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Approximation of minimal surfaces with free boundaries

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In this paper we develop a penalty method to approximate solutions of the free boundary problem for minimal surfaces. To this end we study the problem of finding minimizers of a functional F_λ which is defined as the sum of the Dirichlet integral and an appropriate penalty term weighted by a parameter λ. We prove existence of a solution for λ large enough as well as convergence to a solution of the free boundary problem as λ tends to infinity. Additionally regularity at the boundary of these solutions is shown, which is crucial for deriving numerical error estimates. Since every solution is harmonic, the analysis is largely simplified by considering boundary values only and using harmonic extensions.

In a subsequent paper we develop a fully discrete finite element procedure for approximating solutions to this problem and prove an error estimate which includes an order of convergence with respect to the grid size.

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0. Introduction

A “minimal surface with free boundaries” or a “solution of the free boundary problem” is a stationary point of Dirichlet’s integral among all disk type surfaces, whose boundary curves lie on a prescribed support surface S.

One main example is, when S is given as a topological torus. Then there is a stationary surface, which“fills the hole of S”, but there are also more stationary solutions which are positioned inside the tube. In order to specify the position of the solution more precisely and avoid degeneration, one can choose a polygon, which does not meet the surface S, and demand, that the boundary of

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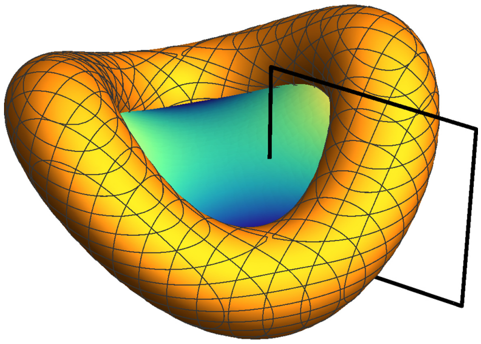

FIG. 1. A minimal surface, whose boundary curve is linked with a polygon: The linking number is nonzero.

the minimal surface on S is "linked" with that polygon (see Figure 1 and see [4, Ch. 1.2] for the topological concept of linking number).

Of course, the problem is also well defined for support surfaces of higher topological type.

The free boundary problem is a variant of the classical "Plateau Problem", which consists in finding a disk type minimal surface spanning a prescribed closed boundary curve Γ. Analytically both problems are investigated very well. There are existence results, valid under weak assumptions on the data, and one can show regularity of solutions up to the boundary under natural assumptions on S or Γ respectively. In the first section we give an overview about the most pertinent analytic results. A comprehensive treatise on minimal surfaces and related topics can be found in the monographs by Dierkes, Hildebrandt, Sauvigny and Tromba [2], [3], [4] and Nitsche [13].

Besides the highly nonlinear nature of the Plateau Problem or the Free Boundary Problem, the numerical approximation of solutions is difficult in both cases for the following reason: the subsidiary condition, namely that the boundary points have to lie on a prescribed set, is a pointwise condition and hence is unfavorable for numerics.

For the Plateau Problem many authors have developed different methods to tackle this difficulty: for the sake of conciseness we refer here only to Dziuk and Hutchinson [5], where a detailed overview on previous results can be found. In the papers by Dziuk and Hutchinson ([5], [6]) the fact is used, that the Dirichlet integral over the unit disk B can be written as integral over ∂B by using the harmonic extension. This functional is defined on the space of reparametrisations of Γ, so the boundary condition has been hidden into the functional.

Dziuk and Hutchinson are the first authors who give a fully discrete, finite element procedure for approximating (minimizing and stationary) minimal surfaces, which also yields an order of convergence with respect to the grid size. Further error estimates for their setting are shown in Pozzi [14].

The only work on the approximation of solutions to the free boundary problem, which we are aware of, is the dissertation of Tchakoutio [16]. He applies the method of [5] and [6] to torus type support surfaces and also obtains convergence results similar as in [6]. Our results here are more general as Tchakoutios' in that we allow arbitrary support surfaces S.

In this paper we develop a new approach to the free boundary problem by using a penalty method. To this end we will set up a new class of "Penalty Problems", which approximate the original free boundary problem in a suitable way and also replace the boundary condition. This is *independent* of the topological type of the support surface S, but only yields minimizers rather than merely stationary surfaces (which need not furnish a minimum for the Dirichlet- or area- functional).

In a subsequent paper (see [10]) we will apply methods of Dziuk and Hutchinson ([5], [6]) to approximate solutions of these penalty problems numerically and prove a convergence result, which yields an order of convergence with respect to the grid size. For this proof, besides the (geometrically motivated) assumption on the non-degeneracy of the second variation, we will not need any further a-priori assumptions.

Our method is based on the following idea: let $B \subset \mathbb{R}^2$ be the unit disk, $\lambda > 0$ a parameter, δ the signed distance function to S, $X : B \to \mathbb{R}^3$ a parametrization of a surface, and $D(X) = \frac{1}{2} \int_B |\nabla X|^2$ the Dirichlet integral. We consider functionals of the form

$$D(X) + \lambda \int_{\partial B} \delta(X)^2 |X_\varphi|.$$

Here the second term "penalizes" by assigning large values if the distance of the boundary of X and the surface S increases. We want to minimize this functional among all functions whose boundary curves lie in a neighbourhood of S and are linked with a prescribed polygon. The procedure is mostly inspired by the classical existence theory.

Because of technical difficulties we will use a slightly modified version F_λ of the above functional (see Definition 4). We can then show that the corresponding variational problems P_λ have a solution, assuming λ is large enough (see Theorem 5).

In this way we have replaced the free boundary condition by adding a penalty term, but solutions X_λ of the penalty problems P_λ are in general no solutions of the free boundary problem. So the next step is to prove, that these solutions X_λ converge to a solution of the free boundary problem as $\lambda \to \infty$ (see Theorem 9). The proof crucially relies on the classical existence theory. Furthermore we can even show an order of convergence with respect to λ (see Theorem 10).

Since we can show that solutions X_λ of P_λ are harmonic functions, $F_\lambda(X_\lambda)$ is uniquely determined by the boundary values $\gamma = X_\lambda|_{\partial B}$. We can therefore write $E_\lambda(\gamma)$ rather than $F_\lambda(X_\lambda)$ and similarly as in the work of Dziuk and Hutchinson ([5], [6]) we may reformulate the penalized problem into one-dimensional variants P_λ^* (see Theorem 15), which in turn is more favorable for numerics.

Additionally we prove that solutions γ of P_λ^* are of class C^2 (see Theorem 14), which will later also be important for the numerical analysis.

Finally we give an outlook to our subsequent paper [10], where we introduce a discretization $E_h^\lambda(\gamma_h)$ of $E_\lambda(\gamma)$ and prove an error estimate of the following form: Let γ be a minimizer of the one-dimensional problem P_λ^*, such that $\delta^2 E_\lambda(\gamma)$ is positive definite, i.e. there is a constant $\tilde{c} > 0$, such that $\delta^2 E_\lambda(\gamma)(\xi, \xi) \geqslant \tilde{c} \|\xi\|^2_{H_2^1}$ for all variations ξ. Then there is a unique solution γ_h of the discrete problem with

$$\|\gamma - \gamma_h\|_{H_2^1} \leqslant ch,$$

where c is independent of h. In section 7 we sketch the employed algorithm and comment on some numerical examples.

To conclude, let us remark that a related problem to the one considered here, is the computation of harmonic maps into surfaces: for this several interesting numerical approaches have been proposed, see Bartels [1], Steinhilber [15] and the references given in there.

1. The free boundary problem

In this section we give a short review of the solution to the free boundary problem for minimal surfaces. Here we present a greatly reduced version of the Chapters 1.1 to 1.3 in [4]. For more comprehensive reasoning we refer to [4].

Consider a closed set $S \subset \mathbb{R}^3$. By $T_\mu = T_\mu(S) := \{x \in \mathbb{R}^3 \mid \mathrm{dist}(x, S) < \mu\}$ we denote its tubular μ-neighbourhood and by $M(S)$ the set of all homotopy classes of closed paths in S.

ASSUMPTION A Suppose there is a number $\mu > 0$ such that the inclusion map $S \to T_\mu$ induces a bijection from $M(S)$ to $M(T_\mu)$.

Let $B \subset \mathbb{R}^2$ be the unit disk and choose a closed polygon Π in $\mathbb{R}^3$ with $\Pi \cap T_\mu = \emptyset$. Then the class of admissible functions is defined as

$$C(\Pi, S) := \left\{X \in H_2^1(B, \mathbb{R}^3) \;\middle|\; X|_{\partial B}(w) \in S \text{ for a.e. } w \in \partial B,\ L(X|_{\partial B}, \Pi) \neq 0\right\}.$$

Here $X|_{\partial B}$ is the L_2-trace of X and L denotes the linking number of two closed curves. (For the exact definition and properties of linking numbers see [4], Ch. 1.2 and the literature cited therein.) Although $X|_{\partial B}$ is not necessarily continuous, one can show that the linking number is well defined (see [4, Ch. 1.1, Theorem 3]).

The main existence result (cp. [4, Ch. 1.3, Theorems 1 and 2]) is

Theorem 1 *Let S be a closed set satisfying assumption A. If there is a closed polygon Π in $\mathbb{R}^3$ with $\Pi \cap T_\mu = \emptyset$, such that the class $C(\Pi, S)$ is not empty, then there is a solution $\tilde{X}$ of the variational problem*

$$P(\Pi, S):\ D(X) \to \min\ in\ C(\Pi, S),$$

where $D(X) = \frac{1}{2}\int_B |\nabla X|^2$ denotes the Dirichlet integral. Any solution $\tilde{X}$ is a minimal surface, i.e., $\tilde{X} \in C^2(B, \mathbb{R}^3) \cap C(\Pi, S)$ is harmonic and conformal

$$\Delta\tilde{X} = \tilde{X}_{uu} + \tilde{X}_{vv} = 0, \quad |\tilde{X}_u|^2 = |\tilde{X}_v|^2, \quad \langle\tilde{X}_u, \tilde{X}_v\rangle = 0$$

in B. Furthermore $\tilde{X}$ minimizes the area functional $A(X)$ in $C(\Pi, S)$ and $\inf_{C(\Pi,S)} A = \inf_{C(\Pi,S)} D$.

REMARK To avoid degeneration of minimizing sequences and exclude trivial solutions, we may specify the topological position of the solution surface relative to the boundary surface S in advance. Here we have chosen to preassign a nontrivial linking number, however also similar topological devices may be equally pertinent, cp. Chapter 1.1 in [4].

For the convenience of the reader we present the main ideas of the existence proof:

Let A and B be closed sets, then $g(A, B) := \sup\{\mathrm{dist}(x, B) \mid x \in A\}$ denotes the greatest distance of A to B. A sequence $X_k \in H_2^1(B, \mathbb{R}^3)$ is said to be a "generalized admissible sequence" (g.a.s.) for $P(\Pi, S)$, if there is a sequence of closed sets S_k such that $X_k \in C(\Pi, S_k)$ and $\lim_{k\to\infty} g(S_k, S) = 0$ holds true.

We set

$$e := \inf\left\{D(X) \mid X \in C(\Pi, S)\right\}$$

and

$$e^* := \inf\left\{\liminf_{k\to\infty} D(X_k) \mid (X_k) \text{ is a g.a.s. for } P(\Pi, S)\right\}.$$

Evidently we have $e^* \leqslant e$.

Now we choose a "g.a.s." Z_k with $\lim_{k\to\infty} D(Z_k) = e^*$, which we call a "generalized minimizing sequence" (g.m.s.). It is possible to pass from the sequence Z_k to a "g.m.s." Y_k of functions with absolutely continuous boundary curves. Then we get a "g.m.s." X_k by taking the harmonic extension of these boundary curves.

Lemma 2 *Let X_k be a "generalized minimizing sequence" for $P(\Pi, S)$ of harmonic functions. Then there is a subsequence, which converges weakly in $H_2^1(B)$ and uniformly on every subset $\Omega \subset\subset B$ to a harmonic function $\tilde{X} \in C(\Pi, S)$.*

(See proof of Theorem 1 in [4].) We denote the subsequence again by X_k. The lower semicontinuity of the Dirichlet integral with respect to weak convergence in H_2^1 yields $D(\tilde{X}) \leqslant \liminf_{k\to\infty} D(X_k) = e^* \leqslant e \leqslant D(\tilde{X})$ and therefore $D(\tilde{X}) = e = e^*$.

