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Sharp interface limit of a Stokes/Cahn–Hilliard system Part I: Convergence result

HELMUT ABELS
Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
E-mail: helmut.abels@ur.de

ANDREAS MARQUARDT
Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
E-mail: andreas.marquardt@ur.de

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We consider the sharp interface limit of a coupled Stokes/Cahn–Hilliard system in a two-dimensional, bounded and smooth domain, i.e., we consider the limiting behavior of solutions when a parameter $\epsilon > 0$ corresponding to the thickness of the diffuse interface tends to zero. We show that for sufficiently short times the solutions to the Stokes/Cahn–Hilliard system converge to solutions of a sharp interface model, where the evolution of the interface is governed by a Mullins–Sekerka system with an additional convection term coupled to a two–phase stationary Stokes system with the Young–Laplace law for the jump of an extra contribution to the stress tensor, representing capillary stresses. We prove the convergence result by estimating the difference between the exact and an approximate solutions. To this end we make use of modifications of spectral estimates shown by X. Chen for the linearized Cahn–Hilliard operator. The treatment of the coupling terms requires careful estimates, the use of the refinements of the latter spectral estimate and a suitable structure of the approximate solutions, which will be constructed in the second part of this contribution.

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1. Introduction and overview

Classically, the transition between two immiscible fluids was considered to be sharp, in the sense of an appearance of a lower-dimensional surface separating the phases. The behavior of a multiphase system is then governed by the intricate interactions between the bulk regions and the interface, mathematically expressed as equations of motion, which hold in each fluid, complemented by boundary conditions at the (free) surface. Models incorporating these ideas – often called *sharp interface models* – and the corresponding free-boundary problems have been widely studied and used to great success in describing a multitude of physical and biological phenomena. However, fundamental problems arise in the analysis and numerical simulation of such problems, whenever the considered interfaces develop singularities. In fluid dynamics, topological changes such as the pinch off of droplets or collisions are non-negligible features of many systems, having a significant impact on the flow. Conversely, *diffuse interface models* turn out to provide a promising, alternative approach to describe such phenomena and overcome the associated difficulties. In these diffuse interface (or *phase field)* methods, a partial mixing of the two phases throughout a thin interfacial layer, heuristically viewed to have a thickness proportional to a length scale parameter $\epsilon > 0$, is taken into account. Naturally, the question of the behavior for the limit $\epsilon \to 0$ arises. This so-called *sharp interface limit* is in fact a question about the connection of sharp and diffuse interface models. Concerning the flow of two macroscopically immiscible, viscous, incompressible Newtonian fluids with matched densities, a fundamental and broadly accepted diffuse interface model is the so-called *model H*. This model consists of a Navier–Stokes system coupled with the Cahn–Hilliard equation and was derived in [15, 16]. The sharp interface limit was studied with the method of formally matched asymptotics in [2] and the existence of solutions for the model H was shown in [1, 10]. Regarding the formal sharp interface limit, short time existence of strong solutions was shown in [7] and existence of weak solutions for long times in [6]. Despite these analytic results and the formal findings for the sharp interface limit, there are only few attempts at rigorously discussing the sharp interface limit for the model H. Using the notion of varifold solutions as discussed in [12] such results for large times were shown in [6] for the model H and in [3] also for the more general situation of fluids with different densities. But the notion of solution for the latter contributions is rather weak and no rates of convergence were obtained and convergence was only shown for a suitable subsequence.

For the Allen–Cahn and Cahn–Hilliard equation another approach is based on the works [18] and [9], where the method of formally matched asymptotics is made rigorous. However, in view of two-phase flow models in fluid mechanics and the arising difficulties therein, the first and so far only convergence result with convergence rates in strong norms is [4]. More precisely, considering a coupled Stokes/Allen–Cahn system in two dimensions, it is shown that smooth solutions of the diffuse interface system converge for short times to solutions of a sharp interface model, where the evolution of the free surface is governed by a convective mean curvature flow coupled to a two-phase Stokes system together with the Young-Laplace law for the jump of the stress tensor, accounting for capillary forces. This contribution builds upon the ideas introduced in [4] and aims to establish the first rigorous result in strong norms for a sharp interface limit of a two phase flow model involving the Cahn–Hilliard equation with convergence rates. In doing so, we hope to build another cornerstone on the way to rigorously showing the sharp interface limit for model H.

More precisely we consider the Stokes/Cahn–Hilliard system

$$-\Delta \mathbf{v}^\epsilon + \nabla p^\epsilon = \mu^\epsilon \nabla c^\epsilon \qquad \text{in } \Omega_T, \tag{1.1}$$
$$\operatorname{div} \mathbf{v}^\epsilon = 0 \qquad \text{in } \Omega_T, \tag{1.2}$$
$$\partial_t c^\epsilon + \mathbf{v}^\epsilon \cdot \nabla c^\epsilon = \Delta \mu^\epsilon \qquad \text{in } \Omega_T, \tag{1.3}$$
$$\mu^\epsilon = -\epsilon \Delta c^\epsilon + \frac{1}{\epsilon} f'(c^\epsilon) \qquad \text{in } \Omega_T, \tag{1.4}$$
$$c^\epsilon|_{t=0} = c_0^\epsilon \qquad \text{in } \Omega, \tag{1.5}$$
$$(-2D_s\mathbf{v}^\epsilon + p^\epsilon \mathbf{I}) \cdot \mathbf{n}_{\partial\Omega} = \alpha_0 \mathbf{v}^\epsilon, \quad \mu^\epsilon = 0, \quad c^\epsilon = -1 \qquad \text{on } \partial\Omega \times (0,T). \tag{1.6}$$

Here $T > 0$, $\Omega \subset \mathbb{R}^2$ is a bounded and smooth domain, $\Omega_T := \Omega \times (0,T)$ and $\alpha_0 > 0$ is fixed. $\mathbf{v}^\epsilon: \Omega_T \to \mathbb{R}^2$ and $p^\epsilon: \Omega_T \to \mathbb{R}$ represent the mean velocity and pressure, $D_s\mathbf{v}^\epsilon := \frac{1}{2}\big(\nabla \mathbf{v}^\epsilon + (\nabla \mathbf{v}^\epsilon)^T\big)$, $c^\epsilon: \Omega \to \mathbb{R}$ is an order parameter representing the concentration difference of the fluids and $\mu^\epsilon: \Omega_T \to \mathbb{R}$ is the chemical potential of the mixture. Moreover, $c_0^\epsilon: \Omega \to \mathbb{R}$ is a suitable initial value, specified in Theorem 1.1 and $f: \mathbb{R} \to \mathbb{R}$ is a double well potential. The system corresponds to the model H if one would add the convection term $\partial_t \mathbf{v}^\epsilon + \mathbf{v}^\epsilon \cdot \nabla \mathbf{v}^\epsilon$ on the left-hand side to (1.1).

Existence of smooth solutions to (1.1)–(1.6) can be shown with similar methods as in [1]. A word is in order about the choice of boundary conditions (1.6). The reason we prescribe such boundary conditions for $\mathbf{v}^\epsilon$ instead of periodic, no-slip or Navier boundary conditions, are major difficulties which arise in the construction of the approximate solutions for $\mathbf{v}^\epsilon$. A more detailed account is given in [5, Remark 3.9]. Classically, the Cahn–Hilliard system is complemented with Neumann boundary conditions for c^ϵ and μ^ϵ. While it is rather unproblematic to adapt the present work to Neumann boundary conditions for c^ϵ, major issues arise when considering $\partial_{\mathbf{n}_{\partial\Omega}} \mu^\epsilon = 0$ instead of $\mu^\epsilon = 0$, see Remark 3.16 below. To circumvent these problems and as the focus of our interest and analysis lies in the obstacles and difficulties occurring close to the interface Γ_t, we decided on the present choice of boundary conditions. We will show that the sharp interface limit of (1.1)–(1.6) is given by the system

$$
\begin{aligned}
-\Delta \mathbf{v} + \nabla p &= 0 && \text{in } \Omega^\pm(t), t \in [0, T_0]\,, && (1.7)\\
\operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega^\pm(t), t \in [0, T_0]\,, && (1.8)\\
\Delta \mu &= 0 && \text{in } \Omega^\pm(t), t \in [0, T_0]\,, && (1.9)\\
(-2D_s \mathbf{v} + p\mathbf{I})\, \mathbf{n}_{\partial\Omega} &= \alpha_0 \mathbf{v} && \text{on } \partial_{T_0}\Omega, && (1.10)\\
\mu &= 0 && \text{on } \partial_{T_0}\Omega, && (1.11)\\
[2D_s \mathbf{v} - p\mathbf{I}]\, \mathbf{n}_{\Gamma_t} &= -2\sigma H_{\Gamma_t} \mathbf{n}_{\Gamma_t} && \text{on } \Gamma_t, t \in [0, T_0]\,, && (1.12)\\
\mu &= \sigma H_{\Gamma_t} && \text{on } \Gamma_t, t \in [0, T_0]\,, && (1.13)\\
-V_{\Gamma_t} + \mathbf{n}_{\Gamma_t} \cdot \mathbf{v} &= \frac{1}{2} [\mathbf{n}_{\Gamma_t} \cdot \nabla \mu] && \text{on } \Gamma_t, t \in [0, T_0]\,, && (1.14)\\
[\mathbf{v}] &= 0 && \text{on } \Gamma_t, t \in [0, T_0]\,, && (1.15)\\
\Gamma(0) &= \Gamma_0. && && (1.16)
\end{aligned}
$$

Here $T_0 > 0$, Ω is the disjoint union of smooth domains $\Omega^+(t)$, $\Omega^-(t)$ and a curve $\Gamma_t \subseteq \Omega$ for every $t \in [0, T_0]$, where $\Gamma_t = \partial\Omega^+(t)$, $\mathbf{n}_{\Gamma_t}$ is the exterior normal with respect to $\Omega^-(t)$, and H_{Γ_t} and V_{Γ_t} denote the mean curvature and normal velocity of the interface Γ_t. Furthermore, $\partial_{T_0}\Omega := \partial\Omega \times (0, T_0)$, Γ_0 is a given initial surface and we use the definitions

$$
\begin{aligned}
[g]\,(p,t) &:= \lim_{h \searrow 0} \Big(g\big(p + \mathbf{n}_{\Gamma_t}(p)h\big) - g\big(p - \mathbf{n}_{\Gamma_t}(p)h\big) \Big) \text{ for } p \in \Gamma_t,\\
\sigma &:= \frac{1}{2} \int_{-\infty}^{\infty} \theta_0'(s)^2 \, \mathrm{d}s, && (1.17)
\end{aligned}
$$

where $\theta_0 : \mathbb{R} \to \mathbb{R}$ is the so-called optimal profile, i.e., the unique solution to the ordinary differential equation

$$
-\theta_0'' + f'(\theta_0) = 0 \quad \text{in } \mathbb{R}, \qquad \theta_0(0) = 0, \quad \lim_{\rho \to \pm\infty} \theta_0(\rho) = \pm 1. \tag{1.18}
$$

Regarding the existence of local strong solutions of (1.7)–(1.16), the proof in [7] may be adapted, where a coupled Navier–Stokes/Mullins–Sekerka system was treated. Regularity theory for parabolic equations and the Stokes equation may then be used to show smoothness of the solution for smooth initial values.

Assuming that suitable approximate solutions $\left(c_A^\epsilon, \mu_A^\epsilon, \mathbf{v}_A^\epsilon, p_A^\epsilon\right)_{\epsilon>0}$ to (1.1)–(1.6) are constructed we show the existence of some $T_1 > 0$ such that the difference between c^ϵ and c_A^ϵ goes to zero in $L^\infty\left(0, T_1; H^{-1}(\Omega)\right)$ with $H^{-1}(\Omega) := \left(H_0^1(\Omega)\right)'$, $L^2(\Omega_{T_1})$, $L^2\left(0, T_1; H^1(\Omega)\right)$ and many other norms as $\epsilon \to 0$ with explicit convergence rates, for some small $T_1 > 0$. These rates will depend on the order up to which the approximate solutions have been constructed. Moreover, we will also present convergence rates for the error $\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon$ in $L^1\left(0, T_1; L^q(\Omega)\right)$ for $q \in (1, 2)$. This result is stated in Theorem 1.1. The key to this endeavors will be a modification of the spectral estimate for the linearized Cahn–Hilliard operator as given in [11], see Theorem 2.13 below. As in [4], the main difficulties which arise in the treatment of the Stokes/Cahn–Hilliard system are due to the appearance of the capillary term $\mu^\epsilon \nabla c^\epsilon$ in (1.1) and the convective term $\mathbf{v}^{\text{ffl}} \cdot \nabla c^\epsilon$ in (1.3). Although we may build upon the insights gained in the cited article, several new and severe obstacles arise in the context of system (1.1)–(1.6) which have to be overcome. A novelty in this context is the introduction of terms of fractional order in the asymptotic expansions. The necessity of such terms is at its core a consequence of our treatment of the convective term $\mathbf{v}^\epsilon \cdot \nabla c^\epsilon$. Where [4] relied on the intricate analysis of a second order, parabolic, degenerate partial differential equation in the construction of the highest order terms, the introduction of fractional order terms renders such considerations unnecessary. The caveat being, that while the produced fractional order terms are smooth, they may not be estimated uniformly in ϵ in arbitrarily strong norms. This is the cause for many technical subtleties in [5], where the construction is discussed and where estimates for the remainder are shown. See also the second author's PhD-thesis [17], which contains the results of this contribution and [5].

Throughout this work we make the following assumptions: Let $\Omega \subset \mathbb{R}^2$ be a smooth domain, $\Gamma_0 \subset\subset \Omega$ be a given, smooth, non-intersecting, closed initial curve. Let moreover $(\mathbf{v}, p, \mu, \Gamma)$ be a smooth solution to (1.7)–(1.16) and $(c^\epsilon, \mu^\epsilon, \mathbf{v}^\epsilon, p^\epsilon)$ be a smooth solution to (1.1)–(1.6) for some $T_0 > 0$. We assume that $\Gamma = \cup_{t\in[0,T_0]} \Gamma_t \times \{t\}$ is a smoothly evolving hypersurface in $\mathbb{R}^2$, where $(\Gamma_t)_{t\in[0,T_0]}$ are compact, non-intersecting, closed curves in Ω. We define $\Omega^+(t)$ to be the inside of Γ_t and set $\Omega^-(t)$ such that Ω is the disjoint union of $\Omega^+(t)$, $\Omega^-(t)$ and Γ_t. Moreover we define $\Omega_T^\pm = \cup_{t\in[0,T]} \Omega^\pm(t) \times \{t\}$, $\Omega_T := \Omega \times (0, T)$ and also $\partial_T \Omega := \partial\Omega \times (0, T)$ for $T \in [0, T_0]$. We define $\mathbf{n}_{\Gamma_t}(p)$ for $p \in \Gamma_t$ as the exterior normal with respect to $\Omega^-(t)$ and V_{Γ_t}, and H_{Γ_t} as the normal velocity and mean curvature of Γ_t with respect to $\mathbf{n}_{\Gamma_t}$, $t \in [0, T_0]$. Let

$$d_\Gamma : \Omega_{T_0} \to \mathbb{R},\ (x, t) \mapsto \begin{cases} \operatorname{dist}\left(\Omega^-(t), x\right) & \text{if } x \notin \Omega^-(t), \\ -\operatorname{dist}\left(\Omega^+(t), x\right) & \text{if } x \in \Omega^-(t) \end{cases}$$

denote the signed distance function to Γ such that d_Γ is positive inside $\Omega_{T_0}^+$. We write

$$\Gamma_t(\alpha) := \left\{x \in \Omega \,|\, |d_\Gamma(x, t)| < \alpha\right\}$$

for $\alpha > 0$ and set $\Gamma(\alpha; T) := \bigcup_{t\in[0,T]} \Gamma_t(\alpha) \times \{t\}$ for $T \in [0, T_0]$. Moreover, we assume that $\delta > 0$ is a small positive constant such that $\operatorname{dist}(\Gamma_t, \partial\Omega) > 5\delta$ for all $t \in [0, T_0]$ and such that the orthogonal projection $\mathrm{Pr}_{\Gamma_t} : \Gamma_t(3\delta) \to \Gamma_t$ is well-defined and smooth for all $t \in [0, T_0]$. In the following we often use the notation $\Gamma(\alpha) := \Gamma(\alpha; T_0)$ as a simplification. We also define a tubular neighborhood around $\partial\Omega$: For this let $d_{\mathbf{B}}: \Omega \to \mathbb{R}$ be the signed distance function to $\partial\Omega$ such that $d_{\mathbf{B}} < 0$ in Ω. As for Γ_t we define a tubular neighborhood by $\partial\Omega(\alpha) := \{x \in \Omega \,|\, -\alpha < d_{\mathbf{B}}(x) < 0\}$ and $\partial_T\Omega\,(\alpha) := \{(x, t) \in \Omega_T \,|\, d_{\mathbf{B}}(x) \in (-\alpha, 0)\}$ for $\alpha > 0$ and $T \in (0, T_0]$. Moreover, we denote the outer unit normal to Ω by $\mathbf{n}_{\partial\Omega}$ and denote the normalized tangent by $\tau_{\partial\Omega}$, which is fixed by the relation

$$\mathbf{n}_{\partial\Omega}(p) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tau_{\partial\Omega}(p)$$

for $p \in \partial\Omega$. Finally, we assume that $\delta > 0$ is chosen small enough such that the orthogonal projection $\mathrm{Pr}_{\partial\Omega} : \partial\Omega(\delta) \to \partial\Omega$ along the normal $\mathbf{n}_{\partial\Omega}$ is also well-defined and smooth.

