Asymptotic stability of local Helfrich minimizers

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We show that local minimizers of the Canham–Helfrich energy are asymptotically stable with respect to a model for relaxational fluid vesicle dynamics that we already studied in previous papers ([13, 14]). The proof is based on a Łojasiewicz–Simon inequality.

Introduction

In [13, 14] we started the analysis of a basic model of fluid vesicle dynamics. In this model we describe the evolution of biological vesicles by considering a homogeneous, Newtonian surface fluid ([2, 23]) subject to suitable elastic stresses that is immersed in a homogeneous, Newtonian bulk fluid. For a detailed introduction to the physics and mathematics of fluid vesicles we refer the reader to [13, 14] and the references therein. There we showed that for most applications one can safely neglect inertial forces and, hence, restrict the model to purely relaxational dynamics. In this case the model takes the form

$$
\begin{aligned}
\operatorname{div} S &= 0 && \text{in } \Omega \setminus \Gamma_t, \\
\operatorname{div} u &= 0 && \text{in } \Omega \setminus \Gamma_t, \\
\operatorname{Div} {}^f T + [\![S]\!] \nu_t &= -\operatorname{Div} {}^e T && \text{on } \Gamma_t, \\
\operatorname{Div} u &= 0 && \text{on } \Gamma_t, \\
u &= 0 && \text{on } \partial\Omega, \\
\Gamma_t|_{t=0} &= \Gamma_0.
\end{aligned}
\tag{1}
$$

Here, Ω is a smooth bounded domain in $\mathbb{R}^3$ containing a closed moving vesicle Γ_t, ν_t is the outer unit normal on Γ_t, and u is the velocity of the bulk fluid in $\Omega \setminus \Gamma_t$ and the velocity of the vesicle on Γ_t; this velocity field is assumed to be continuous across Γ_t. Furthermore, $S = 2\mu_b Du - \pi I$ is the Newtonian bulk stress tensor with the constant dynamic viscosity μ_b of the bulk fluid, the symmetric part Du of the gradient of u, and the bulk pressure π, $[\![S]\!]$ is the jump of the bulk stress tensor across the membrane (subtracting the outer limit from the inner limit), Div is the surface divergence (see below), and $T = {}^f T + {}^e T$ is the surface stress tensor which is composed of a fluid

part $^f T$ and an elastic part $^e T$. In coordinates we have $^f T^i_\alpha = {}^f \tilde{T}^\beta_\alpha \, \partial^i_\beta$ with (see [2, 13, 14, 23])

$$^f \tilde{T}^\beta_\alpha = -q \, \delta^\beta_\alpha + 2\mu \, (\mathfrak{D}u)^\beta_\alpha = -q \, \delta^\beta_\alpha + \mu \, g^{\beta\gamma} (v_{\alpha;\gamma} + v_{\gamma;\alpha} - 2w \, k_{\alpha\gamma})$$

and

$$^h T^i_\alpha = \kappa \left((H - C_0)^2/2 \, \partial^i_\alpha - (H - C_0) \, k^\beta_\alpha \, \partial^i_\beta - (H - C_0)_{,\alpha} v^i_t \right).$$

Here, q is the surface pressure acting as a Lagrange multiplier with respect to the constraint $\text{Div}\, u = 0$, μ is the constant dynamic viscosity of the surface fluid, $\mathfrak{D}u$ is the *surface rate-of-strain tensor*, k, H, and K are the second fundamental form, twice the mean curvature, and the Gauss curvature of Γ_t, respectively, κ is the bending rigidity, C_0 is the spontaneous curvature, ∂_α denotes the α-th coordinate vector field, and the semicolon denotes covariant differentiation while the comma indicates usual partial differentiation. Furthermore, on Γ_t we decomposed the function $u = v + w \, v_t$ into its tangential and its normal part. Throughout the paper, Latin indices refer to Cartesian coordinates in $\mathbb{R}^3$ while Greek indices refer to arbitrary coordinates on Γ_t. In particular, we note that the surface stress tensors are instances of *hybrid tensor fields* ([2, 23]) taking a tangential direction and returning a force density that is, in general, not tangential. The surface divergences for the non-tangential vector field u and the hybrid tensor field T can be written as

$$\text{Div}\, u = g^{\alpha\beta} \langle \partial_\alpha u, \partial_\beta \rangle_e,$$
$$(\text{Div}\, T)^i = g^{\alpha\beta} T^i_{\alpha;\beta},$$

where g denotes the Riemannian metric on Γ_t induced by the Euclidean metric e in $\mathbb{R}^3$, and the semicolon denotes the corresponding covariant differentiation of the covectors $(T^i_\alpha)_{\alpha=1,2}$ (for fixed i). We showed in [14] that

$$\begin{aligned}
\text{Div}\, u = &\, \text{div}_g \, v - w \, H, \\
\text{Div}\, {}^f T = &\, -\text{grad}_g \, q - q \, H v_t + \mu \left(\Delta_g v + \text{grad}_g (w \, H) + K v - 2 \, \text{div}_g (w \, k) \right) \\
&\, + 2\mu \left(\langle \nabla^g v, k \rangle_g - w \, (H^2 - 2K) \right) v_t \\
\text{Div}\, {}^e T = &\, -\kappa \left(\Delta_g H + H(H^2/2 - 2K) + C_0(2K - HC_0/2) \right) v_t.
\end{aligned} \tag{2}$$

Here, grad_g, div_g, ∇^g, Δ_g denote the differential operators (acting on tangential tensor fields) corresponding to the metric g, and, with a slight abuse of notation, we write $\langle \nabla^g v, k \rangle_g$ for the contraction of the tensor fields $\nabla^g v$ and k using g.

At first sight, the basic structure of our system (1) might not be so clear. Note that $\text{Div}\, {}^e T$ can be computed from Γ_t alone. Hence, we have to solve the Stokes-type system defined by the left hand side of (1) with $-\text{Div}\, {}^e T$ as a right hand side for the fluid velocity u. Then, the normal part w of u on Γ_t tells us how the vesicle will move in the next instant. Furthermore, it is easy to conclude from $(1)_{2,4}$ that area and the enclosed volume of each connected component Γ^i_t of Γ_t are preserved under this evolution; see [14]. Hence, the phase space N of our system consists of embedded surfaces $\Gamma \subset \Omega$ with a fixed number k of connected components Γ^i and with fixed area and enclosed volume of each Γ^i. Moreover, we showed in [14] that (1) can be formulated as a gradient flow of the *Canham–Helfrich energy* (see [3, 9, 10])

$$F(\Gamma) = \int_\Gamma \frac{\kappa}{2} (H - C_0)^2 \, dA$$

with respect to a suitably defined Riemannian metric on N. This observation will be particularly important for the present paper. For this reason we will repeat the details. For $\Gamma \in N$, the tangent space $T_\Gamma N$ can be identified with the space of scalar fields w on Γ such that the linearized constraints

$$\int_{\Gamma^i} w \, dA = 0 \quad \text{and} \quad \int_{\Gamma^i} w \, H \, dA = 0 \tag{3}$$

hold for all $i = 1, \ldots, m$. For $w \in T_\Gamma N$, consider the system

$$\begin{aligned}
\operatorname{div} S &= 0 & &\text{in } \Omega \setminus \Gamma, \\
\operatorname{div} u &= 0 & &\text{in } \Omega \setminus \Gamma, \\
P_\Gamma \big(\operatorname{Div}{}^f T + [\![S]\!]v \big) &= 0 & &\text{on } \Gamma, \\
\operatorname{Div} u &= 0 & &\text{on } \Gamma, \\
u \cdot v &= w & &\text{on } \Gamma, \\
u &= 0 & &\text{on } \partial\Omega.
\end{aligned} \tag{4}$$

Here, P_Γ denotes the field of orthogonal projections onto the tangent spaces of Γ. Note that the conditions (3) are necessary for the solvability of these equations, due to the incompressibility constraints. For $w_1, w_2 \in T_\Gamma N$, define the Riemannian metric on N associated with fluid vesicle dynamics by

$$\langle w_1, w_2 \rangle_V := 2\mu_b \int_{\Omega \setminus \Gamma} \langle Du_1, Du_2 \rangle_e \, dx + 2\mu \int_\Gamma \langle \mathfrak{D}u_1, \mathfrak{D}u_2 \rangle_g \, dA, \tag{5}$$

where u_1, u_2 solve the system (4) with data w_1, w_2. Note that the length of a curve in N endowed with this metric is given by the energy dissipated during the corresponding forced deformation of the membrane. The representation of $-dF_\Gamma$ with respect to the metric (5) is given by $[u]_\Gamma \cdot v$, where u solves (1) and where we use the notation $[u]_\Gamma$ to emphasize that we are taking the trace of u on Γ. Indeed, for all $w \in T_\Gamma N$ and corresponding solutions $\tilde{u}$ of (4) we have

$$\begin{aligned}
\langle u \cdot v, w \rangle_V &= 2\mu_b \int_{\Omega \setminus \Gamma} \langle Du, D\tilde{u} \rangle_e \, dx + 2\mu \int_\Gamma \langle \mathfrak{D}u, \mathfrak{D}\tilde{u} \rangle_g \, dA \\
&= -\int_\Gamma \langle [\![S]\!]v, \tilde{u} \rangle_e \, dA - \int_\Gamma \langle \operatorname{Div}{}^f T, \tilde{u} \rangle_e \, dA = \int_\Gamma \langle \operatorname{Div}{}^e T, \tilde{u} \rangle_e \, dA \\
&= -\int_\Gamma \operatorname{grad}_{L_2} F_\Gamma \, w \, dA = -dF_\Gamma(w).
\end{aligned} \tag{6}$$

