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Erratum to: A homogenization result in the gradient theory of phase transitions

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We would like to thank Will Feldman and Peter S. Morfe for pointing out the typo in the cell formula of Definition 1.3 in [1]. Here we present the correct formula for the energy density of the limiting functional, together with the minor modifications needed to adjust accordingly the proofs of the results presented in the paper.

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DEFINITION 1 We define the function $\sigma : \mathbb{S}^{N-1} \to [0,\infty)$ as

$$\sigma(\nu) := \lim_{T\to\infty} g(\nu, T)\,,$$

where

$$g(\nu,T) := \frac{1}{T^{N-1}} \inf\left\{ \int_{TQ_\nu} \Big[W\big(y,u(y)\big) + |\nabla u|^2 \Big] \mathrm{d}y \;:\; Q_\nu \in \mathcal{Q}_\nu,\ u \in \mathcal{C}(\rho, Q_\nu, T) \right\},$$

and

$$\mathcal{C}(\rho, Q_\nu, T) := \left\{ u \in H^1(TQ_\nu;\mathbb{R}^d) \;:\; u = \widetilde{u}_{\rho,1,\nu} \text{ on } \partial\,(TQ_\nu) \right\}.$$

REMARK 2 In the previous version of the paper, the boundary datum was taken to be $\widetilde{u}_{\rho,T,\nu}$. The mistake tracks back to forgetting to rescaling back from the formula we obtained in the liminf inequality. To be precise, at the end of the proof of Proposition 6.1, the function v_k defined by $v_k(z) := \overline{w}(\eta_k z)$ satisfies $v_k = \widetilde{u}_{\rho,1,\nu}$ on $\partial\big(\frac{1}{\eta_k} Q_\nu\big)$.

The only result whose proof needs to be adjusted is Lemma 4.1. Despite the fact that only a couple of modifications are needed, for the sake of the completeness we present here the entire proof.

Lemma 3 *Let $\nu \in \mathbb{S}^{N-1}$. Then $\sigma(\nu)$ is well defined and is finite.*

Proof. Let $\nu \in \mathbb{S}^{N-1}$. For $T > \sqrt{N}$ let $Q_T \in \mathbb{Q}_\nu$ and $u_T \in \mathcal{C}(Q_T, T)$ be such that

$$\frac{1}{T^{N-1}} \int_{TQ_T} W\big(y, u_T(y)\big) + |\nabla u_T(y)|^2 dy \leqslant g(T) + \frac{1}{T}, \tag{1}$$

where, for simplicity of notation, we write $g(T)$ for $g(\nu, T)$. Let $\{\nu_T^{(1)}, \dots, \nu_T^{(N)}\}$ be an orthonormal basis of $\mathbb{R}^N$ normal to the faces of Q_T such that $\nu = \nu_T^{(N)}$. We define an oriented rectangular prism centered at 0 via

$$P(\alpha, \beta) := \big\{x \in \mathbb{R}^N : |x \cdot \nu| \leqslant \beta \text{ and } |x \cdot \nu_T^{(i)}| \leqslant \alpha \text{ for } 1 \leqslant i \leqslant N-1\big\}.$$

Let $S > T + 3 + \sqrt{N}$. We claim that for all $m \in \mathbb{N}$ with $2 \leqslant m < T$, we have

$$g(S) \leqslant g(T) + R(m, S, T)\,, \tag{2}$$

where the quantity $R(m, S, T)$ does not depend on ν and is such that

$$\lim_{m\to\infty} \lim_{T\to\infty} \lim_{S\to\infty} R(m, S, T) = 0.$$

Note that if this holds then

$$\limsup_{S\to\infty} g(S) \leqslant \liminf_{T\to\infty} g(T),$$

and this ensures the existence of the limit in the definition of σ. Therefore, the remainder of Step 1 is dedicated to proving (2).