Concerning the potential f, we assume that it is a fourth order polynomial, satisfying

$$f(\pm 1) = f'(\pm 1) = 0,\ f''(\pm 1) > 0,\ f(s) = f(-s) > 0 \quad \text{for all } s \in \mathbb{R} \tag{1.19}$$

and fulfilling $f^{(4)} > 0$. Then the ordinary differential equation (1.18) allows for a unique, monotonically increasing solution $\theta_0 : \mathbb{R} \to (-1, 1)$. This solution furthermore satisfies the decay estimate

$$\left|\theta_0^2(\rho) - 1\right| + |\theta_0^{(n)}(\rho)| \leqslant C_n e^{-\alpha|\rho|} \quad \text{for all } \rho \in \mathbb{R},\ n \in \mathbb{N}\setminus\{0\} \tag{1.20}$$

for constants $C_n > 0$, $n \in \mathbb{N}\setminus\{0\}$, and fixed $\alpha \in \left(0, \min\{\sqrt{f''(-1)}, \sqrt{f''(1)}\}\right)$. We denote by $\xi \in C^\infty(\mathbb{R})$ a cut-off function such that

$$\xi(s) = 1 \text{ if } |s| \leqslant \delta,\ \xi(s) = 0 \text{ if } |s| > 2\delta,\ \text{and } 0 \geqslant s\xi'(s) \geqslant -4 \quad \text{if } \delta \leqslant |s| \leqslant 2\delta. \tag{1.21}$$

The following theorem is the main theorem of this article (for an explanation of the used notations see the preliminaries section):

Theorem 1.1 *Let $M \in \mathbb{N}$ with $M \geqslant 4$, ξ be a cut-off function satisfying (1.21), $\gamma(x) := \xi(4d_{\mathbf{B}}(x))$ for all $x \in \Omega$ and let for $\epsilon \in (0,1)$ a smooth function $\psi_0^\epsilon : \Omega \to \mathbb{R}$ be given, which satisfies $\|\psi_0^\epsilon\|_{C^1(\Omega)} \leqslant C_{\psi_0}\epsilon^M$ for some $C_{\psi_0} > 0$ independent of ϵ. Then there are smooth functions $c_A^\epsilon : \Omega \times [0, T_0] \to \mathbb{R}$, $\mathbf{v}_A^\epsilon : \Omega \times [0, T_0] \to \mathbb{R}^2$ for $\epsilon \in (0,1)$ such that the following holds:*

Let $(\mathbf{v}^\epsilon, p^\epsilon, c^\epsilon, \mu^\epsilon)$ be smooth solutions to (1.1)–(1.6) with initial value

$$c_0^\epsilon(x) = c_A^\epsilon(x, 0) + \psi_0^\epsilon(x) \tag{1.22}$$

for all $x \in \Omega$. Then there are some $\epsilon_0 \in (0,1]$, $K > 0$, $T \in (0, T_0]$ such that

$$\left\|c^\epsilon - c_A^\epsilon\right\|_{L^2(\Omega_T)} + \left\|\nabla^\Gamma\left(c^\epsilon - c_A^\epsilon\right)\right\|_{L^2(\Gamma(\delta,T))} \leqslant K\epsilon^{M-\frac{1}{2}}, \tag{1.23a}$$

$$\epsilon\left\|\nabla\left(c^\epsilon - c_A^\epsilon\right)\right\|_{L^2(\Omega_T\setminus\Gamma(\delta,T))} + \left\|c^\epsilon - c_A^\epsilon\right\|_{L^2(\Omega_T\setminus\Gamma(\delta,T))} \leqslant K\epsilon^{M+\frac{1}{2}}, \tag{1.23b}$$

$$\epsilon^{\frac{3}{2}}\left\|\partial_{\mathbf{n}}\left(c^\epsilon - c_A^\epsilon\right)\right\|_{L^2(\Gamma(\delta,T))} + \left\|c^\epsilon - c_A^\epsilon\right\|_{L^\infty(0,T;H^{-1}(\Omega))} \leqslant K\epsilon^M, \tag{1.23c}$$

$$\int_{\Omega_T} \epsilon\left|\nabla\left(c^\epsilon - c_A^\epsilon\right)\right|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right)\left(c^\epsilon - c_A^\epsilon\right)^2 \,\mathrm{d}(x,t) \leqslant K^2\epsilon^{2M}, \tag{1.23d}$$

$$\left\|\gamma\left(c^\epsilon - c_A^\epsilon\right)\right\|_{L^\infty(0,T;L^2(\Omega))} + \epsilon^{\frac{1}{2}}\left\|\gamma\Delta\left(c^\epsilon - c_A^\epsilon\right)\right\|_{L^2(\Omega_T)} \leqslant K\epsilon^{M-\frac{1}{2}}, \tag{1.23e}$$

$$\left\|\gamma\nabla\left(c^\epsilon - c_A^\epsilon\right)\right\|_{L^2(\Omega_T)} + \left\|\gamma\left(c^\epsilon - c_A^\epsilon\right)\nabla\left(c^\epsilon - c_A^\epsilon\right)\right\|_{L^2(\Omega_T)} \leqslant K\epsilon^M, \tag{1.23f}$$

and for $q \in (1,2)$

$$\left\|\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon\right\|_{L^1(0,T;L^q(\Omega))} \leqslant C(K,q)\,\epsilon^{M-\frac{1}{2}}, \tag{1.24}$$

hold for all $\epsilon \in (0, \epsilon_0)$ *and some* $C(K, q) > 0$*. Moreover, we have*

$$\lim_{\epsilon \to 0} c_A^\epsilon = \pm 1 \text{ in } L^\infty\left(\Omega_T'\right) \tag{1.25}$$

and

$$\lim_{\epsilon \to 0} \mathbf{v}_A^\epsilon = \mathbf{v}^\pm \text{ in } L^6\left((s,t); H^2\left(\Omega'\right)^2\right) \tag{1.26}$$

for every $\Omega' \times (s,t) \subset\subset \Omega_T^\pm$*.*

Throughout this work we will often consider the following assumptions.

Assumption 1.2 Let $M \in \mathbb{N}$ with $M \geqslant 4$ and $\gamma(x) := \xi(4d_{\mathbf{B}}(x))$ for all $x \in \Omega$. We assume that $c_A \colon \Omega \times [0, T_0] \to \mathbb{R}$ is a smooth function and that there are $\epsilon_0 \in (0,1)$, $K \geqslant 1$ and a family $(T_\epsilon)_{\epsilon \in (0,\epsilon_0)} \subset (0, T_0]$ such that the following holds: If c^ϵ is given as in Theorem 1.1 with $c_0^\epsilon(x) = c_A(x, 0)$, then it holds for $R := c^\epsilon - c_A^\epsilon$

$$\|R\|_{L^2(\Omega_{T_\epsilon})} + \|\nabla^\Gamma R\|_{L^2(\Gamma(T_\epsilon,\delta))} + \left\|\left(\tfrac{1}{\epsilon}R, \nabla R\right)\right\|_{L^2(\Omega_{T_\epsilon}\setminus\Gamma(T_\epsilon,\delta))} \leqslant K\epsilon^{M-\frac{1}{2}}, \tag{1.27a}$$

$$\epsilon^{\frac{3}{2}}\|\partial_{\mathbf{n}} R\|_{L^2(\Gamma(T_\epsilon,\delta))} + \|R\|_{L^\infty(0,T_\epsilon;H^{-1}(\Omega))} \leqslant K\epsilon^M, \tag{1.27b}$$

$$\int_{\Omega_{T_\epsilon}} \epsilon|\nabla R|^2 + \tfrac{1}{\epsilon} f''(c_A^\epsilon)R^2 \, \mathrm{d}(x,t) \leqslant K^2\epsilon^{2M}, \tag{1.27c}$$

$$\epsilon^{\frac{1}{2}}\|\gamma R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} + \left\|\left(\epsilon\gamma\Delta R, \gamma\nabla R, \gamma R(\nabla R)\right)\right\|_{L^2(\Omega_{T_\epsilon})} \leqslant K\epsilon^M \tag{1.27d}$$

for all $\epsilon \in (0, \epsilon_0)$. Moreover, we assume that there exist $\epsilon_0 > 0$ and a constant $C_0 > 0$ independent of ϵ, such that

$$E^\epsilon\left(c_0^\epsilon\right) + \|c_0^\epsilon\|_{L^\infty(\Omega)} \leqslant C_0 \tag{1.28}$$

for all $\epsilon \in (0, \epsilon_0)$.

As a first result, we give an energy estimate for (1.1)–(1.6). We consider for $\epsilon > 0$ the free energy

$$E^\epsilon(c^\epsilon)(t) = \frac{\epsilon}{2}\int_\Omega |\nabla c^\epsilon(x,t)|^2 \, \mathrm{d}x + \frac{1}{\epsilon}\int_\Omega f(c^\epsilon(x,t)) \, \mathrm{d}x \text{ for } t \in [0, T_0]. \tag{1.29}$$

Then one derives

$$\sup_{0\leqslant t\leqslant T} E^\epsilon(c^\epsilon(t)) + \int_0^T \int_\Omega \left(|\nabla \mathbf{v}^\epsilon|^2 + |\nabla\mu^\epsilon|^2\right) dx\, dt + \alpha_0 \int_0^t \int_{\partial\Omega} |v|^2 \, d\sigma\, dt \leqslant C_0. \tag{1.30}$$

in a standard manner from testing (1.1) with $\mathbf{v}^\epsilon$, (1.3) with μ^ϵ and (1.4) with $\partial_t c^\epsilon$ and integration by parts. As a corollary we obtain:

Lemma 1.3 *Let* $(c^\epsilon, \mu^\epsilon, \mathbf{v}^\epsilon, p^\epsilon)$ *be a classical solution to* (1.1)–(1.6) *and let* $\epsilon_0 > 0$ *and* $C_0 > 0$ *be given such that* (1.27) *and* (1.28) *hold true. Then there is some* $\epsilon_1 \in (0, \epsilon_0)$ *and some constant* $C > 0$*, depending only on* T_0, C_0 *and* ϵ_0*, such that*

$$\epsilon^7\|\Delta c^\epsilon\|_{L^2(\Omega_t)}^2 + \epsilon \sup_{\tau\in[0,t]}\|\nabla c^\epsilon(.,\tau)\|_{L^2(\Omega)}^2 + \|(\nabla\mu^\epsilon, \nabla\mathbf{v}^\epsilon)\|_{L^2(\Omega_t)}^2 + \alpha_0\|\mathbf{v}^\epsilon\|_{L^2(\partial_t\Omega)}^2 \leqslant C$$

for all $t \in [0, T_\epsilon]$ *and* $\epsilon \in (0, \epsilon_1)$*.*

Proof. All estimates apart from the one for $\epsilon^7 \|\Delta c^\epsilon\|^2_{L^2(\Omega_t)}$ follow directly from (1.30). Because of the Dirichlet boundary condition of μ^ϵ we get

$$\begin{aligned}\|\Delta c^\epsilon\|_{L^2(\Omega_t)} &\leqslant \frac{1}{\epsilon}\|\mu^\epsilon\|_{L^2(\Omega_t)} + \frac{1}{\epsilon^2}\left\|f'(c^\epsilon)\right\|_{L^2(\Omega_t)} \leqslant \frac{1}{\epsilon}C\|\nabla\mu^\epsilon\|_{L^2(\Omega_t)} + \frac{1}{\epsilon^2}\left\|f'(c^\epsilon)\right\|_{L^2(\Omega_t)}\\ &\leqslant \frac{C}{\epsilon^2}\left(1+\|c^\epsilon\|^3_{L^6(\Omega_t)}\right) \leqslant \frac{C}{\epsilon^2}\left(1+\|\nabla c^\epsilon\|^3_{L^\infty(0,t;L^2(\Omega))}\right) \leqslant \frac{C}{\epsilon^2}\left(1+\epsilon^{-\frac{3}{2}}\right)\end{aligned}$$

for ϵ small enough, where we used Poincaré's inequality in the second inequality, and the fact that f is a polynomial of fourth order in the third inequality. □

The contribution is organized as follows: Section 2 summarizes the needed mathematical tools, in particular existence results for stationary Stokes equations with relevant boundary conditions and we discuss a modified spectral estimate, which is key for the proof of Theorem 1.1. Section 3 is then devoted to showing Theorem 1.1. First we will state a result on existence of approximate solutions, cf. Theorem 3.1 below. This result and all subsequently discussed properties of the approximate solutions which are needed in this work, are shown in [5], see also [17]. A key result in Subsection 3.1 is Lemma 3.4, which provides an estimate for the leading term of the error in the velocity $\mathbf{v}_A^\epsilon - \mathbf{v}^\epsilon$. In order to show this, a spectral decomposition of $c^\epsilon - c_A^\epsilon$ is needed. In Subsection 3.2, we collect many important statements which are essential to the proof of Theorem 1.1, many of which are concerned with dealing with the aforementioned error in the velocity. These results enable us to effectively deal with the problems arising due to the presence of the convective term in the Cahn–Hilliard equation. Finally, a list of notation can be found at the end of the manuscript.

2. Preliminaries

2.1 *Stationary stokes equation in one phase*

We consider the one-phase stationary Stokes equation

$$-\Delta\mathbf{v} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div}\mathbf{v} = g \quad \text{in } \Omega, \tag{2.2}$$

$$(-2D_s\mathbf{v} + p\mathbf{I})\,\mathbf{n}_{\partial\Omega} = \alpha_0\mathbf{v} \quad \text{on } \partial\Omega \tag{2.3}$$

for given $\mathbf{f} \in V_g'(\Omega)$ and $g \in L^2(\Omega)$. We denote $C_\sigma^\infty(\overline{\Omega}) := \{\mathbf{u} \in C^\infty(\overline{\Omega})^2 \,|\, \operatorname{div}\mathbf{u} = 0\}$, $H_\sigma^1(\Omega) := \overline{C_\sigma^\infty(\overline{\Omega})}^{H_1(\Omega)}$ and set

$$V_g(\Omega) := \begin{cases} H_\sigma^1(\Omega) & \text{if } g = 0,\\ H^1(\Omega)^2 & \text{else,}\end{cases} \qquad H_g(\Omega) := \begin{cases} L_\sigma^2(\Omega) & \text{if } g \equiv 0,\\ L^2(\Omega)^2 & \text{else}\end{cases} \tag{2.4}$$

and let $V_g'(\Omega)$ denote the dual space of $V_g(\Omega)$.

We call $\mathbf{v} \in V_g(\Omega)$ a weak solution of (2.1)–(2.3) if

$$2\int_\Omega D_s\mathbf{v} : D_s\psi\,\mathrm{d}x + \alpha_0\int_{\partial\Omega}\mathbf{v}\cdot\psi\,\mathrm{d}\mathcal{H}^1(s) = \langle\mathbf{f},\psi\rangle_{V_g',V_g} \tag{2.5}$$

holds for all $\psi \in C_\sigma^\infty\left(\overline{\Omega}\right)$ and

$$\operatorname{div} \mathbf{v} = g \text{ in } L^2(\Omega). \tag{2.6}$$

Note that in the case $g = 0$ the condition (2.6) is already included in the definition of the space V_0 and can thus be omitted. Moreover, a classical solution to (2.1)–(2.3) is a weak solution.

