Interfaces Free Bound. 23 (2021), 323–351
DOI 10.4171/IFB/456

Coarse graining and large-N behavior of the d-dimensional N-clock model

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[Received 6 April 2020 and in revised form 19 December 2020]

We study the asymptotic behavior of the N-clock model, a nearest neighbors ferromagnetic spin model on the d-dimensional cubic ε-lattice in which the spin field is constrained to take values in a discretization $\mathcal{S}_N$ of the unit circle $\mathbb{S}^1$ consisting of N equispaced points. Our Γ-convergence analysis consists of two steps: we first fix N and let the lattice spacing $\varepsilon \to 0$, obtaining an interface energy in the continuum defined on piecewise constant spin fields with values in $\mathcal{S}_N$; at a second stage, we let $N \to +\infty$. The final result of this two-step limit process is an anisotropic total variation of $\mathbb{S}^1$-valued vector fields of bounded variation.

2020 Mathematics Subject Classification: 49J45, 26B30, 82B20.

Keywords: Γ-convergence, XY model, N-clock model, vector fields of bounded variation with values in the unit circle.

Contents

1. Introduction

In this paper we are interested in the variational analysis of the N-clock model (also known as planar Potts model or $\mathbb{Z}_N$-model) in the d-dimensional setting. The N-clock model is a nearest neighbors ferromagnetic spin model on the cubic lattice in which the spin field is constrained to take values in a set of N equispaced points of the unit circle $\mathbb{S}^1$. It plays a fundamental role in understanding phase transition phenomena in the theory of classical ferromagnetic spin fields, as it is closely related to the XY (planar rotator) model, for which the spin field is allowed to attain all the values of $\mathbb{S}^1$. In fact, the N-clock model is considered as an approximation of the XY model, as for N large enough it predicts Berezinskii–Kosterlitz–Thouless transitions [30], i.e., phase transitions mediated by the formation and interaction of topological singularities, the so-called vortices [15, 31, 32].

With the aim of describing the relation between the N-clock model and the XY model, probabilistic methods have been used in [33, 39], while a variational analysis at zero temperature has been only very recently carried out in [26, 27]. There the authors study the effective behavior of (suitably rescaled versions of) the energy of the N-clock model on the 2-dimensional square lattice $\varepsilon\mathbb{Z}^2$, examining the case when the number $N = N_\varepsilon$ of equi-spaced points on $\mathbb{S}^1$ depends on ε and diverges as $\varepsilon \to 0$. The coarse grained model, which describes the microscopic/mesoscopic geometry of the spin field, is strongly affected by the rate of divergence of $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

In this paper we advance the variational analysis of the N-clock model by considering the model on a d-dimensional cubic lattice $\varepsilon\mathbb{Z}^d$, with $d \geqslant 2$, in the case where the number N is fixed and independent of ε. We shall first identify the limit of the N-clock model as $\varepsilon \to 0$ keeping N fixed and, at a second stage, we will let $N \to +\infty$. In contrast to the energy of the XY model, the energy resulting from this two-step limit process is by nature unfit to describe the concentration of energy around vortex-like singularities, indicating that the dependence of N on ε seems inevitable with the intent to approximate the XY model at zero temperature. To the best of our knowledge, the explicit identification of the limit energies in the $\varepsilon \to 0$ and $N \to \infty$ regimes and in any dimension makes the result contained in this paper the first quantitative answer to the question whether the N-clock model approximates the XY model at zero temperature. We shall see that the result is rather analogous to the limiting energy of the N_ε-clock model in a specific rate of divergence $N_\varepsilon \to +\infty$, chosen among those examined in the two-dimensional setting in [27]. To present in detail the results in this paper, we first recall the Γ-convergence result on the d-dimensional XY-model. Given a bounded open set $\Omega \subset \mathbb{R}^d$, the energy associated to a spin field $u : \varepsilon\mathbb{Z}^d \cap \Omega \to \mathbb{S}^1$ is given by

$$XY_\varepsilon(u) = \frac{1}{2} \sum_{\langle i,j \rangle \text{ in } \Omega} \varepsilon^d |u(\varepsilon i) - u(\varepsilon j)|^2,$$

where the sum is taken over ordered pairs of nearest neighbors $\langle i, j \rangle$, i.e., $(i, j) \in \mathbb{Z}^d \times \mathbb{Z}^d$ such that $|i - j| = 1$ and $\varepsilon i, \varepsilon j \in \Omega$. As observed in [5] the relevant scaling of the energy is $\varepsilon^2 |\log \varepsilon|$ in the sense that a bound of the type $XY_\varepsilon(u_\varepsilon) \leqslant C\varepsilon^2|\log \varepsilon|$ implies compactness of the Jacobians $J\hat{u}_\varepsilon$ of the continuous piecewise affine interpolations $\hat{u}_\varepsilon$ of u_ε on the cells of the lattice. More precisely, the Jacobians, seen as $(d - 2)$ dimensional currents concentrate on rectifiable sets of codimension 2. For $d = 2, 3$ the latter are the so-called vortices and vortex lines of the spin field, respectively. For $d = 2$ the Γ-limit of XY_ε at the logarithmic scaling is, with a slight abuse of notation, given by

$$\Gamma\text{-}\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2 |\log \varepsilon|} XY_\varepsilon(u) = 2\pi |\mu|(\Omega),$$

where $\mu = \sum_{i=1}^N d_i \delta_{x_i}$ is the measure encoding the location $x_i \in \Omega$ and the multiplicity $d_i \in \mathbb{Z}$ of the vortex-like singularities (cf. [4–6, 10, 13, 14, 20, 25, 36] in the discrete setting and [1, 11, 17, 38] in the continuum setting for more details).

Next we summarize the analysis of [27] (valid only in dimension $d = 2$), starting with some notation. Given $N \in \mathbb{N}$, we consider the set of N equispaced points on the unit circle

$$\mathcal{S}_N := \big\{\exp\big(\iota \tfrac{2\pi}{N} k\big) \ : \ k = 0, \dots, N-1\big\},$$

where ι is the imaginary unit. Given a bounded, open set $\Omega \subset \mathbb{R}^2$, the energy of a spin field $u \colon \varepsilon\mathbb{Z}^2 \to \mathcal{S}_N$ is given by

$$E_\varepsilon^N(u) := \frac{1}{2} \sum_{\langle i,j\rangle \text{ in } \Omega} \varepsilon^2 |u(\varepsilon i) - u(\varepsilon j)|^2,$$

We recall that a wide range of phenomena has been observed in [26, 27] when exploring the possible regimes of $N = N_\varepsilon$. Here we outline the one pertaining to the discussion in the present paper, namely $N = N_\varepsilon \ll \frac{1}{\varepsilon|\log\varepsilon|}$. The relevant scaling of the energy in this regime is $\frac{N_\varepsilon}{2\pi\varepsilon} E_\varepsilon^{N_\varepsilon}$. Sequences of spin fields u_ε with equibounded energy accumulate to vector fields in $BV(\Omega;\mathbb{S}^1)$, and the scaled energy $\frac{N_\varepsilon}{2\pi\varepsilon} E_\varepsilon^{N_\varepsilon}$ approximates an anisotropic total variation for maps in $BV(\Omega;\mathbb{S}^1)$.

In the next theorem we state the result in the regime $N_\varepsilon \ll \frac{1}{\varepsilon|\log\varepsilon|}$ rigorously. We denote by $|\cdot|_1$ the 1-norm on vectors, by $|\cdot|_{2,1}$ the anisotropic norm on matrices given by the sum of the Euclidean norms of the columns, and by $\mathrm{d}_{\mathbb{S}^1}$ the geodesic distance on $\mathbb{S}^1$. For the notation concerning functions of bounded variation we refer to Section 2.1.

Theorem 1.1 ([27]) *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open set with Lipschitz boundary. Assume that $N_\varepsilon \ll \frac{1}{\varepsilon|\log\varepsilon|}$. Then the following results hold true:*

(i) *(Compactness) Let $u_\varepsilon \colon \Omega \cap \varepsilon\mathbb{Z}^2 \to \mathcal{S}_{N_\varepsilon}$ be such that $\frac{N_\varepsilon}{2\pi\varepsilon} E_\varepsilon^{N_\varepsilon}(u_\varepsilon) \leqslant C$. Then there exists a subsequence (not relabeled) and a function $u \in BV(\Omega;\mathbb{S}^1)$ such that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$.*

(ii) *(Γ-liminf inequality) Assume that $u_\varepsilon \colon \Omega \cap \varepsilon\mathbb{Z}^2 \to \mathcal{S}_{N_\varepsilon}$ and $u \in BV(\Omega;\mathbb{S}^1)$ satisfy $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$. Then*

$$\liminf_{\varepsilon\to 0} \frac{N_\varepsilon}{2\pi\varepsilon} E_\varepsilon^{N_\varepsilon}(u_\varepsilon) \geqslant \int_\Omega |\nabla u|_{2,1}\,\mathrm{d}x + |\mathrm{D}^{(c)} u|_{2,1}(\Omega) + \int_{\Omega\cap J_u} \mathrm{d}_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1\,\mathrm{d}\mathcal{H}^1.$$

(iii) *(Γ-limsup inequality) Let $u \in BV(\Omega;\mathbb{S}^1)$. Then there exists a sequence $u_\varepsilon \colon \Omega \cap \varepsilon\mathbb{Z}^2 \to \mathcal{S}_{N_\varepsilon}$ such that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$ and*

$$\limsup_{\varepsilon\to 0} \frac{N_\varepsilon}{2\pi\varepsilon} E_\varepsilon^{N_\varepsilon}(u_\varepsilon) \leqslant \int_\Omega |\nabla u|_{2,1}\,\mathrm{d}x + |\mathrm{D}^{(c)} u|_{2,1}(\Omega) + \int_{\Omega\cap J_u} \mathrm{d}_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1\,\mathrm{d}\mathcal{H}^1.$$

We are now in a position to present the two main results in this paper. We shall consider $\Omega \subset \mathbb{R}^d$ a bounded, open set with Lipschitz boundary and the energy defined for admissible spin fields on the d-dimensional cubic lattice $u \colon \Omega \cap \varepsilon\mathbb{Z}^d \to \mathcal{S}_N$ by

$$E_\varepsilon^N(u) := \frac{1}{2} \sum_{\langle i,j\rangle \text{ in } \Omega} \varepsilon^d |u(\varepsilon i) - u(\varepsilon j)|^2,$$

where the sum is taken over ordered pairs of nearest neighbors $\langle i,j\rangle$, i.e., $(i,j) \in \mathbb{Z}^d \times \mathbb{Z}^d$ such that $|i-j| = 1$ and $\varepsilon i, \varepsilon j \in \Omega$ (the factor $\frac{1}{2}$ accounts for the fact that each pair is counted twice).

We state the first result concerning the limit of E_ε^N as $\varepsilon \to 0$. For N fixed, the physical system is expected to behave like a classical Ising-type system with N phases. (See also [3, 7, 8, 12, 18, 21–24, 28, 29] for the analysis of spin systems in the surface scaling.) According to the results proven for the Ising system, we expect the limit energy to be finite on functions of bounded variation with values in the finite set $\mathcal{S}_N$. In the next theorem we identify precisely the surface energy concentrated on the interfaces between the phases of the spin field. We denote by $\theta_N := \frac{2\pi}{N}$ the smallest angle between two different vectors in $\mathcal{S}_N$.

Theorem 1.2 (Limit as $\varepsilon \to 0$) *Let $\Omega \subset \mathbb{R}^d$ be a bounded, open set with Lipschitz boundary. Let $N \geqslant 2$ and $\theta_N := 2\pi/N$. Then the following results hold true:*

(i) *(Compactness) Let $u_\varepsilon\colon \Omega \cap \varepsilon\mathbb{Z}^d \to \mathcal{S}_N$ be such that $\frac{N}{2\pi\varepsilon} E_\varepsilon^N(u_\varepsilon) \leqslant C$. Then there exists a subsequence (not relabeled) and a function $u \in BV(\Omega;\mathcal{S}_N)$ such that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$ as $\varepsilon \to 0$.*

(ii) *(Γ-liminf inequality) Assume that $u_\varepsilon\colon \Omega \cap \varepsilon\mathbb{Z}^d \to \mathcal{S}_N$ and $u \in BV(\Omega;\mathcal{S}_N)$ satisfy $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$ as $\varepsilon \to 0$. Then*

$$\liminf_{\varepsilon\to 0} \frac{N}{2\pi\varepsilon} E_\varepsilon^N(u_\varepsilon) \geqslant \frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N^2} \int_{\Omega\cap J_u} d_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1 \,d\mathcal{H}^{d-1}.$$

(iii) *(Γ-limsup inequality) Let $u \in BV(\Omega;\mathcal{S}_N)$. Then there exists a sequence $u_\varepsilon\colon \Omega \cap \varepsilon\mathbb{Z}^d \to \mathcal{S}_N$ such that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$ as $\varepsilon \to 0$ and*

$$\limsup_{\varepsilon\to 0} \frac{N}{2\pi\varepsilon} E_\varepsilon^N(u_\varepsilon) \leqslant \frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N^2} \int_{\Omega\cap J_u} d_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1 \,d\mathcal{H}^{d-1}.$$

To clarify the expression of the limit functional in Theorem 1.2, we sketch here the proof of the Γ-limsup inequality in a very simple setting. Assume that Ω is the unit cube $Q = (-1/2,1/2)^d$ and u is the pure-jump function with constant value $u^- = (1,0)$ in $Q^- = (-1/2,1/2)^{d-1}\times(-1/2,0)$ and constant value $u^+ = \exp(\iota k^+\theta_N)$ in $Q^+ = (-1/2,1/2)^{d-1}\times(0,1/2)$, where $k^+ \in \mathbb{N}$ is such that $0 \leqslant k^+\theta_N \leqslant \pi$. In this case, the jump set is given by $J_u = (-1/2,1/2)^{d-1}\times\{0\}$. Then u_ε is constructed by rotating k^+ times of an angle θ_N starting from u^- up to u^+ on hyperplanes parallel to the jump set, cf. Figure 1. More precisely, for $0 \leqslant k \leqslant k^+$ we define

$$u_\varepsilon(\varepsilon i) := \exp(\iota k\theta_N) \quad \text{if} \quad \varepsilon i \cdot e_d = k\varepsilon$$

and we put $u_\varepsilon(\varepsilon i) = (1,0)$ if $\varepsilon i \cdot e_d < 0$ and $u_\varepsilon(\varepsilon i) = \exp(\iota k^+\theta_N)$ if $\varepsilon i \cdot e_d > k^+\varepsilon$, instead. Between two hyperplanes there are $\frac{1}{\varepsilon^{d-1}}$ interacting pairs of nearest neighbors. For two such points $\varepsilon i, \varepsilon j$ we have by a simple geometric argument $|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| = 2\sin(\frac{\theta_N}{2})$. Summing over all interactions we conclude that

$$\begin{aligned}\frac{N}{2\pi\varepsilon} E_\varepsilon^N(u_\varepsilon) &= \frac{1}{2\theta_N} \sum_{\langle i,j\rangle \text{ in } Q} \varepsilon^{d-1}|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2 = \frac{1}{\theta_N}\sum_{k=0}^{k^+} 4\sin^2\left(\frac{\theta_N}{2}\right)\\ &= \frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N^2} k^+\theta_N.\end{aligned}$$

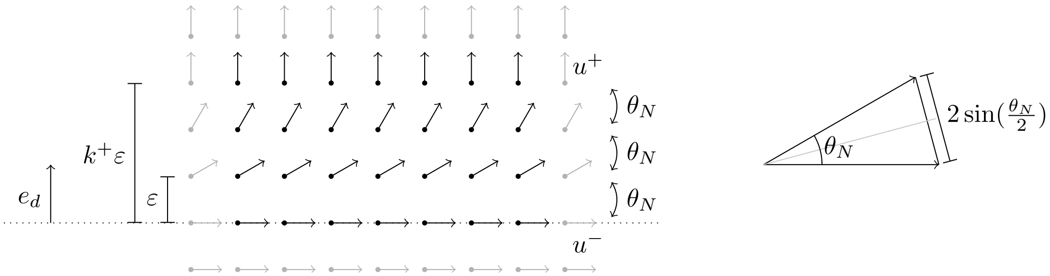

FIG. 1. On the left: A recovery sequence in the case of a jump set aligned with the lattice. The spin makes a transition from u^- to u^+ jumping with the smallest possible non-zero angle θ_N. On the right: Euclidean distance between two vectors of length 1 with angle θ_N between them.