The idea is to construct a competitor u_S for the infimum problem defining $g(S)$ by taking $\lfloor \frac{S}{T} \rfloor^{N-1}$ copies of $TQ_\nu \cap \nu^\perp$ centered on $\nu^\perp \cap SQ_\nu$ in each of which we define u_S to be (a translation of) u_T. In order to compare the energy of u_S to the energy of u_T, we need the copies of the cube TQ_ν to be integer translations of the original. Moreover, we also have to ensure that the boundary conditions render u_S admissible for the infimum problem defining $g(S)$. For this reason, we need the centers of the translated copies of $TQ_\nu \cap \nu^\perp$ to be close to $\nu^\perp \cap SQ_\nu$ (recall that the mollifiers $\rho_{T,\nu}$ and $\rho_{S,\nu}$ only depend on the direction ν).

Set

$$M_{T,S} := \left\lfloor \frac{S - \frac{1}{T}}{T + \sqrt{N} + 2} \right\rfloor^{N-1},$$

and notice that

$$\lim_{T\to\infty} \lim_{S\to\infty} \frac{T^{N-1}}{S^{N-1}} M_{T,S} = 1\,. \tag{3}$$

We can tile $\left(S - \frac{1}{T}\right) Q_T$ with disjoint prisms $\left\{p_i + P\left(T + \sqrt{N} + 2, S - \frac{1}{T}\right)\right\}_{i=1}^{M_{T,S}}$ so that

$$p_i + P\left(T + \sqrt{N} + 2, S - \frac{1}{T}\right) \subset \left(S - \frac{1}{T}\right) Q_T, \qquad p_i \in \nu^\perp,$$

for each $i \in \{1, \ldots, M_{T,S}\}$. In each cube $p_i + \sqrt{N} Q_T$ we can find $x_i \in \mathbb{Z}^N$ since $\operatorname{dist}(\cdot, \mathbb{Z}^N) \leqslant \sqrt{N}$ in $\mathbb{R}^N$, and we have

$$x_i + (T+2)Q_T \subset p_i + (T + \sqrt{N} + 2)Q_T.$$

Consider, for $m \in \mathbb{N}$ and $i \in \{1, \ldots, M_{T,S}\}$ cut-off functions $\varphi_{m,i} \in C_c(x_i + (T + \frac{1}{m})Q_T; [0,1])$ be such that

$$\varphi_{m,i}(x) = \begin{cases} 0 & \text{if } x \in \partial\Big(x_i + \Big(T + \frac{1}{m}\Big)Q_T\Big), \\ 1 & \text{if } x \in x_i + TQ_T, \end{cases} \qquad \|\nabla \varphi_{m,i}\|_{L^\infty} \leqslant Cm, \tag{4}$$

for some $C > 0$. Define $u_S : SQ_T \to \mathbb{R}^d$ by

$$u_S(x) := \begin{cases} u_T(x - x_i) & \text{if } x \in x_i + TQ_T, \\ \varphi_{m,i}(x)(\rho * u_{0,\nu})(x + p_i - x_i) + \big(1 - \varphi_{m,i}(x)\big)(\rho * u_{0,\nu})(x) & \\ & \text{if } x \in \big(x_i + (T + \frac{1}{m})Q_T\big) \setminus (x_i + TQ_T), \\ (\rho * u_{0,\nu})(x) & \text{otherwise.} \end{cases}$$

Notice that since $p_i \cdot \nu = 0$, if $x \in \partial(x_i + TQ_T)$ we have

$$u_T(x - x_i) = (\rho * u_{0,\nu})(x - x_i) = (\rho * u_{0,\nu})(x + p_i - x_i).$$

Thus $u_S \in H^1(SQ_T; \mathbb{R}^d)$ and, if $x \in \partial(SQ_T)$ then $u_S(x) = (\rho * u_{0,\nu})(x)$, so u_S is admissible