Theorem 2.1 *For each $g \in L^2(\Omega)$ and $\mathbf{f} \in V_g'(\Omega)$ there is a unique weak solution $\mathbf{v} \in V_g(\Omega)$ of (2.1)–(2.3). Moreover there exists a constant $C(\Omega, \alpha_0) > 0$, which is independent of $\mathbf{f}$, such that*

$$\|\mathbf{v}\|_{H^1(\Omega)} \leqslant C(\Omega, \alpha_0)\big(\|\mathbf{f}\|_{V_g'(\Omega)} + \|g\|_{L^2(\Omega)}\big). \tag{2.7}$$

Proof. In the case $g = 0$ the result is a direct consequence of the Lax–Milgram Lemma. The case $g \neq 0$ can be easily reduced to the latter case by considering $\tilde{\mathbf{v}} = \mathbf{v} - \nabla q$, where $q \in H^2(\Omega) \cap H_0^1(\Omega)$ is such that $\Delta q = g$. □

The following corollary yields existence of a pressure term.

Corollary 2.2 *Let $g \in L^2(\Omega)$ and $\mathbf{f} \in L^2(\Omega)^2$. Then there is a unique $(\mathbf{v}, p) \in V_g \times L^2(\Omega)$ of (2.1)–(2.3) such that*

$$2\int_\Omega D_s\mathbf{v} : D_s\psi - p\operatorname{div}\psi \,\mathrm{d}x + \alpha_0 \int_{\partial\Omega} \mathbf{v}\cdot\psi \,\mathrm{d}\mathcal{H}^1(s) = \int_\Omega \mathbf{f}\cdot\psi\,\mathrm{d}x \quad \text{for all } \psi \in H^1(\Omega)$$

and (2.6) holds. Moreover, there is a constant $C > 0$, independent of $\mathbf{v}$ and p, such that

$$\|(\mathbf{v}, p)\|_{H^1(\Omega)\times L^2(\Omega)} \leqslant C\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}\right).$$

Proof. Let $\mathbf{v}$ be the weak solution to (2.5)–(2.6) as given by Theorem 2.1. Elliptic theory implies that

$$\Delta_D : \mathcal{D}(\Delta_D) := H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega) : u \mapsto \Delta u$$

is bijective. Thus, the adjoint operator $(\Delta_D)' : L^2(\Omega)' \to (H^2(\Omega) \cap H_0^1(\Omega))'$ is also bijective. Using the continuity of the trace operator and Hölder's inequality we find that the functional $F : \mathcal{D}(\Delta_D) \to \mathbb{R}$

$$F(\varphi) := \int_\Omega (2D_s\mathbf{v} : D_s(\nabla\varphi) - \mathbf{f}\cdot\nabla\varphi)\,\mathrm{d}x + \alpha_0\int_{\partial\Omega} \mathbf{v}\cdot\nabla\varphi\,\mathrm{d}\mathcal{H}^1(s) \quad \forall\varphi \in \mathcal{D}(\Delta_D)$$

is bounded and linear. Thus the Riesz representation theorem yields the existence of $p \in L^2(\Omega)$ such that

$$(p, \Delta\varphi)_{L^2} = \left\langle \Delta_D'\left((p, .)_{L^2}\right), \varphi\right\rangle_{\mathcal{D}(\Delta_D)', \mathcal{D}(\Delta_D)} = F(\varphi) \tag{2.8}$$

for all $\varphi \in \mathcal{D}(\Delta_D)$. Since the operator $\left((\Delta_D)'\right)^{-1}$ is bounded, we find

$$\|p\|_{L^2(\Omega)} \leqslant C\,\|F\|_{(H^2(\Omega)\cap H_0^1(\Omega))'} \leqslant C\left(\|\mathbf{v}\|_{H^1(\Omega)} + \|\mathbf{f}\|_{L^2(\Omega)}\right) \leqslant C\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}\right),$$

where we used (2.7) in the last line.

Now let $\psi \in H^1(\Omega)^2$ be arbitrary and let $q \in \mathcal{D}(\Delta_D)$ be such that $\Delta q = \operatorname{div}\psi$. Moreover set $\psi_0 := \psi - \nabla q$. Then $\operatorname{div}\psi_0 = 0$ and

$$\begin{aligned}
&\int_\Omega 2D_s\mathbf{v} : D_s\psi - p\operatorname{div}\psi\,\mathrm{d}x + \alpha_0\int_{\partial\Omega}\mathbf{v}\cdot\psi\,\mathrm{d}\mathcal{H}^1(s)\\
&\quad = \int_\Omega \mathbf{f}\cdot\psi_0\,\mathrm{d}x + \int_\Omega (2D_s\mathbf{v} : D_s(\nabla q) - p\Delta q)\,\mathrm{d}x + \alpha_0\int_{\partial\Omega}\mathbf{v}\cdot\nabla q\,\mathrm{d}\mathcal{H}^1(s) = \int_\Omega \mathbf{f}\cdot\psi\,\mathrm{d}x,
\end{aligned}$$

where we used (2.5) and (2.8). As $\psi \in H^1(\Omega)^2$ was arbitrary, this yields the claim. □

Theorem 2.3 (Existence of strong solutions) *Let $g \equiv 0$ and $\mathbf{f} \in L^2(\Omega)^2$. Then there exists a unique solution $(\mathbf{v}, p) \in H^2(\Omega)^2 \times H^1(\Omega)$ to (2.1)–(2.3), which satisfies the estimate*

$$\|\mathbf{v}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \leqslant C \|\mathbf{f}\|_{L^2(\Omega)} .$$

Moreover, if $\mathbf{f}$ is smooth, then $\mathbf{v}$ and p are smooth as well.

Proof. For $q \in (1, \infty)$, Theorem 3.1 in [20] implies that there is $\lambda > 0$ such that for every $\mathbf{g} \in L^q(\Omega)^2$ and $\mathbf{a} \in W^1_q(\Omega)^2$ the problem

$$\begin{aligned} \lambda\mathbf{u} - \Delta\mathbf{u} + \nabla q &= \mathbf{g} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ (-2D_s\mathbf{u} + q\mathbf{I})\,\mathbf{n}_{\partial\Omega} &= \mathbf{a}|_{\partial\Omega} && \text{on } \partial\Omega \end{aligned} \tag{2.9}$$

admits for a unique solution $(\mathbf{u}, q) \in W^2_q(\Omega)^2 \times W^1_q(\Omega)$. Additionally, the estimate

$$\|\mathbf{u}\|_{W^2_q(\Omega)} + \|q\|_{W^1_q(\Omega)} \leqslant C\left(\|\mathbf{g}\|_{L^q(\Omega)} + \|\mathbf{a}\|_{W^1_q(\Omega)}\right) \tag{2.10}$$

holds. Considering a weak solution $(\mathbf{v}, p) \in V_0 \times L^2(\Omega)$ of (2.1)–(2.3) as given in Corollary 2.2 and defining $\mathbf{g} := \mathbf{f} + \lambda\mathbf{v} \in L^2(\Omega)^2$ and $\mathbf{a} := \alpha_0\mathbf{v} \in H^1(\Omega)^2$, we now introduce $(\mathbf{u}, q) \in H^2(\Omega) \times H^1(\Omega)$ as the strong solution to (2.9) regarding these data. Writing $\mathbf{w} := \mathbf{u} - \mathbf{v}$ and $r := q - p$ we easily find that $(\mathbf{w}, r) \in H^1(\Omega)^2 \times L^2(\Omega)$ is a weak solution to

$$\begin{aligned} \lambda\mathbf{w} - \Delta\mathbf{w} + \nabla r &= 0 && \text{in } \Omega, \\ \operatorname{div} \mathbf{w} &= 0 && \text{in } \Omega, \\ (-2D_s\mathbf{w} + r\mathbf{I})\,\mathbf{n}_{\partial\Omega} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Testing with $\psi = \mathbf{w}$ we immediately find that $\mathbf{w} \equiv 0$ a.e. and thus $\mathbf{u} = \mathbf{v}$, in particular $\mathbf{v} \in H^2(\Omega)^2$. Furthermore, $\mathbf{w} = 0$ implies $\nabla r = 0$ in Ω and $r = 0$ on $\partial\Omega$, so that we can conclude $r \equiv 0$ a.e. in Ω leading to $p = q$ and $p \in H^1(\Omega)$. The estimate follows from (2.10) and (2.7). For higher regularity one may use results on existence of solutions with higher regularity, e.g., due to Grubb and Solonnikov [14] in a similar manner to obtain smoothness of the solution for smooth boundaries and smooth data. □

Lemma 2.4 *Let $g \equiv 0$ and $\mathbf{f} \in V_0'$, and let $\mathbf{v} \in H^1_\sigma(\Omega)$ be the weak solution to (2.1)–(2.3). Then for all $q' \in (1, 2)$*

$$\|\mathbf{v}\|_{L^{q'}(\Omega)} \leqslant C_q \sup_{\psi \in W^2_q(\Omega)^2, \psi \neq 0} \frac{|\mathbf{f}(\psi)|}{\|\psi\|_{W^2_q(\Omega)}},$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $C_q > 0$ is independent of $\mathbf{v}$ and f.

Proof. For this we introduce $T(\mathbf{u}, p) := -2D_s\mathbf{u} + p\mathbf{I}$ for $\mathbf{u} \in W^1_q(\Omega)$, $p \in L^2(\Omega)$ and set

$$D(A_S) = \left\{\mathbf{u} \in W^2_q(\Omega) \middle| \operatorname{div} \mathbf{u} = 0, \exists p \in W^1_q(\Omega) : T(\mathbf{u}, p)\,\mathbf{n}|_{\partial\Omega} = \alpha_0\mathbf{u}|_{\partial\Omega}\right\}.$$

We define the operator

$$A_S \colon D(A_S) \subset L^q_\sigma(\Omega) \to L^q_\sigma(\Omega),\ \mathbf{u} \mapsto P_\sigma\left(-\Delta\mathbf{u} + \nabla p\right),$$

for p as in the definition of $D(A_S)$ and where P_σ denotes the Helmholtz projection given by

$$P_\sigma : L^q(\Omega)^2 \to L^q_\sigma(\Omega),\ \psi \mapsto P_\sigma(\psi) = \psi - \nabla r,$$

where $r \in W^1_{q,0}(\Omega)$ is the unique weak solution to

$$\begin{aligned} \Delta r &= \operatorname{div}\psi && \text{in } \Omega, \\ r &= 0 && \text{on } \partial\Omega. \end{aligned}$$

One can verify in a straight-forward manner that A_S is well defined. Moreover,

$$\int_\Omega (A_S\mathbf{u})\cdot\mathbf{u}\,dx = \int_\Omega 2\,|D_s\mathbf{u}|^2\,dx + \alpha_0 \int_{\partial\Omega} \mathbf{u}^2\,d\mathcal{H}^1(s) \geqslant C\,\|\mathbf{u}\|^2_{L^2(\Omega)} \tag{2.11}$$

for some $C > 0$ and $\mathbf{u} \in \mathcal{D}(A_S)$, where we used [8, Corollary 5.8] in the last line. This immediately shows the injectivity of A_S. Concerning surjectivity, let $\tilde{\mathbf{f}} \in L^q_\sigma(\Omega)$. As $q > 2$, Theorem 2.3 implies that there is a unique strong solution $(\tilde{\mathbf{v}}, p) \in H^2(\Omega) \times H^1(\Omega)$ to (2.1)–(2.3) (with $\mathbf{f}$ replaced by $\tilde{\mathbf{f}}$ and $g \equiv 0$). Choosing $\lambda > 0$ as in the proof of Theorem 2.3, we find that $\mathbf{g} := \tilde{\mathbf{f}} + \lambda\tilde{\mathbf{v}}$ and $\mathbf{a} := \alpha_0\tilde{\mathbf{v}}$ satisfy $\mathbf{g} \in L^q(\Omega)$ and $\mathbf{a} \in W^1_q(\Omega)$ as a consequence of the Sobolev embedding theorem. Thus, Theorem 3.1. in [20] implies the existence of a unique solution $(\mathbf{u}, r) \in W^2_q(\Omega) \times W^1_q(\Omega)$ to (2.9) and an analogous argumentation as in the proof of Theorem 2.3 leads to $\tilde{\mathbf{v}} = \mathbf{u}$ and $p = r$ along with the estimate

$$\|\tilde{\mathbf{v}}\|_{W^2_q(\Omega)} + \|p\|_{W^1_q(\Omega)} \leqslant C\,\|\tilde{\mathbf{f}}\|_{L^q(\Omega)}. \tag{2.12}$$

In particular, $T(\tilde{\mathbf{v}}, p)\,\mathbf{n}|_{\partial\Omega} = \alpha_0\tilde{\mathbf{v}}|_{\partial\Omega}$ is satisfied. So $\tilde{\mathbf{v}} \in \mathcal{D}(A_s)$ and, since $-\Delta\tilde{\mathbf{v}} + \nabla p = \tilde{\mathbf{f}}$ holds in $L^q(\Omega)$, we have $A_s(\tilde{\mathbf{v}}) = \tilde{\mathbf{f}}$. In fact, this not only implies surjectivity, but also the existence of a bounded inverse A_S^{-1} as a result of (2.12). Consequently, $\left(\mathcal{D}(A_s), \|.\|_{A_s}\right)$ is a Banach space, where $\|.\|_{A_s}$ denotes the graph norm. All these considerations result in the fact that the adjoint $A'_S : \left(L^q_\sigma(\Omega)\right)' \to (\mathcal{D}(A_S))'$ is an invertible and bounded operator.

Let now $\mathbf{v} \in H^1_\sigma(\Omega)$ be the given weak solution to (2.1)–(2.3) and fix $q > 2$. Then $\mathbf{v} \in L^{q'}_\sigma(\Omega) \cong \left(L^q_\sigma(\Omega)\right)'$ and we have for $\psi \in \mathcal{D}(A_S)$

$$\left\langle A'_S\mathbf{v}, \psi\right\rangle_{(\mathcal{D}(A_S))',\mathcal{D}(A_S)} = 2\int_\Omega D_s\mathbf{v} : D_s\psi\,dx + \alpha_0\int_{\partial\Omega} \psi\cdot\mathbf{v}\,dx = \langle \mathbf{f}, \psi\rangle_{(\mathcal{D}(A_S))',\mathcal{D}(A_S)}\,.$$

As a result $A'_S\mathbf{v} = \mathbf{f}$ in $(\mathcal{D}(A_S))'$ and thus $\mathbf{v} = \left(A'_S\right)^{-1}\mathbf{f}$ in $\left(L^q_\sigma(\Omega)\right)'$ which enables us to estimate

$$\|\mathbf{v}\|_{L^{q'}(\Omega)} = \left\|(A'_S)^{-1}\mathbf{f}\right\|_{(L^q_\sigma(\Omega))'} \leqslant C\,\|\mathbf{f}\|_{(\mathcal{D}(A_S))'} \leqslant C\,\|\mathbf{f}\|_{(W^2_q(\Omega))'}\,. \qquad \square$$

2.2 *Differential-geometric background*

We use a similar notation as in [4]. We parameterize the curves $(\Gamma_t)_{t\in[0,T_0]}$ by choosing a family of smooth diffeomorphisms

$$X_0 \colon \mathbb{T}^1 \times [0, T_0] \to \Omega \tag{2.13}$$

such that $\partial_s X_0(s,t) \neq 0$ for all $s \in \mathbb{T}^1$, $t \in [0, T_0]$. In particular, $\bigcup_{t\in[0,T_0]} X_0(\mathbb{T}^1\times\{t\})\times\{t\} = \Gamma$. Moreover, we define the tangent and normal vectors on Γ_t at $X_0(s,t)$ as

$$\boldsymbol{\tau}(s,t) := \frac{\partial_s X_0(s,t)}{|\partial_s X_0(s,t)|} \quad \text{and} \quad \mathbf{n}(s,t) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{\tau}(s,t) \tag{2.14}$$

for all $(s,t) \in \mathbb{T}^1 \times [0,T_0]$. We choose X_0 (and thereby the orientation of Γ_t) such that $\mathbf{n}(.,t)$ is the exterior normal with respect to $\Omega^-(t)$. Thus, for a point $p \in \Gamma_t$ with $p = X_0(s,t)$ it holds $\mathbf{n}_{\Gamma_t}(p) = \mathbf{n}(s,t)$

Furthermore, we define $V(s,t) := V_{\Gamma_t}(X_0(s,t))$ and $H(s,t) := H_{\Gamma_t}(X_0(s,t))$ and note that $V(s,t) = \partial_t X_0(s,t) \cdot \mathbf{n}(s,t)$ for all $(s,t) \in \mathbb{T}^1 \times [0,T_0]$ by definition of the normal velocity. We also introduce the pull-back and write for a function $\mathbf{v}\colon \Gamma \to \mathbb{R}^d$, $d \in \mathbb{N}$

$$\left(X_0^* \mathbf{v}\right)(s,t) := \mathbf{v}\big(X_0(s,t),t\big) \quad \text{for all } (s,t) \in \mathbb{T}^1 \times [0,T_0]. \tag{2.15}$$

On the other hand, we define for a function $h\colon \mathbb{T}^1 \times [0,T_0]$

$$\big(X_0^{*,-1} h\big)(p) := h\big(X_0^{-1}(p)\big) \quad \text{for all } p \in \Gamma_t, t \in [0,T_0]. \tag{2.16}$$

Choosing $\delta > 0$ small enough, the orthogonal projection $\mathrm{Pr}_{\Gamma_t}\colon \Gamma_t(3\delta) \to \Gamma_t$ is well defined and smooth for all $t \in [0,T_0]$ and the mapping

$$\phi_t\colon \Gamma_t(3\delta) \to (-3\delta, 3\delta) \times \Gamma_t,\ x \mapsto \big(d_\Gamma(x,t), \mathrm{Pr}_{\Gamma_t}(x)\big)$$

is a diffeomorphism. Its inverse is given by $\phi_t^{-1}(r,p) = p + r\mathbf{n}_{\Gamma_t}(p)$. Although Pr_{Γ_t} and ϕ_t are well defined in $\Gamma_t(3\delta)$, almost all computations later on are performed in $\Gamma_t(2\delta)$, which is why, for the sake of readability, we work on $\Gamma_t(2\delta)$ in the following.