Since $k^+\theta_N = \mathrm{d}_{\mathbb{S}^1}(u^-, u^+)$, the previous expression reduces to the one in Theorem 1.2 and makes clear the role of $4\sin^2(\frac{\theta_N}{2})/\theta_N^2$: it is the correcting factor which allows us to pass from the Euclidean distance between vectors to their geodesic distance. The proof of the upper bound is based on the construction in a more general setting of a recovery sequence which mimics the one presented here in the introduction, cf. Proposition 3.4. The proof of the lower bound is based on Lemma 3.1, which shows that the behavior described above is always the most convenient from an energetical point of view.

We further emphasize that the proofs of Theorems 1.1 and 1.2 are significantly different. Theorem 1.1 is obtained in [27] as a by-product of the more involved analysis in the scaling regime $N_\varepsilon \gg \frac{1}{\varepsilon|\log\varepsilon|}$ using Cartesian currents. Theorem 1.2 is proven here with a different approach that allows us to circumvent the use of Cartesian currents and to prove the result in any dimension d.

In Section 5 we also study the Γ-convergence of the functionals E_ε^N as $\varepsilon \to 0$ under volume constraints on the phases of the spin fields or under Dirichlet boundary conditions.

We are now interested in the limit as $N \to +\infty$ of the energy defined by

$$E_N(u) := \frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N^2}\int_{\Omega\cap J_u} \mathrm{d}_{\mathbb{S}^1}(u^-, u^+)|\nu_u|_1 \,\mathrm{d}\mathcal{H}^{d-1}, \quad \text{for } u \in BV(\Omega; \mathcal{S}_N),$$

where $\theta_N := 2\pi/N$, i.e., the energy resulting from the limit process $\varepsilon \to 0$ in Theorem 1.2. Up to the factor $4\big(\sin^2\left(\frac{\theta_N}{2}\right)/\theta_N^2\big)$, which is close to 1 for N large, the energy E_N coincides (for $d = 2$) with the limiting energy of Theorem 1.1 restricted to Caccioppoli partitions taking values in $\mathcal{S}_N$. In the second result of this paper we show that the Γ-limit of E_N as $N \to +\infty$ agrees with the limiting energy of Theorem 1.1. This is rigorously proved in the next theorem, which holds for any dimension d.

Theorem 1.3 (Limit as $N \to +\infty$) *Let $\Omega \subset \mathbb{R}^d$ be a bounded, open set with Lipschitz boundary. Then the following results hold:*

(i) *(Compactness) Let $u_N : \Omega \to \mathcal{S}_N$ be such that $E_N(u_N) \leqslant C$. Then there exists a subsequence (not relabeled) and a function $u \in BV(\Omega; \mathbb{S}^1)$ such that $u_N \to u$ in $L^1(\Omega; \mathbb{R}^2)$ as $N \to +\infty$.*

(ii) *(Γ-liminf inequality) Assume that* $u_N\colon \Omega \to \mathcal{S}_N$ *and* $u \in BV(\Omega;\mathbb{S}^1)$ *satisfy* $u_N \to u$ *in* $L^1(\Omega;\mathbb{R}^2)$ *as* $N \to +\infty$. *Then*

$$\liminf_{N\to+\infty} E_N(u_N) \geqslant \int_\Omega |\nabla u|_{2,1}\,\mathrm{d}x + |\mathrm{D}^{(c)}u|_{2,1}(\Omega) + \int_{\Omega\cap J_u} \mathrm{d}_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1\,\mathrm{d}\mathcal{H}^{d-1}.$$

(iii) *(Γ-limsup inequality) Let* $u \in BV(\Omega;\mathbb{S}^1)$. *Then there exists a sequence* $u_N\colon \Omega \to \mathcal{S}_N$ *such that* $u_N \to u$ *in* $L^1(\Omega;\mathbb{R}^2)$ *as* $N \to +\infty$ *and*

$$\limsup_{N\to+\infty} E_N(u_N) \leqslant \int_\Omega |\nabla u|_{2,1}\,\mathrm{d}x + |\mathrm{D}^{(c)}u|_{2,1}(\Omega) + \int_{\Omega\cap J_u} \mathrm{d}_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1\,\mathrm{d}\mathcal{H}^{d-1}.$$

The proof of the upper bound in Theorem 1.3 is based on the following remark: a map $u \in BV(\Omega;\mathbb{S}^1)$ can be approximated in energy by maps $W^{1,1}(\Omega;\mathbb{S}^1)$ which are smooth outside manifolds of codimension 2; such maps can be suitably sampled far from the singularities to define a $u_N \in BV(\Omega;\mathcal{S}_N)$; a crucial observation is that the precise definition of u_N close to the singularities is not important, as the energy $E_N(u_N)$ does not concentrate close to manifolds of codimension 2. It is worth noticing that the latter feature is peculiar of this regime: in the other regimes studied in [26] where $N = N_\varepsilon$ depends on ε and $N_\varepsilon \gg \frac{1}{\varepsilon|\log\varepsilon|}$ the behavior of the recovery sequence around the singularities becomes relevant and makes the generalization to the d-dimensional setting of the results in [26] more delicate and out of the scope of the present paper.

2. Notation and preliminary results

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ be the unit sphere. If $u, v \in \mathbb{S}^1$, their geodesic distance on $\mathbb{S}^1$ is denoted by $\mathrm{d}_{\mathbb{S}^1}(u,v)$. It is given by the angle in $[0,\pi]$ between the vectors u and v, i.e., $\mathrm{d}_{\mathbb{S}^1}(u,v) = \arccos(u\cdot v)$. Observe that

$$\tfrac{1}{2}|u-v| = \sin\big(\tfrac{1}{2}\mathrm{d}_{\mathbb{S}^1}(u,v)\big). \tag{2.1}$$

We denote the imaginary unit by ι. When it is convenient we will tacitly identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Given a vector $a = (a_i)_{i=1}^d \in \mathbb{R}^d$, its 1-norm is $|a|_1 = \sum_{i=1}^d |a_i|$. We define the $(2,1)$-norm of a matrix $A = (a_{ij})_{i,j=1}^d \in \mathbb{R}^{d\times d}$ as the sum of the Euclidean norms of its columns, i.e.,

$$|A|_{2,1} := \sum_{j=1}^d \Big(\sum_{i=1}^d |a_{ij}|^2\Big)^{1/2}.$$

Given a unit vector $\nu \in \mathbb{S}^{d-1}$, we denote by Q_ν a cube with two faces orthogonal to ν, namely, we consider an orthonormal basis $(\nu,\nu_2,\ldots,\nu_d)$ of $\mathbb{R}^d$ and we define

$$Q_\nu = \Big\{x \in \mathbb{R}^d : |x\cdot\nu| < \tfrac{1}{2},\ |x\cdot\nu_i| < \tfrac{1}{2}\Big\}. \tag{2.2}$$

For two sequences α_ε and β_ε of positive numbers, we write $\alpha_\varepsilon \ll \beta_\varepsilon$ if $\lim_{\varepsilon\to0}\frac{\alpha_\varepsilon}{\beta_\varepsilon} = 0$.

2.1 *BV-functions*

In this section we recall basic facts about functions of bounded variation. For more details we refer to the monograph [2].

Let $O \subset \mathbb{R}^d$ be an open set. A function $u \in L^1(O;\mathbb{R}^n)$ is a function of bounded variation if its distributional derivative $\mathrm{D}u$ is given by a finite matrix-valued Radon measure on O. In that case, we write $u \in BV(O;\mathbb{R}^n)$.

The space $BV_{\mathrm{loc}}(O;\mathbb{R}^n)$ is defined as usual. The space $BV(O;\mathbb{R}^n)$ becomes a Banach space when endowed with the norm $\|u\|_{BV(O)} = \|u\|_{L^1(O)} + |\mathrm{D}u|(O)$, where $|\mathrm{D}u|$ denotes the total variation measure of $\mathrm{D}u$. The total variation with respect to the anisotropic norm $|\cdot|_{2,1}$ is denoted by $|\mathrm{D}u|_{2,1}$. When O is a bounded Lipschitz domain, then $BV(O;\mathbb{R}^n)$ is compactly embedded in $L^1(O;\mathbb{R}^n)$. We say that a sequence u_n converges weakly* in $BV(O;\mathbb{R}^n)$ to u if $u_n \to u$ in $L^1(O;\mathbb{R}^n)$ and $\mathrm{D}u_n \overset{*}{\rightharpoonup} \mathrm{D}u$ in the sense of measures.

We state some fine properties of BV-functions. To this end, we need some definitions. A function $u \in L^1(O;\mathbb{R}^n)$ is said to have an approximate limit at $x \in O$ whenever there exists $z \in \mathbb{R}^n$ such that

$$\lim_{\rho\to 0} \frac{1}{\rho^d} \int_{B_\rho(x)} |u(y) - z| \,\mathrm{d}y = 0.$$

Next we introduce so-called approximate jump points. Given $x \in O$ and $\nu \in \mathbb{S}^{d-1}$ we set

$$B_\rho^\pm(x,\nu) = \{ y \in B_\rho(x) : \pm(y-x)\cdot\nu > 0 \}.$$

We say that $x \in O$ is an approximate jump point of u if there exist $a \neq b \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{d-1}$ such that

$$\lim_{\rho\to 0} \frac{1}{\rho^d} \int_{B_\rho^+(x,\nu)} |u(y) - a| \,\mathrm{d}y = \lim_{\rho\to 0} \frac{1}{\rho^d} \int_{B_\rho^-(x,\nu)} |u(y) - b| \,\mathrm{d}y = 0.$$

The triplet (a,b,ν) is determined uniquely up to the change to $(b,a,-\nu)$. We denote it by $(u^+(x), u^-(x), \nu_u(x))$ and we let J_u be the set of approximate jump points of u. The triplet (u^+, u^-, ν_u) can be chosen as a Borel function on the Borel set J_u. Denoting by ∇u the approximate gradient of u, we can decompose the measure $\mathrm{D}u$ as the sum

$$\mathrm{D}u(B) = \int_B \nabla u \,\mathrm{d}x + \int_{J_u\cap B} (u^+ - u^-)\otimes \nu_u \,\mathrm{d}\mathcal{H}^{d-1} + \mathrm{D}^{(c)}u(B),$$

where $\mathrm{D}^{(c)}u$ is the so-called Cantor part and $\mathrm{D}^{(j)}u = (u^+ - u^-)\otimes \nu_u \mathcal{H}^{d-1} \llcorner J_u$ is the so-called jump part. If $\mathcal{S} \subset \mathbb{R}^n$, we define the space $BV(O;\mathcal{S})$ as the space of those functions $u \in BV(O;\mathbb{R}^n)$ such that $u(x) \in \mathcal{S}$ for $\mathcal{L}^d$-a.e. $x \in O$.

We will need the slicing properties of BV-functions. Given a unit vector $\xi \in \mathbb{S}^{d-1}$, we denote by Π^ξ the hyperplane orthogonal to ξ. For every set $E \subset \mathbb{R}^d$ and $z \in \Pi^\xi$, the section of E corresponding to z is the set $E_z^\xi := \{t \in \mathbb{R} : z + t\xi \in E\}$. Accordingly, for any function $u\colon E \to \mathbb{R}^n$, the function $u_z^\xi\colon E_z^\xi \to \mathbb{R}^n$ is defined by $u_z^\xi(t) := u(z + t\xi)$.

We recall a characterization of BV functions by slicing [2, Remark 3.104]. Let us fix an open set $O \subset \mathbb{R}^d$ and $u \in L^1(O;\mathbb{R}^n)$. Then $u \in BV(O;\mathbb{R}^n)$ if and only if for every $\xi \in \mathbb{S}^{d-1}$ we have $u_z^\xi \in BV(O_z^\xi;\mathbb{R}^n)$ for $\mathcal{H}^{d-1}$-a.e. $z \in \Pi^\xi$ and

$$\int_{\Pi^\xi} |\mathrm{D}u_z^\xi|(O_z^\xi) \,\mathrm{d}\mathcal{H}^{d-1}(z) < \infty.$$

Moreover, it is possible to reconstruct the distributional gradient $\mathrm{D}u$ from the gradients of the slices $\mathrm{D}u_z^\xi$ through the formula $\mathrm{D}u\,\xi = \mathcal{H}^{d-1} \llcorner \Pi^\xi \otimes \mathrm{D}u_z^\xi$, i.e.,

$$\mathrm{D}u\,\xi(B) = \int_{\Pi^\xi} \mathrm{D}u_z^\xi(B_z^\xi) \,\mathrm{d}\mathcal{H}^{d-1}(z),$$

for every Borel set $B \subset \mathbb{R}^d$. More precisely, the same decomposition holds true for each part of the decomposition of $\mathrm{D}u$, namely

$$\int_B \nabla u\,\xi\,\mathrm{d}x = \int_{\Pi^\xi} \nabla u_z^\xi(B_z^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(z),$$
$$\mathrm{D}^{(c)}u\,\xi(B) = \int_{\Pi^\xi} \mathrm{D}^{(c)}u_z^\xi(B_z^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(z),$$
$$\mathrm{D}^{(j)}u\,\xi(B) = \int_{\Pi^\xi} \mathrm{D}^{(j)}u_z^\xi(B_z^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(z),$$

for every Borel set $B \subset \mathbb{R}^d$. Moreover, $J_{u_z^\xi} = (J_u)_z^\xi$ for $\mathcal{H}^{d-1}$-a.e. $z \in \Pi^\xi$ and $(u_z^\xi)^\pm(t) = (u^\pm)_z^\xi(t)$ ($= (u^\mp)_z^\xi(t)$, respectively) for every $t \in (J_u)_z^\xi$ if $\xi \cdot \nu_u(z+t\xi) > 0$ (if $\xi \cdot \nu_u(z+t\xi) < 0$, respectively).