Here, S and ${}^f T$ denote the stress tensors with respect to u, and we used integration by parts for the second identity (see [14]), (1)$_3$ for the third identity, and (2)$_3$ for fourth identity. We conclude that, indeed, (1) is the gradient flow of the Canham–Helfrich energy on N endowed with the Riemannian metric (5). In particular, the energy F is a strict Lyapunov functional, and, along the flow,

$$\frac{d}{dt} F(\Gamma_t) = dF_{\Gamma_t}(u \cdot v) = -\langle u \cdot v, u \cdot v \rangle_V = -2\mu_b \int_{\Omega \setminus \Gamma_t} |Du|_e^2 \, dx - 2\mu \int_{\Gamma_t} |\mathfrak{D}u|_g^2 \, dA.$$

In [14] we concluded from this identity that the equilibria of (1) must satisfy the well-known Helfrich equation

$$\operatorname{grad}_{L_2} F_\Gamma + [\![\pi]\!] + q \, H = 0 \tag{7}$$

with the pressure jump and the surface pressure acting as Lagrange multipliers with respect to the volume and area constraints.

We refer the reader to the introduction in [14] for an overview of what is known concerning equilibrium configurations and dynamics of fluid vesicles from a physical point of view on the one hand and from the point of view of a rigorous mathematical analysis on the other hand. We only remark that in [14] we presented a thorough L_2-analysis of the Stokes-type systems defined by the left hand sides of (1) and (4) that takes into account geometric variations of the membrane, while in [13] we proved a local well-posedness result for the dynamical system (1); both results will turn out to be fundamental ingredients in the present article. Here, we will show that surfaces that minimize the Canham–Helfrich energy locally are asymptotically stable, that is, solutions starting near such a minimizer Γ_0 exist for all times, remain nearby, and converge to a possibly different minimizer. The limit will in general be different from Γ_0 because the Canham–Helfrich energy admits continuous symmetries, and the equilibria therefore constitute a finite-dimensional manifold in phase space. In fact, these symmetries do not only include the rather trivial Euclidean group but, in the case of vanishing spontaneous curvature and higher genus, also *special conformal transformations*; see [24] and the references therein.[1] For parabolic (that is, purely relaxational) systems, stability usually follows in a more or less straightforward manner from the well-known principle of linearized stability provided that the spectrum of the linearization is strictly negative; see for instance [1, 17]. However, due to the symmetries mentioned above, in our case the kernel of the linearization will be non-trivial; this is a typical situation in geometric problems. One can deal with this difficulty by center manifold theory (which is technical; see for instance [7, 8, 11, 17]) or by the *generalized principle of linearized stability* (see [21, 22]). For the latter, however, one needs to know quite a lot about the equilibria and the symmetry group; more precisely, one needs to assure that the linearization's kernel not only contains the tangent space to the manifold of equilibria (which is always true) but actually coincides with it. Usually, this is shown by direct computations. Proving this in our case is difficult for two reasons. Firstly, as mentioned above, the symmetry group can be rather complicated, and secondly, almost all local minimizers have a highly non-trivial configuration; in fact, while the round sphere is a solution of (7) for any choice of C_0, it is the only known solution of spherical topology for $C_0 = 0$ which admits an analytical expression. Fortunately, for gradient-type systems there exists a third method for proving stability, the Łojasiewicz inequality ([15, 16]) and its infinite-dimensional analogue, the Łojasiewicz–Simon inequality ([12, 25]). The coincidence of the tangent space to the manifold of equilibria and the linearization's kernel essentially says that transversely to the manifold of equilibria the energy grows quadratically; this leads, in fact, to exponential convergence.[2] However, as long as the energy grows to some (uniformly bounded) power in transverse directions, one at least has algebraic convergence; this essentially is the content of the Łojasiewicz inequality and its application to stability. In finite dimensions an analytic energy always grows to some power in transverse directions since otherwise it would be constant in this direction. Now, the essential step towards an application in infinite dimensions is to note that all the critical directions are contained in the linearization's kernel which in applications usually is finite-dimensional. An abstract result following such a Lyapunov–Schmidt type reduction which is sufficiently general for our purpose can be found in [5].

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[1] These symmetries can in fact be observed under the microscope where thermal excitation continuously drives the vesicle along the manifold of equilibria; this phenomenon is called conformal diffusion.

[2] It might be interesting to understand in general under what circumstances geometric energies grow quadratically in *all* transverse directions.

The present article is organized as follows. In Section 1 we prove a suitable Łojasiewicz–Simon inequality for our system by applying results from [5]. In Section 2 we will combine this inequality with the local well-posedness result from [13] and a quantitative form of parabolic regularization to prove asymptotic stability. In the appendix we derive the second variation of the Canham–Helfrich energy and the linearization of its L_2-gradient.

Before we proceed, let us fix some notation. Throughout the article, let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. For a closed surface $\Gamma \subset \Omega$ we write ν for outer unit normal, Γ^i, $i = 1, \ldots, m$, for the connected components of Γ, Ω^i for the open set enclosed by Γ^i, and we define

$$\Omega^0 := \Omega \setminus \left(\bigcup_{i=1}^m \Gamma^i \cup \Omega^i \right).$$

We denote by $[u]_\Gamma$ the trace of the bulk field u on Γ; however, when there is no danger of confusion we will sometimes omit the brackets. Sometimes we use the notation $u \cdot v$ instead of $\langle u, v \rangle_e$ for $u, v \in \mathbb{R}^3$. We denote by $r(a)$ generic tensor fields that are polynomial or analytic functions of their argument a. For tensor fields r_1 and r_2 we write $r_1 * r_2$ for any tensor field that depends in a bilinear way on r_1 and r_2, and we use the abbreviations $r * (r_1, \ldots, r_k) = r * r_1 + \ldots + r * r_k$ and $r^k = r * \ldots * r$ (with k factors on the right hand side). For $p \in (1, \infty)$, $k \in \mathbb{N}$, and $s \in \mathbb{R}_+ \setminus \mathbb{N}$ we denote by W_p^k the usual Sobolev spaces and by W_p^s the Sobolev-Slobodetskij spaces. For an arbitrary smooth, d-dimensional Riemannian manifold $(M, \tilde{e})$ the norm of the latter spaces is given by

$$\|T\|_{W_p^s(M)} = \|T\|_{W_p^k(M)} + |(\nabla^{\tilde{e}})^k T|_{W_p^{s-k}(M)},$$

where k is the largest integer smaller than s and

$$\left|(\nabla^{\tilde{e}})^k T\right|_{W_p^{s-k}(M)}^p := \int_M \int_M \frac{\left|(\nabla^{\tilde{e}})^k T(x) - (\nabla^{\tilde{e}})^k T(y)\right|_e^p}{d_{\tilde{e}}(x, y)^{d+(s-k)p}} \, dV_{\tilde{e}}(x) \, dV_{\tilde{e}}(y).$$

In this formula $d_{\tilde{e}}$ is the Riemannian distance function while $dV_{\tilde{e}}$ is the volume element corresponding to $\tilde{e}$. Finally, we define $H^s := W_2^s$.

