Interfaces Free Bound. 23 (2021), 1–58
DOI 10.4171/IFB/449

Existence and uniqueness of axially symmetric compressible subsonic jet impinging on an infinite wall

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[Received 10 October 2019 and in revised form 16 September 2020]

This paper is concerned with the well-posedness theory of the impact of a subsonic axially symmetric jet emerging from a semi-infinitely long nozzle, onto a rigid wall. The fluid motion is described by the steady isentropic Euler system. We showed that there exists a critical value $M_{cr} > 0$, if the given mass flux is less than M_{cr}, there exists a unique smooth subsonic axially symmetric jet issuing from the given semi-infinitely long nozzle and hitting a given uneven wall. The surface of the axially symmetric impinging jet is a free boundary, which detaches from the edge of the nozzle smoothly. It is showed that a unique suitable choice of the pressure difference between the chamber and the atmosphere guarantees the continuous fit condition of the free boundary. Moreover, the asymptotic behaviors and the decay properties of the impinging jet and the free surface in downstream were also obtained. The main results in this paper solved the open problem on the well-posedness of the compressible axially symmetric impinging jet, which has proposed by A. Friedman in Chapter 16 in [26]. The key ingredient of our proof is based on the variational method to the quasilinear elliptic equation with the Bernoulli's type free boundaries.

2020 Mathematics Subject Classification: Primary 76N10, 76G25; Secondary 35Q31, 35J25.

Keywords: Existence and uniqueness, free streamline, compressible Euler system, subsonic impinging flow.

1. Introduction

The problem of a compressible jet falling from a channel and impacting on a wall is a fascinating one, with very practical applications. The canonical problem is of interest in a number of areas, such a flow is produced by the downwards-directed jet from a vertical take-off aircraft spreading out over the ground, or by a jet of water form a tap falling into a full sink. The monographs of Birkhoff and Zarantonello in [10], Jacob in [31], Gurevich in [30] and Milne-Thomson in [33] gave good surveys of these flows.

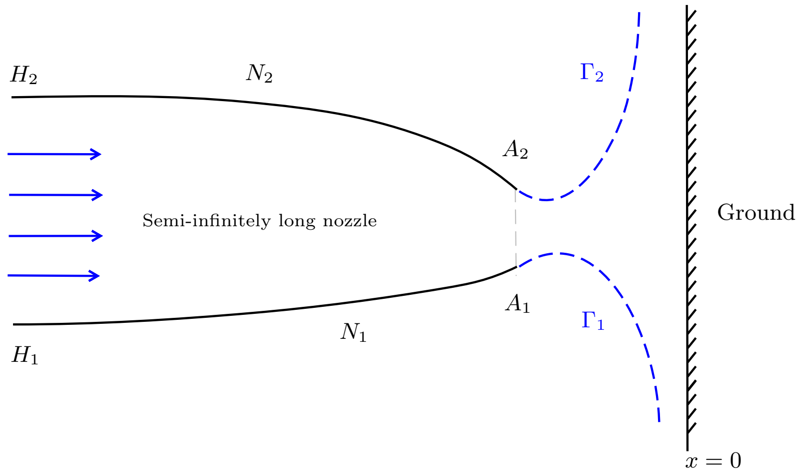

FIG. 1. Subsonic impinging jet in two dimensions

The two-dimensional impinging jets have been considered by Helmholtz and Kirchhoff in 1968. They constructed a solution to the steady irrotational flows of ideal incompressible weightless fluid, bounded by the walls and free streamlines. A first systematic well-posedness result on the incompressible impinging jet was mentioned in Page 364 and Page 416 in [25] that A. Friedman and L. Caffarelli established the existence of the incompressible irrotational jet issuing from a two-dimensional semi-infinite channel and impinging on an infinite plate (see also in Chapter 16.3 in [26]), see [2–4, 13, 15] in different settings. Furthermore, A. Friedman investigated the compressible subsonic free surface flow theory on the sharped charge jets in [26] and proposed several open problems on the impinging jets in two dimensions. The one is

Problem (4). Do the same for the compressible case.

In the recent work [14], the authors established the existence result on the subsonic jets issuing from a convergent nozzle and impact on a flat plate for some special atmospheric pressure (Please see Figure 1).

As A. Friedman pointed out in [26], "... the compressible axially symmetric case is quite open ...". In this paper, we will focus on another open problem pointed out by A. Friedman in [26]:

Problem (5). Extend the results to the axially symmetric flows.

We will establish the existence and uniqueness of the compressible impinging jet in axially symmetric case, and solve the open problem (5) pointed out by A. Friedman. Many numerical simulations of the impact of a compressible flows from a cylinder on a rigid wall are referred to [27, 29, 32, 34, 35].

The present paper treats the compressible impinging jet problem created by the impingement of a subsonic axially symmetric jet emerging from a semi-infinitely long nozzle on a solid curved wall (see Figure 2). The geometry considered here is a semi-infinite nozzle in the form of a circular cylinder, in an unbounded space. The infinite uneven wall is solid and undeformable. The fluid is assumed to be steady, inviscid and irrotational throughout, and the jet emerges from the orifice of

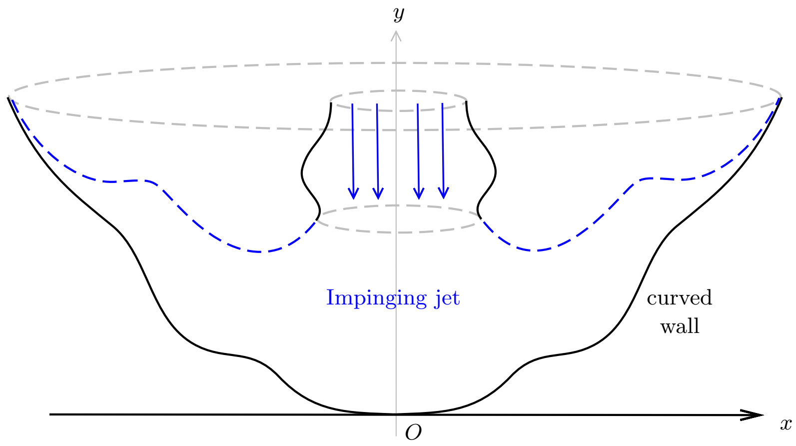

FIG. 2. Subsonic axially symmetric impinging jet

the nozzle of circular cross-section bounded by a stream surface, the nozzle wall and the curved wall.

1.1 *Formulation of the physical problem*

The steady isentropic compressible flow is governed by the following three-dimensional Euler system

$$\begin{cases} \nabla \cdot (\rho U) = 0, \\ (\rho U \cdot \nabla) U + \nabla p = 0, \end{cases} \tag{1.1}$$

with the irrotational condition

$$\nabla \times U = 0. \tag{1.2}$$

Here, $U = (u_1, u_2, u_3)$ is the velocity, $\rho = \rho(x_1, x_2, x_3)$ is the density and $p = p(\rho)$ denotes the pressure, $(x_1, x_2, x_3) \in \mathbb{R}^3$ is the space variable. Without loss of generality, we assume that the flow is perfect polytropic gas satisfying the γ-law

$$p = \mathcal{A}\rho^\gamma, \tag{1.3}$$

with $\mathcal{A} > 0$ and the adiabatic exponent $\gamma > 1$. The sound speed of the flow is defined as $c(\rho) = \sqrt{p'(\rho)} = \sqrt{\mathcal{A}\gamma\rho^{\gamma-1}}$, and the flow is subsonic if and only if $|U| < c(\rho)$.

Here, we consider the axially symmetric flow in this paper, and take $y = x_3$ to be the axis of symmetry and

$$x = \sqrt{x_1^2 + x_2^2}.$$

Let the fluid density and velocity be $\rho(x, y)$ and $u(x, y), v(x, y), w(x, y)$ in cylindrical coordinates, where u, v, w are radial velocity, axially velocity and swirl velocity respectively. Furthermore, we look for such an axisymmetric flow without swirl in this paper, one has

$$u_1(x_1, x_2, x_3) = u(x, y)\frac{x_1}{x}, \ u_2(x_1, x_2, x_3) = u(x, y)\frac{x_2}{x}, \ u_3(x_1, x_2, x_3) = v(x, y).$$

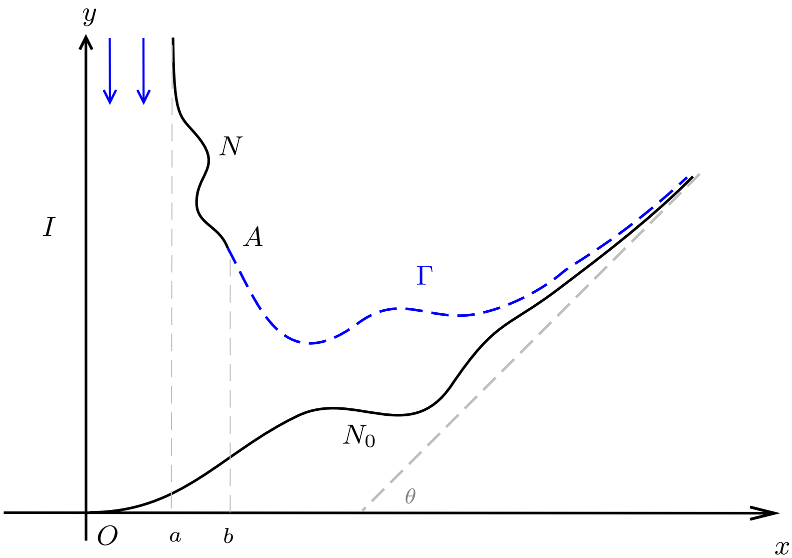

FIG. 3. Axially symmetric impinging jet

Then the governing system (1.1) and (1.2) are written in the cylindrical coordinates as

$$\begin{cases} (x\rho u)_x + (x\rho v)_y = 0, \\ (x\rho u^2)_x + (x\rho uv)_y + xp_x = 0, \\ (x\rho uv)_x + (x\rho v^2)_y + xp_y = 0, \end{cases} \tag{1.4}$$

with the irrotational condition

$$u_y - v_x = 0. \tag{1.5}$$

In order to clarify the physical problem, we start with the notation and the assumptions on the geometry of the nozzle and the impermeable wall as follows. As shown in Figure 3, we denote the semi-infinite nozzle as

$$N : y = g(x) \in C^{2,\alpha}\big((a,b]\big), \quad g(b) = 1 \quad \text{and} \quad \lim_{x\to a+} g(x) = +\infty, \tag{1.6}$$

with $A = (b, 1)$ being the endpoint of the nozzle, and $g(x)$ is decreasing in $(a, a+\varepsilon_0)$ for any small $\varepsilon_0 > 0$. Denote the uneven wall as $N_0 :\ y = g_0(x)$ for $x \geqslant 0$ satisfying

$$g_0(x) \in C^{2,\alpha}\big([0,+\infty)\big), \quad g_0(0) = 0, \;\; g(x) > g_0(x) \quad \text{for any } x \in (a,b], \tag{1.7}$$

and there exist a $\theta \in \big[0, \frac{\pi}{2}\big)$, and a $R_0 > b$, such that

$$g_0'(x) \to \tan\theta \quad \text{as } x \to +\infty, \text{ and } \; g_0''(x) \geqslant 0 \quad \text{for any } x > R_0. \tag{1.8}$$

The boundary conditions require that the nozzle wall N and the uneven wall N_0 are assumed to be impermeable, thus

$$(u, v)\cdot\vec{n} = 0 \quad \text{on } N \cup N_0, \tag{1.9}$$

where $\vec{n}$ is the unit outward normal to $N \cup N_0$. We denote the incoming mass flux as M_0 in the axially symmetric nozzle, namely

$$-\int_0^{x_0} 2\pi x\rho(x, y_0)v(x, y_0)dx = M_0 > 0, \tag{1.10}$$

for any $y_0 \in (1, +\infty)$, where $x_0 = \inf\{x \mid g(x) = y_0\}$.

The well-known Bernoulli's law gives that

$$\frac{q^2}{2} + \frac{\mathcal{A}\gamma}{\gamma - 1}\rho^{\gamma-1} = \mathcal{B} \text{ in the fluid field,} \tag{1.11}$$

where $q = \sqrt{u^2 + v^2}$ is the flow speed and $\mathcal{B}$ is the Bernoulli's constant.

The free surface Γ is defined as an interface between the fluid issuing from the nozzle wall N and the fluid outside. And then the fluid still satisfies the slip boundary condition on the free surface Γ. Moreover, the pressure on Γ balances to the atmospheric pressure p_{atm}, and thus we assume that

$$p = p_{atm} \quad \text{on } \Gamma. \tag{1.12}$$

Hence, we can formulate the compressible subsonic impinging jet problem into the following free boundary problem (FBP).

DEFINITION 1.1 (The free boundary problem (FBP)) Given a semi-infinitely long nozzle wall N, an uneven wall N_0, for some appropriate incoming mass flux $M_0 > 0$, whether there exists a unique axially symmetric subsonic impinging jet flow, such that the free surface Γ detaches smoothly from the endpoint of the nozzle wall N, and goes to infinity in x-direction, and the pressure balances to the atmospheric pressure p_{atm} on the free surface?

Next, we give the definition of the subsonic solution to the FBP.

DEFINITION 1.2 (A subsonic solution to the FBP) A vector (u, v, ρ, Γ) is called a subsonic solution to the FBP, provided that

(1) the free surface Γ is given by a C^1-smooth function $y = k(x)$ for $x \in (b, +\infty)$ with

$$k(b + 0) = g(b - 0) = 1, \quad k'(b + 0) = g'(b - 0), \tag{1.13}$$

and

$$k'(x) \to \tan\theta, \quad k(x) - g_0(x) \to 0 \text{ as } x \to +\infty.$$

(2) $(u, v, \rho) \in C^{1,\alpha}(\Omega_0) \cap C(\bar{\Omega}_0)$ solves the compressible Euler system (1.4) in Ω_0, where Ω_0 is the flow field bounded by N, N_0, I and Γ;

(3) $\sup_{(x,y)\in\bar{\Omega}_0} \frac{\sqrt{u^2+v^2}}{c(\rho)} < 1$ and $p = p_{atm}$ on Γ.

REMARK The conditions (1.13) are so-called *continuous fit condition* and *smooth fit condition* to the impinging jet, which imply that the free surface Γ initiates smoothly from the endpoint A of the nozzle wall N.

1.2 *Main results*

Before we state the main results in this paper, we would like to emphasize that the atmospheric pressure p_{atm} is an arbitrary constant here. Once it is fixed, we found that there exists an interval

(p_1, p_2) for the constant pressure p_{in} in the inlet, and then our results reveal that we can impose a unique $p_{in} \in (p_1, p_2)$ to guarantee the unique existence of the axially symmetric impinging jet. And the critical values p_1, p_2 depend on the atmospheric pressure p_{atm} and the mass flux M_0, which can be determined uniquely by the following formulas,

$$\frac{M_0^2}{2\pi^2 a^4 \left(\frac{p_1}{\mathcal{A}}\right)^{\frac{2}{\gamma}}} + \frac{\mathcal{A}\gamma}{\gamma - 1}\left(\frac{p_1}{\mathcal{A}}\right)^{\frac{\gamma-1}{\gamma}} = \frac{\mathcal{A}\gamma}{\gamma - 1}\left(\frac{p_{atm}}{\mathcal{A}}\right)^{\frac{\gamma-1}{\gamma}}$$

and

$$\frac{M_0^2}{2\pi^2 a^4 \left(\frac{p_2}{\mathcal{A}}\right)^{\frac{2}{\gamma}}} + \frac{\mathcal{A}\gamma}{\gamma - 1}\left(\frac{p_2}{\mathcal{A}}\right)^{\frac{\gamma-1}{\gamma}} = \frac{\mathcal{A}\gamma(\gamma+1)}{2(\gamma - 1)}\left(\frac{p_{atm}}{\mathcal{A}}\right)^{\frac{\gamma-1}{\gamma}}.$$

Obviously, $p_2 > p_1$ and the interval (p_1, p_2) is well-defined.

The main results in this paper are stated as follows.

Theorem 1.3 *Assume that the semi-infinitely long nozzle wall N and the uneven wall N_0 satisfy the conditions (1.6)–(1.8), for any given atmospheric pressure $p_{atm} > 0$, then there exists a critical mass flux M_{cr}, such that for any $M_0 \in (0, M_{cr})$, there exist a unique incoming pressure $p_{in} \in (p_1, p_2)$ and a unique subsonic solution (u, v, ρ, Γ) to the FBP. Moreover,*

(1) *the subsonic impinging jet flow satisfies the asymptotic behavior in upstream as follows,*

$$(u, v, \rho)(x, y) \to (0, v_{in}, \rho_{in}) \quad \text{and} \quad \nabla(u, v, \rho) \to 0,$$

uniformly in any compact subset of $(0, a)$ as $y \to +\infty$, where $v_{in} = -\frac{M_0}{\pi a^2 \rho_{in}}$ and $\rho_{in} = (\frac{p_{in}}{\mathcal{A}})^{\frac{1}{\gamma}}$.

Similarly, the subsonic impinging jet flow satisfies the asymptotic behavior in downstream as follows,

$$(u, v, \rho)(x, y) \to (q_0 \cos\theta, q_0 \sin\theta, \rho_0), \quad (x, y) \in \Omega_0,$$

as $x^2 + y^2 \to +\infty$, where $\rho_0 = \left(\frac{p_{atm}}{\mathcal{A}}\right)^{\frac{1}{\gamma}}$ and $q_0 = \sqrt{v_{in}^2 + \frac{2\mathcal{A}\gamma}{\gamma-1}\left(\rho_{in}^{\gamma-1} - \rho_0^{\gamma-1}\right)}$;

(2) *the free boundary Γ converges to N_0 in the far field, and*

$$x\big(k(x) - g_0(x)\big) \to \frac{M_0}{2\rho_0 q_0 \pi \cos\theta} \quad \text{as } x \to +\infty. \tag{1.14}$$

Furthermore, the free boundary Γ is analytic;

(3) *the radial velocity $u > 0$ in $\bar{\Omega}_0 \setminus I$;*

(4) *M_{cr} is the upper critical value of mass flux for the existence of subsonic jet flow in the following sense: either*

$$\sup_{\bar{\Omega}_0}\big(u^2 + v^2 - c^2(\rho)\big) \to 0 \ \text{ as } \ M_0 \to M_{cr}^-,$$

or there is no $\sigma > 0$, such that for any $M_0 \in (M_{cr}, M_{cr} + \sigma)$, there exists a subsonic solution to the compressible jet flow problem and

$$\sup_{M_0 \in (M_{cr}, M_{cr}+\sigma)}\left(\sup_{\bar{\Omega}_0}\big(u^2 + v^2 - c^2(\rho)\big)\right) < 0.$$

Remark In view of the statement (4) in Theorem 1.3, we have that either

$$\sup_{\bar{\Omega}_0}\big(u^2 + v^2 - c^2(\rho)\big) \to 0 \quad \text{as} \ \ M_0 \to M_{cr}^-,$$

or for any $M_0 > M_{cr}$, there exists an incoming mass flux $\tilde{M}_0 \in [M_{cr}, M_0]$, such that there are no an incoming pressure $p_{in} \in (p_1, p_2)$ and a subsonic solution to the compressible jet flow problem satisfying

$$\sup_{\bar{\Omega}_0}\big(u^2 + v^2 - c^2(\rho)\big) < 0.$$

Remark Our result indicates that the chamber pressure p_{in} in the inlet is determined uniquely by the continuous fit condition, provided that the atmospheric pressure p_{atm} is imposed. On another hand, the result implies that there exists a unique pressure difference between the chamber and the outside, such that there exists a unique axially symmetric impinging jet with continuous fit condition.

Remark The result (1.14) in fact gives the convergence rate of the distance between the free surface Γ and the curved wall N_0 in the downstream, which is quite different from the two-dimensional case.

Remark Here, we restrict the magnitude of the incoming mass flux M_0 to guarantee the global subsonicity of the impinging jet, this idea is motivated by the recent works [16, 17, 21–23, 36–38] on the global subsonic flow in an infinitely long nozzle. This is also quite different from the recent results on two-dimensional subsonic impinging jets in [14].

Based on the significant work [7] by Alt, Caffarelli and Friedman, we can obtain the higher regularity of the free boundary near the end point A.

Theorem 1.4 *If N is $C^{3,\alpha}$ near A, then the solution (u, v, ρ, Γ) established in Theorem 1.3 satisfies that either*

(1) *$N \cup \Gamma$ is C^2 at A or,*
(2) *the optimal regularity of $N \cup \Gamma$ at A is only $C^{1,\frac{1}{2}}$ and the curvature of Γ goes to $\pm\infty$ as $x \to b^+$.*

Remark The results of Theorem 1.4 imply that either

the curvature along Γ tends to the curvature of N at A,

or

the curvature of Γ tends to $\pm\infty$ in absolute value as one approaches A along Γ.

The second case is so-called *abrupt separation*. The proof of Theorem 1.4 follows from Theorem 1.1 in [7] directly.

To investigate the well-posedness of the compressible subsonic impinging jet in axially symmetric case, from the mathematical point of view, there are at least three difficulties and key points here. The one is how to discover a mechanism to guarantee the smoothness and the global subsonicity of the impinging jet in the whole fluid field. In the first well-posedness result [6] on compressible subsonic jet, the authors suggested to constrain the atmospheric pressure with subsonic condition and the convex geometry condition of the nozzle wall. With the aid of geometry property, they can conclude that the compressible jet achieves its maximal speed on the free boundary, and then the subsonic condition on the free boundary implies the global subsonicity of the jet. A similar idea has been adapted in the recent work on subsonic impinging jet in two dimensions in [14]. However, in the present work, our idea is quite different from the one in [6]. We do not restrict the condition on the atmospheric pressure and the geometry condition on the nozzle wall, and we find an upper critical value of the incoming mass flux and show the regularity and global subsonicity of the impinging jet provided that the incoming mass flux is less than the upper critical value.

The second difficulty is how to fulfill the continuous fit condition between the nozzle wall and the free boundary. In the pioneer work [6], the continuous fit condition was fulfilled for special choice of the incoming mass flux. Namely, they showed that there exists a unique incoming mass flux, such that the free boundary connects smoothly at the endpoint of the nozzle wall. Here, we choose the pressure in the inlet as a parameter and show that there exists a unique pressure in the inlet lying in an appropriate interval (p_1, p_2), such that the continuous fit condition holds. As mentioned in the second Remark, the result implies that there exists a pressure difference between the inlet and the outlet, such that the continuous fit condition is fulfilled. This makes the result more reasonable from the physical point of view. The third key point here is that for the 3D axially symmetric impinging jet there is no uniform positive distance between the free surface Γ and the uneven wall N_0. Our proof firstly focuses on the decay estimates of the solution in far field. Moreover, with the optimal decay rate in hand, we get the convergence rate of the distance between the free surface and the ground. And then rescaling the impinging jet in downstream obtains many important facts, such as the asymptotic behavior of the impinging jet in downstream.

The rest of the paper is organized as follows. In Section 2, we introduce a variational problem to solve the free boundary problem, and moreover, establish some properties of the minimizer, such as the bounded gradient lemma and the non-degeneracy lemma. The Section 3 is about the free boundary of the minimizer, we prove the continuous dependence of the minimizer and the free boundary with respect to the parameter λ, and obtain the continuous fit condition of the free boundary. In Section 4, we establish the existence and uniqueness of the subsonic solution to the axially symmetric impinging flow problem, provided that the incoming mass flux is small enough. The existence of critical mass flux is obtained in Section 5. In the final section, we give a summary of the proof to the main results in this paper.

2. Mathematical setting on the FBP

2.1 *Stream function setting*

Based on the continuity equation, the stream function ψ can be introduced such that

$$u = \frac{1}{x\rho}\psi_y \quad \text{and} \quad v = -\frac{1}{x\rho}\psi_x. \tag{2.1}$$

Without loss of generality, we can impose the boundary condition as

$$\psi = m_0 \ \text{ on } N \cup \Gamma \text{ and } \psi = 0 \ \text{ on } N_0 \cup I,$$

where $m_0 = \frac{M_0}{2\pi}$. Denote Ω as the possible fluid field and $E = \Omega \cap \{x > b\}$. Define the free boundary of the stream function as follows

$$\Gamma = E \cap \partial\{\psi < m_0\}.$$

Since the pressure is equal to the constant atmospheric pressure $p_{atm} > 0$ on the free boundary Γ, it follows from Bernoulli's law (1.11) that the density ρ and the momentum ρq are also constants on Γ, denote

$$\rho_0 = \left(\frac{p_{atm}}{\mathcal{A}}\right)^{\frac{1}{\gamma}}, \quad q_0 = \left(2\mathcal{B} - \frac{2\mathcal{A}\gamma}{\gamma-1}\rho_0^{\gamma-1}\right)^{\frac{1}{2}} \quad \text{and} \quad \lambda = \rho_0 q_0 \ \text{ on } \ \Gamma. \tag{2.2}$$

It is easy to see that

$$\left|\frac{\nabla\psi}{x}\right| = \frac{1}{x}\frac{\partial\psi}{\partial\nu} = \lambda = \rho q = \rho_0 q_0 = \rho_0\left(2\mathcal{B} - \frac{2\mathcal{A}\gamma}{\gamma-1}\rho_0^{\gamma-1}\right)^{\frac{1}{2}} \quad \text{on } \ \Gamma, \tag{2.3}$$

where ν is the outer normal vector of Γ. Moreover, one has

$$\frac{2m_0^2}{a^4\rho_{in}^2} + \frac{\mathcal{A}\gamma}{\gamma-1}\rho_{in}^{\gamma-1} = \frac{\lambda^2}{2\rho_0^2} + \frac{\mathcal{A}\gamma}{\gamma-1}\rho_0^{\gamma-1} = \mathcal{B}. \tag{2.4}$$

By virtue of (2.4), λ is uniquely determined by the density ρ_{in} in upstream, once m_0 and ρ_0 are fixed. Therefore, we can take λ as a parameter to solve the free boundary problem firstly, and denote

$$\mathcal{B}(\lambda^2) = \frac{\lambda^2}{2\rho_0^2} + \frac{\mathcal{A}\gamma}{\gamma-1}\rho_0^{\gamma-1}. \tag{2.5}$$

As we know, there exist some critical quantities,

$$q_{\lambda,cr} = \left(2\mathcal{B}(\lambda^2)\frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{2}}, \quad \rho_{\lambda,cr} = \left(\frac{2\mathcal{B}(\lambda^2)}{\mathcal{A}\gamma}\frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}, \quad \rho_{\lambda,max} = \left(\frac{\mathcal{B}(\lambda^2)}{\mathcal{A}\gamma}(\gamma-1)\right)^{\frac{1}{\gamma-1}},$$

such that the flow is subsonic if and only if $q < q_{\lambda,cr}$ or $\rho_{\lambda,cr} < \rho \leqslant \rho_{\lambda,max}$ (see also in [9] and [18]).