2.2 *Known results for general models with finite phases*

We recall here some results that were proved for more general energies defined for functions taking values in a given finite set. In [19], Braides together with the first and third author consider energies $\mathcal{E}_\varepsilon$ defined for spin variables $u\colon \varepsilon\mathcal{L} \to \mathcal{S}$, where $\mathcal{S}$ is a finite set and $\mathcal{L}$ is a so-called thin stochastic lattice. In general, these points sets are located in a fixed neighborhood of a lower-dimensional subspace such that there is a minimal distance between points and there are no arbitrarily large holes in the neighborhood of the subspace. The energies in [19] can be of the form

$$\mathcal{E}_\varepsilon(u) = \sum_{(\varepsilon x,\varepsilon y)\in(\varepsilon\mathcal{L}\cap\Omega)^2} \varepsilon^{d-1} f\big(x-y, u(\varepsilon x), u(\varepsilon y)\big),$$

where the energy density $f\colon \mathbb{R}^d \times \mathcal{S}^2 \to [0,+\infty)$ has to satisfy certain growth and decay conditions. We do not state them explicitly here, but we mention that they cover in particular the case when $\mathcal{L} = \mathbb{Z}^d$ is a periodic lattice that is completely contained in the subspace $\mathbb{R}^d$ and

$$f(x,m_1,m_2) = \begin{cases} c\,|m_1-m_2|^2 & \text{if } |x| = 1,\\ 0 & \text{otherwise.}\end{cases}$$

With $c = \frac{N}{4\pi}$ and $\mathcal{S} = \mathcal{S}_N$ we recover the energy $\frac{N}{2\pi\varepsilon}E_\varepsilon^N$, so that all results of [19] can be applied. In particular, we can use an integral representation result and the characterization of the corresponding integrand through an asymptotic cell formula. Indeed, by [19, Theorem 5.8] we know that in the case of spatially homogeneous interactions the Γ-limit as $\varepsilon \to 0$ of $\frac{N}{2\pi\varepsilon}E_\varepsilon^N$ exists, is finite only on $BV(\Omega;\mathcal{S}_N)$, and for $u \in BV(\Omega;\mathcal{S}_N)$ it is of the form

$$\int_{\Omega\cap J_u} \varphi(u^-,u^+,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{2.3}$$

where the integrand is given by an asymptotic minimization problem with suitable boundary conditions. More precisely, denoting by $u_\nu^{s,r}:\mathbb{R}^d \to \mathbb{R}$ ($\nu \in \mathbb{S}^{d-1}$ and $s,r \in \mathcal{S}_N$) the function

$$u_\nu^{s,r}(x) = \begin{cases} s & \text{if } x\cdot\nu > 0,\\ r & \text{if } x\cdot\nu \leqslant 0,\end{cases}$$

then in the case of just nearest neighbor interactions the function $\varphi(s, r, \nu)$ is given by

$$\varphi(s, r, \nu) = \lim_{\varepsilon \to 0} \min \left\{ \frac{N}{2\pi\varepsilon} E_\varepsilon^N(v, Q_\nu) : \, v(\varepsilon i) = u_\nu^{s,r}(\varepsilon i) \quad \forall\, \varepsilon i \in \varepsilon\mathbb{Z}^d \text{ s.t. } \mathrm{dist}(\varepsilon i, \partial Q_\nu) \leqslant 2\varepsilon \right\}, \tag{2.4}$$

cf. [19, Remarks 5.9 and 4.2(i)] for the fact that the width of the discrete boundary layer can be taken as 2ε. In the above formula, Q_ν denotes a unit cube centered at the origin with two faces orthogonal to ν as in (2.2). The energy $E_\varepsilon^N(u, Q_\nu)$ denotes the energy restricted to the set Q_ν. More in general, for any non-empty set $A \subset \mathbb{R}^d$ and $u \colon \varepsilon\mathbb{Z}^d \to \mathcal{S}_N$ let us introduce for later purposes the localized functional

$$E_\varepsilon^N(u, A) = \frac{1}{2} \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^d \cap A \\ |i-j|=1}} \varepsilon^d |u(\varepsilon i) - u(\varepsilon j)|^2.$$

3. Continuum limit for fixed N as lattice spacing vanishes

In this section we identify the variational limit of the N-clock model as $\varepsilon \to 0$ for the scaled energy $\frac{N}{2\pi\varepsilon} E_\varepsilon^N$. We start with the following auxiliary result that will be crucial to establish the lower bound.

Lemma 3.1 *Let $k \in \mathbb{N}$. Then for all $\theta \in [0, \pi/k]$ it holds that*

$$\sin^2\left(\frac{k\theta}{2}\right) \geqslant k \sin^2\left(\frac{\theta}{2}\right).$$

Proof. We can assume that $k \geqslant 2$. Setting $y = \frac{k\theta}{2}$ we have that $y \in [0, \pi/2]$ and the claim reduces to $\sin^2(y) \geqslant k \sin^2(y/k)$ for all $y \in [0, \pi/2]$. Since for $y \in [0, \pi/2]$ both $\sin(y)$ and $\sin(y/k)$ are non-negative, we can alternatively show that

$$\sin(y) \geqslant \sqrt{k} \sin(y/k) \quad \text{for all } y \in [0, \pi/2]. \tag{3.1}$$

Let us define the auxiliary function $f_k(y) = \sin(y) - \sqrt{k}\sin(y/k)$. We show that it is strictly concave on $[0, \pi/2]$, so that its minimum is achieved at $y = 0$ or $y = \pi/2$. Indeed, for $y \in [0, \pi/2]$ we have by the monotonicity of the sinus function that

$$f_k''(y) = -\sin(y) + k^{-\frac{3}{2}} \sin(y/k) \leqslant -\sin(y) + k^{-\frac{3}{2}} \sin(y) \leqslant -\frac{1}{2}\sin(y),$$

so that $f_k''(y) < 0$ whenever $y \in (0, \pi/2]$. Hence

$$\min_{y \in [0, \pi/2]} f_k(y) = \min\bigl\{f_k(0), f_k(\pi/2)\bigr\} = \min\bigl\{0, 1 - \sqrt{k}\sin\bigl(\pi/(2k)\bigr)\bigr\}.$$

We conclude the proof once we show that $\sqrt{k}\sin(\pi/(2k)) \leqslant 1$ for all $k \geqslant 2$. Using that $\sin(x) < x$ for all $x > 0$, for $k \geqslant 3$ we can bound the left hand side by

$$\sqrt{k}\sin\bigl(\pi/(2k)\bigr) \leqslant \frac{\pi}{2\sqrt{k}} \leqslant \frac{\pi}{2\sqrt{3}} < 1,$$

while for $k = 2$ we have $\sqrt{2}\sin(\pi/4) = 1$. Thus $f_k(y) \geqslant 0$ for all $y \in [0, \pi/2]$ which yields (3.1) and concludes the proof. $\square$

Next we establish a lower-semicontinuity result which helps to prove the lower bound.

Lemma 3.2 *For an open set $A \subset \Omega$ let $E(\cdot, A)\colon L^1(A;\mathbb{R}^2) \to [0,+\infty]$ be the functional defined by*

$$E(u,A) = \int_A |\nabla u|_{2,1}\,\mathrm{d}x + |\mathrm{D}^{(c)}u|_{2,1}(A) + \int_{J_u\cap A} \mathrm{d}_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1\,\mathrm{d}\mathcal{H}^{d-1}$$

for $u \in BV(A;\mathbb{S}^1)$ and extended to $+\infty$ otherwise. Then $u \mapsto E(u,A)$ is $L^1(A;\mathbb{R}^2)$-lower semicontinuous.

Proof. For an open set $I \subset \mathbb{R}$ let $E^{1d}(\cdot, I)\colon L^1(I;\mathbb{R}^2) \to [0,+\infty]$ be defined by

$$E^{1d}(w,I) := \begin{cases} \displaystyle\int_I |w'|\,\mathrm{d}t + |\mathrm{D}^{(c)}w|(I) + \sum_{t\in J_w\cap I} \mathrm{d}_{\mathbb{S}^1}\big(w^+(t),w^-(t)\big)\,, & \text{if } w \in BV(I;\mathbb{S}^1),\\ +\infty\,, & \text{otherwise.}\end{cases}$$

By [9, Theorem 3.1] (see also [9, Remark 4.3]), the functional $E^{1d}(\cdot, I)$ is the relaxation of

$$\int_I |w'|\,\mathrm{d}t, \quad w \in W^{1,1}(I;\mathbb{S}^1)$$

with respect to the strong topology of $L^1(I;\mathbb{R}^2)$. In particular, it is lower semicontinuous.

We next fix an open set $A \subset \Omega$ and $v_n, v \in L^1(A;\mathbb{R}^2)$ such that $v_n \to v$ strongly in $L^1(A;\mathbb{R}^2)$. We want to prove that

$$E(v;A) \leqslant \liminf_{n\to+\infty} E(v_n;A). \tag{3.2}$$

Without loss of generality, we assume that the right-hand side in (3.2) is finite and that the lim inf is actually a limit. Since $|\mathrm{D}v_n|(A) \leqslant E(v_n;A)$ we obtain $v \in BV(A;\mathbb{S}^1)$ and $v_n \overset{*}{\rightharpoonup} v$ weakly* in $BV(A;\mathbb{R}^2)$. Note further that

$$E(v_n,A) = \sum_{\ell=1}^d \Big\{ \int_A |\nabla v_n\, e_\ell|\,\mathrm{d}x + |\mathrm{D}^{(c)}v_n\, e_\ell|(A) + \int_{J_{v_n}\cap A} \mathrm{d}_{\mathbb{S}^1}(v_n^+,v_n^-)|\nu_{v_n}\cdot e_\ell|\,\mathrm{d}\mathcal{H}^{d-1}\Big\}. \tag{3.3}$$

Let us fix a direction $\xi \in \mathbb{S}^1$, which plays the role of one of the coordinate directions e_ℓ. In the following we use the notation and the properties of slicing recalled in Section 2.1. We start by extracting a subsequence of n (possibly depending on ξ and which we do not relabel) such that the liminf

$$\liminf_{n\to+\infty} \int_A |\nabla v_n\,\xi|\,\mathrm{d}x + |\mathrm{D}^{(c)}v_n\,\xi|(A) + \int_{J_{v_n}\cap A} \mathrm{d}_{\mathbb{S}^1}(v_n^+,v_n^-)|\nu_{v_n}\cdot\xi|\,\mathrm{d}\mathcal{H}^{d-1}$$

is actually a limit. Moreover, since $v_n \to v$ strongly in $L^1(A;\mathbb{R}^2)$, by Fubini's Theorem we extract a further subsequence (possibly depending on ξ and which we do not relabel) such that

$$(v_n)^\xi_z \to v^\xi_z \quad \text{strongly in } L^1(A^\xi_z;\mathbb{R}^2)\,, \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } z \in \Pi^\xi.$$

Moreover, we know that $v^\xi_z \in BV(A^\xi_z;\mathbb{S}^1)$ for $\mathcal{H}^{d-1}$-a.e. $z \in \Pi^\xi$.

We observe now that the coarea formula (cf. [2, formula (272)] with $g = \mathrm{d}_{\mathbb{S}^1}(v_n^+, v_n^-)$, $E = J_{v_n} \cap A$, and f the projection onto the orthogonal complement of ξ) implies

$$\int_{J_{v_n}\cap A} \mathrm{d}_{\mathbb{S}^1}(v_n^+, v_n^-)|\nu_{v_n}\cdot\xi|\,\mathrm{d}\mathcal{H}^{d-1} = \int_{\Pi^\xi}\Big[\sum_{t\in J_{(v_n)_z^\xi}\cap A_z^\xi} \mathrm{d}_{\mathbb{S}^1}\Big(\big((v_n)_z^\xi\big)^+(t), \big((v_n)_z^\xi\big)^-(t)\Big)\Big]\,\mathrm{d}\mathcal{H}^{d-1}(z).$$

Hence, by the equality above and by Fatou's Lemma, we deduce that

$$\begin{aligned}
&\lim_{n\to+\infty}\int_A |\nabla v_n\,\xi|\,\mathrm{d}x + |\mathrm{D}^{(c)}v_n\,\xi|(A) + \int_{J_{v_n}\cap A}\mathrm{d}_{\mathbb{S}^1}(v_n^+, v_n^-)|\nu_{v_n}\cdot\xi|\,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad= \lim_{n\to+\infty}\int_{\Pi^\xi}\Big[\int_{A_z^\xi}\big|\big((v_n)_z^\xi\big)'\big|\,\mathrm{d}t + |\mathrm{D}^{(c)}(v_n)_z^\xi|(A_z^\xi)\\
&\qquad+\sum_{t\in J_{(v_n)_z^\xi}\cap A_z^\xi}\mathrm{d}_{\mathbb{S}^1}\Big(\big((v_n)_z^\xi\big)^+(t), \big((v_n)_z^\xi\big)^-(t)\Big)\Big]\,\mathrm{d}\mathcal{H}^{d-1}(z)\\
&\quad\geqslant \int_{\Pi^\xi}\liminf_{n\to+\infty}\Big[\int_{A_z^\xi}\big|\big((v_n)_z^\xi\big)'\big|\,\mathrm{d}t + |\mathrm{D}^{(c)}(v_n)_z^\xi|(A_z^\xi)\\
&\qquad+\sum_{t\in J_{(v_n)_z^\xi}\cap A_z^\xi}\mathrm{d}_{\mathbb{S}^1}\Big(\big((v_n)_z^\xi\big)^+(t), \big((v_n)_z^\xi\big)^-(t)\Big)\Big]\,\mathrm{d}\mathcal{H}^{d-1}(z).
\end{aligned}\tag{3.4}$$

From the one-dimensional lower semicontinuity result we infer that

$$\begin{aligned}
&\liminf_{n\to+\infty}\int_{A_z^\xi}\big|\big((v_n)_z^\xi\big)'\big|\,\mathrm{d}t + |\mathrm{D}^{(c)}(v_n)_z^\xi|(A_z^\xi) + \sum_{t\in J_{(v_n)_z^\xi}\cap A_z^\xi}\mathrm{d}_{\mathbb{S}^1}\Big(\big((v_n)_z^\xi\big)^+(t), \big((v_n)_z^\xi\big)^-(t)\Big)\\
&\quad= \liminf_{n\to+\infty} E^{1d}\big((v_n)_z^\xi, A_z^\xi\big)\\
&\quad\geqslant E^{1d}(v_z^\xi, A_z^\xi) = \int_{A_z^\xi}\big|\big(v_z^\xi\big)'\big|\,\mathrm{d}t + |\mathrm{D}^{(c)}v_z^\xi|(A_z^\xi) + \sum_{t\in J_{v_z^\xi}\cap A_z^\xi}\mathrm{d}_{\mathbb{S}^1}\big((v_z^\xi)^+(t), (v_z^\xi)^-(t)\big)
\end{aligned}$$

for $\mathcal{H}^{d-1}$-a.e. $z\in\Pi^\xi$. Integrating the inequality above with respect to $z\in\Pi^\xi$, again by the coarea formula, and by (3.4) we obtain that

$$\begin{aligned}
&\lim_{n\to+\infty}\int_A |\nabla v_n\,\xi|\,\mathrm{d}x + |\mathrm{D}^{(c)}v_n\,\xi|(A) + \int_{J_{v_n}\cap A}\mathrm{d}_{\mathbb{S}^1}(v_n^+, v_n^-)|\nu_{v_n}\cdot\xi|\,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad\geqslant \int_A |\nabla v\,\xi|\,\mathrm{d}x + |\mathrm{D}^{(c)}v\,\xi|(A) + \int_{J_v\cap A}\mathrm{d}_{\mathbb{S}^1}(v^+, v^-)|\nu_v\cdot\xi|\,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}\tag{3.5}$$

We conclude the proof of (3.2) by evaluating the last inequality for $\xi = e_1,\dots,e_d$, by (3.3), and employing the superadditivity of the lim inf. □

Now we can prove the lower bound for the Γ-limit of the functionals $\frac{N}{2\pi\varepsilon}E_\varepsilon^N$.