1. Łojasiewicz–Simon Inequality

Let M be the (formal) manifold of closed, embedded surfaces of class $H^{7/2}$ contained in Ω. We introduce local coordinates near an arbitrary element of M by approximating it by a smooth $\Gamma \in M$ (see, for instance, [21]) and writing nearby surfaces as graphs over Γ. Let us make this more precise. We denote by S_α, $\alpha > 0$, the open set of points in Ω whose distance from Γ is less than α. It's a well-known fact from elementary differential geometry that there exists a maximal $\kappa_\Gamma > 0$ such that the mapping

$$\Lambda : \Gamma \times (-\kappa_\Gamma, \kappa_\Gamma) \to S_{\kappa_\Gamma}, \quad (x, d) \mapsto x + d\,\nu(x)$$

is a diffeomorphism. For $h \in U = U(\Gamma) := \{h \in H^{7/2}(\Gamma) \mid |h| < \kappa_\Gamma/2 \text{ on } \Gamma\}$, let

$$\varphi_h : \Gamma \to \mathbb{R}^3, \ x \mapsto x + h(x)\nu(x)$$

and $\Gamma_h := \varphi_h(\Gamma) \subset \mathbb{R}^3$. Then, the bijection

$$\Psi : U \to \Psi(V) \subset M, \quad h \mapsto \Gamma_h$$

defines a local chart. We consider the Canham–Helfrich energy F as a function on U, that is, for $h \in U$, we let

$$\tilde{F}(h) := (F \circ \Psi)(h) = \frac{\kappa}{2} \int_{\Gamma} (H(h) - C_0)^2 \sqrt{g(h)} \, dA.$$

Here and in the following, we denote by $H(h)$, $g_{\alpha\beta}(h)$, etc. the geometric quantities on Γ_h pulled back by φ_h and $\sqrt{g(h)} := \sqrt{\det(g_{\alpha\beta}(h))}$. Furthermore, for $i = 1, \ldots, m$ we define $\Gamma_h^i := \varphi_h(\Gamma^i)$,

$$A_i(h) := \int_{\Gamma_h^i} dA_h, \quad \text{and} \quad V_i(h) := \frac{1}{3} \int_{\Gamma_h^i} x \cdot \nu_h \, dA_h(x),$$

where ν_h is the outer unit normal and dA_h is the area element on Γ_h. Note that, by the divergence theorem, $V_i(h)$ is nothing but the volume of the set enclosed by Γ_h^i.

Lemma 1.1 *Let X, Y be smooth tangent vector fields on Γ. Then, the following functions are well-defined and analytic:*

  (i) $U \to H^{5/2}(\Gamma) : \quad h \mapsto g(h)(X, Y)$ and $h \mapsto \sqrt{g(h)}$,
  (ii) $U \to H^{3/2}(\Gamma) : \quad h \mapsto k(h)(X, Y)$, $h \mapsto H(h)$, and $h \mapsto K(h)$,
  (iii) $U \to \mathbb{R} : \quad h \mapsto A_i(h)$, $h \mapsto V_i(h)$ (for $i = 1, \ldots, m$), and $h \mapsto \tilde{F}(h)$,
  (iv) $U \to (H^{1/2}(\Gamma))' : \quad h \mapsto d\tilde{F}_h$.

Proof. We proved in Appendix B in [14] that

$$g_{\alpha\beta}(h) = r_{\alpha\beta}(h/\kappa_\Gamma, hk, \nabla h),$$
$$g^{\alpha\beta}(h) = r^{\alpha\beta}(h/\kappa_\Gamma, hk, \nabla h),$$
$$k_{\alpha\beta}(h) = r_{\alpha\beta}(h/\kappa_\Gamma, hk, \nabla h) * \left(k, \nabla h/\kappa_\Gamma, h\nabla^g k, (\nabla^g)^2 h\right).$$

Since $\det(g_{\alpha\beta}(h))$ is uniformly positive and $H^s(\Gamma)$ is an algebra for $s > 1$, claim (i) follows. From these considerations and $H(h) = g^{\alpha\beta}(h) k_{\alpha\beta}(h)$, $K(h) = \det(k_{\alpha\beta}(h))$ we infer that claim (ii) is true. Since the map

$$L_1(\Gamma) \to \mathbb{R}, \quad f \mapsto \int_{\Gamma} f \, dA$$

is linear and bounded, from (i) and (ii) we can deduce the first and the third claim in (iii). Furthermore, by a straightforward computation one can check that the normal ν_h to Γ_h is given by

$$\nu_h \circ \varphi_h = \frac{\nu - \text{grad}_{\bar{g}} \, h}{(1 + |\text{grad}_{\bar{g}} \, h|_g^2)^{1/2}} = r(hk, \nabla h) \tag{8}$$

with the metric $\bar{g}_{\alpha\beta} = g_{\alpha\beta} - 2hk_{\alpha\beta} + h^2 k_{\alpha\gamma} k_\beta^\gamma$. In view of

$$V_i(h) = \frac{1}{3} \int_{\Gamma_h^i} x \cdot \nu_h \, dA_h = \frac{1}{3} \int_{\Gamma^i} (y + h\nu) \cdot (\nu_h \circ \varphi_h) \sqrt{g(h)} \, dA(y),$$

this proves the second claim in (iii). Finally, note that for $w \in H^{7/2}(\Gamma)$ we have

$$d\Psi_h(w) = \nu_h \cdot \left((w\nu) \circ \varphi_h^{-1}\right)$$

and thus for sufficiently smooth $h \in U$

$$
\begin{aligned}
d\tilde{F}_h(w) &= dF_{\Gamma_h} d\Psi_h(w) \\
&= \kappa \int_\Gamma \Bigg(\Delta_{g(h)} H(h) + H(h)\Big((H(h)^2/2 - 2K(h)) \\
&\qquad\qquad + C_0\big(2K(h) - H(h)\,C_0/2\big)\Big)\Bigg)(\nu_h \circ \varphi_h)\cdot v\, w\, \sqrt{g(h)}\, dA.
\end{aligned}
\tag{9}
$$

In view of (i), (ii), (8), and the fact that $H^{3/2}(\Gamma)$ is an algebra, we see that the summands in (9) not involving the Laplacian give rise to an analytic map from U to $H^{3/2}(\Gamma)$. We proved in Appendix B in [14] that

$$
\begin{aligned}
\Delta_{g(h)} H(h) &= r(h/\kappa_\Gamma, hk, \nabla h) * (\nabla^g)^2 H(h) \\
&\quad + r(h/\kappa_\Gamma, hk, \nabla h) * \big(k, \nabla h/\kappa_\Gamma, h\nabla^g k, (\nabla^g)^2 h\big) * \nabla^g H(h).
\end{aligned}
\tag{10}
$$

Since

$$
H^{3/2}(\Gamma) \cdot H^{1/2}(\Gamma) \hookrightarrow H^{1/2}(\Gamma),
$$

that is, pointwise multiplication is continuous in the indicated function spaces, the expressions in (9) involving the second summand in (10) give rise to an analytic map from U to $H^{1/2}(\Gamma)$. Finally, concerning the expressions involving the first summand in (10), we note that, on the one hand, since

$$
H^{5/2}(\Gamma) \cdot L_2(\Gamma) \hookrightarrow L_2(\Gamma),
$$

the map

$$
H^{7/2}(\Gamma) \to \mathcal{B}(H^2(\Gamma), L_2(\Gamma)), \quad h \mapsto r(h/\kappa_\Gamma, hk, \nabla h) * (\nabla^g)^2
$$

is analytic. On the other hand, for scalar fields f integration by parts gives

$$
\begin{aligned}
\int_\Gamma &r(h/\kappa_\Gamma, hk, \nabla h) * (\nabla^g)^2 f\, w\, dA \\
&= \int_\Gamma r(h/\kappa_\Gamma, hk, \nabla h) * \nabla f * \nabla w\, dA \\
&\qquad + \int_\Gamma r(h/\kappa_\Gamma, hk, \nabla h) * \big(k, \nabla h/\kappa_\Gamma, h\nabla^g k, (\nabla^g)^2 h\big) * \nabla f\, w\, dA,
\end{aligned}
$$

proving that

$$
H^{7/2}(\Gamma) \to \mathcal{B}\big(H^1(\Gamma), (H^1(\Gamma))'\big), \quad h \mapsto r(h/\kappa_\Gamma, hk, \nabla h) * (\nabla^g)^2
$$

is also analytic. Thus, by interpolation, this map is analytic from $H^{7/2}(\Gamma)$ to $\mathcal{B}(H^{3/2}(\Gamma), (H^{1/2}(\Gamma))')$. This together with the second assertion in (ii) proves (iv). $\qquad\square$

REMARK 1.2 The preceding proof shows that for $h \in U$ the (formal) derivative of $d\Psi_h$ defines linear isomorphisms from $H^s(\Gamma)$ to $H^s(\Gamma_h)$ for all $s \in [0, 5/2]$ which are uniformly bounded in both directions for $h \in U$ being uniformly bounded in $H^{7/2}(\Gamma)$.