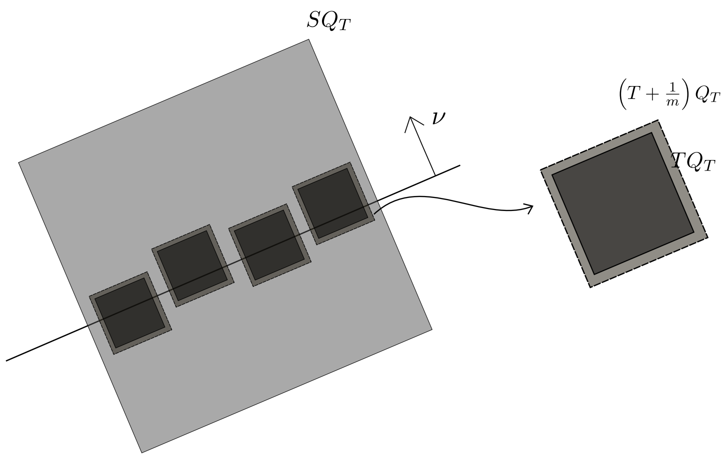

FIG. 1. Construction of the function u_S: In each black cube $x_i + TQ_T$ we defined it as a copy of u_T and we use the grey region $(x_i + (T + \frac{1}{m})Q_T) \setminus (x_i + TQ_T)$ around it to adjust the boundary conditions and make them match the value of u_S in the light grey region where we define it to be $\rho * u_{0,\nu}$.

for the infimum in the definition of $g(S)$. In particular,

$$\begin{aligned} g(S) &\leqslant \frac{1}{S^{N-1}} \int_{SQ_T} \Big[W\big(x, u_S(x)\big) + |\nabla u_S(x)|^2 \Big] dx \\ &= \frac{1}{S^{N-1}} \mathfrak{F}_1(u_S, SQ_T) \\ &=: I_1(T,S) + I_2(T,S,m) + I_3(T,S,m), \end{aligned} \tag{5}$$

where

$$\begin{aligned} I_1(T,S) &:= \frac{1}{S^{N-1}} \sum_{i=1}^{M_{T,S}} \mathfrak{F}_1(u_S, x_i + TQ_T), \\ I_2(T,S,m) &:= \frac{1}{S^{N-1}} \sum_{i=1}^{M_{T,S}} \mathfrak{F}_1 \left(u_S, \left(x_i + \left(T + \frac{1}{m}\right) Q_T \right) \setminus (x_i + TQ_T) \right), \\ I_3(T,S,m) &:= \frac{1}{S^{N-1}} \mathfrak{F}_1(u_S, E_{T,S,m}), \end{aligned}$$

and we set

$$E_{T,S,m} := SQ_T \setminus \bigcup_{i=1}^{M_{T,S}} \left(x_i + \left(T + \frac{1}{m}\right) Q_T \right).$$

We now bound each of terms I_1, I_2, I_3 separately. We start with $I_1(T,S)$. Since $x_i \in \mathbb{Z}^N$, the periodicity of W together with (1) yield

$$\begin{aligned} I_1(T,S) &= \frac{1}{S^{N-1}} M_{S,T} \int_{TQ_T} \Big[W\big(x, u_T(x)\big) + |\nabla u_T(x)|^2 \Big] dx \\ &\leqslant \frac{1}{S^{N-1}} M_{T,S} T^{N-1} \left(g(T) + \frac{1}{T} \right). \end{aligned} \tag{6}$$

In order to estimate $I_2(T,S,m)$, notice that, since for every $x \in \mathbb{R}^N$ the function $t \mapsto (\rho * u_{0,\nu})(x + t\nu)$ is constant outside of an interval of size 1, we have that for every $i \in \{1, \ldots, M_{T,S}\}$ it holds

$$\begin{aligned} &\int_{(x_i + (T + \frac{1}{m}) Q_T) \setminus (x_i + TQ_T)} |\nabla(\rho_1 * u_{0,\nu})(x + p_i - x_i)|^2 \, \mathrm{d}x \\ &\qquad\qquad \leqslant \|\nabla(\rho * u_{0,\nu})\|^2_{L^\infty} \left[\left(T + \frac{1}{m} \right)^{N-1} - T^{N-1} \right]. \end{aligned} \tag{7}$$