Proposition 3.3 *Let $u_\varepsilon\colon \varepsilon\mathbb{Z}^d \to \mathcal{S}_N$ and $u \in BV(\Omega;\mathcal{S}_N)$ be such that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$. Then*

$$\liminf_{\varepsilon\to 0} \frac{N}{2\pi\varepsilon} E_\varepsilon^N(u_\varepsilon) \geqslant \frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N^2} \int_{\Omega\cap J_u} \mathrm{d}_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1 \,\mathrm{d}\mathcal{H}^{d-1}.$$

Proof. To simplify the notation we denote θ_N by θ. Let $A \subset\subset \Omega$ be an open set. By (2.1) it holds that

$$|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| = 2\sin\left(\tfrac{1}{2}\mathrm{d}_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j))\right).$$

Since u_ε takes values in $\mathcal{S}_N$, the geodesic distance $\mathrm{d}_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j))$ is an integer multiple of θ, i.e., there exists a $k \in \mathbb{N}$ (depending on i, j, and ε) such that $\mathrm{d}_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j)) = k\theta$. Note that $k\theta \leqslant \pi$. Hence from Lemma 3.1 we infer that

$$\begin{aligned}\frac{1}{2}|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2 &= 2\sin^2\left(\tfrac{1}{2}\mathrm{d}_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j))\right) = 2\sin^2\left(\frac{k\theta}{2}\right) \geqslant 2k\sin^2\left(\frac{\theta}{2}\right)\\ &= 2\mathrm{d}_{\mathbb{S}^1}\left(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j)\right)\frac{\sin^2(\frac{\theta}{2})}{\theta}.\end{aligned}$$

Since u_ε is piecewise constant on cubes of the form $Q = (-\varepsilon/2, \varepsilon/2)^d + z$, $z \in \mathbb{Z}^d$, with faces of length ε that are parallel to the coordinate axes (so that the outer normal vector satisfies $|\nu|_1 = 1$), we obtain that for ε small enough

$$\begin{aligned}\frac{N}{2\pi\varepsilon} E_\varepsilon^N(u_\varepsilon) &\geqslant \frac{1}{\theta}\frac{2\sin^2\left(\frac{\theta}{2}\right)}{\theta} \sum_{\langle i,j\rangle \text{ in } \Omega} \varepsilon \mathrm{d}_{\mathbb{S}^1}\left(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j)\right)\\ &\geqslant \frac{4\sin^2\left(\frac{\theta}{2}\right)}{\theta^2} \int_{A\cap J_u} \mathrm{d}_{\mathbb{S}^1}(u_\varepsilon^-, u_\varepsilon^+)|\nu_{u_\varepsilon}|_1 \,\mathrm{d}\mathcal{H}^{d-1},\end{aligned}$$

where we also used that $N = 2\pi/\theta$ and that the discrete energy counts each interaction twice. Note that by Lemma 3.2 the functional

$$u \mapsto \int_{A\cap J_u} \mathrm{d}_{\mathbb{S}^1}(u^-,u^+)|\nu_u| \,\mathrm{d}\mathcal{H}^{d-1}$$

is $L^1(A;\mathbb{R}^2)$-lower semicontinuous on $BV(A;\mathcal{S}_N)$, as it is the restriction of a lower semicontinuous functional to a closed subset of $BV(A;\mathcal{S}_N)$. Thus letting $\varepsilon \to 0$ we deduce that

$$\liminf_{\varepsilon\to 0} \frac{N}{2\pi\varepsilon} E_\varepsilon^N(u_\varepsilon) \geqslant \frac{4\sin^2(\frac{\theta}{2})}{\theta^2} \int_{A\cap J_u} \mathrm{d}_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1 \,\mathrm{d}\mathcal{H}^{d-1}.$$

The claim now follows from the arbitrariness of $A \subset\subset \Omega$. □

We next prove that the corresponding upper bound for the Γ-limit.

Proposition 3.4 *Let $u \in BV(\Omega;\mathcal{S}_N)$. Then there exists a sequence $u_\varepsilon\colon \varepsilon\mathbb{Z}^d \to \mathcal{S}_N$ such that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$ and*

$$\limsup_{\varepsilon\to 0} \frac{N}{2\pi\varepsilon} E_\varepsilon^N(u_\varepsilon) = \frac{4\sin^2(\frac{\theta_N}{2})}{\theta_N^2} \int_{\Omega\cap J_u} \mathrm{d}_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1 \,\mathrm{d}\mathcal{H}^{d-1}.$$

Proof. To simplify the notation we denote θ_N by θ. Due to the discussion in Section 2.2, the Γ-limit of $\frac{N}{2\pi\varepsilon}E_\varepsilon^N$ has the form (2.3). To prove the upper bound it suffices to define a suitable candidate for the minimum problem (2.4) whose energy can be bounded in the limit as $\varepsilon \to 0$ by $4\sin^2(\frac{\theta}{2})\theta^{-2}\mathrm{d}_{\mathbb{S}^1}(s,r)|\nu|_1$. Write $s = \exp(\iota k_s\theta)$ and $r = \exp(\iota k_r\theta)$ with $0 \leqslant k_s, k_r \leqslant N-1$. We will treat the case when $k_r = 0$, i.e., $r = (1,0)$, and $0 < k_s\theta \leqslant \pi$. The construction we provide can then be composed with a rotation in the co-domain to cover the general case. The idea is to define a candidate whose angular variable jumps by θ along the discretization of k_s parallel hyperplanes orthogonal to ν, where all hyperplanes are $\mathcal{O}(\varepsilon)$-close to the hyperplane $\Pi_\nu := \{x \in \mathbb{R}^d : x\cdot\nu = 0\}$. The correction in order to satisfy the boundary conditions of the minimum problem (2.4) will be of lower order. In formulas, let $u_\varepsilon\colon \varepsilon\mathbb{Z}^d \to \mathcal{S}_N$ be defined by

$$u_\varepsilon(\varepsilon i) := \begin{cases} \exp\left(\iota \min\left\{k_s, \max\{0, \lfloor i\cdot\nu\rfloor\}\right\}\theta\right) & \text{if } \mathrm{dist}(\varepsilon i, \partial Q_\nu) > 2\varepsilon, \\ u_\nu^{s,r}(\varepsilon i) & \text{if } \mathrm{dist}(\varepsilon i, \partial Q_\nu) \leqslant 2\varepsilon, \end{cases}$$

where $\lfloor x\rfloor$ denotes the integer part of x. Hence for all $\varepsilon i \in \varepsilon\mathbb{Z}^d \cap Q_\nu$ such that $\varepsilon i\cdot\nu \leqslant 0$ we have $u_\varepsilon(\varepsilon i) = r$, while for all $\varepsilon i \in \varepsilon\mathbb{Z}^d$ with $\varepsilon i\cdot\nu \geqslant k_s\varepsilon$ we have $u_\varepsilon(\varepsilon i) = s$, so that for non-vanishing interactions at least one point belongs to the set

$$H_\varepsilon^{k_s} := \left\{x \in Q_\nu : x\cdot\nu \in (0, \varepsilon k_s)\right\}.$$

Note that we have the volume bound

$$\left|H_{2\varepsilon}^{k_s} \cap \left\{\mathrm{dist}(x, \partial Q_\nu) \leqslant 4\varepsilon\right\}\right| \leqslant Ck_s\varepsilon^2,$$

where C depends only on the dimension. Hence, for ε small enough,

$$\#\left\{z \in \mathbb{Z}^d : \varepsilon z \in H_{2\varepsilon}^{k_s} \cap \left\{\mathrm{dist}(x, \partial Q_\nu) \leqslant 3\varepsilon\right\}\right\} \leqslant Ck_s\varepsilon^{2-d} \tag{3.6}$$

To simplify notation, we also define the auxiliary function $v_\varepsilon\colon \varepsilon\mathbb{Z}^d \to \mathcal{S}_N$ by

$$v_\varepsilon(\varepsilon i) := \exp\left(\iota \min\left\{k_s, \max\{0, \lfloor i\cdot\nu\rfloor\}\right\}\theta\right).$$

As $\mathcal{S}_N \subset \mathbb{S}^1$, it holds that $|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2 \leqslant 4$, so due to the almost additivity of the set function $A \mapsto E_\varepsilon^N(u, A)$ the energy of u_ε can be estimated by

$$\begin{aligned} \frac{N}{2\pi\varepsilon}E_\varepsilon^N(u_\varepsilon, Q_\nu) &\leqslant \frac{N}{2\pi\varepsilon}E_\varepsilon^N\left(u_\varepsilon, H_{2\varepsilon}^{k_s} \cap \{\mathrm{dist}(x, \partial Q_\nu) \leqslant 3\varepsilon\}\right) + \frac{N}{2\pi\varepsilon}E_\varepsilon^N(v_\varepsilon, Q_\nu) \\ &\leqslant CNk_s\varepsilon + \frac{N}{2\pi\varepsilon}E_\varepsilon^N(v_\varepsilon, Q_\nu) \leqslant CN^2\varepsilon + \frac{N}{2\pi\varepsilon}E_\varepsilon^N(v_\varepsilon, Q_\nu). \end{aligned}$$

As N is fixed, the first term in the right-hand side vanishes when $\varepsilon \to 0$. Since u_ε is admissible for the minimum problem (2.4) it suffices to show that

$$\limsup_{\varepsilon\to 0} \frac{N}{2\pi\varepsilon}E_\varepsilon^N(v_\varepsilon, Q_\nu) \leqslant \frac{4\sin^2\left(\frac{\theta}{2}\right)}{\theta^2}k_s\theta|\nu|_1 = \frac{4\sin^2\left(\frac{\theta}{2}\right)}{\theta}k_s|\nu|_1. \tag{3.7}$$

We start by noticing that when $\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^d \cap Q_\nu$ are such that $|i - j| = 1$ and $v_\varepsilon(\varepsilon i) \neq v_\varepsilon(\varepsilon j)$, then $\varepsilon i\cdot\nu \neq \varepsilon j\cdot\nu$. Without loss of generality, we assume $\varepsilon i\cdot\nu > \varepsilon j\cdot\nu$. Note that $j\cdot\nu \geqslant 0$. Indeed, if instead $j \cdot \nu < 0$, then $i \cdot \nu < 1$ and thus $v_\varepsilon(\varepsilon i) = v_\varepsilon(\varepsilon j)$, which contradicts $v_\varepsilon(\varepsilon i) \neq v_\varepsilon(\varepsilon j)$. Moreover, by a similar argument we also know that $k_s + 1 > i \cdot \nu$. To sum up, we have that

$$0 \leqslant \varepsilon j \cdot \nu < \varepsilon i \cdot \nu < (k_s + 1)\varepsilon. \tag{3.8}$$

Finally, we have the estimate $|(\varepsilon i - \varepsilon j) \cdot \nu| \leqslant \varepsilon$, so that by (2.1)

$$|v_\varepsilon(\varepsilon i) - v_\varepsilon(\varepsilon j)|^2 = 4\sin^2\big(\tfrac{\theta}{2}\big). \tag{3.9}$$

It remains to count the interactions. We will first split them according to their jump between $\varepsilon j \cdot \nu$ and $\varepsilon i \cdot \nu$. More precisely, for a natural number $k \in \{1, \dots, k_s\}$ we set

$$I_{k,\varepsilon} := \Big\{(\varepsilon i, \varepsilon j) \in \big(\varepsilon\mathbb{Z}^d \cap Q_\nu\big)^2 : \ |i - j| = 1\,,\ \lfloor j \cdot \nu \rfloor = k - 1\,,\ \lfloor i \cdot \nu \rfloor = k\Big\}.$$

Note that a pair $(\varepsilon i, \varepsilon j) \in I_{k,\varepsilon}$ is only counted once. Since each pair of interactions in the energy is counted twice, we deduce from (3.9) and the equality $N/2\pi = 1/\theta$ that

$$\frac{N}{2\pi\varepsilon} E_\varepsilon^N(v_\varepsilon, Q_\nu) \leqslant \frac{4\sin^2\big(\frac{\theta}{2}\big)}{\theta} \sum_{k=1}^{k_s} \varepsilon^{d-1} \# I_{k,\varepsilon}.$$

We deduce then (3.7) from the asymptotic formula

$$\limsup_{\varepsilon \to 0} \varepsilon^{d-1} \# I_{k,\varepsilon} \leqslant |\nu|_1. \tag{3.10}$$

The above formula can be justified as follows: First further subdivide the set $I_{k,\varepsilon}$ into the d disjoint sets $(I^\ell_{k,\varepsilon})_{\ell=1}^d$ defined by

$$I^\ell_{k,\varepsilon} := \big\{(\varepsilon i, \varepsilon j) \in I_{k,\varepsilon} : \ (i - j) \text{ is parallel to } e_\ell\big\} \quad \text{for } \ell = 1, \dots, d.$$

Observe that $I_{k,\varepsilon} = \bigcup_{\ell=1}^d I^\ell_{k,\varepsilon}$ and that if there exists a pair $(\varepsilon i, \varepsilon j) \in I^\ell_{k,\varepsilon}$, then $\nu_\ell \neq 0$. Indeed, in that case the hyperplane $H_\nu = \{x \cdot \nu = 0\}$ does not contain $(i - j)$, and in turn e_ℓ, by definition of $I_{k,\varepsilon}$. Next we estimate where the line $\varepsilon j + \mathbb{R}e_\ell$ intersects the hyperplane $H_\nu = \{x \cdot \nu = 0\}$. It does in a unique point $\varepsilon j + \lambda e_\ell$ when $I^\ell_{k,\varepsilon} \neq \emptyset$. Since $0 \leqslant \varepsilon j \cdot \nu \leqslant k\varepsilon$ it follows that

$$|\lambda| \leqslant \frac{k\varepsilon}{|\nu_\ell|}.$$

Therefore, given $t > 1$, for $\varepsilon = \varepsilon(t)$ small enough the intersection point is contained in $tQ_\nu \cap H_\nu$. Since by definition the mapping $I^\ell_{k,\varepsilon} \ni (\varepsilon i, \varepsilon j) \mapsto \varepsilon j - (\varepsilon j \cdot e_\ell)e_\ell$ is injective, we obtain that

$$\# I^\ell_{k,\varepsilon} \leqslant \#\big\{\varepsilon i \in \varepsilon\mathbb{Z}^d : \ \varepsilon i \in \Pi_{x_\ell = 0}(tQ_\nu \cap H_\nu)\big\},$$

where $\Pi_{x_\ell=0}$ denotes the projection onto the subspace $\{x_\ell = 0\}$. In particular, it holds that

$$\varepsilon^{d-1} \# I_{k,\varepsilon} = \sum_{\ell=1}^d \varepsilon^{d-1} \# I^\ell_{k,\varepsilon} \leqslant \sum_{\ell=1}^d \varepsilon^{d-1} \#\big(\varepsilon\mathbb{Z}^d \cap \Pi_{x_\ell=0}(tQ_\nu \cap H_\nu)\big).$$