For solutions of $P(\Pi, S)$ the following regularity result holds:

Theorem 3 *Let S be a 2-dimensional compact submanifold of $\mathbb{R}^3$ of class C^m or $C^{m,\beta}$, $m \geqslant 3$, $\beta \in (0,1)$. Then any stationary point of the Dirichlet integral in $C(\Pi, S)$ is of class $C^{m-1,\alpha}(\overline{B})$ for any $\alpha \in (0,1)$ or of class $C^{m,\beta}(\overline{B})$ respectively.*

This is mainly the statement of Theorem 1 in Chapter 2.8. of [4]. One can easily see that S is an "admissible support surface" (in the sense of Definition 1 in Ch. 2.6 of [4]) since – by assumption – S is a compact submanifold. Furthermore the statement of Theorem 1 loc. cit. is formulated for solutions of semifree boundary problems, but all considerations in the proof are strictly local and can hence be carried over in essentially the same way (see the remarks at the end of Chapter 2.4 in [4]).

2. The Penalty Functional

From now on we will always assume that S fulfills

Assumption B Let $S \subset \mathbb{R}^3$ be a 2-dimensional compact submanifold of class C^3.

Clearly, Assumption B implies Assumption A.

Consider a suitably small neighborhood $U \subset \mathbb{R}^3$ of S and a C^3-function $G : U \to \mathbb{R}$ such that

$$S = \{x \in U : G(x) = 0\},$$

i.e., S is the zero level set of the function G. It is well known that we may choose G as the signed distance function relative to S (cp. for example Appendix in [8]), however we do not explicitly require $\nabla G \neq 0$ on S, hence also other choices for G might be appropriate.

Definition 4 Let $\lambda > 0$ be a parameter and $G \in C^3(U)$ such that $S = G^{-1}(0)$. Furthermore consider a point $P \in \mathbb{R}^3$ and a mapping $\varrho \in H_2^1(\partial B)$ with $\varrho \geqslant 0$, $\varrho(0) = 1$ and $\varrho(\pi) = 0$. For $X \in H_2^1(B, \mathbb{R}^3) \cap H_2^1(\partial B, \mathbb{R}^3)$ with $X(\partial B) \subset U$ we define the penalty functional

$$F_\lambda(X) := D(X) + \lambda \int_{\partial B} G(X)^2 \sqrt{|X_\varphi|^2 + \frac{1}{\lambda}} + \frac{1}{\lambda} \int_{\partial B} \left(|X_\varphi|^2 + \varrho|X - P|^2\right). \tag{1}$$

By (r, φ) we denote polar coordinates in $\overline{B}$.

REMARK The main idea behind the definition of the functional F_λ above consists in adding a term to Dirichlet's integral which penalizes the functional for not assuming boundary values on the prescribed surface S. Additionally we add a small third term which allows us to work in the class $H_2^1(\partial B)$ and a fourth one which (for an appropriate choice of the point P) removes the invariance with respect to rotations of B. This property is crucial for the numerical analysis carried out in the subsequent paper [10].

We now want to choose a $\mu > 0$, such that the μ-neighbourhood T_μ of S fulfills some important properties. At first we choose $\mu > 0$ so small that $\overline{T_\mu} \subset U$. Because $G \in C^3(U)$, there is an open set $\tilde{U}$ with $\overline{T_\mu} \subset \tilde{U} \subset U$, in which G as well as its first, second and third derivatives are bounded. Therefore we may assume that G as well as its first and second derivatives are Lipschitz continuous in $\overline{T_\mu}$. Furthermore let μ be so small that the inclusion map $S \to T_{2\mu}$ induces a bijection from $M(S)$ to $M(T_{2\mu})$.

This choice of μ would be sufficient for most of the following results. For later purposes in Section 5 we select $\mu > 0$ so small that - in addition to the other requirements - we have that the absolute values of principle curvatures of S are bounded by $\frac{1}{2\mu}$ and that for all $\mu_0 < 2\mu$ the parallel surface to S at distance μ_0 is again of class C^3.

Let Π be a closed polygon, such that $\Pi \cap T_{2\mu} = \emptyset$. Now we can define the class of admissible functions:

$$C := \left\{ X \in H_2^1(B, \mathbb{R}^3) \cap H_2^1(\partial B, \mathbb{R}^3) \;\middle|\; X(\partial B) \subset T_\mu,\ L(X|_{\partial B}, \Pi) \neq 0 \right\} \tag{2}$$

Again, L is the linking number of two closed curves, which can be defined here in a classical way as $X|_{\partial B}$ is continuous (recall that $H_2^1(\partial B)$ is embedded in $C^{0,\frac{1}{2}}(\partial B)$). We remark that the class C does not depend on λ. Furthermore it is not empty, if we assume that $C(\Pi, S)$ is not empty (see remark following Theorem 5).

We are now concerned with the following questions:

Is the variational problem

$$P_\lambda : \ F_\lambda(X) \to \min \ in\ C \tag{3}$$

solvable, at least if λ is large enough? And in case this is true, do solutions X_λ of the problems P_λ converge to a solution $\tilde{X}$ of $P(\Pi, S)$ for $\lambda \to \infty$ in a suitable way?

In the next sections we will show that these questions can be answered in the affirmative. Hence as far as a numerical treatment is concerned, it is convenient to consider problem P_λ rather that the original problem $P(\Pi, S)$. A main advantage of our approach is that - in this way - we do not need to enforce the pointwise boundary condition $X|_{\partial B}(\partial B) \subset S$.

3. Existence of a solution

For $0 \leqslant t \leqslant \mu$ we set

$$h(t) := \min_{z \in \overline{T_\mu} \setminus T_t} G(z)^2,$$

where $T_0 := \emptyset$. Then $h(0) = 0$ and $h(t) > 0$, if $t > 0$. By definition h is continuous and increasing. If G is the signed distance function, we have $h(t) = t^2$.

Our main goal in this section is to prove the following existence theorem:

Theorem 5 *Let the class $C(\Pi, S)$ be non empty and consider a solution $\tilde{X}$ of $P(\Pi, S)$. Then for all $\lambda > \max\left\{\frac{D(\tilde{X})+2}{\mu h(\frac{\mu}{2})}, \int_{\partial B}\left(|\tilde{X}_\varphi|^2 + \varrho|\tilde{X} - P|^2\right)\right\} =: \lambda_0(\mu, \tilde{X})$ the variational problem P_λ is solvable. Every solution of P_λ is a harmonic function.*

First we note that the class C (recall equation (2)) is not empty, since by Theorem 1 there is a solution $\tilde{X}$ of $P(\Pi, S)$, which, by Theorem 3, is smooth up to the boundary.

We set $e_\lambda := \inf\{F_\lambda(X) |\; X \in C\}$ and pick a minimizing sequence for P_λ, i.e. a sequence $X_k \subset C$ with $\lim_{k\to\infty} F_\lambda(X_k) = e_\lambda$.

There is a constant M_λ, such that $D(X_k) \leqslant F_\lambda(X_k) \leqslant M_\lambda$ for all k. This implies that there is a subsequence X_k, that converges weakly in $H_2^1(B)$ to a function $X \in H_2^1(B)$.

Furthermore the trace $X_k|_{\partial B}$ converges in $L_2(\partial B)$ to $X|_{\partial B}$. Because X_k is a minimizing sequence, we have $\int_{\partial B} |(X_k)_\varphi|^2 \leqslant \lambda F_\lambda(X_k) \leqslant \lambda M_\lambda$. So there is another subsequence X_k, whose boundary values $X_k|_{\partial B}$ converge weakly in $H_2^1(\partial B)$ to $X|_{\partial B}$ whence $X \in H_2^1(B) \cap H_2^1(\partial B)$.

Lemma 6 *For $0 < \mu_0 < \mu$ and a solution $\tilde{X}$ of $P(\Pi, S)$ let $\lambda \geqslant \lambda_0(\mu_0, \tilde{X})$. Consider a minimizing sequence X_k for P_λ, that converges weakly in $H_2^1(B)$ to a function $X \in H_2^1(B) \cap H_2^1(\partial B)$. Then $X(\partial B) \subset \overline{T_{\mu_0}}$ holds true.*

Proof. Assume there was a subsequence X_k with $X_k(1, \varphi_k) \notin T_{\mu_0}$ for some $\varphi_k \in [0, 2\pi]$. If $\partial T_{\frac{\mu_0}{2}} \cap X_k(\partial B) = \emptyset$ we obtain

$$\lambda \int_{\partial B} G(X_k)^2 \sqrt{|(X_k)_\varphi|^2 + \frac{1}{\lambda}} \geqslant \lambda \int_{\partial B} G(X_k)^2 |(X_k)_\varphi| \geqslant \lambda h\left(\frac{\mu_0}{2}\right) 2\pi\mu,$$

because $X_k|_{\partial B}$ lies completely in $T_\mu \backslash T_{\frac{\mu_0}{2}}$ (recall, that h is increasing) and, because of $\Pi \cap T_{2\mu} = \emptyset$ and $L(X_k|_{\partial B}, \Pi) \neq 0$, the boundary curve $X_k|_{\partial B}$ has to wind around a μ-neighbourhood of Π, so the length of $X_k|_{\partial B}$ has to be at least $2\pi\mu$ (see Figure 2).

If $\partial T_{\frac{\mu_0}{2}} \cap X_k(\partial B) \neq \emptyset$ (see Figure 3) we obtain

$$\lambda \int_{\partial B} G(X_k)^2 \sqrt{|(X_k)_\varphi|^2 + \frac{1}{\lambda}} \geqslant \lambda \int_{\partial B} G(X_k)^2 |(X_k)_\varphi| \geqslant \lambda h\left(\frac{\mu_0}{2}\right) 2\frac{\mu_0}{2},$$

because $X_k|_{\partial B}$ takes values in $\overline{T_{\frac{\mu_0}{2}}}$ as well as in $T_\mu \backslash T_{\mu_0}$ and therefore two parts of $X_k|_{\partial B}$ with length $\frac{\mu_0}{2}$ have to be contained in $T_{\mu_0} \backslash T_{\frac{\mu_0}{2}}$. These two estimates yield

$$\lambda \int_{\partial B} G(X_k)^2 \sqrt{|(X_k)_\varphi|^2 + \frac{1}{\lambda}} \geqslant \lambda\mu_0 h\left(\frac{\mu_0}{2}\right).$$

As already seen, we have $\tilde{X} \in C$. Additionally $G(z) = 0$ holds for all $z \in \tilde{X}(\partial B)$. For $\lambda \geqslant \lambda_0(\mu_0, \tilde{X}) = \max\left\{\frac{D(\tilde{X})+2}{\mu_0 h(\frac{\mu_0}{2})}, \int_{\partial B}\left(|\tilde{X}_\varphi|^2 + \varrho|\tilde{X} - P|^2\right)\right\}$ and every k we get

$$\begin{aligned} F_\lambda(X_k) &\geqslant \lambda \int_{\partial B} G(X_k)^2 \sqrt{|(X_k)_\varphi|^2 + \frac{1}{\lambda}} \geqslant \lambda\mu_0 h\left(\frac{\mu_0}{2}\right) \geqslant D(\tilde{X}) + 2 \\ &\geqslant D(\tilde{X}) + \frac{1}{\lambda}\int_{\partial B}\left(|\tilde{X}_\varphi|^2 + \varrho|\tilde{X} - P|^2\right) + 1 = F_\lambda(\tilde{X}) + 1 \geqslant e_\lambda + 1. \end{aligned}$$

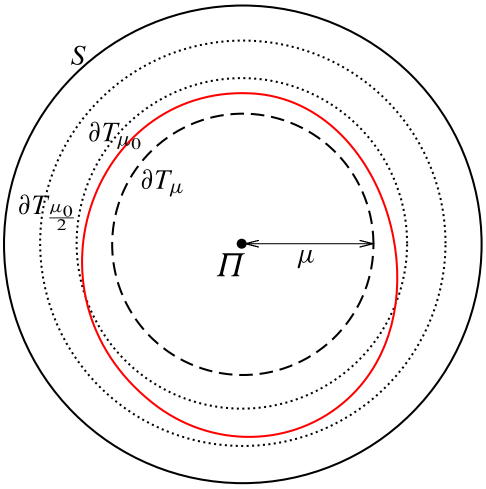

FIG. 2. The case $\partial T_{\frac{\mu_0}{2}} \cap X_k(\partial B) = \emptyset$

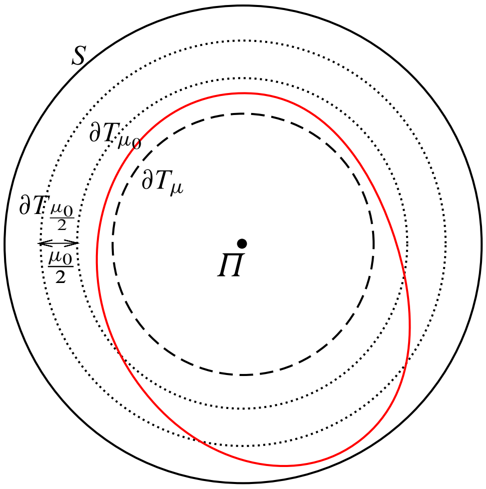

FIG. 3. The case $\partial T_{\frac{\mu_0}{2}} \cap X_k(\partial B) \neq \emptyset$

This is a contradiction as X_k is a minimizing sequence. We conclude, that our assumption was wrong and therefore $X_k(\partial B) \subset T_{\mu_0}$ holds true for every $k > k_0(\lambda)$.
Since X_k converges weakly in $H_2^1(B)$ to X, the boundary values $X_k|_{\partial B}$ converge in $L_2(\partial B)$ to $X|_{\partial B}$. So there is a subsequence, which converges almost everywhere and we get $X|_{\partial B}(w) \in \overline{T_{\mu_0}}$ for almost all $w \in \partial B$. As $X|_{\partial B}$ is continuous, $X(\partial B) \subset \overline{T_{\mu_0}}$ holds true. □

The following corollary will be used later.