For $A = (A_1, \ldots, A_k)$, and $V = (V_1, \ldots, V_k)$ with $A_i, V_i > 0$, let $N = N_{A,V} \subset M$ denote the (formal) submanifold of closed, embedded surfaces of class $H^{7/2}$ contained in Ω which consist of k connected components Γ^i of fixed area A_i and fixed enclosed volume V_i. For fixed smooth $\Gamma \in N$, in a neighbourhood of Γ we can consider N as an analytic submanifold of $U(\Gamma)$. By Lemma 1.1 (iii), the map

$$f : U \to \mathfrak{R}_1 \times \ldots \times \mathfrak{R}_k, \quad h \mapsto \big(f_1(h), \ldots, f_k(h)\big),$$

where $f_i(h) = (A_i(h), V_i(h))$, $\mathfrak{R}_i = \mathbb{R}^2$ if Γ^i is not a round sphere and $f_i(h) = A_i(h)$, $\mathfrak{R}_i = \mathbb{R}$ else, is analytic, and its differential at $h = 0$ is surjective. Indeed, for $w \in H^{7/2}(\Gamma)$ we have

$$d(f_i)_{h=0}(w) = \left(\int_{\Gamma^i} w H \, dA, \int_{\Gamma^i} w \, dA \right)$$

if Γ_i is not a round sphere, and else

$$d(f_i)_{h=0}(w) = \int_{\Gamma_i} w \, dA;$$

note that in the first case the functions 1 and H are linearly independent. Thus, by the implicit function theorem (see [6], for instance), there exist a closed complement B of $T_\Gamma N := \ker df_{h=0}$ in $H^{7/2}(\Gamma)$, bounded open neighbourhoods $\tilde{U}$ and $\hat{U}$ of the origin in $T_\Gamma N$ and B, respectively, and an analytic function $\gamma : \tilde{U} \to \hat{U}$ such that $\gamma(0) = 0$, $d\gamma(0) = 0$, and

$$\{h \in \tilde{U} \times \hat{U} \mid f(h) = f(0)\} = \{w + \gamma(w) \mid w \in \tilde{U}\} =: G.$$

Concatenating the map $1 + \gamma : \tilde{U} \to G$, which is bianalytic, with the map $\Psi : U \to M$ yields local coordinates for N.

For $\Gamma \in N$ and $s \geqslant 0$ let

$$H_n^s(\Gamma) := \{w \in H^s(\Gamma) \quad \text{such that (3) holds}\}.$$

Furthermore, let $L_{2,n}(\Gamma) := H_n^0(\Gamma)$, $Y_\Gamma := H_n^{1/2}(\Gamma)$, and note that $T_\Gamma N = H_n^{7/2}(\Gamma)$.

REMARK 1.3 For $w_0 \in \tilde{U}$, let us have a closer look at the map

$$d(1 + \gamma)_{w_0} : T_\Gamma N \to T_{(1+\gamma)(w_0)} G,$$

which is a linear isomorphism. Let $b_1, \ldots, b_n$ be a basis for B. From the definition of the functional f it is not hard to see that there exist analytic maps $g_i : U \to H^{3/2}$, $i = 1, \ldots, n$, such that

$$T_{(1+\gamma)(w_0)} G = \ker df_{(1+\gamma)(w_0)} = Z_{w_0} \cap H^{7/2}(\Gamma)$$

for the L_2-orthogonal complement $Z_{w_0} = (\mathrm{span}\{g_1(w_0), \ldots, g_n(w_0)\})^\perp \subset L_2(\Gamma)$. Thus, we have

$$d(1 + \gamma)_{w_0} : w \mapsto w + a^i(w) b_i,$$

where the real coefficients a^i are determined by the linear system

$$a^i(w) \big(b_i, g_j(w_0)\big)_{L_2(\Gamma)} = -\big(w, g_j(w_0)\big)_{L_2(\Gamma)}$$

for $j = 1, \ldots, n$. The matrix on the left hand side is invertible, all coefficients depend analytically on w_0, and the right hand side as a function in w is a continuous functional on $L_1(\Gamma)$. In particular, for all $s \geqslant 0$, $d(1 + \gamma)$ can be extended to an analytic map

$$d(1 + \gamma) : \tilde{U} \to \mathcal{B}\big(H^s(\Gamma), H^s(\Gamma)\big),$$

and for all $w_0 \in \tilde{U}$

$$d(1 + \gamma)_{w_0} : H_n^s(\Gamma) \to Z_{w_0} \cap H^s(\Gamma)$$

is an isomorphism with uniformly bounded continuity constants. Furthermore, from Remark 1.2 we infer that $d\Psi_{(1+\gamma)(w_0)} : Z_{w_0} \cap H^s(\Gamma) \to H_n^s(\Gamma_{(1+\gamma)(w_0)})$, $s \in [0, 5/2]$, defines isomorphisms which are uniformly bounded in both directions for $w_0 \in \tilde{U}$.

Now we are ready to state one of the main results of this section.

Theorem 1.4 (Łojasiewicz–Simon inequality) *Let $\Gamma_0 \in N$ be a smooth stationary point for F in N, that is, $dF_{\Gamma_0} = 0$. Then there exists a neighbourhood $\tilde{U}$ of Γ_0 in N (in the above sense), a constant $c > 0$, and a number $\theta \in (0, 1/2]$ such that*

$$|F(\Gamma) - F(\Gamma_0)|^{1-\theta} \leqslant c\|dF_\Gamma\|_{Y_\Gamma'}$$

for all $\Gamma \in \tilde{U}$.

In order to prove the theorem, we have to analyze the second variation of F in N.

Proposition 1.5 *For smooth $\Gamma \in N$ the second variation $d^2 F_\Gamma : T_\Gamma N \to Y_\Gamma'$ is a Fredholm operator of index 0.*[3] *In particular, we have*

$$\operatorname{rg} d^2 F_\Gamma = (\ker d^2 F_\Gamma)^\perp \cap Y_\Gamma'. \tag{11}$$

Proof. In the appendix we show that the second variation has the form

$$d^2 F_\Gamma(w, \tilde{w}) = \kappa \int_\Gamma \big(\Delta_g w \, \Delta_g \tilde{w} + B w \, \tilde{w}\big) \, dA,$$

where $w, \tilde{w} \in T_\Gamma N$ and $B w = (a^{\alpha\beta} w_{,\beta})_{;\alpha} + b \, w$. The operator B obviously maps $T_\Gamma N$ compactly into Y_Γ'. Hence, it suffices to consider the biharmonic operator alone. By standard arguments based on L_2-theory for the Laplacian on Γ and Riesz' representation theorem, for some $\eta > 0$ the operator $\Delta_g^2 + \eta : H_n^2(\Gamma) \to (H_n^2(\Gamma))'$ is an isomorphism. From this and again L_2-theory for the Laplacian, we have that $\Delta_g^2 + \eta : H_n^4(\Gamma) \to (L_{2,n}(\Gamma))'$ is an isomorphism, too. Indeed, each element of $(L_{2,n}(\Gamma))'$ is an element of $(H_n^2(\Gamma))'$ and hence the image of some $H_n^2(\Gamma)$ function. However, $(\Delta_g^2 + \eta)u \in (L_{2,n}(\Gamma))'$ means that $(\Delta_g^2 + \eta)u \in L_{2,n}(\Gamma)$ modulo some linear combination of the functions 1 and H on each connected component of Γ, thus, $\Delta_g^2 u \in L_2(\Gamma)$. Now, L_2-theory tells us that, in fact, $u \in H_n^4(\Gamma)$, and hence, $\Delta_g^2 + \eta : H_n^4(\Gamma) \to (L_{2,n}(\Gamma))'$ is surjective. On the other hand, injectivity is obvious. Interpolating these results we obtain that $\Delta_g^2 + \eta : T_\Gamma N \to Y_\Gamma'$ is an isomorphism. Thus, by Fredholm's alternative, for $K := \eta(\Delta_g^2 + \eta)^{-1}$ we have that $\operatorname{id} - K$ is a Fredholm operator of index 0 on $T_\Gamma N$. However, it is easy to see that $\ker(\operatorname{id} - K) = \ker \Delta_g^2 \subset T_\Gamma N$

---

[3] The condition of smoothness can be weakened of course. It is assumed here only for simplicity.

and $(\Delta_g^2 + \eta)(\mathrm{rg}(\mathrm{id} - K)) = \mathrm{rg}\,\Delta_g^2 \subset Y_\Gamma'$. Hence, $\Delta_g^2 : TN_\Gamma \to Y_\Gamma'$ is Fredholm of index 0, and, thus, the same is true for $d^2 F_\Gamma$.

Finally, the symmetry of $d^2 F_\Gamma$ implies that

$$\mathrm{rg}\,d^2 F_\Gamma \subset (\ker d^2 F_\Gamma)^\perp \cap Y_\Gamma'. \tag{12}$$

But, by

$$\ker d^2 F_\Gamma \cap (\ker d^2 F_\Gamma)^\perp = \{0\} \quad \text{in } Y_\Gamma'$$

and the vanishing Fredholm index, the inclusion in (12) must be an identity, that is, (11) holds.   □

Proof of Theorem 1.4. We consider F as a function on $\tilde{U}$, that is, for $w \in \tilde{U}$, we let $\hat{F}(w) := \tilde{F}(w + \gamma(w))$. By Lemma 1.1 (iii), $\hat{F}$ is analytic. Furthermore, for $w_0 \in \tilde{U}$ and $w_1 \in T_\Gamma N$ we have

$$\begin{aligned}
d\hat{F}_{w_0}(w_1) &= d\tilde{F}_{(1+\gamma)(w_0)} d(1 + \gamma)_{w_0}(w_1) \\
&= dF_{\Gamma_{(1+\gamma)(w_0)}} d\Psi_{(1+\gamma)(w_0)} d(1 + \gamma)_{w_0}(w_1).
\end{aligned} \tag{13}$$

Thus, by Remark 1.3 it is sufficient to show that there exists a neighbourhood $\tilde{U} \subset T_{\Gamma_0} N$ of the origin and a number $\theta \in (0, 1/2]$ such that

$$\left| \hat{F}(w_0) - \hat{F}(0) \right|^{1-\theta} \leqslant c\, \|d\hat{F}_{w_0}\|_{Y_\Gamma'}$$

for all $w_0 \in \tilde{U}$.