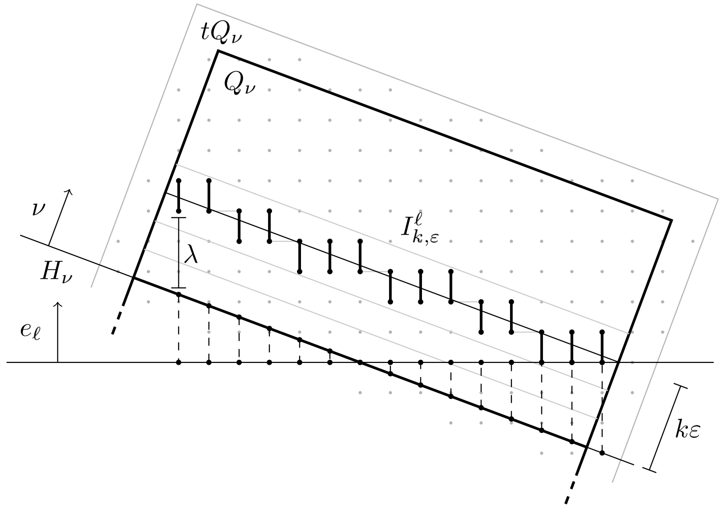

FIG. 2. Counting the number of points in $I^\ell_{k,\varepsilon}$

By elementary geometric considerations we can bound the cardinality via a $(d-1)$-dimensional volume as

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon^{d-1}\big(\#\varepsilon\mathbb{Z}^d\cap\Pi_{x_\ell=0}(tQ_\nu\cap H_\nu)\big) &= \mathcal{H}^{d-1}\big(\Pi_{x_\ell=0}(tQ_\nu\cap H_\nu)\big)\\ &= t^{d-1}\mathcal{H}^{d-1}\big(\Pi_{x_\ell=0}(Q_\nu\cap H_\nu)\big).\end{aligned}$$

Since $t>1$ was arbitrary we deduce that

$$\limsup_{\varepsilon\to 0}\varepsilon^{d-1}\#I^\ell_{k,\varepsilon}\leqslant\mathcal{H}^{d-1}\big(\Pi_{x_\ell=0}(Q_\nu\cap H_\nu)\big).$$

We claim that the right-hand side term equals $|\nu_\ell|$, which then concludes the proof summing over ℓ. This is a consequence of the coarea formula in the form [2, Theorem 2.93] taking f to be the projection $\Pi_{x_\ell=0}$ and $E=Q_\nu\cap H_\nu$ and using the fact that the $(d-1)$-dimensional coarea factor of the projection $\Pi_{x_\ell=0}$ on the tangent space H_ν is given by $|\nu_\ell|$ (cf. [2, formula (3.110)]). □

4. Limit of the continuum functional for large N

In this section we study the Γ-convergence of the limit functionals E_N defined on $L^1(\Omega;\mathbb{R}^2)$ by

$$E_N(u):=\begin{cases}\dfrac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N^2}\displaystyle\int_{\Omega\cap J_u}\mathrm{d}_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1\,\mathrm{d}\mathcal{H}^{d-1} & \text{if } u\in BV(\Omega;\mathcal{S}_N),\\ +\infty & \text{otherwise,}\end{cases}\tag{4.1}$$

as $N\to+\infty$, where we write θ_N to stress the dependence on N of the minimal angle between vectors in $\mathcal{S}_N$. We show that the Γ-limit of E_N coincides with the functional derived in [27] in the regime $N = N_\varepsilon \ll \frac{1}{\varepsilon|\log\varepsilon|}$ and $d = 2$. More precisely, we define the functional

$$E(u) := \begin{cases} \int_\Omega |\nabla u|_{2,1}\,\mathrm{d}x + |\mathrm{D}^{(c)}u|_{2,1}(\Omega) + \int_{\Omega\cap J_u} \mathrm{d}_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1\,\mathrm{d}\mathcal{H}^{d-1}, & \text{if } u \in BV(\Omega;\mathbb{S}^1),\\ +\infty & \text{otherwise,}\end{cases} \tag{4.2}$$

for $u \in L^1(\Omega;\mathbb{R}^2)$.

We first state and prove the lower bound together with a compactness result.

Proposition 4.1 (Lower bound and compactness) *Let $u_N \in BV(\Omega;\mathcal{S}_N)$ be a sequence such that*

$$\sup_N E_N(u_N) < +\infty.$$

Then up to subsequences $u_N \to u \in BV(\Omega;\mathbb{S}^1)$ strongly in $L^1(\Omega;\mathbb{R}^2)$. Moreover, for any sequence $u_N \in BV(\Omega;\mathcal{S}_N)$ and $u \in BV(\Omega;\mathbb{S}^1)$ such that $u_N \to u$ in $L^1(\Omega;\mathbb{R}^2)$ it holds that

$$\liminf_{N\to+\infty} E_N(u_N) \geqslant E(u).$$

Proof. Since $\mathrm{d}_{\mathbb{S}^1}(u,v) \geqslant |u-v|$ and $|\nu|_1 \geqslant 1$ for any unit vector ν, the functionals E_N satisfy

$$E_N(u) \geqslant \frac{4\sin^2(\frac{\theta_N}{2})}{\theta_N^2}\int_{\Omega\cap J_u} |u^+ - u^-|\,\mathrm{d}\mathcal{H}^{d-1} = \frac{4\sin^2(\frac{\theta_N}{2})}{\theta_N^2}|\mathrm{D}u|(\Omega).$$

Note that $\theta_N = 2\pi/N$ implies $\theta_N \to 0$ as $N \to +\infty$. Hence

$$\lim_{N\to+\infty} \frac{4\sin^2(\frac{\theta_N}{2})}{\theta_N^2} = 1. \tag{4.3}$$

Thus the compactness statement follows from the inclusion $\mathcal{S}_N \subset \mathbb{S}^1$ and standard compactness results in $BV(\Omega;\mathbb{R}^2)$.

In order to prove the lower bound, note that

$$\begin{aligned} E_N(u) &\geqslant \frac{4\sin^2(\frac{\theta_N}{2})}{\theta_N^2}\left(\int_\Omega |\nabla u|_{2,1}\,\mathrm{d}x + |\mathrm{D}^{(c)}u|_{2,1}(\Omega) + \int_{\Omega\cap J_u} \mathrm{d}_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1\,\mathrm{d}\mathcal{H}^{d-1}\right)\\ &= \frac{4\sin^2(\frac{\theta_N}{2})}{\theta_N^2}E(u)\end{aligned}$$

for all $u \in BV(\Omega;\mathbb{S}^1)$, cf. (4.1)–(4.2). The functional E is $L^1(\Omega;\mathbb{R}^2)$-lower semicontinuous by Lemma 3.2. Hence, the claim follows from (4.3). □

We now establish the upper bound via several approximations combined with a relaxation result for integral functionals defined on $W^{1,1}(\Omega;\mathbb{S}^1)$.

We recall here the density result proven in [16]. Let

$$\mathcal{R}_1^\infty(\Omega;\mathbb{S}^1) := \Big\{u \in W^{1,1}(\Omega;\mathbb{S}^1) : \text{ there exists } \Sigma = \textstyle\bigcup_{i=1}^m \Sigma_h\,,\ m \in \mathbb{N},\ \Sigma_h \text{ closed subset of a } (d-2)\text{-dimensional manifold, such that } u \in C^\infty(\Omega\setminus\Sigma;\mathbb{S}^1)\Big\}.$$

Theorem 4.2 *The class $\mathcal{R}_1^\infty(\Omega;\mathbb{S}^1)$ is dense in $W^{1,1}(\Omega;\mathbb{S}^1)$ with respect to the strong convergence in $W^{1,1}(\Omega;\mathbb{R}^2)$.*

Proposition 4.3 (Upper bound) *For every function $u \in BV(\Omega;\mathbb{S}^1)$ there exists a sequence $u_N \in BV(\Omega;\mathcal{S}_N)$ such that $u_N \to u$ strongly in $L^1(\Omega;\mathbb{R}^2)$ and*

$$\lim_{N\to+\infty} E_N(u_N) = E(u).$$

Proof. Thanks to Proposition 3.3, it is enough to prove that for every $u \in BV(\Omega;\mathbb{S}^1)$ there exists a sequence $u_N \in BV(\Omega;\mathcal{S}_N)$ such that $u_N \to u$ strongly in $L^1(\Omega;\mathbb{R}^2)$ and

$$\limsup_{N\to+\infty} E_N(u_N) \leqslant E(u). \tag{4.4}$$

Step 1. (Reducing to the case $u \in W^{1,1}(\Omega;\mathbb{S}^1)$). Let us start by considering the functional given by

$$\int_\Omega |\nabla u|_{2,1}\,\mathrm{d}x\,, \quad \text{if } u \in W^{1,1}(\Omega;\mathbb{S}^1) \tag{4.5}$$

and by $+\infty$ otherwise in $L^1(\Omega;\mathbb{R}^2)$. This functional satisfies all the assumptions of the functionals studied in [9], cf. assumptions (H1)–(H5) therein. Then, by [9, Theorem 3.1], its relaxation is given by

$$\int_\Omega |\nabla u|_{2,1}\,\mathrm{d}x + |\mathrm{D}^{(c)}u|_{2,1}(\Omega) + \int_{\Omega\cap J_u} K(u^-,u^+,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}, \quad \text{if } u \in BV(\Omega;\mathbb{S}^1)$$

and by $+\infty$ otherwise in $L^1(\Omega;\mathbb{R}^2)$. The density of the surface energy $K\colon\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^{d-1} \to [0,+\infty)$ is characterized by the formula

$$K(a,b,\nu) := \inf\left\{\int_{Q_\nu} |\nabla\psi|_{2,1}\,\mathrm{d}x \,:\, \psi \in \mathcal{P}(a,b,\nu)\right\},$$

where Q_ν is a unit cube centered at the origin with two faces orthogonal to ν and $\mathcal{P}(a,b,\nu)$ is the collection of all $\psi \in W^{1,1}(Q_\nu;\mathbb{S}^1)$ with $\psi(x) = a$ if $x\cdot\nu = -\frac{1}{2}$, $\psi(x) = b$ if $x\cdot\nu = \frac{1}{2}$, and ψ is periodic with period 1 in the direction orthogonal to ν. In particular, $\mathcal{P}(a,b,\nu)$ contains the collection of functions with a one-dimensional profile in the direction ν, i.e., functions $\psi \in W^{1,1}(Q_\nu;\mathbb{S}^1)$ such that there exists a curve $\gamma \in W^{1,1}((-\frac{1}{2},\frac{1}{2});\mathbb{S}^1)$ with $\gamma(-\frac{1}{2}) = a$, $\gamma(\frac{1}{2}) = b$ satisfying $\psi(x) = \gamma(x\cdot\nu)$. For such functions we have $\nabla\psi(x) = \gamma'(x\cdot\nu)\otimes\nu$ and therefore, since $|\gamma'(x\cdot\nu)\otimes\nu|_{2,1} = |\gamma'(x\cdot\nu)|\,|\nu|_1$,

$$K(a,b,\nu) \leqslant \int_{Q_\nu} |\nabla\psi|_{2,1}\,\mathrm{d}x = |\nu|_1 \int_{Q_\nu} |\gamma'(x\cdot\nu)|\,\mathrm{d}x = |\nu|_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} |\gamma'(t)|\,\mathrm{d}t.$$

Taking the infimum over all such curves $\gamma \in W^{1,1}\big((-\frac{1}{2},\frac{1}{2});\mathbb{S}^1\big)$ with $\gamma(-\frac{1}{2}) = a$, $\gamma(\frac{1}{2}) = b$, we conclude that

$$K(a,b,\nu) \leqslant \mathrm{d}_{\mathbb{S}^1}(a,b)|\nu|_1.$$

In particular, the relaxation of (4.5) is smaller than E, cf. (4.2). This entails that for every $u \in BV(\Omega;\mathbb{S}^1)$ there exists a sequence $u_j \in W^{1,1}(\Omega;\mathbb{S}^1)$ such that $u_j \to u$ in $L^1(\Omega;\mathbb{R}^2)$ and

$$\limsup_{j\to+\infty} \int_\Omega |\nabla u_j|_{2,1}\,\mathrm{d}x \leqslant E(u).$$

Thanks to this property and to a diagonal argument, it is enough to prove the upper bound (4.4) assuming $u \in W^{1,1}(\Omega;\mathbb{S}^1)$.

Step 2. (Extending outside Ω). Let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$. There exists $t > 0$ and a bi-Lipschitz map $\Gamma\colon (\partial\Omega\times(-t,t)) \to \Gamma(\partial\Omega\times(-t,t))$ such that $\Gamma(x,0) = x$ for all $x \in \partial\Omega$, $\Gamma(\partial\Omega\times(-t,t))$ is an open neighborhood of $\partial\Omega$ and

$$\Gamma\big(\partial\Omega\times(-t,0)\big) \subset \Omega, \qquad \Gamma\big(\partial\Omega \times (0,t)\big) \subset \mathbb{R}^2 \setminus \overline{\Omega}. \tag{4.6}$$

This result is a consequence of [35, Theorem 7.4 & Corollary 7.5]; details can be found for instance in [34, Theorem 2.3]. The extension of u is then achieved via reflection. More precisely, for a sufficiently small $\tilde{t} > 0$ we define it on $\tilde{\Omega}$ with $\tilde{\Omega} = \Omega + B_{\tilde{t}}(0)$ by

$$\tilde{u}(x) = \begin{cases} u\Big(\Gamma\Big(P\big(\Gamma^{-1}(x)\big)\Big)\Big) & \text{if } x \notin \Omega, \\ u(x) & \text{otherwise}, \end{cases} \tag{4.7}$$

where $P(x,\tau) = (x,-\tau)$. Since Γ is bi-Lipschitz, we have that $\tilde{u} \in W^{1,1}(\tilde{\Omega};\mathbb{S}^1)$ and by a change of variables we can bound the L^1-norm of its gradient via

$$\int_{\tilde{\Omega}} |\nabla\tilde{u}|\,\mathrm{d}x \leqslant \int_{\Omega} |\nabla u|\,\mathrm{d}x + C_\Gamma \int_{\tilde{\Omega}\setminus\Omega} |(\nabla u)\circ\Gamma\circ P\circ\Gamma^{-1}|\,\mathrm{d}x \leqslant C_\Gamma \int_{\Omega} |\nabla u|\,\mathrm{d}x, \tag{4.8}$$

where the constant C_Γ depends only on the bi-Lipschitz properties of Γ and the dimension. With an abuse of notation we will denote the extended function $\tilde{u} \in W^{1,1}(\tilde{\Omega};\mathbb{S}^1)$ again by u.

Step 3. (Reducing to the case $u \in \mathcal{R}^\infty_1(\tilde{\Omega};\mathbb{S}^1)$). Given $u \in W^{1,1}(\Omega;\mathbb{S}^1)$, we extend it to a function in $W^{1,1}(\tilde{\Omega};\mathbb{S}^1)$ as in the previous step. By Theorem 4.2 there exists a sequence $u_j \in \mathcal{R}^\infty_1(\tilde{\Omega};\mathbb{S}^1)$ such that $u_j \to u$ strongly in $W^{1,1}(\tilde{\Omega};\mathbb{R}^2)$. In particular,

$$\lim_{j\to+\infty} \int_{\Omega} |\nabla u_j|_{2,1}\,\mathrm{d}x = \int_{\Omega} |\nabla u|_{2,1}.$$

Hence, by a diagonal argument it is enough to prove the upper bound (4.4) assuming $u \in \mathcal{R}^\infty_1(\tilde{\Omega};\mathbb{S}^1)$.