Corollary 7 *Let* $0 < \mu_0 < \mu$ *and* $\tilde{X}$ *be a solution of* $P(\Pi, S)$. *Further let* $\lambda \geqslant \lambda_0(\mu_0, \tilde{X})$ *and* X_λ *be a solution of* P_λ. *Then* $X_\lambda(\partial B) \subset \overline{T_{\mu_0}}$ *holds true.*

Proof. This is a direct consequence of Lemma 6 applied to the constant sequence $X_k := X_\lambda$ for all k. □

By Lemma 6 we now have $X(\partial B) \subset \overline{T_{\mu_0}} \subset T_\mu$, for λ large enough. Hölder's inequality yields

$$|X_k(\varphi_2) - X_k(\varphi_1)| \leqslant \int_{\varphi_1}^{\varphi_2} |(X_k)_\varphi| \leqslant \sqrt{\varphi_2 - \varphi_1}\sqrt{\int_{\varphi_1}^{\varphi_2} |(X_k)_\varphi|^2} \leqslant \sqrt{\varphi_2 - \varphi_1}\sqrt{\lambda M_\lambda}.$$

Therefore the $X_k|_{\partial B}$ are equicontinuous and because of $X_k(\partial B) \subset T_\mu$ they are uniformly bounded. Applying the theorem of Arzelà–Ascoli we can extract another subsequence X_k, which converges uniformly on ∂B. As $L(X_k|_{\partial B}, \Pi) \neq 0$ holds for all k, we get $L(X|_{\partial B}, \Pi) \neq 0$. We conclude that $X \in C$ and therefore

$$e_\lambda \leqslant F_\lambda(X).$$

We will now make use of a well known lower semicontinuity result (see for example Theorem 1.8.2 in [12]):

Lemma 8 *Let $f(x,z,p) \in C^0(\mathbb{R}^n, U, \mathbb{R}^{nN})$, where $U \subset \mathbb{R}^N$. Let f be convex in p for all (x,z), $f(x,z,p) \geqslant 0$ for all (x,z,p) and $f_p \in C^0(\mathbb{R}^n, U, \mathbb{R}^{nN})$. Consider a domain $\Omega \subset \mathbb{R}^n$ with $z_k(\Omega), z(\Omega) \subset U$ and $z_k \rightharpoonup z$ in $H_1^1(\Omega)$. Then we have*

$$\int_\Omega f(x,z,\nabla z)dx \leqslant \liminf_{k\to\infty} \int_\Omega f(x,z_k,\nabla z_k)dx.$$

We apply this lemma with $n = 1$, $N = 3$, $f(x,z,p) = \lambda G(z)^2\sqrt{|p|^2 + \frac{1}{\lambda}} + \frac{1}{\lambda}\big(|p|^2 + \varrho(x)|z - P|^2\big)$ and $U = T_\mu$. In connection with the lower semicontinuity of the Dirichlet integral we obtain

$$F_\lambda(X) \leqslant \liminf_{k\to\infty} F_\lambda(X_k) = e_\lambda$$

and therefore $F_\lambda(X) = e_\lambda$. We conclude that X is a solution of P_λ.

Every solution X of P_λ is a minimizer of the Dirichlet integral among all variations with compact support in B, because these do not change the boundary integrals. Therefore X is a weak solution of the Laplace equation and a standard regularity result (see, for example, [8, Corollary 8.11]) yields that X is a classical solution and hence harmonic.

4. Convergence for $\lambda \to \infty$

In this section we investigate the connection between solutions of P_λ and those of $P(\Pi, S)$. Invoking Corollary 7 the following convergence result follows from the classical theory.

Theorem 9 *Consider a solution $\tilde{X}$ of the free boundary problem $P(\Pi, S)$ and solutions X_{λ_n} of the problems P_{λ_n}, where $\lambda_n \to \infty$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} D(X_{\lambda_n}) = \lim_{n\to\infty} F_{\lambda_n}(X_{\lambda_n}) = D(\tilde{X})$$

holds true and there is a subsequence of X_{λ_n}, which converges in $H_2^1(B)$ and uniformly on every subset $\Omega \subset\subset B$ to a solution of $P(\Pi, S)$.

Proof. By Theorem 3 we obtain $\tilde{X} \in C$ and thus

$$\begin{aligned}\limsup_{n\to\infty} D(X_{\lambda_n}) &\leqslant \limsup_{n\to\infty} F_{\lambda_n}(X_{\lambda_n}) \leqslant \limsup_{n\to\infty} F_{\lambda_n}(\tilde{X}) \\ &= \limsup_{n\to\infty}\left(D(\tilde{X}) + \frac{1}{\lambda_n}\int_{\partial B}\big(|\tilde{X}_\varphi|^2 + \varrho|\tilde{X} - P|^2\big)\right) = D(\tilde{X}). \qquad (4)\end{aligned}$$

For all $n \geqslant N_0$ we have $\lambda_n > \max\left\{\frac{D(\tilde{X})+2}{\mu h(\frac{\mu}{2})}, \int_{\partial B}\big(|\tilde{X}_\varphi|^2 + \varrho|\tilde{X} - P|^2\big)\right\}$. From the properties of the function h it follows that the function $f(\tau) := \frac{D(\tilde{X})+2}{\tau h(\frac{\tau}{2})}$ is continuous and strictly decreasing on $(0, \mu]$. Furthermore we have $\lim_{\tau\to 0} f(\tau) = \infty$, so for every $n \geqslant N_0$ there exists exactly one $\mu_0 = \mu_0(n) < \mu$ with $\lambda_n = f(\mu_0(n))$ and we obtain $\lambda_n = \max\left\{\frac{D(\tilde{X})+2}{\mu_0 h(\frac{\mu_0}{2})}, \int_{\partial B}\big(|\tilde{X}_\varphi|^2 + \varrho|\tilde{X} - P|^2\big)\right\}$. By applying Corollary 7 we get $X_{\lambda_n}(\partial B) \subset \overline{T_{\mu_0(n)}}$ and thus $X_{\lambda_n} \in C(\Pi, \overline{T_{\mu_0(n)}})$ for all $n \geqslant N_0$.

Using the notation introduced in Theorem 1, we infer from $\lim_{n\to\infty} \mu_0(n) = 0$ the relation $\lim_{n\to\infty} g(\overline{T_{\mu_0(n)}}, S) = 0$. Whence X_{λ_n} is a "generalized admissible sequence" for $P(\Pi, S)$ and

$$D(\tilde{X}) = e = e^* \leqslant \liminf_{n\to\infty} D(X_{\lambda_n}) \leqslant \liminf_{n\to\infty} F_{\lambda_n}(X_{\lambda_n}). \qquad (5)$$

The estimates (4) and (5) prove the first assertion.

Since $\lim_{n\to\infty} D(X_{\lambda_n}) = D(\tilde{X}) = e$ holds true, the harmonic functions X_{λ_n} form a "generalized minimizing sequence". Therefore by Lemma 2, there is a subsequence of X_{λ_n}, which converges weakly in $H_2^1(B)$ and uniformly on every subset $\Omega \subset\subset B$ to a solution $P(\Pi, S)$. From weak convergence in $H_2^1(B)$ we get strong convergence in $L_2(B)$ by Rellich's theorem. Since also the Dirichlet integrals converge, we obtain strong convergence in $H_2^1(B)$. □

Theorem 10 *Let G be the signed distance function of S, $\tilde{X}$ a solution of $P(\Pi, S)$, $\lambda > \max\left\{\frac{4}{\mu^3}(D(\tilde{X})+2), \int_{\partial B}\left(|\tilde{X}_\varphi|^2+\varrho|\tilde{X}-P|^2\right)\right\}$ and X_λ a solution of P_λ. Then there is a constant C that does not depend on λ such that*

$$|D(\tilde{X}) - D(X_\lambda)| \leqslant \frac{C}{\sqrt[3]{\lambda}}, \quad |D(\tilde{X}) - F_\lambda(X_\lambda)| \leqslant \frac{C}{\sqrt[3]{\lambda}}.$$

REMARK In order to simplify the notation in the proof, we have assumed that G is the signed distance function. With the same arguments it can be shown that, if G is a function with polynomial growth of order n, the order of convergence is $\frac{1}{2n+1}$.

Proof. Similarly to the proof of the previous theorem we infer

$$\begin{aligned} D(X_\lambda) - D(\tilde{X}) &\leqslant F_\lambda(X_\lambda) - D(\tilde{X}) \leqslant F_\lambda(\tilde{X}) - D(\tilde{X}) \\ &= \frac{1}{\lambda}\int_{\partial B}\left(|\tilde{X}_\varphi|^2 + \varrho|\tilde{X}-P|^2\right) = \frac{C_1}{\lambda}. \end{aligned} \tag{6}$$

Setting $\mu_0 := \sqrt[3]{\frac{4}{\lambda}(D(\tilde{X})+2)}$, we have $\mu_0 < \mu$. According to Theorem 1 there is a solution $\tilde{Y}_{\mu_0}$ of the problem $P(\Pi, \overline{T_{\mu_0}})$, because the closed set $\overline{T_{\mu_0}}$ fulfills Assumption A and we have $\tilde{X} \in C(\Pi, \overline{T_{\mu_0}})$. By the definition of μ_0 we have $\lambda \geqslant \max\left\{\frac{4}{\mu_0^3}(D(\tilde{X})+2), \int_{\partial B}\left(|\tilde{X}_\varphi|^2+\varrho|\tilde{X}-P|^2\right)\right\}$. Corollary 7 yields $X_\lambda(\partial B) \subset \overline{T_{\mu_0}}$ and thus $X_\lambda \in C(\Pi, \overline{T_{\mu_0}})$. It follows that

$$D(\tilde{X}) - F_\lambda(X_\lambda) \leqslant D(\tilde{X}) - D(X_\lambda) \leqslant D(\tilde{X}) - D(\tilde{Y}_{\mu_0}). \tag{7}$$

As $\tilde{X}$ and $\tilde{Y}_{\mu_0}$ are minimal surfaces by Theorem 1, we obtain

$$D(\tilde{X}) - D(\tilde{Y}_{\mu_0}) = A(\tilde{X}) - A(\tilde{Y}_{\mu_0}). \tag{8}$$

In order to continue estimating this term, we have to show, that $\tilde{Y}_{\mu_0}$ is also a solution of the problem $P(\Pi, \partial T_{\mu_0})$.

