $y = \chi^{k-1,1}(x)$ and $y = \chi^{k-1,5}(x)$ intersect $y = \chi(x)$ and $\partial\Omega$ at $(X'_{k-1}, \chi^{k-1,1}(X'_{k-1}))$ and $(X''_{k-1}, \chi^{k-1,5}(X''_{k-1}))$ respectively for some X'_{k-1} and X''_{k-1}. Thus, by Lemma 4.3, there exists a constant $X^* > 0$, independent of X^*_{k-1}, such that $X^*_{k-1} + X^*$ is greater than X'_{k-1} and X''_{k-1}. Then we get a sequence $\{X^*_k\}_{k=0}^{\infty}$ by setting $X^*_k = X^*_{k-1} + X^*$.

We denote by Ω^*_{k-1} the domain with the boundaries that $x = X^*_{k-1}$, $x = X^*_k$, $\partial\Omega_h$, and $y = \chi_{h,\theta}(x)$. Let $\tilde{L}_{1,h}(X-)$ (or $L_{1,h}(X-)$) be the summation of all the strength of the weak 1-waves after (or before) the reaction step on the line $x = X$. Obviously for $ih < X \leqslant (i+1)h$,

$$\tilde{L}_{1,h}(X-) - L_{1,h}(X-) \leqslant Me^{-lih}h\|Z_0\|_\infty.$$

Then by the equations that the approximate solutions satisfy, we can deduce in the same way as the one in [27, 34] that on the line $x = X^*_k$, if h_j is sufficiently small, then

$$\tilde{L}_{1,h_j}(X^*_k-) = O(1)\Big(dL^b_{h_j,\theta}(\Lambda^*_{b,k-1}) + dQ_{h_j,\theta}(\Lambda^*_{k-1}) + \big(e^{-lX^*_{k-1}} - e^{-lX^*_k}\big)\|Z_0\|_\infty\Big),$$

where $\Lambda^*_{b,k-1}$ consists of the diamonds covering $\partial\Omega^*_{k-1} \cap \partial\Omega_h$, Λ^*_{k-1} consists of the diamonds in the interior of Ω^*_{k-1}, and the bound of $O(1)$ is independent of U_{h_j} and h_j.

Similarly, let $\tilde{L}_{5,h_j}(X-)$ stand for the summation of all the strength of the weak 5-waves on the line $x = X$ after the reaction step, then on the line $x = X^*_k$, if h_j is sufficiently small, then

$$\tilde{L}_{5,h_j}(X^*_k-) = O(1)\Big(dL^c_{h_j,\theta}(\Lambda^*_{c,k-1}) + dQ_{h_j,\theta}(\Lambda^*_{k-1}) + \big(e^{-lX^*_{k-1}} - e^{-lX^*_k}\big)\|Z_0\|_\infty\Big),$$

where $\Lambda^*_{c,k-1}$ consists of the diamonds covering the strong contact discontinuity $y = \chi_{h,\theta}(x)$ in Ω^*_{k-1}.

Next, for $x \in (X^*_{k-1}, X^*_k)$, it follows from the local estimates in Section 2 that

$$\begin{aligned}\tilde{L}_{1,h_j}(x-) = O(1)\Big(&\tilde{L}_{1,h_j}(X^*_{k-1}+) + dL^b_{h_j,\theta}(\Lambda^*_{b,k-1}) + dQ_{h_j,\theta}(\Lambda^*_{k-1}) \\ &+ \big(e^{-lX^*_{k-1}} - e^{-lX^*_k}\big)\|Z_0\|_\infty\Big),\end{aligned}$$

and

$$\tilde{L}_{5,h_j}(x-) = O(1)\Big(\tilde{L}_{5,h_j}(X^*_{k-1}+) + dL^c_{h_j,\theta}(\Lambda^*_{c,k-1}) + dQ_{h_j,\theta}(\Lambda^*_{k-1}) + \big(e^{-lX^*_{k-1}} - e^{-lX^*_k}\big)\|Z_0\|_\infty\Big).$$

