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Free boundary regularity for a degenerate problem with right hand side

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We consider a one-phase free boundary problem for p-Laplacian with non-zero right hand side. We use the approach present in [6] to prove that flat free boundaries and Lipschitz free boundaries are $C^{1,\gamma}$.

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1. Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^n$ and $p \geqslant 2$ we consider the degenerate problem

$$\begin{cases} \Delta_p u = f, & \text{in} \quad \Omega_+(u), \\ |\nabla u|^p = Q, & \text{on} \quad \mathfrak{F}(u). \end{cases} \tag{1.1}$$

Here, as usually $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $Q \geqslant 0$ is a $C^{0,\alpha}$-continuous function, $f \in L^\infty(\Omega) \cap C(\Omega)$ and

$$\Omega^+(u) := \{x \in \Omega : u(x) > 0\} \quad \text{and} \quad \mathfrak{F}(u) := \partial\Omega^+(u) \cap \Omega.$$

The study of the regularity of the free boundary $\mathfrak{F}(u)$ to the problem (1.1) has a large literature:

1. *Variational approach.* The case $f = 0$ and $p = 2$ was studied in the seminal work of Alt and Caffarelli [1]. Danielli and Petrosyan in [5] established the regularity of $\mathfrak{F}(u)$ for $f = 0$ and $p > 2$. Recently, Lederman and Wolansky in [9] completed the study of the regularity of the free boundary for the case $f \neq 0$ and $p > 2$.
2. *Non-variational approach.* The case $f = 0$ and $p = 2$ was studied in [2–4] and for $p = 2$ and $f \neq 0$ the regularity of $\mathfrak{F}(u)$ for the problem (1.1) was obtained in [6]. In [11, 12] a theory for general two-phase free boundary problems for the p-Laplace operator was developed in the homogenous case $f = 0$.

In this paper we will develop the regularity theory of $\mathcal{F}(u)$ through a non-variational approach. Precisely, we will apply the technique presented in [6] to prove that flat free boundaries are $C^{1,\gamma}$ (see Section 2 for the definition of viscosity solutions):

Theorem 1.1 *Let u be a viscosity solution to (1.1) in ball $B_1(0)$. Suppose that $0 \in \mathfrak{F}(u)$ and $Q(0) = 1$. There exists a universal constant $\tilde{\varepsilon} > 0$ such that, if the graph of u is $\tilde{\varepsilon}$-flat in $B_1(0)$, i.e.*

$$(x_n - \tilde{\varepsilon})^+ \leqslant u(x) \leqslant (x_n + \tilde{\varepsilon})^+ \ \textit{for} \ x \in B_1(0),$$

and

$$\|f\|_{L^\infty(B_1(0))} \leqslant \tilde{\varepsilon}, \ \ [Q]_{C^{0,\beta}(B_1(0))} \leqslant \tilde{\varepsilon},$$

then $F(u)$ is $C^{1,\gamma}$ in $B_{\frac{1}{2}}(0)$.

As in [6], the strategy of the proof of Theorem 1.1 is to obtain the *improvement of flatness* property for the graph of a solution u: if the graph of u oscillates ε away from a hyperplane in B_1 then in $B_{\delta_0}(0)$ it oscillates $\frac{\delta_0\varepsilon}{2}(0)$ away from possibly a different hyperplane. The fundamental steps to achieve this property are: Harnack type Inequality and Limiting solution. In our problem, the structure of the operator Δ_p requires some changes. In next section, we comment on the main difficulties we came across and how to overcome them.

Moreover, through a blow-up from Theorem 1.1 and the approach used in [2], we obtain the our second main result:

Theorem 1.2 (Lipschitz implies $C^{1,\gamma}$) *Let u be a viscosity solution for the free boundary problem*

$$\begin{cases} \Delta_p u = f, & \textit{in} \quad \Omega_+(u), \\ |\nabla u|^p = Q, & \textit{on} \quad \mathfrak{F}(u). \end{cases}$$

Assume that $0 \in \mathfrak{F}(u)$, $f \in L^\infty(B_1)$ is continuous in $B_1^+(u)$ and $Q(0) > 0$. If $\mathfrak{F}(u)$ is a Lipschitz graph in a neighborhood of 0, then $\mathfrak{F}(u)$ is $C^{1,\gamma}$ in a (smaller) neighborhood of 0.

In Theorem 1.2, the size of the neighborhood where $\mathfrak{F}(u)$ is $C^{1,\gamma}$ depends on the radius r of the ball B_r where $\mathfrak{F}(u)$ is Lipschitz, the Lipschitz norm of $\mathfrak{F}(u)$, β, $\|f\|_\infty$ and dimension n.

Finally, we further mention that Theorem 1.1 can be established, with minor modifications, to more general operator. For example, if we consider the problem

$$\begin{cases} \operatorname{div}(A|\nabla u|^{p-2}\nabla u) = f & \text{in} \quad \Omega_+(u) \\ \langle A|\nabla u|^{p-2}\nabla u, \nabla u\rangle = Q & \text{on} \quad \mathfrak{F}(u), \end{cases}$$

where the matrix A is assumed to be Lipschitz and positive definite, we extend the results obtained in [10].

The paper is organized as follows. In Section 2 we define the notion of viscosity solution to the free boundary problem (1.1) and gather few tools that we shall use in the proofs of Theorem 1.1 and Theorem 1.2. In Section 3 we present the proof of Harnack type inequality. Section 4 is devoted to the proof of improvement of flatness and in Section 5 we establish the regularity of the free boundary $\mathfrak{F}(u)$.

2. Mathematical set-up

Let us move towards the hypotheses, set-up and main notations used in this article. For B_1 we denote the open unit ball in the Euclidean space $\mathbb{R}^n$. We start by gathering some basic information of the limiting configuration. We shall use viscosity solution setting to access the free boundary regularity theory.

DEFINITION 2.1 Given two continuous functions u and ϕ defined in an open Ω and a point $x_0 \in \Omega$, we say that ϕ touches u by below (resp. above) at x_0 whenever $u(x_0) = \phi(x_0)$

$$u(x) \geqslant \phi(x) \text{ (resp. } u(x) \leqslant \phi(x)) \text{ in a neighborhood } \mathcal{O} \text{ of } x_0.$$

If this inequality is strict in $\mathcal{O} \setminus \{x_0\}$, we say that ϕ touches u strictly by below (resp. above).

DEFINITION 2.2 Let $u \in C(\Omega)$ nonnegative. We say that u is a viscosity solution to

$$\begin{cases} \Delta_p u = f, & \text{in} \quad \Omega_+(u), \\ |\nabla u|^p = Q, & \text{on} \quad \mathfrak{F}(u). \end{cases} \tag{2.1}$$

if and only if the following conditions are satisfied:

(F1) If $\phi \in C^2(\Omega^+(u))$ touches u by below (resp. above) at $x_0 \in \Omega^+(u)$ then

$$\Delta_p \phi(x_0) \leqslant f(x_0) \ \big(\text{resp. } \Delta_p \phi(x_0) \geqslant f(x_0)\big).$$

(F2) If $\phi \in C^2(\Omega)$ and ϕ^+ touches u below (resp. above) at $x_0 \in \mathfrak{F}(u)$ and $|\nabla \phi|(x_0) \neq 0$ then

$$|\nabla \phi|^p(x_0) \leqslant Q(x_0) \ \big(\text{resp. } |\nabla \phi|^p(x_0) \geqslant Q(x_0)\big).$$

We refer to the usual definition of subsolution, supersolution and solution of a degenerate PDE. Let us introduce the notion of comparison subsolution/supersolution.