Since the closed set $\overline{T_{\mu_0}}$ fulfills a chord-arc condition, $\tilde{Y}_{\mu_0} \in C^0(\overline{B})$ holds (see Chapter 2.5, Theorem 4 in [4]). Assume there was a $w \in \partial B$ with $\tilde{Y}_{\mu_0}(w) \in T_{\mu_0}$. Then, because $\tilde{Y}_{\mu_0}$ is continuous, there is an $\epsilon > 0$ such that $\tilde{Y}_{\mu_0}(B \cap B_\epsilon(w)) \in T_{\mu_0}$. From the linking condition we infer that the minimal surface $\tilde{Y}_{\mu_0}$ is certainly not a constant and therefore $D_{B\cap B_\epsilon(w)}(\tilde{Y}_{\mu_0}) > 0$. According to the Riemann mapping theorem we choose a conformal mapping σ from B to $B\backslash\overline{B_\epsilon(w)}$. Because of $D_{B\backslash\overline{B_\epsilon(w)}}(\tilde{Y}_{\mu_0}) = D_B(\tilde{Y}_{\mu_0}(\sigma))$ we get a function $\tilde{Y}_{\mu_0}(\sigma) \in C(\Pi, \overline{T_{\mu_0}})$ with $D(\tilde{Y}_{\mu_0}(\sigma)) < D(\tilde{Y}_{\mu_0})$, which is not possible. This shows that $\tilde{Y}_{\mu_0}$ cannot take boundary values in T_{μ_0} and we obtain $\tilde{Y}_{\mu_0}(\partial B) \subset \partial T_{\mu_0}$.

Again because of continuity, $\tilde{Y}_{\mu_0}(\partial B)$ is contained in a connected component of ∂T_{μ_0}, which we denote by $\partial T^1_{\mu_0}$. This is a parallel surface of S with distance $\mu_0 < \mu$ and thus, according to our choice of μ in Section 2, a compact submanifold of class C^3. In particular, by Theorem 3 we have $\tilde{Y}_{\mu_0} \in C^2(\overline{B})$. Now we define a mapping Y_{μ_0} by

$$Y_{\mu_0} = \begin{cases} \tilde{Y}_{\mu_0}(2r, \varphi) & in\ B_{\frac{1}{2}} \\ \tilde{Y}_{\mu_0}(1, \varphi) + \mu_0(2r-1)N(\tilde{Y}_{\mu_0}(1, \varphi)) & in\ B \backslash B_{\frac{1}{2}} \end{cases},$$

where N denotes the inner normal of $\partial T^1_{\mu_0}$. By construction we have $Y_{\mu_0} \in C^2(\overline{B_{\frac{1}{2}}}) \cap C^2(\overline{B \backslash B_{\frac{1}{2}}}) \cap C^0(\overline{B})$ and thus $Y_{\mu_0} \in C(\Pi, S)$. We obtain

$$\begin{aligned} A(\tilde{X}) - A(\tilde{Y}_{\mu_0}) &= A(\tilde{X}) - A_{B_{1/2}}(Y_{\mu_0}) = A(\tilde{X}) - A(Y_{\mu_0}) + A_{B\backslash B_{1/2}}(Y_{\mu_0}) \\ &\leqslant A_{B\backslash B_{1/2}}(Y_{\mu_0}), \end{aligned} \tag{9}$$

because $\tilde{X}$ also minimizes the area functional according to Theorem 1. We denote the principal curvatures of S by κ_i, $i = 1, 2$. By our choice of μ their absolute values are bounded by $\frac{1}{2\mu}$. For the principal curvatures $\tilde{\kappa}_i$ of the parallel surfaces with distance $\mu_0 \in [0, \mu]$

$$\tilde{\kappa}_i = \frac{\kappa_i}{1 \pm \kappa_i \mu_0}$$

holds, depending on the choice of orientation (see Ch. 14.6 in [8]). We obtain

$$|\tilde{\kappa}_i| \leqslant \frac{|\kappa_i|}{1 - |\kappa_i \mu_0|} \leqslant \frac{1}{2\mu\left(1 - \frac{|\mu_0|}{2\mu}\right)} \leqslant \frac{1}{\mu}.$$

Therefore the absolute values of the principal curvatures of $\partial T^1_{\mu_0}$ are bounded by $\frac{1}{\mu}$. Setting $\gamma(\varphi) := \tilde{Y}_{\mu_0}(1, \varphi)$ we have for all $r \in (\frac{1}{2}, 1)$:

$$(Y_{\mu_0})_r = 2\mu_0 N\big(\gamma(\varphi)\big), \quad (Y_{\mu_0})_\varphi = \gamma'(\varphi) + \mu_0(2r-1)\nabla N\big(\gamma(\varphi)\big) \cdot \gamma'(\varphi).$$

It follows that

$$\begin{aligned} A_{B\backslash B_{1/2}}(Y_{\mu_0}) &= \int_0^{2\pi} \int_{\frac{1}{2}}^1 |(Y_{\mu_0})_r \times (Y_{\mu_0})_\varphi| dr d\varphi \\ &\leqslant \int_0^{2\pi} \int_{\frac{1}{2}}^1 |(Y_{\mu_0})_r||(Y_{\mu_0})_\varphi| dr d\varphi \\ &= \int_0^{2\pi} \int_{\frac{1}{2}}^1 2\mu_0 |\gamma'(\varphi) + \mu_0(2r-1)\nabla N\big(\gamma(\varphi)\big) \cdot \gamma'(\varphi)| dr d\varphi \\ &= \int_0^{2\pi} \int_{\frac{1}{2}}^1 2\mu_0 \Big(|\gamma'(\varphi)|^2 + 2\mu_0(2r-1)\langle \gamma'(\varphi), \nabla N\big(\gamma(\varphi)\big) \cdot \gamma'(\varphi)\rangle \\ &\qquad + \mu_0^2(2r-1)^2 |\nabla N(\gamma(\varphi)) \cdot \gamma'(\varphi)|^2\Big)^{\frac{1}{2}} dr d\varphi \\ &\leqslant \int_0^{2\pi} \mu_0 \Big(|\gamma'(\varphi)|^2 + 2\mu_0 |\langle \gamma'(\varphi), \nabla N\big(\gamma(\varphi)\big) \cdot \gamma'(\varphi)\rangle| \\ &\qquad + \mu_0^2 |\nabla N\big(\gamma(\varphi)\big) \cdot \gamma'(\varphi)|^2\Big)^{\frac{1}{2}} d\varphi. \end{aligned}$$

Depending on the choice of orientation, the Weingarten mapping of $\partial T^1_{\mu_0}$ is $\pm\nabla N$. Note that the identity $KI - 2HII + III = 0$ holds for the three fundamental forms I, II and III, where I and II are the first and the second fundamental forms resp. and the third fundamental form III is essentially given by the square of the Weingarten map. For details see Chapter 1.2 in [2], especially Definition (9) and Equation (26). Therefore $III \leqslant |K|I + 2|H||II|$ and we obtain

$$\left|\nabla N\big(\gamma(\varphi)\big)\cdot\gamma'(\varphi)\right|^2 \leqslant \left|\tilde\kappa_1\tilde\kappa_2\right|\left|\gamma'(\varphi)\right|^2 + \left|\tilde\kappa_1+\tilde\kappa_2\right|\left|\langle\gamma'(\varphi),\nabla N(\gamma(\varphi))\cdot\gamma'(\varphi)\rangle\right|,$$

and thus with $|\tilde\kappa_1|, |\tilde\kappa_2| \leqslant \frac{1}{\mu}$ and $\mu_0 < \mu$:

$$\begin{aligned} A_{B\setminus B_{1/2}}(Y_{\mu_0}) &\leqslant \int_0^{2\pi} \mu_0\Big((1+\mu_0^2|\tilde\kappa_1\tilde\kappa_2|)|\gamma'(\varphi)|^2 \\ &\qquad + (2\mu_0+\mu_0^2|\tilde\kappa_1+\tilde\kappa_2|)|\langle\gamma'(\varphi),\nabla N(\gamma(\varphi))\cdot\gamma'(\varphi)\rangle|\Big)^{\frac12}d\varphi \\ &\leqslant \int_0^{2\pi}\mu_0\Big(2|\gamma'(\varphi)|^2 + 4\mu_0|\langle\gamma'(\varphi),\nabla N(\gamma(\varphi))\cdot\gamma'(\varphi)\rangle|\Big)^{\frac12}d\varphi. \end{aligned}$$

For $\gamma'(\varphi) \neq 0$ we have

$$\left|\langle\gamma'(\varphi),\nabla N\big(\gamma(\varphi)\big)\cdot\gamma'(\varphi)\rangle\right| = \frac{\left|\langle\gamma'(\varphi),\nabla N\big(\gamma(\varphi)\big)\cdot\gamma'(\varphi)\rangle\right|}{|\gamma'(\varphi)|^2}|\gamma'(\varphi)|^2 \leqslant \frac{1}{\mu}|\gamma'(\varphi)|^2,$$

since the normal curvature is bounded by the principal curvatures (see Chapter 1.2, Definitions (20) and (21) in [2]). For $\gamma'(\varphi) = 0$ this inequality is trivial and it follows that

$$\begin{aligned} A_{B\setminus B_{1/2}}(Y_{\mu_0}) &\leqslant \int_0^{2\pi}\mu_0\big(6|\gamma'(\varphi)|^2\big)^{\frac12}d\varphi = \sqrt6\mu_0\int_0^{2\pi}|\gamma'(\varphi)|d\varphi \\ &= \sqrt6\mu_0 l(\gamma) = \sqrt6\mu_0 l(\tilde Y_{\mu_0}|_{\partial B}). \end{aligned} \tag{10}$$

According to Chapter 4.6, Remark 10 in [4] the inequality

$$l(\tilde Y_{\mu_0}|_{\partial B}) \leqslant \frac{2}{\mu}D(\tilde Y_{\mu_0}) \tag{11}$$

holds, as $\partial T^1_{\mu_0}$ fulfills a 'μ-sphere condition'. Collecting (7), (8), (9), (10) and (11) we infer

$$\begin{aligned} D(\tilde X) - F_\lambda(X_\lambda) &\leqslant D(\tilde X) - D(X_\lambda) \leqslant 2\sqrt6\frac{\mu_0}{\mu}D(\tilde Y_{\mu_0}) \\ &\leqslant 2\sqrt6\frac{\mu_0}{\mu}D(X_\lambda) \leqslant 2\sqrt6\frac{\mu_0}{\mu}F_\lambda(X_\lambda) \leqslant 2\sqrt6\frac{\mu_0}{\mu}F_\lambda(\tilde X) \\ &= 2\sqrt6\frac{\mu_0}{\mu}\left(D(\tilde X) + \frac{1}{\lambda}\int_{\partial B}\big(|\tilde X_\varphi|^2 + \varrho|\tilde X - P|^2\big)\right) \\ &\leqslant 2\sqrt6\frac{\mu_0}{\mu}\big(D(\tilde X)+1\big) = \frac{2\sqrt6}{\mu}\sqrt[3]{\frac{4}{\lambda}(D(\tilde X)+2)}\big(D(\tilde X)+1\big) = \frac{C_2}{\sqrt[3]{\lambda}}. \end{aligned} \tag{12}$$

Because λ is bounded from below, the assertion follows from (6) and (12). □

5. Regularity

In this section we show that the boundary values $X_\lambda|_{\partial B}$ of a solution X_λ of P_λ are of class C^2: this will be important later for the numerical analysis (see Jenschke [10]).

We recall some basic results about fractional order Sobolev spaces (see [6, § 3]).

The spaces $H_2^s(\partial B)$ and $H_2^s(B)$ can be defined for all real s, but apart from the classical cases $s \in \mathbb{N}$ (and $H_2^0 := L_2$) we will only need the following cases.

For $f\colon \partial B \mapsto \mathbb{R}$ the $H_2^{1/2}(\partial B)$-seminorm is defined by

$$|f|^2_{H_2^{1/2}(\partial B)} = \int_{\partial B}\int_{\partial B} \frac{|f(\varphi) - f(\tilde\varphi)|^2}{|\varphi - \tilde\varphi|^2} d\varphi d\tilde\varphi$$

and for $u\colon B \mapsto \mathbb{R}$ the $H_2^{1/2}(B)$-seminorm is defined by

$$|u|^2_{H_2^{1/2}(B)} = \int_B\int_B \frac{|u(x) - u(\tilde x)|^2}{|x - \tilde x|^3} dx d\tilde x.$$

In both cases the corresponding norm is given by

$$\|\cdot\|^2_{H_2^{1/2}} = \|\cdot\|^2_{L_2} + |\cdot|^2_{H_2^{1/2}}.$$

Furthermore we have

$$|f|_{H_2^{3/2}(\partial B)} = |f_\varphi|_{H_2^{1/2}(\partial B)}, \quad |u|_{H_2^{3/2}(B)} = |\nabla u|_{H_2^{1/2}(B)}$$

and

$$\|\cdot\|^2_{H_2^{3/2}} = \|\cdot\|^2_{H_2^1} + |\cdot|^2_{H_2^{3/2}}.$$

If $u \in H_2^{s+1/2}(B)$ for $s \in \{\frac{1}{2}, 1, \frac{3}{2}\}$, then u has a well defined trace f on ∂B and

$$\|f\|_{H_2^s(\partial B)} \leqslant c\|u\|_{H_2^{s+1/2}(B)}.$$

(See Chapter 1, Section 9.2 in [11].) Such an estimate is in general not true for $s = 0$, but if a function $u \in H_2^{1/2}(B)$ fulfills $\Delta u \in L_2(B)$, then u again has a trace f and

$$\|f\|_{L_2(\partial B)} \leqslant c\big(\|u\|_{H_2^{1/2}(B)} + \|\Delta u\|_{L_2(B)}\big). \tag{13}$$

(See Chapter 2, Section 7.3, Theorem 7.3 in [11].) Conversely, if $f \in H_2^s(\partial B)$ for $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}\}$, then there is a unique harmonic function $\Phi(f)$ defined on B with trace f as before, and in particular

$$\|\Phi(f)\|_{H_2^{s+1/2}(B)} \leqslant c\|f\|_{H_2^s(\partial B)}. \tag{14}$$

(See Chapter 2, Section 7.3, Theorem 7.4 in [11].)