Step 4. (Reducing to the case of piecewise constant $\mathbb{S}^1$-valued maps). Let $u \in \mathcal{R}^\infty_1(\tilde{\Omega};\mathbb{S}^1)$. Then there exists $\Sigma = \bigcup_{h=1}^m \Sigma_h$ with Σ_h closed subset of a smooth $(d-2)$-dimensional manifold such that $u \in C^\infty(\tilde{\Omega}\setminus\Sigma;\mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega};\mathbb{S}^1)$. We construct now an approximation of u through $\mathbb{S}^1$-valued maps which are piecewise constant on a lattice of spacing $\lambda > 0$. Let us consider the family of half-open cubes

$$I_\lambda(\lambda z) = \lambda z + \lambda[0,1)^d\,, \quad z \in \mathbb{Z}^d$$

and the set

$$\Omega^\lambda := \bigcup\big\{I_\lambda(\lambda z) : z \in \mathbb{Z}^d \text{ such that } I_\lambda(\lambda z)\cap\Omega \neq \emptyset\big\}.$$

Let Ω' be such that $\Omega \subset\subset \Omega' \subset\subset \tilde{\Omega}$. For λ small enough we have $\Omega^\lambda \subset\subset \Omega' \subset\subset \tilde{\Omega}$. We now define the piecewise constant function $u_\lambda\colon \Omega^\lambda \to \mathbb{S}^1$ as follows. Let $z \in \mathbb{Z}^d$ be such that

$I_\lambda(\lambda z) \subset \Omega^\lambda$. If $\overline{I_\lambda(\lambda z)} \cap \Sigma = \emptyset$, the map u is C^∞ in the interior of $I_\lambda(\lambda z)$ and thus it admits a lifting φ_z (unique up to a multiple integer of 2π), which is C^∞ in the interior of $I_\lambda(\lambda z)$, namely $u = \exp(\iota\varphi_z)$ in $I_\lambda(\lambda z)$. We consider the average

$$\overline{\varphi}_z := \frac{1}{\lambda^d} \int_{I_\lambda(\lambda z)} \varphi_z(x)\,\mathrm{d}x$$

and we set $u_\lambda(x) := \exp(\iota\overline{\varphi}_z)$ for $x \in I_\lambda(\lambda z)$. If, instead, $\overline{I_\lambda(\lambda z)} \cap \Sigma \neq \emptyset$ we put $u_\lambda(x) := e_1$ for $x \in I_\lambda(\lambda z)$ (the precise value e_1 being not relevant).

We remark that $u_\lambda \to u$ strongly in $L^1(\Omega;\mathbb{R}^2)$. Indeed, let B be a ball such that $B \subset\subset \Omega \setminus \Sigma$. Since B is simply connected and $u \in C^\infty(\overline{B};\mathbb{S}^1)$, there exists a lifting $\varphi \in C^\infty(B;\mathbb{R})$, namely, $u = \exp(\iota\varphi)$ in B. If $I_\lambda(\lambda z) \cap B \neq \emptyset$, then $\overline{I_\lambda(\lambda z)} \cap \Sigma = \emptyset$ for λ small enough. In particular, we can consider the lifting φ_z of u in $I_\lambda(\lambda z)$ used in the definition of u_λ. By uniqueness of the liftings up to integer multiples of 2π, there exists a $k_z \in \mathbb{Z}$ such that $\varphi_z = \varphi + 2\pi k_z$. This entails

$$\overline{\varphi}_z = \frac{1}{\lambda^d} \int_{I_\lambda(\lambda z)} \varphi(y)\,\mathrm{d}y + 2\pi k_z.$$

Given $x \in B$, we consider a family of cubes $I_\lambda(\lambda z_\lambda) \ni x$. By Lebesgue's differentiation theorem

$$\frac{1}{\lambda^d} \int_{I_\lambda(\lambda z_\lambda)} \varphi(y)\,\mathrm{d}y \to \varphi(x)$$

for $\mathcal{L}^d$-a.e. $x \in B$. Then $u_\lambda \to u$ a.e. in Ω and by dominated convergence we obtain $u_\lambda \to u$ in $L^1(\Omega;\mathbb{R}^2)$.

Let us prove that

$$\limsup_{\lambda\to 0} \int_{\Omega^\lambda \cap J_{u_\lambda}} \mathrm{d}_{\mathbb{S}^1}(u_\lambda^-, u_\lambda^+)|\nu_{u_\lambda}|_1\,\mathrm{d}\mathcal{H}^{d-1} \leqslant \int_\Omega |\nabla u|_{2,1}\,\mathrm{d}x. \tag{4.9}$$

For $i \in \{1,\dots,d\}$ we define the families of indices

$$\mathcal{Z}_i(\lambda) := \Big\{z \in \mathbb{Z}^d \,:\, I_\lambda(\lambda z) \cup I_\lambda\big(\lambda(z+e_i)\big) \subset \Omega^\lambda\Big\},$$

$$\mathcal{G}_i(\lambda) := \Big\{z \in \mathcal{Z}_i(\lambda) : \overline{I_\lambda(\lambda z)} \cap \Sigma = \emptyset \quad \text{and} \quad \overline{I_\lambda\big(\lambda(z+e_i)\big)} \cap \Sigma = \emptyset\Big\},$$

$$\mathcal{B}_i(\lambda) := \Big\{z \in \mathcal{Z}_i(\lambda) : \overline{I_\lambda(\lambda z)} \cap \Sigma \neq \emptyset \quad \text{or} \quad \overline{I_\lambda\big(\lambda(z+e_i)\big)} \cap \Sigma \neq \emptyset\Big\}.$$

Let $z \in \mathcal{G}_i(\lambda)$. As in the definition of u_λ, we let φ_z and φ_{z+e_i} be the liftings of u in $I_\lambda(\lambda z)$ and $I_\lambda(\lambda(z+e_i))$, respectively. Moreover, since u is C^∞ in the interior of the rectangle $I_\lambda(\lambda z) \cup I_\lambda(\lambda(z+e_i))$, it admits a C^∞ lifting φ such that $u = \exp(\iota\varphi)$ in $I_\lambda(\lambda z) \cup I_\lambda(\lambda(z+e_i))$. By uniqueness of the liftings up to integer multiples of 2π, there exist $k_z, k_{z+e_i} \in \mathbb{Z}$ such that $\varphi_z = \varphi + 2\pi k_z$ in $I_\lambda(\lambda z)$ and $\varphi_{z+e_i} = \varphi + 2\pi k_{z+e_i}$ in $I_\lambda(\lambda(z+e_i))$. Note that

$$\overline{\varphi}_z = \frac{1}{\lambda^d} \int_{I_\lambda(\lambda z)} \varphi(x)\,\mathrm{d}x + 2\pi k_z\,, \quad \overline{\varphi}_{z+e_i} = \frac{1}{\lambda^d} \int_{I_\lambda(\lambda(z+e_i))} \varphi(x)\,\mathrm{d}x + 2\pi k_{z+e_i}\,.$$

Now we are in a position to estimate

$$
\begin{aligned}
\mathrm{d}_{\mathbb{S}^1}\big(u_\lambda\big(\lambda(z+e_i)\big),u_\lambda(\lambda z)\big) &= \mathrm{d}_{\mathbb{S}^1}\big(\exp(\iota\overline{\varphi}_{z+e_i}),\exp(\iota\overline{\varphi}_z)\big) \\
&\leqslant \frac{1}{\lambda^d}\Big|\int_{I_\lambda(\lambda(z+e_i))}\varphi(x)\,\mathrm{d}x-\int_{I_\lambda(\lambda z)}\varphi(x)\,\mathrm{d}x\Big| \\
&= \frac{1}{\lambda^d}\int_{I_\lambda(\lambda(z))}\big|\varphi(x+\lambda e_i)-\varphi(x)\big|\,\mathrm{d}x \\
&\leqslant \frac{1}{\lambda^{d-1}}\int_{I_\lambda(\lambda(z))}\int_0^1\big|\partial_i\varphi(x+t\lambda e_i)\big|\,\mathrm{d}t\,\mathrm{d}x \\
&= \frac{1}{\lambda^{d-1}}\int_0^1\int_{I_\lambda(\lambda(z))}\big|\partial_i u(x+t\lambda e_i)\big|\,\mathrm{d}x\,\mathrm{d}t. \qquad (4.10)
\end{aligned}
$$

Using the fact that $\Omega^\lambda\subset\subset\Omega'$, for λ small enough we obtain

$$
\begin{aligned}
\sum_{i=1}^d\sum_{z\in\mathcal{G}_i(\lambda)}\lambda^{d-1}\mathrm{d}_{\mathbb{S}^1}\big(u_\lambda\big(\lambda(z+e_i)\big),u_\lambda(\lambda z)\big) &\leqslant \sum_{i=1}^d\sum_{z\in\mathcal{G}_i(\lambda)}\int_0^1\int_{I_\lambda(\lambda(z))}\big|\partial_i u(x+t\lambda e_i)\big|\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \int_0^1\sum_{i=1}^d\int_{\Omega^\lambda}\big|\partial_i u(x+t\lambda e_i)\big|\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \sum_{i=1}^d\int_{\Omega'}\big|\partial_i u(x)\big|\,\mathrm{d}x=\int_{\Omega'}\big|\nabla u(x)\big|_{2,1}\,\mathrm{d}x.
\end{aligned}
$$

Let $z\in\mathcal{B}_i(\lambda)$. Since $\overline{I_\lambda(\lambda z)}\cap\Sigma\neq\emptyset$ or $\overline{I_\lambda(\lambda(z+e_i))}\cap\Sigma\neq\emptyset$, we have that $\overline{I_\lambda(\lambda z)}\subset B_{4\lambda\sqrt{d}}(\Sigma)$. By [2, Theorem 2.104], the Minkowski content of Σ equals its Hausdorff measure, namely $\frac{\mathcal{L}^d(B_\rho(\Sigma))}{\omega_2\rho^2}\to\mathcal{H}^{d-2}(\Sigma)$ as $\rho\to 0$. This implies that

$$
\#\mathcal{B}_i(\lambda)\leqslant\frac{1}{\lambda^d}\mathcal{L}^d\big(B_{4\lambda\sqrt{d}}(\Sigma)\big)\leqslant\frac{1}{\lambda^d}2\mathcal{H}^{d-2}(\Sigma)\omega_2(4\lambda\sqrt{d})^2\leqslant C_{\Sigma,d}\frac{1}{\lambda^{d-2}}
$$

for λ small enough. Using the rough estimate $\mathrm{d}_{\mathbb{S}^1}\big(u_\lambda(\lambda(z+e_i)),u_\lambda(\lambda z)\big)\leqslant\pi$ we deduce that

$$
\sum_{i=1}^d\sum_{z\in\mathcal{B}_i(\lambda)}\lambda^{d-1}\mathrm{d}_{\mathbb{S}^1}\Big(u_\lambda\big(\lambda(z+e_i)\big),u_\lambda(\lambda z)\Big)\leqslant C_{\Sigma,d}\lambda, \qquad (4.11)
$$

the constant $C_{\Sigma,d}$ being larger than the previous one.

From (4.10) and (4.11) it follows that

$$
\begin{aligned}
&\int_{\Omega^\lambda\cap J_{u_\lambda}}\mathrm{d}_{\mathbb{S}^1}(u_\lambda^+,u_\lambda^-)|\nu_{u_\lambda}|_1\,\mathrm{d}\mathcal{H}^{d-1} \\
&\leqslant\sum_{i=1}^d\Big(\sum_{z\in\mathcal{G}_i(\lambda)}\lambda^{d-1}\mathrm{d}_{\mathbb{S}^1}\Big(u_\lambda\big(\lambda(z+e_i)\big),u_\lambda(\lambda z)\Big)+\sum_{z\in\mathcal{B}_i(\lambda)}\lambda^{d-1}\mathrm{d}_{\mathbb{S}^1}\Big(u_\lambda\big(\lambda(z+e_i)\big),u_\lambda(\lambda z)\Big)\Big) \\
&\leqslant\int_{\Omega'}|\nabla u|_{2,1}\,\mathrm{d}x+C_{\Sigma,d}\lambda
\end{aligned}
$$

and hence, letting $\lambda \to 0$ and $\Omega' \searrow \Omega$, (4.9). Thanks to this step, it suffices to prove the upper bound assuming that the $\mathbb{S}^1$-valued map is constant on each of the cubes $I_\lambda(\lambda z) \subset \Omega^\lambda$.

Step 5. (Construction of u_N). Let $u_\lambda : \Omega^\lambda \to \mathbb{S}^1$ be a map that is constant on each of the cubes $I_\lambda(\lambda z)$. We consider the discretization map $\mathfrak{P}_N : \mathbb{S}^1 \to \mathcal{S}_N$ defined as follows: given $a \in \mathbb{S}^1$, we let $\varphi_a \in [0, 2\pi)$ be the unique angle such that $a = \exp(\iota\varphi)$ and we set

$$\mathfrak{P}_N(a) := \exp\big(\iota\theta_N \lfloor \varphi_a/\theta_N \rfloor\big).$$

Note that $\mathrm{d}_{\mathbb{S}^1}(\mathfrak{P}_N(a), a) = |\theta_N \lfloor \varphi_a/\theta_N \rfloor - \varphi_a| \leqslant \theta_N$. We put $u_N := \mathfrak{P}_N(u_\lambda) \in BV(\Omega; \mathcal{S}_N)$.

Then, by the triangle inequality

$$\begin{aligned}
&\int_{\Omega \cap J_{u_N}} \mathrm{d}_{\mathbb{S}^1}(u_N^-, u_N^+)|\nu_{u_N}|_1 \,\mathrm{d}\mathcal{H}^{d-1} \\
&\quad \leqslant \sum_{z \in \mathcal{Z}_i(\lambda)} \lambda^{d-1} \mathrm{d}_{\mathbb{S}^1}\Big(u_N\big(\lambda(z+e_i)\big), u_N(\lambda z)\Big) \\
&\quad \leqslant \int_{\Omega^\lambda \cap J_{u_\lambda}} \mathrm{d}_{\mathbb{S}^1}(u_\lambda^+, u_\lambda^-)|\nu_{u_\lambda}|_1 \,\mathrm{d}\mathcal{H}^{d-1} \\
&\qquad\quad + \sum_{z \in \mathcal{Z}_i(\lambda)} \lambda^{d-1} \Big(\mathrm{d}_{\mathbb{S}^1}\Big(u_N\big(\lambda(z+e_i)\big), u_\lambda\big(\lambda(z+e_i)\big)\Big) + \mathrm{d}_{\mathbb{S}^1}\big(u_N(\lambda z), u_\lambda(\lambda z)\big)\Big) \\
&\quad \leqslant \int_{\Omega^\lambda \cap J_{u_\lambda}} \mathrm{d}_{\mathbb{S}^1}(u_\lambda^+, u_\lambda^-)|\nu_{u_\lambda}|_1 \,\mathrm{d}\mathcal{H}^{d-1} + \sum_{z \in \mathcal{Z}_i(\lambda)} \lambda^{d-1} 2\theta_N \\
&\quad \leqslant \int_{\Omega^\lambda \cap J_{u_\lambda}} \mathrm{d}_{\mathbb{S}^1}(u_\lambda^+, u_\lambda^-)|\nu_{u_\lambda}|_1 \,\mathrm{d}\mathcal{H}^{d-1} + 2\theta_N \mathcal{H}^{d-1}(\Omega^\lambda \cap J_{u_\lambda}).
\end{aligned}$$

Letting $N \to +\infty$ and by (4.3) we conclude the proof. □

5. Constrained problems

In this final section we apply the results for the discrete-to-continuum limit to some constrained minimization problem. Again here we can use the more abstract results of [19]. We consider the case of discrete Dirichlet boundary conditions and discrete phase constraints. We start with the latter. Note that in both cases we do not state separately the convergence of minimizers which is a standard consequence of the general theory of Γ-convergence.