DEFINITION 2.3 We say $u \in C(\Omega)$ is a strict (comparison) subsolution (resp. supersolution) to (1.1) in Ω, if only if $u \in C^2(\overline{\Omega^+(v)})$ and the following conditions are satisfied:

(G1) $\Delta_p u > f(x)$ (resp. $< f$) in $\Omega^+(u)$;
(G2) If $x_0 \in \mathfrak{F}(u)$, then

$$|\nabla u|^p(x_0) > Q(x_0) \ \big(\text{resp. } 0 < |\nabla u|^p(x_0) < Q(x_0)\big).$$

The next lemma provides a basic comparison principle for solutions to the free boundary problem (1.1). The Lemma below yields the crucial tool in the proof of main result.

Lemma 2.4 *The following remark is a consequence of the definitions above: Let u, v be respectively a solution and a strict subsolution to* (1.1) *in Ω. If $u \geqslant v^+$ in Ω then $u > v^+$ in $\Omega^+(v) \cup \mathfrak{F}(v)$.*

Difficulties and changes

1. Harnack type Inequality. When we consider the problem (1.1) for $p \geqslant 2$ the main difficulty lies in the following fact: if p is an affine function and u is a solution to the problem

$$\Delta_p u = f, \quad \text{in } B_r(x_0), \tag{2.2}$$

we can not conclude that $u + p$ is a solution to the equation (2.2). For $p = 2$ we know $u + p$ is still solution for the problem (2.2). In [6], this fact is important because it allows us to apply Harnack Inequality for $v = u + p$ which is crucial to reach a *improvement of flatness* for the graph of u . We overcome this difficulty by observing that $u + p$ is a solution for the problem

$$\mathfrak{L}_{p,e} v := \operatorname{div}\big(|\nabla v + e|^{p-2}(\nabla v + e)\big) = f, \quad \text{in } B_r\big(x_0\big),$$

where $e \in \mathbb{R}^n$ with $|e| = 1$ and the operator $\mathfrak{L}_{p,e}$ behaves like the Δ_p. Precisely, we use the following result:

Lemma 2.5 (Harnack) *Let u be a nonnegative viscosity solution to*

$$div\big(|\nabla v + e|^{p-2}(\nabla v + e)\big) = f, \quad in\ B_{2r}, \tag{2.3}$$

where $|e| = 1$. *Then, there exists a constant* C *depending only on* n *and* p *such that*

$$\sup_{B_r} v \leqslant C\Big\{\inf_{B_r} v + r(\|f\|_{L^\infty(B_{2r})} + C)\Big\}.$$

Proof. Define $\mathfrak{A} : \mathbb{R}^n \to \mathbb{R}^n$

$$\mathfrak{A}(\eta) = |\eta + e|^{p-2}(\eta + e). \tag{2.4}$$

Notice that

$$|\mathfrak{A}(\eta)| = |\eta + e|^{p-1} \leqslant C(p)|\eta|^{p-1} + C(p), \quad \text{for all } \eta \in \mathbb{R}^n. \tag{2.5}$$

On the other hand, we have

$$\begin{aligned} \langle \mathfrak{A}(\eta), \eta\rangle &= |\eta + e|^{p-2}\langle(\eta + e), \eta\rangle \\ &= |\eta + e|^{p-2}\langle(\eta + e), \eta + e - e\rangle \\ &= |\eta + e|^{p} - |\eta + e|^{p-2}\langle\eta + e, e\rangle \\ &\geqslant |\eta + e|^{p} - |\eta + e|^{p-1}, \quad \text{for all } \eta \in \mathbb{R}^n. \end{aligned} \tag{2.6}$$

Thus, if $|\eta + e|^{p-1} \leqslant 2$ we obtain

$$\begin{aligned} \langle \mathfrak{A}(\eta), \eta\rangle &\geqslant |\eta + e|^{p} - 2^{p-1} \\ &\geqslant c(p)|\eta|^{p} - (1 + 2^{p-1}), \end{aligned} \tag{2.7}$$

and for $|\eta + e|^{p-1} > 2$ we have

$$\begin{aligned} \langle \mathfrak{A}(\eta), \eta\rangle &\geqslant |\eta + e|^{p} - |\eta + e|^{p-1} \\ &= |\eta + e|^{p}(1 - |\eta + e|^{-1}) \\ &\geqslant \frac{1}{2}|\eta + e|^{p} \\ &\geqslant \frac{1}{2}\big(c(p)|\eta|^{p} - 1\big). \end{aligned} \tag{2.8}$$

Then, using (2.6), (2.7) and (2.8) we estimate

$$\langle \mathfrak{A}(\eta), \eta\rangle \geqslant c_1(p)|\eta|^{p} - C_1(p), \quad \text{for all } \eta \in \mathbb{R}^n. \tag{2.9}$$

Hence, by classical theory (see [8] and [14]) the result follows. □

2. Limiting solution. In what follows, we denote by

$$B_\rho^+ := \big\{x \in \mathbb{R}^n : |x| < \rho, x_n > 0\big\} \tag{2.10}$$

$$\Upsilon_\rho := \big\{x \in \mathbb{R}^n : |x| < \rho, x_n = 0\big\} \tag{2.11}$$

Our main result will follow from the regularity properties of solutions to the classical Neumann problem for the constant coefficient linear equation

$$\begin{cases} \mathfrak{L}_p u_\infty = 0, & \text{in} \quad B_\rho^+, \\ \frac{\partial u_\infty}{\partial \nu} = 0, & \text{on} \quad \Upsilon_\rho, \end{cases} \tag{2.12}$$

where $\nu := (0, 0, \ldots, 0, 1)$ and $\mathfrak{L}_p u := \Delta u + (p-2)\partial_{nn} u$. We use the notion of viscosity solution to (2.12).

DEFINITION 2.6 Let $u_\infty \in C(B_\rho \cap \{x_n \geqslant 0\})$. We say that u_∞ is a viscosity solution to (2.12) if given $P(x)$ a quadratic polynomial touching u_∞ by below (i. above) at $x_0 \in B_\rho \cap \{x_n \geqslant 0\}$, then

(i) if $x_0 \in B_\rho^+$ then $\mathfrak{L}_p P(x_0) \leqslant 0$ (resp. $\mathfrak{L}_p P(x_0) \geqslant 0$);
(ii) if $x_0 \in \Upsilon_\rho$ then $\frac{\partial P(x_0)}{\partial \nu} \leqslant 0$ (resp. $\frac{\partial P(x_0)}{\partial \nu} \geqslant 0$)

REMARK 2.7 Notice that, in the definition above we can choose polynomials P that touch u_∞ strictly by below/above. Also, it suffices to verify that (ii) holds for polynomials $\tilde{P}$ with $\mathfrak{L}_p \tilde{P} > 0$ (see [6]).

The proof this result is classical and will be omitted (see for example [6]).

Lemma 2.8 *Let u_∞ be a viscosity solution to*

$$\begin{cases} \mathfrak{L}_p u_\infty = 0, & \text{in} \quad B_\rho^+ \\ \frac{\partial u_\infty}{\partial \nu} = 0, & \text{on} \quad \Upsilon_\rho \end{cases} \tag{2.13}$$

with $\|u_\infty\|_{L^\infty} \leqslant 1$. There exists a universal constant $C_0 > 0$ such that

$$\left|u_\infty(x) - u_\infty(0) - \nabla u_\infty(0) \cdot x\right| \leqslant C_0 \rho^2 \quad \text{in } B_\rho \cap \{x_n \geqslant 0\}.$$

REMARK 2.9 When $p = 2$ the operator $\mathfrak{L}_2$ coincides with the Laplacian, i.e., we obtain the DeSilva's limiting equation.