Collecting the above results we are able to prove the following lemma:

Lemma 11 *Let $u \in H_2^1(B) \cap H_2^1(\partial B)$ be a harmonic function. Then the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ have L_2-traces on ∂B and*

$$\int_B \nabla u \cdot \nabla \Psi = \int_{\partial B} \nabla u \cdot \nu \Psi$$

holds true for all $\Psi \in C^\infty(\overline{B})$, where ν denotes the outer normal of ∂B.

Proof. As $u|_{\partial B} \in H_2^1$, it follows from (14) and the uniqueness of the harmonic extension that $u \in H_2^{3/2}(B)$. Evidently we have $\frac{\partial u}{\partial x} \in H_2^{1/2}(B)$ and $\frac{\partial u}{\partial y} \in H_2^{1/2}(B)$. Furthermore these derivatives are again harmonic functions, so according to (13) $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ have L_2-traces on ∂B. Choosing a sequence of functions $g_k \in C^\infty(\partial B)$ with $\lim_{k\to\infty} \|u - g_k\|_{H_2^1(\partial B)} = 0$, we obtain from the linearity of Φ and (14)

$$\|\nabla u - \nabla \Phi(g_k)\|_{H_2^{1/2}(B)} \leqslant \|u - \Phi(g_k)\|_{H_2^{3/2}(B)} \leqslant c\|u - g_k\|_{H_2^1(\partial B)}.$$

Thus $\lim_{k\to\infty} \|\nabla u - \nabla \Phi(g_k)\|_{L_2(B)} \leqslant \lim_{k\to\infty} \|\nabla u - \nabla \Phi(g_k)\|_{H_2^{1/2}(B)} = 0$. Furthermore from the linearity of the trace operator and (13) we infer

$$\|\nabla u \cdot \nu - \nabla \Phi(g_k) \cdot \nu\|_{L_2(\partial B)} \leqslant \|\nabla u - \nabla \Phi(g_k)\|_{L_2(\partial B)} \leqslant c\|\nabla u - \nabla \Phi(g_k)\|_{H_2^{1/2}(B)}$$

and therefore $\lim_{k\to\infty} \|\nabla u \cdot \nu - \nabla \Phi(g_k) \cdot \nu\|_{L_2(\partial B)} = 0$. Clearly

$$\int_B \nabla \Phi(g_k) \cdot \nabla \Psi = \int_{\partial B} \nabla \Phi(g_k) \cdot \nu \Psi$$

is fulfilled for all $\Psi \in C^\infty(\overline{B})$, so the assertion follows by approximation. □

Using the result above, we can derive a natural boundary condition for solutions of P_λ.

Lemma 12 *Let X_λ be a solution of P_λ, then there is a function $f \in H_2^1(\partial B, \mathbb{R}^3)$, such that*

$$\frac{\lambda G(X_\lambda)^2 (X_\lambda)_\varphi}{\sqrt{|(X_\lambda)_\varphi|^2 + \frac{1}{\lambda}}} + \frac{2}{\lambda}(X_\lambda)_\varphi = f \tag{15}$$

almost everywhere on ∂B. If additionally $(X_\lambda)_\varphi \in C^{0,\frac{1}{2}}(\partial B)$ holds, we even have $f \in C^{1,\frac{1}{2}}(\partial B, \mathbb{R}^3)$.

Proof. Consider $\Psi \in C^\infty(\overline{B}, \mathbb{R}^3)$, then

$$\begin{aligned} 0 = \frac{\partial}{\partial t} F_\lambda(X_\lambda + t\Psi)\Big|_{t=0} &= \int_B \langle \nabla X_\lambda, \nabla \Psi\rangle \\ &+ \lambda \int_{\partial B} \left(G(X_\lambda)^2 \frac{\langle (X_\lambda)_\varphi, \Psi_\varphi\rangle}{\sqrt{|(X_\lambda)_\varphi|^2 + \frac{1}{\lambda}}} + 2G(X_\lambda)\langle \nabla G(X_\lambda), \Psi\rangle \sqrt{|(X_\lambda)_\varphi|^2 + \frac{1}{\lambda}} \right) \\ &+ \frac{2}{\lambda} \int_{\partial B} \big(\langle (X_\lambda)_\varphi, \Psi_\varphi\rangle + \langle \varrho(X_\lambda - P), \Psi\rangle \big). \end{aligned}$$

According to Theorem 5 the map $X_\lambda \in H_2^1(B, \mathbb{R}^3) \cap H_2^1(\partial B, \mathbb{R}^3)$ is a harmonic function and thus Lemma 11 yields

$$\int_B \langle \nabla X_\lambda, \nabla \Psi\rangle = \int_{\partial B} \langle \nabla X_\lambda \cdot \nu, \Psi\rangle =: \int_{\partial B} \langle (X_\lambda)_r, \Psi\rangle.$$

Therefore we have

$$\int_{\partial B}\left(\langle (X_\lambda)_r,\Psi\rangle + 2\lambda G(X_\lambda)\sqrt{|(X_\lambda)_\varphi|^2+\frac{1}{\lambda}}\langle \nabla G(X_\lambda),\Psi\rangle + \frac{2}{\lambda}\langle \varrho(X_\lambda - P),\Psi\rangle\right)$$
$$+\lambda\int_{\partial B} G(X_\lambda)^2\frac{\langle (X_\lambda)_\varphi,\Psi_\varphi\rangle}{\sqrt{|(X_\lambda)_\varphi|^2+\frac{1}{\lambda}}} + \frac{2}{\lambda}\int_{\partial B}\langle (X_\lambda)_\varphi,\Psi_\varphi\rangle = 0.$$

Integration by parts over the interval $[\varphi_0,\varphi_0+2\pi]$ yields for all Ψ with $\Psi|_{\partial B}\in C_c^\infty(\partial B\backslash\{\varphi_0\},\mathbb{R}^3)$

$$-\int_{\partial B}\left\langle \int_{\varphi_0}^{\varphi}(X_\lambda)_r + 2\lambda G(X_\lambda)\sqrt{|(X_\lambda)_\varphi|^2+\frac{1}{\lambda}}\nabla G(X_\lambda) + \frac{2}{\lambda}\varrho(X_\lambda - P)d\tilde{\varphi},\Psi_\varphi\right\rangle$$
$$+\lambda\int_{\partial B} G(X_\lambda)^2\frac{\langle (X_\lambda)_\varphi,\Psi_\varphi\rangle}{\sqrt{|(X_\lambda)_\varphi|^2+\frac{1}{\lambda}}} + \frac{2}{\lambda}\int_{\partial B}\langle (X_\lambda)_\varphi,\Psi_\varphi\rangle = 0.$$

(Here we have to keep in mind that the indefinite integral of a periodic function need not be periodic as well.) By Du Bois-Reymond's Lemma there is a constant $C(\varphi_0)\in\mathbb{R}^3$, such that

$$-\int_{\varphi_0}^{\varphi}(X_\lambda)_r + 2\lambda G(X_\lambda)\sqrt{|(X_\lambda)_\varphi|^2+\frac{1}{\lambda}}\nabla G(X_\lambda) + \frac{2}{\lambda}\varrho(X_\lambda - P)d\tilde{\varphi}$$
$$+\lambda G(X_\lambda)^2\frac{(X_\lambda)_\varphi}{\sqrt{|(X_\lambda)_\varphi|^2+\frac{1}{\lambda}}} + \frac{2}{\lambda}(X_\lambda)_\varphi = C(\varphi_0)$$

almost everywhere on $[\varphi_0,\varphi_0+2\pi]$. Therefore we set

$$f_{\varphi_0} = C(\varphi_0) + \int_{\varphi_0}^{\varphi}(X_\lambda)_r + 2\lambda G(X_\lambda)\sqrt{|(X_\lambda)_\varphi|^2+\frac{1}{\lambda}}\nabla G(X_\lambda) + \frac{2}{\lambda}\varrho(X_\lambda - P)d\tilde{\varphi}.$$

Then $f_{\varphi_0}\in H_2^1(\partial B\backslash\{\varphi_0\},\mathbb{R}^3)$ holds, because the integrand is of class $L_2(\partial B)$. Moreover f_{φ_0} satisfies equation (15) almost everywhere on ∂B. As φ_0 was arbitrary, we can consider the functions f_0 and f_π. They coincide almost everywhere on ∂B, because both satisfy equation (15) and therefore we have $f_0 = f_\pi =: f\in H_2^1(\partial B,\mathbb{R}^3)$.

If additionally $(X_\lambda)_\varphi\in C^{0,\frac{1}{2}}(\partial B)$ holds, the harmonic function X_λ has boundary values of class $C^{1,\frac{1}{2}}$. Therefore we have $X_\lambda\in C^{1,\frac{1}{2}}(\overline{B})$. (See Theorem 8.34. and Lemma 6.38. in [8].) It follows that $(X_\lambda)_r\in C^{0,\frac{1}{2}}(\partial B)$ and thus $f_{\varphi_0}\in C^{1,\frac{1}{2}}(\partial B\backslash\{\varphi_0\},\mathbb{R}^3)$. With the same argument as above we conclude $f_{\varphi_0} := f\in C^{1,\frac{1}{2}}(\partial B,\mathbb{R}^3)$. □

In order to prove regularity of X_λ, we have to solve the equation for $(X_\lambda)_\varphi$. For that we will use a very specific version of the Implicit Function Theorem. (In the following the norm of a matrix always denotes the operator norm.)

Theorem 13 *Let $F(x,y)\colon\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}^m$ be continuous and continuously differentiable with respect to y. Assume that $|F|_{C^{0,\alpha}(\mathbb{R}\times\mathbb{R}^m)}<\infty$ holds. Let $(a,b)\in\mathbb{R}\times\mathbb{R}^m$ be a point with* $\det\frac{\partial F}{\partial y}(a,b)\neq 0$ *and $F(a,b)=0$. Then there is an open neighbourhood $V_1\subset\mathbb{R}$ of a, a* *neighbourhood $V_2 \subset \mathbb{R}^m$ of b as well as a uniquely determined mapping $g \in C^0(V_1, V_2)$ with $|g|_{C^{0,\alpha}(V_1)} < \infty$ and $g(a) = b$ such that $F(x, g(x)) = 0$ for all $x \in V_1$.*

If additionally $\frac{\partial F}{\partial y}$ is uniformly continuous on $\mathbb{R} \times \mathbb{R}^m$ and $\left\| \left(\frac{\partial F}{\partial y}(a, b) \right)^{-1} \right\| \leqslant C$ for all $(a, b) \in \Omega$, where $\Omega \subset \{(x, y) \in \mathbb{R} \times \mathbb{R}^m \mid F(x, y) = 0,\ \det \frac{\partial F}{\partial y}(x, y) \neq 0\}$, we can choose $V_1 = B_\epsilon(a)$, where ϵ is independent of $(a, b) \in \Omega$.

REMARK 1 In contrast to the main assertion it is of course crucial for the additional assertion to hold that F is defined on the whole space $\mathbb{R} \times \mathbb{R}^m$ and has a global Hölder constant. Notice that we do not need boundedness of F.

REMARK 2 We only need and prove the version of the theorem containing the additional assertion. It is quite obvious from the proof that the main assertion holds as well.

Proof. We follow the reasoning in [7], only adding and changing arguments where it is necessary.