Let $t = \left|\frac{\nabla\psi}{x}\right|^2$ be the square norm of the momentum and the Bernoulli's law (1.11) gives that

$$\frac{t}{2\rho^2} + \frac{\mathcal{A}\gamma}{\gamma-1}\rho^{\gamma-1} = \frac{\lambda^2}{2\rho_0^2} + \frac{\mathcal{A}\gamma}{\gamma-1}\rho_0^{\gamma-1} = \mathcal{B}(\lambda^2).$$

Moreover, set

$$\Pi_\lambda = \rho_{\lambda,cr} q_{\lambda,cr}$$

and

$$\mathcal{H}(t,\rho;\lambda^2) = \frac{t}{2\rho^2} + \frac{\mathcal{A}\gamma}{\gamma-1}\rho^{\gamma-1} - \mathcal{B}(\lambda^2) = 0, \tag{2.6}$$

for $t > 0$, $\rho_{\lambda,cr} < \rho \leqslant \rho_{\lambda,max}$ and $\lambda \in (0, \Pi_\lambda)$.

A simple manipulation leads to that

$$\frac{\partial\mathcal{H}}{\partial(\lambda^2)}(t,\rho;\lambda^2) = -\frac{1}{2\rho_0^2} \quad \text{and} \quad \frac{\partial\mathcal{H}}{\partial t}(t,\rho;\lambda^2) = \frac{1}{2\rho^2}, \tag{2.7}$$

and

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial \rho}(t,\rho;\lambda^2) &= \frac{1}{\rho}\left(\mathcal{A}\gamma\rho^{\gamma-1} - \frac{t}{\rho^2}\right)\\ &= \frac{1}{\rho}\left(\frac{\gamma+1}{\gamma-1}\mathcal{A}\gamma\rho^{\gamma-1} - 2\mathcal{B}(\lambda^2)\right)\\ &= \frac{1}{\rho}\frac{\gamma+1}{\gamma-1}\mathcal{A}\gamma\left(\rho^{\gamma-1} - \rho_{\lambda,cr}^{\gamma-1}\right)\\ &> 0, \end{aligned} \tag{2.8}$$

for any $\rho_{\lambda,cr} < \rho \leqslant \rho_{\lambda,max}$ and $\lambda \in (0, \Pi_\lambda)$, where we used the equality (2.6). Thus, noticing (2.6), the density ρ can be described as a function of t with a parameter $\lambda \in (0, \Pi_\lambda)$ saying $\rho(t;\lambda^2)$, provided that $\rho_{\lambda,cr} < \rho \leqslant \rho_{\lambda,max}$. Furthermore, it follows from (2.7) and (2.8) that

$$\frac{\partial \rho(t;\lambda^2)}{\partial t} = -\frac{\frac{\partial \mathcal{H}(t,\rho;\lambda^2)}{\partial t}}{\frac{\partial \mathcal{H}(t,\rho;\lambda^2)}{\partial \rho}} = -\frac{1}{2\rho^2\frac{\partial \mathcal{H}(t,\rho;\lambda^2)}{\partial \rho}} < 0, \tag{2.9}$$

and

$$\frac{\partial \rho(t;\lambda^2)}{\partial (\lambda^2)} = -\frac{\frac{\partial \mathcal{H}(t,\rho;\lambda^2)}{\partial (\lambda^2)}}{\frac{\partial \mathcal{H}(t,\rho;\lambda^2)}{\partial \rho}} = \frac{1}{2\rho_0^2\frac{\partial \mathcal{H}(t,\rho;\lambda^2)}{\partial \rho}} > 0, \tag{2.10}$$

for any $\rho_{\lambda,cr} < \rho \leqslant \rho_{\lambda,max}$ and $\lambda \in (0, \Pi_\lambda)$. In view of (2.9), it is easy to check that for any $\lambda \in (0, \Pi_\lambda)$,

$$\rho_{\lambda,cr} < \rho \leqslant \rho_{\lambda,max} \quad \text{if and only if} \quad t \in (0, \Pi_\lambda^2). \tag{2.11}$$

Thus we can conclude that $\rho(t;\lambda^2)$ is a decreasing smooth function with respect to $t \in [0, \Pi_\lambda^2)$ for $\lambda \in (0, \Pi_\lambda)$. Moreover, the density ρ for subsonic flow has the following uniform estimates

$$\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\rho_0 \leqslant \rho_{\lambda,cr} < \rho \leqslant \rho_{\lambda,max} \leqslant \left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}}\rho_0. \tag{2.12}$$

After a direct computation, there exists a $\lambda_{cr} = \left(\mathcal{A}\gamma\rho_0^{\gamma+1}\right)^{\frac{1}{2}} = (\mathcal{A}\gamma)^{\frac{1}{2}}\left(\frac{p_{atm}}{\mathcal{A}}\right)^{\frac{\gamma+1}{2\gamma}}$, such that

$$\lambda < \Pi_\lambda \quad \text{for any} \quad \lambda < \lambda_{cr}, \quad \text{and} \quad \lambda_{cr} = \Pi_{\lambda_{cr}}.$$

In this paper, we assume that the parameter $\lambda < \lambda_{cr}$. In the following, denote

$$\rho_1(t;\lambda^2) = \frac{\partial \rho(t;\lambda^2)}{\partial t} \quad \text{and} \quad \rho_2(t;\lambda^2) = \frac{\partial \rho(t;\lambda^2)}{\partial (\lambda^2)}$$

the derivatives of the function $\rho(t;\lambda^2)$ for any $t \in (0, \Pi_\lambda^2)$ and $\lambda \in (0, \Pi_\lambda)$.

Furthermore, the irrotational condition (1.5) deduces to the governing equation for the stream function in the flow field that

$$Q_\lambda\psi = \nabla\cdot\left(\frac{\nabla\psi}{x\rho(|\frac{\nabla\psi}{x}|^2;\lambda^2)}\right) = 0 \quad \text{in } \Omega_0, \tag{2.13}$$

where $\nabla = (\partial_x, \partial_y)$ and $\Omega_0 = \Omega \cap \{\psi < m_0\}$ is the fluid field.

It is not difficult to see that the equation in (2.13) becomes degenerate as $\frac{|\nabla\psi|}{x} \to \Pi_\lambda$, in order to guarantee the uniform ellipticity, at first we consider the following modified problem. The essence of this idea has already been illustrated in the compressible subsonic problem, seeing [6, 8, 9, 11, 19–23, 36–38].

Let $\tilde{\rho}(t;\lambda^2)$ be a smooth decreasing function satisfying

$$\tilde{\rho}(t;\lambda^2) = \begin{cases} \rho(t;\lambda^2), & \text{for } t \leqslant (\Pi_\lambda - 2\tilde{\varepsilon})^2, \\ \rho((\Pi_\lambda - \tilde{\varepsilon})^2;\lambda^2), & \text{for } t \geqslant (\Pi_\lambda - \tilde{\varepsilon})^2, \end{cases} \tag{2.14}$$

for any small $\tilde{\varepsilon} > 0$, and

$$-\frac{\tilde{\rho}_1(t;\lambda^2)}{\tilde{\rho}^2(t;\lambda^2)} \leqslant \frac{C_{\tilde{\varepsilon}}}{1+t} \quad \text{for } C_{\tilde{\varepsilon}} > 0. \tag{2.15}$$

Here, $\tilde{\rho}_1(t;\lambda^2)$ denotes the derivative function of $\tilde{\rho}(t;\lambda^2)$ with respect to t.

Hence, we first consider the following modified free boundary problem

$$\begin{cases} \tilde{Q}_\lambda\psi = \nabla\cdot\left(\frac{\nabla\psi}{x\tilde{\rho}(|\frac{\nabla\psi}{x}|^2;\lambda^2)}\right) = 0 & \text{in } \Omega\cap\{\psi < m_0\}, \\ \frac{1}{x}\frac{\partial\psi}{\partial\nu} = \lambda & \text{on } \Gamma, \\ \psi = 0 \quad \text{on } N_0\cup I, \quad \psi = m_0 & \text{on } N\cup\Gamma. \end{cases} \tag{2.16}$$

In the end of this section, we will verify that $\frac{|\nabla\psi|}{x} \leqslant \Pi_\lambda - 2\tilde{\varepsilon}$ in $\bar{\Omega}_0$, thus the subsonic cut-off can be taken away and $\tilde{\rho}\big(\big|\frac{\nabla\psi}{x}\big|^2;\lambda^2\big) = \rho\big(\big|\frac{\nabla\psi}{x}\big|^2;\lambda^2\big)$.

2.2 *Variational approach*

To solve the free boundary value problem (2.16) with any parameter $\lambda < \Pi_\lambda$, we will introduce the variational method, which has been adapted to solve the compressible jet problem in [6]. Next, we give the corresponding variational problem as follows. Firstly, define an admissible set (see Figure 3) as

$$K = \big\{\psi \in H^1_{loc}(\mathbb{R}^2) \mid \quad \psi \leqslant m_0 \text{ a.e. in } \mathbb{R}^2,\ \psi = m_0 \text{ lies above } N, \\ \psi = 0 \text{ lies below } N_0 \text{ and lies left } I\big\}.$$

Denote

$$F(t;\lambda) = \int_0^t \frac{1}{\tilde{\rho}(\tau;\lambda^2)}d\tau, \quad F_1(t;\lambda) = \frac{\partial F(t;\lambda)}{\partial t} \quad \text{and} \quad F_{11}(t;\lambda) = \frac{\partial^2 F(t;\lambda)}{\partial t^2},$$

which together with (2.15) yield that

$$F(0;\lambda) = 0, \quad F(\lambda^2;\lambda) = \frac{1}{\rho_0}, \quad 0 \leqslant F_{11}(t;\lambda) \leqslant \frac{C_{\tilde{\varepsilon}}}{1+t}. \tag{2.17}$$

Set

$$\Lambda = \Lambda(\lambda^2) = \sqrt{2F_1(\lambda^2;\lambda)\lambda^2 - F(\lambda^2;\lambda)}.$$

In view of (2.12), it is easy to check that

$$\frac{\Lambda^2(\lambda^2)}{\lambda^2} = \frac{2}{\rho_0} - \frac{1}{\lambda^2}\int_0^{\lambda^2} \frac{1}{\tilde{\rho}(\tau;\lambda^2)}d\tau \leqslant \frac{1}{\rho_0}\left(2-\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\right), \tag{2.18}$$

and

$$\frac{\Lambda^2(\lambda^2)}{\lambda^2} = \frac{2}{\rho_0} - \frac{1}{\lambda^2}\int_0^{\lambda^2} \frac{1}{\tilde{\rho}(\tau;\lambda^2)}d\tau \geqslant \frac{1}{\rho_0}\left(2-\left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}}\right) > 0, \tag{2.19}$$

for any $\lambda \in [0, \Pi_\lambda)$. Furthermore, it follows from (2.10) that

$$\frac{d\Lambda^2(\lambda^2)}{d(\lambda^2)} = \frac{1}{\rho_0} + \int_0^{\lambda^2} \frac{\rho_2(\tau;\lambda^2)}{\rho^2(\tau;\lambda^2)}d\tau > \frac{1}{\rho_0}, \tag{2.20}$$

for any $\lambda \in [0, \Pi_\lambda - 2\tilde{\varepsilon})$, which implies that $\Lambda(\lambda^2)$ is uniquely determined by $\lambda \in (0, \Pi_\lambda - 2\tilde{\varepsilon})$.

Set

$$f(\eta;\lambda) = F(|\eta|^2;\lambda) \quad \text{with} \quad \eta = (\eta_1, \eta_2) \in \mathbb{R}^2, \tag{2.21}$$

it follows from (2.17) that $f(\eta;\lambda)$ is convex with respect to η, and there exists a constant ϑ depending on λ_{cr} and $\tilde{\varepsilon}$, such that

$$\vartheta|\xi|^2 \leqslant \sum_{i,j=1}^{2} \frac{\partial^2 f(\eta;\lambda)}{\partial\eta_i\partial\eta_j}\xi_i\xi_j \leqslant \vartheta^{-1}|\xi|^2 \quad \text{for any} \quad \xi = (\xi_1,\xi_2) \in \mathbb{R}^2,$$

and

$$\vartheta|\eta|^2 \leqslant f_\eta(\eta;\lambda)\cdot\eta \quad \text{and} \quad \vartheta|\eta|^2 \leqslant f(\eta;\lambda) \leqslant \vartheta^{-1}|\eta|^2.$$

Define a function with any parameter $\lambda \in (0, \Pi_\lambda)$ as follows,

$$G(\nabla\psi,\psi,x;\lambda) = xF\left(\left|\frac{\nabla\psi}{x}\right|^2;\lambda\right) + \left(x\Lambda^2 - 2\lambda F_1(\lambda^2;\lambda)\nabla\psi\cdot e\right)\chi_{\{\psi<m_0\}\cap E}, \tag{2.22}$$

where χ_D is the indicator function of the set D and $e = (-\sin\theta, \cos\theta)$. By virtue of the convexity of $F(t;\lambda)$ with respect to t, one has

$$\begin{aligned} G(\nabla\psi,\psi,x;\lambda) \geqslant{}& x\left(F\left(\left|\frac{\nabla\psi}{x}\right|^2;\lambda\right) - F(\lambda^2;\lambda) - F_1(\lambda^2;\lambda)\left(\left|\frac{\nabla\psi}{x}\right|^2 - \lambda^2\right)\right)\chi_{\{\psi<m_0\}\cap E} \\ &+ xF_1(\lambda^2;\lambda)\left|\frac{\nabla\psi}{x} - \lambda e\right|^2\chi_{\{\psi<m_0\}\cap E} \\ \geqslant{}& xF_1(\lambda^2;\lambda)\left|\frac{\nabla\psi}{x} - \lambda e\right|^2\chi_{\{\psi<m_0\}\cap E}, \end{aligned} \tag{2.23}$$

and

$$G(\nabla\psi,\psi,x;\lambda) \leqslant Cx\left|\frac{\nabla\psi}{x} - \lambda e\right|^2\chi_{\{\psi<m_0\}\cap E} + xF_1\left(\left|\frac{\nabla\psi}{x}\right|^2;\lambda\right)\chi_{\{\psi<m_0\}\setminus E}. \tag{2.24}$$

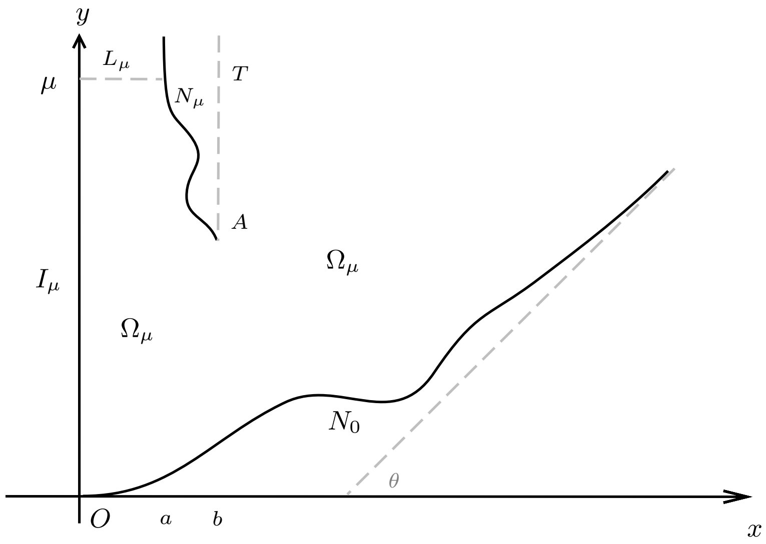

FIG. 4. Truncated domain

Hence, we define a functional

$$J_\lambda(\psi) = \int_\Omega G(\nabla\psi, \psi, x; \lambda)dxdy.$$

It follows from (2.23) that the functional $J_\lambda(\psi)$ is non-negative for any $\psi \in K$. Obviously, $J_\lambda(\psi)$ is unbounded for any $\psi \in K$. Thus we will truncate the domain as Ω_μ for any $\mu > 1$ (see Figure 4), which is bounded by N_μ, I_μ, N_0, L_μ and $T = \{(b, y) \mid y > 1\}$, where

$$N_\mu = N \cap \{x \geqslant x_\mu\}, \quad I_\mu = I \cap \{y \leqslant \mu\}, \quad \text{and} \quad L_\mu = \{(x, y) \mid 0 \leqslant x \leqslant x_\mu, y = \mu\},$$

with $x_\mu = \min\{x \mid g(x) = \mu\}$. Define the following functional in the truncated domain Ω_μ,

$$J_{\lambda,\mu}(\psi) = \int_{\Omega_\mu} G(\nabla\psi, \psi, x; \lambda)dxdy.$$

To overcome the singularity of the functional $J_{\lambda,\mu}$ near y-axis, we first consider the following variational problem.

The truncated variational problem $(P^\delta_{\lambda,\mu})$

For any $\lambda \leqslant \Pi_\lambda - 3\tilde{\varepsilon}$, $\mu > 1$ and small $\delta > 0$, find a $\psi^\delta_{\lambda,\mu} \in K^\delta_\mu$ such that

$$J^\delta_{\lambda,\mu}(\psi^\delta_{\lambda,\mu}) = \min_{\psi \in K^\delta_\mu} J^\delta_{\lambda,\mu}(\psi),$$

where

$$J^\delta_{\lambda,\mu}(\psi) = \int_{\Omega_\mu} G(\nabla\psi, \psi, x + \delta; \lambda)dxdy$$

and

$$K_\mu^\delta = \left\{ \psi \in K \mid \psi = \min\left\{ \frac{m_0}{x_\mu^2}((x+\delta)^2 - \delta^2), m_0 \right\} \quad \text{on } L_\mu \right\}.$$

Lemma 2.1 *The variational problem $(P_{\lambda,\mu}^\delta)$ has a minimizer $\psi_{\lambda,\mu}^\delta$ and $\psi_{\lambda,\mu}^\delta \in C^{0,1}(\Omega_\mu)$. Furthermore, the minimizer $\psi_{\lambda,\mu}^\delta$ satisfies that*

$$\int_{\Omega_\mu} f_\eta \left(\frac{\nabla \psi_{\lambda,\mu}^\delta}{x+\delta}; \lambda \right) \cdot \nabla \xi dxdy \geqslant 0 \quad \text{for any } \xi \in C_0^\infty(\Omega_\mu) \text{ and } \xi \geqslant 0,$$

and

$$\int_{\Omega_\mu} f_\eta \left(\frac{\nabla \psi_{\lambda,\mu}^\delta}{x+\delta}; \lambda \right) \cdot \nabla \zeta dxdy = 0 \quad \text{for any } \zeta \in C_0^\infty(\Omega_\mu \cap \{\psi_{\lambda,\mu}^\delta < m_0\}), \tag{2.25}$$

where $f(\eta;\lambda)$ is defined in (2.21). Furthermore,

$$0 \leqslant \psi_{\lambda,\mu}^\delta(x,y) \leqslant \min\left\{ \frac{m_0}{x_\mu^2}\big((x+\delta)^2 - \delta^2\big), m_0 \right\} \quad \text{in } \Omega_\mu. \tag{2.26}$$

Proof. Define

$$\psi_0(x,y) = \min\left\{ \lambda \max\left\{ (x+\delta)\Big(\big(y - g_0(R_0)\big)\cos\theta - (x+\delta-R_0)\sin\theta \Big), 0 \right\}, m_0 \right\},$$

it follows from (1.8) that $\psi_0 = 0$ on $N_0 \cap \{x \geqslant R_0\}$. Then we can extend ψ_0 into the domain $\Omega_\mu \setminus \{x \leqslant R_0\}$ so that it belongs to the admissible set K_μ^δ and $\{\psi_0 < m_0\} \cap (\Omega_\mu \setminus \{x \leqslant R_0\})$ is bounded. Hence, it suffices to verify that

$$\int_{\Omega_\mu \cap \{x > R_0\}} G(\nabla\psi_0, \psi_0, x+\delta; \lambda) dxdy < +\infty.$$

In fact, it follows from (2.24) that

$$\begin{aligned}
&\int_{\Omega_\mu \cap \{x > R_0\}} G(\nabla\psi_0, \psi_0, x+\delta; \lambda) dxdy \\
&\leqslant C \int_{R_0}^{+\infty} \int_{g_0(R_0)+(x+\delta-R_0)\tan\theta}^{g_0(R_0)+(x+\delta-R_0)\tan\theta + \frac{m_0}{\lambda(x+\delta)\cos\theta}} (x+\delta) \left| \frac{\nabla\psi_0}{x+\delta} - \lambda e \right|^2 dydx \\
&\leqslant C \int_{R_0+\delta}^{+\infty} \int_{g_0(R_0)+(x-R_0)\tan\theta}^{g_0(R_0)+(x-R_0)\tan\theta + \frac{m_0}{\lambda x\cos\theta}} \frac{\lambda^2 \Big(\big(y - g_0(R_0)\big)\cos\theta - (x-R_0)\sin\theta \Big)^2}{x} dydx \\
&\leqslant \frac{C}{(R_0+\delta)^3}.
\end{aligned}$$

The existence of the minimizer to the variational problem $(P_{\lambda,\mu}^\delta)$ can be obtained via the similar arguments in Lemma 1.1 in [4] and Theorem 1.1 in [5], and denote $\psi^\delta = \psi_{\lambda,\mu}^\delta$ as the minimizer to the variational problem $(P_{\lambda,\mu}^\delta)$ for simplicity.

For any nonnegative function $\xi\in C_0^\infty(\Omega_\mu)$ and $\varepsilon>0$, it is easy to check that $\min\{\psi^\delta+\varepsilon\xi,m_0\}\in K_\mu$ and $\{\min\{\psi^\delta+\varepsilon\xi,m_0\}<m_0\}\subset\{\psi^\delta<m_0\}$. Thus, we have

$$\begin{aligned}
0&\leqslant J^\delta_{\lambda,\mu}\left(\min\{\psi^\delta+\varepsilon\xi,m_0\}\right)-J^\delta_{\lambda,\mu}(\psi^\delta)\\
&=\int_{\Omega_\mu}(x+\delta)F\left(\frac{|\nabla\min\{\psi^\delta+\varepsilon\xi,m_0\}|^2}{(x+\delta)^2};\lambda\right)-(x+\delta)F\left(\frac{|\nabla\psi^\delta|^2}{(x+\delta)^2};\lambda\right)dxdy\\
&\leqslant\int_{\Omega_\mu\cap\{\psi^\delta+\varepsilon\xi\leqslant m_0\}}(x+\delta)F\left(\frac{|\nabla(\psi^\delta+\varepsilon\xi)|^2}{(x+\delta)^2};\lambda\right)-(x+\delta)F\left(\frac{|\nabla\psi^\delta|^2}{(x+\delta)^2};\lambda\right)dxdy\\
&\leqslant 2\varepsilon\int_{\Omega_\mu\cap\{\psi^\delta+\varepsilon\xi\leqslant m_0\}}F_1\left(\frac{|\nabla(\psi^\delta+\varepsilon\xi)|^2}{(x+\delta)^2};\lambda\right)\frac{\nabla\psi^\delta\cdot\nabla\xi}{x+\delta}dxdy+o(\varepsilon),
\end{aligned}$$

which implies that

$$0\leqslant\int_{\Omega_\mu\cap\{\psi^\delta+\varepsilon\xi\leqslant m_0\}}F_1\left(\frac{|\nabla(\psi^\delta+\varepsilon\xi)|^2}{(x+\delta)^2};\lambda\right)\frac{\nabla\psi^\delta\cdot\nabla\xi}{x+\delta}dxdy.$$

Taking $\varepsilon\to0$ in above inequality, we have

$$0\leqslant\int_{\Omega_\mu}F_1\left(\frac{|\nabla\psi^\delta|^2}{(x+\delta)^2};\lambda\right)\frac{\nabla\psi^\delta\cdot\nabla\xi}{x+\delta}dxdy.$$

Similarly, we can verify that (2.25) holds.