3. Harnack type inequality

In this section, based on comparison principle granted in Lemma 2.4, we prove a Harnack type inequality for a solution u to the problem (1.1) with the following conditions:

$$\|f\|_{L^\infty(\Omega)} \leqslant \varepsilon^2, \tag{3.1}$$

$$\|Q - 1\|_{L^\infty(\Omega)} \leqslant \varepsilon^p, \tag{3.2}$$

for $0 < \varepsilon < 1$.

Lemma 3.1 *Let u be a viscosity solution to (1.1) in Ω, under assumptions (3.1)–(3.2). There exists a universal constant $\tilde{\varepsilon} > 0$ such that if $0 < \varepsilon \leqslant \tilde{\varepsilon}$ and u satisfies*

$$p^+(x) \leqslant u(x) \leqslant \left(p(x) + \varepsilon\right)^+, \quad |\sigma| < \frac{1}{20} \quad \text{in } B_1(0), \quad p(x) = x_n + \sigma, \tag{3.3}$$

then if at $x_0 = \frac{1}{10} e_n$

$$u(x_0) \geqslant \left(p(x_0) + \frac{\varepsilon}{2}\right)^+, \tag{3.4}$$

then

$$u \geqslant (p + c\varepsilon)^{+} \quad \textit{in} \ \overline{B}_{\frac{1}{2}}(0), \tag{3.5}$$

for some $0 < c < 1$. *Analogously, if*

$$u(x_0) \leqslant \big(p(x_0) + \frac{\varepsilon}{2}\big)^{+}, \tag{3.6}$$

then

$$u \leqslant \big(p + (1-c)\varepsilon\big)^{+} \quad \textit{in} \ \overline{B}_{\frac{1}{2}}(0). \tag{3.7}$$

Proof. We verify (3.5). The proof of (3.7) is analogous. Notice that

$$B_{\frac{1}{20}}(x_0) \subset B_1^{+}(u). \tag{3.8}$$

From [15] we know that u is $C^{1,\alpha}$ in $B_{\frac{1}{40}}(x_0)$ with

$$[u]_{1+\alpha, B_{1/40}(x_0)} \leqslant C,$$

where $\alpha = \alpha(n, p) \in (0,1)$ and $C = C(n, p) > 1$. Now we consider two cases:

Case 1: $|\nabla u(x_0)| < \frac{1}{4}$. Choose $r_1 = r_1(n, p) > 0$ such that

$$|\nabla u| \leqslant \frac{1}{2} \quad \text{in } B_{r_1}(x_0). \tag{3.9}$$

There exists a constant $r_2 = r_2(r_1) = r_2(n, p) > 0$ that satisfies

$$(x - r_2 e_n) \in B_{r_1}(x_0), \quad \text{for all } x \in B_{\frac{r_1}{2}}(x_0).$$

Notice that $v = u - p$ satisfies

$$\mathfrak{L}_{p,e_n} v = f, \quad \text{in} \ B_{1/20}(x_0),$$

where $e_n := (0, \ldots, 1) \in \mathbb{R}^n$. In particular, for $r_3 = \min\left\{\frac{r_1}{4}, \frac{r_2}{8}\right\}$ we apply the Lemma 2.5 in $B_{2r_3}(x_0)$ to obtain

$$u(x) - p(x) \geqslant c_0\big(u(x_0) - p(x_0)\big) - 4r_3 \geqslant \frac{c_0\varepsilon}{2} - 4r_3 \tag{3.10}$$

for all $x \in B_{r_3}(x_0)$. From (3.9) and (3.10) we can write

$$\begin{aligned}
\frac{c_0\varepsilon}{2} - 4r_3 &\leqslant u(x) - p(x) \\
&= u\big((x - r_2e_n) + r_2e_n\big) - p\big((x - r_2e_n) + r_2e_n\big) \\
&= u\big((x - r_2e_n) + r_2e_n\big) - p\big((x - r_2e_n)\big) - r_2 \\
&\leqslant u\big((x - r_2e_n)\big) - p\big((x - r_2e_n)\big) + \frac{r_2}{2} - r_2,
\end{aligned}$$

for all $x \in B_{r_3}(x_0)$. Thus, we find

$$\begin{aligned}\frac{c_0\varepsilon}{2} &\leqslant \frac{c_0\varepsilon}{2} - 4r_3 + \frac{r_2}{2} \\ &\leqslant u(x) - p(x),\end{aligned} \tag{3.11}$$

for all $x \in B_{r_3}(\overline{x}_0)$, where $\overline{x}_0 = x_0 - r_2 e_n$. Let $w\colon \overline{D} \to \mathbb{R}$ be defined by

$$w(x) = c\left(|x - \overline{x}_0|^{-\gamma} - \left(\frac{4}{5}\right)^{-\gamma}\right), \tag{3.12}$$

where $D := B_{\frac{4}{5}}(\overline{x}_0) \setminus \overline{B}_{r_3}(\overline{x}_0)$. We choose $c = c(n, p, \gamma) > 0$ such that

$$w = \begin{cases} 0, & \text{on} \quad \partial B_{\frac{4}{5}}(\overline{x}_0), \\ 1, & \text{on} \quad \partial B_{r_3}(\overline{x}_0). \end{cases} \tag{3.13}$$

Now define

$$v(x) = p(x) + \frac{c_0\varepsilon}{2}\big(w(x) - 1\big), \quad x \in \overline{B}_{\frac{4}{5}}(\overline{x}_0), \tag{3.14}$$

and for $t \geqslant 0$,

$$v_t(x) = v(x) + t, \quad x \in \overline{B}_{\frac{4}{5}}(\overline{x}_0). \tag{3.15}$$

By choice of c we have $w \leqslant 1$ in D. Then, extending w to 1 in $B_{r_3}(\overline{x}_0)$ we find

$$v_0(x) = v(x) \leqslant p(x) \leqslant u(x), \quad x \in \overline{B}_{\frac{4}{5}}(\overline{x}_0). \tag{3.16}$$

Consider

$$t_0 = \sup\big\{t \geqslant 0 : v_t \leqslant u \ \text{ in } \ \overline{B}_{\frac{4}{5}}(\overline{x}_0)\big\}.$$

Assume, for the moment, that we have already verified $t_0 \geqslant \frac{c_0\varepsilon}{2}$. From definition of v we have

$$u(x) \geqslant v(x) + t_0 \geqslant p(x) + \frac{c_0\varepsilon}{2} w(x), \quad \forall x \in B_{\frac{4}{5}}(\overline{x}_0).$$

Notice that $B_{\frac{1}{2}}(0) \subset B_{\frac{3}{5}}(\overline{x}_0)$ and

$$w(x) \geqslant \begin{cases} c\left[(\frac{3}{5})^{-\gamma} - (\frac{4}{5})^{-\gamma}\right], & \text{in} \quad B_{\frac{3}{5}}(\overline{x}_0) \setminus B_{r_3}(\overline{x}_0), \\ 1, & \text{on} \quad B_{r_3}(\overline{x}_0). \end{cases}$$

Hence, we conclude (ε small) that

$$u(x) - p(x) \geqslant c_1\varepsilon, \quad \text{in} \quad B_{1/2}(0),$$

and the result is proved. Let us now prove that indeed $t_0 \geqslant \frac{c_0\varepsilon}{2}$. For that, we suppose for the sake of contradiction that $t_0 < \frac{c_0\varepsilon}{2}$. Then there would exist $y_0 \in \overline{B}_{\frac{4}{5}}(\overline{x}_0)$ such that

$$v_t(y_0) = u(y_0).$$

In the sequel, we show that $y_0 \in B_{r_3}(\overline{x}_0)$. From definition of v_t and by the fact that w has zero boundary data on $\partial B_{4/5}(\overline{x}_0)$ we have

$$v_t = p - \frac{c_0\varepsilon}{2} + t_0 < u \ \text{ in } \ \partial B_{4/5}(\overline{x}_0),$$

where we have used that $u \geqslant p$ and $t_0 < \frac{c_0\varepsilon}{2}$. We compute directly,

$$\partial_i w = -\gamma\big(x_i - \overline{x}_0^i\big)|x - \overline{x}_0|^{-\gamma-2} \tag{3.17}$$

and

$$\partial_{ij} w = \gamma|x - \overline{x}_0|^{-\gamma-2}\big\{(\gamma+2)(x_i - \overline{x}_0^i)(x_j - \overline{x}_0^j)|x - \overline{x}_0|^{-2} - \delta_{ij}\big\}. \tag{3.18}$$