Volume constraints in the N-clock model. Let $V = (V_k)_{k=1}^N \in (0,1)^N$ be such that $\sum_{k=1}^N V_k = 1$ and for $k = 1, \ldots, N$ let $V_{k,\varepsilon} \in \big(\#(\varepsilon\mathbb{Z}^d \cap \Omega)^{-1}\mathbb{N}\big) \cap [0,1]$ be such that

$$\lim_{\varepsilon \to 0} V_{k,\varepsilon} = V_k \qquad \forall 1 \leqslant k \leqslant N. \tag{5.1}$$

We define a new set of constrained spin configurations by

$$\mathcal{PC}_\varepsilon(V) := \left\{u : \varepsilon\mathbb{Z}^d \cap \Omega \to \mathcal{S}_N : \frac{\#\{u = \exp(ik\theta)\}}{\#(\varepsilon\mathbb{Z}^d \cap \Omega)} = V_{k,\varepsilon} \quad \forall 1 \leqslant k \leqslant N\right\}.$$

Define then the constrained functional

$$E^N_{\varepsilon,V}(u) = \begin{cases} E^N_\varepsilon(u) & \text{if } u \in \mathcal{PC}_\varepsilon(V), \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^2). \end{cases}$$

Then by [19, Theorem 6.2] we have the following Γ-convergence result.

Corollary 5.1 *Let $N \in \mathbb{N}$ and for $1 \leqslant k \leqslant N$ let $V_{k,\varepsilon} \in (0,1)$ satisfy (5.1). Then as $\varepsilon \to 0$ the sequence of functionals $\frac{N_\varepsilon}{2\pi\varepsilon} E^N_{\varepsilon,V}$ Γ-converge with respect to the strong $L^1(\Omega;\mathbb{R}^2)$ to the functional $E_{N,V}\colon L^1(\Omega;\mathbb{R}^2) \to [0,+\infty]$ defined by*

$$E_{N,V}(u) := \begin{cases} \dfrac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N^2} \displaystyle\int_{\Omega\cap J_u} \mathrm{d}_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1 \, \mathrm{d}\mathcal{H}^{d-1} & \text{if } u \in BV(\Omega;\mathcal{S}_N) \text{ and } |\{u = \exp(ik\theta)\}| = V_k \ \forall 1 \leqslant k \leqslant N, \\ +\infty & \text{otherwise.} \end{cases}$$

Dirichlet Boundary conditions. In order to define discrete Dirichlet boundary conditions and to derive a convergence result, we need to assume some well-preparedness of the boundary condition. For the sake of simplicity we assume that $u_0 \in BV_{\mathrm{loc}}(\mathbb{R}^d;\mathcal{S}_N)$ is such that there exists a locally finite partition of $\mathbb{R}^d$ into Lipschitz domains $(E_i)_{i\in\mathbb{N}}$ such that u_0 is constant on each E_i. In particular, the closure of the jump set $\overline{J}_{u_0}$ is a locally finite union of Lipschitz graphs. When the sets $(E_i)_{i\in\mathbb{N}}$ are polyhedra, we call u_0 a polyhedral partition. We further assume that

$$\mathcal{H}^{d-1}(\partial\Omega \cap \overline{J}_{u_0}) = 0.\text{[1]} \tag{5.2}$$

We define the set of configurations satisfying a discrete Dirichlet boundary condition $u = u_0$ by

$$\mathcal{PC}_{\varepsilon,u_0} = \left\{u\colon \varepsilon\mathbb{Z}^d \cap \Omega \to \mathcal{S}_N : u(\varepsilon i) = u_0(\varepsilon i) \text{ if } \mathrm{dist}(\varepsilon i,\partial\Omega) \leqslant 2\varepsilon\right\}.$$

As for the case of volume constraints, we define the constrained functionals

$$E^N_{\varepsilon,u_0}(u) := \begin{cases} E^N_\varepsilon(u) & \text{if } u \in \mathcal{PC}_{\varepsilon,u_0}, \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^2). \end{cases}$$

Then we have the following result.

Corollary 5.2 *Let $u_0 \in BV_{\mathrm{loc}}(\mathbb{R}^d;\mathcal{S}_N)$ be a polyhedral partition satisfying (5.2). Then as $\varepsilon \to 0$ the sequence of functionals $\frac{N_\varepsilon}{2\pi\varepsilon} E^N_{\varepsilon,u_0}$ Γ-converge with respect to the strong $L^1(\Omega;\mathbb{R}^2)$ to the functional $E_{N,u_0}\colon L^1(\Omega;\mathbb{R}^2) \to [0,+\infty]$ that is finite only on $BV(\Omega;\mathcal{S}_N)$, where it is given by*

$$E_{N,u_0}(u) := \frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N^2}\left(\int_{\Omega\cap J_u} \mathrm{d}_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial\Omega} \mathrm{d}_{\mathbb{S}^1}(u^-,u_0^+)|\nu_x|_1\,\mathrm{d}\mathcal{H}^{d-1}\right),$$

where ν_x denotes the unit outer normal vector at $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$.

[1] This is a technical assumption that we need in the proof. In general, this condition can be ensured by a local reflection argument which does not change u_0 inside Ω, but one would need to prove that this reflection keeps the level sets Lipschitz regular. While this should follow from construction, we avoid such technical details here.

Proof. *Step 1. Proof of the* $\liminf$-*inequality.* Without loss of generality let $u_\varepsilon \to u$ in $L^1(\Omega)$ be such that

$$\liminf_{\varepsilon\to 0} \frac{N_\varepsilon}{2\pi\varepsilon} E^N_{\varepsilon,u_0}(u_\varepsilon) \leqslant C.$$

By Theorem 1.2(i) we obtain that $u \in BV(\Omega;\mathcal{S}_N)$. Passing to a subsequence we can assume that $u_\varepsilon \in \mathcal{PC}_{\varepsilon,u_0}$ for all ε. Given $\delta > 0$, using property (5.2), we find Lipschitz sets $\Omega_1 \subset\subset \Omega \subset\subset \Omega_2$ such that

$$\mathcal{H}^{d-1}((\overline{\Omega}_2 \setminus \Omega_1) \cap \overline{J}_{u_0}) \leqslant \delta. \tag{5.3}$$

Define $u_{\varepsilon,0} \in \mathcal{PC}_{\varepsilon,u_0}$ by $u_{\varepsilon,0}(\varepsilon x) := u_0(\varepsilon x)$. Since the level sets of u_0 are Lipschitz sets, it is not difficult to prove that $u_{\varepsilon,0} \to u_0$ in $L^1(\Omega)$. Moreover, note that for $\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^d$ with $|i-j| = 1$ we have that $u_{\varepsilon,0}(\varepsilon i) \neq u_{\varepsilon,0}(\varepsilon j)$ only if

$$\{\varepsilon i, \varepsilon j\} \in \overline{J}_{u_0} + B_\varepsilon(0).$$

In order to bound the energy of $u_{\varepsilon,0}$ on the set $\Omega_2 \setminus \overline{\Omega}_1$, one can use suitable level sets of the signed distance function to $\partial\Omega_h$, $h = 1, 2$, to find sequences of Lipschitz sets $\Omega_{1,n}, \Omega_{2,n}$ such that $\Omega_{1,n} \subset\subset \Omega_1$, $\Omega_2 \subset\subset \Omega_{2,n}$ and $\Omega_{1,n} \uparrow \Omega_1$ and $\Omega_{2,n} \downarrow \Omega_2$. Additionally, we can suppose that

$$\mathcal{H}^{d-1}\big((\partial\Omega_{1,n} \cup \partial\Omega_{2,n}) \cap \overline{J}_{u_0}\big) = 0 \qquad \forall n \in \mathbb{N}. \tag{5.4}$$

For fixed n and ε small enough we then have

$$\begin{aligned} E^N_\varepsilon(u_{\varepsilon,0}, \Omega_2 \setminus \overline{\Omega}_1) &\leqslant C\varepsilon^d \#\big\{\varepsilon i \in \varepsilon\mathbb{Z}^d \cap \Omega_2 \setminus \overline{\Omega}_1 : \varepsilon i \in \overline{J}_{u_0} + B_\varepsilon(0)\big\} \\ &\leqslant C\,\big|\big(\Omega_{2,n} \setminus \overline{\Omega}_{1,n}\big) \cap \big(\overline{J}_{u_0} + B_{C\varepsilon}(0)\big)\big|. \end{aligned}$$

Since $\overline{J}_{u_0}$ is a locally finite union of Lipschitz graphs which satisfies (5.4), it follows by the theory of Minkowski contents that

$$\begin{aligned} \limsup_{\varepsilon\to 0} \frac{N}{2\pi\varepsilon} E^N_\varepsilon(u_{\varepsilon,0}, \Omega_2 \setminus \overline{\Omega}_1) &\leqslant \limsup_{\varepsilon\to 0} C \frac{\big|\big(\Omega_{2,n} \setminus \overline{\Omega}_{1,n}\big) \cap \big(\overline{J}_{u_0} + B_{C\varepsilon}(0)\big)\big|}{\varepsilon} \\ &\leqslant C\mathcal{H}^{d-1}\big((\Omega_{2,n} \setminus \overline{\Omega}_{1,n}) \cap \overline{J}_{u_0}\big). \end{aligned}$$

Letting $n \to +\infty$, it follows from (5.3) that

$$\limsup_{\varepsilon\to 0} \frac{N}{2\pi\varepsilon} E^N_\varepsilon(u_{0,\varepsilon}, \Omega_2 \setminus \overline{\Omega}_1) \leqslant C\mathcal{H}^{d-1}\big((\overline{\Omega}_2 \setminus \Omega_1) \cap \overline{J}_{u_0}\big) \leqslant C\delta. \tag{5.5}$$

Define $\tilde{u}_\varepsilon \in \mathcal{PC}_{\varepsilon,u_0}$ by

$$\tilde{u}_\varepsilon(\varepsilon i) = \mathbb{1}_\Omega(\varepsilon i) u_\varepsilon(\varepsilon i) + \big(1 - \mathbb{1}_\Omega(\varepsilon i)\big) u_{\varepsilon,0}(\varepsilon i).$$

Observe that

$$\tilde{u}_\varepsilon \to \tilde{u} := \mathbb{1}_\Omega u + (1 - \mathbb{1}_\Omega) u_0 \quad \text{ in } L^1(\Omega_2). \tag{5.6}$$

The energy of $\tilde{u}_\varepsilon$ in the larger set Ω_2 can be estimated by

$$E^N_{\varepsilon,u_0}(\tilde{u}_\varepsilon, \Omega_2) \leqslant E^N_{\varepsilon,u_0}(u_\varepsilon, \Omega) + E_\varepsilon(u_{\varepsilon,0}, \Omega_2 \setminus \Omega) + 2 \sum_{\substack{i,j \in \varepsilon\mathbb{Z}^d,\, |i-j|=1 \\ \varepsilon i \in \Omega, \varepsilon j \in \Omega_2 \setminus \Omega}} \frac{1}{2}\varepsilon^d |\tilde{u}_\varepsilon(\varepsilon i) - \tilde{u}_\varepsilon(\varepsilon j)|^2, \tag{5.7}$$

where we used that the discrete energy counts each interaction twice. Note that when $\varepsilon i \in \Omega$ and $\varepsilon j \in \Omega_2 \setminus \Omega$, then $\operatorname{dist}(\varepsilon i, \partial\Omega) \leqslant |\varepsilon i - \varepsilon j| = \varepsilon$, so that by the boundary conditions of u_ε imply that $\tilde{u}_\varepsilon(\varepsilon i) = u_{\varepsilon,0}(\varepsilon i)$. Moreover, by definition $\tilde{u}_\varepsilon(\varepsilon j) = u_{\varepsilon,0}(\varepsilon j)$, so that the last two terms in the right-hand side of (5.7) can be estimated via

$$E_\varepsilon^N(u_{\varepsilon,0}, \Omega_2 \setminus \Omega) + 2 \sum_{\substack{i,j \in \varepsilon\mathbb{Z}^d, |i-j|=1 \\ \varepsilon i \in \Omega, \varepsilon j \in \Omega_2 \setminus \Omega}} \frac{1}{2}\varepsilon^d |\tilde{u}_\varepsilon(\varepsilon i) - \tilde{u}_\varepsilon(\varepsilon j)|^2 \leqslant E_\varepsilon^N(u_{\varepsilon,0}, \Omega_2 \setminus \overline{\Omega}_1).$$

Combining this bound with (5.7) and (5.5) we conclude that

$$\liminf_{\varepsilon\to 0} \frac{N}{2\pi\varepsilon} E^N_{\varepsilon,u_0}(\tilde{u}_\varepsilon, \Omega_2) \leqslant \liminf_{\varepsilon\to 0} \frac{N}{2\pi\varepsilon} E^N_{\varepsilon,u_0}(u_\varepsilon, \Omega) + C\delta.$$

Since Theorem 1.2 holds for any bounded Lipschitz-domain, we can in particular choose Ω_2 so that with (5.6) we obtain

$$\frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N^2} \int_{\Omega_2 \cap J_{\tilde{u}}} \mathrm{d}_{\mathbb{S}^1}(\tilde{u}^-, \tilde{u}^+)|\nu_{\tilde{u}}|_1 \, \mathrm{d}\mathcal{H}^{d-1} \leqslant \liminf_{\varepsilon\to 0} \frac{N}{2\pi\varepsilon} E^N_{\varepsilon,u_0}(u_\varepsilon, \Omega) + C\delta.$$

Now letting $\Omega_2 \downarrow \overline{\Omega}$ and then $\delta \to 0$ we obtain the lim inf-inequality using the structure of $\tilde{u}$ given by (5.6).