Next, we will show that

$$\psi^\delta(x,y)\geqslant0\quad\text{in}\quad\Omega_\mu.\tag{2.27}$$

Denote $\psi^\delta_\varepsilon=\psi^\delta-\varepsilon\min\{\psi^\delta,0\}$ for $\varepsilon\in(0,1)$. It is easy to check that $\psi^\delta_\varepsilon\in K^\delta_\mu$,

$$\psi^\delta_\varepsilon>0\quad\text{if and only if}\quad\psi^\delta>0\quad\text{and}\quad\psi^\delta_\varepsilon\geqslant\psi^\delta\quad\text{in}\quad\Omega_\mu.$$

Since ψ^δ is the minimizer to the truncated variational problem $(P^\delta_{\lambda,\mu})$, one has

$$0\leqslant J^\delta_{\lambda,\mu}(\psi^\delta_\varepsilon)-J^\delta_{\lambda,\mu}(\psi^\delta).\tag{2.28}$$

For any sufficiently large $R>0$, denote $\Omega_{\mu,R}=\Omega_\mu\cap\{y<R\}$ and $E_R=\Omega_{\mu,R}\cap\{x>b\}$, we have

$$\int_{\Omega_{\mu,R}} G(\nabla\psi_\varepsilon^\delta,\psi_\varepsilon^\delta,x+\delta;\lambda)dxdy-\int_{\Omega_{\mu,R}} G(\nabla\psi^\delta,\psi^\delta,x+\delta;\lambda)dxdy$$
$$=\int_{\Omega_{\mu,R}}(x+\delta)F\left(\left|\frac{\nabla\psi_\varepsilon^\delta}{x+\delta}\right|^2;\lambda\right)-(x+\delta)F\left(\left|\frac{\nabla\psi^\delta}{x+\delta}\right|^2;\lambda\right)dxdy$$
$$-2\lambda F_1(\lambda^2;\lambda)\int_{\Omega_{\mu,R}\cap E}\nabla\psi_\varepsilon^\delta\cdot e\chi_{\{\psi_\varepsilon^\delta<m_0\}}-\nabla\psi^\delta\cdot e\chi_{\{\psi^\delta<m_0\}}dxdy$$
$$\leqslant\int_{\Omega_{\mu,R}}F_1\left(\left|\frac{\nabla\psi_\varepsilon^\delta}{x+\delta}\right|^2;\lambda\right)\frac{|\nabla\psi_\varepsilon^\delta|^2-|\nabla\psi^\delta|^2}{x+\delta}dxdy$$
$$-2\lambda F_1(\lambda^2;\lambda)\int_{\partial E_R}(\psi_\varepsilon^\delta-\psi^\delta)e\cdot\nu dS$$
$$\leqslant\int_{\Omega_{\mu,R}}F_1\left(\left|\frac{\nabla\psi_\varepsilon^\delta}{x+\delta}\right|^2;\lambda\right)\frac{((1-\varepsilon)^2-1)|\nabla\min\{\psi^\delta,0\}|^2}{x+\delta}dxdy. \tag{2.29}$$

Here, we have used the fact

$$\int_{\partial E_R}(\psi_\varepsilon^\delta-\psi_\varepsilon)e\cdot\nu dS=\sin\theta\int_{\partial E_R\cap\{x=b\}}(\psi_\varepsilon^\delta-\psi_\varepsilon)dS+\cos\theta\int_{\partial E_R\cap\{y=R\}}(\psi_\varepsilon^\delta-\psi_\varepsilon)dS\geqslant 0.$$

Taking $R\to+\infty$ in (2.29), it follows from (2.28) that

$$0\leqslant\left((1-\varepsilon)^2-1\right)\int_{\Omega_\mu}F_1\left(\left|\frac{\nabla\psi_\varepsilon^\delta}{x+\delta}\right|^2;\lambda\right)\frac{|\nabla\min\{\psi^\delta,0\}|^2}{x+\delta}dxdy,$$

which implies that (2.27) holds.

Since $0\leqslant\psi_{\lambda,\mu}^\delta\leqslant m_0$ in Ω_μ, it suffices to show that

$$\psi_{\lambda,\mu}^\delta\leqslant\frac{m_0}{x_\mu^2}((x+\delta)^2-\delta^2)\quad\text{in}\quad\Omega_\mu\cap\left\{x\leqslant\sqrt{x_\mu^2+\delta^2}-\delta\right\}. \tag{2.30}$$

In view of (2.25), the maximum principle gives that the inequality (2.26) holds. □

With the aid of Lemma 2.1, we consider the following truncated variational problem.

The truncated variational Problem $(P_{\lambda,\mu})$

For any $\lambda\leqslant\Pi_\lambda-3\tilde{\varepsilon}$ and $\mu>1$, find a $\psi\in K_\mu$ such that

$$J_{\lambda,\mu}(\psi_{\lambda,\mu})=\min_{\psi\in K_\mu}J_{\lambda,\mu}(\psi),$$

where

$$K_\mu=\left\{\psi\in K\mid\psi=\frac{m_0}{x_\mu^2}x^2\quad\text{on }L_\mu\quad\text{and}\quad\psi\leqslant\min\left\{\frac{m_0}{x_\mu^2}x^2,m_0\right\}\quad\text{a.e. in}\quad\Omega_\mu\right\}.$$

2.3 *Existence and fundamental properties of minimizer*

Lemma 2.2 *There exists a minimizer $\psi_{\lambda,\mu}$ to the variational problem $(P_{\lambda,\mu})$ and $\psi_{\lambda,\mu} \in C^{0,1}(\Omega_\mu)$. Moreover,*

(1) *the minimizer $\psi_{\lambda,\mu}$ satisfies that*

$$\int_{\Omega_\mu} f_\eta\left(\frac{\nabla\psi_{\lambda,\mu}}{x};\lambda\right)\cdot\nabla\xi dxdy \geqslant 0 \quad \textit{for any} \quad \xi\in C_0^\infty(\Omega_\mu) \quad \textit{and} \quad \xi\geqslant 0,$$

and

$$\tilde{Q}_\lambda\psi_{\lambda,\mu}=0 \quad \textit{in} \quad \Omega_\mu\cap\{\psi_{\lambda,\mu}<m_0\} \quad \textit{and} \quad \psi_{\lambda,\mu}\in C^{2,\alpha}(\Omega_\mu\cap\{\psi_{\lambda,\mu}<m_0\}).$$

Furthermore,

$$\psi_{\lambda,\mu}(x,y)\geqslant 0 \quad \textit{in} \quad \Omega_\mu. \tag{2.31}$$

(2) *The free boundary $\Gamma_{\lambda,\mu}=E\cap\partial\{\psi_{\lambda,\mu}<m_0\}$ is analytic, and*

$$\frac{1}{x}|\nabla\psi_{\lambda,\mu}|=\lambda \quad \textit{on} \quad \Gamma_{\lambda,\mu},$$

and

$$\frac{1}{x}|\nabla\psi_{\lambda,\mu}|\geqslant\lambda \quad \textit{on} \quad l,$$

where l is a segment with $l\subset T\cap\partial\{\psi_{\lambda,\mu}<m_0\}$.

Proof. (1) Along the similar arguments in the proof of Lemma 1.1 in [4] and Theorem 1.1 in [5], one has that there exists a sequence $\{\delta_n\}$ with $\delta_n\to 0$ as $n\to+\infty$, such that

$$\psi_{\lambda,\mu}^{\delta_n}\rightharpoonup\psi_0 \text{ in } H^1_{loc}(\Omega_\mu) \text{ and } \psi_{\lambda,\mu}^{\delta_n}\to\psi_0 \text{ uniformly in any compact subset of } \Omega_\mu.$$

It follows from (2.26) that

$$0\leqslant\psi_0(x,y)\leqslant\min\left\{\frac{x^2}{x_\mu^2}m_0,m_0\right\} \text{ in } \Omega_\mu,$$

which together with (2.25) gives that

$$\nabla\cdot f_\eta\left(\frac{\nabla\psi_0}{x};\lambda\right)=0 \text{ in } \Omega_\mu\cap\left\{x<\frac{x_\mu}{2}\right\}.$$

Next, we will check that $J_{\lambda,\mu}(\psi_0)<+\infty$. By virtue of the proof of Lemma 2.1, it suffices to show that

$$\frac{|\nabla\psi_0(X)|}{x}\leqslant C \text{ near } I.$$

For any $X_0=(x_0,y_0)\in\Omega_\mu$ with $x_0<\frac{x_\mu}{4}$, denote $\phi(X)=\frac{\psi_0(X_0+r_0X)}{r_0^2}$ with $r_0=\frac{x_0}{2}$. It is easy to check that

$$\nabla\cdot f_\eta\left(\frac{\nabla\phi}{2+x};\lambda\right)=0 \quad \text{and} \quad 0\leqslant\phi\leqslant(2+x)^2m_0 \text{ in } B_1(0).$$

Thanks to the gradient estimate in Chapter 12 in [28], one has

$$|\nabla\phi(0)| \leqslant C,$$

where the constant C is independent of x_0. This gives that

$$|\nabla\psi_0(X_0)| = r_0|\nabla\phi(0)| \leqslant Cx_0.$$

Since $J_{\lambda,\mu}(\psi_0) < +\infty$ and $\psi_0 \in K_\mu$, the minimal functional $J_{\lambda,\mu}(\psi)$ is finite. By using the proof of Lemma 1.1 in [4], we can conclude that there exists a minimizer to the variational problem $(P_{\lambda,\mu})$.
Denote $\psi_{\lambda,\mu}$ be the minimizer to the variational problem $(P_{\lambda,\mu})$ and $\Gamma_{\lambda,\mu} = E\cap\partial\{\psi_{\lambda,\mu} < m_0\}$ as the free boundary of $\psi_{\lambda,\mu}$. Thanks to Lemma 2.1, we can show that $\psi_{\lambda,\mu} \in C^{0,1}(\Omega_\mu)$ satisfies the assertion (1) of this lemma.

(2) Since $F(t;\lambda)$ is C^2-smooth with respect to $t \in [0,+\infty)$, it follows from Theorem 6.3 in [5] that the free boundary $\Gamma_{\lambda,\mu}$ is $C^{1,\alpha}$, and thus $\psi_{\lambda,\mu}$ is $C^{1,\alpha}$ up to the free boundary. Since $\lambda \leqslant \Pi_\lambda - 3\tilde{\varepsilon}$, the subsonic cut-off can be removed near the free boundary. Then $F(t;\lambda)$ is analytic near $\Gamma_{\lambda,\mu}$, the Remark 6.4 in [5] gives that the free boundary $\Gamma_{\lambda,\mu}$ is analytic. By using the similar arguments in the proof of Lemma 9.1 in [14], we can conclude that

$$\frac{1}{x}|\nabla\psi_{\lambda,\mu}| = \lambda \ \text{ on } \ \Gamma_{\lambda,\mu}, \quad \text{and} \quad \frac{1}{x}|\nabla\psi_{\lambda,\mu}| \geqslant \lambda \ \text{ on } \ l,$$

where l is a segment with $l \subset T \cap \partial\{\psi_{\lambda,\mu} < m_0\}$.

□

Next, we will give the bounded gradient lemma in the following.

Lemma 2.3 *Let $X_0 = (x_0, y_0)$ be a free boundary point and let $B_r(X_0) \subset B_R(X_0) \subset E$ with $r < R$. Then*

$$|\nabla\psi_{\lambda,\mu}(X)| \leqslant C\Lambda x \quad \text{in } B_r(X_0),$$

where the constant C depends only on ϑ, N, N_0 and $\left(1-\frac{r}{R}\right)^{-1}$, but not on m_0.

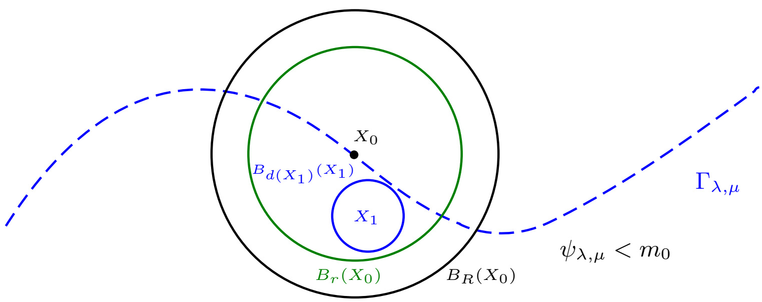

FIG. 5. $B_{d(X_1)}(X_1)$ and $B_R(X_0)$

Proof. *Step* 1. In this step, we will show that

$$m_0 - \psi_{\lambda,\mu}(X) \leqslant C\Lambda x d(X) \quad \text{for any } X \in B_r(X_0), \tag{2.32}$$

where $d(X) = dist(X, \Gamma_{\lambda,\mu})$ and the constant C depends only on ϑ, b and $\left(1-\frac{r}{R}\right)^{-1}$.

Denote $d_0 = R - r$. Suppose that $X_1 = (x_1, y_1) \in B_r(X_0) \cap \{\psi < m_0\}$ and $d(X_1) < d_0$. Thus $B_{d(X_1)}(X_1) \subset B_R(X_0) \cap \{\psi_{\lambda,\mu} < m_0\}$ (please see Figure 5). Next, we assume that

$$m_0 - \psi_{\lambda,\mu}(X_1) > Md(X_1)x_1, \tag{2.33}$$

and we will derive an upper bound of M in the following. Denote

$$\phi(X) = \frac{m_0 - \psi_{\lambda,\mu}(X_1 + dX)}{dx_1} \quad \text{with } d = d(X_1), \tag{2.34}$$

and one has

$$\nabla \cdot f_\eta\left(\frac{x_1 \nabla\phi}{x_1 + dx}; \lambda\right) = 0 \quad \text{in } B_1(0).$$

It follows from (2.33) that

$$\phi(0) > M.$$

It follows from Harnack's inequality (see Theorem 8.20 in [28]) that

$$\phi(X) \geqslant cM \quad \text{in } B_{\frac{3}{4}}(0) \text{ for some } c > 0, \tag{2.35}$$

where the constant c is independent of d and x_1. On another hand, there exists a $\tilde{X} = (\tilde{x}, \tilde{y}) \in \partial B_1(0) \cap \Gamma_{\lambda,\mu}$. Define a function Ψ, which satisfies that

$$\begin{cases} \nabla \cdot f_\eta\left(\frac{x_1 \nabla\Psi}{x_1 + dx}; \lambda\right) = 0 & \text{in } B_1(\tilde{X}), \\ \Psi = \phi & \text{outside } B_1(\tilde{X}). \end{cases}$$

Since $\nabla \cdot f_\eta\left(\frac{x_1 \nabla\phi}{x_1 + dx}; \lambda\right) \geqslant 0$ in $B_1(\tilde{X})$, the maximum principle gives that

$$\phi \leqslant \Psi \quad \text{in} \quad B_1(\tilde{X}). \tag{2.36}$$

Then we have

$$\begin{aligned}
0 \leqslant & \int_{B_1(\tilde{X})} (x_1 + dx)\left\{F\left(\frac{|x_1\nabla\Psi|^2}{(x_1+dx)^2}; \lambda\right) - F\left(\frac{|x_1\nabla\phi|^2}{(x_1+dx)^2}; \lambda\right)\right\} dxdy \\
& \quad - 2\lambda F_1(\lambda^2;\lambda) \int_{B_1(\tilde{X})} \nabla(\Psi - \phi)\cdot e dxdy + \Lambda^2 \int_{B_1(\tilde{X})} (x_1 + dx)(\chi_{\{\Psi>0\}} - \chi_{\{\phi>0\}}) dxdy \\
\leqslant & \int_{B_1(\tilde{X})} -\vartheta \frac{x_1^2 |\nabla(\Psi - \phi)|^2}{x_1 + dx} + f_\eta\left(\frac{x_1\nabla\Psi}{x_1+dx}; \lambda\right)\cdot\nabla(\Psi - \phi) + \Lambda^2 (x_1 + dx)\chi_{\{\phi=0\}} dxdy \\
= & -\vartheta \int_{B_1(\tilde{X})} \frac{x_1^2|\nabla(\Psi-\phi)|^2}{x_1+dx} dxdy + \Lambda^2 \int_{B_1(\tilde{X})} (x_1 + dx)\chi_{\{\phi=0\}} dxdy,
\end{aligned}$$

which implies that

$$\begin{aligned}\int_{B_1(\tilde{X})}|\nabla(\Psi-\phi)|^2dxdy &\leqslant C\int_{B_1(\tilde{X})}\frac{x_1|\nabla(\Psi-\phi)|^2}{x_1+dx}dxdy\\ &\leqslant \frac{C\Lambda^2}{x_1}\int_{B_1(\tilde{X})}(x_1+dx)\chi_{\{\phi=0\}}dxdy\\ &\leqslant C\Lambda^2\int_{B_1(\tilde{X})}\chi_{\{\phi=0\}}dxdy,\end{aligned}\tag{2.37}$$

where C is a constant depending only on ϑ and $\left(1-\frac{r}{R}\right)^{-1}$.

It follows from (2.35) and (2.36) that

$$\Psi(X)\geqslant\phi(X)\geqslant cM\quad\text{in}\quad B_{\frac{3}{4}}(0)\cap B_1(\tilde{X}).$$

Applying Harnack's inequality for Ψ in $B_1(\tilde{X})$, one has

$$\Psi(X)\geqslant C_0\quad\text{in}\quad B_{\frac{1}{2}}(\tilde{X}),\quad C_0=cM.\tag{2.38}$$

Define $\varphi(X)=C_0\left(e^{-\nu|X-\tilde{X}|^2}-e^{-\nu}\right)$, after a direct computation, we have

$$\nabla\cdot f_\eta\left(\frac{x_1\nabla\varphi}{x_1+dx};\lambda\right)=\frac{2\nu C_0x_1e^{-\nu|\xi|^2}}{x_1+dx}\left(f_{\eta_i\eta_j}(2\nu\xi_i\xi_j-\delta_{ij})+\frac{d(f_{\eta_1\eta_1}\xi_1+f_{\eta_1\eta_2}\xi_2)}{x_1+dx}\right)>0$$

in $B_1(\tilde{X})\setminus B_{\frac{1}{2}}(\tilde{X})$, provided that ν is large enough, where $f_{\eta_i\eta_j}=f_{\eta_i\eta_j}\left(\frac{x_1\nabla\varphi}{x_1+dx};\lambda\right)$ and $\xi=(\xi_1,\xi_2)=X-\tilde{X}$ for $i,j=1,2$.

It is easy to check that

$$\Psi\geqslant\varphi\ \text{ on }\ \partial\big(B_1(\tilde{X})\setminus B_{\frac{1}{2}}(\tilde{X})\big).$$

The maximum principle gives that

$$\Psi(X)\geqslant\varphi(X)=C_0\big(e^{-\nu|X-\tilde{X}|^2}-e^{-\nu}\big)\geqslant cC_0(1-|X-\tilde{X}|)\quad\text{in}\quad B_1(\tilde{X})\setminus B_{\frac{1}{2}}(\tilde{X}),$$

which together with (2.38) gives that

$$\Psi(X)\geqslant cM(1-|X-\tilde{X}|)\quad\text{in}\quad B_1(\tilde{X})\setminus B_{\frac{1}{2}}(\tilde{X}).\tag{2.39}$$

With the aid of (2.37) and (2.39), along the similar arguments in the proof of Lemma 3.2 in [1] and Lemma 2.2 in [5], one has

$$M^2\leqslant C\Lambda^2,$$

where the constant C depends only on ϑ, N, N_0 and $\left(1-\frac{r}{R}\right)^{-1}$. This implies that

$$m_0-\psi_{\lambda,\mu}(X_1)\leqslant C\Lambda d(X_1)x_1.\tag{2.40}$$

Take any point $X_2=(x_2,y_2)\in B_r(X_0)$ such that $d(X_2)>d_0$ and there exists a point $X_1\in B_{\frac{d_0}{2}}(X_2)$ with $d(X_1)<d_0$. By using Harnack's inequality for $m_0-\psi_{\lambda,\mu}$ in $B_{d_0}(X_2)$ and (2.40), one has

$$m_0-\psi_{\lambda,\mu}(X_2)\leqslant C\big(m_0-\psi_{\lambda,\mu}(X_1)\big)\leqslant C\Lambda d(X_1)x_1\leqslant C\Lambda d(X_2)x_2.$$

For any $X \in B_r(X_0)$, we can repeat this argument step by step, and after a finite steps k (depending only on ϑ, N, N_0 and $\left(1-\frac{r}{R}\right)^{-1}$), such that

$$m_0 - \psi_{\lambda,\mu}(X) \leqslant C\Lambda d(X)x.$$

Hence, we complete the proof of (2.32).

Step 2. In this step, we will complete the proof of this lemma. For any $X_1 \in B_r(X_0)$, denote $d_0 = R - r$ and $d(X) = dist(X, \Gamma_{\lambda,\mu})$, and we consider the following two cases.

Case 1. $d(X_1) < d_0$. Then it follows from (2.32) that

$$\nabla \cdot f_\eta\left(\frac{x_1 \nabla\phi}{x_1 + dx}; \lambda\right) = 0 \text{ and } 0 \leqslant \phi \leqslant \frac{C\Lambda(x_1 + dx)d(X_1 + dX)}{dx_1} \leqslant C\Lambda \text{ in } B_1(0),$$

where ϕ and d are defined in (2.34), the constant C depends only on ϑ, N, N_0 and $\left(1-\frac{r}{R}\right)^{-1}$. Applying the elliptic estimate for the quasilinear equation in [28], one has

$$|\nabla\phi(0)| \leqslant C,$$

which gives that

$$|\nabla\psi_{\lambda,\mu}(X_1)| = x_1|\nabla\phi(0)| \leqslant Cx_1.$$

Case 2. $d(X_1) \geqslant d_0$. Obviously, $B_{d_0}(X_1) \subset B_R(X_0) \cap \{\psi_{\lambda,\mu} < m_0\}$. Denote $\phi_0(X_1) = \frac{m_0 - \psi_{\lambda,\mu}(X_1 + d_0 X)}{d_0 x_1}$, it follows from (2.34) that

$$\nabla\cdot f_\eta\left(\frac{x_1\nabla\phi_0}{x_1 + d_0 x}; \lambda\right) = 0 \text{ and } 0 \leqslant \phi_0 \leqslant \frac{C\Lambda(x_1 + d_0 x)d(X_1 + d_0 X)}{d_0 x_1} \leqslant C\Lambda \text{ in } B_1(0).$$

By using the elliptic estimate for ϕ_0 in $B_1(0)$, one has

$$|\nabla\phi_0(0)| \leqslant C \text{ and } |\nabla\psi_{\lambda,\mu}(X_1)| = x_1|\nabla\phi_0(0)| \leqslant Cx_1. \qquad \square$$

With the aid of Lemma 2.3, applying the similar arguments in the proof of Lemma 2.4 in [5], we can obtain the following lemma.

Lemma 2.4 *There exists a positive constant* C^**, such that for any disc* $B_r(X_0) \subset\subset \Omega_\mu$ *with* $X_0 = (x_0, y_0)$*,* $r \leqslant \frac{x_0}{2}$*, then*

$$\frac{1}{r}\fint_{\partial B_r(X_0)} (m_0 - \psi_{\lambda,\mu})dS \geqslant C^*\Lambda x_0,$$

implies that

$$\psi_{\lambda,\mu} < m_0 \quad in \;\; B_r(X_0).$$

We next establish a non-degeneracy lemma.

Lemma 2.5 *There is a universal constant* $c^* > 0$ *such that for any disc* $B_r(X_0)$ *with center* $X_0 = (x_0, y_0) \in E$ *and* $r \leqslant \frac{x_0}{2}$*, then*

$$\frac{1}{r}\left(\fint_{B_r(X_0)} (m_0 - \psi_{\lambda,\mu})^2 dxdy\right)^{\frac{1}{2}} \leqslant c^*\Lambda x_0, \tag{2.41}$$

implies that

$$\psi_{\lambda,\mu} = m_0 \quad in \;\; B_{\frac{r}{8}}(X_0) \cap E.$$

Proof. It is easy to check that the set

$$B_{\frac{r}{8}}(X_0)\cap E$$

can be covered by discs of the form

$$B_{r_1}(X_1)\subset B_{\frac{r}{4}}(X_0)\cap E\ \text{ with }\ r_1=\frac{r}{16}.$$

Thus, it suffices to show that $\psi_{\lambda,\mu}=m_0$ in any discs $B_{r_1}(X_1)\subset B_{\frac{r}{4}}(X_0)\cap E$, provided that the assumption (2.41) holds. Let ψ_0 solves the following boundary value problem

$$\begin{cases}\nabla\cdot f_\eta\left(\frac{\nabla\psi_0}{x};\lambda\right)=0 & \text{in } B_{2r_1}(X_1)\setminus \overline{B_{r_1}(X_1)},\\ \psi_0=m_0 \text{ in } \overline{B_{r_1}(X_1)},\ \ \psi_0=\psi_{\lambda,\mu} & \text{outside of } B_{2r_1}(X_1).\end{cases}\tag{2.42}$$

Obviously, $\max\{\psi_0,\psi_{\lambda,\mu}\}\in K_\mu$, and thus

$$\begin{aligned}0&\leqslant J_{\lambda,\mu}\big(\max\{\psi_{\lambda,\mu},\psi_0\}\big)-J_{\lambda,\mu}(\psi_{\lambda,\mu})\\ &=\int_{B_{2r_1}(X_1)}xF\left(\left|\frac{\nabla\max\{\psi_{\lambda,\mu},\psi_0\}}{x}\right|^2;\lambda\right)-xF\left(\left|\frac{\nabla\psi_{\lambda,\mu}}{x}\right|^2;\lambda\right)dxdy\\ &\qquad -2F_1(\lambda^2;\lambda)\lambda\int_{B_{2r_1}(X_1)}\nabla\max\{\psi_0-\psi_{\lambda,\mu},0\}\cdot e dxdy\\ &\qquad +\Lambda^2\int_{B_{2r_1}(X_1)}x\chi_{\{\max\{\psi_{\lambda,\mu},\psi_0\}<m_0\}}-x\chi_{\{\psi_{\lambda,\mu}<m_0\}}dxdy\\ &=I_1+I_2+I_3.\end{aligned}\tag{2.43}$$

For the first term on the right hand side of (2.43), one has

$$\begin{aligned}I_1&\leqslant\int_{B_{2r_1}(X_1)\setminus B_{r_1}(X_1)}\nabla\max\{\psi_0-\psi_{\lambda,\mu},0\}\cdot f_\eta\left(\frac{\nabla\max\{\psi_{\lambda,\mu},\psi_0\}}{x};\lambda\right)dxdy\\ &\qquad-\int_{B_{r_1}(X_1)}xF\left(\left|\frac{\nabla\psi_{\lambda,\mu}}{x}\right|^2;\lambda\right)dxdy\\ &\leqslant-\int_{B_{r_1}(X_1)}xF\left(\left|\frac{\nabla\psi_{\lambda,\mu}}{x}\right|^2;\lambda\right)dxdy\\ &\qquad+2\int_{\partial B_{r_1}(X_1)}(m_0-\psi_{\lambda,\mu})F_1\left(\left|\frac{\nabla\psi_0}{x}\right|^2;\lambda\right)\frac{\nabla\psi_0\cdot\nu}{x}dS.\end{aligned}\tag{2.44}$$

It is easy to check that $I_2=0$ and

$$I_3=-\Lambda^2\int_{B_{2r_1}(X_1)}x\chi_{\{\psi_{\lambda,\mu}<\psi_0=m_0\}}dxdy\leqslant-\Lambda^2\int_{B_{r_1}(X_1)}x\chi_{\{\psi_{\lambda,\mu}<m_0\}}dxdy,$$

which together with (2.43) and (2.44) gives that

$$\int_{B_{r_1}(X_1)} xF\left(\left|\frac{\nabla\psi_{\lambda,\mu}}{x}\right|^2;\lambda\right)dxdy+\Lambda^2\int_{B_{r_1}(X_1)} x\chi_{\{\psi_{\lambda,\mu}<m_0\}}dxdy$$
$$\leqslant 2\int_{\partial B_{r_1}(X_1)}(m_0-\psi_{\lambda,\mu})F_1\left(\left|\frac{\nabla\psi_0}{x}\right|^2;\lambda\right)\frac{\nabla\psi_0\cdot\nu}{x}dS. \quad (2.45)$$