Moreover, since $|\nabla w| \leqslant C(n, p, \gamma)$ in D we find

$$\frac{1}{2} \leqslant |\nabla v_t| \leqslant 2 \quad \text{in } \ D,$$

if $\varepsilon > 0$ is small. Thus, if $\gamma = \gamma(n, p) > 1$ is large, from (3.17) and (3.18) we have

$$\begin{aligned}
\Delta_p v_t &= \operatorname{div}\big(|\nabla v_t|^{p-2}(c_0\varepsilon\nabla w)\big) + \operatorname{div}(|\nabla v_t|^{p-2}e_n)\\
&= c_0\varepsilon\sum_{i=1}^{n}|\nabla v_t|^{p-2}\partial_{ii}w + c_0^3\varepsilon^3\sum_{i,j=1}^{n}(p-2)|\nabla v_t|^{p-4}(\partial_i w)(\partial_j w)\partial_{ij}w\\
&\quad + c_0^2\varepsilon^2(p-2)|\nabla v_t|^{p-4}\sum_{i=1}^{n}\partial_i w\partial w_n\partial_{in}w + c_0^2\varepsilon^2(p-2)|\nabla v_t|^{p-4}\sum_{i=1}^{n}\partial_i w\partial_{in}w\\
&\quad + c_0\varepsilon|\nabla v_t|^{p-4}(p-2)\partial_{nn}w\\
&= \gamma|x - \overline{x}_0|^{-(\gamma+2)}c_0\varepsilon|\nabla v_t|^{p-2}\Big\{\gamma + 2 + (\gamma+2)\frac{(x_n - \overline{x}_0^n)^2}{|x - \overline{x}_0|^2|\nabla v_t|^2}(p-2) - (p-2) - n\Big\}\\
&\quad + c_0^2\varepsilon^2(p-2)|\nabla v_t|^{p-4}\Big\{c_0\varepsilon\sum_{i,j=1}^{n}(\partial_i w)(\partial_j w)\partial_{ij}w + \sum_{i=1}^{n}(1 + \partial w_n)\partial_i w\partial_{in}w\Big\}\\
&\geqslant (c_1 - C_2\varepsilon)\varepsilon\\
&> \varepsilon^2 \quad \text{in } \ D.
\end{aligned}$$

On the other hand, we have

$$|\nabla v_{t_0}| \geqslant |\partial_n v| = |1 + (c_0/2)\varepsilon\partial_n w|, \ \text{ in } \ D. \tag{3.19}$$

By radial symmetry of w, we have

$$\partial_n w(x) = |\nabla w(x)|\langle \nu_x, e_n\rangle, \ \ x \in D, \tag{3.20}$$

where ν_x is the unit vector in the direction of $x - \overline{x}_0$. From (3.17) we have

$$\begin{aligned}
|\nabla w| &= c\gamma|x - \overline{x}_0|^{-(\gamma+2)}|x - \overline{x}_0|\\
&= c\gamma|x - \overline{x}_0|^{-(\gamma+1)}\\
&\geqslant c_6 > 0, \quad \text{in } \ D.
\end{aligned}$$

Also we have $\langle \nu_x, e_n\rangle \geqslant c$ in $\{v_{t_0} \leqslant 0\} \cap D$ (for ε small enough). In fact, if ε is small enough

$$\left\{v_{t_0} \leqslant 0\right\} \cap D \subset \left\{p \leqslant \frac{c_0\varepsilon}{2}\right\} = \left\{x_n \leqslant \frac{c_0\varepsilon}{2} - \sigma\right\} \subset \{x_n < 1/20\}.$$

We therefore conclude that

$$\begin{aligned}\langle \nu_x, e_n\rangle &= \frac{1}{|\overline{x}_0 - x|}\langle \overline{x}_0 - x, e_n\rangle \\ &\geqslant \frac{5}{4}\langle \overline{x}_0 - x, e_n\rangle \\ &= \frac{5}{4}(\frac{1}{10} - r_2 - x_n + \frac{1}{20} - \frac{1}{20}) \\ &> c_7, \quad \text{in } \{v_{t_0} \leqslant 0\} \cap D.\end{aligned}$$

From (3.19) and (3.20) we obtain

$$\begin{aligned}|\nabla v_{t_0}|^2 &\geqslant |\partial_n v_{t_0}|^2 \\ &= 1 + 2\tilde{c}\varepsilon + \tilde{c}\varepsilon^2|\nabla w|^2 \\ &\geqslant 1 + 2c_9\varepsilon + c_{10}\varepsilon^2 \\ &\geqslant 1 + \varepsilon^2.\end{aligned}$$

Hence, we find

$$|\nabla v_{t_0}|^p \geqslant 1 + \varepsilon^p > Q \ \text{ in } \ D \cap \mathfrak{F}(v_{t_0})$$

in $\{v_{t_0} \leqslant 0\} \cap D$. In particular, we have

$$|\nabla v_{t_0}|^p > Q \ \text{ in } \ D \cap \mathfrak{F}(v_{t_0}).$$

Thus, v_{t_0} is a strict subsolution in D and by Lemma 2.4 (u is a viscosity solution of problem (1.1) in $B_1(0)$) we conclude that $y_0 \in B_{r_3}(x_0)$. This is a contradiction. In fact, we would get

$$u(y_0) = v_{t_0}(y_0) = v(y_0) + t_0 \leqslant p(y_0) + t_0 < p(y_0) + c_0\varepsilon.$$

which drives us to a contradiction to (3.11). The Lemma 3.1 is concluded.

Case 2: $|\nabla u(x_0)| \geqslant \frac{1}{4}$. Since u is $C^{1,\alpha}$ and Lipschitz continuous in $B_{\frac{1}{40}}(x_0)$, there exist constants $r_0 = r_0(n, p) > 0$ and $C = C(n, p) > 1$ such that

$$\frac{1}{8} \leqslant |\nabla u| \leqslant C \quad \text{in } B_{2r_0}(x_0). \tag{3.21}$$

Thus, u satisfies the uniformly elliptic equation

$$\sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij} v = f \quad \text{in } B_{2r_0}(x_0), \tag{3.22}$$

where

$$a_{ij} = |\nabla u|^{p-2}\left\{\delta_{ij} + (p-2)\frac{\partial_i u \partial_j u}{|\nabla u|^2}\right\}.$$

Then, by classical Harnack Inequality we obtain

$$\begin{aligned} u(x) - p(x) &\geqslant c_0\big(u(x_0) - p(x_0)\big) - C\|f\|_\infty \\ &\geqslant \frac{c_0\varepsilon}{2} - C_1\varepsilon^2 \\ &\geqslant c_1\varepsilon, \end{aligned}$$

for all $x \in B_{r_0}(x_0)$, if $\varepsilon > 0$ is sufficiently small. Now, we consider the barrie

$$w(x) = \begin{cases} c\left[|x - x_0|^{-\gamma} - (\frac{4}{5})^{-\gamma}\right], & \text{in} \quad B_{\frac{4}{5}}(x_0) \setminus B_{r_0}(x_0), \\ 1, & \text{on} \quad B_{r_0}(x_0), \end{cases}$$

and the Lemma 3.1 follows as in Case 1. □

Now we establish the main tool in the proof of Theorem 1.1.