We want to apply Corollary 3.11 in [5] with $X = V = T_\Gamma N$ and $Y = W = Y_\Gamma'$. From the first equality in (13), Remark 1.3, and Lemma 1.1 (iv), it follows that the map $\tilde{U} \to Y_\Gamma'$, $w_0 \mapsto d\tilde{F}_{w_0}$ is analytic. Moreover, since Γ is a stationary point for F in N and $d\gamma(0) = 0$, for $w_0, w_1 \in T_\Gamma N$ we have

$$d^2 \hat{F}_{w=0}(w_0, w_1) = d^2 F_\Gamma(w_0, w_1).$$

Thus, by Proposition 1.5, $\ker d^2 \hat{F}_{w=0}$ is finite-dimensional and

$$\mathrm{rg}\,d^2 \hat{F}_{w=0} = (\ker d^2 \hat{F}_{w=0})^\perp \cap Y_\Gamma'. \tag{14}$$

Let $\tilde{P} : Y_\Gamma \to \ker d^2 \hat{F}_{w=0}$ be a continuous projection and consider its restriction P to $T_\Gamma N$, that is, the projection $P : T_\Gamma N \to \ker d^2 \hat{F}_{w=0}$ along $\ker P \cap T_\Gamma N$. For $w \in T_\Gamma N$ and $y' \in Y_\Gamma'$ we have

$$\langle P'y', w\rangle_{Y_\Gamma} = \langle y', \tilde{P}w\rangle_{Y_\Gamma} \leqslant \|y'\|_{Y_\Gamma'} \|\tilde{P}w\|_{Y_\Gamma} \leqslant c\,\|y'\|_{Y_\Gamma'} \|w\|_{Y_\Gamma},$$

which, by denseness of $T_\Gamma N$ in Y_Γ, proves that P' leaves Y_Γ' invariant. Since $\ker P' = (\mathrm{rg}\,P)^\perp$, equation (14) takes the form

$$\mathrm{rg}\,d^2 \hat{F}_{w=0} = \ker P' \cap Y_\Gamma'.$$

Hence, we checked all assumptions in Corollary 3.11 in [5]. This finishes the proof.   □

2. Asymptotic stability

For $T > 0$ we define $I := (0, T)$. In [13] we proved the following theorem with the exception of (15) which expresses parabolic regularization in a quantitative form.

Theorem 2.1 *Let $\Gamma \subset \Omega$ be a smooth, closed surface that contains no round sphere. For all $p \in (3, \infty) \setminus \{4\}$ and sufficiently small $\epsilon > 0$ there exists a time $T > 0$ such that for all height functions $h_0 \in \bar{B}_\epsilon(0) \subset W_p^{5-4/p}(\Gamma)$ there exists a unique*

$$h \in W_p^1\big(I, W_p^{2-1/p}(\Gamma)\big) \cap L_p\big(I, W_p^{5-1/p}(\Gamma)\big)$$

with $\|h\|_{L_\infty(I \times \Gamma)} \leqslant \kappa_\Gamma/2$ as well as suitable hydrodynamic fields (u, π, q) such that $\Gamma_t = \Gamma_{h(t)}$ and (u, π, q) solve (1) *in the time interval I with initial value Γ_{h_0}. The map*

$$\bar{B}_\epsilon \subset W_p^{5-4/p}(\Gamma) \to W_p^1\big(I, W_p^{2-1/p}(\Gamma)\big) \cap L_p\big(I, W_p^{5-1/p}(\Gamma)\big), \quad h_0 \mapsto h$$

is Lipschitz continuous. Furthermore, for sufficiently small $\delta > 0$ and all $t' \in (0, T)$ there exists a constant $c > 0$ such that

$$\|h\|_{C([t', T], W_p^{5-4/p+\delta}(\Gamma))} \leqslant c. \tag{15}$$

For details concerning the hydrodynamic fields see [13]; alternatively, these can be constructed (in the L_2-scale) by applying Theorem 3.6 or Theorem 3.7 from [14].

If Γ consists only of round spheres, then (1) is uniquely solved by the constant-in-time solution with $\Gamma_t = \Gamma$, $u = 0$, and suitably chosen pressure functions; global well-posedness is trivial in this case. So far, we cannot prove the well-posedness of our system in the case that Γ contains both round spheres and non-spheres. The reason for this lies in the different degrees of gauge freedom of the pressure functions for round spheres on the one hand and non-spheres on the other hand; see [13].

Proof. In order to show (15) we have to repeat parts the proof of Theorem 3.1 in [13] in a time-weighted setting. We will only briefly sketch the procedure. Without further explanation we will use the notation from [13]. Let $p \in [2, \infty)$ and $\mu \in (1/p, 1)$ such that

$$2 + 3\mu - 4/p \in (2 - 1/p, 5 - 4/p)$$

is not a natural number.

First, we prove that the linearization

$$
\begin{aligned}
\mu_b \Delta u - \operatorname{grad} \pi &= f_1 && \text{in } \Omega \setminus \Gamma, \\
\operatorname{div} u &= f_2 && \text{in } \Omega \setminus \Gamma, \\
\mu\big(\Delta_g v + \operatorname{grad}_g(w\, H) + K\, v - 2 \operatorname{div}_g(w\, k)\big) & && \\
- \operatorname{grad}_g q + P_\Gamma [\![S]\!] v &= f_3^\top && \text{on } \Gamma, \\
2\mu\big(\langle \nabla^g v, k \rangle_g - w\,(H^2 - 2K)\big) - q\, H + [\![S]\!] v \cdot v - A h &= f_3^\perp && \text{on } \Gamma, \\
\operatorname{div}_g v - w\, H &= f_4 && \text{on } \Gamma, \\
u - v - w\, v &= f_5 && \text{on } \Gamma, \\
\partial_t h - w &= f_6 && \text{on } \Gamma
\end{aligned}
\tag{16}
$$

admits a unique solution (u, v, w, π, q, h) with

$$
\begin{aligned}
t^{1-\mu} u &\in L_p\big(I, W_p^2(\Omega \setminus \Gamma, \mathbb{R}^3) \cap W_p^1(\Omega, \mathbb{R}^3)\big), & t^{1-\mu} v &\in L_p\big(I, W_p^2(\Gamma, T\Gamma)\big), \\
t^{1-\mu} w &\in L_p\big(I, W_p^{2-1/p}(\Gamma)\big), & t^{1-\mu} \pi &\in L_p\big(I, W_p^1(\Omega \setminus \Gamma)\big), & t^{1-\mu} q &\in L_p\big(I, W_p^1(\Gamma)\big), \\
t^{1-\mu} h &\in L_p\big(I, W_p^{5-1/p}(\Gamma)\big), & t^{1-\mu} \partial_t h &\in L_p\big(I, W_p^{2-1/p}(\Gamma)\big)
\end{aligned}
$$

and $\int_\Omega \pi \, dx = 0$ provided that the data $(f_1, \ldots, f_6, h_0)$ satisfy

$$
\begin{aligned}
t^{1-\mu} f_1 &\in L_p\big(I, L_p(\Omega \setminus \Gamma, \mathbb{R}^3)\big), & t^{1-\mu} f_2 &\in L_p\big(I, W_p^1(\Omega \setminus \Gamma)\big), \\
t^{1-\mu} f_3^\top &\in L_p\big(I, L_p(\Gamma, T\Gamma)\big), & t^{1-\mu} f_3^\perp &\in L_p\big(I, W_p^{1-1/p}(\Gamma)\big), \\
t^{1-\mu} f_4 &\in L_p\big(I, W_p^1(\Gamma)\big), & t^{1-\mu} f_5, t^{1-\mu} f_6 &\in L_p\big(I, W_p^{2-1/p}(\Gamma, \mathbb{R}^3)\big), \\
h_0 &\in W_p^{2+3\mu-4/p}(\Gamma)
\end{aligned}
$$