Set $\tilde{\psi}(X)=\frac{m_0-\psi_{\lambda,\mu}(X_1+r_1X)}{r_1x_0}$ and $\tilde{\psi}_0(X)=\frac{m_0-\psi_0(X_1+r_1X)}{r_1x_0}$, one has

$$\nabla\cdot f_\eta\left(\frac{x_0\nabla\tilde{\psi}}{x_1+r_1x};\lambda\right)\geqslant 0\ \text{ in } B_{16}\left(\frac{X_0-X_1}{r_1}\right).$$

Moreover, it follows from the assumption (2.41) that

$$\left(\fint_{B_{16}\left(\frac{X_0-X_1}{r_1}\right)}\tilde{\psi}^2dxdy\right)^{\frac{1}{2}}\leqslant\delta\Lambda,$$

where δ is to be chosen later on. By using the L^∞ estimate in Theorem 8.17 in [28], one has

$$\sup_{X\in B_8\left(\frac{X_0-X_1}{r_1}\right)}\tilde{\psi}(X)\leqslant C\left(\fint_{B_{16}\left(\frac{X_0-X_1}{r_1}\right)}\tilde{\psi}^2dxdy\right)^{\frac{1}{2}}\leqslant C\delta\Lambda, \quad (2.46)$$

where C is a constant depending only on ϑ and a. Since $B_2(0)\subset B_8\left(\frac{X_0-X_1}{r_1}\right)$, it follows from (2.46) that

$$0\leqslant\tilde{\psi}_0=\tilde{\psi}\leqslant C\Lambda\delta\ \text{ on }\ \partial B_2(0)\quad\text{and}\quad\tilde{\psi}_0=0\ \text{ on }\ \partial B_1(0). \quad (2.47)$$

It is easy to check that

$$\nabla\cdot f_\eta\left(\frac{x_0\nabla\tilde{\psi}_0}{x_1+r_1x};\lambda\right)=0\quad\text{in}\quad B_2(0)\setminus B_1(0).$$

Applying the boundary elliptic estimate in Lemma 6.10 in [28], one has

$$|\nabla\tilde{\psi}_0\cdot\nu|\leqslant C\delta\Lambda\quad\text{on}\quad\partial B_1.$$

In view of (2.45), (2.47) and the trace theorem, one has

$$
\begin{aligned}
&\int_{B_1(0)} (x_1+r_1x)F\left(\frac{x_0^2|\nabla\tilde{\psi}|^2}{(x_1+r_1x)^2};\lambda\right)+\Lambda^2(x_1+r_1x)\chi_{\{\tilde{\psi}>0\}}dxdy\\
&\quad\leqslant 2x_0\int_{\partial B_1(0)}\tilde{\psi}F_1\left(\frac{x_0^2|\nabla\tilde{\psi}_0|^2}{|x_1+r_1x|^2};\lambda\right)\frac{x_0|\nabla\psi_0\cdot\nu|}{x_1+r_1x}dS\\
&\quad\leqslant Cx_0\delta\Lambda\int_{\partial B_1(0)}\tilde{\psi}dS\\
&\quad\leqslant Cx_0\delta\Lambda\left(\int_{B_1(0)}\tilde{\psi}dxdy+\int_{B_1(0)}|\nabla\tilde{\psi}|dxdy\right)\\
&\quad\leqslant Cx_0\delta\Lambda\left\{\left(C\delta\Lambda+\frac{1}{4\varepsilon}\right)\int_{B_1(0)}\chi_{\{\tilde{\psi}>0\}}dxdy+\varepsilon\int_{B_1(0)}|\nabla\tilde{\psi}|^2dxdy\right\},
\end{aligned}
\tag{2.48}
$$

where we have used the fact

$$
|\nabla\tilde{\psi}|\leqslant\varepsilon|\nabla\tilde{\psi}|^2+\frac{1}{4\varepsilon}\ \text{ a.e. in } B_1\cap\{\tilde{\psi}>0\}.
$$

On the other hand, we have

$$
\begin{aligned}
&x_0\int_{B_1(0)}|\nabla\tilde{\psi}|^2dxdy+x_0\int_{B_1(0)}\Lambda^2\chi_{\{\tilde{\psi}>0\}}dxdy\\
&\quad\leqslant C\int_{B_1(0)}(x_1+r_1x)F\left(\frac{x_0^2|\nabla\tilde{\psi}|^2}{(x_1+r_1x)^2};\lambda\right)+\Lambda^2(x_1+r_1x)\chi_{\{\tilde{\psi}>0\}}dxdy\\
&\quad\leqslant Cx_0\delta\Lambda\left\{\left(C\delta\Lambda-\frac{1}{4\varepsilon}\right)\int_{B_1(0)}\chi_{\{\tilde{\psi}>0\}}dxdy+\varepsilon\int_{B_1(0)}|\nabla\tilde{\psi}|^2dxdy\right\}.
\end{aligned}
\tag{2.49}
$$

Taking $\varepsilon=\frac{1}{C\delta\Lambda}$, it follows from (2.48) and (2.49) that

$$
\left(1-C\delta^2\right)x_0\Lambda^2\int_{B_1(0)}\chi_{\{\tilde{\psi}>0\}}dxdy\leqslant 0,
$$

which implies that

$$
\tilde{\psi}=0\ \text{ in }\ B_1(0),
$$

provided that $\delta<\sqrt{\frac{1}{C}}$, where the constant C depends on ϑ and b. The proof is completed. □

Theorem 2.6 *The minimizer $\psi_{\lambda,\mu}$ is Lipschitz continuous in any compact subset of $\bar{\Omega}_\mu$ that does not contain A or the points where $\partial\Omega_\mu$ is not $C^{1,\alpha}$.*

Proof. Denote $\psi=\psi_{\lambda,\mu}$ and $\Gamma=\Gamma_{\lambda,\mu}$ for simplicity. The Lipschitz continuity of ψ in any compact subset of Ω_μ follows from the proof of Lemma 2.3. On another hand, the Lipschitz continuity of ψ near $L_\mu\cup N_\mu\cup\left(N_{0,\mu}\cap\left\{x\leqslant\frac{b}{2}\right\}\right)$ can be obtained by using the elliptic estimate. Along the similar arguments in the proof of Lemma 2.2, we can obtain the Lipschitz continuity of ψ near the symmetric axis I.

We next consider the Lipschitz continuity of ψ near T or near the wall $N_0\cap\left\{x\geqslant\frac{b}{2}\right\}$.

For $X=(x,y)\in\Omega_\mu$ with $y-1>\delta$, denote $X_0=(b,y)$, $d(X)=dist(X,\Gamma_{\lambda,\mu})$ and $d_1(X)=dist(X,T)$. If $d(X)\leqslant d_1(X)$, by using the similar arguments in the proof Lemma 2.3, we have $|\nabla\psi(X)|\leqslant C$, where the constant C depends on Λ and ϑ.

For the case $d(X)>d_1(X)=x-b$, set $r_0=\min\{\frac{b}{2},y-1\}$ and $B_{r_0}=B_{r_0}(X_0)$.

Consider a function ϕ, which solves the following boundary value problem

$$\begin{cases}\nabla\cdot f_\eta\left(\frac{\nabla\phi}{x};\lambda\right)=0 & \text{in } B_{r_0}\cap\{x>b\},\\ \phi=0 & \text{on } B_{r_0}\cap\{x=b\},\ \ \phi=m_0-\psi \ \text{ on } \partial B_{r_0}\cap\{x>b\}.\end{cases}$$

The maximum principle gives that

$$m_0-\psi\leqslant\phi \text{ in } B_{r_0}\cap\{x>b\}. \tag{2.50}$$

Set $\tilde\phi(\tilde X)=\frac{\phi(X_0+r_0\tilde X)}{r_0}$ with $\tilde X=(\tilde x,\tilde y)$. Noting $0\leqslant\phi\leqslant m_0$, one has

$$\begin{cases}\nabla\cdot f_\eta\left(\frac{\nabla\tilde\phi}{b+r_0\tilde x};\lambda\right)=0 & \text{in } B_1(0)\cap\{\tilde x>0\},\\ 0\leqslant\tilde\phi\leqslant\frac{m_0}{r_0} & \text{on } \partial\big(B_1(0)\cap\{\tilde x>0\}\big).\end{cases}$$

Applying the elliptic estimates for $\tilde\phi$ in $B_1(0)\cap\{\tilde x>0\}$, one has

$$\tilde\phi(\tilde X)\leqslant C\frac{\tilde x}{r_0}\quad \text{in } B_{\frac12}(0)\cap\{\tilde x>0\},$$

which gives that

$$\phi(X)=\tilde\phi\left(\frac{X-X_0}{r_0}\right)\leqslant C\frac{x-b}{r_0}\quad\text{in}\quad B_{\frac{r_0}{2}}(X_0)\cap\{x>b\}. \tag{2.51}$$

If $r=d_1(x)=x-b<\frac{r_0}{4}$, we have $B_r(X)\subset B_{\frac{r_0}{2}}(X_0)\cap\{x>b\}$. Set $\tilde\psi(\tilde X)=\frac{m_0-\psi(X+r\tilde X)}{r}$ with $\tilde X=(\tilde x,\tilde y)$, it follows from (2.50) and (2.51) that

$$\tilde\psi(\tilde X)\leqslant\frac{\phi(X+r\tilde X)}{r}\leqslant\frac{C(x+r\tilde x-b)}{rr_0}\leqslant\frac{C}{r_0}\quad\text{in}\quad B_1(0).$$

By using the elliptic estimate, one has

$$|\nabla\psi(X)|=|\nabla\tilde\psi(0)|\leqslant\frac{C}{r_0}.$$

If $r=d_1(x)=x-b\geqslant\frac{r_0}{4}$, the elliptic estimate gives the desired uniform bound for $\nabla\psi(X)$.

Finally, we consider the Lipschitz continuity of ψ near $N_0\cap\{x\geqslant b\}$. Since N_0 is $C^{2,\alpha}$ and $\psi_{\lambda,\mu}=0$ on N_0, the Harnack's inequality is still valid up to the boundary $N_0\cap\{x\geqslant\frac{b}{2}\}$. It follows from the similar arguments in the proof of Lemma 2.3 that

$$\frac{|\nabla\psi(X)|}{x}\leqslant C\Lambda\quad\text{near}\quad N_0\cap\left\{x\geqslant\frac{b}{2}\right\}.$$

□

3. The free boundary of the minimizer $\psi_{\lambda,\mu}$

In this section, we will show some important properties of the free boundary, such as the continuity of the graph and the continuous fit condition.

3.1 *Uniqueness and monotonicity of the minimizer*

To obtain the continuous fit condition of the free boundary, we construct the uniqueness and the monotonicity of the minimizer to the truncated variational problem $(P_{\lambda,\mu})$.

Lemma 3.1 *For any $\lambda \leqslant \Pi_\lambda - 3\tilde{\varepsilon}$ and $\mu > 1$, the minimizer $\psi_{\lambda,\mu}$ to the truncated variational problem $(P_{\lambda,\mu})$ is unique, and $\psi_{\lambda,\mu}(x, y_1) \geqslant \psi_{\lambda,\mu}(x, y_2)$ for any $y_1 > y_2$.*

Proof. Suppose that ψ_1 and ψ_2 are two minimizers to the truncated variational problem $(P_{\lambda,\mu})$. Set

$$\psi_1^\varepsilon(x, y) = \psi_1(x, y - \varepsilon) \quad \text{for any} \quad \varepsilon > 0.$$

Notice that $\psi_1^\varepsilon(x, y)$ is a minimizer of the functional $J_{\lambda,\mu}^\varepsilon$ in Ω_μ^ε with the corresponding admissible set K_μ^ε as follows

$$\Omega_\mu^\varepsilon = \{(x, y) | \, (x, y - \varepsilon) \in \Omega_\mu\} \quad \text{and} \quad K_\mu^\varepsilon = \{\psi^\varepsilon(x, y - \varepsilon) \in K_\mu | \, (x, y) \in \Omega_\mu^\varepsilon\}.$$

Extend $\psi_2(x, y) = \frac{m_0}{x_\mu^2} x^2$ in $\{(x, y) \mid 0 < x \leqslant x_\mu, \mu < y \leqslant \mu + \varepsilon\}$ and denote

$$\varphi_1 = \min\{\psi_1^\varepsilon, \psi_2\} \quad \text{and} \quad \varphi_2 = \max\{\psi_1^\varepsilon, \psi_2\}.$$

Obviously, $\varphi_1 \in K_\mu^\varepsilon$ and $\varphi_2 \in K_\mu$. For any sufficiently large $R > R_0$, denote $\Omega_{\mu,R} = \Omega_\mu \cap \{y < R\}$ and $\Omega_{\mu,R}^\varepsilon = \Omega_\mu^\varepsilon \cap \{y < R\}$. Since $\varphi_1 = \psi_1^\varepsilon$ in $\Omega_{\mu,R} \setminus \Omega_{\mu,R}^\varepsilon$ and $\varphi_1 = \psi_1^\varepsilon$ in $\Omega_{\mu,R}^\varepsilon \setminus \Omega_{\mu,R}$, it is easy to check that

$$\begin{aligned}
&\int_{\Omega_{\mu,R}^\varepsilon} xF\left(\frac{|\nabla\varphi_1|^2}{x^2}; \lambda\right) dxdy + \int_{\Omega_{\mu,R}} xF\left(\frac{|\nabla\varphi_2|^2}{x^2}; \lambda\right) dxdy \\
&= \int_{\Omega_{\mu,R}^\varepsilon} xF\left(\frac{|\nabla\psi_1^\varepsilon|^2}{x^2}; \lambda\right) dxdy + \int_{\Omega_{\mu,R}} xF\left(\frac{|\nabla\psi_2|^2}{x^2}; \lambda\right) dxdy,
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
&\int_{E_R^\varepsilon} x\chi_{\{\varphi_1 < m_0\}} dxdy + \int_{E_R} x\chi_{\{\varphi_2 < m_0\}} dxdy \\
&= \int_{E_R^\varepsilon} x\chi_{\{\psi_1^\varepsilon < m_0\}} dxdy + \int_{E_R} x\chi_{\{\psi_2 < m_0\}} dxdy,
\end{aligned} \tag{3.2}$$

where $E_R = \Omega_{\mu,R} \cap \{x > b\}$ and $E_R^\varepsilon = \Omega_{\mu,R}^\varepsilon \cap \{x > b\}$.

Integration by parts, one has

$$\begin{aligned}&\int_{E_R^\varepsilon} \nabla\varphi_1\cdot e\chi_{\{\varphi_1<m_0\}}dxdy+\int_{E_R}\nabla\varphi_2\cdot e\chi_{\{\varphi_2<m_0\}}dxdy\\&\qquad-\int_{E_R^\varepsilon}\nabla\psi_1^\varepsilon\cdot e\chi_{\{\varphi_1<m_0\}}dxdy+\int_{E_R}\nabla\psi_2\cdot e\chi_{\{\psi_2<m_0\}}dxdy\\&=\int_{\partial E_R^\varepsilon}(\varphi_1-\psi_1^\varepsilon)e\cdot\nu dS+\int_{E_R}(\varphi_2-\psi_2)e\cdot\nu dS\\&=\int_{\partial E_R^\varepsilon\cap\partial E_R}(\varphi_1+\varphi_2-\psi_1^\varepsilon-\psi_2)e\cdot\nu dS\\&=0,\end{aligned}\tag{3.3}$$

where we have used the facts $\varphi_2=\psi_2=0$ in $E_R\setminus E_R^\varepsilon$.

In view of (3.1)–(3.3), one has

$$\begin{aligned}&\int_{\Omega_{\mu,R}^\varepsilon}G(\nabla\varphi_1,\varphi_1,x;\lambda)dxdy+\int_{\Omega_{\mu,R}}G(\nabla\varphi_1,\varphi_1,x;\lambda)dxdy\\&=\int_{\Omega_{\mu,R}^\varepsilon}G(\nabla\psi_1^\varepsilon,\psi_1^\varepsilon,x;\lambda)dxdy+\int_{\Omega_{\mu,R}}G(\nabla\psi_2,\psi_2,x;\lambda)dxdy.\end{aligned}\tag{3.4}$$

Taking $R\to+\infty$ in (3.4) yields that

$$J_{\lambda,\mu}^\varepsilon(\psi_1^\varepsilon)+J_{\lambda,\mu}(\varphi_2)=J_{\lambda,\mu}^\varepsilon(\varphi_1)+J_{\lambda,\mu}(\psi_2).\tag{3.5}$$

Since ψ_1^ε and ψ_2 are minimizers, it follows from (3.5) that

$$J_{\lambda,\mu}^\varepsilon(\psi_1^\varepsilon)=J_{\lambda,\mu}^\varepsilon(\varphi_1)\quad\text{and}\quad J_{\lambda,\mu}(\psi_2)=J_{\lambda,\mu}(\varphi_2).\tag{3.6}$$

Next, we claim that

$$\psi_1^\varepsilon(x,y)<\psi_2(x,y)\quad\text{in } D,\tag{3.7}$$

where D is the maximal connected component of $\Omega_\mu\cap\{\psi_2<m_0\}$, which contains an Ω_μ-neighborhood of N_μ.

Suppose not, note that $\psi_1^\varepsilon<m_0=\psi_2$ on N_μ, then there exists a dist B_1, such that

$$\psi_1^\varepsilon<\psi_2\quad\text{in}\quad B_1,\qquad \bar B_1\subset\Omega_\mu\cap\{\psi_2<m_0\},$$

and

$$\psi_1^\varepsilon=\psi_2\quad\text{at some points}\quad X_0\in\partial B_1\cap\big(\Omega_\mu\cap\{\psi_2<m_0\}\big).$$

Thanks to Hopf's lemma, one has

$$\frac{\partial(\psi_1^\varepsilon-\psi_2)}{\partial\nu}>0\quad\text{at }X_0,$$

where ν is the outer normal vector of ∂B_1 at X_0. This implies that the level set $\{\psi_1^\varepsilon=\psi_2=\psi_2(X_0)\}$ is smooth curve in a neighborhood of X_0. Then there exists a smooth curve $\Gamma_0=\{X\mid\psi_1^\varepsilon(X)=\psi_2(X)=\psi_2(X_0)\}$ passing through X_0 and a disc B_2, such that

$$\psi_1^\varepsilon>\psi_2\quad\text{in }B_2\quad\text{and}\quad X_0\in\Gamma_0\cap\partial B_2\cap\partial B_1.$$

Hence, one has

$$\frac{\partial(\varphi_1-\psi_2)(X)}{\partial\nu}=\frac{\partial(\psi_1^\varepsilon-\psi_2)(X)}{\partial\nu}\to\frac{\partial(\psi_1^\varepsilon-\psi_2)(X_0)}{\partial\nu}>0\quad\text{as }\ X\to X_0,\,X\in B_1,$$

and

$$\frac{\partial(\varphi_1-\psi_2)(X)}{\partial\nu}=\frac{\partial(\psi_2-\psi_2)(X)}{\partial\nu}=0,\quad X\in B_2,$$

which implies that φ_1 is not C^1-smooth in a neighborhood of X_0, due to that ψ_2 is smooth at X_0. On the other hand, it follows from (3.6) that φ_1 is a minimizer, and $\varphi_1(X_0)<m_0$. By virtue of the elliptic regularity, we can conclude that φ_1 is smooth in a neighborhood of X_0. This leads a contradiction. Hence, we complete the proof of the claim (3.7).

We next show that

$$\psi_1(x,y)\text{ is monotone increasing with respect to }y\text{ in }\Omega_\mu. \tag{3.8}$$

Choosing $\psi_1=\psi_2$ in (3.7), it implies that

$$\frac{\partial\psi_1}{\partial y}\geqslant 0\quad\text{in }\ D. \tag{3.9}$$

To obtain (3.8), it suffices to show that

$$D=\Omega_\mu\cap\{\psi_1<m_0\}.$$

Suppose not, it follows from (3.7) that $D\cap\{x>b\}=\{x>b,y<\phi(x)\}$. As a part of the free boundary of the graph ϕ, we can conclude that $\phi(x)$ is continuous (see the proof of Lemma 3.3 later). Define $\psi_0=\psi_1$ in D and $\psi_0=m_0$ in $\Omega_\mu\cap\{x>b,y\geqslant\phi(x)\}$, it is easy to check that $\psi\in K_\mu$. Therefore, we have

$$J_{\lambda,\mu}(\psi_0)-J_{\lambda,\mu}(\psi_1)=-\int_{\Omega_\mu\backslash D}G(\nabla\psi_1,\psi_1,x;\lambda)dxdy<0,$$

which leads a contradiction.

Similarly, we can obtain that

$$\psi_2(x,y)\text{ is monotone increasing with respect to }y\text{ in }\Omega_\mu,$$

which together with (3.8) gives that

$$\Omega_\mu\cap\{\psi_1<m_0\}\text{ and }\Omega_\mu\cap\{\psi_2<m_0\}\text{ are connected.}$$

In view of (3.7), one has

$$\psi_1^\varepsilon(x,y)\leqslant\psi_2(x,y)\quad\text{in }\ \Omega_\mu.$$

Taking $\varepsilon\to 0$ in above inequality, we have

$$\psi_1\leqslant\psi_2\quad\text{in }\ \Omega_\mu.$$

Similarly, we can obtain that

$$\psi_1\geqslant\psi_2\quad\text{in }\Omega_\mu.$$

Hence, $\psi_1=\psi_2$ and the minimizer to the variational problem $(P_{\lambda,\mu})$ is unique. □

3.2 *Fundamental properties of the free boundary*

In this section, we show some significant properties of the free boundary $\Gamma_{\lambda,\mu}$ to the truncated variational problem $(P_{\lambda,\mu})$. Thanks to the monotonicity of the minimizer $\psi_{\lambda,\mu}(x,y)$ with respect to y, there exists a mapping $y = k_{\lambda,\mu}(x)$, such that

$$E \cap \{\psi_{\lambda,\mu} < m_0\} = \{(x,y) \mid b < x < \infty, g_0(x) < y < k_{\lambda,\mu}(x)\}.$$

To obtain the continuity of the function $k_{\lambda,\mu}(x)$, we need the following non-oscillation lemma and the proof can be found in Lemma 4.4 in [6].

Lemma 3.2 *Let G be a domain in $E \cap \{\psi_{\lambda,\mu} < m_0\}$, bounded by two disjointed arcs γ_1, γ_2 of the free boundary, $y = \beta_1$, $y = \beta_2$. Suppose that the arcs γ_i ($i = 1, 2$) lie in $\{\beta_1 < y < \beta_2\}$ with the endpoints (α_i, β_1) and (ζ_i, β_2). Suppose the distant $d = dist(G, B) > 0$, then*

$$|\beta_2 - \beta_1| \leqslant C \max\{|\alpha_1 - \alpha_2|, \ |\zeta_1 - \zeta_2|\},$$

where C is a constant depending only on Λ, ϑ, d, N, N_0 and m_0.

REMARK The nonoscillation Lemma 3.2 remains true provided that one of the arcs γ_2 is a line segment on $T = \{(b, y) \mid y \geqslant 1\}$, and

$$\frac{1}{x}\frac{\partial \psi_{\lambda,\mu}}{\partial \nu} \geqslant \lambda \quad \text{on } \gamma_2.$$

Lemma 3.3 *The function $y = k_{\lambda,\mu}(x)$ is continuous for $x \in (b, +\infty)$. Moreover, $k_{\lambda,\mu}(b) = \lim_{x\to b^+} k_{\lambda,\mu}(x)$ exists and is finite.*

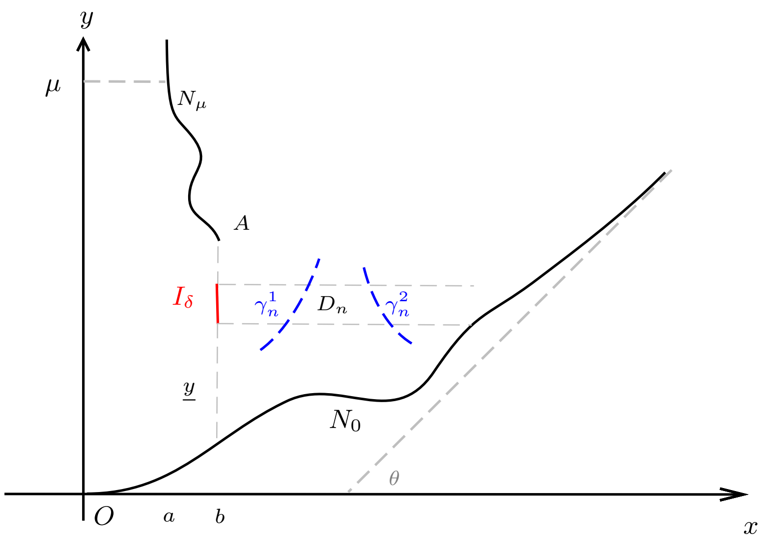

FIG. 6. The domain D_n

Proof. We first consider the existence of the limit $\lim_{x\to b^+} k_{\lambda,\mu}(x)$.

Step 1. $\lim_{x\to b^+} k_{\lambda,\mu}(x) = \lim_{x\to b^+} k_{\lambda,\mu}(x)$.

Suppose not, one has that $\liminf_{x\to b^+} k_{\lambda,\mu}(x) < \limsup_{x\to b^+} k_{\lambda,\mu}(x)$, then we consider the following two cases for $\underline{y} = \lim_{x\to b^+} k_{\lambda,\mu}(x)$.