Theorem 3.2 *Let u be a viscosity solution to (1.1) in Ω under assumptions (3.1)–(3.2). There exists a universal constant $\tilde{\varepsilon} > 0$ such that, if u satisfies at some $x_0 \in \Omega^+(u) \cup F(u)$,*

$$(x_n + a_0)^+ \leqslant u(x) \leqslant (x_n + d_0)^+ \ \ in \ \ B_r(x_0) \subset \Omega, \tag{3.23}$$

with $|a_0| < \frac{1}{20}$ and

$$d_0 - a_0 \leqslant \varepsilon r, \quad \varepsilon \leqslant \tilde{\varepsilon} \tag{3.24}$$

then

$$(x_n + a_1)^+ \leqslant u(x) \leqslant (x_n + d_1)^+ \ \ in \ \ B_{\frac{r}{40}}(x_0) \tag{3.25}$$

with

$$a_0 \leqslant a_1 \leqslant d_1 \leqslant d_0, \quad d_1 - a_1 \leqslant (1 - c)\varepsilon r, \tag{3.26}$$

and $0 < c < 1$ universal.

Proof. With no loss of generality, we can assume $x_0 = 0$ and $r = 1$. We put $p(x) = x_n + a_0$ and by (3.23)

$$p^+(x) \leqslant u(x) \leqslant (p(x) + \varepsilon)^+ \quad (d_0 \leqslant a_0 + \varepsilon). \tag{3.27}$$

Then, since

$$u\Big(\frac{1}{10}e_n\Big) \geqslant \Big(p\Big(\frac{1}{10}e_n\Big) + \frac{\varepsilon}{2}\Big)^+ \quad \text{or} \quad u\Big(\frac{1}{10}e_n\Big) < \Big(p\Big(\frac{1}{10}e_n\Big) + \frac{\varepsilon}{2}\Big)^+$$

we can apply Lemma 3.1 to obtain the result. □

From Harnack inequality, Theorem 3.2, precisely as in [6], we obtain the following key estimate for flatness improvement.

Corollary 3.3 *Let u be a viscosity solution to (1.1) in Ω under assumptions (3.1)–(3.2). If u satisfies (3.23) then in $B_1(x_0)$ the function $\tilde{u}_\varepsilon := \frac{u - x_n}{\varepsilon}$ has a Hölder modulus of continuity at X_0 outside of ball of radius $\varepsilon/\tilde{\varepsilon}$, i.e., for all $x \in (\Omega^+(u) \cup F(u)) \cap B_1(x_0)$ with $|x - x_0| \geqslant \varepsilon/\tilde{\varepsilon}$*

$$\big|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)\big| \leqslant C|x - x_0|^\gamma.$$

4. Improvement of flatness

In this section we prove the *improvement of flatness* lemma, from which the proof of main Theorem 1.1 will follow via an interactive argument. Next, we present the basic induction step towards $C^{1,\gamma}$ regularity at 0.

Theorem 4.1 (Improvement of flatness) *Let $u \in C(\Omega)$ be a viscosity solution to*

$$\begin{cases} \Delta_p u = f, & in \quad \Omega_+(u), \\ |\nabla u|^p = Q, & on \quad \mathfrak{F}(u). \end{cases} \tag{4.1}$$

with $(0 < \epsilon < 1)$

$$\max\left\{\|f\|_{L^\infty(\Omega)},\ \|Q-1\|_{L^\infty(\Omega)}\right\} \leqslant \epsilon^p. \tag{4.2}$$

Suppose that u satisfies

$$(x_n - \epsilon)^+ \leqslant u(x) \leqslant (x_n + \epsilon)^+ \quad for\ x \in B_1 \tag{4.3}$$

with $0 \in \mathfrak{F}(u)$. If $0 < r \leqslant r_0$ for r_0 a universal constant and $0 < \epsilon \leqslant \epsilon_0$ for some ϵ_0 depending on r, then

$$\left(\langle x, \nu\rangle - r\frac{\epsilon}{2}\right)^+ \leqslant u(x) \leqslant \left(\langle x, \nu\rangle + r\frac{\epsilon}{2}\right)^+ \quad x \in B_r, \tag{4.4}$$

with $|\nu| = 1$, and $|\nu - e_n| \leqslant C\epsilon^2$ for a universal constant $C > 0$.

Proof. We divide the proof of this Lemma into three steps. We use the following notation:

$$\Omega_\rho(u) := \left(B_1^+(u) \cup \mathfrak{F}(u)\right) \cap B_\rho.$$

Step 1 – Compactness Lemma: Fix $r \leqslant r_0$ with r_0 universal (the precise r_0 will be given in Step 3). Assume by contradiction that we can find a sequence $\epsilon_k \to 0$ and a sequence $\{u_k\}_{k\geqslant 1} \subset C(\Omega)$ be a sequence of viscosity solution to

$$\begin{cases} \Delta_p u_k = f_k & \text{in} \quad \Omega_1^+(u_k) \\ |\nabla u_k|^p = Q_k(x) & \text{on} \quad \mathfrak{F}(u_k) \end{cases} \tag{4.5}$$

with

$$\max\{\|f_k\|_{L^\infty},\ \|Q_k - 1\|_{L^\infty}\} \leqslant \epsilon_k^p, \tag{4.6}$$

as $k \to \infty$, such that

$$(x_n - \epsilon_k)^+ \leqslant u_k(x) \leqslant (x_n + \epsilon_k)^+ \quad \text{for } x \in B_1,\ 0 \in \mathfrak{F}(u_k) \tag{4.7}$$

but it does not satisfy the conclusion (4.4) of the Lemma. Let $v_k : \Omega_1(u_k) \to \mathbb{R}$ defined by

$$v_k(x) := \frac{u_k(x) - x_n}{\epsilon_k}.$$

Then (4.7) gives,

$$-1 \leqslant v_k(x) \leqslant 1 \quad \text{for } x \in \Omega_1(u_k). \tag{4.8}$$

From Corollary 3.3, it follows that the function v_k satisfies

$$|v_k(x) - v_k(y)| \leqslant C|x-y|^\gamma, \tag{4.9}$$

for C universal and

$$|x-y| \geqslant \epsilon_k/\bar{\epsilon}, \quad x, y \in \Omega_{1/2}(u_k).$$

From (4.7) it clearly follows that $\mathfrak{F}(u_k) \to B_1 \cap \{x_n = 0\}$ in the Hausdorff distance. This fact and (4.9) together with Arzelà–Ascoli give that as $\epsilon_k \to 0$ the graphs of the v_k over $\Omega_{1/2}(u_k)$ converge(up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function u_∞ over $B_{1/2} \cap \{x_n \geqslant 0\}$.

Step 2 – Limiting Solution: We claim that u_∞ is a solution of the problem

$$\begin{cases} \mathfrak{L}_p u_\infty = 0 \ \text{ in } \ B^+_{\frac{1}{2}} \\ \partial_n u_\infty = 0 \ \text{ on } \ \Upsilon_{1/2} \end{cases} \tag{4.10}$$

in viscosity sense. In fact, given a quadratic polynomial $P(x)$ touching u_∞ at $x_0 \in B_{\frac{1}{2}}(0) \cap \{x_n \geqslant 0\}$ strictly by below we need to prove that

(i) If $x_0 \in B_{\frac{1}{2}}(0) \cap \{x_n > 0\}$ then $\mathfrak{L}_p P \leqslant 0$;
(ii) If $x_0 \in B_{\frac{1}{2}}(0) \cap \{x_n = 0\}$ then $\partial_n P(x_0) \leqslant 0$.