with $\int_\Omega f_2 \, dx = 0$. To this end, we eliminate h_0 by choosing an extension $\tilde{h}$ such that $t^{1-\mu}\tilde{h} \in L_p(I, W_p^{5-1/p}(\Gamma))$ and $t^{1-\mu}\partial_t \tilde{h} \in L_p(I, W_p^{2-1/p}(\Gamma))$; see Proposition 3.1 in [20]. Next, we eliminate $(f_1, \ldots, f_5)$ by solving the stationary system

$$
\begin{aligned}
\mu_b \Delta u - \operatorname{grad} \pi &= f_1 & &\text{in } \Omega \setminus \Gamma, \\
\operatorname{div} u &= f_2 & &\text{in } \Omega \setminus \Gamma, \\
\mu\big(\Delta_g v + \operatorname{grad}_g (w \, H) + K \, v - 2\operatorname{div}_g (w \, k)\big) & & & \\
- \operatorname{grad}_g q + P_\Gamma [\![S]\!] v &= f_3^\top & &\text{on } \Gamma, \\
2\mu\big(\langle \nabla^g v, k \rangle_g - w \, (H^2 - 2K)\big) - q \, H + [\![S]\!] v \cdot v &= f_3^\perp & &\text{on } \Gamma, \\
\operatorname{div}_g v - w \, H &= f_4 & &\text{on } \Gamma, \\
u - v - w \, v &= f_5 & &\text{on } \Gamma
\end{aligned}
\tag{17}
$$

for almost all $t \in I$. In the proof of Theorem 3.2 in [13] we showed unique L_p-solvability of the principal linearization of this system in the double half-space; combining this result with standard localization and transformation techniques as well as the L_2-theory proved in [14], we easily obtain unique L_p-solvability for (17). Thus, it is sufficient to solve (16) for h with all data vanishing except for f_6. We can write this system in the form

$$
\partial_t h + L h = f_6, \quad h(0) = 0.
\tag{18}
$$

Here, $L : D(L) \to X$ with $X = W_p^{2-1/p}(\Gamma)$ and $D(L) = W_p^{5-1/p}(\Gamma)$ is the linear operator that maps $h \in D(L)$ to $w = [u]_\Gamma \cdot v \in X$, where u solves (17) with $f_3^\perp = Ah$ and all other data vanishing. This operator is closed as can be seen from the L_p-theory for (17) and for the Laplacian on Γ. Furthermore, we proved in [14] that for $f_6 \in L_p(I, X)$ equation (18) admits a unique solution $h \in L_p(I, D(L))$ with $\partial_t h \in L_p(I, X)$. From this, however, using a summation argument one easily deduces that the same assertion holds with $I = (0, \infty)$ if we replace L by $L + \lambda$ for a sufficiently large $\lambda > 0$. Now, from Theorem 2.4 in [20] we finally obtain the existence of a unique solution h of (18) such that $t^{1-\mu} h \in L_p(I, D(L))$ and $t^{1-\mu}\partial_t h \in L_p(I, X)$ provided that $t^{1-\mu} f_6 \in L_p(I, X)$.

Next, we have to repeat the contraction mapping argument from Section 4 in [13] in a time-weighted setting. Essentially, throughout the proof we simply replace the spaces $\mathbb{E}_p(T)$, $\mathbb{G}_p(T)$, and $\mathbb{F}_p(T)$ by the corresponding time-weighted spaces, which we denote by $\mathbb{E}_{p,\mu}(T)$, etc., and correspondingly the time-trace space $W_p^{5-4/p}(\Gamma)$ by $W_p^{2+3\mu-4/p}(\Gamma)$. However, we have to prove that for $p > 3$ and $\mu > 3/p$ the statement analogous to Lemma 4.1 in [13] holds. The analogue of assertion (i) in Lemma 4.1 follows from Proposition 3.1 in [20] and the existence of a bounded extension operator $\mathbb{E}_{p,\mu}(T) \to \mathbb{E}_{p,\mu}(\infty)$; the latter can be constructed quite simply via reflection. The analogue of assertion (ii) in Lemma 4.1 follows from Theorem 4.2 in [18]. Furthermore, noting

that for $\mu \geqslant 1/p + 2/3$ we have $W_p^{-1+3\mu-4/p}(\Gamma) \hookrightarrow W_p^{1-1/p}(\Gamma)$, the assertion analogous to Lemma 4.2 in [13] can be shown. Now, we can follow the proof of Theorem 1.1 in [13] line by line to see that the first part of Theorem 2.1 holds with $W_p^{5-4/p}(\Gamma)$ replaced by $W_p^{2+3\mu-4/p}(\Gamma)$ and $W_p^1(I, W_p^{2-1/p}(\Gamma)) \cap L_p(I, W_p^{5-1/p}(\Gamma))$ replaced by the corresponding time-weighted space.

Finally, let us fix a $p \in (3, \infty) \setminus \{4\}$. We choose $\tilde{p} \in (p, \infty)$ and $\mu \in (1/\tilde{p}, 1)$ such that the well-posedness result we just proved holds and such that the spatial regularities of the time-trace spaces of $\mathbb{E}_p(T)$ and $\mathbb{E}_{\tilde{p},\mu}(T)$ coincide, that is, $2 + 3\mu - 4/\tilde{p} = 5 - 4/p$. Thus, we can apply the well-posedness theorem in the time-weighted setting to all $h_0 \in \bar{B}_\epsilon \subset W_p^{5-4/p}(\Gamma)$ for sufficiently small ϵ and obtain a solution $h \in \mathbb{E}_p(T)$ such that

$$\|h\|_{\mathbb{E}_{\tilde{p},\mu}(T)} \leqslant c$$

for some constant $c > 0$. However, for all $t' \in (0, T)$ the space $\mathbb{E}_{\tilde{p},\mu}(T)$ obviously embeds into $C([t', T], W_{\tilde{p}}^{5-4/\tilde{p}}(\Gamma))$; this proves (15). □

Now, we can prove the asymptotic stability of local Helfrich minimizers.

Theorem 2.2 *Let $\Gamma \in N$ be a smooth local minimizer for F in N that contains no round sphere, and let $\tilde{U} \subset T_\Gamma N$ be the set from Theorem 1.4. For all $\sigma > 0$ there exists an $\epsilon > 0$ such that for all*

$$h_0 \in \left(\bar{B}_\epsilon(0) \cap (1 + \gamma)(\tilde{U})\right) \subset W_p^{5-4/p}(\Gamma)$$

the corresponding solution h from Theorem 2.1 exists for all times and satisfies $\|h(t)\|_{W_p^{5-4/p}(\Gamma)} \leqslant \sigma$ as well as

$$\|h(t) - h_\infty\|_{W_p^{5-4/p}(\Gamma)} \leqslant c\, t^{-\beta} \tag{19}$$

for all $t > 0$, constants $c, \beta > 0$, and some $h_\infty \in (1 + \gamma)(\tilde{U})$ with $F(\Gamma_{h_\infty}) = F(\Gamma)$.

Proof. The proof proceeds in two steps. First, we use Theorem 1.4 to show that if our solution exists for all times then its energy will converge to the local minimum and it will satisfy an arbitrarily small bound in a low norm. Then, combining these insights and parabolic regularization in a bootstrap argument we can prove that the solution will indeed exist for all times and converge in phase space.

By the Lipschitz continuity of the solution map from Theorem 2.1, we can choose ϵ so small that for all h_0 as in the assertion of the present theorem the corresponding solution h satisfies $\|h(t)\|_{W_p^{5-4/p}(\Gamma)} \leqslant \sigma$ for all $t \in \bar{I}$. Let $\Gamma_t := \Gamma_{h(t)}$. Let us fix an instant $t \in I$ and consider the space

$$X_{\Gamma_t} := \left\{u \in H_0^1(\Omega; \mathbb{R}^3) \,|\, \operatorname{div} u = 0 \text{ in } \Omega \setminus \Gamma_t, \quad \operatorname{Div} u = 0 \text{ on } \Gamma_t, \quad P_{\Gamma_t}[u]_{\Gamma_t} \in H^1(\Gamma_t; T\Gamma_t)\right\}$$

endowed with the canonical scalar product and with the bilinear form

$$B_{\Gamma_t}(u, \varphi) = 2\mu_b \int_{\Omega \setminus \Gamma_t} \langle Du, D\varphi \rangle_e \, dx + 2\mu \int_{\Gamma_t} \langle \mathfrak{D}u, \mathfrak{D}\varphi \rangle_g \, dA.$$

We saw in the proof of Theorem 3.11 in [14] that B_{Γ_t} defines a uniformly equivalent scalar product on X_{Γ_t} for $h(t)$ being uniformly bounded in $W_p^{5-4/p}(\Gamma)$ with $\|h(t)\|_{L_\infty(\Gamma)} \leqslant \kappa_\Gamma/2$. Also recall from [14] that the weak solution $u \in X_{\Gamma_t}$ of (1) is characterized by the equation $B_{\Gamma_t}(u, \varphi) =$