We set $B := \frac{\partial F}{\partial y}(a, b) \in GL(m, \mathbb{R})$ and define the mapping $G: \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ by $G(x, y) := y - B^{-1} F(x, y)$. Then since $F(x, y) = 0 \iff y = G(x, y)$, we will therefore consider a fixed-point problem.

Set $K := |F|_{C^{0,\alpha}(\mathbb{R} \times \mathbb{R}^m)}$. Because $\frac{\partial G}{\partial y}(x, y) = I - B^{-1} \frac{\partial F}{\partial y}(x, y)$, where I denotes the $m \times m$ unit matrix, we obtain $\frac{\partial G}{\partial y}(a, b) = 0$. It follows that

$$\left\| \frac{\partial G}{\partial y}(x, y) \right\| = \left\| \frac{\partial G}{\partial y}(x, y) - \frac{\partial G}{\partial y}(a, b) \right\| = \left\| B^{-1} \left(\frac{\partial F}{\partial y}(x, y) - \frac{\partial F}{\partial y}(a, b) \right) \right\|$$
$$\leqslant \|B^{-1}\| \left\| \frac{\partial F}{\partial y}(x, y) - \frac{\partial F}{\partial y}(a, b) \right\| \leqslant C \left\| \frac{\partial F}{\partial y}(x, y) - \frac{\partial F}{\partial y}(a, b) \right\|.$$

Since all components of the matrix $\frac{\partial F}{\partial y}$ are uniformly continuous, there exist $\tilde{\epsilon} > 0$ and $\tilde{r} > 0$ which are independent of $(a, b) \in \Omega$, such that $\left\| \frac{\partial G}{\partial y}(x, y) \right\| \leqslant \frac{1}{2+KC}$ for all $(x, y) \in B_{\tilde{\epsilon}}(a) \times B_{\tilde{r}}(b) \subset \mathbb{R} \times \mathbb{R}^m$.

We choose $r > 0$ with $r < \tilde{r}$. As $G(a, b) = b$ and F is uniformly continuous, there is an $\epsilon > 0$ which is independent of (a, b) with $\epsilon \leqslant \tilde{\epsilon}$ such that

$$\left| G(x, b) - b \right| = \left| B^{-1} \left(F(x, b) - F(a, b) \right) \right| \leqslant C \left| F(x, b) - F(a, b) \right| \leqslant \frac{r}{2}$$

for all $x \in B_\epsilon(a)$. We set $V_1 = B_\epsilon(a)$ and $V_2 = \overline{B_r(b)}$, then according to the mean value theorem it follows for all $x \in V_1$ and $y, \tilde{y} \in V_2$ that

$$|G(x, y) - G(x, \tilde{y})| \leqslant \frac{1}{2 + KC} |y - \tilde{y}| \leqslant \frac{1}{2} |y - \tilde{y}|. \tag{16}$$

Setting $\tilde{y} = b$ we have for all $x \in V_1$

$$|y - b| \leqslant r \Longrightarrow |G(x, y) - b| \leqslant r. \tag{17}$$

For every fixed $x \in V_1$ the mapping $y \mapsto G(x, y)$ is therefore a mapping from the closed ball V_2 to itself, which by (16) is also a contraction. The Banach Fixed-Point Theorem yields for all $x \in V_1$ the existence of exactly one $y =: g(x) \in V_2$, such that $G(x, y) = y$ or $F(x, y) = 0$ respectively.

Consider a function $\varphi(x) \in C^0(V_1, \mathbb{R}^m)$ with $\|\varphi - b\|_{C^0(V_1)} \leqslant r$ and $|\varphi|_{C^{0,\alpha}(V_1)} \leqslant 1 + KC$, then for the mapping defined by $\psi(x) := G(x, \varphi(x))$ we again obtain $\psi(x) \in C^0(V_1, \mathbb{R}^m)$ with $\|\psi - b\|_{C^0(V_1)} \leqslant r$ because of (17). Furthermore according to (16) we have

$$\begin{aligned}\frac{|\psi(x) - \psi(\tilde{x})|}{|x - \tilde{x}|^\alpha} &= \frac{\big|G\big(x, \varphi(x)\big) - G\big(\tilde{x}, \varphi(\tilde{x})\big)\big|}{|x - \tilde{x}|^\alpha} \\ &\leqslant \frac{\big|G\big(x, \varphi(x)\big) - G\big(x, \varphi(\tilde{x})\big)\big|}{|x - \tilde{x}|^\alpha} + \frac{\big|G\big(x, \varphi(\tilde{x})\big) - G\big(\tilde{x}, \varphi(\tilde{x})\big)\big|}{|x - \tilde{x}|^\alpha} \\ &\leqslant \frac{1}{2 + KC} \frac{|\varphi(x) - \varphi(\tilde{x})|}{|x - \tilde{x}|^\alpha} + \|B^{-1}\| \frac{\big|F\big(x, \varphi(\tilde{x})\big) - F\big(\tilde{x}, \varphi(\tilde{x})\big)\big|}{|x - \tilde{x}|^\alpha} \\ &\leqslant \frac{1}{2 + KC}(1 + KC) + KC \leqslant 1 + KC,\end{aligned}$$

which shows that $|\psi|_{C^{0,\alpha}(V_1)} \leqslant 1 + KC$ holds, too. The mapping $\varphi \mapsto \psi$ is therefore a mapping Φ from the closed subset

$$A := \big\{\varphi \in C^0(V_1, \mathbb{R}^m)\colon \|\varphi - b\|_{C^0(V_1)} \leqslant r,\ |\varphi|_{C^{0,\alpha}(V_1)} \leqslant 1 + KC\big\}$$

of the Banach space consisting of all bounded functions of class $C^0(V_1, \mathbb{R}^m)$ to itself. From (16) we infer for $\varphi_1, \varphi_2 \in A$

$$\begin{aligned}\|\Phi(\varphi_1) - \Phi(\varphi_2)\|_{C^0(V_1)} &= \sup_{x \in V_1} \big|G\big(x, \varphi_1(x)\big) - G\big(x, \varphi_2(x)\big)\big| \\ &\leqslant \frac{1}{2} \sup_{x \in V_1} |\varphi_1(x) - \varphi_2(x)| = \frac{1}{2}\|\varphi_1 - \varphi_2\|_{C^0(V_1)}.\end{aligned}$$

The mapping $\Phi\colon A \to A$ is therefore a contraction and hence has exactly one fixed point $g \in A \subset C^0(V_1, \mathbb{R}^m)$. This function $g \in C^{0,\alpha}(V_1, V_2)$ satisfies $G(x, g(x)) = g(x)$ or $F(x, g(x)) = 0$ respectively for all $x \in V_1$ and coincides with the mapping obtained above. □

Theorem 14 *Let X_λ be a solution of P_λ, then we have $X_\lambda \in C^2(\partial B)$.*

Proof. We define $F(\varphi, z)\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(\varphi, z) := \frac{\lambda G\big(X_\lambda(1, \varphi)\big)^2 z}{\sqrt{|z|^2 + \frac{1}{\lambda}}} + \frac{2}{\lambda} z - f(\varphi),$$

where f is the function from Lemma 12. (We have identified the functions defined on ∂B with 2π-periodic functions on $\mathbb{R}$.) Then for almost all $\varphi \in \mathbb{R}$ we obtain $F(\varphi, (X_\lambda)_\varphi(1, \varphi)) = 0$.

We have $X_\lambda(1, \varphi) \in H_2^1(\mathbb{R}) \subset C^{0,\frac{1}{2}}(\mathbb{R})$ and $f(\varphi) \in H_2^1(\mathbb{R}) \subset C^{0,\frac{1}{2}}(\mathbb{R})$. The function G^2 is Lipschitz continuous on $\overline{T_\mu}$ and therefore $G(X_\lambda(1, \varphi))^2 \in C^{0,\frac{1}{2}}(\mathbb{R})$. Since all these functions are periodic, they even have globally finite Hölder constants. The functions $\frac{\lambda z}{\sqrt{|z|^2 + \frac{1}{\lambda}}}$ and $\frac{2}{\lambda} z$ are globally Lipschitz continuous. As $\frac{\lambda z}{\sqrt{|z|^2 + \frac{1}{\lambda}}}$ is also bounded, $F(\varphi, z)$ is globally $\frac{1}{2}$-Hölder continuous. Furthermore $F(\varphi,z)$ is continuously differentiable with respect to z and we derive

$$\frac{\partial F_i}{\partial z_i}=\frac{2}{\lambda}-\frac{\lambda G(X_\lambda)^2 z_i^2}{\left(|z|^2+\frac{1}{\lambda}\right)^{\frac{3}{2}}}+\frac{\lambda G(X_\lambda)^2}{\sqrt{|z|^2+\frac{1}{\lambda}}},\qquad \frac{\partial F_i}{\partial z_j}=-\frac{\lambda G(X_\lambda)^2 z_i z_j}{\left(|z|^2+\frac{1}{\lambda}\right)^{\frac{3}{2}}},\ i\neq j.$$

As above it can be shown that the derivatives are globally $\frac{1}{2}$-Hölder continuous and therefore uniformly continuous. A straightforward calculation shows

$$\det\frac{\partial F}{\partial z}=\left(\frac{2}{\lambda}+\frac{\lambda G(X_\lambda)^2}{\sqrt{|z|^2+\frac{1}{\lambda}}}\right)^2\left(\frac{2}{\lambda}+\frac{G(X_\lambda)^2}{\left(|z|^2+\frac{1}{\lambda}\right)^{\frac{3}{2}}}\right)\geqslant\left(\frac{2}{\lambda}\right)^3.$$

Therefore $\frac{\partial F}{\partial z}$ is invertible. Because of $\left|\frac{\partial F_i}{\partial z_j}\right|\leqslant\frac{2}{\lambda}+\lambda^{\frac{3}{2}}G(X_\lambda)^2$ for all $i,j\in\{1,2,3\}$ and $X_\lambda(\partial B)\subset T_\mu$ we have

$$\left|\left(\frac{\partial F}{\partial z}\right)^{-1}_{ij}\right|=\frac{1}{\det\frac{\partial F}{\partial z}}\left|\left(\operatorname{Adj}\frac{\partial F}{\partial z}\right)_{ij}\right|\leqslant\left(\frac{2}{\lambda}\right)^{-3}2\left(\frac{2}{\lambda}+\lambda^{\frac{3}{2}}G(X_\lambda)^2\right)^2\leqslant C.$$

Now all conditions of Theorem 13 are fulfilled and we infer that for almost all $a\in\mathbb{R}$ there is a neighbourhood $B_\epsilon(a)$ and a mapping $g_a\in C^0(B_\epsilon(a))$ with $|g_a|_{C^{0,\alpha}(B_\epsilon(a))}<\infty$ and $g_a(a)=(X_\lambda)_\varphi(1,a)$ such that $F(\varphi,g_a(\varphi))=0$. These local solutions are unique only in a small neighbourhood of $(X_\lambda)_\varphi(a)$, so it remains to show that they coincide with $(X_\lambda)_\varphi$.

Assume that $F(\varphi,z)=0$ and $F(\varphi,\tilde{z})=0$, then we get

$$f(\varphi)=\frac{2}{\lambda}z+\frac{\lambda G\big(X_\lambda(\varphi)\big)^2 z}{\sqrt{|z|^2+\frac{1}{\lambda}}}\ \text{and}\ f(\varphi)=\frac{2}{\lambda}\tilde{z}+\frac{\lambda G\big(X_\lambda(\varphi)\big)^2\tilde{z}}{\sqrt{|\tilde{z}|^2+\frac{1}{\lambda}}},$$

which yields

$$\left(\frac{2}{\lambda}+\frac{\lambda G\big(X_\lambda(\varphi)\big)^2}{\sqrt{|z|^2+\frac{1}{\lambda}}}\right)z=\left(\frac{2}{\lambda}+\frac{\lambda G\big(X_\lambda(\varphi)\big)^2}{\sqrt{|\tilde{z}|^2+\frac{1}{\lambda}}}\right)\tilde{z}.$$

As the terms in brackets are positive, we have $\tilde{z}=tz$ with $t>0$. If $z=0$, it follows that $\tilde{z}=0$. If $z\neq 0$, we consider the function

$$h(s):=\left(\frac{2}{\lambda}+\frac{\lambda G\big(X_\lambda(\varphi)\big)^2}{\sqrt{s^2|z|^2+\frac{1}{\lambda}}}\right)s.$$

Its derivative fulfills

$$h'(s)=\frac{G\big(X_\lambda(\varphi)\big)^2}{\sqrt{s^2|z|^2+\frac{1}{\lambda}}^{\,3}}+\frac{2}{\lambda}>0$$

and therefore $h(s)$ is injective. From

$$\left(\frac{2}{\lambda}+\frac{\lambda G\big(X_\lambda(\varphi)\big)^2}{\sqrt{|z|^2+\frac{1}{\lambda}}}\right)z=\left(\frac{2}{\lambda}+\frac{\lambda G\big(X_\lambda(\varphi)\big)^2}{\sqrt{|tz|^2+\frac{1}{\lambda}}}\right)tz$$

and $z \neq 0$ we infer $h(1) = h(t)$. We obtain $t = 1$ and therefore $z = tz = \tilde{z}$.