Combining ϕ_t^{-1} and X_0 we may define a diffeomorphism

$$\begin{aligned} X\colon &(-2\delta, 2\delta) \times \mathbb{T}^1 \times [0,T_0] \to \Gamma(2\delta), \\ &(r,s,t) \mapsto \Big(\phi_t^{-1}\big(r, X_0(s,t)\big), t\Big) = \big(X_0(s,t) + r\mathbf{n}(s,t), t\big) \end{aligned} \tag{2.17}$$

with inverse given by

$$X^{-1}\colon \Gamma(2\delta) \to (-2\delta, 2\delta) \times \mathbb{T}^1 \times [0,T_0],\ (x,t) \mapsto \big(d_\Gamma(x,t), S(x,t), t\big), \tag{2.18}$$

where we define

$$S(x,t) := \Big(X_0^{-1}\big(\mathrm{Pr}_{\Gamma_t}(x)\big)\Big)_1 \tag{2.19}$$

for $(x,t) \in \Gamma(2\delta)$ and where $(.)_1$ signifies that we take the first component. In particular it holds $S(x,t) = S(\mathrm{Pr}_{\Gamma_t}(x), t)$. In the following we will write $\mathbf{n}(x,t) := \mathbf{n}(S(x,t),t)$ and $\boldsymbol{\tau}(x,t) := \boldsymbol{\tau}(S(x,t),t)$ for $(x,t) \in \Gamma(3\delta)$.

Proposition 2.5 *For every $t \in [0,T_0]$, $x \in \Gamma_t(2\delta)$, $s \in \mathbb{T}^1$, $r \in (-2\delta, 2\delta)$ it holds*

$$\begin{aligned} |\nabla d_\Gamma(x,t)| &= 1, & \Delta d_\Gamma\big(X_0(s,t),t\big) &= -H(s,t), \\ -\partial_t d_\Gamma\big(X(r,s,t)\big) &= V(s,t), & \nabla d_\Gamma\big(X(r,s,t)\big) &= \mathbf{n}(s,t), \\ \nabla S(x,t) \cdot \nabla d_\Gamma(x,t) &= 0. \end{aligned}$$

Proof. We refer to [19, Chapter 2.3] and [13, Chapter 4.1] for the proofs. □

For a function $\phi\colon \Gamma(2\delta) \to \mathbb{R}$ we define $\tilde{\phi}(r,s,t) := \phi(X(r,s,t))$ and often write $\phi(r,s,t)$ instead of $\tilde{\phi}(r,s,t)$. In the case that ϕ is twice continuously differentiable, we introduce the notations

$$
\begin{aligned}
\partial_t^\Gamma \tilde{\phi}(r,s,t) &:= \big(\partial_t + \partial_t S\big(X(r,s,t)\big)\partial_s\big)\,\tilde{\phi}(r,s,t),\\
\nabla^\Gamma \tilde{\phi}(r,s,t) &:= \nabla S\big(X(r,s,t)\big)\partial_s \tilde{\phi}(r,s,t),\\
\Delta^\Gamma \tilde{\phi}(r,s,t) &:= \Big(\Delta S\big(X(r,s,t)\big)\partial_s + (\nabla S \cdot \nabla S)\,\big(X(r,s,t)\big)\partial_{ss}\Big)\tilde{\phi}(r,s,t).
\end{aligned}
\tag{2.20}
$$

Similarly, if $\mathbf{v}\colon \Gamma(2\delta) \to \mathbb{R}^2$ is continuously differentiable, we will also write $\tilde{\mathbf{v}}\,(r,st) := \mathbf{v}\,(X(r,s,t))$ and introduce

$$
\operatorname{div}^\Gamma \tilde{\mathbf{v}}(r,s,t) = \nabla S\big(X(r,s,t)\big)\cdot \partial_s \tilde{\mathbf{v}}(r,s,t). \tag{2.21}
$$

For later use we introduce

$$
\begin{aligned}
\nabla^\Gamma \phi(x,t) &:= \nabla S(x,t)\partial_s \tilde{\phi}\big(d_\Gamma(x,t), S(x,t), t\big),\\
\operatorname{div}^\Gamma \mathbf{v}(x,t) &:= \nabla S(x,t)\partial_s \tilde{\mathbf{v}}\big(d_\Gamma(x,t), S(x,t), t\big)
\end{aligned}
$$

for $(x,t) \in \Gamma(2\delta)$. With these notations we have the decompositions

$$
\nabla\phi(x,t) = \partial_{\mathbf{n}}\phi(x,t)\mathbf{n} + \nabla^\Gamma \phi(x,t), \tag{2.22}
$$

$$
\operatorname{div}\mathbf{v}(x,t) = \partial_{\mathbf{n}}\mathbf{v}(x,t)\cdot\mathbf{n} + \operatorname{div}^\Gamma \mathbf{v}(x,t) \tag{2.23}
$$

for all $(x,t) \in \Gamma(2\delta)$, as

$$
\frac{d}{dr}\,(\phi\circ X)\,\big|_{(r,s,t)=(d_\Gamma(x,t),S(x,t),t)} = \partial_{\mathbf{n}}\phi(x,t).
$$

REMARK 2.6 If $h\colon \mathbb{T}^1 \times [0,T_0] \to \mathbb{R}$ is a function that is independent of $r \in (-2\delta, 2\delta)$, the functions $\partial_t^\Gamma h$, $\nabla^\Gamma h$ and $\Delta^\Gamma h$ will nevertheless depend on r via the derivatives of S. To connect the presented concepts with the classical surface operators we introduce the following notations:

$$
D_{t,\Gamma}h(s,t) := \partial_t^\Gamma h(0,s,t), \quad \nabla_\Gamma h(s,t) := \nabla^\Gamma h(0,s,t), \quad \Delta_\Gamma h(s,t) := \Delta^\Gamma h(0,s,t). \tag{2.24}
$$

Later in this work, we will often consider a concatenation $h\,(S(x,t),t)$ and thus will write for simplicity

$$
\begin{aligned}
\partial_t^\Gamma h(x,t) &:= \big(\partial_t + \partial_t S(x,t)\partial_s\big)h\big(S(x,t),t\big),\\
\nabla^\Gamma h(x,t) &:= \big(\nabla S(x,t)\partial_s\big)h\big(S(x,t),t\big),\\
\Delta^\Gamma h(x,t) &:= \big(\Delta S(x,t)\partial_s + \nabla S(x,t)\cdot\nabla S(x,t)\partial_{ss}\big)h\big(S(x,t),t\big)
\end{aligned}
\tag{2.25}
$$

for $(x,t) \in \Gamma(2\delta)$. As a consequence we obtain the identity

$$
\partial_t^\Gamma h(x,t) = X_0^*\left(\partial_t^\Gamma h\right)(s,t) = \partial_t^\Gamma h(0,s,t) = D_{t,\Gamma}h(s,t) \tag{2.26}
$$

for $(s,t) \in \mathbb{T}^1 \times [0,T_0]$ and $(X_0(s,t),t) = (x,t) \in \Gamma$. This might seem cumbersome but turns out to be convenient throughout this work.

In later parts of this article, we will introduce stretched coordinates of the form

$$\rho^\epsilon(x,t) = \frac{d_\Gamma(x,t) - \epsilon h\big(S(x,t),t\big)}{\epsilon} \tag{2.27}$$

for $(x,t) \in \Gamma(2\delta)$, $\epsilon \in (0,1)$ and for some smooth function $h\colon \mathbb{T}^1 \times [0,T_0] \to \mathbb{R}$ (which will later on also depend on ϵ). Writing $\rho = \rho^\epsilon$, the relation between the regular and the stretched variables can be expressed as

$$\hat{X}(\rho,s,t) := X\Big(\epsilon\big(\rho + h(s,t)\big), s, t\Big) = \Big(X_0(s,t) + \epsilon\big(\rho + h(s,t)\big)\mathbf{n}(s,t), t\Big). \tag{2.28}$$

Lemma 2.7 *Let $\phi\colon \mathbb{R} \times \Gamma(2\delta) \to \mathbb{R}$ be twice continuously differentiable and let ρ be given as in (2.27). Then the following formulas hold for $(x,t) \in \Gamma(2\delta)$ and $\epsilon \in (0,1)$*

$$\begin{aligned}
\partial_t\big(\phi\,(\rho(x,t),x,t)\big) &= \big(-\tfrac{1}{\epsilon}V(s,t) - \partial_t^\Gamma h(x,t)\big)\,\partial_\rho\phi\,(\rho,x,t) + \partial_t\phi\,(\rho,x,t)\,,\\
\nabla\big(\phi\,(\rho(x,t),x,t)\big) &= \big(\tfrac{1}{\epsilon}\mathbf{n}(s,t) - \nabla^\Gamma h(x,t)\big)\,\partial_\rho\phi\,(\rho,x,t) + \nabla_x\phi\,(\rho,x,t)\,,\\
\Delta\big(\phi\,(\rho(x,t),x,t)\big) &= \big(\tfrac{1}{\epsilon^2} + |\nabla^\Gamma h(x,t)|^2\big)\partial_{\rho\rho}\phi\,(\rho,x,t)\\
&\quad + \big(\epsilon^{-1}\Delta d_\Gamma(x,t) - \Delta^\Gamma h(x,t)\big)\partial_\rho\phi\,(\rho,x,t)\\
&\quad + 2\big(\epsilon^{-1}\mathbf{n}(s,t) - \nabla^\Gamma h(x,t)\big)\cdot\nabla_x\partial_\rho\phi\,(\rho,x,t) + \Delta_x\phi\,(\rho(x,t),x,t)\,,
\end{aligned}$$

where $s = S(x,t)$ and $\rho = \rho(x,t)$. Here ∇_x and Δ_x operate solely on the x-variable of ϕ.

Proof. This follows from the chain rule, Proposition 2.5 and the notations introduced in Remark 2.6. □

By (2.22) and (2.23) we have

$$\nabla^\Gamma u(x,t) = \Big(\mathbf{I} - \mathbf{n}\big(S(x,t),t\big) \otimes \mathbf{n}\,(S(x,t),t)\Big)\nabla u(x,t) \quad \text{and} \tag{2.29}$$

$$\operatorname{div}^\Gamma \mathbf{v}(x,t) = \Big(\mathbf{I} - \mathbf{n}\big(S(x,t),t\big) \otimes \mathbf{n}\,(S(x,t),t)\Big) : \nabla\mathbf{v}(x,t) \tag{2.30}$$

for suitable $u\colon \Gamma(2\delta) \to \mathbb{R}$, $\mathbf{v}\colon \Gamma(2\delta) \to \mathbb{R}^2$. A consequence is:

Lemma 2.8 *Let $t \in [0,T_0]$ and $\mathbf{v} \in H^1(\Gamma_t(\delta))^2$, $u \in H^1(\Gamma_t(\delta))$. Then it holds*

$$\int_{\Gamma_t(\delta)} u \operatorname{div}^\Gamma \mathbf{v}\,\mathrm{d}x = -\int_{\Gamma_t(\delta)} \nabla^\Gamma u \cdot \mathbf{v}\,\mathrm{d}x - \int_{\Gamma_t(\delta)} u\mathbf{v}\cdot\mathbf{n}\kappa\,\mathrm{d}x + \int_{\partial(\Gamma_t(\delta))} u\big((\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\cdot\mathbf{v}\big)\cdot\nu\,\mathrm{d}\mathcal{H}^1(s),$$

where $\kappa := -\operatorname{div}(\mathbf{n}\,(S(x,t),t))$ and $\nu(s)$ is the outer unit normal to $\Gamma_t(\delta)$ for $s \in \partial(\Gamma_t(\delta))$.

Proof. This is a consequence of (2.29), (2.30), and the divergence theorem. □

For later use we define

$$\big[\partial_\mathbf{n}, \nabla^\Gamma\big]u := \partial_\mathbf{n}\big((\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\,\nabla u\big) - (\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\,\nabla\,(\partial_\mathbf{n} u) \tag{2.31}$$

and compute

$$\big[\partial_\mathbf{n}, \nabla^\Gamma\big]u = -\nabla S\,\big(\partial_s\mathbf{n}\cdot\nabla^\Gamma u\big)\,. \tag{2.32}$$

2.3 *Remainder terms*

We introduce the following function spaces. For $t \in [0, T_0]$ and $1 \leqslant p < \infty$ we define

$$L^{p,\infty}(\Gamma_t(2\delta)) := \left\{ f : \Gamma_t(2\delta) \to \mathbb{R} \text{ measurable} \mid \|f\|_{L^{p,\infty}(\Gamma_t(2\delta))} < \infty \right\},$$

where

$$\|f\|_{L^{p,\infty}(\Gamma_t(2\delta))} := \left(\int_{\mathbb{T}^1} \operatorname{esssup}_{|r|\leqslant 2\delta} |f((X(r,s,t))_1)|^p \, ds \right)^{\frac{1}{p}}.$$

Here $X_1(r,s,t) := X_0(s,t) + r\mathbf{n}(s,t)$ denotes the first component of X. The following embedding was already remarked in [4, Subsection 2.5].

Lemma 2.9 *We have $H^1(\Gamma_t(2\delta)) \hookrightarrow L^{4,\infty}(\Gamma_t(2\delta))$ with operator norm uniformly bounded with respect to $t \in [0, T_0]$.*

Proof. This is a consequence of the Gagliardo–Nirenberg interpolation and the fact that Γ_t is one-dimensional. □

For $T \in [0, T_0]$, $1 \leqslant p, q < \infty$ and $\alpha \in (0, 3\delta)$ we set

$$L^q\Big(0,T;L^p\big(\Gamma_t(\alpha)\big)\Big) := \left\{ f : \Gamma(\alpha, T) \to \mathbb{R} \text{ measurable} \mid \|f\|_{L^q(0,T;L^p(\Gamma_t(\alpha)))} < \infty \right\},$$

$$\|f\|_{L^q(0,T;L^p(\Gamma_t(\alpha)))} := \left(\int_0^T \left(\int_{\Gamma_t(\alpha)} |f(x,t)|^p \, dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}.$$

In a similar way, we define $L^q(0,T;L^p(\Omega\backslash\Gamma_t(\alpha)))$ and $L^q\big(0,T;L^p\big(\Omega^{\pm}(t)\big)\big)$ and the corresponding norms. Moreover, for $m \in \mathbb{N}_0$ we denote for $U(t) = \Omega^{\pm}(t)$ or $U(t) = \Gamma_t(\alpha)$

$$L^p\Big(0,T;H^m\big(U(t)\big)\Big) := \Big\{ f \in L^p\Big(0,T;L^2\big(\Omega^{\pm}(t)\big)\Big) : \partial_x^{\alpha} f \in L^p\Big(0,T;L^2\big(U(t)\big)\Big) \forall |\alpha| \leqslant m \Big\},$$

$$\|f\|_{L^p(0,T;H^m(U(t)))} := \sum_{|\alpha|\leqslant m} \|\partial_x^{\alpha} f\|_{L^p(0,T;L^2(U(t)))}.$$

For future use, we introduce a concept of remainder terms, similar to [4, Definition 2.5].