Step 2. Proof of the lim sup-*inequality.* We start assuming that $u = u_0$ in a neighborhood of $\partial\Omega$. Let $u_\varepsilon \colon \varepsilon\mathbb{Z}^d \to \mathcal{S}_N$ be a recovery sequence for u given by Theorem 1.2, so that

$$\lim_{\varepsilon\to 0} \frac{N}{2\pi\varepsilon} E_\varepsilon^N(u_\varepsilon, \Omega) = \frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N^2} \int_{\Omega \cap J_u} \mathrm{d}_{\mathbb{S}^1}(u^-, u^+)|\nu_u|_1 \, \mathrm{d}\mathcal{H}^{d-1}. \tag{5.8}$$

We will modify u_ε such that it fulfills the discrete boundary conditions but without changing its energy too much. First we redefine $u_\varepsilon(\varepsilon i) = u_0(\varepsilon i)$ for all $\varepsilon i \notin \Omega$. Since $u = u_0$ in a neighborhood of $\partial\Omega$, we can find a Lipschitz set $\Omega_1 \subset\subset \Omega$ such that

$$\begin{aligned} &u = u_0 \text{ on } \Omega \setminus \Omega_1, \\ &\mathcal{H}^{d-1}(\partial\Omega_1 \cap \overline{J}_{u_0}) = 0. \end{aligned} \tag{5.9}$$

Fix $0 < \eta \leqslant \frac{1}{2}\operatorname{dist}(\Omega_1, \partial\Omega)$. Setting $M_\varepsilon = \lfloor \frac{\eta}{2\sqrt{d}\varepsilon} \rfloor$ (i.e., the integer part of $\frac{\eta}{2\sqrt{d}\varepsilon}$), for $1 \leqslant k \leqslant M_\varepsilon$ we introduce the sets

$$\Omega_{\varepsilon,k} := \left\{x \in \Omega : \operatorname{dist}(x, \Omega_1) < \sqrt{d}k\varepsilon\right\}.$$

We further define $u_\varepsilon^k \in \mathcal{PC}_{\varepsilon,u_0}$ via interpolation by

$$u_\varepsilon^k(\varepsilon i) := \mathbb{1}_{\Omega_{\varepsilon,k}}(\varepsilon i) u_\varepsilon(\varepsilon i) + \left(1 - \mathbb{1}_{\Omega_{\varepsilon,k}}(\varepsilon i)\right) u_0(\varepsilon i).$$

Next we compute the energy of such a spin field. It holds that

$$\begin{aligned} E_\varepsilon^N(u_\varepsilon^k, \Omega) &\leqslant E_\varepsilon^N(u_\varepsilon, \Omega_{\varepsilon,k}) + E_\varepsilon^N(u_{\varepsilon,0}, \Omega \setminus \Omega_{\varepsilon,k}) + \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^d, |i-j|=1 \\ \varepsilon i \in \Omega_{\varepsilon,k}, \varepsilon j \in \Omega \setminus \Omega_{\varepsilon,k}}} \varepsilon^d \left|u_\varepsilon^k(\varepsilon i) - u_\varepsilon^k(\varepsilon j)\right|^2 \\ &\leqslant E_\varepsilon^N(u_\varepsilon, \Omega) + E_\varepsilon^N(u_{\varepsilon,0}, \Omega \setminus \overline{\Omega}_1) + \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^d, |i-j|=1 \\ \varepsilon i \in \Omega_{\varepsilon,k}, \varepsilon j \in \Omega \setminus \Omega_{\varepsilon,k}}} \varepsilon^d \left|u_\varepsilon^k(\varepsilon i) - u_\varepsilon^k(\varepsilon j)\right|^2. \end{aligned} \tag{5.10}$$

Note that by a discrete product rule we have

$$u_\varepsilon^k(\varepsilon i) - u_\varepsilon^k(\varepsilon j) = \mathbb{1}_{\Omega_{\varepsilon,k}}(\varepsilon i)\big(u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)\big) + \big(\mathbb{1}_{\Omega_{\varepsilon,k}}(\varepsilon i) - \mathbb{1}_{\Omega_{\varepsilon,k}}(\varepsilon j)\big)\big(u_\varepsilon(\varepsilon j) - u_0(\varepsilon j)\big) + \big(1 - \mathbb{1}_{\Omega_{\varepsilon,k}}(\varepsilon i)\big)\big(u_0(\varepsilon i) - u_0(\varepsilon j)\big).$$

Using this inequality, the terms in the last sum above can be bounded by

$$|u_\varepsilon^k(\varepsilon i) - u_\varepsilon^k(\varepsilon j)|^2 \leqslant 2\left(|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2 + |u_\varepsilon(\varepsilon j) - u_0(\varepsilon j)|^2 + |u_0(\varepsilon i) - u_0(\varepsilon j)|^2\right).$$

Hence, we can control the energy on the boundary layer of $\partial\Omega_{\varepsilon,k}$ by

$$\sum_{\substack{\varepsilon i,\varepsilon j\in\varepsilon\mathbb{Z}^d,\,|i-j|=1\\ \varepsilon i\in\Omega_{\varepsilon,k},\varepsilon j\in\Omega\setminus\Omega_{\varepsilon,k}}} \varepsilon^d|u_\varepsilon^k(\varepsilon i)-u_\varepsilon^k(\varepsilon j)|^2 \leqslant 4E_\varepsilon^N(u_\varepsilon,\Omega_{\varepsilon,k+1}\setminus\Omega_{\varepsilon,k-1})+4E_\varepsilon^N(u_{\varepsilon,0},\Omega_{\varepsilon,k+1}\setminus\Omega_{\varepsilon,k-1}) + C\sum_{\varepsilon j\in\varepsilon\mathbb{Z}^d\cap\Omega_{\varepsilon,k+1}\setminus\Omega_{\varepsilon,k-1}} \varepsilon^d|u_\varepsilon(\varepsilon j) - u_0(\varepsilon j)|^2.$$

We can estimate the last sum as an integral over the slightly larger set $S_{\varepsilon,k} := \Omega_{\varepsilon,k+2}\setminus\Omega_{\varepsilon,k-2}$, so that we can continue the estimate (5.10) by

$$E_\varepsilon^N(u_\varepsilon^k,\Omega) \leqslant E_\varepsilon^N(u_\varepsilon,\Omega) + E_\varepsilon^N(u_{\varepsilon,0},\Omega\setminus\overline{\Omega}_1) + C\left(E_\varepsilon^N(u_\varepsilon,S_{\varepsilon,k}) + E_\varepsilon^N(u_{\varepsilon,0},S_{\varepsilon,k})\right) + C\|u_\varepsilon - u_{\varepsilon,0}\|^2_{L^2(S_{\varepsilon,k})}.$$

Now we can use a classical averaging argument. Note that each point $\varepsilon i \in \varepsilon\mathbb{Z}^d$ can belong to at most 4 different sets $S_{\varepsilon,k}$. Since $S_{\varepsilon,k}\subset\Omega\setminus\overline{\Omega}_1$ for all $3\leqslant k\leqslant M_\varepsilon-2$, the superadditivity of the energy yields

$$\sum_{k=3}^{M_\varepsilon-2}\left(E_\varepsilon^N(u_\varepsilon,S_{\varepsilon,k}) + E_\varepsilon^N(u_{\varepsilon,0},S_{\varepsilon,k}) + \|u_\varepsilon - u_{\varepsilon,0}\|^2_{L^2(S_{\varepsilon,k})}\right) \leqslant 4\left(E_\varepsilon^N\big(u_\varepsilon,\Omega\setminus\overline{\Omega}_1\big) + E_\varepsilon^N\big(u_{\varepsilon,0},\Omega\setminus\overline{\Omega}_1\big) + \|u_\varepsilon - u_{\varepsilon,0}\|^2_{L^2(\Omega\setminus\overline{\Omega}_1)}\right).$$

Now choose $k_\varepsilon\in\{3,\dots,M_\varepsilon-2\}$ such that the term in the above sum is minimal. Then

$$\begin{aligned} E_\varepsilon^N\big(u_\varepsilon^{k_\varepsilon},\Omega\big) &\leqslant E_\varepsilon^N(u_\varepsilon,\Omega) + E_\varepsilon^N\big(u_{\varepsilon,0},\Omega\setminus\overline{\Omega}_1\big) \\ &\quad + \frac{C}{M_\varepsilon-4}\sum_{k=3}^{M_\varepsilon-2}\left(E_\varepsilon^N(u_\varepsilon,S_{\varepsilon,k}) + E_\varepsilon^N(u_{\varepsilon,0},S_{\varepsilon,k}) + \|u_\varepsilon - u_{\varepsilon,0}\|^2_{L^2(S_{\varepsilon,k})}\right) \\ &\leqslant \left(1+\frac{C}{M_\varepsilon}\right)\left(E_\varepsilon^N(u_\varepsilon,\Omega) + E_\varepsilon^N(u_{\varepsilon,0},\Omega\setminus\overline{\Omega}_1)\right) + \frac{C}{M_\varepsilon}\|u_\varepsilon - u_{\varepsilon,0}\|^2_{L^2(\Omega\setminus\overline{\Omega}_1)}. \end{aligned}$$

Multiplying the above estimate by $\frac{N}{2\pi\varepsilon}$, we note that $(\varepsilon M_\varepsilon)^{-1}$ remains bounded when $\varepsilon\to 0$. Moreover, as in Step 1 (cf. (5.5) and (5.2), (5.9)) one shows that

$$\limsup_{\varepsilon\to 0}\frac{N}{2\pi\varepsilon}E_\varepsilon^N\big(u_{\varepsilon,0},\Omega\setminus\overline{\Omega}_1\big) \leqslant C\mathcal{H}^{d-1}\big((\overline{\Omega}\setminus\Omega_1)\cap\overline{J}_{u_0}\big) = C\mathcal{H}^{d-1}\big((\Omega\setminus\overline{\Omega}_1)\cap\overline{J}_{u_0}\big),$$

while for the term $\frac{N}{2\pi\varepsilon}E_\varepsilon^N(u_\varepsilon,\Omega)$ we can use (5.8). In order to control the L^2-norm in the last but one estimate, note that on $\Omega\setminus\Omega_1$ the two sequences u_ε and $u_{\varepsilon,0}$ converge in L^1 (and thus in L^2 by uniform boundedness) to the same limit since $u=u_0$ on $\Omega\setminus\Omega_1$. Consequently,

$$\limsup_{\varepsilon\to 0}\frac{N}{2\pi\varepsilon}E_\varepsilon^N(u_\varepsilon^{k_\varepsilon},\Omega)\leqslant\limsup_{\varepsilon\to 0}\frac{N}{2\pi\varepsilon}E_\varepsilon^N(u_\varepsilon,\Omega)+C\mathcal{H}^{d-1}\big((\Omega\setminus\overline{\Omega}_1)\cap\overline{J}_{u_0}\big).$$

In order to conclude, we remark that on Ω_1 it holds that $u_\varepsilon^{k_\varepsilon}\to u$ in L^1, while as explained above, on $\Omega\setminus\Omega_1$ the sequences u_ε and $u_{\varepsilon,0}$ have the same limit, so that $u_\varepsilon^{k_\varepsilon}$, being a varying convex-combination of those sequences, converges also to u on $\Omega\setminus\Omega_1$. We conclude that $u_\varepsilon^{k_\varepsilon}\to u$ in $L^1(\Omega)$. Hence from the definition of the Γ-$\limsup$ we deduce that

$$\Gamma\text{-}\limsup_{\varepsilon\to 0}\left(\frac{N}{2\pi\varepsilon}E_{\varepsilon,u_0}^N(\cdot,\Omega)\right)(u)\leqslant\limsup_{\varepsilon\to 0}\frac{N}{2\pi\varepsilon}E_\varepsilon^N(u_\varepsilon^{k_\varepsilon},\Omega)+C\,\mathcal{H}^{d-1}\big((\Omega\setminus\overline{\Omega}_1)\cap\overline{J}_{u_0}\big).$$

As the choice of Ω_1 with the properties (5.9) was arbitrary, we can take a sequence $\Omega_{1,n}$ such that $\Omega_{1,n}\uparrow\Omega$ which makes the last term negligible. Hence

$$\Gamma\text{-}\limsup_{\varepsilon\to 0}\left(\frac{N}{2\pi\varepsilon}E_{\varepsilon,u_0}^N(\cdot,\Omega)\right)(u)\leqslant\frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N^2}\int_{\Omega\cap J_u}\mathrm{d}_{\mathbb{S}^1}(u-,u^+)|\nu_u|_1\,\mathrm{d}\mathcal{H}^{d-1}.\tag{5.11}$$

Note that due to (5.2) the right-hand side coincides with the functional E_{N,u_0} if the function u coincides with u_0 in a neighborhood of $\partial\Omega$. The general case can be deduced with a density argument as follows.

Given any $u\in BV(\Omega;\mathcal{S}_N)$ we extend it to $\mathbb{R}^d$ by setting $u|_{\mathbb{R}^d\setminus\Omega}:=e_1$ and we let $\Omega_2\supset\supset\Omega$ be a Lipschitz set. Set $\tilde{u}=\mathbb{1}_\Omega u+(1-\mathbb{1}_\Omega)u_0$. Due to (5.2) we can apply [19, Lemma B.1] to deduce that there exists a sequence $\Omega_n\subset\subset\Omega$ of sets of finite perimeter such that $u_n:=\mathbb{1}_{\Omega_n}u+(1-\mathbb{1}_{\Omega_n})u_0$ converges to $\tilde{u}$ in $L^1(\Omega_2)$ and in addition $\mathcal{H}^{d-1}(\Omega_2\cap J_{u_n})\to\mathcal{H}^{d-1}(\Omega_2\cap J_{\tilde{u}})$. Those properties allow us to use the Reshetnyak continuity result for partitions [37, Theorem 3.1] which yields that

$$\lim_{n\to+\infty}\frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N^2}\int_{\Omega_2\cap J_{u_n}}\mathrm{d}_{\mathbb{S}^1}(u_n^-,u_n^+)|\nu_{u_n}|_1\,\mathrm{d}\mathcal{H}^{d-1}=\frac{4\sin^2\left(\frac{\theta_N}{2}\right)}{\theta_N^2}\int_{\Omega_2\cap J_{\tilde{u}}}\mathrm{d}_{\mathbb{S}^1}(\tilde{u}^-,\tilde{u}^+)|\nu_{\tilde{u}}|_1\,\mathrm{d}\mathcal{H}^{d-1}.$$

Moreover, note that $u_n=u_0$ in a neighborhood of $\partial\Omega$. Hence by the previous reasoning and lower semicontinuity of the Γ-lim sup we obtain

$$\begin{aligned}\Gamma\text{-}\limsup_{\varepsilon}\Big(\frac{N}{2\pi\varepsilon}E^N_{\varepsilon,u_0}(\cdot,\Omega)\Big)(u) &\leqslant \liminf_{n\to+\infty}\Big(\Gamma\text{-}\limsup_{\varepsilon\to 0}\Big(\frac{N}{2\pi\varepsilon}E^N_{\varepsilon,u_0}(\cdot,\Omega)\Big)(u_n)\Big)\\ &\leqslant \liminf_{n\to+\infty}\frac{4\sin^2(\frac{\theta_N}{2})}{\theta_N^2}\int_{\Omega\cap J_{u_n}} \mathrm{d}_{\mathbb{S}^1}(u_n^-,u_n^+)|\nu_{u_n}|_1\,\mathrm{d}\mathcal{H}^{d-1}\\ &\leqslant \liminf_{n\to+\infty}\frac{4\sin^2(\frac{\theta_N}{2})}{\theta_N^2}\int_{\Omega_2\cap J_{u_n}} \mathrm{d}_{\mathbb{S}^1}(u_n^-,u_n^+)|\nu_{u_n}|_1\,\mathrm{d}\mathcal{H}^{d-1}\\ &= \frac{4\sin^2(\frac{\theta_N}{2})}{\theta_N^2}\int_{\Omega_2\cap J_{\tilde u}} \mathrm{d}_{\mathbb{S}^1}(\tilde u^-,\tilde u^+)|\nu_{\tilde u}|_1\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

Letting $\Omega_2 \downarrow \overline{\Omega}$ yields the upper bound using the structure of $\tilde u$. □

Acknowledgments. The work of M. Cicalese was supported by the DFG Collaborative Research Center TRR 109, "Discretization in Geometry and Dynamics". G. Orlando has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 792583. G. Orlando thanks the TU München, where he was affiliated while working on the paper.

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