Case 1. $\underline{y} < 1$. Denote $\delta = \frac{1-\underline{y}}{4}$ and $I_\delta = \left\{(1,y) \mid 1-\frac{5}{2}\delta \leqslant y \leqslant 1-\frac{3}{2}\delta\right\}$. Then there exist two sequences $\{x_n\}$ and $\{\tilde{x}_n\}$ with $x_n \downarrow b$ and $\tilde{x}_n \downarrow b$, such that

$$k_{\lambda,\mu}(x_n) \to 1-\delta \quad \text{and} \quad k_{\lambda,\mu}(x_n) \to 1-3\delta, \ \ \delta > 0, \tag{3.10}$$

and

$$\psi_{\lambda,\mu}(x_n, y) = m_0, \quad \psi_{\lambda,\mu}(\tilde{x}_n, y) < m_0 \ \text{ for } |y-1+2\delta| \leqslant \frac{\delta}{2}, \quad x_{n+1} < \tilde{x}_n < x_n. \tag{3.11}$$

By virtue of Lemma 2.3 and the monotonicity of $\psi_{\lambda,\mu}$, we have that $\psi_{\lambda,\mu}$ is Lipschitz continuous in an $\bar{E}$-neighborhood of I_δ and $\psi_{\lambda,\mu} = m_0$ on I_δ.

It follows from (3.11) that there exists a domain $D_n \subset E \cap \{\psi_{\lambda,\mu} < m_0\}$ (please see Figure 6), which is bounded by the arcs $y_1 = 1-\frac{3}{2}\delta$, $y_2 = 1-\frac{5}{2}\delta$, γ_n^1 and γ_n^2. Here, γ_n^1 and γ_n^2 are parts of free boundary $\Gamma_{\lambda,\mu} \cap \{x \leqslant x_{n-1}\}$, and the curve γ_n^1 lies the right of the curve γ_n^2. Denote $h_n = \operatorname{dist}(\gamma_n^1, \gamma_n^2)$. By virtue of that $x_n \to b^+$, one has

$$h_n \to 0 \ \text{ as } \ n \to +\infty. \tag{3.12}$$

Thanks to the non-oscillation Lemma 3.2, we have

$$0 < \delta \leqslant Ch_n,$$

which contradicts to (3.12), provided that n is sufficiently large.

Case 2. $\underline{y} \geqslant 1$. Take a constant $\delta > 0$, such that $\delta \leqslant \frac{\limsup_{x\to b^+} k_{\lambda,\mu}(x) - \underline{y}}{4}$. Denote $\gamma_\delta = \{(b,y) \mid \underline{y}+\frac{3}{2}\delta \leqslant y \leqslant \underline{y}+\frac{5}{2}\delta\}$. In an E-neighborhood of γ_δ, we can obtain a contradiction by using the non-oscillation Lemma 3.2.

Similarly, we can show that the limits $\lim_{x\to x_0^+} k_{\lambda,\mu}(x)$ and $\lim_{x\to x_0^-} k_{\lambda,\mu}(x)$ exist for any $x_0 \in (b,+\infty)$.

Step 2. $\lim_{x\to x_0^+} k_{\lambda,\mu}(x) = \lim_{x\to x_0^-} k_{\lambda,\mu}(x)$ for any $x_0 \in (b,+\infty)$.

Suppose that there exists a $x_0 \in (b,+\infty)$, such that $\lim_{x\to x_0^+} k_{\lambda,\mu}(x) \neq \lim_{x\to x_0^-} k_{\lambda,\mu}(x)$. Without loss of generality, we assume that $\lim_{x\to x_0^+} k_{\lambda,\mu}(x) > \lim_{x\to x_0^-} k_{\lambda,\mu}(x)$. Denote $\gamma = \{(x_0,y) \mid y_1 \leqslant y \leqslant y_2\}$ with $\lim_{x\to x_0^-} k_{\lambda,\mu}(x) < y_1 < y_2 < \lim_{x\to x_0^-} k_{\lambda,\mu}(x)$. The monotonicity and Lipschitz continuity of $\psi_{\lambda,\mu}$ give that γ is the free boundary of $\psi_{\lambda,\mu}$ and $\psi_{\lambda,\mu} < m_0$ in $E_\varepsilon = \{(x,y) \mid x_0 < x < x_0+\varepsilon, y_1 < y < y_2\}$ for small $\varepsilon > 0$. Then we have

$$\tilde{Q}_\lambda \psi_{\lambda,\mu} = 0 \ \text{ in } \ E_\varepsilon, \quad \psi_{\lambda,\mu} = m_0 \quad \text{and} \quad \frac{1}{x}\frac{\psi_{\lambda,\mu}}{\partial x} = -\lambda \ \text{ on } \ \gamma.$$

Since $\psi_{\lambda,\mu}$ is analytic in E_ε for small $\varepsilon > 0$, thanks to Cauchy–Kovalevskaya theorem, one has

$$\psi_{\lambda,\mu}(x,y) = -\lambda(x^2 - x_0^2) + m_0 \ \text{ in } \ \Omega_\mu \cap \{x_0 < x < x_0+\varepsilon\},$$

which contradicts to $\psi_{\lambda,\mu} = 0$ on N_0.

Step 3. $k_{\lambda,\mu}(x) < +\infty$ for any $x \in [b, +\infty)$. We first show that

$$\text{the free boundary } \Gamma_{\lambda,\mu} \text{ is non-empty in } E. \tag{3.13}$$

Suppose that $\Gamma_{\lambda,\mu}$ is empty, it implies that

$$\psi_{\lambda,\mu} < m_0 \ \text{ in } \ E. \tag{3.14}$$

For any $R > 0$, there exists a disc $B_R(X_0) \subset E$ with $X_0 = (x_0, y_0)$, such that

$$\frac{1}{R}\left(\fint_{B_R(X_0)} (m_0 - \psi_{\lambda,\mu})^2 dxdy\right)^{\frac{1}{2}} \leqslant \frac{m_0}{R} \leqslant c^* \Lambda x_0,$$

for sufficiently large R. It follows from non-degeneracy Lemma 2.5 that $\psi_{\lambda,\mu} = m_0$ in $B_{\frac{R}{8}}(X_0)$. This contradicts to (3.14). With the aid of (3.13), we can take a maximal interval $(\gamma_1, \gamma_2) \subset (b, +\infty)$, such that

$$k_{\lambda,\mu}(x) \ \text{ is finite in } (\gamma_1, \gamma_2) \text{ and } k_{\lambda,\mu}(\gamma_1 + 0) = k_{\lambda,\mu}(\gamma_2 - 0) = +\infty.$$

We first show that

$$\gamma_2 = +\infty. \tag{3.15}$$

If not, then $\gamma_2 < +\infty$. By using the proof of (3.13), we can conclude that $\Gamma_{\lambda,\mu}$ is non-empty in $E \cap \{x > \gamma_2\}$, and there exists a $\gamma_3 \in [\gamma_2, +\infty)$, such that

$$\Gamma_{\lambda,\mu} \cap \{\gamma_2 < x < \gamma_3\} = \varnothing \ \text{ and } k_{\lambda,\mu}(x) \text{ is finite in } (\gamma_3, \gamma_3 + \varepsilon) \text{ for small } \varepsilon > 0.$$

Denote $D_R = \left\{(x, y) \mid \frac{\gamma_1+\gamma_2}{2} < x < \gamma_3 + \frac{\varepsilon}{2}, R < y < 2R\right\}$ for large R, applying the non-oscillation Lemma 3.2 for $\psi_{\lambda,\mu}$ in D_R, one has

$$R \leqslant C \left|\frac{\gamma_1 + \gamma_2}{2} - \gamma_3 - \frac{\varepsilon}{2}\right|,$$

where the constant C is independent of R. This leads a contradiction for sufficiently large $R > 0$.

Next, we will show that

$$\gamma_1 = b.$$

If not, then $\gamma_1 > b$ and we consider the following two cases.

Case 1. $k_{\lambda,\mu}(x) = \infty$ for any $x \in (b, \gamma_1)$. It follows from Lemma 2.2 that $\frac{1}{x}\frac{\partial \psi_{\lambda,\mu}}{\partial \nu} \geqslant \lambda$ on $T_R = \{(b, y) \mid R < y < 2R\}$ for large $R > 0$. Denote $D_R = \{(x, y) \mid b < x < \frac{\gamma_1+\gamma_2}{2}, R < y < 2R\}$, by using the non-oscillation Lemma 3.2 and the Remark in Section 3.2 for $\psi_{\lambda,\mu}$ in D_R, one has

$$R \leqslant C\frac{\gamma_1 + \gamma_2 - 2b}{2},$$

which leads a contradiction for sufficiently large $R > 0$.

Case 2. There exists a $\gamma_0 \in (b, \gamma_1]$, such that

$$\Gamma_{\lambda,\mu} \cap \{\gamma_0 < x < \gamma_1\} = \varnothing \text{ and } k_{\lambda,\mu}(x) \text{ is finite in } (\gamma_0 - \varepsilon, \gamma_0) \text{ for small } \varepsilon > 0.$$

Using the non-oscillation Lemma 3.2 for $\psi_{\lambda,\mu}$ in D_R leads a contradiction for sufficiently large $R > 0$, where $D_R = \{(x, y) \mid \gamma_0 - \frac{\varepsilon}{2} < x < \frac{\gamma_1+\gamma_2}{2}, R < y < 2R\}$.

Finally, we can show that $k_{\lambda,\mu}(b) < +\infty$ by using the non-oscillation Lemma 3.2 and the Remark in Section 3.2. □

In the following, we will show some important properties, such as, the optimal decay rate of the free boundary, the convergence rate and the asymptotic behavior of the subsonic impinging jet in downstream.

Lemma 3.4 *The minimizer $\psi_{\lambda,\mu}$ and the free boundary $y = k_{\lambda,\mu}(x)$ satisfy that*

(1) *for any sufficiently large $x_0 > b$, there exists a constant C (independent of x_0) such that*

$$\int_{\Omega_\mu \cap \{x \geqslant x_0\}} G(\nabla\psi, \psi, x; \lambda) dxdy \leqslant \frac{C}{x_0^3}, \tag{3.16}$$

where

$$G(\nabla\psi, \psi, x; \lambda) = xF\left(\left|\frac{\nabla\psi}{x}\right|^2; \lambda\right) + \left(x\Lambda^2 - 2\lambda F_1(\lambda^2; \lambda)\nabla\psi \cdot e\right) \chi_{\{\psi<m_0\}\cap E}.$$

(2) *In the downstream,*

$$x(k_{\lambda,\mu}(x) - g_0(x)) \to \frac{m_0}{\lambda \cos\theta} \quad \text{as} \quad x \to +\infty, \tag{3.17}$$

and

$$\frac{\nabla\psi_{\lambda,\mu}(x, y)}{x} \to (-\lambda \sin\theta, \lambda \cos\theta) \text{ for } (x, y) \in \Omega_\mu \cap \{\psi_{\lambda,\mu} < m_0\}, \text{ as } x \to +\infty. \tag{3.18}$$

Proof. (1) By using the inequality (2.24), we have

$$\int_{\Omega_\mu \cap \{x \geqslant x_0\}} G(\nabla\psi, \psi, x; \lambda) dxdy \leqslant C \int_{\Omega_\mu \cap \{x \geqslant x_0\}} x \left|\frac{\nabla\psi_{\lambda,\mu}}{x} - \lambda e\chi_{\{\psi<m_0\}}\right|^2 dxdy, \tag{3.19}$$

for $x_0 > b$.

It follows from the proof of Proposition 4.4 in [12] that

$$\int_{\Omega_\mu \cap \{x \geqslant x_0\}} x \left|\frac{\nabla\psi_{\lambda,\mu}}{x} - \lambda e\chi_{\{\psi<m_0\}}\right|^2 dxdy \leqslant \frac{C}{x_0^3},$$

for sufficiently large $x_0 > b$, where C is a constant independent of x_0. This together with (3.19) gives (3.16).

(2) For a sequence $\{x_n\}$ with $x_n \to +\infty$, set $\psi_n(\tilde{X}) = \psi_{\lambda,\mu}(X_n + \frac{\tilde{X}}{x_n})$ with $X_n = (x_n, g_0(x_n))$ and $\tilde{X} = (\tilde{x}, \tilde{y})$. Obviously, $\nabla\psi_n(\tilde{X}) = \frac{1}{x_n}\nabla\psi_{\lambda,\mu}(X_n + \frac{\tilde{X}}{x_n})$. For any $R > 0$, thanks to (3.16), we have

$$\begin{aligned} I_n &= \int_{\{|\tilde{x}|<Rx_n\}} \left(1 + \frac{\tilde{x}}{x_n^2}\right) \left| \frac{\nabla\psi_n}{1 + \frac{\tilde{x}}{x_n^2}} - \lambda e \chi_{\{\psi_n < m_0\}} \right|^2 d\tilde{x} d\tilde{y} \\ &= x_n \int_{\Omega_\mu \cap \{x_n - R| < x < x_n + R\}} x \left| \frac{\nabla\psi_{\lambda,\mu}}{x} - \lambda e \chi_{\{\psi_{\lambda,\mu} < m_0\}} \right|^2 dxdy \\ &\leqslant \frac{Cx_n}{(x_n - R)^3} \to 0 \text{ as } x_n \to +\infty. \end{aligned}$$

Without loss of generality, we may assume that

$$\psi_n \to \psi_0 \text{ weakly in } H^1_{loc}(\mathbb{R}^2) \text{ and } a.e. \text{ in } \mathbb{R}^2, \tag{3.20}$$

and

$$\chi_{\{\psi_n < m_0\}} \to \gamma \text{ weakly star in } L^\infty_{loc}(\mathbb{R}^2) \text{ and } \chi_{\{\psi_0 < m_0\}} \leqslant \gamma \leqslant 1.$$

Furthermore, one has

$$\int_{\mathbb{R}^2} \left|\nabla\psi_0 - \lambda e \chi_{\{\psi_0 < m_0\}}\right|^2 d\tilde{x} d\tilde{y} \leqslant \int_{\mathbb{R}^2} |\nabla\psi_0 - \lambda e \gamma|^2 d\tilde{x} d\tilde{y} \leqslant \liminf_{n\to+\infty} I_n = 0,$$

which gives that

$$\nabla\psi_0 = \lambda\chi_{\{\psi_0 < m_0\}} e \quad \text{a.e. in } \mathbb{R}^2 \text{ and } \psi_0(0) = 0. \tag{3.21}$$

Denote $\omega(s,t) = \psi_0(\tilde{x}, \tilde{y})$ with $s = \tilde{x}\cos\theta + \tilde{y}\sin\theta$ and $t = \tilde{y}\cos\theta - \tilde{x}\sin\theta$, one has

$$\frac{\partial\omega}{\partial s} = \frac{\partial\psi_0}{\partial\tilde{x}}\cos\theta + \frac{\partial\psi_0}{\partial\tilde{y}}\sin\theta \quad \text{and} \quad \frac{\partial\omega}{\partial t} = -\frac{\partial\psi_0}{\partial\tilde{x}}\sin\theta + \frac{\partial\psi_0}{\partial\tilde{y}}\cos\theta.$$

By virtue of (3.21), one has

$$\frac{\partial\omega(s,t)}{\partial s} = 0 \quad \text{and} \quad \frac{\partial\omega(s,t)}{\partial t} = \lambda\chi_{\{\omega < m_0\}} \text{ a.e. in } \mathbb{R}^2, \tag{3.22}$$

which imply that $\omega(s,t)$ is only a function of t. Moreover, $\omega(s,t)$ is monotone increasing with respect to t. In view of $\omega(0,0) = 0$ and $0 \leqslant \omega \leqslant m_0$, it follows from (3.22) that

$$\psi_0(\tilde{x}, \tilde{y}) = \omega(t) = \min\{\max\{\lambda t, 0\}, m_0\} = \min\{\max\{\lambda(\tilde{y}\cos\theta - \tilde{x}\sin\theta), 0\}, m_0\} \text{ in } \mathbb{R}^2.$$

To obtain the asymptotic behavior of the free boundary $\Gamma_{\lambda,\mu}$, we first show that

$$\partial\{\psi_n < m_0\} \text{ converges to } \partial\{\psi_0 < m_0\} \text{ locally in Hausdorff metric, as } n \to +\infty. \tag{3.23}$$

Definition of Hausdorff distance $d(D, F)$ between two sets D and F is as follows

$$d(D,F) = \inf\left\{\varepsilon > 0 \mid D \subset \bigcup_{X\in F} B_\varepsilon(X) \text{ and } F \subset \bigcup_{X\in D} B_\varepsilon(X)\right\}.$$

For any $X_0 = (x_0, y_0) \notin \partial\{\psi_0 < m_0\}$, the continuity of ψ_0 gives that $\psi_0(X_0) < m_0$ or $\psi_0(X_0) = m_0$. If $\psi_0 < m_0$ in $B_r(X_0)$, it follows from (3.20) that

$$\lim_{n\to+\infty} \fint_{\partial B_r(X_0)} (m_0 - \psi_n) dS > 0,$$

which implies that

$$\frac{1}{r}\fint_{\partial B_{\frac{r}{x_n}}\left(X_n+\frac{X_0}{x_n}\right)} (m_0 - \psi_{\lambda,\mu}) dS = \frac{1}{r}\fint_{\partial B_r(X_0)} (m_0 - \psi_n) dS > C\Lambda,$$

for sufficiently large n and small $r > 0$. Thanks to Lemma 2.4, one has

$$\psi_{\lambda,\mu} < m_0 \text{ in } B_{\frac{r}{x_n}}\left(X_n + \frac{X_0}{x_n}\right), \text{ namely, } \psi_n < m_0 \text{ in } B_r(X_0),$$

for sufficiently large n and small $r > 0$.
If $\psi_0 = m_0$ in $B_r(X_0)$, it follows from (3.20) that for a.e $r > 0$,

$$\lim_{n\to+\infty} \frac{2}{r}\left(\fint_{B_{\frac{r}{2}}(X_0)} (m_0 - \psi_n)^2 dxdy\right)^{\frac{1}{2}} = 0,$$

which together with Lemma 2.5 gives that $\psi_n = m_0$ in $B_{\frac{r}{8}}(X_0)$ for sufficiently large n.
Hence, we have the convergence of the free boundary in the Hausdorff distance.
For any $R > 0$ and small $\varepsilon > 0$, it follows from (3.23) that there exists a large $N_{\varepsilon,R}$, such that the free boundary $B_R(0) \cap \partial\{\psi_n < m_0\}$ and $B_R(0) \cap \partial\{\psi_0 < m_0\}$ lie each within an ε-neighborhood of one another, provided that $n > N_{\varepsilon,R}$. Thus we can check that the free boundary $B_R(0) \cap \partial\{\psi_n < m_0\}$ satisfies the flatness condition (see Section 5 in [5]), it follows from Theorem 6.3 in [5] that

$$B_R(0) \cap \partial\{\psi_n < m_0\} \to B_R(0) \cap \partial\{\psi_0 < m_0\} \text{ in } C^{1,\alpha},$$

which implies that

$$k'_{\lambda,\mu}(x_n + \tilde{x}) \to \tan\theta \text{ as } x_n \to +\infty.$$

Since $\tilde{y} = x_n\left(k_{\lambda,\mu}(x_n + \frac{\tilde{x}}{x_n}) - g_0(x_n)\right)$ is the free boundary of ψ_n, we have

$$\frac{d\tilde{y}}{d\tilde{x}} = k'_{\lambda,\mu}\left(x_n + \frac{\tilde{x}}{x_n}\right) \to \tan\theta \text{ and } x_n\left(k_{\lambda,\mu}\left(x_n + \frac{\tilde{x}}{x_n}\right) - g_0(x_n)\right) \to \tilde{x}\tan\theta + \frac{m_0}{\lambda\cos\theta},$$

which imply that

$$k'_{\lambda,\mu}(x_n) \to \tan\theta \text{ and } x_n\left(k_{\lambda,\mu}(x_n) - g_0(x_n)\right) \to \frac{m_0}{\lambda\cos\theta}.$$

The compactness of ψ_n gives that

$$\nabla\psi_n \to \nabla\psi_0 = \lambda(-\cos\theta, \sin\theta) \text{ uniformly in any compact subset of } S,$$

where $S = \left\{(\tilde{x}, \tilde{y}) \mid 0 < \tilde{y}\cos\theta - \tilde{x}\sin\theta < \frac{m_0}{\lambda}\right\}$, and thus (3.18) holds. □

Finally, we will give the gradient estimate of $\psi_{\lambda,\mu}$ near the initial point of $\Gamma_{\lambda,\mu}$ in the following.

Lemma 3.5 *Let $P = (b, k_{\lambda,L}(b))$, then*

$$|\nabla\psi_{\lambda,\mu}(X)| \leqslant C \quad \text{in } B_r(P),$$

for some $r > 0$, where C is constant depending only on Λ, b and ϑ, but not on m_0.

Proof. Without loss of generality, we assume that $P = A$. For any small $\gamma > 0$, it suffices to show that

$$|\nabla\psi_{\lambda,\mu}(X)| \leqslant C \ \text{ in } \ \Omega \cap \{\gamma < r < 2\gamma\},$$

where $r = |X - A|$. Denote

$$\psi_\gamma(X) = \frac{m_0 - \psi_{\lambda,\mu}(A + \gamma X)}{\gamma} \ \text{ in } \ D_1 = \left\{X \mid \frac{1}{2} < |X| < \frac{5}{2}, A + \gamma X \in \Omega\right\},$$

and

$$D_2 = \left\{X \mid 1 < |X| < 2, A + \gamma X \in \Omega\right\}.$$

Then we have

$$\nabla \cdot f_\eta\left(\frac{\nabla\psi_\gamma}{b + \gamma x}; \lambda\right) = 0 \ \text{ in } \ D_1 \cap \{\psi_\gamma > 0\}.$$

It follows from Lemma 2.1 that

$$\psi_\gamma = 0, \quad \frac{1}{b + \gamma x}\frac{\partial\psi_\gamma}{\partial\nu} = \lambda \ \text{ on the free boundary of } \psi_\gamma, \text{ and } 0 \leqslant \psi_\gamma \leqslant \frac{m_0}{\gamma}.$$

Since $\partial D_1 \cap \{X \mid A + \gamma X \in \partial\Omega_\mu\}$ is $C^{2,\alpha}$-smooth and $\psi_\gamma = 0$ on $\partial D_1 \cap \{X \mid A + \gamma X \in \partial\Omega_\mu\}$, the Harnack's inequality is still valid up to this part of the boundary. By using the similar arguments in the proof of Lemma 2.3, we have

$$|\nabla\psi_{\lambda,\mu}(A + \gamma X)| = |\nabla\psi_\gamma(X)| \leqslant C \ \text{ in } \ D_2,$$

where C is a constant depending on Λ, b and ϑ, but not on $\frac{m_0}{\gamma}$. Therefore, we obtain the assertion of this lemma. □

3.3 *Continuous dependence of $\psi_{\lambda,\mu}$ and $\Gamma_{\lambda,\mu}$ with respect to λ*

To obtain the continuous fit condition, we will show that the minimizer $\psi_{\lambda,\mu}$ and the free boundary $\Gamma_{\lambda,\mu}$ are continuous dependence with respect to the parameter $\lambda \leqslant \Pi_\lambda - 4\tilde{\varepsilon}$.

Lemma 3.6 *Let $\psi_{\lambda_n,\mu}$ be a minimizer to the variational problem $(P_{\lambda_n,\mu})$ with the admissible set K_μ and $\lambda_n \leqslant \Pi_{\lambda_n} - 4\tilde{\varepsilon}$, then we have*

$$\psi_{\lambda_n,\mu} \rightharpoonup \psi_{\lambda,\mu} \ \text{ in } \ H^1_{loc}(\Omega_\mu) \text{ and } \psi_{\lambda_n,\mu} \to \psi_{\lambda,\mu} \ \text{ a.e. in } \ \Omega_\mu,$$

as $\lambda_n \to \lambda \leqslant \Pi_\lambda - 4\tilde{\varepsilon}$, where $\psi_{\lambda,\mu}$ is the minimizer to the variational problem $(P_{\lambda,\mu})$.

Proof. Denote $\psi_n = \psi_{\lambda_n,\mu}$ for simplicity. By virtue of Proposition 2.1, we have $|\nabla\psi_n| \leqslant C$ in any compact subset of Ω_μ, where the constant C is independent of n. Then there exists a subsequence $\{\psi_n\}$ such that

$$\psi_n \rightharpoonup \omega \ \text{ weakly in } \ H^1_{loc}(\Omega_\mu) \text{ and } \psi_n \to \omega \text{ in } C^{\alpha}_{loc}(\Omega_\mu) \ \text{ for all } \ 0<\alpha<1, \tag{3.24}$$

and

$$\nabla\psi_n \to \nabla\omega \ \text{ weakly star in } \ L^\infty_{loc}(\Omega_\mu).$$

Step 1. $E\cap\partial\{\psi_n<m_0\}\to E\cap\partial\{\omega<m_0\}$ locally in the Hausdorff distance in Ω_μ.

For any $X_0=(x_0,y_0)\notin E\cap\partial\{\omega<m_0\}$, the continuity of ω gives that there exists a small $r>0$, such that $B_r(X_0)\cap\partial\{\omega<m_0\}=\varnothing$. We next claim that

$$B_{\frac{r}{16}}(X_0)\cap\partial\{\psi_n<m_0\}=\varnothing \text{ for sufficiently large } n. \tag{3.25}$$

If $\omega<m_0$ in $B_r(X_0)$, (3.24) implies that the claim (3.25) is valid. If $\omega=m_0$ in $B_r(X_0)$, for any small $\varepsilon>0$, it follows from (3.24) that there exists a N_ε such that

$$|m_0-\psi_n(X)|<\varepsilon \ \text{ in } \ B_{\frac{r}{2}}(X_0) \ \text{ for } \ n>N_\varepsilon,$$

which implies that

$$\frac{2}{r}\left(\fint_{B_{\frac{r}{2}}(X_0)}(m_0-\psi_n)^2dxdy\right)^{\frac{1}{2}}<\frac{2\varepsilon}{r}\leqslant\lambda c^*\Lambda x_0 \text{ for } n>N_\varepsilon.$$

Thanks to Lemma 2.5 for ψ_n, one has $\psi_n = m_0$ in $B_{\frac{r}{16}}(X_0)$ for sufficiently large n, and the claim (3.25) holds.