DEFINITION 2.10 Let $n \in \mathbb{N}$, $\epsilon_0 > 0$. For $\alpha > 0$ let $\mathcal{R}_\alpha$ denote the vector space of all families $(\hat{r}_\epsilon)_{\epsilon\in(0,\epsilon_0)}$ of continuous functions $\hat{r}_\epsilon : \mathbb{R} \times \Gamma(2\delta) \to \mathbb{R}^n$ which satisfy

$$|\hat{r}_\epsilon(\rho, x, t)| \leqslant Ce^{-\alpha|\rho|} \quad \text{for all } \rho \in \mathbb{R}, (x,t) \in \Gamma(2\delta), \epsilon \in (0,1).$$

Moreover, let $\mathcal{R}_\alpha^0$ be the subspace of all $(\hat{r}_\epsilon)_{\epsilon\in(0,\epsilon_0)} \in \mathcal{R}_\alpha$ such that

$$\hat{r}_\epsilon(\rho, x, t) = 0 \quad \text{for all } \rho \in \mathbb{R}, (x,t) \in \Gamma.$$

2.4 *Spectral theory*

The results in this chapter are adapted from [11]. For detailed proofs concerning the changed stretched variable see [17, Chapter 3]. Moreover, we define

$$J(r,s,t) := \det\big(D_{(r,s)}X(r,s,t)\big) \tag{2.33}$$

The statements in this section are made under the following assumptions:

ASSUMPTION 2.11 Let $\epsilon \in (0, \epsilon_0)$, $T \in (0, T_0]$ and ξ be a cut-off function satisfying (1.21). We assume that $c_A^\epsilon : \Omega_T \to \mathbb{R}$ is a smooth function, which has the structure

$$c_A^\epsilon(x,t) = \xi\big(d_\Gamma(x,t)\big)\Big(\theta_0\big(\rho(x,t)\big) + \epsilon p^\epsilon\big(\mathrm{Pr}_{\Gamma_t}(x),t\big)\theta_1\big(\rho(x,t)\big)\Big) + \xi\big(d_\Gamma(x,t)\big)\epsilon^2 q^\epsilon(x,t) \\ + \Big(1-\xi\big(d_\Gamma(x,t)\big)\Big)\big(c_A^{\epsilon,+}(x,t)\chi_{\Omega^+_{T_0}}(x,t) + c_A^{\epsilon,-}(x,t)\chi_{\Omega^-_{T_0}}(x,t)\big) \tag{2.34}$$

for all $(x,t) \in \Omega_T$, where $\rho(x,t) := \frac{d_\Gamma(x,t)}{\epsilon} - h^\epsilon(S(x,t),t)$. The occurring functions are supposed to be smooth and satisfy for some $C^* > 0$ the following properties:

$\theta_1 : \mathbb{R} \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\mathbb{R}} \theta_1(\rho)\theta_0'(\rho)^2 f^{(3)}\big(\theta_0(\rho)\big)\,\mathrm{d}\rho = 0. \tag{2.35}$$

Furthermore, $p^\epsilon : \Gamma \to \mathbb{R}$, $q^\epsilon : \Gamma(2\delta) \to \mathbb{R}$ satisfy

$$\sup_{\epsilon\in(0,\epsilon_0)} \sup_{(x,t)\in\Gamma(2\delta;T)} \left(|p^\epsilon(\mathrm{Pr}_{\Gamma_t}(x),t)| + \frac{\epsilon}{\epsilon + \big|d_\Gamma(x,t) - \epsilon h^\epsilon\big(S(x,t),t\big)\big|}|q^\epsilon(x,t)|\right) \leqslant C^*, \tag{2.36}$$

$h^\epsilon : \mathbb{T}^1 \times [0,T] \to \mathbb{R}$ fulfills

$$\sup_{\epsilon\in(0,\epsilon_0)} \sup_{(s,t)\in\mathbb{T}^1\times[0,T]} \big(|h^\epsilon(s,t)| + |\partial_s h^\epsilon(s,t)|\big) \leqslant C^* \tag{2.37}$$

and $c_A^{\epsilon,\pm} : \Omega_T^\pm \to \mathbb{R}$ satisfy

$$\pm\, c_A^{\epsilon,\pm} > 0 \text{ in } \Omega_T^\pm. \tag{2.38}$$

Additionally, we suppose that there is some C^* such that

$$\sup_{\epsilon\in(0,\epsilon_0)} \left(\sup_{(x,t)\in\Omega_T} \big|c_A^\epsilon(x,t)\big| + \sup_{x\in\Gamma(\delta)} \big|\nabla^\Gamma c_A^\epsilon(x,t)\big|\right) \leqslant C^*, \tag{2.39}$$

$$\inf_{\epsilon\in(0,\epsilon_0)} \inf_{(x,t)\in\Omega_T\setminus\Gamma(\delta;T)} f''\left(c_A^\epsilon(x,t)\right) \geqslant \frac{1}{C^*}. \tag{2.40}$$

Corollary 2.12 *Let Assumptions 2.11 hold true and let $t \in [0,T]$, let $\psi \in H^1(\Gamma_t(\delta))$ and $\Lambda_\epsilon \in \mathbb{R}$ be such that*

$$\int_{\Gamma_t(\delta)} \epsilon\,|\nabla\psi(x)|^2 + \epsilon^{-1} f''\left(c_A^\epsilon(x,t)\right)\psi(x)^2\,\mathrm{d}x \leqslant \Lambda_\epsilon$$

and denote $I_\epsilon^{s,t} := \big(-\frac{\delta}{\epsilon} - h^\epsilon(s,t), \frac{\delta}{\epsilon} - h^\epsilon(s,t)\big)$. Then, for $\epsilon > 0$ small enough, there exist functions $Z \in H^1(\mathbb{T}^1)$, $\psi^{\mathbf{R}} \in H^1(\Gamma_t(\delta))$ and smooth $\Psi : I_\epsilon^{s,t} \times \mathbb{T}^1 \to \mathbb{R}$ such that

$$\psi(r,s) = \epsilon^{-\frac{1}{2}} Z(s)\Big(\beta(s)\theta_0'\big(\rho(r,s)\big) + \Psi\big(\rho(r,s),s\big)\Big) + \psi^{\mathbf{R}}(r,s) \tag{2.41}$$

for almost all $(r,s) \in (-\delta,\delta)\times\mathbb{T}^1$, where $\rho(r,s) = \frac{r}{\epsilon} - h^\epsilon(s,t)$ and $\beta(s) = \left(\int_{I_\epsilon^{s,t}} \left(\theta_0'(\rho)\right)^2 \mathrm{d}\rho\right)^{-\frac{1}{2}}$. Moreover,

$$\|\psi^{\mathbf{R}}\|^2_{L^2(\Gamma_t(\delta))} \leqslant C\big(\epsilon\Lambda_\epsilon + \epsilon^2\|\psi\|^2_{L^2(\Gamma_t(\delta))}\big), \tag{2.42}$$

$$\|Z\|_{H^1(\mathbb{T}^1)}^2 + \left\|\nabla^\Gamma \psi\right\|_{L^2(\Gamma_t(\delta))}^2 + \left\|\psi^{\mathbf{R}}\right\|_{H^1(\Gamma_t(\delta))}^2 \leqslant C\left(\|\psi\|_{L^2(\Gamma_t(\delta))}^2 + \frac{\Lambda_\epsilon}{\epsilon}\right), \tag{2.43}$$

and

$$\sup_{s\in\mathbb{T}^1}\left(\int_{I_\epsilon^{s,t}}\left(\Psi(\rho,s)^2 + \Psi_\rho(\rho,s)^2\right) J\left(\epsilon\left(\rho + h^\epsilon(s,t)\right),s\right) \mathrm{d}\rho\right) \leqslant C\epsilon^2. \tag{2.44}$$

Proof. We define $\tilde{\psi} := \frac{\psi}{\|\psi\|_{L^2(\Gamma_t(\delta))}}$. Then we have

$$\int_{\Gamma_t(\delta)} \epsilon\left|\nabla\tilde{\psi}\right|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right)\tilde{\psi}^2 \,\mathrm{d}x \leqslant \frac{\Lambda_\epsilon}{\|\psi\|_{L^2(\Gamma_t(\delta))}^2}$$

and may use Lemma 2.2 and Lemma 2.4 in [11] adapted to the case of the stretched variable $\rho = \frac{r}{\epsilon} - h^\epsilon(s,t)$ instead of $z = \frac{r}{\epsilon}$, where $r \in I_1$, $s \in \mathbb{T}^1$, see [17, Chapter 3] for the details. This yields existence of some functions $\tilde{Z} \in H^1(\mathbb{T}^1)$, $\tilde{\psi}^{\mathbf{R}} \in H^1(\Gamma_t(\delta))$ and Ψ such that

$$\tilde{\psi}(r,s) = \epsilon^{-\frac{1}{2}}\tilde{Z}(s)\Big(\beta(s)\theta_0'\big(\rho(r,s)\big) + \Psi\big(\rho(r,s),s\big)\Big) + \tilde{\psi}^{\mathbf{R}}(r,s) \tag{2.45}$$

with

$$\|\tilde{Z}\|_{H^1(\mathbb{T}^1)}^2 + \|\nabla^\Gamma\tilde{\psi}\|_{L^2(\Gamma_t(\delta))}^2 + \|\tilde{\psi}^{\mathbf{R}}\|_{H^1(\Gamma_t(\delta))}^2 \leqslant C\left(1 + \frac{\Lambda_\epsilon}{\epsilon\|\psi\|_{L^2(\Gamma_t(\delta))}^2}\right), \tag{2.46}$$

$$\|\tilde{\psi}^{\mathbf{R}}\|_{L^2(\Gamma_t(\delta))}^2 \leqslant C\left(\epsilon\frac{\Lambda_\epsilon}{\|\psi\|_{L^2(\Gamma_t(\delta))}^2} + \epsilon^2\right), \tag{2.47}$$

and such that Ψ satisfies (2.44). Furthermore, if we define

$$\psi_1(r,s) := \epsilon^{-\frac{1}{2}}\Big(\beta(s)\theta_0'\big(\rho(r,s)\big) + \Psi\big(\rho(r,s),s\big)\Big),$$
$$Z(s) := (\psi_1,\psi)_J \quad \text{and} \quad \psi^{\mathbf{R}}(r,s) := \psi(r,s) - Z(s)\psi_1(r,s),$$

we have the identities

$$Z(s) = (\psi_1,\psi)_J = \left(\psi_1, \tilde{\psi}\,\|\psi\|_{L^2(\Gamma_t(\delta))}\right) = \tilde{Z}(s)\,\|\psi\|_{L^2(\Gamma_t(\delta))}$$

and

$$\psi^{\mathbf{R}}(r,s) = \tilde{\psi}(r,s)\,\|\psi\|_{L^2(\Gamma_t(\delta))} + \tilde{Z}(s)\,\|\psi\|_{L^2(\Gamma_t(\delta))}\,, \psi_1(r,s) = \tilde{\psi}^{\mathbf{R}}(r,s)\,\|\psi\|_{L^2(\Gamma_t(\delta))}$$

for almost all $(r,s) \in (-\delta,\delta)\times\mathbb{T}^1$. Thus, (2.41), (2.42), (2.43) follow immediately. □

In the following we consider $H_0^1(\Omega)$ equipped with the scalar product $(u,v)_1 = \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x$. The induced norm $|.|_1$ is equivalent to the usual H^1-norm by Poincaré's inequality.

Theorem 2.13 (Spectral estimate) *Let Assumption 2.11 hold true and let $t \in [0,T]$. There exist constants $C_1 > 0$, $C_2 \geqslant 0$ and $\epsilon_1 > 0$, independent of t, such that for all $\psi \in H_0^1(\Omega)$ it holds*

$$\begin{aligned}\int_\Omega \epsilon\left|\nabla\psi\right|^2 + \epsilon^{-1}f''(c_A^\epsilon)\psi^2\,\mathrm{d}x &\geqslant C_1\left(\epsilon\left\|\psi\right\|_{L^2(\Omega)}^2 + \epsilon^{-1}\left\|\psi\right\|_{L^2\left(\Omega\setminus\Gamma_t(\delta)\right)} + \epsilon\left\|\nabla^\Gamma\psi\right\|_{L^2\left(\Gamma_t(\delta)\right)}^2\right)\\ &\quad + C_1\left(\epsilon^3\left\|\nabla\psi\right\|_{L^2(\Omega)}^2 + \epsilon\left\|\nabla\psi\right\|_{L^2\left(\Omega\setminus\Gamma_t(\delta)\right)}^2\right) - C_2\left\|\psi\right\|_{H^{-1}(\Omega)}^2.\end{aligned} \tag{2.48}$$

for all $\epsilon \in (0,\epsilon_1)$.

Proof. Due to (2.40) we may estimate

$$\begin{aligned}
&\int_\Omega \epsilon\,|\nabla\psi|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right)\psi^2\,\mathrm{d}x\\
&\geqslant \int_{\Omega\backslash\Gamma_t(\delta)} \epsilon\,|\nabla\psi|^2 + C\epsilon^{-1}\,|\psi|^2\,\mathrm{d}x + \int_{\mathbb{T}^1}\int_{-\delta}^{\delta} \left(\epsilon\,|\psi_r|^2 + \epsilon\,|\nabla_\tau\psi|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right)\psi^2\right) J\,\mathrm{d}r\,\mathrm{d}s\\
&\geqslant \int_{\Omega\backslash\Gamma_t(\delta)} \epsilon\,|\nabla\psi|^2 + C_1\epsilon^{-1}\,|\psi|^2\,\mathrm{d}x + \epsilon\int_{\Gamma_t(\delta)} |\nabla_\tau\psi|^2\,\mathrm{d}x - C_2\epsilon\int_\Omega \psi^2\,\mathrm{d}x, \qquad (2.49)
\end{aligned}$$

where the last inequality is a consequence of [11, Lemma 2.2], adapted to the case of the stretched variable $\rho = \frac{r}{\epsilon} - h^\epsilon(s,t)$ instead of $z = \frac{r}{\epsilon}$, where $r \in (-\delta,\delta)$, $s \in \Gamma$, cf. [17, Proof of Theorem 3.12] for more details. We observe that we may now use (2.49) to derive

$$\begin{aligned}
&\int_\Omega \epsilon\,|\nabla\psi|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right)\psi^2\,\mathrm{d}x\\
&\qquad\geqslant C_1\left(\epsilon\|\nabla^\Gamma\psi\|^2_{L^2(\Gamma_t(\delta))} + \epsilon^{-1}\|\psi\|^2_{L^2(\Omega\backslash\Gamma_t(\delta))} + \epsilon\|\nabla\psi\|^2_{L^2(\Omega\backslash\Gamma_t(\delta))}\right)\\
&\qquad\qquad + C_1\epsilon^3\|\nabla\psi\|^2_{L^2(\Omega)} - C_2\epsilon\|\psi\|^2_{L^2(\Omega)} \qquad (2.50)
\end{aligned}$$

for $C_1, C_2 > 0$ and all $\epsilon \in (0,\epsilon_1)$, after choosing ϵ_1 so small that $\epsilon_1 \leqslant \frac{1}{2}$ is fulfilled. Now, in order to prove (2.48) we fix a constant $c > C_2$ and $\epsilon \in (0,\epsilon_0)$ and consider two different cases: First, we assume

$$\int_\Omega \epsilon\,|\nabla\psi|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right)\psi^2\,\mathrm{d}x > c\epsilon\,\|\psi\|^2_{L^2(\Omega)}$$

which leads to the claim immediately, with $C_2 = 0$. In the case

$$\int_\Omega \epsilon\,|\nabla\psi|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right)\psi^2\,\mathrm{d}x \leqslant c\epsilon\,\|\psi\|^2_{L^2(\Omega)}$$

let $w \in H^2(\Omega) \cap H_0^1(\Omega)$ be the unique solution to $-\Delta w = \psi$. Then [11, Theorem 3.1] implies

$$\tilde{C}\epsilon\,\|\psi\|^2_{L^2(\Omega)} \leqslant \|\nabla w\|^2_{L^2(\Omega)}\,. \qquad (2.51)$$

Moreover, $\|\psi\|^2_{H^{-1}(\Omega)} = \|\nabla w\|^2_{L^2(\Omega)}$ and thus we get

$$\begin{aligned}
\int_\Omega \epsilon|\nabla\psi|^2 + \epsilon^{-1} f'\left(c_A^\epsilon\right)\psi^2\,\mathrm{d}x &\geqslant C\left(\epsilon\|\psi\|^2_{L^2(\Omega)} + \epsilon^{-1}\|\psi\|_{L^2(\Omega\backslash\Gamma_t(\delta))} + \epsilon\|\nabla^\Gamma\psi\|^2_{L^2(\Gamma_t(\delta))}\right)\\
&\quad + C\left(\epsilon\|\nabla\psi\|^2_{L^2(\Omega\backslash\Gamma_t(\delta))} + \epsilon^3\|\nabla\psi\|^2_{L^2(\Omega)}\right) - \tilde{C}\|\psi\|^2_{H^{-1}(\Omega)}.
\end{aligned}$$

This proves the assertion. □

3. Proof of Theorem 1.1

3.1 *The approximate solutions*

A major ingredient of this work is the construction of an approximate solution, which satisfies (1.1)–(1.6) up to a sufficiently high order. In the following we present a collection of properties of the approximations, which are necessary to prove Theorem 1.1 and are constructed in [5], alternatively see [17].