$-dF_{\Gamma_t}([\varphi]_{\Gamma_t} \cdot v_t)$ for all $\varphi \in X_{\Gamma_t}$; cf. (6). Thus, we have

$$-\frac{d}{dt}\big(F(\Gamma_t) - F(\Gamma)\big) = -dF_{\Gamma_t}\big([u]_{\Gamma_t} \cdot v_t\big) = B_{\Gamma_t}(u, u)$$

$$= \|dF_{\Gamma_t}\|_{B'_{\Gamma_t}}^2 \geqslant \frac{1}{c}\|dF_{\Gamma_t}\|_{X'_{\Gamma_t}}^2 \geqslant \frac{1}{c}\|dF_{\Gamma_t}\|_{Y'_{\Gamma_t}}^2 \tag{20}$$

for some uniform constant $c > 0$. Here, the third identity reflects the fact that Riesz' isomorphism is an isometry, for the first estimate we employed the inequality $\|\cdot\|_{B_{\Gamma_t}} \leqslant c\|\cdot\|_{X_{\Gamma_t}}$, and the second estimate follows from the inequality $\|\tilde{u}\|_{X_{\Gamma_t}} \leqslant c\|\tilde{w}\|_{Y_{\Gamma_t}}$, where $\tilde{u} \in X_{\Gamma_t}$ is a suitable extension of $\tilde{w} \in Y_{\Gamma_t}$, with a uniform constant $c > 0$ (this follows from Theorem 3.12 in [14]). Without restriction we can choose σ so small that Theorem 1.4 can be applied, and thus we infer from (20) that

$$\frac{d}{dt}\big(F(\Gamma_t) - F(\Gamma)\big) \leqslant -\frac{1}{c}\big(F(\Gamma_t) - F(\Gamma)\big)^{2(1-\theta)};$$

here, we additionally assumed σ to be so small that $F(\Gamma_t) \geqslant F(\Gamma)$. This shows that for all $t \in I$ we have

$$F(\Gamma_t) - F(\Gamma) \leqslant c \begin{cases} t^{-1/(1-2\theta)} & \text{if } \theta < 1/2, \\ e^{-c_0 t} & \text{if } \theta = 1/2. \end{cases} \tag{21}$$

Moreover, we can apply Theorem 1.4 once more to obtain

$$-\frac{d}{dt}\big(F(\Gamma_t) - F(\Gamma)\big)^\theta = \theta\big(F(\Gamma_t) - F(\Gamma)\big)^{\theta-1} B_{\Gamma_t}(u, u)$$

$$\geqslant \theta/c\,\big(F(\Gamma_t) - F(\Gamma)\big)^{\theta-1}\|dF_{\Gamma_t}\|_{Y'_{\Gamma_t}}\|[u]_{\Gamma_t} \cdot v_t\|_{Y_{\Gamma_t}} \tag{22}$$

$$\geqslant \theta/c\,\|[u]_{\Gamma_t} \cdot v_t\|_{Y_{\Gamma_t}}.$$

Here, for the first estimate we used the identity $B_{\Gamma_t}(u, u) = \|dF_{\Gamma_t}\|_{B'_{\Gamma_t}}\|u\|_{B_{\Gamma_t}}$, the inequalities in (20), the uniform equivalence of the scalar products on X_{Γ_t}, and the uniform continuity of the trace operator $X_{\Gamma_t} \to Y_{\Gamma_t}$, $u \mapsto [u]_{\Gamma_t} \cdot v_t$ (see Lemma 3.8 in [14]). Since $d\Psi_{h(t)}^{-1}([u]_{\Gamma_t} \cdot v_t) = \partial_t h$, by Remark 1.3 it follows that

$$\int_0^t \|\partial_t h\|_{H^{1/2}(\Gamma)}\,ds \leqslant c\int_0^t \|[u]_{\Gamma_t} \cdot v_t\|_{Y_{\Gamma_t}}\,ds \leqslant c\,\big(F(\Gamma_{h_0}) - F(\Gamma)\big)^\theta \tag{23}$$

for all $t \in I$; hence

$$\|h(t)\|_{H^{1/2}(\Gamma)} \leqslant \|h_0\|_{H^{1/2}(\Gamma)} + c\,\big(F(\Gamma_{h_0}) - F(\Gamma)\big)^\theta. \tag{24}$$

Furthermore, for sufficiently small σ we deduce from Theorem 2.1 that our solution h exists at least as long as $h(t)$ remains in $\bar{B}_\sigma(0) \subset W_p^{5-4/p}(\Gamma)$, and if the latter is true on a time interval $I' = (0, T')$ then

$$\|h\|_{C([t', T'], W_p^{5-4/p+\delta}(\Gamma))} \leqslant c \tag{25}$$

for arbitrarily small $t' > 0$ and some constant $c > 0$. Now, let

$$T^* := \sup\big\{T' > 0 \,|\, \|h\|_{C(\bar{I}', W_p^{5-4/p}(\Gamma))} \leqslant \sigma\big\}.$$

By interpolation and (25) there exists a $\theta' \in (0, 1)$ such that

$$\|h(t)\|_{W_p^{5-4/p}(\Gamma)} \lesssim \|h(t)\|_{W_p^{5-4/p+\delta}(\Gamma)}^{\theta'} \|h(t)\|_{H^{1/2}(\Gamma)}^{1-\theta'} \leqslant c \, \|h(t)\|_{H^{1/2}(\Gamma)}^{1-\theta'}$$

for all $t \in [0, T^*]$. However, from this estimate and (24), for sufficiently small ϵ we obtain that $\|h(t)\|_{W_p^{5-4/p}(\Gamma)} \leqslant \sigma/2$. Hence, $T^* = \infty$ and the solution exists for all times. Furthermore, from (23) we obtain $\partial_t h \in L_1((0, \infty); H^{1/2}(\Gamma))$, and thus $h(t) \to h_\infty$ in $H^{1/2}(\Gamma))$ for $t \to \infty$ and some $h_\infty \in H^{1/2}(\Gamma))$. Integrating (22) over (t, ∞) yields

$$\|h(t) - h_\infty\|_{H^{1/2}(\Gamma)} \leqslant c \int_t^\infty \|\partial_s h(s)\|_{H^{1/2}(\Gamma)} \, ds \leqslant c \, (F(\Gamma_t) - F(\Gamma))^\theta \leqslant c \, t^{-\beta'}$$

for some $\beta' > 0$. By weak compactness, we have $h_\infty \in W_p^{5-4/p+\delta}(\Gamma)$ and thus we can use interpolation once more to obtain (19). Finally, from (19) and (21) we conclude that $F(\Gamma_{h_\infty}) = F(\Gamma)$. $\qquad\square$

Appendix. Second variation of the Canham–Helfrich energy

We give a brief derivation of the second variation of the Canham–Helfrich energy and of the linearization of its L_2-gradient. Of course, this issue has been addressed before in the literature; see, for instance, [4, 19, 26]. Our result agrees with the one in [19] (which, however, is not explicitly covariant; see (42), (43)), but it does not agree with the ones in [4] (which lacks one of the terms in our expression) and in [26] (which seems to contain typos; expressions like $Hg^{\alpha\beta} - Kk^{\alpha\beta}$ do not make sense from the point of view of physical dimensions).

Let $\Gamma_t \subset \mathbb{R}^3$ be a closed vesicle moving with velocity $u = w \, \nu_t$. Consider the family of diffeomorphisms $\varphi_{t,s} : \Gamma_t \to \Gamma_s$ associated with the vector field u, that is, $\varphi_{t,t} = \mathrm{id}_{\Gamma_t}$ and $\partial_s \varphi_{t,s} = u \circ \varphi_{t,s}$. We denote by Df/Dt the *material derivative* of a scalar field f on Γ_t with respect to the vector field u, that is,

$$\frac{Df}{Dt}\Big|_t := \partial_s|_{s=t} f \circ \varphi_{t,s}.$$

Throughout this appendix we shall work in *convected coordinates* (x^α) on Γ_t, that is, $x^\alpha|_t = x^\alpha|_s \circ \varphi_{t,s}$. Taking the material derivative of (tangential) tensor components in such coordinates yields the Lie derivative of the corresponding tensor field which is again a tensor field. In [14] we proved the identities

$$\frac{D}{Dt} g_{\alpha\beta} = -2w \, k_{\alpha\beta}, \quad \frac{D}{Dt} g^{\alpha\beta} = -g^{\alpha\mu} g^{\beta\nu} \frac{D}{Dt} g_{\mu\nu} = 2w \, k^{\alpha\beta},$$

$$\frac{D}{Dt} k_{\alpha\beta} = w_{;\alpha\beta} - w \, k_{\alpha\gamma} k_\beta^\gamma, \quad \frac{D}{Dt} k^{\alpha\beta} = w_{;}^{\alpha\beta} + 3w \, k^{\alpha\gamma} k_\gamma^\beta,$$

$$\frac{D}{Dt} dA = -w \, H \, dA, \quad \frac{D}{Dt} H = \Delta_g w + w(H^2 - 2K)$$