That is, for every $\varphi \in \mathbb{R}$ there is at most one solution z of the equation $F(\varphi, z) = 0$. Therefore the functions $g_a \in C^{0,\frac{1}{2}}(B_\epsilon(a))$ have to coincide with $(X_\lambda)_\varphi$ almost everywhere. As ϵ can be chosen independently of a, the open sets $B_\epsilon(a)$ cover the interval $[0, 2\pi]$, and since $[0, 2\pi]$ is compact, only a finite number of the $B_\epsilon(a)$ are needed. We infer that $(X_\lambda)_\varphi \in C^{0,\frac{1}{2}}(\partial B, \mathbb{R}^3)$.

Lemma 12 even yields $f(\varphi) \in C^{1,\frac{1}{2}}(\partial B)$ and therefore $F(\varphi, z)$ is continuously differentiable. We can now apply the standard version of the Implicit Function Theorem and obtain $(X_\lambda)_\varphi \in C^1(\partial B)$ or $X_\lambda \in C^2(\partial B)$ respectively. □

6. The one-dimensional functional

We now reformulate the variational problem (3) into a one-dimensional problem.

Theorem 15 *We define the class of functions*

$$C^* := \left\{\gamma \in H_2^1(\partial B, \mathbb{R}^3) \;\middle|\; \gamma(\partial B) \subset T_\mu,\ L(\gamma, \Pi) \neq 0\right\} \tag{18}$$

and for $\gamma \in C^*$ we define the functional

$$E_\lambda(\gamma) := D(\Phi(\gamma)) + \lambda \int_{\partial B} G(\gamma)^2 \sqrt{|\gamma'|^2 + \frac{1}{\lambda}} + \frac{1}{\lambda} \int_{\partial B} \left(|\gamma'|^2 + \varrho|\gamma - P|^2\right), \tag{19}$$

where $\Phi(\gamma)$ is the harmonic extension of γ. Then P_λ is equivalent to the variational problem

$$P_\lambda^* :\ E_\lambda(\gamma) \to \min\ in\ C^*. \tag{20}$$

Proof. As solutions of P_λ are harmonic due to Theorem 5, they are exactly the minimizers of F_λ in the class $\{X \in C \mid \Delta X = 0\}$. Because of uniqueness, the boundary values of these solutions are exactly the minimizers of E_λ in the class $\{\gamma \in C^* \mid \Phi(\gamma) \in H_2^1(B, \mathbb{R}^3)\}$. But by (14) $\Phi(\gamma) \in H_2^1(B, \mathbb{R}^3)$ holds for all $\gamma \in C^*$, which yields the assertion. □

Notice, that solutions of P_λ^* are of class C^2 by Theorem 14. One can show $E_\lambda(\gamma) \in C^1(C^*, \mathbb{R})$, but it is not true that $E_\lambda(\gamma) \in C^2(C^*, \mathbb{R})$. However, the second variation exists for all $\gamma \in C^*$. Fixing $\gamma \in C^*$ and $\xi, \eta \in H_2^1(\partial B, \mathbb{R}^3)$, and using the linearity of Φ, we derive:

$$\begin{aligned}\delta E_\lambda(\gamma)(\xi) = \frac{d}{d\epsilon} E_\lambda(\gamma + \epsilon\xi)\Big|_{\epsilon=0} &= \int_B \langle \nabla\Phi(\gamma), \nabla\Phi(\xi)\rangle \\ &+ \lambda \int_{\partial B} \left(2G(\gamma)\langle \nabla G(\gamma), \xi\rangle \sqrt{|\gamma'|^2 + \frac{1}{\lambda}} + G(\gamma)^2 \frac{\langle \gamma', \xi'\rangle}{\sqrt{|\gamma'|^2 + \frac{1}{\lambda}}}\right) \\ &+ \frac{2}{\lambda} \int_{\partial B} \left(\langle \gamma', \xi'\rangle + \langle \varrho(\gamma - P), \xi\rangle\right)\end{aligned} \tag{21}$$

and

$$\begin{aligned}\delta^2 E_\lambda(\gamma)(\xi,\eta) &= \frac{d}{d\epsilon}\frac{d}{d\tau}E_\lambda(\gamma+\epsilon\xi+\tau\eta)\Big|_{\epsilon=0}\Big|_{\tau=0} = \int_B \langle\nabla\Phi(\xi),\nabla\Phi(\eta)\rangle\\
&+\lambda\int_{\partial B}\Bigg(2\langle\nabla G(\gamma),\xi\rangle\langle\nabla G(\gamma),\eta\rangle\sqrt{|\gamma'|^2+\frac{1}{\lambda}}+2G(\gamma)\langle\xi,\nabla^2 G(\gamma)\eta\rangle\sqrt{|\gamma'|^2+\frac{1}{\lambda}}\\
&+2G(\gamma)\langle\nabla G(\gamma),\xi\rangle\frac{\langle\gamma',\eta'\rangle}{\sqrt{|\gamma'|^2+\frac{1}{\lambda}}}+2G(\gamma)\langle\nabla G(\gamma),\eta\rangle\frac{\langle\gamma',\xi'\rangle}{\sqrt{|\gamma'|^2+\frac{1}{\lambda}}}\\
&+G(\gamma)^2\frac{\langle\xi',\eta'\rangle\big(|\gamma'|^2+\frac{1}{\lambda}\big)-\langle\gamma',\xi'\rangle\langle\gamma',\eta'\rangle}{\big(|\gamma'|^2+\frac{1}{\lambda}\big)^{\frac{3}{2}}}\Bigg)\\
&+\frac{2}{\lambda}\int_{\partial B}\big(\langle\xi',\eta'\rangle+\varrho\langle\xi,\eta\rangle\big). \qquad (22)\end{aligned}$$

7. Implementation and numerical results

It is beyond the scope of this article to show in detail how the one-dimensional version (20) of the penalty functional can be treated numerically and we refer the reader to the subsequent paper (see [10]) for a comprehensive presentation. There we will show how the algorithm given in this section can be transformed to matrix vector form, prove an error estimate for solutions of the discrete problem and discuss various aspects of the numerical results in some detail. Here we only sketch the approach.

The main ideas of the discretization are taken from the papers of Dziuk and Hutchinson ([5] and [6]), and we use their notation as far as possible. Let G_h be a triangulation of B with the following properties: Every triangle $G \in G_h$ has diameter at most h and at least σh for some $\sigma > 0$ independent of h and has angles bounded away from zero independently of h.

We define

$$B_h = \bigcup\{G \mid G \in G_h\}, \quad \partial B_h = \bigcup\{E_j \mid 1 \leqslant j \leqslant M\},$$

where the E_j are the boundary edges. We denote the boundary nodes by $e^{i\varphi_j}$, $1 \leqslant j \leqslant M$ or just φ_j if convenient. Furthermore we define the projection $\pi : \partial B \to \partial B_h$ by

$$\pi\big(e^{i((1-t)\varphi_{j-1}+t\varphi_j)}\big) = (1-t)e^{i\varphi_{j-1}} + te^{i\varphi_j}$$

for all $0 \leqslant t \leqslant 1$ and all $j \in \{1,\dots,M\}$. (Of course we will always identify φ_0 with φ_M.) We work in the following discrete function space:

$$C_h = \big\{\gamma_h \in C^0(\partial B,\mathbb{R}^3)\ \big|\ \gamma_h \in P_1(\pi^{-1}[E_j])\ \forall j,\ \gamma_h(\partial B) \subset T_\mu,\ L(\gamma_h,\Pi) \neq 0\big\}, \qquad (23)$$

where $P_1(\pi^{-1}[E_j])$ is the set of polynomials of degree one over the arc $\pi^{-1}[E_j]$. Thus we have $C_h \subset C^*$. The space of discrete variations is defined by

$$H_h = \big\{\xi_h \in C^0(\partial B,\mathbb{R}^3)\ \big|\ \xi_h \in P_1(\pi^{-1}[E_j])\ \forall j\big\} \qquad (24)$$

and

$$\overline{H}_h = \left\{ \xi_h \in C^0(\partial B, \mathbb{R}) \,\middle|\, \xi_h \in P_1(\pi^{-1}[E_j]) \;\forall j \right\}.$$

Then $\overline{H}_h$ is an M-dimensional vector space and C_h is an open subset of the $3M$-dimensional vector space H_h. Consider $f \in C^0(\partial B, \mathbb{R}^n)$, $n \in \{1, 3\}$. Then we define the piecewise linear interpolant $I_h f \in H_h$ or $I_h f \in \overline{H}_h$ respectively by

$$I_h f\big((1-t)\varphi_{j-1} + t\varphi_j\big) = (1-t) f(\varphi_{j-1}) + t f(\varphi_j)$$

for all $0 \leqslant t \leqslant 1$ and all j, where we identified the function f defined on ∂B with a 2π-periodic function defined on $\mathbb{R}$. (Notice, that this differs from the definition in [5] and [6], where this operator is defined on ∂B_h rather than on ∂B.)

Clearly the class C_h is not empty, if h is small enough and C^* is not empty. Note that Theorem 5 yields the existence of a solution γ_λ of P^*_λ, which belongs to $\overline{T_{\mu_0}}$ by Corollary 7. Thus for h small enough we have $I_h \gamma_\lambda \in C_h$.

DEFINITION 16 For $\gamma_h \in C_h$ the discrete functional is defined by

$$E_h^\lambda(\gamma_h) := D_h\left(\Phi_h(\gamma_h(\pi^{-1}))\right) + \lambda \int_{\partial B} I_{\frac{h}{2}}\left(G(\gamma_h)^2\right) \sqrt{|\gamma_h'|^2 + \frac{1}{\lambda}} + \frac{1}{\lambda} \int_{\partial B} \left(|\gamma_h'|^2 + (I_h \varrho)|\gamma_h - P|^2\right). \quad (25)$$

Here $D_h(\Phi_h(f_h)) = \frac{1}{2}\int_{B_h} |\nabla \Phi_h(f_h)|^2$ is the Dirichlet integral of the discrete harmonic extension Φ_h of a suitable discrete function f_h.

REMARK The discrete harmonic extension is defined in the usual way (see for instance [5, § 4.1]). In order to obtain a good algorithm, we have chosen a more refined interpolation $I_{\frac{h}{2}}$ instead of I_h for the function G^2 (see [9] or [10] for details). This allows us to use coarser grids.