On the other hand, for any $X_0=(x_0,y_0)\notin E\cap\partial\{\psi_n<m_0\}$. Then $B_r(X_0)\cap\partial\{\psi_n<m_0\}=\varnothing$ for small $r>0$, and we claim that

$$B_{\frac{r}{2}}(X_0)\cap\partial\{\omega<m_0\}=\varnothing. \tag{3.26}$$

If $\psi_n<m_0$ in $B_r(X_0)$ for a subsequence $\{\psi_n\}$, one has

$$\tilde{Q}_{\lambda_n}\psi_n=0 \ \text{ in } \ B_r(X_0),$$

which implies that

$$\tilde{Q}_{\lambda}\omega=0 \ \text{ in } \ B_r(X_0), \ \omega\leqslant m_0 \ \text{ in } \ B_r(X_0).$$

The strong maximum principle yields that $\omega=m_0$ or $\omega<m_0$ in $B_r(X_0)$. Thus, the claim (3.26) holds.

It is easy to check that the claim (3.26) is valid, if $\psi_n=m_0$ in $B_r(X_0)$ for a subsequence $\{\psi_n\}$. Hence, we complete the proof of the convergence of the free boundary in the Hausdorff distance.

Step 2. $\chi_{\{\psi_n<m_0\}\cap E}\to\chi_{\{\omega<m_0\}\cap E}$ locally in $L^1(\Omega_\mu)$.

In view of (3.24), we can deduce that Lemma 2.4 and Lemma 2.5 still hold for ω by taking the limit $n\to\infty$. By using Theorem 2.8 in [5], one has

$$\mathcal{L}^2(E\cap\partial\{\omega<m_0\}\cap B_R)=0, \quad \text{for any } R>0,$$

where $\mathcal{L}^2$ is the Lebesgue measure in $\mathbb{R}^2$.

Applying the results in Step 1, there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$, such that

$$E \cap \{\psi_n < m_0\} \subset O_{\varepsilon_n},$$

where O_{ε_n} be an ε_n-neighborhood of $E \cap \partial\{\omega < m_0\}$. Then we have

$$\int_{\Omega_\mu \cap B_R} \left|\chi_{\{\psi_n < m_0\} \cap E} - \chi_{\{\omega < m_0\} \cap E}\right| dxdy \leqslant \int_{\Omega_\mu \cap B_R \cap O_{\varepsilon_n}} dxdy \to 0,$$

for any $R > 0$.

Step 3. $\nabla\psi_n \to \nabla\omega$ a.e. locally in Ω_μ.

Let D be any compact subset of $\Omega_\mu \cap \{\omega < m_0\}$. Then one has

$$\tilde{Q}_{\lambda_n}\psi_n = 0 \ \text{ in } \ D \ \text{ for sufficiently large } n.$$

The elliptic estimates for ψ_n gives that

$$\nabla\psi_n \to \nabla\omega \ \text{ in } \ D. \tag{3.27}$$

Next, we claim that

$$\nabla\psi_n \to \nabla\omega \ \text{ a.e. in } \Omega_\mu \cap \{\omega = m_0\}. \tag{3.28}$$

Since $\{\omega = m_0\}$ is $\mathcal{L}^2$-measurable, it follows from Corollary 3 in [24] that

$$\lim_{r\to 0} \frac{\mathcal{L}^2\big(B_r(X) \cap \{\omega = m_0\}\big)}{\mathcal{L}^2\big(B_r(X)\big)} = 1 \quad \text{for} \quad \mathcal{L}^2 \text{ a.e. } X \in \{\omega = m_0\}.$$

Denote

$$\mathcal{S} = \left\{X \in \{\omega = m_0\} \mid \lim_{r\to 0} \frac{\mathcal{L}^2\big(B_r(X) \cap \{\omega = m_0\}\big)}{\mathcal{L}^2\big(B_r(X)\big)} = 1\right\}.$$

We next show that

$$m_0 - \omega(X_0 + X) = o(|X|) \quad \text{for any} \quad X_0 \in \mathcal{S}. \tag{3.29}$$

In fact, if $m_0 - \omega(Y) > kr$ for some $Y \in B_r(X_0)$ with $r \to 0$ and $k > 0$. The Lipschitz continuity of ω gives that

$$m_0 - \omega(X) > \frac{k}{2}r \quad \text{in} \quad B_{\varepsilon kr}(Y) \quad \text{for some small } \ \varepsilon > 0,$$

which implies that $\{\omega < m_0\}$ has positive density at X_0, and then it contradicts to $X_0 \in \mathcal{S}$.

With the aid of (3.24) and (3.29), for any $\varepsilon > 0$, we have

$$\frac{m_0 - \psi_n}{r} < \varepsilon \quad \text{in} \quad B_r(X_0) \quad \text{for small } \ r,$$

provided that n is sufficiently large, that is $n > N(\varepsilon, r)$. It follows from the non-degeneracy Lemma 2.5 that $\psi_n \equiv m_0$ in $B_{\frac{r}{8}}(X_0)$, which implies that $\omega \equiv m_0$ in $B_{\frac{r}{10}}(X_0)$. Then we have that the set $\mathcal{S}$ is open. Furthermore, one has

$$\psi_n \equiv \omega \text{ in any compact subset of } \mathcal{S}, \text{ provided that } n \text{ is sufficiently large.}$$

This completes the proof of the claim (3.28).

Since $\mathcal{L}^2(E \cap \partial\{\omega < m_0\}) = 0$, it follows from (3.27) and (3.28) that $\nabla\psi_n \to \nabla\omega$ a.e. in $\mathbb{R}^2$.

Step 4. $\omega = \psi$, where $\psi = \psi_{\lambda,\mu}$ is the minimizer to the truncated variational problem $(P_{\lambda,\mu})$.

First, we will show that

$$J_R(\omega) \leqslant J_R(\phi) \text{ for any } \phi \in K_\mu \text{ and } \phi = \omega \text{ on } \partial\Omega_R, \tag{3.30}$$

where $J_R(\phi) = \int_{\Omega_R} G(\nabla\phi, \phi, x; \lambda)dxdy$ and $\Omega_R = \Omega_\mu \cap B_R(0)$ for sufficiently large $R > 0$.

For any $\phi \in K_\mu$ and $\phi = \omega$ on $\partial\Omega_R$, set

$$\phi_n = \phi + (1 - \xi_\delta)(\psi_n - \omega),$$

where $\xi_\delta(X) = \min\{\frac{\operatorname{dist}(X, \mathbb{R}^2 \backslash \Omega_R)}{\delta}, 1\}$. Obviously, $\phi_n = \psi_n$ on $\partial\Omega_R$ and extend $\phi_n = \psi_n$ outside Ω_R, such that $\phi_n \in K_\mu$. Then one has

$$\int_{\Omega_R} G(\nabla\psi_n, \psi_n, x; \lambda_n)dxdy \leqslant \int_{\Omega_R} G(\nabla\phi_n, \phi_n, x; \lambda_n)dxdy.$$

By using the convergence results in Step 2 and Step 3, taking $n \to \infty$ in the above inequality, we have

$$\begin{aligned}
&\int_{\Omega_R} G(\nabla\omega, \omega, x; \lambda)dxdy \\
&\qquad \leqslant \int_{\Omega_R \cap \{\xi_\delta = 1\}} G(\nabla\phi, \phi, x; \lambda)dxdy + \int_{\Omega_R \cap \{\xi_\delta < 1\}} G(\nabla\phi, \phi, x; \lambda)dxdy.
\end{aligned} \tag{3.31}$$

Taking $\delta \to 0$ in (3.31) yields that

$$\textstyle\int_{\Omega_R} G(\nabla\omega, \omega, x; \lambda)dxdy \leqslant \int_{\Omega_R} G(\nabla\phi, \phi, x; \lambda)dxdy. \tag{3.32}$$

Hence, (3.30) can be obtained by taking $R \to +\infty$ in (3.32).

It follows from (3.16) that

$$\int_{\Omega_\mu \cap \{x \geqslant x_0\}} G(\nabla\omega, \omega, x; \lambda)dxdy \leqslant \frac{C}{x_0^3},$$

for sufficiently large $x_0 > 0$, which implies that the results (2) in Lemma 3.4 still be valid for ω.

For any $\varepsilon > 0$, Set $\psi_\varepsilon(x, y) = \psi(x, y - \varepsilon)$ and $\Omega_\mu^\varepsilon = \{(x, y) \mid (x, y - \varepsilon) \in \Omega_\mu\}$. Extend $\omega(x, y) = \frac{m_0}{x_\mu}x^2$ in $\{(x, y) \mid 0 \leqslant x \leqslant x_\mu, \mu \leqslant y \leqslant \mu + \varepsilon\}$. It is easy to check that $\min\{\psi_\varepsilon, \omega\} \in K_\mu^\varepsilon$ and $\max\{\psi_\varepsilon, \omega\} \in K_\mu$. Therefore, one has

$$J_{\lambda,\mu}^\varepsilon(\psi_\varepsilon) \leqslant J_{\lambda,\mu}^\varepsilon\big(\min\{\psi_\varepsilon, \omega\}\big). \tag{3.33}$$

Similar to the proof of Theorem 4.1 in [12], we can check that

$$J_{\lambda,\mu}(\omega) + J_{\lambda,\mu}^\varepsilon(\psi_\varepsilon) = J_{\lambda,\mu}\big(\max\{\psi_\varepsilon, \omega\}\big) + J_{\lambda,\mu}^\varepsilon\big(\min\{\psi_\varepsilon, \omega\}\big). \tag{3.34}$$

With the aid of the asymptotic behaviors of ψ and ω in Lemma 3.4, we have

$$\psi_\varepsilon(x, y) \leqslant \omega(x, y) \text{ in } \Omega_\mu \setminus B_R \text{ for any sufficiently large } R > 0.$$

Thus $\max\{\psi_\varepsilon, \omega\} = \omega$ on $\partial\Omega_R$, it follows from (3.30) that

$$J_R(\omega) \leqslant J_R\big(\max\{\psi_\varepsilon, \omega\}\big).$$

Taking $R \to +\infty$ in the above inequality yields that

$$J_{\lambda,\mu}(\omega) \leqslant J_{\lambda,\mu}\big(\max\{\psi_\varepsilon, \omega\}\big),$$

which together with (3.33) and (3.34) gives that

$$J^{\varepsilon}_{\lambda,\mu}(\psi_\varepsilon) = J^{\varepsilon}_{\lambda,\mu}\big(\min\{\psi_\varepsilon, \omega\}\big).$$

Since the minimizer ψ_ε to the variational problem $(P^{\varepsilon}_{\lambda,\mu})$ is unique, one has

$$\psi(x, y-\varepsilon) = \psi_\varepsilon(x, y) = \min\{\psi_\varepsilon, \omega\} \leqslant \omega(x, y) \ \text{ in } \ \Omega_\mu. \tag{3.35}$$

Similarly, we can show that

$$\psi(x, y+\varepsilon) \geqslant \omega(x, y) \ \text{ in } \ \Omega_\mu. \tag{3.36}$$

Taking $\varepsilon \to 0$ in (3.35) and (3.36), we have that $\psi = \omega$ in Ω_μ. □

Next, we will obtain the continuous dependence of the free boundary $\Gamma_{\lambda,\mu}$ with respect to λ. We remark that even though the ideas of the proof borrow from the one for incompressible jet as done in Theorem 3.1 in [4] and Theorem 6.1 in Chapter 3 in [25], due to the difference of the governing equations and the functional, we have to overcome several additional difficulties. Actually, the stream function satisfies the linear elliptic equation for the incompressible flows, and here we have to deal with a quasilinear elliptic equation.

Lemma 3.7 *The free boundary $y = k_{\lambda_n,\mu}(x)$ of the minimizer $\psi_{\lambda_n,\mu}$ with $\lambda_n \leqslant \Pi_{\lambda_n} - 4\tilde{\varepsilon}$ satisfies that*

$$k_{\lambda_n,\mu}(x) \to k_{\lambda,\mu}(x) \ \textit{ for any } x \in (b, +\infty),$$

as $\lambda_n \to \lambda$, where $y = k_{\lambda,\mu}(x)$ is the free boundary of the minimizer $\psi_{\lambda,\mu}$.

Proof. For any fixed $x \in (b, +\infty)$, the convergence of $k_{\lambda_n,\mu}(x)$ can be obtained by using the Step 1 in the proof of Lemma 3.6.

Next, we will consider the initial point of the free boundary and show that $k_{\lambda_n,\mu}(b) \to k_{\lambda,\mu}(b)$ as $\lambda_n \to \lambda$. Suppose not, then there exists a subsequence $\{k_{\lambda_n,\mu}(b)\}$, such that $k_{\lambda_n,\mu}(b) \to k_{\lambda,\mu}(b) + \beta$ and $\beta \neq 0$. We will show that it is impossible case by case based on the sign of β.

Case 1. $\beta < 0$. The monotonicity of $\psi_{\lambda,\mu}(x, y)$ with respect to y gives that $k_{\lambda,\mu}(b) + \beta \geqslant 1$. In fact, if $k_{\lambda,\mu}(b) + \beta < 1$, it follows from Lemma 3.6 that

$$\psi_{\lambda,\mu}(b, y) = m_0 \quad \text{for} \quad k_{\lambda,\mu}(b) + \beta \leqslant y \leqslant \min\{k_{\lambda,\mu}(b), 1\},$$

which contradicts to the fact $\psi_{\lambda,\mu} < m_0$ for $y < \min\{k_{\lambda,\mu}(b), 1\}$, due to the fact that $\psi_{\lambda,\mu}(x, y)$ is monotone increasing with respect to y.

Denote

$$T_\beta = \left\{(b, y) \mid k_{\lambda,\mu}(b) + \frac{3\beta}{4} < y < k_{\lambda,\mu}(b) + \frac{\beta}{4}\right\}.$$

Next, we claim that

$$\frac{\partial \psi_{\lambda,\mu}(b+0,y)}{\partial x} = -\lambda \quad \text{on} \quad T_\beta. \tag{3.37}$$

For small $\varepsilon > 0$, set

$$U_\varepsilon = \left\{(x,y) \mid b-\varepsilon < x < b+\varepsilon, k_{\lambda,\mu}(b) + \frac{3\beta}{4} < y < k_{\lambda,\mu}(b) + \frac{\beta}{4}\right\}.$$

It is easy to check that

$$T_\beta = U_\varepsilon \cap \partial\{\psi_{\lambda,\mu} < m_0\}.$$

Set $\phi_n = m_0 - \psi_{\lambda_n,\mu}$ and $\phi = m_0 - \psi_{\lambda,\mu}$, then one has

$$\begin{cases} \tilde{Q}_{\lambda_n}\phi_n = 0 & \text{in } U_\varepsilon \cap \{\phi_n > 0\}, \\ \frac{1}{x}\frac{\partial \phi_n}{\partial \nu_n} = \lambda_n & \text{on } U_\varepsilon \cap \partial\{\phi_n > 0\}, \end{cases}$$

where ν_n is the inner normal vector to $U_\varepsilon \cap \partial\{\phi_n > 0\}$. Now, in order to show the claim (3.37), it suffices to check that ϕ satisfies the following boundary value problem,

$$\begin{cases} \tilde{Q}_{\lambda}\phi = 0 & \text{in } U_\varepsilon \cap \{\phi > 0\}, \\ \frac{1}{x}\frac{\partial \phi}{\partial \nu} = \lambda & \text{on } U_\varepsilon \cap \partial\{\phi > 0\}, \end{cases} \tag{3.38}$$

where $\nu = (1,0)$ is the inner normal vector to $\partial\{\phi > 0\}$ at T_β.

We divide the proof into two steps to show that (3.38) holds.

Step 1. In this step, we will verify that

$$\frac{1}{x}\frac{\partial \phi}{\partial \nu} \geqslant \lambda \quad \text{on} \quad U_\varepsilon \cap \partial\{\phi > 0\}. \tag{3.39}$$

It follows from (3.24) that for $\alpha \in (0,1)$,

$$\phi_n \to \phi \quad \text{uniformly in } C^\alpha(U_\varepsilon),$$

and $U_\varepsilon \cap \{\phi_n > 0\} \to U_\varepsilon \cap \{\phi > 0\}$ in the Hausdorff metric space, and

$$\nabla\phi_n \to \nabla\phi \quad \text{weakly in } L^2(U_\varepsilon),$$

as $n \to +\infty$. Moreover, by virtue of the bounded gradient Lemma 2.3 and the Step 4 in the proof of Lemma 3.6, one has

$$|\nabla\phi_n| \leqslant C\Lambda x \quad \text{in } U_\varepsilon \text{ and } \nabla\phi_n \to \nabla\phi \quad \text{a.e. in } U_\varepsilon, \text{ as } n \to \infty, \tag{3.40}$$

where the constant C is independent of n.

Since $F_1(t;\lambda)=\frac{1}{\tilde\rho(t;\lambda^2)}$ is $C^{1,\alpha}$, it follows from (3.40) that

$$\begin{aligned}\lim_{n\to+\infty} F_1(\lambda_n^2;\lambda_n)\lambda_n\int_{U_\varepsilon\cap\partial\{\phi_n>0\}}\xi dS\\ &=-\lim_{n\to+\infty}\int_{U_\varepsilon\cap\{\phi_n>0\}}\frac{1}{x}F_1\left(\left|\frac{\nabla\phi_n}{x}\right|^2;\lambda_n\right)\nabla\phi_n\cdot\nabla\xi dxdy\\ &=-\int_{U_\varepsilon\cap\{\phi>0\}}\frac{1}{x}F_1\left(\left|\frac{\nabla\phi}{x}\right|^2;\lambda\right)\nabla\phi\cdot\nabla\xi dxdy\\ &=\int_{U_\varepsilon\cap\partial\{\phi>0\}}\frac{1}{x}F_1\left(\left|\frac{\nabla\phi}{x}\right|^2;\lambda\right)\frac{\partial\phi}{\partial\nu}\xi dS,\end{aligned}\tag{3.41}$$

for any non-negative $\xi\in C_0^\infty(U_\varepsilon)$.

On other hand, it follows from (3.6) in Chapter 3 in [25] that

$$\int_{U_\varepsilon\cap\partial\{\phi>0\}}\xi dS\leqslant\liminf_{n\to+\infty}\int_{U_\varepsilon\cap\partial\{\phi_n>0\}}\xi dS.\tag{3.42}$$

In view of (3.41) and (3.42), one has

$$F_1(\lambda^2;\lambda)\lambda\int_{U_\varepsilon\cap\partial\{\phi>0\}}\xi dS\leqslant\int_{U_\varepsilon\cap\partial\{\phi>0\}}\frac{1}{x}F_1\left(\left|\frac{\nabla\phi}{x}\right|^2;\lambda\right)\frac{\partial\phi}{\partial\nu}\xi dS,\tag{3.43}$$

for any non-negative $\xi\in C_0^\infty(U)$. Since ϕ is $C^{1,\alpha}$ up to the boundary $\partial\{\phi>0\}$, it follows from (3.43) that

$$F_1\left(\left|\frac{\nabla\phi}{x}\right|^2;\lambda\right)\left|\frac{\nabla\phi}{x}\right|=F_1\left(\left|\frac{\nabla\phi}{x}\right|^2;\lambda\right)\frac{1}{x}\frac{\partial\phi}{\partial\nu}\geqslant F_1\left(\lambda^2;\lambda\right)\lambda\quad\text{on}\quad U_\varepsilon\cap\partial\{\phi>0\}.\tag{3.44}$$

Since $F_1(t^2;\lambda)t$ is increasing with respect to t, (3.44) gives that (3.39) holds.

Step 2. In this step, we will check that

$$\frac{1}{x}\frac{\partial\phi}{\partial\nu}\leqslant\lambda\quad\text{on}\quad U_\varepsilon\cap\partial\{\phi>0\}.\tag{3.45}$$

By virtue of the non-oscillation Lemma (3.2) and the flatness of the free boundary in Section 5 in [5], we have that

the free boundary $\Gamma_{\lambda_n,\mu}$ is a y-graph in U_ε for sufficiently large n,

where we denote $\Gamma_{\lambda_n,\mu}:x=f_{\lambda_n,\mu}(y)$ in the region U_ε. It follows from the result in Theorem 6.3 and Remark 6.4 in [5] that

$$\left|f^{(j)}_{\lambda_n,\mu}(y)\right|\leqslant C\quad\text{for}\quad k_{\lambda,\mu}(b)+\frac{3\beta}{4}<y<k_{\lambda,\mu}(b)+\frac{\beta}{4},\quad j=1,2.$$

Then we have

$$U_\varepsilon \cap \partial\{\phi_n > 0\} \to U_\varepsilon \cap \partial\{\phi > 0\} \text{ in } C^{1,\alpha} \text{ for some } \alpha \in (0,1).$$

For any fixed $X_0 = (x_0, y_0) \in U_\varepsilon \cap \partial\{\phi > 0\}$, it follows from Lemma 3.6 that there exists a sequence $X_n \in U_\varepsilon$ with $\phi_n(X_n) = 0$, such that $X_n \to X_0$ as $n \to +\infty$. Take a small $r > 0$ and domain $E_n \subset B_r(X_0) \subset U_\varepsilon$, such that

$$B_r(X_0) \cap \partial E_n \to B_r(X_0) \cap \partial\{\phi > 0\} \text{ in } C^{1,\gamma},\ \phi_n > 0 \text{ in } E_n \text{ and } X_n \notin E_n, \tag{3.46}$$

for $0 < \gamma < \alpha$. We can take a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$, such that

$$E_n = B_r(X_0) \cap \{x > b + \varepsilon_n\}.$$

Define a function $f_{\delta,n}(y)$ as follows

$$f_{\delta,n}(y) = b + \varepsilon_n - \delta\eta\left(\frac{2(y - y_0)}{r}\right),\quad \delta > 0,$$

where

$$\eta(y) = \begin{cases} e^{-\frac{y^2}{1-y^2}} & \text{for } |y| < 1, \\ 0 & \text{for } |y| \geqslant 1. \end{cases}$$

Denote the domain $E_{\delta,n} = B_r(X_0) \cap \{x > f_{\delta,n}(y)\}$. It is easy to check that $E_{0,n} = E_n$. By virtue of the definitions of E_n and $E_{\delta,n}$, there exists the largest number $\delta = \delta_n$, such that $\phi_n > 0$ in $E_{\delta_n,n}$, and $B_r(X_0) \cap \partial E_{\delta_n,n}$ contains a point of the free boundary of ϕ_n, which is denoted as $\tilde{X}_n = (\tilde{x}_n, \tilde{y}_n)$ with $\tilde{x}_n = f_{\delta_n,n}(\tilde{y}_n)$. Furthermore,

$$\delta_n \to 0 \text{ as } n \to +\infty.$$

Let ω_n be the solution of the following Dirichlet problem

$$\begin{cases} \tilde{Q}_{\lambda_n}\omega_n = 0 \text{ in } E_{\delta_n,n}, \\ \omega_n = 0 \text{ on } \partial E_{\delta_n,n} \cap B_{\frac{r}{2}}(X_0),\ \omega_n = \zeta\phi_n \text{ on } \partial E_{\delta_n,n} \cap \big(B_r(X_0) \setminus B_{\frac{r}{2}}(X_0)\big), \\ \omega_n = \phi_n \text{ on } \partial E_{\delta_n,n} \cap \partial B_r(X_0), \end{cases}$$

where $\zeta(X) = \min\big\{\max\big\{\frac{2|X-X_0|-r}{r}, 0\big\}, 1\big\}$. Since ϕ_n satisfies the quasilinear equation $\tilde{Q}_{\lambda_n}\phi_n = 0$ in $E_{\delta_n,n}$ and $\omega_n \leqslant \phi_n$ on $\partial E_{\delta_n,n}$, the maximum principle implies that $\omega_n \leqslant \phi_n$ in $E_{\delta_n,n}$. Hence, one has

$$\lambda_n = \frac{1}{\tilde{x}_n}\frac{\partial\phi_n(\tilde{X}_n)}{\partial\nu_n} \geqslant \frac{1}{\tilde{x}_n}\frac{\partial\omega_n(\tilde{X}_n)}{\partial\nu_n}, \tag{3.47}$$

where ν_n is the inner normal vector to $\partial E_{\delta_n,n}$ at $\tilde{X}_n$.

It follows from the fact (3.46) that

$$f_{\delta_n,n}(y) \to b \text{ in } C^{1,\gamma}.$$

Thanks to the standard estimates of the solutions of the elliptic equation of second order, we conclude that ω_n in $E_{\delta_n,n} \cap B_{\frac{r}{2}}(X_0)$ converges to ϕ in $\{\phi > 0\} \cap B_{\frac{r}{2}}(X_0)$ in $C^{1,\gamma}$-sense. Suppose that

$$\tilde{X}_n \to \tilde{X} = (\tilde{x}, \tilde{y}) \in \partial\{\phi > 0\} \cap B_{\frac{r}{2}}(X_0).$$

This together with (3.47) gives that

$$\frac{1}{\tilde{x}_n}\frac{\partial \omega_n(\tilde{X}_n)}{\partial \nu} \to \frac{1}{\tilde{x}}\frac{\partial \phi(\tilde{X})}{\partial \nu} \quad \text{and} \quad \frac{1}{\tilde{x}}\frac{\partial \phi(\tilde{X})}{\partial \nu} \leqslant \lambda.$$

Taking $r \to 0$, we have $\tilde{X} \to X_0$ and

$$\lambda \geqslant \frac{1}{\tilde{x}}\frac{\partial \phi(\tilde{X})}{\partial \nu} \to \frac{1}{x_0}\frac{\partial \phi(X_0)}{\partial \nu}, \quad X_0 \in U_\varepsilon \cap \partial\{\phi > 0\},$$

which gives the inequality (3.45) holds.

Therefore, the claim (3.38) follows from (3.39) and (3.45).

With the aid of the claim (3.37), $\psi_{\lambda,\mu}$ satisfies

$$\tilde{Q}_\lambda \psi_{\lambda,\mu} = 0 \text{ in } E_\varepsilon, \ \frac{1}{x}\frac{\partial \psi_{\lambda,\mu}}{\partial x} = -\lambda \text{ and } \psi_{\lambda,\mu} = m_0 \text{ on } \partial E_\varepsilon \cap \{x = b\},$$

for small $\varepsilon > 0$, where $E_\varepsilon = \{(x, y) \mid b < x < b + \varepsilon, k_{\lambda,\mu}(b) + \frac{3\beta}{4} < y < k_{\lambda,\mu}(b) + \frac{\beta}{4}\}$.