As in [6], there exist points $x_j \in \Omega_{\frac{1}{2}}(u_j)$, $x_j \to x_0$, and constants $c_j \to 0$ such that

$$u_j(x_j) = \tilde{P}(x_j)$$

and

$$u_j(x) \geqslant \tilde{P}(x) \ \text{ in a neighborhood of } \ x_j$$

where

$$\tilde{P}(x) = \varepsilon_j(P(x) + c_j) + x_n.$$

We have two possibilities:

(a) If $x_0 \in B_{\frac{1}{2}} \cap \{x_n > 0\}$ then, since P touches u_j by below at x_j, we estimate

$$\begin{aligned} \varepsilon_j^p &\geqslant f_j(x_j) \\ &\geqslant \Delta_p \tilde{P} \\ &= \varepsilon_j \sum_{i,k=1}^n (p-2)|\nabla \tilde{P}|^{p-4}(\partial_i \tilde{P})(\partial_k \tilde{P})\partial_{ik} P + \varepsilon_j \sum_{i=1}^n |\nabla \tilde{P}|^{p-2}\partial_{ii} P. \end{aligned}$$

Using that $\nabla \tilde{P} = \varepsilon_j \nabla P + e_n$ and taking $\varepsilon_j \longrightarrow 0$ we obtain

$$\mathfrak{L}_p P \leqslant 0.$$

(b) If $x_0 \in B_{\frac{1}{2}} \cap \{x_n = 0\}$ we can assume, see [6], that

$$\mathfrak{L}_p P > 0 \tag{4.11}$$

Notice that for j sufficiently large we have $x_j \in \mathfrak{F}(u_j)$. In fact, suppose by contradiction that there exists a subsequence $x_{j_n} \in B_1^+(u_{j_n})$ such that $x_{j_n} \to x_0$. Then arguing as in (i) we obtain

$$\mathfrak{L}_p P \leqslant C\varepsilon_j,$$

which contradicts (4.11) as $j_n \to \infty$. Therefore, there exists $j_0 \in \mathbb{N}$ such that $x_j \in \mathfrak{F}(u_j)$ for $j \geqslant j_0$. Moreover,

$$|\nabla \tilde{P}| \geqslant 1 - \varepsilon_j |\nabla P| > 0,$$

for j sufficiently large (we can assume that $j \geqslant j_0$). Since that $\tilde{P}^+$ touches u_j by below we have

$$|\nabla \tilde{P}|^p \leqslant Q_j(x_j) \leqslant (1 + \varepsilon_j^p).$$

Then, we obtain

$$|\nabla \tilde{P}|^2 \leqslant (1 + \varepsilon_j^2).$$

Moreover,

$$|\nabla \tilde{P}|^2 = \varepsilon_j^2 \big|\nabla P(x_j)\big|^2 + 1 + 2\varepsilon_j \partial_n P(x_j),$$

where we have used $|\nabla \tilde{P}|^2 \leqslant C$. In conclusion, we obtain

$$\varepsilon_j^2 \big|\nabla P(x_j)\big|^2 + 1 + 2\varepsilon_j \partial_n P(x_j) \leqslant 1 + \varepsilon_j^2. \tag{4.12}$$

Hence, dividing (4.12) by ε_j and taking $j \to \infty$ we obtain $\partial_n P(x_0) \leqslant 0$.

The choice of r_0 and the conclusion of the Theorem 1.1 follows from the regularity of $\tilde{u}$:

Step 3 – Improvement of flatness: From the previous step, u_∞ solve (4.10) and from (4.8),

$$-1 \leqslant u_\infty \leqslant 1 \quad \text{in } B_{1/2} \cap \{x_n \geqslant 0\}.$$

From Lemma 2.8 and the bound above we obtain that, for the given r,

$$\big|u_\infty(x) - u_\infty(0) - \langle \nabla u_\infty(0), x\rangle\big| \leqslant C_0 r^2 \quad \text{in } B_r \cap \{x_n \geqslant 0\},$$

for a universal constant C_0. In particular, since $0 \in \mathfrak{F}(u_\infty)$ and $\frac{\partial u_\infty(0)}{\partial \mu} = 0$, we estimate

$$\langle \tilde{x}, \tilde{\nu}\rangle - C_1 r^2 \leqslant u_\infty(x) \leqslant \langle \tilde{x}, \tilde{\nu}\rangle + C_0 r^2 \quad \text{in } B_r \cap \{x_n \geqslant 0\},$$

where $\tilde{\nu}_i = \langle \nabla u_\infty(0), e_i\rangle$, $i = 1, \ldots, n-1$, $|\tilde{\nu}| \leqslant \tilde{C}$ and $\tilde{C}$ is a universal constant. Therefore, for k large enough we get,

$$\langle \tilde{x}, \tilde{\nu}\rangle - C_1 r^2 \leqslant v_k(x) \leqslant \langle \tilde{x}, \tilde{\nu}\rangle + C_1 r^2 \quad \text{in } \Omega_r(u_k).$$

From the definition of v_k the inequality above reads

$$\epsilon_k \tilde{x} \cdot \tilde{\nu} + x_n - \epsilon_k C_1 r^2 \leqslant u_k \leqslant \epsilon_k \langle \tilde{x}, \tilde{\nu}\rangle + x_n + \epsilon_k C_1 r^2 \quad \text{in } \Omega_r(u_k). \tag{4.13}$$

Define

$$\nu := \frac{1}{\sqrt{1 + \epsilon_k^2}} (\epsilon_k \tilde{\nu}, 1).$$

Since, for k large,

$$1 \leqslant \sqrt{1+\epsilon_k^2} \leqslant 1 + \frac{\epsilon_k^2}{2},$$

we conclude from (4.13) that

$$\langle x, \nu\rangle - \frac{\epsilon_k^2}{2} r - C_1 r^2 \epsilon_k \leqslant u_k \leqslant \langle x, \nu\rangle + \frac{\epsilon_k^2}{2} r + C_1 r^2 \epsilon_k \quad \text{in } \Omega_r(u_k).$$

In particular, if r_0 is such that $C_1 r_0 \leqslant \frac{1}{4}$ and also k is large enough so that $\epsilon_k \leqslant \frac{1}{2}$ we find

$$\langle x, \nu\rangle - \frac{\epsilon_k}{2} r \leqslant u_k \leqslant \langle x, \nu\rangle + \frac{\epsilon_k}{2} r \quad \text{in } \Omega_r(u_k),$$

which together with (4.7) implies that

$$\Big(\langle x, \nu\rangle - \frac{\epsilon_k}{2} r\Big)^+ \leqslant u_k \leqslant \Big(\langle x, \nu\rangle + \frac{\epsilon_k}{2} r\Big)^+ \quad \text{in } B_r.$$

Thus the u_k satisfy the conclusion of the Lemma, and we reached a contradiction. □

5. Regularity of the free boundary

In this section we will prove the Theorem 1.1 and via a blow-up from Theorem 1.1 we will present the proof of Theorem 1.2. The proof of Theorem 1.1 is based on flatness improvement coming from Harnack type estimates and it follows closely the work of [6]. Hereafter, we will assume

$$|Q(x) - Q(y)| \leqslant \tau(|x-y|) \quad \text{for } x, y \in B_1, \tag{5.1}$$

where the modulus of continuity τ satisfies

$$\tau(t) \lesssim C t^\beta, \tag{5.2}$$

for some $0 < \beta < 1$ and $C > 0$.