Theorem 3.1 *For every* $\epsilon \in (0,1)$ *there are* $\mathbf{v}_A^\epsilon, \mathbf{w}_1^\epsilon : \Omega_{T_0} \to \mathbb{R}^2$ $c_A^\epsilon, \mu_A^\epsilon, p_A^\epsilon : \Omega_{T_0} \to \mathbb{R}$ *and* $\mathbf{r}_{\mathrm{S}}^\epsilon : \Omega_{T_0} \to \mathbb{R}^2$, $r_{\mathrm{div}}^\epsilon, r_{\mathrm{CH1}}^\epsilon, r_{\mathrm{CH2}}^\epsilon : \Omega_{T_0} \to \mathbb{R}$ *such that*

$$-\Delta \mathbf{v}_A^\epsilon + \nabla p_A^\epsilon = \mu_A^\epsilon \nabla c_A^\epsilon + \mathbf{r}_{\mathrm{S}}^\epsilon, \tag{3.1}$$

$$\mathrm{div}\mathbf{v}_A^\epsilon = r_{\mathrm{div}}^\epsilon, \tag{3.2}$$

$$\partial_t c_A^\epsilon + \left(\mathbf{v}_A^\epsilon + \epsilon^{M-\frac{1}{2}} \mathbf{w}_1^\epsilon|_\Gamma \xi(d_\Gamma)\right) \cdot \nabla c_A^\epsilon = \Delta \mu_A^\epsilon + r_{\mathrm{CH1}}^\epsilon, \tag{3.3}$$

$$\mu_A^\epsilon = -\epsilon \Delta c_A^\epsilon + \epsilon^{-1} f'\left(c_A^\epsilon\right) + r_{\mathrm{CH2}}^\epsilon, \tag{3.4}$$

in Ω_{T_0} *and*

$$c_A^\epsilon = -1, \quad \mu_A^\epsilon = 0, \quad \left(-2D_s \mathbf{v}_A^\epsilon + p_A^\epsilon \mathbf{I}\right)\mathbf{n}_{\partial\Omega} = \alpha_0 \mathbf{v}_A^\epsilon, \quad r_{\mathrm{div}}^\epsilon = 0 \tag{3.5}$$

are satisfied on $\partial_{T_0}\Omega$. *If additionally Assumption 1.2 holds for* $\epsilon_0 \in (0,1)$, $K \geqslant 1$ *and a family* $(T_\epsilon)_{\epsilon\in(0,\epsilon_0)} \subset (0, T_0]$, *then there are some* $\epsilon_1 \in (0, \epsilon_0]$, $C(K) > 0$ *depending on* K *and* $C_K : (0, T_0] \times (0,1] \to (0,\infty)$ *(also dependent on* K*), which satisfies* $C_K(T,\epsilon) \to 0$ *as* $(T,\epsilon) \to 0$, *such that*

$$\int_0^{T_\epsilon} \left| \int_\Omega r_{\mathrm{CH1}}^\epsilon(x,t)\varphi(x,t)\,\mathrm{d}x \right| \mathrm{d}t \leqslant C_K(T_\epsilon,\epsilon)\,\epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}, \tag{3.6}$$

$$\int_0^{T_\epsilon} \left| \int_\Omega r_{\mathrm{CH2}}^\epsilon(x,t)\left(c^\epsilon(x,t) - c_A^\epsilon(x,t)\right) \mathrm{d}x \right| \mathrm{d}t \leqslant C_K(T_\epsilon,\epsilon)\,\epsilon^{2M}, \tag{3.7}$$

$$\left\|\mathbf{r}_{\mathrm{S}}^\epsilon\right\|_{L^2\left(0,T_\epsilon;(H^1(\Omega))'\right)} + \left\|r_{\mathrm{div}}^\epsilon\right\|_{L^2(\Omega_{T_\epsilon})} \leqslant C(K)\epsilon^M, \tag{3.8}$$

$$\left\|r_{\mathrm{CH2}}^\epsilon \nabla c_A^\epsilon\right\|_{L^2\left(0,T_\epsilon;(H^1(\Omega)^2)'\right)} \leqslant C(K) C(T_\epsilon,\epsilon)\,\epsilon^M \tag{3.9}$$

$$\left\|r_{\mathrm{CH1}}^\epsilon\right\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))} \leqslant C(K)\epsilon^M \tag{3.10}$$

for all $\epsilon \in (0,\epsilon_1)$ *and* $\varphi \in L^\infty\left(0, T_\epsilon; H^1(\Omega)\right)$.

In the following we will need a more intricate knowledge of the approximate solutions. Let ξ be a cut-off function satisfying (1.21), and we denote $\mathbf{v}^\pm := \mathbf{v}|_{\Omega_{T_0}^\pm}$, $\mu^\pm := \mu|_{\Omega_{T_0}^\pm}$ for solutions $\mu, \mathbf{v}$ of (1.7)–(1.16). We assume that $\mathbf{v}^\pm$, $\mu^\pm$ are smoothly extended to $\Omega_{T_0}^\pm \cup \Gamma(2\delta; T_0)$, where $\mathbf{v}^\pm$ is moreover divergence free in that region. We refer to [5, Remark 3.1], for more details on this extension and [5, Remark 4.3] for more information on the structural details discussed below. We have

$$\begin{aligned}
c_A^\epsilon(x,t) &= \xi\big(d_\Gamma(x,t)\big)c_I(x,t) + \Big(1 - \xi\big(d_\Gamma(x,t)\big)\Big)c_{O,\mathbf{B}}(x,t),\\
\mu_A^\epsilon(x,t) &= \xi\big(d_\Gamma(x,t)\big)\mu_I(x,t) + \Big(1 - \xi\big(d_\Gamma(x,t)\big)\Big)\mu_{O,\mathbf{B}}(x,t) + \epsilon^{M-\frac{1}{2}}\mu_{A,M-\frac{1}{2}}^\epsilon(x,t),\\
\mathbf{v}_A^\epsilon(x,t) &= \xi\big(d_\Gamma(x,t)\big)\mathbf{v}_I(x,t) + \Big(1 - \xi\big(d_\Gamma(x,t)\big)\Big)\mathbf{v}_{O,\mathbf{B}}(x,t) + \epsilon^{M-\frac{1}{2}}\mathbf{v}_{A,M-\frac{1}{2}}^\epsilon(x,t)
\end{aligned}$$

for $(x,t) \in \Omega_{T_0}$. Here $c_{O,\mathbf{B}} = \pm 1 + \mathcal{O}(\epsilon)$ in $C^1\left(\Omega_{T_0}^\pm\right)$ as $\epsilon \to 0$, with $\|c_{O,\mathbf{B}}\|_{C^2\left(\Omega_{T_0}^\pm\right)} \leqslant C$ and

$\mu_{O,\mathbf{B}} = \mu^{\pm} + \mathcal{O}(\epsilon)$ and $\mathbf{v}_{O,\mathbf{B}} = \mathbf{v}^{\pm} + \mathcal{O}(\epsilon)$ in $L^\infty\left(\Omega^{\pm}_{T_0}\right)$ as $\epsilon \to 0$. Moreover,

$$c_I(x,t) = \sum_{k=0}^{M+1} \epsilon^k c_k\big(\rho(x,t), x, t\big) \quad \forall (x,t) \in \Gamma(2\delta; T_0),$$

where $c_k : \mathbb{R} \times \Gamma(2\delta; T_0) \to \mathbb{R}$, $k \in \{0, \dots, M+1\}$, are smooth and bounded functions, which do not depend on ϵ and have bounded derivatives. Here

$$\rho(x,t) = \frac{d_\Gamma(x,t)}{\epsilon} - h^\epsilon_A(S(x,t),t) \quad \forall (x,t) \in \Gamma(2\delta; T_0), \tag{3.11}$$

where $h^\epsilon_A(s,t) = \sum_{k=0}^{M} \epsilon^k h_{k+1}(s,t) + \epsilon^{M-\frac{3}{2}} h^\epsilon_{M-\frac{1}{2}}(s,t)$ for $(s,t) \in \mathbb{T}^1 \times [0,T_0]$ and h_k are smooth and bounded functions independent of ϵ with bounded derivatives, for $k \in \{1, \dots, M+1\}$. Moreover, μ_I and $\mathbf{v}_I$ have the same kind of expansion with coefficients μ_k and $\mathbf{v}_k$, $k \in \{0, \dots, M+1\}$. In particular, we have

$$\begin{aligned} c_0(\rho,x,t) &= \theta_0(\rho),\ \mathbf{v}_0(\rho,x,t) = \mathbf{v}^+(x,t)\eta(\rho) - \mathbf{v}^-(x,t)\big(1-\eta(\rho)\big) \\ \mu_0(\rho,x,t) &= \mu^+(x,t)\eta(\rho) - \mu^-(x,t)\big(1-\eta(\rho)\big) \end{aligned} \tag{3.12}$$

for $(\rho,x,t) \in \mathbb{R} \times \Gamma(2\delta; T_0)$, where $\eta : \mathbb{R} \to [0,1]$ is smooth and satisfies $\eta = 0$ in $(-\infty,-1]$, $\eta = 1$ in $[1,\infty)$ and $\eta' \geqslant 0$ in $\mathbb{R}$. The so-called inner terms satisfy moreover $\nabla^l_x \partial^m_t \partial^i_\rho u \in \mathcal{R}_\alpha$ for some $\alpha > 0$, where $i \geqslant 1, m, l \geqslant 0$ and $u = c_k, \mu_k, \mathbf{v}_k$ for $k \in \{0, \dots, M+1\}$. Additionally, we note for later use $h^\epsilon_A(s,0) = 0$ for all $s \in \mathbb{T}^1$.

Regarding the structure of the fractional order terms, we have

$$\begin{aligned} \mu^\epsilon_{A,M-\frac{1}{2}} &= \xi(d_\Gamma)\,\mu^\epsilon_{M-\frac{1}{2}} + \big(1-\xi(d_\Gamma)\big)\Big(\mu^{+,\epsilon}_{M-\frac{1}{2}} \overline{\chi_{\Omega^+_{T_0}}} + \mu^{-,\epsilon}_{M-\frac{1}{2}} \chi_{\overline{\Omega}_{T_0}}\Big), \\ \mathbf{v}^\epsilon_{A,M-\frac{1}{2}}(x,t) &= \xi(d_\Gamma)\,\mathbf{v}^\epsilon_{M-\frac{1}{2}} + \big(1-\xi(d_\Gamma)\big)\Big(\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}} \overline{\chi_{\Omega^+_{T_0}}} + \mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}} \chi_{\overline{\Omega}_{T_0}}\Big) \end{aligned}$$

in Ω_{T_0}, where $\mu^{\pm,\epsilon}_{M-\frac{1}{2}}, \mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}$ are functions defined on $\Omega^{\pm}_{T_0} \cup \Gamma(2\delta; T_0)$ and $\mu^\epsilon_{M-\frac{1}{2}} := \mu^{+,\epsilon}_{M-\frac{1}{2}}\eta - \mu^{-,\epsilon}_{M-\frac{1}{2}}(1-\eta)$ and $\mathbf{v}^\epsilon_{M-\frac{1}{2}} := \mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}}\eta - \mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}(1-\eta)$ in $\Gamma(2\delta; T_0)$. As technical details, we remark that

$$r^\epsilon_{\mathrm{CH2}} = \epsilon^{M-\frac{1}{2}} \mu^-_{M-\frac{1}{2}} + \mathcal{O}\left(\epsilon^{M+1}\right) \text{ in } L^\infty\left(\Omega_{T_0} \backslash \Gamma(2\delta)\right) \text{ as } \epsilon \to 0, \tag{3.13}$$

which is a direct consequence of [5, Remark 4.4] and that $\mu^-_{M-\frac{1}{2}} = 0$ on $\partial_{T_0}\Omega$, which is discussed in [5, Remark 4.3].

A key element in the proof of Theorem 1.1 is an understanding of the term $\mathbf{w}^\epsilon_1$ mentioned in Theorem 3.1 and also of the appearing fractional order terms, which are in the end a consequence of the appearance of $\mathbf{w}^\epsilon_1$. This motivates the following analysis: For $T \in (0,T_0]$ we consider weak solutions $\tilde{\mathbf{w}}^\epsilon_1 : \Omega_T \to \mathbb{R}^2$ and $q^\epsilon_1 : \Omega_T \to \mathbb{R}$ of

$$-\Delta \tilde{\mathbf{w}}^\epsilon_1 + \nabla q^\epsilon_1 = -\epsilon \operatorname{div}\big((\nabla c^\epsilon_A - \mathbf{h}) \otimes_s \nabla R\big) \quad \text{in } \Omega_T, \tag{3.14}$$

$$\operatorname{div} \tilde{\mathbf{w}}^\epsilon_1 = 0 \quad \text{in } \Omega_T, \tag{3.15}$$

$$\big(-2D_s \tilde{\mathbf{w}}^\epsilon_1 + q^\epsilon_1 \mathbf{I}\big) \cdot \mathbf{n}_{\partial\Omega} = \alpha_0 \tilde{\mathbf{w}}^\epsilon_1 \quad \text{on } \partial_T\Omega \tag{3.16}$$

in the sense of (2.5). Here we denote $R := c^\epsilon - c_A^\epsilon$ and we define $\mathbf{h}$ by

$$\mathbf{h}(x,t) := -\xi\big(d_\Gamma(x,t)\big) \sum_{k=0}^{M+1} \epsilon^k \partial_\rho c_k\big(\rho(x,t),x,t\big)\epsilon^{M-\frac{3}{2}}\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}(x,t) \tag{3.17}$$

and $\otimes_s$ as $\mathbf{a} \otimes_s \mathbf{b} := \mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a}$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. We calculate

$$\begin{aligned}\left(\nabla c_A^\epsilon - \mathbf{h}\right)(x,t) &= \xi'\big(d_\Gamma(x,t)\big)\nabla d_\Gamma(x,t) c_I(x,t) + \xi\big(d_\Gamma(x,t)\big)\left(\sum_{k=0}^{M+1} \epsilon^k \nabla_x c_k\big(\rho(x,t),x,t\big)\right)\\ &\quad + \xi\big(d_\Gamma(x,t)\big)\left(\sum_{k=0}^{M+1} \epsilon^k \partial_\rho c_k\big(\rho(x,t),x,t\big)\left(\rho(x,t),x,t\right)\right.\\ &\qquad\qquad \left.\left(\frac{1}{\epsilon}\nabla d_\Gamma(x,t) - \sum_{i=0}^{M} \epsilon^i \nabla^\Gamma h_{i+1}(x,t)\right)\right)\\ &\quad + \nabla\Big(\Big(1-\xi\big(d_\Gamma(x,t)\big)\Big)c_{O,\mathbf{B}}(x,t)\Big)\end{aligned} \tag{3.18}$$

for $(x,t) \in \Omega_{T_0}$. We understand the right-hand side of (3.14) as a functional in $(V_0)'$ given by

$$\mathbf{f}^\epsilon(\psi) := \int_\Omega \epsilon\big((\nabla c_A^\epsilon - \mathbf{h}) \otimes \nabla R + \nabla R \otimes (\nabla c_A^\epsilon - \mathbf{h})\big) : \nabla\psi \, dx \tag{3.19}$$

for $\psi \in V_0$ and fixed $t \in [0,T]$. $\mathbf{w}_1^\epsilon$ as introduced in Theorem 3.1 is just a rescaling of $\tilde{\mathbf{w}}_1^\epsilon$, i.e.,

$$\mathbf{w}_1^\epsilon = \frac{\tilde{\mathbf{w}}_1^\epsilon}{\epsilon^{M-\frac{1}{2}}} \tag{3.20}$$

holds. Furthermore, we introduce

$$X_T = L^2\big(0,T;H^{\frac{7}{2}}(\mathbb{T}^1)\big) \cap H^1\big(0,T;H^{\frac{1}{2}}(\mathbb{T}^1)\big) \tag{3.21}$$

for $T \in \mathbb{R}_+ \cup \{\infty\}$, where we equip X_T with the norm

$$\|h\|_{X_T} = \|h\|_{L^2(0,T;H^{\frac{7}{2}}(\mathbb{T}^1))} + \|h\|_{H^1(0,T;H^{\frac{1}{2}}(\mathbb{T}^1))} + \|h|_{t=0}\|_{H^2(\mathbb{T}^1)}.$$

Note that $X_T \hookrightarrow C^0\left([0,T];H^2\left(\mathbb{T}^1\right)\right)$, where the operator norm of the embedding can be bounded independently of T.