Notice that E_h^λ is not the restriction of E_λ to C_h. In contrast to the functional (19) we have $E_h^\lambda(\gamma_h) \in C^2(C_h, \mathbb{R})$. For $\gamma_h \in C_h$, $\xi_h, \eta_h \in H_h$ we derive the first and second variation:

$$\begin{aligned} \delta E_h^\lambda(\gamma_h)(\xi_h) = & \int_{B_h} \left\langle \nabla \Phi_h(\gamma_h(\pi^{-1})), \nabla \Phi_h(\xi_h(\pi^{-1})) \right\rangle \\ & + \lambda \int_{\partial B} \left(2 I_{\frac{h}{2}}\left(G(\gamma_h)\langle \nabla G(\gamma_h), \xi_h\rangle\right) \sqrt{|\gamma_h'|^2 + \frac{1}{\lambda}} + I_{\frac{h}{2}}\left(G(\gamma_h)^2\right) \frac{\langle \gamma_h', \xi_h' \rangle}{\sqrt{|\gamma_h'|^2 + \frac{1}{\lambda}}} \right) \\ & + \frac{2}{\lambda} \int_{\partial B} \left(\langle \gamma_h', \xi_h' \rangle + (I_h \varrho) \langle \gamma_h - P, \xi_h \rangle \right), \end{aligned} \quad (26)$$

$$\delta^2 E_h^\lambda(\gamma_h)(\xi_h,\eta_h) = \int_{B_h} \left\langle \nabla \Phi_h(\xi_h(\pi^{-1})), \nabla \Phi_h(\eta_h(\pi^{-1}))\right\rangle$$
$$+ \lambda \int_{\partial B} \left(2I_{\frac{h}{2}}\left(\langle \nabla G(\gamma_h), \xi_h\rangle \langle \nabla G(\gamma_h), \eta_h\rangle\right) \sqrt{|\gamma_h'|^2 + \frac{1}{\lambda}} \right.$$
$$+ 2I_{\frac{h}{2}}\left(G(\gamma_h)\langle \xi_h, \nabla^2 G(\gamma_h)\eta_h\rangle\right) \sqrt{|\gamma_h'|^2 + \frac{1}{\lambda}}$$
$$+ 2I_{\frac{h}{2}}\left(G(\gamma_h)\langle \nabla G(\gamma_h), \xi_h\rangle\right) \frac{\langle \gamma_h', \eta_h'\rangle}{\sqrt{|\gamma_h'|^2 + \frac{1}{\lambda}}} + 2I_{\frac{h}{2}}\left(G(\gamma_h)\langle \nabla G(\gamma_h), \eta_h\rangle\right) \frac{\langle \gamma_h', \xi_h'\rangle}{\sqrt{|\gamma_h'|^2 + \frac{1}{\lambda}}}$$
$$\left. + I_{\frac{h}{2}}\left(G(\gamma_h)^2\right) \frac{\langle \xi_h', \eta_h'\rangle \left(|\gamma_h'|^2 + \frac{1}{\lambda}\right) - \langle \gamma_h', \xi_h'\rangle \langle \gamma_h', \eta_h'\rangle}{\left(|\gamma_h'|^2 + \frac{1}{\lambda}\right)^{\frac{3}{2}}} \right)$$
$$+ \frac{2}{\lambda} \int_{\partial B} \left(\langle \xi_h', \eta_h'\rangle + (I_h \varrho)\langle \xi_h, \eta_h\rangle \right).$$

DEFINITION 17 A function $\gamma_h \in C_h$ is called a "Solution of the Discrete Problem", if $\delta E_h^\lambda(\gamma_h)(\xi_h) = 0$ for all $\xi_h \in H_h$.

To compute solutions of the discrete problem we use a damped Newton Algorithm:

Algorithm

0. Choose an initial parametrization γ_h and a tolerance ϵ.
1. Compute $\delta E_h^\lambda(\gamma_h)$.
2. If $\|\delta E_h^\lambda(\gamma_h)\|_{H_h'} \leqslant \epsilon$, then go to Step 7.
3. Compute $\delta^2 E_h^\lambda(\gamma_h)$.
4. Solve the linear problem $\delta^2 E_h^\lambda(\gamma_h)(\eta_h, \xi_h)$=-$\delta E_h^\lambda(\gamma_h)(\xi_h)$ $\forall \xi_h \in H_h$.
5. If $\|\delta E_h^\lambda(\gamma_h)\|_{H_h'} \leqslant \|\delta E_h^\lambda(\gamma_h + \eta_h)\|_{H_h'}$, set $\eta_h := \frac{\eta_h}{2}$ and do Step 5 again.
6. Update the solution: $\gamma_h := \gamma_h + \eta_h$ and go to Step 1.
7. Compute the discrete surface $\Phi_h(\gamma_h(\pi^{-1}))$ and the value $E_h^\lambda(\gamma_h)$ and stop.

For the implementation we have used Mathematica (Version 10.2). In particular the meshes were produced with the Mathematica function "DiscretizeRegion". It is convenient to use the functional

$$F_{\lambda,\eta}(X) = D(X) + \lambda \int_{\partial B} G(X)^2 \sqrt{|X_\varphi|^2 + \frac{1}{\eta}} + \frac{1}{\eta} \int_{\partial B} \left(|X_\varphi|^2 + \varrho |X - P|^2\right)$$

instead of (1) as well as a discrete version $E_h^{\lambda,\eta}$ instead of (25). By setting $\eta = C\lambda$ with a fixed constant $C > 0$, η directly depends on λ. As before we again write F_λ, X_λ etc. In the following examples we have set $P = (1, 0, 0)$ and

$$\varrho(\varphi) = \begin{cases} 1 - \frac{\varphi}{\pi} & \text{in } [0, \pi] \\ \frac{\varphi}{\pi} - 1 & \text{in } [\pi, 2\pi] \end{cases}.$$

Example 1: We consider the function

$$G(x, y, z) = (x^2 + y^2 + z^2 + 2)^2 - 9(x^2 + y^2)$$

whose zero level set defines a torus in $\mathbb{R}^3$, and are interested in the order of convergence which can be observed numerically (see Theorem 10). The solution of the free boundary problem (see Theorem 1) is $\tilde{X}(x, y, z) = (x, y, 0)$ and therefore $D(\tilde{X}) = \pi$. For λ_1, λ_2 and discrete solutions $\gamma_h^{\lambda_1}$ of $E_h^{\lambda_1}$ and $\gamma_h^{\lambda_2}$ of $E_h^{\lambda_2}$ respectively we define the experimental order of convergence *eoc* by

$$eoc(E_h^\lambda) = \ln\left(\frac{\left|E_h^{\lambda_1}(\gamma_h^{\lambda_1}) - D(\tilde{X})\right|}{\left|E_h^{\lambda_2}(\gamma_h^{\lambda_2}) - D(\tilde{X})\right|}\right) \Big/ \ln\left(\frac{\lambda_2}{\lambda_1}\right)$$

and

$$eoc(D_h) = \ln\left(\frac{\left|D_h\left(\Phi_h(\gamma_h^{\lambda_1}(\pi^{-1}))\right) - D(\tilde{X})\right|}{\left|D_h\left(\Phi_h(\gamma_h^{\lambda_2}(\pi^{-1}))\right) - D(\tilde{X})\right|}\right) \Big/ \ln\left(\frac{\lambda_2}{\lambda_1}\right).$$

Note that since we deal here with the discrete functionals D_h and E_h^λ we cannot expect convergence for fixed h as $\lambda \to \infty$. We take a fine grid ($h = 0.07$), the tolerance $\epsilon = 0.001$ and $C = 100$ (i.e., $\eta = 100\lambda$).

The results are shown in Table 1 and one can observe that the experimental order of convergence lies about 1. This is notably better than the order $\frac{1}{9}$, which we can prove for a polynomial of degree four (see Remark after Theorem 10). Some heuristic calculations indicate that the order of convergence in λ derived in Theorem 10 can indeed be improved when the supporting surface S is a torus: however, whether this can be done for different S is still not clear.

Comparing the two graphics in Figure 4, we can observe convergence: for $\lambda = 1$ there is still a visible gap between the boundary of the solution and the supporting surface S, whereas the gap seems to disappear for $\lambda = 16$.

Both pictures show also the effect of the fourth term in the functional E_h^λ: the solution is pulled to the right side (in direction of the point P) where the triangles are smaller than at the left side.

Example 2: By

$$G(x, y, z) = \left(x^2 + y^2 - z^2 - 10\right)\left(x^2 + z^2 - 1\right) - 10$$

a non-torus-type support surface is given. We choose $\lambda = 0.1$, $C = 10000$, $h = 0.13443$ and $\epsilon = 0.001$ and get $D_h = 42.9789$ and $E_h^\lambda = 43.2267$ (see Figure 5).

TABLE 1. EOCs for the test problem described in Example 1

λ	$E_h^\lambda(\gamma_h)$	$eoc(E_h^\lambda)$	$D_h(\gamma_h)$	$eoc(D_h)$
1	3.18966	—	3.05301	—
2	3.16601	0.97715	3.09751	1.00681
4	3.15383	0.99662	3.11951	0.99730
8	3.14768	1.00741	3.13045	0.98682
16	3.14464	0.99826	3.13592	0.97400

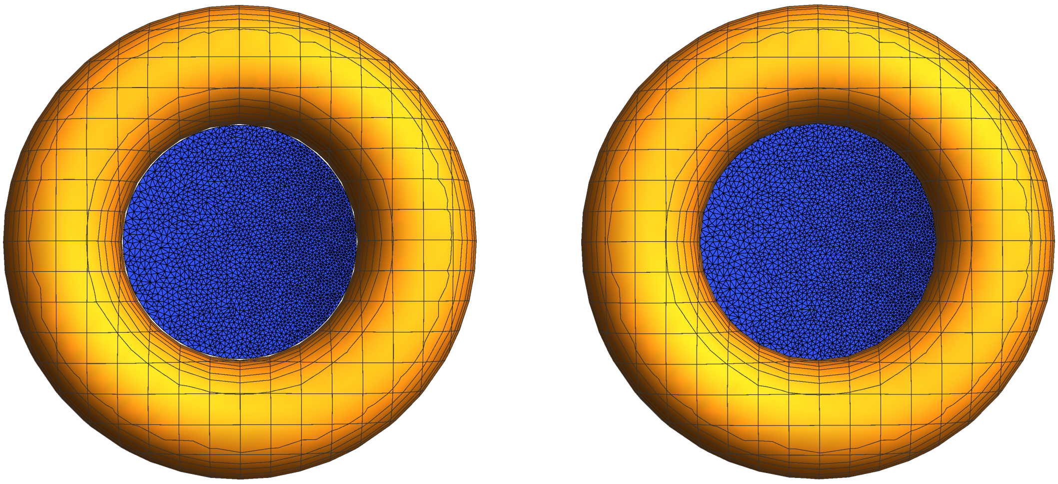

FIG. 4. Discrete solutions for Example 1 for $\lambda = 1$ and $\lambda = 16$

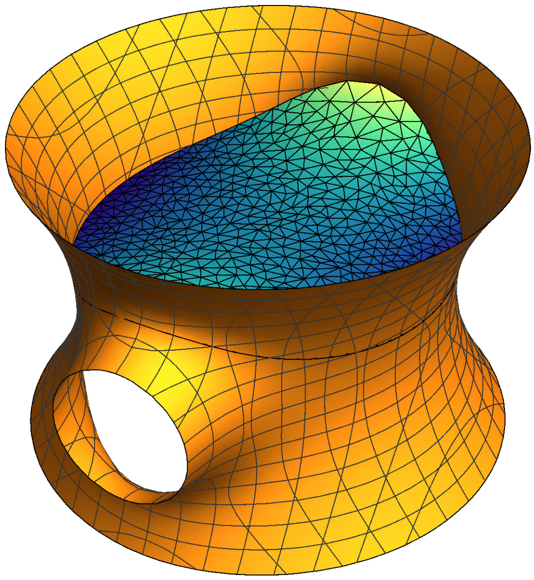

FIG. 5. A discrete solution for Example 2

Example 3: One can also simulate the classical Plateau Problem or the partially free boundary problem by using a thin torus-type surface or a part of this. By

$$G(x,y,z) = \Big((x^2+y^2+z^2+4)^2 - 16(x^2+y^2)\Big)\Big((x^2+y^2+z^2-1)x + \frac{1}{10}\Big) - \delta,$$

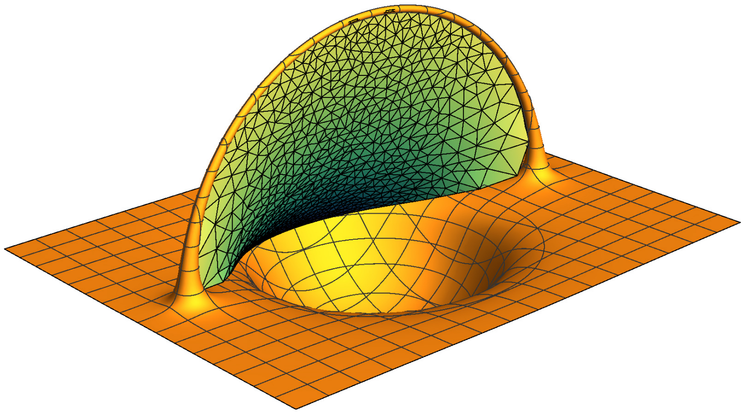

FIG. 6. A discrete solution for Example 3

where $\delta = \frac{1}{10}$ a surface is given, which "almost defines" a partially free boundary problem. Choosing $\epsilon = 0.001$, $C = 100$, $h = 0.13443$ and $\lambda = 16$ we get $D_h = 7.45652$ and $E_h^\lambda = 7.51755$ (see Figure 6). Note that for $\delta \to 0$ the thickness of the thin torus around the semi-arc tends to zero.

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