It follows from the Cauchy–Kovalevskaya theorem that

$$\psi_{\lambda,\mu} = -\lambda(x^2 - b^2) + m_0 \text{ in } \{(x, y) \mid b < x < b + \varepsilon, g_0(b) < y < +\infty\} \cap \Omega_\mu.$$

This contradicts to the fact $\psi_{\lambda,\mu} = 0$ on N_0.

Case 2. $\beta > 0$ and $k_{\lambda,\mu}(b) < 1$. By using the similar arguments in Case 1, we can conclude that

$$\frac{1}{b}\frac{\partial \psi_{\lambda,\mu}(b - 0, y)}{\partial x} = \lambda \text{ if } k_{\lambda,\mu}(b) + \frac{\beta}{4} < y < \min\left\{k_{\lambda,\mu}(b) + \frac{3\beta}{4}, 1\right\}.$$

We can obtain a contradiction by using Cauchy–Kovalevskaya theorem as in Case 1 again.

Case 3. $\beta > 0$ and $k_{\lambda,\mu}(b) \geqslant 1$. It follows from the arguments in Lemma 2.2 that

$$\frac{1}{b}\frac{\partial \psi_{\lambda_n,\mu}(b + 0, y)}{\partial x} \leqslant -\lambda_n \quad \text{on} \quad \left\{x = b, k_{\lambda,\mu}(b) + \frac{\beta}{4} < y < k_{\lambda,\mu}(b) + \frac{3\beta}{4}\right\},$$

for sufficiently large n. Let D_n be bounded by $x = b$, $y = k_{\lambda_n,\mu}(x)$, $y = k_{\lambda,\mu}(b) + \frac{\beta}{4}$ and $y = k_{\lambda,\mu}(b) + \frac{3\beta}{4}$. Furthermore, we have

$$x_n = \min\left\{x \mid k_{\lambda_n,\mu}(x) = k_{\lambda,\mu}(b) + \frac{3\beta}{4}\right\} \to b \quad \text{as} \quad n \to +\infty.$$

Thanks to the non-oscillation Lemma 4.4 in [6] for $\psi_{\lambda_n,\mu}$ in D_n, one has

$$\frac{\beta}{2} \leqslant C(x_n - b), \ \text{ the constant } C \text{ is independent of } n,$$

which leads a contradiction for sufficiently large n. □

3.4 *The continuous fit condition of the free boundary $\Gamma_{\lambda,\mu}$*

In the following, we will consider the continuous fit condition of the free boundary at A.

Proposition 3.8 *For any $\mu > 1$ and small $m_0 > 0$, there exists a $\lambda_\mu \leqslant \Pi_{\lambda_\mu} - 4\tilde{\varepsilon}$, such that the free boundary $\Gamma_{\lambda_\mu,\mu}$ satisfies the continuous fit condition at A, namely, $k_{\lambda_\mu,\mu}(b) = 1$. Moreover,*

$$\lambda_\mu \leqslant C_0 m_0, \quad \text{the constant } C_0 \text{ is independent of } \mu \text{ and } m_0.$$

Proof. *Step* 1. In this step, we will show that

$$\text{for } m_0 > 0, \text{ if } \lambda > 0 \text{ is sufficiently small, then } k_{\lambda,\mu}(b) > 1.$$

Suppose that there exists a small λ_0, such that $k_{\lambda_0,\mu}(b) \leqslant 1$. Let S be a ring centered at $P = (b, k_{\lambda_0,\mu}(b))$ with some suitable radius R_1 and R_2 which are independent of m_0 and $R_1 < R_2$, such that $\Gamma_{\lambda_0,\mu} \cap S \cap \{x > b\}$ and $N_0 \cap \bar{S} \cap \{x > b\}$ are nonempty.

It follows from Lemma 2.3 that there exists a constant C_0 depending on N, N_0, ϑ (independent of m_0), such that

$$\left|D\psi_{\lambda_0,\mu}\right| \leqslant C_0 \Lambda(\lambda_0) \ \text{ in } \ S \cap \{x > b\},$$

where $\Lambda(\lambda_0) = \sqrt{2F_1(\lambda_0^2;\lambda_0)\lambda_0^2 - F(\lambda_0^2;\lambda_0)}$.

Choosing $X_1 \in \Gamma_{\lambda_0,\mu} \cap \bar{S} \cap \{x > b\}$, $X_2 \in N_0 \cap \bar{S} \cap \{x > b\}$ with $|X_1 - P| = |X_2 - P|$, shows that

$$m_0 = \int_\gamma \left|\frac{\partial \psi_{\lambda_0}}{\partial s}\right| dS \leqslant |D\psi_{\lambda,\mu}| 2\pi R_2 \leqslant C\Lambda(\lambda_0^2), \tag{3.48}$$

where $\gamma \subset S$ is disc curve which connects X_1 and X_2, s is the unit tangent vector of γ. On another hand, it follows from (2.18) that

$$\Lambda(\lambda_0) \leqslant \lambda_0 \sqrt{\frac{1}{\rho_0}\left(2 - \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\right)},$$

which together with (3.48) gives that

$$0 < m_0 \leqslant C\lambda_0.$$

This is impossible for sufficiently small λ_0.

Step 2. For $\lambda = \frac{\lambda_{cr}}{2}$, we will show that

$$\text{if } m_0 > 0 \text{ is sufficiently small, then } k_{\lambda,\mu}(b) < 1.$$

Suppose that $k_{\lambda,\mu}(b) \geqslant 1$ for some small $m_0 > 0$, then there exists a disc $B_R(X_0) \subset \Omega_\mu$ (R is fixed), such that $X_0 \in E$ and $B_{\frac{R}{8}}(X_0) \cap \Gamma_{\lambda,\mu} \neq \varnothing$. According to the non-degeneracy Lemma 2.5, we have

$$\frac{m_0}{R} \geqslant \frac{1}{R}\left(\fint_{B_R(X_0)} (m_0 - \psi_{\lambda,\mu})^2 dxdy\right)^{\frac{1}{2}} \geqslant c\Lambda(\lambda^2),$$

which together with (2.19) gives that

$$m_0 \geqslant cR\Lambda(\lambda^2) \geqslant C\lambda = \frac{C\lambda_{cr}}{2} > 0.$$

This leads a contradiction for sufficiently small m_0.

Step 3. In this step, we will show that there exists a $\lambda_\mu \leqslant \Pi_{\lambda_\mu} - 4\tilde{\varepsilon}$, such that $k_{\lambda_\mu,\mu}(b) = 1$.

For any small $m_0 > 0$, with the aid of the results in Step 1, we can define a set

$$\Sigma_\mu = \{\lambda \mid k_{\lambda,\mu}(b) > 1\}.$$

Define

$$\lambda_\mu = \sup_{\lambda \in \Sigma_\mu} \lambda. \tag{3.49}$$

The result in Step 2 gives that

$$\lambda_\mu \leqslant C_0 m_0, \tag{3.50}$$

where C_0 is a constant depending on N, N_0 and ϑ, independent of m_0 and μ. It is easy to check that there exists a $C_0 > 0$ (independent of μ), such that

$$\lambda_\mu \leqslant C_0 m_0 \leqslant \Pi_{\lambda_\mu} - 4\tilde{\varepsilon},$$

for sufficiently small m_0. The continuous dependence of $k_{\lambda,\mu}(b)$ with respect to λ gives that

$$k_{\lambda_\mu,\mu}(b) = 1.$$

If not, the definition of λ_μ implies that $k_{\lambda_\mu,\mu}(b) > 1$. By using the continuous dependence of $k_{\lambda,\mu}(b)$ with respect to λ in Lemma 3.7, there exists a $\lambda \in (\lambda_\mu, \Pi_\lambda - 3\tilde{\varepsilon})$, such that

$$\lambda - \lambda_\mu \text{ is small and } k_{\lambda,\mu}(b) > 1.$$

Therefore, $\lambda \in \Sigma_\mu$, which contradicts to the definition of λ_μ in (3.49). □

4. The existence of subsonic solution to the impinging jet flow problem

To establish the existence of subsonic solution to the impinging jet flow problem, we will take limit $\mu \to \infty$ to the solution $\psi_{\lambda_\mu,\mu}$ of the truncated variational problem $(P_{\lambda_\mu,\mu})$ and show the limit ψ_λ is indeed a solution to the following variational problem (P_λ) stated as follows in the whole domain Ω.

The variational problem (P_λ)

For any bounded domain $D \subset \Omega$, find a $\psi_\lambda \in K$ such that

$$J_D(\psi_\lambda) \leqslant J_D(\psi),$$

for any $\psi \in K_0$ with $\psi = \psi_\lambda$ on ∂D, where $J_D(\psi) = \int_D G(\nabla\psi, \psi, x; \lambda)dxdy$ and

$$K_0 = \left\{\psi \in K \mid \psi \leqslant \min\left\{\frac{m_0}{a^2}x^2, m_0\right\} \ \text{ a.e. in } \ \Omega_\mu\right\}.$$

Theorem 4.1 *If $m_0 > 0$ is sufficiently small, there exist a $\lambda \leqslant \Pi_\lambda - 4\tilde{\varepsilon}$ and a subsonic solution $(\psi_\lambda, \Gamma_\lambda)$ to the compressible impinging jet flow.*

Proof. By virtue of the uniform gradient estimate $|\nabla\psi_{\lambda_{\mu_n},\mu_n}| \leqslant C$ in any compact subset of $\mathbb{R}^2$, it follows from the similar arguments in the proof of Lemma 3.6 that there exist subsequences $\{\psi_{\lambda_{\mu_n},\mu_n}\}$ and $\{\lambda_{\mu_n}\}$ with $\lambda_{\mu_n} \leqslant \Pi_{\lambda_{\mu_n}} - 4\tilde{\varepsilon}$ and $k_{\lambda_{\mu_n},\mu_n}(b) = 1$, such that

$$\lambda_{\mu_n} \to \lambda \leqslant \Pi_\lambda - 4\tilde{\varepsilon},$$

and

$$\psi_{\lambda_{\mu_n},\mu_n} \to \psi_\lambda \text{ weakly in } H^1_{loc}(\mathbb{R}^2) \text{ and uniformly in any compact subset of } \mathbb{R}^2,$$

as $n \to \infty$. Moreover, it follows from (3.50) that

$$\lambda \leqslant C_0 m_0, \quad \text{the constant } C_0 \text{ depends on } N, N_0, \vartheta, \text{ not on } m_0. \tag{4.1}$$

Step 1. ψ_λ is a subsonic solution of the free boundary problem (2.16).

By using the similar arguments in Step 4 in the proof of Lemma 3.6, we can check that ψ_λ is a minimizer to the variational problem (P_λ). Moreover, the monotonicity of $\psi_{\lambda_{\mu_n},\mu_n}(x,y)$ in Lemma 3.1 gives that $\psi_\lambda(x,y)$ is monotonic increasing with respect to y, which implies that the free boundary of ψ_λ is x-graph. Applying the similar arguments in the proof of Lemma 3.3, there exists a continuous function $k_\lambda(x)$, such that the free boundary Γ_λ of the minimizer ψ_λ can be described as

$$\Gamma_\lambda = E \cap \partial\{\psi_\lambda < m_0\} : y = k_\lambda(x) \quad \text{for } x \in (b, +\infty).$$

Furthermore, it follows from the similar arguments in Lemma 3.7 that

$$k_\lambda(x) = \lim_{n\to\infty} k_{\lambda_{\mu_n},\mu_n}(x) \quad \text{for any } x \in [b, +\infty).$$

In particular, one has

$$k_\lambda(b) = 1,$$

which is the continuous fit condition to the axially symmetric compressible subsonic impinging jet flow.

In Theorem 6.1 in [5] and Section 3.11 in [25], Alt, Caffarelli and Friedman obtained that the continuous fit condition implies the smooth fit condition at the detachment point A, namely, $N \cup \Gamma_\lambda$ is C^1 at A. Since ψ_λ is a minimizer to the variational problem (P_λ), it follows from Theorem 6.3 in [5] that the free boundary Γ_λ is $C^{1,\alpha}$, and thus ψ_λ is $C^{1,\alpha}$-smooth up to the free boundary Γ_λ. In view of $\lambda \leqslant \Pi_\lambda - 3\tilde{\varepsilon}$, the subsonic cut-off can be removed near Γ_λ. Then $F(t;\lambda)$ is analytic with respect to t, near the free boundary Γ_λ. Recalling Remark 6.4 in [5], we can conclude that the free boundary Γ_λ is analytic. By using the similar arguments in the proof of Theorem 9.1 in [14], we can conclude that

$$\frac{1}{x}\frac{\partial\psi_\lambda}{\partial\nu} = \left|\frac{\nabla\psi_\lambda}{x}\right| = \lambda \text{ on } \Gamma_\lambda,$$

where ν is outer normal vector to Γ_λ. Utilizing Lemma 6.4 in [6], one has

$$\nabla\psi_\lambda \text{ is uniformly continuous in a } \{\psi_\lambda < m_0\}\text{-neighborhood of } A,$$

and

$$\left|\frac{\nabla\psi_\lambda}{x}\right| = \lambda \text{ at } A.$$

Since ψ_λ is a minimizer to the variational problem (P_λ), applying the results (2) in Lemma 2.2, one has

$$\tilde{Q}_\lambda\psi_\lambda = 0 \ \text{ in } \ \Omega\cap\{\psi_\lambda < m_0\}.$$

Hence, the minimizer ψ_λ is a solution of the truncated free boundary problem (2.16).

Step 2. The asymptotic behavior of ψ_λ will be obtained. For the asymptotic behavior of ψ_λ in upstream, we can use the blow-up arguments in the proof of Lemma 5 in [37], and obtain that

$$\nabla\psi_\lambda(x,y)\to(-v_{in},0),\quad \tilde{\rho}\to\rho_{in}\quad \text{and}\quad \nabla(\partial_x\psi_\lambda,\partial_y\psi_\lambda,\tilde{\rho})\to 0,$$

uniformly in any compact subset of $(0,a)$, as $y\to+\infty$, where $v_{in}=-\frac{2m_0}{\rho_0 a^2}$ and $\rho_{in}=\tilde{\rho}\big(\frac{4m_0^2}{a^4};\lambda^2\big)$.

By virtue of (3.16), there exists a constant $C>0$, such that

$$\int_{\Omega\cap\{x>x_0\}} x\left|\frac{\nabla\psi}{x}-\lambda e\chi_{\{\psi<m_0\}}\right|^2 dxdy \leqslant \frac{C}{x_0^3}, \tag{4.2}$$

for sufficiently large $x_0>b$, where the constant C is independent of x_0.

With the aid of (4.2), by using the similar arguments in the proof of Lemma 3.4, we can show that

$$x\big(k_\lambda(x)-g_0(x)\big)\to\frac{m_0}{\lambda\cos\theta}\quad \text{as } x\to+\infty, \tag{4.3}$$

and

$$\frac{\nabla\psi_\lambda(x,y)}{x}\to(-\lambda\sin\theta,\lambda\cos\theta)\quad \text{for}\quad (x,y)\in\Omega_\mu\cap\{\psi_\lambda<m_0\},\ \text{as } x\to+\infty. \tag{4.4}$$

Step 3. In this step, we will remove the subsonic cut-off for $\tilde{\rho}_\lambda$ in (2.14).

For $\lambda\leqslant\Pi_\lambda-3\tilde{\varepsilon}$, it is easy to check that $\tilde{\rho}\big(\big|\frac{\nabla\psi_\lambda}{x}\big|^2;\lambda^2\big)$ is a monotonic decreasing function of $\big|\frac{\nabla\psi_\lambda}{x}\big|\in(0,\Pi_\lambda)$. Moreover,

$$\frac{|\nabla\psi_\lambda|}{x} \text{ takes the maximum at } X_0 \text{ if and only if } q \text{ takes the maximum at } X_0, \tag{4.5}$$

where $q=\sqrt{u^2+v^2}$ is the speed of the fluid.

By using the similar arguments in Page 114 in [6], one has

$$\mathcal{Q}q^2 = D_i\big(e^{\gamma q^2}a_{ij}(X;\lambda)D_jq^2\big)\geqslant 0\quad \text{in the fluid region } \Omega_0,$$

for some $\gamma>0$, where

$$a_{ij}(X;\lambda)=\tilde{\rho}\big(|\nabla\varphi(X)|^2;\lambda^2\big)\delta_{ij}+2\rho_1\big(|\nabla\varphi(X)|^2;\lambda^2\big)D_i\varphi(X)D_j\varphi(X)$$

and the potential function φ satisfies that $\nabla\varphi = (u, v)$. In view of the maximum principle for q^2 in Ω_0, we have that q^2 cannot take its maximum in Ω_0. Since the flow is assumed to be symmetric with respect to the symmetric axis I, thus I can be regarded as the interior of the fluid field by the even extension of φ. Thus we conclude that the speed q cannot take its maximum at the symmetric axis I. By virtue of (4.5), $\frac{|\nabla\psi_\lambda(X)|}{x}$ takes its maximal value at $N \cup \Gamma_\lambda \cup N_0$ or in the far field. Next, we consider the following three cases for the maximal momentum $\frac{|\nabla\psi_\lambda(X)|}{x}$.

Case 1. $\frac{|\nabla\psi_\lambda(X)|}{x}$ takes its maximum in the far field or on the free boundary Γ_λ. By virtue of (4.1), it follows from the asymptotic behavior and the free boundary condition that

$$\sup_{X\in\bar{\Omega}_0} \frac{|\nabla\psi_\lambda(X)|}{x} = \max\left\{\frac{m_0}{a}, \lambda\right\} \leqslant C_0 m_0, \tag{4.6}$$

where C_0 is a constant depending only on N, N_0 and ϑ.

Case 2. $\frac{|\nabla\psi_\lambda(X)|}{x}$ takes its maximum on walls $N_0 \cap \left\{x \leqslant \frac{a+b}{2}\right\}$ or on $N \cap \left\{x \leqslant \frac{a+b}{2}\right\}$. By using the similar arguments in Section 3 in [37], we have

$$\sup_{X\in\bar{\Omega}_0} \frac{|\nabla\psi_\lambda(X)|}{x} \leqslant C_0 m_0, \tag{4.7}$$

where C_0 is a constant depending only on N, N_0 and ϑ.

Case 3. $\frac{|\nabla\psi_\lambda(X)|}{x}$ takes its maximum at the nozzle wall $N_0 \cap \left\{x \geqslant \frac{a+b}{2}\right\}$ or $N \cap \left\{x \geqslant \frac{a+b}{2}\right\}$. Applying the similar arguments in the proof of Theorem 2.3 and Lemma 3.5, we have

$$\sup_{X\in\bar{\Omega}_0} \frac{|\nabla\psi_\lambda(X)|}{x} \leqslant C\lambda \leqslant C_0 m_0, \tag{4.8}$$

where C_0 is a constant depending only on N, N_0 and ϑ.

It follows from (4.7)–(4.8) that

$$\frac{|\nabla\psi_\lambda(X)|}{x} \leqslant C_0 m_0 \ \text{ in } \Omega_0,$$

which implies that $C_0 m_0 \leqslant \Pi_\lambda - 4\tilde{\varepsilon}$ for sufficiently small $m_0 > 0$. Then the subsonic cut-off can be taken away $\rho(t; \lambda^2) = \tilde{\rho}(t; \lambda^2)$.

Step 4. In this step, the positivity of horizontal velocity will be obtained, namely,

$$\frac{\partial\psi_\lambda}{\partial y} > 0 \ \text{ in } \ \bar{\Omega}_0 \setminus I, \tag{4.9}$$

where $\Omega_0 = \Omega \cap \{\psi_\lambda < m_0\}$.

Set $\omega = \partial_y \psi_\lambda$, which solves

$$\partial_i \left(\frac{\partial_i \omega x^2 \rho - 2\rho_1 \partial_i \psi_\lambda \partial_j \psi_\lambda \partial_j \omega}{x^3 \rho^2} \right) = 0 \quad \text{in } \Omega \cap \{\psi_\lambda < m_0\}.$$

Since $\omega \geqslant 0$ in Ω_0, the strong maximum principle gives that $\omega > 0$ in Ω_0.

Owing to that ψ_λ attains its maximal value m_0 on N, it follows from Hopf's lemma that

$$0 < \frac{\partial \psi_\lambda}{\partial \nu}\big(x, g(x)\big) = \partial_y \psi_\lambda \sqrt{1 + \big(g'(x)\big)^2} = \omega \sqrt{1 + \big(g'(x)\big)^2} \quad \text{on } N \setminus A,$$

where $\nu = \frac{(g'(x),1)}{\sqrt{1+(g'(x))^2}}$ is the outer normal vector to N. Similarly, we can show that

$$\omega > 0 \quad \text{on } N_0 \setminus \big(0, g_0(0)\big).$$

Next, we will show that

$$\omega > 0 \quad \text{on } \Gamma_\lambda. \tag{4.10}$$

Suppose that there exists a free boundary point $X_0 = (x_0, y_0)$, such that $\omega(X_0) = 0$. Without loss of generality, we take $\nu = (0, 1)$ as the outer normal vector of Γ_λ at X_0. Since Γ_λ is analytic at X_0, thanks to Hopf's lemma, one has

$$\partial_{xy}\psi_\lambda = \frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial \nu} < 0 \quad \text{at } X_0. \tag{4.11}$$

On another hand, it follows from $|\nabla\psi_\lambda|^2 = \lambda^2 x^2$ on Γ_λ that

$$0 = \frac{\partial(\lambda^2 x^2)}{\partial s} = \frac{\partial|\nabla\psi_\lambda|^2}{\partial s} = 2\partial_x\psi_\lambda\partial_{xy}\psi_\lambda + 2\partial_y\psi_\lambda\partial_{yy}\psi_\lambda = 2\partial_x\psi_\lambda\partial_{xy}\psi_\lambda,$$

where $s = (0, 1)$ is the tangential vector of Γ_λ at X_0. This contradicts to (4.11).

Recalling that $|g'(b - 0)| < +\infty$, one has

$$\omega = \frac{|\nabla\psi_\lambda|}{\sqrt{1 + \big(g'(x)\big)^2}} = \frac{\lambda x}{\sqrt{1 + \big(g'(x)\big)^2}} > 0 \text{ at } A.$$

Hence, we complete the proof of (4.9). In view of (4.10), the implicit function theorem gives that $k_\lambda(x) \in C^1((b, \infty))$. The analyticity of free boundary Γ_λ gives that $k_\lambda(x)$ is analytic in $(b, +\infty)$. □

4.1 *Uniqueness of the compressible subsonic jet flow*

In this section, we will consider the uniqueness of subsonic solution of the compressible jet flow problem for any given incoming mass flux m_0.

Theorem 4.2 *For any given $m_0 > 0$, suppose that $(\psi_{\lambda_1}, \tilde{\Gamma}_{\lambda_1})$ and $(\tilde{\psi}_{\lambda_2}, \Gamma_{\lambda_2})$ be two subsonic solutions to compressible impinging jet flow problem, respectively. Then $\lambda_1 = \lambda_2$ and $\psi_{\lambda_1} = \tilde{\psi}_{\lambda_2}$.*

Proof. We will divide the proof into two steps.

Step 1. We will show that $\lambda_1 = \lambda_2$. If not, without loss of generality, one may assume that $\lambda_1 > \lambda_2$. In view of the asymptotic behaviors of ψ_{λ_1} and $\tilde{\psi}_{\lambda_2}$ in downstream (see Step 2 in the proof of Theorem 4.1), one has

$$k_{\lambda_1}(x) - g_0(x) \sim \frac{m_0}{\lambda_1 x \cos\theta} \quad \text{and} \quad \tilde{k}_{\lambda_2}(x) - g_0(x) \sim \frac{m_0}{\lambda_2 x \cos\theta}$$

for sufficiently large $x > 0$, which implies

$$k_{\lambda_1}(x) < \tilde{k}_{\lambda_2}(x) \quad \text{for sufficiently large} \quad x > 0. \tag{4.12}$$

Denote $\phi_1 = \frac{m_0-\psi_{\lambda_1}}{\Pi_{\lambda_1}}$ and $\tilde{\phi}_2 = \frac{m_0-\tilde{\psi}_{\lambda_2}}{\Pi_{\lambda_2}}$, where $\Pi_{\lambda_1} = \rho_{\lambda_1,cr}q_{\lambda_1,cr}$ and $\Pi_{\lambda_2} = \rho_{\lambda_2,cr}q_{\lambda_2,cr}$. It is easy to check that

$$\frac{|\nabla\phi_1|^2}{2x^2\rho^2} + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)} \quad \text{in} \quad \Omega \cap \{\phi_1 > 0\}, \tag{4.13}$$

and

$$\frac{|\nabla\tilde{\phi}_2|^2}{2x^2\rho^2} + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)} \quad \text{in} \quad \Omega \cap \{\tilde{\phi}_2 > 0\}, \tag{4.14}$$

where

$$\rho(t) = \frac{\rho\left(\frac{t}{\Pi_{\lambda_1}^2};\lambda_1^2\right)}{\rho_{\lambda_1,cr}} = \frac{\rho\left(\frac{t}{\Pi_{\lambda_2}^2};\lambda_2^2\right)}{\rho_{\lambda_2,cr}}.$$

Moreover, $\rho = \rho\left(\left|\frac{\nabla\phi}{x}\right|^2\right)$ is monotone decreasing with respect to $\left|\frac{\nabla\phi}{x}\right| \in [0,1)$. Thus ϕ_1 and $\tilde{\phi}_2$ satisfy the following quasilinear elliptic equations,

$$Q\phi_1 = \operatorname{div}\left(\frac{\nabla\phi_1}{x\rho(|\frac{\nabla\phi_1}{x}|^2)}\right) = 0 \quad \text{in the fluid} \quad \Omega \cap \{\phi_1 > 0\},$$

and

$$Q\tilde{\phi}_2 = \operatorname{div}\left(\frac{\nabla\tilde{\phi}_2}{x\rho(|\frac{\nabla\tilde{\phi}_2}{x}|^2)}\right) = 0 \quad \text{in the fluid} \quad \Omega \cap \{\tilde{\phi}_2 > 0\},$$

respectively. Denote $\tilde{\phi}_2^{\varepsilon}(x,y) = \tilde{\phi}_2(x, y-\varepsilon)$ for $\varepsilon \geqslant 0$, let $\tilde{\Gamma}_{\lambda_2}^{\varepsilon} : y = \tilde{k}_{\lambda_2}(x) + \varepsilon$ to be the free boundary of $\tilde{\phi}_2^{\varepsilon}$. Choose $\varepsilon_0 \geqslant 0$ to be the smallest one, such that

$$\tilde{\phi}_2^{\varepsilon_0}(X) \geqslant \phi_1(X) \ \text{ in } \Omega, \quad \text{and} \quad \tilde{\phi}_2^{\varepsilon_0}(X_0) = \phi_1(X_0) \quad \text{for some } X_0 \in \overline{\Omega \cap \{\phi_1 > 0\}}.$$

Next, we consider the following two cases for ε_0.