Therefore, in domain Ω^*_{k-1}, we have

$$\begin{aligned}
&\int_{X^*_{k-1}}^{X^*_k} T.V.\left\{\Big(\frac{v_{h_j}(\tau,\cdot)}{u_{h_j}(\tau,\cdot)}, p_{h_j}(\tau,\cdot)\Big)\Big|_{(\chi_{i-1},g_{i-1})}\right\} d\tau \\
&\quad\leqslant O(1)(X^*_k - X^*_{k-1}) \max_{x\in[X^*_{k-1},X^*_k]} \big(\tilde{L}_{1,h_j}(x-) + \tilde{L}_{5,h_j}(x-)\big) \\
&\quad\leqslant O(1)X^*\Big(\tilde{L}_{1,h_j}(X^*_{k-1}-) + \tilde{L}_{5,h_j}(X^*_{k-1}-) + dL^b_{h_j,\theta}(\Lambda^*_{b,k-1}) + dL^c_{h_j,\theta}(\Lambda^*_{c,k-1}) \\
&\qquad + dQ_{h_j,\theta}(\Lambda^*_{k-1}) + \big(e^{-lX^*_{k-1}} - e^{-lX^*_k}\big)\|Z_0\|_\infty\Big).
\end{aligned}$$

Then by (4.33), we complete the proof. □

Now, by applying Theorem 1.4 and Lemma 4.2 and passing the limit $h_j \to 0$, we obtain that the equations satisfied by the integral average of the weak solution of (1.1) are

$$\big(g(x) - \chi(x)\big)\bar{\rho}\bar{u} = -y^{(0)}\bar{\rho}_0\bar{u}_0 + O(1)\delta^2_*, \tag{4.34}$$

$$\big(g(x) - \chi(x)\big)(\bar{\rho}\bar{u}^2 + \bar{p}) = -y^{(0)}(\bar{\rho}_0\bar{u}_0^2 + \bar{p}_0) + \int_0^x \big(g'(\tau) - \chi'(\tau)\big)\bar{p}\,d\tau + O(1)\delta^2_*, \tag{4.35}$$

$$\begin{aligned}
&\big(g(x) - \chi(x)\big)\bar{\rho}\bar{u}\Big(\frac{\gamma\bar{p}}{(\gamma-1)\bar{\rho}} + \frac{1}{2}\bar{u}^2\Big) \\
&\quad = -y^{(0)}\bar{\rho}_0\bar{u}_0\Big(\frac{\gamma\bar{p}_0}{(\gamma-1)\bar{\rho}_0} + \frac{1}{2}\bar{u}_0^2\Big) + q_0\int_0^x \big(g(\tau) - \chi(\tau)\big)\bar{\rho}\phi(\bar{T})\bar{Z}\,d\tau + O(1)\delta^2_*,
\end{aligned} \tag{4.36}$$

and

$$\big(g(x) - \chi(x)\big)(\bar{\rho}\bar{u}\bar{Z}) = -y^{(0)}\bar{\rho}_0\bar{u}_0\bar{Z}_0 - \int_0^x \big(g(\tau) - \chi(\tau)\big)\bar{\rho}\phi(\bar{T})\bar{Z}\,d\tau + O(1)\delta^2_*. \tag{4.37}$$

4.4 *Proof of Theorem* 1.6

Finally, we can show Theorem 1.6 now.

Proof. Let $A(x) = g(x) - \chi(x)$, and let $A(0) = -y^{(0)}$, then equations (4.34)–(4.37) become

$$\begin{cases}
\bar{\rho}\bar{u}A(x) = \bar{\rho}_0\bar{u}_0A(0) + O(1)\delta^2_*, \\
(\bar{\rho}\bar{u}^2 + \bar{p})A(x) = (\bar{\rho}_0\bar{u}_0^2 + \bar{p}_0)A(0) + \int_0^x A'(\tau)\bar{p}\,d\tau + O(1)\delta^2_*, \\
\big(\frac{\gamma\bar{p}}{(\gamma-1)\bar{\rho}} + \frac{1}{2}\bar{u}^2\big)\bar{\rho}\bar{u}A(x) = \big(\frac{\gamma\bar{p}_0}{(\gamma-1)\bar{\rho}_0} + \frac{1}{2}\bar{u}_0^2\big)\bar{\rho}_0\bar{u}_0A(0) + q_0\int_0^x A(\tau)\bar{\rho}\phi(\bar{T})\bar{Z}\,d\tau + O(1)\delta^2_*, \\
\bar{\rho}\bar{u}\bar{Z}A(x) = \bar{\rho}_0\bar{u}_0\bar{Z}_0A(0) - \int_0^x A(\tau)\bar{\rho}\phi(\bar{T})\bar{Z}\,d\tau + O(1)\delta^2_*.
\end{cases} \tag{4.38}$$