Proof of Theorem 1.1. The idea of proof is to iterate the Theorem 4.1 in the appropriate geometric scaling. Let u be a viscosity solution to the free boundary problem

$$\begin{cases} \Delta_p u = f, & \text{in } B_1^+(u), \\ |\nabla u|^p = Q, & \text{on } \mathfrak{F}(u). \end{cases} \tag{5.3}$$

where $B_1^+(u) = \{x \in B_1^+ : u(x) > 0\}$ and $\mathfrak{F}^+(u) := \partial B_1^+(u) \cap B_1$. Let us fix $\bar{r} > 0$ to be a universal constant such that

$$\bar{r}^\beta \leqslant \min\Big\{(\frac{1}{2})^p, r_0\Big\}, \tag{5.4}$$

with r_0 the universal constant in Theorem 4.1. For the chose $\bar{r}$, let $\epsilon_0 := \epsilon_0(\bar{r})$ give by Theorem 4.1. Now, let

$$\bar{\epsilon} := \epsilon_0^p \quad \text{and} \quad \epsilon = \epsilon_k := 2^{-k}\epsilon_0. \tag{5.5}$$

Our choice of $\bar{\epsilon}$ guarantees that

$$(x_n - \epsilon_0)^+ \leqslant u(x) \leqslant (x_n + \epsilon_0)^+ \quad \text{in } B_1. \tag{5.6}$$

Thus by Theorem 4.1

$$\Big(\langle x,\nu_1\rangle-\bar r\frac{\epsilon_0}{2}\Big)^+\leqslant u(x)\leqslant\Big(\langle x,\nu_1\rangle+\bar r\frac{\epsilon_0}{2}\Big)^+\quad\text{in}\ \ B_{\bar r},$$

with $|\nu_1|=1$ and $|\nu_1-\nu_0|\leqslant C{\epsilon_0}^2$ (where $\nu_0=e_n$).

Smallness regime: Consider the sequence of rescalings $u_k:B_1\to\mathbb{R}$

$$u_k(x):=\frac{u(\lambda_k x)}{\lambda_k}$$

with $\lambda_k=\bar r^k$, $k=0,1,2,\ldots$, for a fixed $\bar r$ as in (5.4). Then each u_k satisfies in the following free boundary problem

$$\begin{cases}\Delta_p u_k=f_k, & \text{in}\quad B_1^+(u_k),\\ |\nabla u_k|^p=Q_k, & \text{on}\quad \mathfrak{F}(u_k).\end{cases}\tag{5.7}$$

$$f_k(x):=\lambda_k f(\lambda_k x)\quad\text{and}\quad Q_k(x):=Q(\lambda_k x).$$

We claim that for the choices made in (5.5) the assumption (4.2) are holds. Indeed, in B_1

$$|f_k(x)|\leqslant\|f\|_{L^\infty}\lambda_k\leqslant\bar\epsilon\bar r^k\leqslant\epsilon_0^p2^{-pk}=(\epsilon_0 2^{-k})^p=\epsilon^p,$$
$$|Q_k(x)-1|=|Q(\lambda_k x)-Q_k(0)|\leqslant\tau(1)\lambda_k^\beta\leqslant\bar\epsilon\bar r^{k\beta}\leqslant(\epsilon_0 2^{-k})^p=\epsilon^p$$

Therefore, we can iterate the argument above and obtain that

$$\big(\langle x,\nu_k\rangle-\epsilon_k\big)^+\leqslant u_k(x)\leqslant\big(\langle x,\nu_k\rangle+\epsilon_k\big)^+\quad\text{in}\ \ B_1,\tag{5.8}$$

with $|\nu_k|=1$, $|\nu_k-\nu_{k+1}|\leqslant C\epsilon_k$ ($\nu_0=e_n$), where C is a universal constant. Thus, we have

$$\Big(\langle x,\nu_k\rangle-\frac{\epsilon_0}{2^k}\bar r^k\Big)^+\leqslant u(x)\leqslant\Big(\langle x,\nu_k\rangle+\frac{\epsilon_0}{2^k}\bar r^k\Big)^+\quad\text{in}\quad B_{\bar r^k}\tag{5.9}$$

with

$$|\nu_{k+1}-\nu_k|\leqslant C\frac{\epsilon_0}{2^k}.\tag{5.10}$$

Inequality (5.9) implies that

$$\partial\{u>0\}\cap B_{\bar r^k}\subset\Big\{|\langle x,\nu_k\rangle|\leqslant\frac{\epsilon_0}{2^k}\bar r^k\Big\}\tag{5.11}$$

This implies that $B_{3/4}\cap\mathfrak{F}(u)$ is a $C^{1,\gamma}$ graph. In fact, by (5.10) we have that $\{\nu_k\}_{k\geqslant1}$ is a Cauchy sequence, therefore the limit

$$\nu(0):=\lim_{k\to\infty}\nu_k$$

exists. Yet from (5.10) we conclude

$$\big|\nu_k-\nu(0)\big|\leqslant C\frac{\epsilon_0}{2^k}.$$

From (5.11) we have

$$\left|\langle x, \nu_k\rangle\right| \leqslant \frac{\epsilon_0}{2^k}\overline{r}^k \tag{5.12}$$

Fix $x \in B_{3/4} \cap \partial\{u > 0\}$ and choose k such that

$$\overline{r}^{k+1} \leqslant |x| \leqslant \overline{r}^k$$

Then

$$\begin{aligned}
\left|\langle x, \nu(0)\rangle\right| &\leqslant \left|\langle x, \nu(0) - \nu_k\rangle\right| + \left|\langle x, \nu_k\rangle\right| \\
&\leqslant \left|\nu(0) - \nu_k\right||x| + \frac{\epsilon_0}{2^k}\overline{r}^k \\
&\leqslant C\frac{\epsilon_0}{2^k}|x| + \frac{\epsilon_0}{2^k}\overline{r}^k \\
&\leqslant C\frac{\epsilon_0}{2^k}\left(|x| + \overline{r}^k\right) \\
&\leqslant C\frac{\epsilon_0}{2^k}\left(|x| + \frac{\overline{r}^{k+1}}{\overline{r}}\right) \\
&\leqslant C\frac{\epsilon_0}{2^k}\left(1 + \frac{1}{\overline{r}}\right)|x|
\end{aligned}$$

From the convenient choice of k, we have $|x| \geqslant \overline{r}^{k+1}$. Hence, if we define $0 < \gamma < 1$ such that

$$\frac{1}{2} = \overline{r}^\gamma$$

i.e., define $\gamma := \frac{\ln(2)}{\ln(\overline{r}^{-1})}$. Thus, we have

$$\begin{aligned}
\left|\langle x, \nu(0)\rangle\right| &\leqslant C\left(\frac{1}{2}\right)^k (1 + \overline{r}^{-1})|x| \\
&= C\left(\frac{1}{2}\right)^{k+1} (1 + \overline{r}^{-1})2|x| \\
&\leqslant C(1 + \overline{r}^{-1})\epsilon_0|x|^{1+\gamma} \leqslant C\epsilon_0|x|^{1+\gamma}.
\end{aligned}$$

Finally, we obtain

$$\partial\{u > 0\} \cap B_{\overline{r}^k} \subset \left\{\langle x, \nu(0)\rangle \leqslant C\epsilon_0\overline{r}^{k(1+\gamma)}\right\},$$

which implies that $\partial\{u > 0\}$ is a differentiable surface at 0 with normal $\nu(0)$. Applying this argument at all points in $\partial\{u > 0\} \cap B_{3/4}$ we see that $\partial\{u > 0\} \cap B_{3/4}$ is in fact a $C^{1,\gamma}$ surface. □

The next lemma is a standard result, that is, Lipschitz continuity and non-degeneracy of a solution u to

$$\begin{cases} \Delta_p u = f, & \text{in} \quad \Omega_+(u), \\ |\nabla u|^p = Q, & \text{on} \quad \mathfrak{F}(u). \end{cases} \tag{5.13}$$