Case 1. $\varepsilon_0 = 0$, then we can choose $X_0 = A$. The strong maximum principle gives that

$$\tilde{\phi}_2(X) > \phi_1(X) \quad \text{and} \quad Q\phi_1 = Q\tilde{\phi}_2 = 0 \quad \text{in } \Omega \cap \{\phi_1 > 0\}.$$

Since $\Gamma_{\lambda_1} \cup N$ and $\tilde{\Gamma}_{\lambda_2} \cup N$ are C^1 at A, one has

$$\frac{\lambda_1}{\Pi_{\lambda_1}} = \frac{1}{b}\frac{\partial\phi_1}{\partial\nu} \leqslant \frac{1}{b}\frac{\partial\tilde{\phi}_2}{\partial\nu} = \frac{\lambda_2}{\Pi_{\lambda_2}} \quad \text{at } A, \tag{4.15}$$

where ν is the inner normal vector of Γ_{λ_1} and $\tilde{\Gamma}_{\lambda_2}$ at A. After a direct computation, one has

$$\frac{d}{d\lambda}\left(\frac{\Pi_\lambda}{\lambda}\right) = \frac{\Pi_\lambda(\lambda^2-\lambda_{cr}^2)}{(\gamma-1)\lambda\rho_0^2\mathcal{B}(\lambda^2)} < 0 \quad \text{for } \lambda < \lambda_{cr},$$

which implies that

$$\frac{\Pi_{\lambda_1}}{\lambda_1} < \frac{\Pi_{\lambda_2}}{\lambda_2} \quad \text{for any} \quad 0 < \lambda_2 < \lambda_1 < \lambda_{cr}. \tag{4.16}$$

This contradicts to (4.15).

Case 2. $\varepsilon_0 > 0$. It follows from strong maximum principle that $X_0 \notin \Omega \cap \{\phi_1 > 0\}$. In fact, if there exists a point $X_0 \in \Omega \cap \{\phi_1 > 0\}$, the continuity of $\phi_1(X)$ and $\tilde{\phi}_2^{\varepsilon_0}(X)$ give that there exists a disc $B_r(X_0) \subset \Omega \cap \{\phi_1 > 0\}$ with $r > 0$, such that

$$Q\phi_1 = Q\tilde{\phi}_2^{\varepsilon_0} = 0 \quad \text{in} \quad B_r(X_0), \quad \text{and} \quad \tilde{\phi}_2^{\varepsilon_0}(X) \geqslant \phi_1(X) \quad \text{in } B_r(X_0).$$

Since $\phi_1(X_0) = \tilde{\phi}_2^{\varepsilon_0}(X_0)$, the strong maximum principle implies that $\phi_1(X) \equiv \tilde{\phi}_2^{\varepsilon_0}(X)$ in $B_r(X_0)$. By using the strong maximum principle again, one has

$$\phi_1(X) \equiv \tilde{\phi}_2^{\varepsilon_0}(X) \quad \text{in} \quad \Omega \cap \{\phi_1 > 0\},$$

which leads a contradiction.

In view of $\varepsilon_0 > 0$, it follows from (3.16) that $|X_0| < +\infty$, and thus $X_0 \in \tilde{\Gamma}_{\lambda_2}^{\varepsilon_0} \cap \Gamma_{\lambda_1}$. Moreover,

$$\tilde{\phi}_2^{\varepsilon_0}(X) > \phi_1(X) \quad \text{and} \quad Q\phi_1 = Q\tilde{\phi}_2^{\varepsilon_0} = 0 \quad \text{in} \quad \Omega \cap \{\phi_1 > 0\}.$$

Since Γ_{λ_1} and $\tilde{\Gamma}_{\lambda_2}^{\varepsilon_0}$ are analytic at X_0, it follows from Hopf's lemma that

$$\frac{\lambda_2}{\Pi_{\lambda_2}} = \frac{1}{x}\frac{\partial \tilde{\phi}_2^{\varepsilon_0}}{\partial \nu} > \frac{1}{x}\frac{\partial \phi_1}{\partial \nu} = \frac{\lambda_1}{\Pi_{\lambda_1}} \quad \text{at } X_0,$$

where ν is the inner normal vector to $\tilde{\Gamma}_{\lambda_2}^{\varepsilon_0}$ and Γ_{λ_1} at X_0, which leads a contradiction to the assumption $\lambda_1 > \lambda_2$, due to (4.16).

Hence, we obtain that $\lambda_1 = \lambda_2$, and denote $\lambda = \lambda_1 = \lambda_2$ in the following.

Step 2. $\psi_\lambda = \tilde{\psi}_\lambda$. Suppose that $\psi_\lambda \neq \tilde{\psi}_\lambda$, without loss of the generality, one may assume that there exists some $x_0 \in (0, +\infty)$, such that

$$k_\lambda(x_0) > \tilde{k}_\lambda(x_0) \quad \text{for some} \quad x_0 > 0. \tag{4.17}$$

Consider a function $\psi_\lambda^\varepsilon(x, y) = \psi_\lambda(x, y - \varepsilon)$ for $\varepsilon \geqslant 0$ and $\Gamma_\lambda^\varepsilon : y = k_\lambda(x) + \varepsilon$ is the free boundary of ψ_λ^ε, choosing the smallest $\varepsilon_0 \geqslant 0$ such that

$$\psi_\lambda^{\varepsilon_0}(X) \leqslant \tilde{\psi}_\lambda(X) \quad \text{in } \Omega, \quad \text{and} \quad \psi_\lambda^{\varepsilon_0}(X_0) = \tilde{\psi}_\lambda(X_0) \quad \text{for some} \quad X_0 \in \bar{\Omega}.$$

It follows from (4.17) that $\varepsilon_0 > 0$, which together with the strong maximum principle and the asymptotic behavior imply that $X_0 \notin \Omega \cap \{\tilde{\psi}_\lambda < m_0\}$ and $X_0 \in \Gamma_\lambda^{\varepsilon_0} \cap \tilde{\Gamma}_\lambda$ with $|X_0| < +\infty$. Then we have

$$\psi_\lambda^{\varepsilon_0}(X) < \tilde{\psi}_\lambda(X) \quad \text{and} \quad Q_\lambda \psi_\lambda^{\varepsilon_0} = Q_\lambda \tilde{\psi}_\lambda = 0 \quad \text{in} \quad \Omega \cap \{\tilde{\psi}_\lambda < m_0\}.$$

Thanks to the Hopf's lemma, one has

$$\lambda = \frac{1}{x}\frac{\partial \psi_\lambda^{\varepsilon_0}}{\partial \nu} > \frac{1}{x}\frac{\partial \tilde{\psi}_\lambda}{\partial \nu} = \lambda \quad \text{at } X_0,$$

where ν is the outer normal vector of $\Gamma_\lambda^{\varepsilon_0} \cap \tilde{\Gamma}_\lambda$ at X_0, which leads a contradiction. □

5. The existence of the critical mass flux

For any sufficiently small $m_0 > 0$, we have shown that there exist a unique $\lambda \leqslant \Pi_\lambda - 4\tilde{\varepsilon}$ and a unique solution $(u, v, \rho, \Gamma_\lambda)$ to the free boundary problem in previous section. One key point is that the smallness of m_0 guarantee the global subsonicity of the compressible jet flow. In this section, we will increase m_0 as large as possible, and obtain the critical upper bound of the incoming mass flux m_0.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence with $\varepsilon_n \downarrow 0$. Denote $\psi^n_{\lambda,m}(x, y)$ as the solution of the following free boundary value problem

$$\begin{cases} \nabla \cdot \left(\frac{\nabla \psi}{x\rho^n(|\frac{\nabla\psi}{x}|^2;\lambda^2)} \right) = 0 \ \text{ in } \Omega \cap \{\psi < m\}, \\ \psi = 0 \text{ on } T, \ \psi = m \text{ on } N \cup \Gamma^n_{\lambda,m}, \\ \frac{1}{x}\frac{\partial\psi}{\partial\nu} = \lambda \ \text{ on } \ \Gamma^n_{\lambda,m}, \end{cases} \tag{5.1}$$

for any sufficiently small $m > 0$ and the free boundary $\Gamma^n_{\lambda,m} : y = k^n_{\lambda,m}(x)$ satisfies the continuous fit condition $k^n_{\lambda,m}(b) = 1$, where ν is the outer normal vector. Here, $\rho^n(t; \lambda^2)$ is a smooth function satisfying

$$\rho^n(t;\lambda^2) = \begin{cases} \rho(t;\lambda^2) & \text{if } 0 \leqslant t \leqslant (\Pi_\lambda - 2\varepsilon_n)^2, \\ \rho\left((\Pi_\lambda - \varepsilon_n)^2;\lambda^2\right) & \text{if } t \geqslant (\Pi_\lambda - \varepsilon_n)^2, \end{cases}$$

and $\rho^n(t;\lambda) - 2\rho^n_1(t;\lambda)t < \gamma_n < +\infty$ with some constant $\gamma_n > 0$.

First, we define a set

$$\mathcal{K}_n(m) = \left\{\psi^n_{\lambda,m} \mid \psi^n_{\lambda,m} \text{ is a subsonic solution to free boundary problem (5.1)}\right\},$$

for any small $\varepsilon_n > 0$ and $m > 0$. For any small $m > 0$ and $\varepsilon_n > 0$, it follows from Theorem (4.1) that there exist a $\lambda = \lambda(m)$ and a unique subsonic solution $\psi^n_{\lambda(m),m}$ to the free boundary problem with

$$\lambda(m) \leqslant \Pi_{\lambda(m)} - 4\varepsilon_n \quad \text{and} \quad \sup_{X\in\Omega\cap\{0<\psi^n_{\lambda(m),m}<m\}} \left|\frac{\nabla\psi^n_{\lambda(m),m}}{x}\right|^2 - \Pi_{\lambda(m)} \leqslant -4\varepsilon_n.$$

Thus $m \in \mathcal{K}_n(m)$ for small $m > 0$ and the set $\mathcal{K}_n(m)$ is not empty. The uniqueness of subsonic solution and λ are established in Section 4. Denote $\lambda = \lambda(m)$ and $\psi^n_{\lambda(m),m}$ as the unique subsonic solution to the free boundary problem (5.1) with

$$\lambda(m) \leqslant \Pi_{\lambda(m)} - 4\varepsilon_n \quad \text{and} \quad \sup_{X\in\Omega\cap\{0<\psi^n_{\lambda(m),m}<m\}} \left|\frac{\nabla\psi^n_{\lambda(m),m}}{x}\right| - \Pi_{\lambda(m)} \leqslant -4\varepsilon_n.$$

Denote

$$T_n(m) = \inf_{\psi^n_{\lambda,m}\in\mathcal{K}_n(m)} \left(\sup_{(x,y)\in\Omega^n_{\lambda,m}} \left|\frac{\nabla\psi^n_{\lambda,m}}{x}\right| - \Pi_\lambda \right), \tag{5.2}$$

where $\Omega^n_{\lambda,m} = \Omega \cap \{0 < \psi^n_{\lambda,m} < m\}$.

Define a set

$$\Sigma_n = \{m \mid \text{for any } \tau \in (0,m), \text{ there exists a subsonic solution } \psi^n_{\lambda,m} \text{ to the free boundary problem with } T_n(\tau) \leqslant -4\varepsilon_n\}.$$

Along the above arguments, we have $m \in \Sigma_n$ for small $m > 0$, which implies that the set Σ_n is non-empty. Obviously, $\Sigma_n \subset \Sigma_{n+1}$.

Set

$$m_n = \sup_{m\in\Sigma_n} m. \tag{5.3}$$

The definition of m_n implies that m_n is monotone increasing with respect to n. For $m > 0$, it is easy to check that

$$m = \psi^n_{\lambda,m}(B) - \psi^n_{\lambda,m}\big(b, g_0(b)\big) \leqslant \sup_{X\in\Omega^n_{\lambda,m}} |\nabla\psi^n_{\lambda,m}(X)||1-g_0(b)| \leqslant \lambda_{cr} b|1-g_0(b)|. \tag{5.4}$$

With the aid of (5.4), we can define

$$m_{cr} = \lim_{n\to+\infty} m_n. \tag{5.5}$$

Lemma 5.1 *For any $m \in (0, m_n]$, $T_n(m)$ is left-continuous with respect to m, namely, $T_n(m) = \lim_{\tau\to m^-} T_n(\tau)$.*

Proof. For any $m \in (0, m_n]$, there exists a sequence $\{\tau_k\}$ with $\tau_k \uparrow m$. The definition of Σ_n gives that there exists a subsonic solution ψ^n_{λ,τ_k} to the free boundary problem, which satisfies that

$$T_n(\tau_k) = \inf_{\psi^n_{\lambda,\tau_k}\in\mathcal{K}_n(\tau_k)} \left(\sup_{X\in\Omega^n_{\lambda,\tau_k}} \left| \frac{\nabla\psi^n_{\lambda,\tau_k}(X)}{x} \right| - \Pi_\lambda \right) \leqslant -4\varepsilon_n.$$

By using the uniqueness result in Section 4, we have that $\lambda = \lambda(\tau_k)$ and solution $\psi^n_{\lambda,\tau_k} = \psi^n_{\lambda(\tau_k),\tau_k}$ is the unique subsonic solution to the free boundary problem (5.1). Then one has

$$T_n(\tau_k) = \sup_{X\in\Omega^n_{\lambda(\tau_k),\tau_k}} \left| \frac{\nabla\psi^n_{\lambda(\tau_k),\tau_k}(X)}{x} \right| - \Pi_{\lambda(\tau_k)} \leqslant -4\varepsilon_n. \tag{5.6}$$

By using the similar arguments in the proof of Lemma 2.5, we can take a subsequence $\{\tau_k\}$, such that

$$\lambda(\tau_k) \to \lambda_0 \leqslant \Pi_{\lambda_0} - 4\varepsilon_n,$$

and

$$\psi^n_{\lambda(\tau_k),\tau_k} \to \psi^n_{\lambda_0,m} \text{ weakly in } H^1_{loc}(\Omega) \text{ and uniformly in any compact subset of } \mathbb{R}^2,$$

as $k \to +\infty$. Moreover, the inequality (5.6) gives that

$$\lim_{\tau_k\to m} T_n(\tau_k) = \sup_{X\in\Omega^n_{\lambda_0,m}} \left| \frac{\nabla\psi^n_{\lambda_0,m}(X)}{x} \right| - \Pi_{\lambda_0} \leqslant -4\varepsilon_n.$$

Thus $\psi^n_{\lambda_0,m}$ is a subsonic solution to the free boundary problem (5.1). Applying the uniqueness results in Section 4, we conclude that $\lambda_0 = \lambda(m)$ and $\psi^n_{\lambda_0,m} = \psi^n_{\lambda(m),m} \in \mathcal{K}_n(m)$. It follows from the definition of $T_n(m)$ in (5.2) and the uniqueness result in Section 4 that

$$T_n(m) = \lim_{\tau_k \to m^-} T_n(\tau_k). \qquad \square$$

Lemma 5.2 *There exists a critical mass flux $m_{cr} > 0$, such that for any $m \in (0, m_{cr})$, there exist a unique $\lambda = \lambda(m) < \lambda_{cr}$ and a unique subsonic solution $\psi_{\lambda,m}$ to the free boundary problem (5.1), such that*

$$T(m) = \sup_{X \in \Omega_{\lambda,m}} \left| \frac{\nabla \psi_{\lambda,m}(X)}{x} \right| - \Pi_\lambda < 0, \tag{5.7}$$

where $\Omega_{\lambda,m} = \Omega \cap \{0 < \psi_{\lambda,m} < m\}$. And m_{cr} is the upper critical mass flux for the existence of subsonic solution in the following sense: either

$$T(m) \to 0 \quad as \quad m \to m_{cr}, \tag{5.8}$$

or there is no $\sigma > 0$, such that for any $m \in (m_{cr}, m_{cr} + \sigma)$, there exist a $\lambda < \lambda_{cr}$ and a subsonic solution $\psi_{\lambda,m}$ to the free boundary problem (5.1), and

$$\sup_{m \in (m_{cr}, m_{cr}+\sigma)} T(m) < 0. \tag{5.9}$$

Proof. For any $m \in (0, m_{cr})$, the definition of m_{cr} in (5.5) gives that there exists a N, such that $m < m_n$ for any $n > N$. Therefore, it follows from the definition of m_n that we have

$$T_n(m) = \inf_{\psi^n_{\lambda,m} \in \mathcal{K}_n(m)} \left(\sup_{X \in \Omega_{\lambda,m}} \left| \frac{\nabla \psi^n_{\lambda,m}(X)}{x} \right| - \Pi_\lambda \right) \leqslant -4\varepsilon_n \ \text{ for } \ n > N.$$

By virtue of Theorem 4.1, we can conclude that there exist a unique $\lambda(m) \leqslant \Pi_{\lambda(m)} - 4\varepsilon_n$ and a unique subsonic solution $\psi^n_{\lambda(m),m}$ to the free boundary problem (5.1), such that

$$T_n(m) = \sup_{X \in \Omega_{\lambda(m),m}} \left| \frac{\nabla \psi^n_{\lambda(m),m}(X)}{x} \right| - \Pi_{\lambda(m)} \leqslant -4\varepsilon_n.$$

Taking $\psi_{\lambda,m} = \psi^n_{\lambda(m),m}$, then $\psi_{\lambda,m}$ is the unique subsonic solution to the compressible impinging jet flow problem (5.1) and $T(m) = T_n(m) \leqslant -4\varepsilon_n < 0$.

If $\sup_{m \in (0,m_{cr})} T(m) < 0$, there exists a large N, such that

$$\sup_{m \in (0,m_{cr})} T(m) < -4\varepsilon_n \tag{5.10}$$

for any $n > N$. It is easy to check that $m_{cr} \in \Sigma_n$, and thus $m_{cr} \leqslant m_n$ for any $n > N$.

It follows from Lemma 5.1 that $T_n(m)$ is left-continuous for $m \in \big(0, m_n\big]$, and thus

$$T(m_{cr}) = T_n(m_{cr}) \leqslant -4\varepsilon_n. \tag{5.11}$$

Suppose that there exists a $\sigma > 0$, such that for any $m \in (m_{cr}, m_{cr} + \sigma)$, there exists a subsonic solution $\psi_{\lambda,m}$ with $\lambda < \lambda_{cr}$ to the free boundary problem (5.1) and

$$\sup_{m \in (m_{cr}, m_{cr}+\sigma)} T(m) = \sup_{m \in (m_{cr}, m_{cr}+\sigma)} \left(\sup_{X \in \Omega_{\lambda,m}} \left| \frac{\nabla \psi_{\lambda,m}(X)}{x} \right| - \Pi_\lambda \right) < 0. \tag{5.12}$$

In view of (5.12), there exists a large $k > 0$, such that

$$\sup_{m\in(m_{cr},m_{cr}+\sigma)} T(m) = \sup_{m\in(m_{cr},m_{cr}+\sigma)} \left(\sup_{X\in\Omega_{\lambda,m}} \left| \frac{\nabla\psi_{\lambda,m}(X)}{x} \right| - \Pi_\lambda \right) \leqslant -4\varepsilon_{n+k}. \tag{5.13}$$

By virtue of (5.10), (5.11) and (5.13), one has

$$T_{n+k}(m_{cr}+\sigma) = T(m_{cr}+\sigma) \leqslant \sup_{m\in(0,m_{cr}+\sigma)} T(m) \leqslant 4\varepsilon_{n+k},$$

for any $n > N$, which implies that $m_{cr}+\sigma \in \Sigma_{n+k}$. The definition of m_{n+k} in (5.3) gives that

$$m_{n+k} \geqslant m_{cr}+\sigma > m_{cr}.$$

This leads a contradiction to the definition of m_{cr} in (5.5). □

6. The proof of the main results

Based on the previous sections, we will complete the proof of Theorem 1.3 and Theorem 1.4 in this section.

Proof of Theorem 1.3. For any given atmosphere pressure $p_{atm} > 0$, it follows from Lemma 5.2 that there exists a critical mass flux $M_{cr} > 0$, such that for any $M_0 \in (0, M_{cr})$, there exist a unique $\lambda = \lambda(m_0) < \lambda_{cr}$ and a unique subsonic solution $(\psi_{\lambda,m_0}, \Gamma_{\lambda,m_0})$ to the axially symmetric compressible impinging flow, where

$$M_{cr} = 2\pi m_{cr} \quad \text{and} \quad M_0 = 2\pi m_0.$$

In view of the proof of Theorem 4.1, we conclude that ψ_{λ,m_0} and the free boundary $\Gamma_{\lambda,m_0} : y = k_{\lambda,m_0}(x)$ satisfy that

$$\psi_{\lambda,m_0} \in C^{2,\alpha}(\Omega_0) \cap C^1(\bar{\Omega}_0),\ k_{\lambda,m_0}(x) \in C^1([b,+\infty)),$$

and

$$k_{\lambda,m_0}(b+0) = 1, \quad k'_{\lambda,m_0}(b+0) = g'(b-0),\ k'_{\lambda,m_0}(x) \to \tan\theta, \quad k_{\lambda,m_0}(x) - g_0(x) \to 0$$

as $x \to +\infty$. Moreover, $\sup_{(x,y)\in\bar{\Omega}_0} \frac{|\nabla\psi_{\lambda,m_0}|}{x\Pi_\lambda} < 1$ and $p = p_{atm}$ on Γ_{λ,m_0}.

By virtue of the Bernoulli's law (1.11), one has

$$\frac{M_0^2}{2\pi a^4 \rho_{in}^2} + \frac{\mathcal{A}\gamma}{\gamma-1}\rho_{in}^{\gamma-1} = \frac{\lambda^2}{2\rho_0^2} + \frac{\mathcal{A}\gamma}{\gamma-1}\rho_0^{\gamma-1}, \quad \rho_0 = \left(\frac{p_{atm}}{\mathcal{A}}\right)^{\frac{1}{\gamma}},$$

which implies that the incoming pressure $p_{in} = \mathcal{A}\rho_{in}^\gamma$ is determined uniquely by λ. Moreover, the subsonicity of solution ψ_{λ,m_0} gives that $p_{in} \in (p_1, p_2)$, where p_1 and p_2 satisfy that $p_1 < p_2$ and

$$\frac{M_0^2}{2\pi^2 a^4 \left(\frac{p_1}{\mathcal{A}}\right)^{\frac{2}{\gamma}}} + \frac{\mathcal{A}\gamma}{\gamma-1}\left(\frac{p_1}{\mathcal{A}}\right)^{\frac{\gamma-1}{\gamma}} = \frac{\mathcal{A}\gamma}{\gamma-1}\left(\frac{p_{atm}}{\mathcal{A}}\right)^{\frac{\gamma-1}{\gamma}}$$

and

$$\frac{M_0^2}{2\pi^2 a^4 \left(\frac{p_2}{\mathcal{A}}\right)^{\frac{2}{\gamma}}} + \frac{\mathcal{A}\gamma}{\gamma-1}\left(\frac{p_2}{\mathcal{A}}\right)^{\frac{\gamma-1}{\gamma}} = \frac{\mathcal{A}\gamma(\gamma+1)}{2(\gamma-1)}\left(\frac{p_{atm}}{\mathcal{A}}\right)^{\frac{\gamma-1}{\gamma}}.$$

Denote

$$u = \frac{1}{x\rho(\frac{|\nabla\psi_{\lambda,m_0}|^2}{|x|^2};\lambda)}\frac{\partial\psi_{\lambda,m_0}}{\partial y}, v = -\frac{1}{x\rho(\frac{|\nabla\psi_{\lambda,m_0}|^2}{|x|^2};\lambda)}\frac{\partial\psi_{\lambda,m_0}}{\partial x}, \quad \Gamma = \Omega\cap\{\psi_{\lambda,m_0} < m_0\}$$

and the density ρ is determined uniquely by

$$\frac{|\nabla\psi_{\lambda,m_0}|^2}{2x^2\rho^2} + \frac{\mathcal{A}\gamma}{\gamma-1}\rho^{\gamma-1} = \frac{\lambda^2}{2\rho_0^2} + \frac{\mathcal{A}\gamma}{\gamma-1}\rho_0^{\gamma-1}.$$

Thus, (u, v, ρ, Γ) satisfies the conditions (1)–(3) in Definition 1.1, and (u, v, ρ, Γ) is the unique subsonic solution to FBP.

The statements (1)–(3) of Theorem 1.3 follows from the Step 2 and Step 4 in the proof of Theorem 4.1 directly. The final statement (4) of Theorem 1.3 is proved in Section 5.

Hence, we complete the proof of Theorem 1.3. □

Proof of Theorem 1.4. By virtue of Theorem 1.3, the subsonic solution $(\psi_{\lambda,m_0}, \Gamma_{\lambda,m_0})$ established in Theorem 1.3 satisfies that

$$\nabla\psi_{\lambda,m_0} \text{ is uniformly continuous in a } \{\psi_{\lambda,m_0} < m_0\}\text{-neighborhood of } A, \tag{6.1}$$

and

$$\Gamma_{\lambda,m_0} \cup N \text{ is } C^1 \text{ at } A. \tag{6.2}$$

Under the assumption that N is $C^{3,\alpha}$ near A, with the aid of (6.1) and (6.2), the proof of Theorem 1.4 follows from Theorem 1.1 in [7] directly. □

Acknowledgments. The authors would like to acknowledge the two anonymous reviewers for improving this paper with their comments. Cheng and Du were partially supported by the National Natural Science Foundation of China grants 11971331 and 12001387, and Sichuan Youth Science and Technology Foundation 2021JDTD0024. Zhang was supported by NSFC grant 12001071.

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