The following lemma is shown in [5, Lemma 3.13] and enables us to access the results obtained in Subsection 2.4.

Lemma 3.2 *Let $\epsilon_0 > 0$, $T \in (0,T_0]$ and $(T_\epsilon)_{\epsilon\in(0,\epsilon_0)} \subset (0,T]$ be given. We assume that there is some $\bar{C} > 0$ such that*

$$\sup_{\epsilon\in(0,\epsilon_0)} \|h^\epsilon_{M-\frac{1}{2}}\|_{X_{T_\epsilon}} \leqslant \bar{C}$$

holds. Then there is $\epsilon_1 \in (0,\epsilon_0]$ such that $c_A^\epsilon(.,t)$ satisfies Assumption 2.11 for all $t \in [0,T_\epsilon]$ and $\epsilon \in (0,\epsilon_1)$, where the appearing constant C^* does not depend on ϵ, T_ϵ, $h^\epsilon_{M-\frac{1}{2}}$ or $\bar{C}$.

The following technical proposition is an essential ingredient for many estimates. Essentially it states that an error R can be split into a multiple of θ_0' plus perturbation terms that is of higher order in ϵ.

Proposition 3.3 *Let $\epsilon_0 > 0$, $T \in (0, T_0]$ and a family $(T_\epsilon)_{\epsilon\in(0,\epsilon_0)} \subset (0, T]$ be given. Let Assumption 1.2 hold true for $c_A = c_A^\epsilon$ and we assume that there is some $\bar{C} \geqslant 1$ such that*

$$\sup_{\epsilon\in(0,\epsilon_0)} \left\| h^\epsilon_{M-\frac{1}{2}} \right\|_{X_{T_\epsilon}} \leqslant \bar{C}.$$

We denote

$$I_\epsilon^{s,t} := \left(-\frac{\delta}{\epsilon} - h_A^\epsilon(s,t), \frac{\delta}{\epsilon} - h_A^\epsilon(s,t)\right) \quad \text{and} \quad \beta(s,t) := \left\|\theta_0'\right\|^{-1}_{L^2(I_\epsilon^{s,t})}$$

for $\epsilon \in (0, \epsilon_0)$, $s \in \mathbb{T}^1$ and $t \in [0, T_\epsilon]$. Then there is some $\epsilon_1 \in (0, \epsilon_0]$ and there exist $Z \in L^2\left(0, T_\epsilon; H^1\left(\mathbb{T}^1\right)\right)$, $F_2^{\mathbf{R}} \in L^2\left(0, T_\epsilon; H^1\left(\Gamma_t(\delta)\right)\right)$ and smooth $F_1^{\mathbf{R}}\colon \Gamma\left(\delta; T_\epsilon\right) \to \mathbb{R}$ such that

$$R(x,t) = \epsilon^{-\frac{1}{2}} Z\big(S(x,t),t\big)\big(\beta(S(x,t),t)\theta_0'\big(\rho(x,t)\big) + F_1^{\mathbf{R}}(x,t)\big) + F_2^{\mathbf{R}}(x,t) \tag{3.22}$$

for almost all $(x,t) \in \Gamma\left(\delta; T_\epsilon\right)$ and all $\epsilon \in (0, \epsilon_1)$. Furthermore, there exist $C(K)$, $C > 0$ independent of ϵ, T_ϵ, $h^\epsilon_{M-\frac{1}{2}}$ and $\bar{C}$ such that $\|\beta\|_{L^\infty(\mathbb{T}^1\times(0,T_\epsilon))} \leqslant C$ and

$$\left\|F_2^{\mathbf{R}}\right\|^2_{L^2(\Gamma(\delta;T_\epsilon))} \leqslant C(K)\epsilon^{2M+1}, \tag{3.23}$$

$$\|Z\|^2_{L^2(0,T_\epsilon;H^1(\mathbb{T}^1))} + \left\|F_2^{\mathbf{R}}\right\|^2_{L^2(0,T_\epsilon;H^1(\Gamma_t(\delta)))} \leqslant C(K)\epsilon^{2M-1} \tag{3.24}$$

for all $\epsilon \in (0, \epsilon_1)$ as well as

$$\sup_{t\in[0,T_\epsilon]} \sup_{s\in\mathbb{T}^1} \int_{I_\epsilon^{s,t}} \left(\left|F_1^{\mathbf{R}}(\rho,s,t)\right|^2 + \left|\partial_\rho F_1^{\mathbf{R}}(\rho,s,t)\right|^2\right) J^\epsilon(\rho,s,t)\, \mathrm{d}\rho \leqslant C(K)\epsilon^2 \tag{3.25}$$

for all $\epsilon \in (0, \epsilon_1)$, where

$$F_1^{\mathbf{R}}(\rho,s,t) := F_1^{\mathbf{R}}\Big(X\Big(\epsilon\big(\rho + h_A^\epsilon(s,t)\big), s, t\Big)\Big)$$

for X as in (2.17) and $J^\epsilon(\rho,s,t) := J(\epsilon(\rho + h_A^\epsilon(s,t)), s, t)$ with $J(r,s,t) := \det\left(D_{(r,s)}X\right)(r,s,t)$.

Proof. Let ϵ_1 be chosen as in Lemma 3.2. Then c_A^ϵ satisfies Assumption 2.11 for all $\epsilon \in (0, \epsilon_1)$. Let

$$\Lambda_\epsilon(t) := \int_{\Gamma_t(\delta)} \epsilon\,|\nabla R|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right)(R)^2\, \mathrm{d}x.$$

Then (1.27c) and (2.40) imply

$$\int_0^{T_\epsilon} \Lambda_\epsilon(t) \leqslant CK^2\epsilon^{2M} \qquad \text{for all } \epsilon \in (0, \epsilon_1). \tag{3.26}$$

Hence for each $t \in [0, T_\epsilon]$, Lemma 2.12 implies the existence of functions $Z(.,t) \in H^1(\mathbb{T}^1)$, $F_1^{\mathbf{R}}(.,t) : \Gamma_t(\delta) \to \mathbb{R}$ and $F_2^{\mathbf{R}}(.,t) \in H^1(\Gamma_t(\delta))$ such that (3.22) holds for almost all $x \in \Gamma_t(\delta)$ and all $\epsilon \in (0, \epsilon_1)$. Moreover,

$$\left\| F_2^{\mathbf{R}}(.,t) \right\|_{L^2(\Gamma_t(\delta))}^2 \leqslant C\left(\epsilon \Lambda_\epsilon(t) + \epsilon^2 \left\| R(.,t) \right\|_{L^2(\Gamma_t(\delta))}^2\right)$$

$$\| Z(.,t) \|_{H^1(\mathbb{T}^1)}^2 + \left\| F_2^{\mathbf{R}}(.,t) \right\|_{H^1(\Gamma_t(\delta))}^2 \leqslant C\left(\| R(.,t) \|_{L^2(\Gamma_t(\delta))}^2 + \frac{\Lambda_\epsilon(t)}{\epsilon} \right)$$

for all $\epsilon \in (0, \epsilon_2)$. Note that $C > 0$ is independent of ϵ, T_ϵ and $\bar{C}$ since C^* in Lemma 3.2 is independent of these quantities as well. Since $\|R\|_{L^2(\Omega_{T_\epsilon})}^2 \leqslant CK^2\epsilon^{2M-1}$ and (3.26) hold true due to (1.27), integration over $(0, T_\epsilon)$ yields (3.23) and (3.24). Finally, (3.25) is a direct consequence of (2.44). □

Now we show the main estimate for $\tilde{\mathbf{w}}_1^\epsilon$:

Lemma 3.4 *Let $\epsilon_0 > 0$, $T' \in (0, T_0]$ and a family $(T_\epsilon)_{\epsilon \in (0,\epsilon_0)} \subset (0, T']$ be given. Let Assumption 1.2 hold true for $c_A = c_A^\epsilon$ and we assume that there is $\bar{C} \geqslant 1$ such that*

$$\sup_{\epsilon \in (0,\epsilon_0)} \left\| h_{M-\frac{1}{2}}^\epsilon \right\|_{X_{T_\epsilon}} \leqslant \bar{C}. \tag{3.27}$$

Then there exists a constant $C(K) > 0$, which is independent of ϵ, T_ϵ, $h_{M-\frac{1}{2}}^\epsilon$ and $\bar{C}$, and some $\epsilon_1 \in (0, \epsilon_0)$ such that

$$\|\tilde{\mathbf{w}}_1^\epsilon\|_{L^2(0,T;H^1(\Omega))} \leqslant C(K)\epsilon^{M-\frac{1}{2}} \quad \text{for all } \epsilon \in (0, \epsilon_1), T \in (0, T_\epsilon]. \tag{3.28}$$

Proof. First of all, we note that there exists $\epsilon_1 \in (0, \epsilon_0]$, which depends on $\bar{C}$, such that

$$\left| \frac{d_\Gamma(x,t)}{\epsilon} - h_A^\epsilon(S(x,t),t) \right| \geqslant \frac{\delta}{2\epsilon} \tag{3.29}$$

for all $(x,t) \in \Gamma(2\delta; T_\epsilon) \setminus \Gamma(\delta; T_\epsilon)$ and $\epsilon \in (0, \epsilon_1)$ because of $X_T \hookrightarrow C^0\left([0,T]; C^1\left(\mathbb{T}^1\right)\right)$ and (3.27). After possibly choosing $\epsilon_1 > 0$ smaller, we may ensure that

$$\left|\theta_0\big(\rho(x,t)\big) - \chi_{\Omega^+}(x,t) + \chi_{\Omega^-}(x,t)\right| + \left|\theta_0'\big(\rho(x,t)\big)\right| \leqslant C_1 e^{-C_2\frac{\delta}{2\epsilon}} \tag{3.30}$$

holds true for all $(x,t) \in \Gamma(2\delta; T_\epsilon) \setminus \Gamma(\delta; T_\epsilon)$ and $\epsilon \in (0, \epsilon_1)$, as a consequence of (1.20), where $C_1, C_2 > 0$ can be chosen independently of ϵ_1. As a last condition on ϵ_1 we impose that $\epsilon_1^{M-\frac{3}{2}} \leqslant \frac{1}{\bar{C}}$, which yields

$$\epsilon^{M-\frac{3}{2}} \left\| h_{M-\frac{1}{2}}^\epsilon \right\|_{X_{T_\epsilon}} \leqslant 1 \tag{3.31}$$

for all $\epsilon \in (0, \epsilon_1)$.

Since $\tilde{\mathbf{w}}_1^\epsilon$ is a weak solution to (3.14)–(3.16) in Ω_{T_ϵ}, we have due to Theorem 2.1

$$\|\tilde{\mathbf{w}}_1^\epsilon\|_{L^2(0,T;H^1(\Omega))} \leqslant C \|\mathbf{f}^\epsilon\|_{L^2(0,T;V_0'(\Omega))}$$

for all $T \in (0, T_\epsilon)$, where $\mathbf{f}^\epsilon$ is given as in (3.19). Let in the following $T \in (0, T_\epsilon]$ and $\psi \in L^2(0, T_\epsilon; V_0(\Omega))$, $\psi \neq 0$. As a starting point, we decompose

$$\begin{aligned}\int_{\Omega_T} \epsilon \left(\nabla c_A^\epsilon - \mathbf{h}\right) \otimes \nabla R : \nabla\psi \, \mathrm{d}(x,t) &= \int_{\Gamma(\delta,T)} \epsilon \left(\nabla c_A^\epsilon - \mathbf{h}\right) \otimes \nabla R : \nabla\psi \, \mathrm{d}(x,t) \\ &\quad + \int_{\Omega_T \setminus \Gamma(\delta;T)} \epsilon \left(\nabla c_A^\epsilon - \mathbf{h}\right) \otimes \nabla R : \nabla\psi \, \mathrm{d}(x,t) \end{aligned} \tag{3.32}$$

and estimate the two integrals on the right-hand side separately. The second summand in $\mathbf{f}^\epsilon$ may be treated analogously.

To estimate the second integral in (3.32), note that $c_I, \nabla_x c_k, \partial_\rho c_k, \nabla^\Gamma h_i \in L^\infty(\Gamma(2\delta))$, $i \in \{1, \ldots, M+1\}$, $k \in \{0, \ldots, M+1\}$, $c_{O,\mathbf{B}}, \nabla c_{O,\mathbf{B}} \in L^\infty\left(\Omega^\pm_{T_0}\right)$ and that we may employ (3.30). Thus, $\left|\nabla c_A^\epsilon(x,t) - \mathbf{h}(x,t)\right| \leqslant C_1\left(1 + \frac{1}{\epsilon} e^{-C_2 \frac{\delta}{2\epsilon}}\right)$ for all $(x,t) \in \Omega_{T_\epsilon} \setminus \Gamma(\delta; T_\epsilon)$ and $\epsilon \in (0, \epsilon_1)$ and we may estimate

$$\begin{aligned}\int_0^T \int_{\Omega\setminus\Gamma_t(\delta)} \left|\epsilon \left(\nabla c_A^\epsilon - \mathbf{h}\right) \otimes \nabla R : \nabla\psi\right| \mathrm{d}x\, \mathrm{d}t &\leqslant C\epsilon \, \|\nabla R\|_{L^2(0,T;L^2(\Omega\setminus\Gamma_t(\delta)))} \|\psi\|_{L^2(0,T;H^1(\Omega))} \\ &\leqslant C(K)\epsilon^{M+\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Omega))}\end{aligned}$$

for $T \in (0, T_\epsilon)$, where we used (1.27a) in the last inequality. Dealing with the first integral on the right-hand side of (3.32) is more complicated. We compute

$$\begin{aligned}&\int_{\Gamma(\delta;T)} \epsilon \left(\nabla c_A^\epsilon - \mathbf{h}\right) \otimes \nabla R : \nabla\psi \, \mathrm{d}(x,t) \\ &= \int_{\Gamma(\delta;T)} \theta_0'(\rho) \left(\mathbf{n} - \epsilon\Big(\sum_{i=0}^{M} \epsilon^i \nabla^\Gamma h_{i+1}\Big)\right) \otimes \nabla R : \nabla\psi \, \mathrm{d}(x,t) \\ &\quad + \int_{\Gamma(\delta;T)} \epsilon \left(\nabla\left(c_A^\epsilon - \theta_0(\rho)\right) - \left(\mathbf{h} + \theta_0'(\rho)\epsilon^{M-\frac{3}{2}} \nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\right)\right) \otimes \nabla R : \nabla\psi \, \mathrm{d}(x,t),\end{aligned} \tag{3.33}$$

where we employ the shortened notations $\rho = \rho(x,t)$ and $\mathbf{n} = \mathbf{n}(S(x,t), t)$. As

$$\left(c_A^\epsilon - \theta_0 \circ \rho\right)(x,t) = \sum_{i=1}^{M+1} \epsilon^i c_i\left(\rho(x,t), x, t\right)$$

for all $(x,t) \in \Gamma(\delta; T_\epsilon)$ we find that there exists some $C > 0$ independent of K and ϵ such that

$$\left|\nabla\left(c_A^\epsilon - \theta_0(\rho)\right) - \left(\mathbf{h} + \theta_0'(\rho)\epsilon^{M-\frac{3}{2}} \nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\right)\right| \leqslant C$$

for all $(x,t) \in \Gamma(\delta; T_\epsilon)$. Thus

$$\begin{aligned}&\int_0^T \int_{\Gamma_t(\delta)} \left|\epsilon\left(\nabla\left(c_A^\epsilon - \theta_0(\rho)\right) - \left(\mathbf{h} + \theta_0'(\rho)\epsilon^{M-\frac{3}{2}} \nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\right)\right) \otimes \nabla R : \nabla\psi\right| \mathrm{d}x\, \mathrm{d}t \\ &\qquad \leqslant C\epsilon \, \|\nabla R\|_{L^2(\Gamma(\delta,T))} \|\psi\|_{L^2(0,T;H^1(\Omega))} \leqslant C(K)\epsilon^{M-\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Omega))}\end{aligned}$$

for $T \in (0, T_\epsilon]$ and $\epsilon \in (0, \epsilon_1)$, by (1.27).