Thus, using (4) and (7) we obtain

$$
\begin{aligned}
I_2(T,S,m) &\leqslant \frac{C}{S^{N-1}} M_{T,S}\left[\left(\frac{\sqrt{N}}{2}+1\right)(1+\|\nabla\varphi_{m,i}\|^2_{L^\infty})+\|\nabla(\rho * u_{0,\nu})\|^2_{L^\infty}\right.\\
&\qquad\qquad\left.+\|\nabla(\rho * u_{0,\nu})\|^2_{L^\infty}\right]\left[\left(T+\frac{1}{m}\right)^{N-1}-T^{N-1}\right]\\
&\leqslant \frac{C}{S^{N-1}} M_{T,S}\left(1+m^2\right)\left[\left(T+\frac{1}{m}\right)^{N-1}-T^{N-1}\right]\\
&\leqslant C\frac{T^{N-1}}{S^{N-1}} M_{T,S}\left(1+m^2\right)\left[\left(1+\frac{1}{Tm}\right)^{N-1}-1\right]\\
&\leqslant C\frac{T^{N-1}}{S^{N-1}} M_{T,S}\left(1+m^2\right)\left(\frac{N-1}{Tm}\right) =: J_2(T,S,m)
\end{aligned}
\tag{8}
$$

where in the last step we used the inequality

$$
(1+t)^{N-1} \leqslant 1+C(N-1)t \tag{9}
$$

for $t \ll 1$, that is valid here when $T \gg 1$.

We can finally estimate $I_3(T,S,m)$ as

$$
\begin{aligned}
I_3(T,S,m) &= \frac{1}{S^{N-1}}\int_{E_{T,S,m}}\left[\,W(x,\rho * u_{0,\nu})+|\nabla(\rho * u_{0,\nu})|^2\,\right]\,\mathrm{d}x\\
&\leqslant \frac{C}{S^{N-1}}\left|E_{T,S,m}\cap\left\{|x\cdot\nu|<1\right\}\right|\left(1+\|\nabla(\rho * u_{0,\nu})\|^2_{L^\infty}\right)\\
&\leqslant \frac{C}{S^{N-1}}\left[S^{N-1}-M_{T,S}T^{N-1}\right] =: J_3(T,S,m).
\end{aligned}
\tag{10}
$$

Taking into account (8) and (10) we obtain

$$
\lim_{m\to\infty}\ \lim_{T\to\infty}\ \lim_{S\to\infty}\left[\,J_2(T,S,m)+J_3(T,S,m)\,\right]=0. \tag{11}
$$

Thus, in view of (5), (6), (3) and (1), we conclude (2) with

$$
R(m,S,T) := J_2(T,S,m)+J_3(T,S,m). \tag{12}
$$

Notice that $R(m,S,T)$ does not depend on ν nor on Q_T.

Finally, to prove that $\sigma(\nu)<\infty$ for all $\nu\in\mathbb{S}^{N-1}$ we notice that, by sending $S\to\infty$ in (2) we get

$$
\sigma(\nu) \leqslant g(T)+\lim_{S\to\infty} R(m,S,T).
$$

Since $g(T)<\infty$ and, by (11) and (12), $\lim_{S\to\infty} R(m,S,T)<\infty$ for all $T>0$, we conclude. □

We also take this opportunity to fix some other minor issues:

(i) In Proposition 4.4, without loss of generality (thanks to Lemma 4.3), we can assume the mollifier ρ and the rotations $\mathcal{R}_n$ to be such that $\rho(\mathcal{R}_n y)=\rho(y)$ for all $y\in\mathbb{R}^N$ and $n\in\mathbb{N}$. Notice that it is possible to satisfy this condition because $\mathcal{R}_n\nu_n=\nu$ for all $n\in\mathbb{N}$;

(ii) Below formula (6.9) we need to use the correct boundary conditions, namely $\bar{w}_k = (\widetilde{u_k})_{1/\eta_k,\nu}$ on ∂Q_ν, where $(\widetilde{u_k})_{1/\eta_k,\nu}$ is defined as in (1.6) of [1];