Lemma 5.1 *Let $u \in C(\Omega)$ be a viscosity solution to (5.13). Given $\epsilon \in (0, 1)$, we can find a universal constant $\tilde{\epsilon}$ such that if $\epsilon \in (0, \tilde{\epsilon}]$, $\mathfrak{F}(u) \cap B_1 \neq \emptyset$, $\mathfrak{F}(u)$ is a Lipschitz graph in B_2 and*

$$\max\left\{\|f\|_{L^\infty(\Omega)},\ \|Q - 1\|_{L^\infty(\Omega)}\right\} \leqslant \epsilon^p, \tag{5.14}$$

then u is Lipschitz and non-degenerate in $B_1^+(u)$ *i.e. there exists universal conconstants* $c_0, c_1 > 0$

$$c_0 dist(z, \mathfrak{F}(u)) \leqslant u(z) \leqslant c_1 dist(z, \mathfrak{F}(u)) \quad \textit{for all} \;\; z \in B_1^+(u).$$

Lemma 5.2 (Compactness) *Let* u_k *be a sequence of (Lipschitz) viscosity solutions to*

$$\begin{cases} \Delta u_k &= f_k \quad in \quad \Omega^+(u_k), \\ |\nabla u_k|^p &= Q_k \quad on \quad \mathfrak{F}(u_k) \end{cases}$$

where f_k *and* Q_k *satisfies the assumption* (5.14). *Assume that*

(i) $u_k \to u_\infty$ *uniformly on compacts*;
(ii) $\partial\{u_k > 0\} \to \partial\{u_\infty > 0\}$ *locally in the Hausdorff distance;*
(iii) $\|f_k\|_{L^\infty} + \|Q_k - 1\|_{L^\infty} = o(1)$*, as* $k \to \infty$.

Then u_∞ *be a viscosity solution of*

$$\begin{cases} \Delta_p u_\infty &= 0, \quad in \quad \Omega^+(u_\infty), \\ |\nabla u_\infty| &= 1, \quad on \quad \mathfrak{F}(u_\infty), \end{cases}$$

in the viscosity sense.

Proof. The proof follows the same scheme of the model Lemma 4.1 (see also [6, Lemma 7.3]). □

Although not strictly necessary, we use the following Liouville type result for global viscosity solutions to a one-phase homogeneous free boundary problem, that could be of independent interest. The result is more general, but we will only show the result for one-phase problems.

Lemma 5.3 *Let* $v : \mathbb{R}^n \to \mathbb{R}$ *be a non-negative viscosity solution to*

$$\begin{cases} \Delta_p v &= 0, \quad in \quad \{v > 0\}, \\ \langle \nabla v, \nu \rangle &= 1, \quad on \quad \mathfrak{F}(v) := \partial\{v > 0\}. \end{cases}$$

Assume that $\mathfrak{F}(v) = \{x_n = g(x'), x' \in \mathbb{R}^{n-1}\}$ *with* $Lip(g) \leqslant M$. *Then g is linear and*

$$v(x) = x_n^+.$$

Proof. Let's follow the ideas of [6]. Initially, assume for simplicity, $0 \in \mathfrak{F}(v)$. Also, balls (of radius ρ center at 0) in $\mathbb{R}^{n-1}$ are denote by B_ρ'. By the regularity theory in [2, 11] and [12] since v is a solution in B_2, the free boundary $\mathfrak{F}(v)$ is $C^{1,\gamma}$ in B_1 with a bound depending only on n and on M. Thus,

$$\left|g(x') - g(0) - \nabla g(0) \cdot x'\right| \leqslant C|x'|^{1+\gamma} \quad \text{for } x' \in B_1'$$

with C depending only on n, M. Moreover, since v us a global solution, the rescaling

$$g_\lambda(x') := \frac{1}{\lambda} g(\lambda x'), \;\; x' \in B_2'$$

which preserves the same Lipschitz constant as g, satisfies the same inequality as above, i.e.

$$\left|g_\lambda(x') - g_\lambda(0) - \nabla g_\lambda(0) \cdot x'\right| \leqslant C|x'|^{1+\gamma} \quad \text{for } x' \in B_1'.$$

Thus,

$$\left|g(y') - g(0) - \nabla g(0) \cdot y'\right| \leqslant C \frac{1}{R^\gamma} |y'|^{1+\gamma}, \;\; y' \in B_R'.$$

Passing to the limit as $R \to \infty$ we obtain the desired claim. □

In this section we finally prove of our second main theorem.

Proof of Theorem 1.2. Let $\bar{\epsilon} > 0$ be the universal constant in Theorem 1.1 and u. Without loss of generality, assume $Q(0) = 1$. Consider the re-scaled function

$$u_k := u_{\delta_k}(x) = \frac{u(\delta_k x)}{\delta_k},$$

with $\delta_k \to 0$ as $k \to \infty$. Each u_k solves

$$\begin{cases} \Delta_p u_k &= f_k \quad \text{in} \quad B_1^+(u_k), \\ |\nabla u_k|^p &= Q_k \quad \text{on} \quad \mathfrak{F}(u_k), \end{cases}$$

with

$$f_k(x) := \delta_k f(\delta_k x) \quad \text{and} \quad Q_k(x) := Q(\delta_k x).$$

Furthermore, for k large, the assumption (5.14) are satisfied for the universal constant $\bar{\epsilon}$. In fact, in B_1 we have

$$|f_k(x)| = \delta_k |f(\delta_k x)| \leqslant \delta_k \|f\|_{L^\infty} \leqslant \bar{\epsilon}^p$$
$$|Q_k(x) - 1| = |Q_k(x) - Q_k(0)| \leqslant \tau(1)\delta_k^\beta \leqslant \bar{\epsilon}^p$$

for k large enough. Therefore, using non-degeneracy (see Lemma 5.1) and uniform Lipschitz continuity of the $u_k' s$ (see Lemma (5.1)), standard arguments imply that (up to a subsequence)

(i) There exists $u_\infty \in C(\Omega)$ such that $u_k \to u_\infty$ uniformly on compacts;
(ii) $\partial\{u_k > 0\} \to \partial\{u_\infty > 0\}$ locally in the Hausdorff distance;
(iii) $\|f_k\|_{L^\infty} + \|Q_k - 1\|_{L^\infty} = o(1)$, as $k \to \infty$

and, as in Lemma 5.2, the blow-up limit u_∞ solves the global homogeneous one-phase free boundary problem

$$\begin{cases} \Delta_p u_\infty &= 0, \quad \text{in} \quad \{u_\infty > 0\}, \\ |\nabla u_\infty| &= 1, \quad \text{on} \quad \mathfrak{F}(u_\infty). \end{cases}$$

Since $\mathfrak{F}(u)$ is a Lipschitz graph in a neighborhood of 0 we also have from have (i)–(iii) that $\mathfrak{F}(u_\infty)$ is Lipschitz continuous. Thus, follows the Lemma 5.3 that u_∞ is a so-called one-phase solution, i.e., (up to rotations)

$$u_\infty = x_n^+.$$

Thus, for k large enough we have

$$\|u_k - u_\infty\|_{L^\infty} \leqslant \bar{\epsilon}$$

and the facts thar u_k is $\bar{\epsilon}$-flat say in B_1, i.e,

$$(x_n - \bar{\epsilon})^+ \leqslant u_k(x) \leqslant (x_n + \bar{\epsilon})^+, \quad x \in B_1.$$

Therefore, we can apply our flatness Theorem 4.1 and conclude that $\mathfrak{F}(u_k)$ and hence $\mathfrak{F}(u)$ is $C^{1,\gamma}$, for some $\gamma \in (0, 1)$. □

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