The Christoffel symbols are given by

$$\Gamma_{\alpha\beta}^\gamma = \frac{1}{2} g^{\gamma\delta} (g_{\beta\delta,\alpha} + g_{\alpha\delta,\beta} - g_{\alpha\beta,\delta}).$$

Recall that the difference of two connections is a tensor field and, hence, so is the material derivative of a connection. Thus, we have

$$
\begin{aligned}
\frac{D}{Dt}\Gamma^{\gamma}_{\alpha\beta} &= -\frac{1}{2}\Gamma^{\mu}_{\alpha\beta}g^{\gamma\nu}\frac{D}{Dt}g_{\mu\nu} + \frac{1}{2}g^{\gamma\delta}\left(\left(\frac{D}{Dt}g_{\beta\delta}\right)_{,\alpha} + \left(\frac{D}{Dt}g_{\alpha\delta}\right)_{,\beta} - \left(\frac{D}{Dt}g_{\alpha\beta}\right)_{,\delta}\right) \\
&= \frac{1}{2}g^{\gamma\delta}\left(\left(\frac{D}{Dt}g_{\beta\delta}\right)_{;\alpha} + \left(\frac{D}{Dt}g_{\alpha\delta}\right)_{;\beta} - \left(\frac{D}{Dt}g_{\alpha\beta}\right)_{;\delta}\right) \\
&= -k^{\gamma}_{\alpha}w_{,\beta} - k^{\gamma}_{\beta}w_{,\alpha} + k_{\alpha\beta}w^{,\gamma} - w\,k^{\gamma}_{\alpha;\beta},
\end{aligned}
$$

where the second identity follows by making use of Riemannian normal coordinates in which the Christoffel symbols vanish (at the center). For a scalar field f on Γ_t we have $\Delta_g f = g^{\alpha\beta}(f_{,\alpha\beta} - \Gamma^{\gamma}_{\alpha\beta}f_{,\gamma})$ and thus

$$
\begin{aligned}
\left[\frac{D}{Dt}, \Delta_g\right]f &= 2w\,k^{\alpha\beta}f_{;\alpha\beta} - g^{\alpha\beta}f_{,\gamma}\frac{D}{Dt}\Gamma^{\gamma}_{\alpha\beta} \\
&= 2w\,k^{\alpha\beta}f_{;\alpha\beta} + 2k^{\alpha\beta}f_{,\alpha}w_{,\beta} - Hf_{,\alpha}w^{,\alpha} + wf_{,\alpha}H^{,\alpha},
\end{aligned}
$$

where $[\cdot,\cdot]$ denotes the commutator. We conclude that

$$
\begin{aligned}
\frac{D}{Dt}\Delta_g H &= 2w\,k^{\alpha\beta}H_{;\alpha\beta} + 2k^{\alpha\beta}w_{,\alpha}H_{,\beta} - Hw_{,\alpha}H^{,\alpha} + wH_{,\alpha}H^{,\alpha} \\
&\quad + \Delta_g(\Delta_g w + w(H^2 - 2K)) \\
&= \Delta_g^2 w + \Delta_g w\,(H^2 - 2K) + w_{,\alpha}\left(2k^{\alpha\beta}H_{,\beta} + 3HH^{,\alpha} - 4K^{,\alpha}\right) \\
&\quad + w\left(2k^{\alpha\beta}H_{;\alpha\beta} + H_{,\alpha}H^{,\alpha} + \Delta_g(H^2 - 2K)\right)
\end{aligned}
\tag{26}
$$

Furthermore, we recall that the Riemannian curvature tensor R is given by

$$
R_{\alpha\beta\gamma}{}^{\delta} = \Gamma^{\delta}_{\gamma\beta,\alpha} - \Gamma^{\delta}_{\gamma\alpha,\beta} + \Gamma^{\delta}_{\mu\alpha}\Gamma^{\mu}_{\gamma\beta} - \Gamma^{\delta}_{\mu\beta}\Gamma^{\mu}_{\gamma\alpha},
$$

while the Ricci tensor Rc satisfies $Rc_{\beta\gamma} = R_{\alpha\beta\gamma}{}^{\alpha} = Kg_{\beta\gamma}$. Thus, making again use of Riemannian normal coordinates, we derive

$$
\frac{D}{Dt}Rc_{\beta\gamma} = \left(\frac{D}{Dt}\Gamma^{\delta}_{\gamma\beta}\right)_{;\alpha} - \left(\frac{D}{Dt}\Gamma^{\delta}_{\gamma\alpha}\right)_{;\beta},
$$

and hence

$$
\begin{aligned}
\frac{D}{Dt}K &= \frac{D}{Dt}\frac{1}{2}\left(Rc_{\beta\gamma}g^{\beta\gamma}\right) = w\,k^{\beta\gamma}Rc_{\beta\gamma} + \frac{1}{2}g^{\beta\gamma}\left(\left(\frac{D}{Dt}\Gamma^{\alpha}_{\gamma\beta}\right)_{;\alpha} - \left(\frac{D}{Dt}\Gamma^{\alpha}_{\gamma\alpha}\right)_{;\beta}\right) \\
&= w\,KH + H\Delta_g w - k^{\alpha\beta}w_{;\alpha\beta}.
\end{aligned}
$$

From this we deduce

$$
\begin{aligned}
\frac{D}{Dt}&H(H^2/2 - 2K) \\
&= \left(\Delta_g w + w(H^2 - 2K)\right)(H^2/2 - 2K) \\
&\quad + H\left(H(\Delta_g w + w(H^2 - 2K)) - 2w\,KH - 2H\Delta_g w + 2k^{\alpha\beta}w_{;\alpha\beta}\right) \\
&= w_{;\alpha\beta}\left((-H^2/2 - 2K)g^{\alpha\beta} + 2Hk^{\alpha\beta}\right) + w(3H^4/2 - 7KH^2 + 4K^2)
\end{aligned}
\tag{27}
$$

and

$$\frac{D}{Dt}(2K - HC_0/2) = w_{;\alpha\beta}\big((2H - C_0/2)g^{\alpha\beta} - 2k^{\alpha\beta}\big) + w(2KH - C_0/2\,H^2 + C_0K). \quad (28)$$

Collecting (26), (27), and (28), we finally compute the linearization of $\mathrm{grad}_{L_2}\,F_{\Gamma_t}$ by taking its material derivative, that is,

$$
\begin{aligned}
\frac{1}{\kappa}\frac{D}{Dt}\,\mathrm{grad}_{L_2}\,F_{\Gamma_t} &= \Delta_g^2 w + w_{;\alpha\beta}\big((H^2/2 - 4K + 2HC_0 - C_0^2/2)g^{\alpha\beta} + 2(H - C_0)k^{\alpha\beta}\big)\\
&\quad + w_{,\alpha}\big(2k^{\alpha\beta}H_{,\beta} + 3HH_{,}^{\alpha} - 4K_{,}^{\alpha}\big) + w\big(2k^{\alpha\beta}H_{;\alpha\beta} + \Delta_g(H^2 - 2K)\\
&\quad + H_{,\alpha}H_{,}^{\alpha} + 3H^4/2 - 7KH^2 + 4K^2 + 2C_0KH - C_0^2/2\,H^2 + C_0^2K\big)\\
&=: \Delta_g^2 w + (a^{\alpha\beta}w_{,\alpha})_{;\beta} + \tilde{b}\,w.
\end{aligned}
$$

Since

$$\frac{d^2}{dt^2}F(\Gamma_t) = d^2 F_{\Gamma_t}(w, w) + dF_{\Gamma_t}\Big(\frac{D}{Dt}w\Big),$$

the second variation of the Canham–Helfrich energy is given by

$$
\begin{aligned}
d^2 F_{\Gamma_t}(w, w) &= \int_{\Gamma_t}\frac{D}{Dt}\,\mathrm{grad}_{L_2}\,F_{\Gamma_t}\,w\,dA + \int_{\Gamma_t}\mathrm{grad}_{L_2}\,F_{\Gamma_t}\,w\,\frac{D}{Dt}\,dA\\
&= \kappa\int_{\Gamma_t}\big((\Delta_g w)^2 + a^{\alpha\beta}w_{,\alpha}w_{,\beta} + b\,w^2\big)\,dA,
\end{aligned}
$$

where

$$b = \big((2k^{\alpha\beta} - Hg^{\alpha\beta})H_{,\alpha}\big)_{;\beta} + \Delta_g(H^2 - 2K) + H^4 - 5KH^2 + 4K^2 + C_0^2K;$$

note that the first term on the right hand side is not contained in the analogous expression (84) in [4].

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