(iii) Formulae (7.23) and (7.24) hold for every open and bounded set $U \subset \mathbb{R}^{N-1}$, and not for $U \subset \mathbb{R}^N$;

(iv) In the proof of the continuity of the function $\sigma : \mathbb{S}^{N-1} \to [0,\infty)$ (notice that we are actually proving that its 1-homogeneous extension is convex), we need to modify the construction of the sets E_n as follows. Take $v_0, v_1, v_2 \in \mathbb{R}^N$ such that $v_0 = v_1 + v_2$ and let $E := \{x \in \Omega : x \cdot \nu_0 \leqslant \alpha\}$, where $\alpha \in \mathbb{R}$ is such that $\Omega \setminus E \neq \emptyset$ and $\Omega \cap E \neq \emptyset$. Let $X \subset \mathbb{R}^N$ be the two dimensional space generated by ν_1 and ν_2, consider the unit two dimensional square $Q' \subset X$ and a triangle $T \subset X$ centered at the origin with outer normals $-\frac{\nu_0}{|\nu_0|}$, $\frac{\nu_1}{|\nu_1|}$ and $\frac{\nu_2}{|\nu_2|}$, and such that

$$1, \quad \frac{|\nu_1|}{|\nu_0|}, \quad \frac{|\nu_2|}{|\nu_0|}$$

are the lengths of the side of T orthogonal to ν_0, ν_1, and ν_2 respectively. Let $Q \subset \mathbb{R}^N$ be the unit cube and $\widetilde{Q} := \{x' \in \mathbb{R}^{N-2} : (0,0,x') \in Q\}$. Let $R : \mathbb{R}^N \to \mathbb{R}^N$ be a rotation such that $R\left(\{(x_1,x_2,0,\dots,0) \in \mathbb{R}^N : (x_1,x_2) \in (-1/2,1/2)^2\}\right) = Q'$. Let $z \in \mathbb{R}^N$ and $r > 0$ be such that $z + rQ \subset \Omega \setminus E$. Then there exists λ such that $\lambda T \subset rQ'$. For $n \in \mathbb{N}$, let

$$E_n := E \cup \bigcup_{i=1}^n T_i,$$

where, for each $i = 1,\dots,n$,

$$T_i := z_i + \left(\frac{\lambda T}{n} \times \widetilde{Q}\right)$$

and the z_i's are such that the elements in the second union are pairwise disjoint and $T_i \subset z + rQ$. Therefore

$$\begin{aligned}\mathfrak{F}_0(\chi_{E_n}) &= \mathfrak{F}_0(\chi_E) + \sum_{i=1}^n \mathfrak{F}_0(\chi_{T_i}) \\ &= \mathfrak{F}_0(\chi_E) + \sum_{i=1}^n \frac{1}{n}\left[\sigma\left(-\frac{\nu_0}{|\nu_0|}\right) + \frac{|\nu_1|}{|\nu_0|}\sigma\left(\frac{\nu_1}{|\nu_1|}\right) + \frac{|\nu_2|}{|\nu_0|}\sigma\left(\frac{\nu_2}{|\nu_2|}\right)\right] \\ &\qquad + \sum_{i=1}^n \frac{2\mathcal{H}^2(T)}{n^2}\sigma(\widetilde{\nu}),\end{aligned}$$

where $\widetilde{\nu} := R(e_3)$, where $e_3 := (0,0,1,0,\dots,0)$. Therefore, using the 1-homogeneity of σ, we get

$$0 \leqslant \liminf_{n\to\infty}\Big[\mathfrak{F}_0(\chi_{E_n}) - \mathfrak{F}_0(\chi_E)\Big] = \frac{1}{|\nu_0|}\big[\sigma(\nu_1) + \sigma(\nu_2) - \sigma(\nu_0)\big].$$

This proves the claim.

References

1. Cristoferi, R., Fonseca, I., Hagerty, A., & Popovici, C., A homogenization result in the gradient theory of phase transitions. *Interf. Free Boundaries* **21** (2019), 367–408. Zbl1425.74389 MR4014392