On the other hand, by Lemma 4.1 and Lemma 4.5, system (4.1) admits a unique solution $U_A(x) = (\rho_A, u_A, p_A, Z_A)^\top$ satisfying (4.2).

By the straightforward calculation from the fourth equation of (4.38), we have that

$$\bar{Z} = \bar{Z}_0 \exp(-\frac{1}{\bar{\rho}_0\bar{u}_0 A(0)}\int_0^x A(\tau)\bar{\rho}\phi(\bar{T})d\tau) + O(1)\delta_*^2.$$

Similarly, from the fourth equation of (4.1), we have

$$Z_A = \bar{Z}_0 \exp(-\frac{1}{\bar{\rho}_0\bar{u}_0 A(0)}\int_0^x A(\tau)\rho_A\phi(T_A)d\tau).$$

Then

$$|\bar{Z} - Z_A| \leqslant O(1)\delta_0 \max_{0\leqslant\tau\leqslant x}(|\bar{\rho}-\rho_A| + |\bar{T}-T_A|) + O(1)\delta_*^2.$$

Next, from the first three equations of the two systems (4.1) and (4.38), we have

$$\begin{cases} (\bar{\rho}-\rho_A)\bar{u}A(x) + \rho_A(\bar{u}-u_A)A(x) = O(1)\delta_*^2, \\ (\bar{\rho}-\rho_A)\bar{u}^2A(x) + \rho_A(\bar{u}^2-u_A^2)A(x) + (\bar{p}-p_A)A(x) = \int_0^x A'(\tau)(\bar{p}-p_A)d\tau + O(1)\delta_*^2, \\ \frac{\gamma}{\gamma-1}(\bar{p}-p_A)\bar{u}A(x) + \frac{\gamma}{\gamma-1}p_A(\bar{u}-u_A)A(x) + \frac{1}{2}(\bar{\rho}-\rho_A)\bar{u}^3A(x) + \frac{1}{2}\rho_A(\bar{u}^3-u_A^3)A(x) \\ \quad = q_0\int_0^x A(\tau)(\bar{\rho}\phi(\bar{T})\bar{Z} - \rho_A\phi(T_A)Z_A)d\tau + O(1)\delta_*^2. \end{cases}$$

From Lemma 4.5, we easily have the following fact that

$$\Big|\int_0^x A'(\tau)(\bar{p}-p_A)d\tau\Big| \leqslant O(1)\delta_* \max_{0\leqslant\tau\leqslant x}|\bar{p}-p_A|,$$

and from Lemma 4.3 and the estimates on the error terms in Section 4.3, we also have that

$$\begin{aligned} &\Big|\int_0^x A(\tau)\big(\bar{\rho}\phi(\bar{T})\bar{Z} - \rho_A\phi(T_A)Z_A\big)d\tau\Big| \\ &\qquad \leqslant A(x)\big(\rho_A u_A Z_A - \bar{\rho}\bar{u}\bar{Z}\big) + O(1)\delta_*^2, \\ &\qquad \leqslant O(1)\delta_* \max_{0\leqslant\tau\leqslant x}\big(|\bar{\rho}-\rho_A| + |\bar{T}-T_A| + |\bar{u}-u_A|\big) + O(1)\delta_*^2. \end{aligned}$$

Therefore, it follows from the implicit function theorem that there exists a constant $C > 0$, such that

$$\max_{x\geqslant 0}|\bar{U} - U_A| \leqslant C\delta_*^2,$$

for sufficiently small δ_*. This completes the proof. □

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