Using the boundedness of θ_0' in $L^\infty(\mathbb{R})$ and that of $\nabla^\Gamma h_i$ in $L^\infty(\Gamma(2\delta))$, $i \in \{1, \ldots, M+1\}$, we also find

$$\int_0^T \int_{\Gamma_t(\delta)} \left|\theta_0'(\rho)\mathbf{n} \otimes \nabla^\Gamma R : \nabla\psi\right| \,\mathrm{d}x\,\mathrm{d}t \leqslant C \left\|\nabla^\Gamma R\right\|_{L^2(\Gamma(\delta,T))} \|\psi\|_{L^2(0,T;H^1(\Omega))} \leqslant C(K)\epsilon^{M-\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Omega))},$$

$$\int_0^T \int_{\Gamma_t(\delta)} \left|\epsilon\theta_0'(\rho)\left(\sum_{i=0}^M \epsilon^i \nabla^\Gamma h_{i+1}\right) \otimes \nabla R : \nabla\psi\right| \,\mathrm{d}x\,\mathrm{d}t \leqslant C\epsilon \|\nabla R\|_{L^2(\Gamma(\delta,T))} \|\psi\|_{L^2(0,T;H^1(\Omega))} \leqslant C(K)\epsilon^{M-\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Omega))}$$

by (1.27). Hence, plugging these results into (3.33), we obtain

$$\left|\int_{\Gamma(\delta;T)} \epsilon\left(\nabla c_A^\epsilon - \mathbf{h}\right) \otimes \nabla R : \nabla\psi \,\mathrm{d}(x,t)\right| \leqslant \mathcal{I} + C(K)\epsilon^{M-\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Omega))}$$

for $T \in (0, T_\epsilon)$ and $\epsilon \in (0, \epsilon_1)$, where

$$\mathcal{I} := \left|\int_{\Gamma(\delta;T)} \theta_0'(\rho)\mathbf{n} \otimes \mathbf{n}\partial_{\mathbf{n}} R : \nabla\psi \,\mathrm{d}(x,t)\right|.$$

Since $\psi \in V_0$, we have $\operatorname{div}\psi = 0$, which implies by (2.23) that $\operatorname{div}^\Gamma \psi = -\mathbf{n} \otimes \mathbf{n} : \nabla\psi$ holds. As the assumptions of Proposition 3.3 are satisfied, we may estimate $\mathcal{I}$ using (3.22) and obtain

$$\begin{aligned}
\mathcal{I} &= \left|\int_{\Gamma(\delta;T)} \theta_0'(\rho)\partial_{\mathbf{n}}\left(\epsilon^{-\frac{1}{2}} Z(S(x,t),t)\left(\beta(S(x,t),t)\theta_0'(\rho) + F_1^{\mathbf{R}}\right) + F_2^{\mathbf{R}}\right) \operatorname{div}^\Gamma \psi \,\mathrm{d}(x,t)\right| \\
&\leqslant \left|\int_0^T \int_{\Gamma_t(\delta)} \frac{1}{2}\partial_{\mathbf{n}}\left(\theta_0'(\rho)^2\right)\epsilon^{-\frac{1}{2}} Z(S(x,t),t)\beta(S(x,t),t)\operatorname{div}^\Gamma\psi \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad + C_1\left|\int_0^T \int_{\mathbb{T}^1} \int_{-\frac{\delta}{\epsilon}-h_A^\epsilon(s,t)}^{\frac{\delta}{\epsilon}-h_A^\epsilon(s,t)} \theta_0'(\rho)\epsilon^{-\frac{1}{2}} Z(s,t)\partial_\rho F_1^{\mathbf{R}}(\rho,s,t)\operatorname{div}^\Gamma\psi J^\epsilon(\rho,s,t) \,\mathrm{d}\rho\,\mathrm{d}s\,\mathrm{d}t\right| \\
&\quad + C_2 \left\|F_2^{\mathbf{R}}\right\|_{L^2(0,T;H^1(\Gamma_t(\delta)))} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))} \\
&=: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\end{aligned}$$

Here we used the same notations as in Proposition 3.3 and in the first lines the short notation $\rho = \rho(x,t)$. Now (3.24) implies

$$\mathcal{J}_3 \leqslant C(K)\epsilon^{M-\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))}$$

and we may estimate $\mathcal{J}_2$ by

$$\begin{aligned}
\mathcal{J}_2 &\leqslant C\epsilon^{-1} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))} \left(\int_0^T \int_{\mathbb{T}^1} Z(s,t)^2 \int_{-\frac{\delta}{\epsilon}-h_A^\epsilon(s,t)}^{\frac{\delta}{\epsilon}-h_A^\epsilon(s,t)} \left(\partial_\rho F_1^{\mathbf{R}}(\rho,s,t)\right)^2 J^\epsilon \,\mathrm{d}\rho\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leqslant C(K)\epsilon^{M-\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))},
\end{aligned}$$

where we used (3.25) in the last line. To treat the remaining integral, we may use Lemma 2.8 to get

$$
\begin{aligned}
\mathcal{J}_1 \leqslant & \left| \int_0^T \int_{\Gamma_t(\delta)} \frac{1}{2} \nabla^\Gamma \left(\partial_{\mathbf{n}} \left(\theta_0'(\rho)^2 \right) \epsilon^{-\frac{1}{2}} Z\big(S(x,t),t\big) \beta\big(S(x,t),t\big) \right) \cdot \psi \,\mathrm{d}x\,\mathrm{d}t \right| \\
& + \left| \int_0^T \int_{\Gamma_t(\delta)} \partial_{\mathbf{n}} \left(\theta_0'(\rho)^2 \right) \epsilon^{-\frac{1}{2}} Z\left(S(x,t),t\right) \beta\big(S(x,t),t\big) \psi \cdot \mathbf{n} \kappa(x,t) \,\mathrm{d}x\,\mathrm{d}t \right| \\
& + C \sum_{\pm} \int_0^T \int_{\mathbb{T}^1} \left| \partial_\rho \left(\theta_0' \left(\frac{\pm\delta}{\epsilon} - h_A^\epsilon(s,t) \right)^2 \right) \epsilon^{-\frac{3}{2}} Z(s,t) \beta(s,t) \psi\left(\pm\delta, s, t\right) \right| \mathrm{d}s\,\mathrm{d}t \\
:= {} & \mathcal{J}_1^1 + \mathcal{J}_1^2 + \mathcal{J}_1^{3,+} + \mathcal{J}_1^{3,-}.
\end{aligned}
$$

Now

$$
\begin{aligned}
\mathcal{J}_1^{3,\pm} & \leqslant C_1 \epsilon^{-\frac{3}{2}} e^{-C_2 \frac{\delta}{2\epsilon}} \int_0^T \int_{\mathbb{T}^1} |Z(s,t)| \sup_{r \in [-\delta,\delta]} |\psi(r,s,t)| \,\mathrm{d}s\,\mathrm{d}t \\
& \leqslant C(K) \epsilon^{M-\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))},
\end{aligned}
$$

where we used (3.30) and the uniform bound on β in the first step and $H^1\left(\Gamma_t(\delta)\right) \hookrightarrow L^{2,\infty}\left(\Gamma_t(\delta)\right)$ (cf. Lemma 2.9) in the second step. For $\mathcal{J}_1^2$, we use integration by parts and get

$$
\begin{aligned}
\mathcal{J}_1^2 \leqslant & \left| \int_0^T \int_{\Gamma_t(\delta)} \left(\theta_0'(\rho)\right)^2 \epsilon^{-\frac{1}{2}} Z\big(S(x,t),t\big) \beta\big(S(x,t),t\big) \partial_{\mathbf{n}} \psi \cdot \mathbf{n}\big(S(x,t),t\big) \kappa(x,t) \,\mathrm{d}x\,\mathrm{d}t \right| \\
& + C \int_0^T \int_{\Gamma_t(\delta)} \left| \left(\theta_0'(\rho)\right)^2 \epsilon^{-\frac{1}{2}} Z\left(S(x,t),t\right) \beta\big(S(x,t),t\big) \psi \right| \mathrm{d}x\,\mathrm{d}t \\
& + C(K) e^{-C_2 \frac{\delta}{2\epsilon}} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))} \\
\leqslant & \, C \epsilon^{-\frac{1}{2}} \|Z\|_{L^2(0,T;H^1(\mathbb{T}^1))} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))} \epsilon^{\frac{1}{2}} \left\| \left(\theta_0'\right)^2 \right\|_{L^2(\mathbb{R})} \\
& + C(K) e^{-C_2 \frac{\delta}{2\epsilon}} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))} \\
\leqslant & \, C(K) \epsilon^{M-\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))},
\end{aligned}
$$

where the exponential decaying term in the first inequality is a consequence of the appearing boundary integral, which may be estimated as in the case of $\mathcal{J}_1^{3,\pm}$. Moreover, we used a change of variables $r \mapsto \frac{r}{\epsilon} - h_A^\epsilon$ in the second step and (3.24) in the last step.

Now we discuss $\mathcal{J}_1^1$ – the last term we need to estimate. Note that by the definition of β in Proposition 3.3, we have

$$
\nabla^\Gamma \beta(s,t) = -\frac{1}{\|\theta_0'\|^2_{L^2(I_\epsilon^{s,t})}} \int_{-\frac{\delta}{\epsilon} - h_A^\epsilon(s,t)}^{\frac{\delta}{\epsilon} - h_A^\epsilon(s,t)} \frac{1}{2} \frac{d}{d\rho} \left(\theta_0'(\rho)^2 \right) \mathrm{d}\rho \left(-\nabla^\Gamma h_A^\epsilon \right) \leqslant C_1 e^{-C_2 \frac{\delta}{2\epsilon}}
$$

for all $\epsilon \in (0, \epsilon_1)$, due to (3.30) and $\epsilon^{M-\frac{3}{2}} \left\| h^\epsilon_{M-\frac{1}{2}} \right\|_{X_{T_\epsilon}} \leqslant 1$, cf. (3.31). Thus, we compute

$$
\begin{aligned}
\mathcal{J}_1^1 \leqslant & \left| \int_0^T \int_{\Gamma_t(\delta)} \frac{1}{2} \partial_{\mathbf{n}} \nabla^\Gamma \left(\theta_0'(\rho(x,t))^2 \right) \epsilon^{-\frac{1}{2}} Z\left(S(x,t),t\right) \beta\big(S(x,t),t\big) \cdot \psi \, \mathrm{d}x \, \mathrm{d}t \right| \\
& + \left| \int_0^T \int_{\Gamma_t(\delta)} \frac{1}{2} \left[\partial_{\mathbf{n}}, \nabla^\Gamma \right] \left(\theta_0'(\rho(x,t))^2 \right) \epsilon^{-\frac{1}{2}} Z\big(S(x,t),t\big) \beta\big(S(x,t),t\big) \cdot \psi \, \mathrm{d}x \, \mathrm{d}t \right| \\
& + \left| \int_0^T \int_{\Gamma_t(\delta)} \frac{1}{2} \partial_{\mathbf{n}} \left(\theta_0'(\rho(x,t))^2 \right) \epsilon^{-\frac{1}{2}} \nabla^\Gamma \left(Z\big(S(x,t),t\big) \beta\left(S(x,t),t\right) \right) \cdot \psi \, \mathrm{d}x \, \mathrm{d}t \right| \\
\leqslant & \, C_1 \int_0^T \int_{\Gamma_t(\delta)} \left| \partial_\rho \left(\theta_0'(\rho(x,t))^2 \right) \nabla^\Gamma h_A^\epsilon \epsilon^{-\frac{1}{2}} Z\big(S(x,t),t\big) \beta\left(S(x,t),t\right) \cdot \partial_{\mathbf{n}} \psi \right| \mathrm{d}x \, \mathrm{d}t \\
& + C_2 \int_0^T \int_{\Gamma_t(\delta)} \left| \partial_\rho \left(\theta_0'(\rho(x,t))^2 \right) \nabla^\Gamma h_A^\epsilon \epsilon^{-\frac{1}{2}} Z\big(S(x,t),t\big) \beta\left(S(x,t),t\right) \cdot \psi \right| \mathrm{d}x \, \mathrm{d}t \\
& + C_3 \int_0^T \int_{\Gamma_t(\delta)} \left| \left(\theta_0'(\rho(x,t))^2 \right) \epsilon^{-\frac{1}{2}} \nabla^\Gamma Z\left(S(x,t),t\right) \beta\big(S(x,t),t\big) \cdot \partial_{\mathbf{n}} \psi \right| \mathrm{d}x \, \mathrm{d}t \\
& + C_4 \int_0^T \int_{\Gamma_t(\delta)} \left| \left(\theta_0'(\rho(x,t))^2 \right) \epsilon^{-\frac{1}{2}} \partial_s Z\left(S(x,t),t\right) \beta\big(S(x,t),t\big) \psi \right| \mathrm{d}x \, \mathrm{d}t \\
& + C_5 e^{-C_6 \frac{\delta}{2\epsilon}} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))} \|Z\|_{L^2(0,T;H^1(\mathbb{T}^1))} \\
\leqslant & \, C(K) \epsilon^{M-\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))} .
\end{aligned}
$$

Here we used the definition of $\left[\partial_{\mathbf{n}}, \nabla^\Gamma\right]$ in the first estimate (cf. (2.31)), integration by parts, (2.32) and the exponential decay of $\nabla^\Gamma \beta$ and the boundary terms in the second step. In the third step we again used $\epsilon^{M-\frac{3}{2}} \|h^\epsilon_{M-\frac{1}{2}}\|_{X_{T_\epsilon}} \leqslant 1$. This concludes the proof. □

Regarding the fractional order terms, we have the following bounds, which are a result of [5], Theorem 3.15. This enables us to use (3.28), whenever Assumption 1.2 is satisfied.

Lemma 3.5 *Let $\epsilon_0 \in (0, 1)$. If Assumption 1.2 holds for $c_A = c_A^\epsilon$, then there exist $\epsilon_1 \in (0, \epsilon_0]$ and a constant $C(K) > 0$ independent of ϵ such that*

$$
\|h^\epsilon_{M-\frac{1}{2}}\|_{X_{T_\epsilon}} + \|\mu^{\pm,\epsilon}_{M-\frac{1}{2}}\|_{Z_{T_\epsilon}} + \|\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}\|_{L^6(0,T_\epsilon;H^2(\Omega^\pm(t)))} \leqslant C(K) \tag{3.34}
$$

for all $\epsilon \in (0, \epsilon_1)$, where $Z_{T_\epsilon} := L^2\left(0, T_\epsilon; H^2(\Omega^\pm(t))\right) \cap L^6\left(0, T_\epsilon; H^1(\Omega^\pm(t))\right)$.

As a direct consequence of (3.34) and $X_T \hookrightarrow C^0([0,T]; C^1(\mathbb{T}^1))$, we remark

$$
\left\|h_A^\epsilon\right\|_{C^0(0,T_\epsilon;C^1(\mathbb{T}^1))} \leqslant C(K) \tag{3.35}
$$

for all $\epsilon \in (0, \epsilon_1)$. Finally, concerning the relation between c_I and $c_{O,\mathbf{B}}$, we have in the case that $\epsilon_0 \in (0, 1)$ and Assumption 1.2 holds for $c_A = c_A^\epsilon$ that

$$
\left\| D_x^l \left(c_I - c_{O,\mathbf{B}}\right) \right\|_{L^\infty(\Gamma(2\delta;T_\epsilon)\setminus\Gamma(\delta;T_\epsilon))} \leqslant C(K) e^{-\frac{C}{\epsilon}} \tag{3.36}
$$

for $l \in \{0, 1\}$ and constants $C(K), C > 0$. This is discussed in [5, Corollary 4.9].

3.2 *Auxiliary results*

Without repeating it, we will consider the following assumptions throughout this section.

ASSUMPTION 3.6 We assume that Assumption 1.2 holds true holds for $c_A = c_A^\epsilon$, $\epsilon_0 \in (0,1)$, $K \geqslant 1$ and a family $(T_\epsilon)_{\epsilon\in(0,\epsilon_0)} \subset (0, T_0]$. Moreover, we assume that $\epsilon_1 \in (0, \epsilon_0]$ is chosen small enough, such that (3.6)–(3.10), the statement of Lemma 3.5, (3.28) and (3.36) hold true.

Finally, we denote $R := c^\epsilon - c_A^\epsilon$.

The following proposition guarantees that Lemma 1.3 may be used.

Proposition 3.7 *Let $\epsilon_0 \in (0,1)$ and $\psi_0^\epsilon : \Omega \to \mathbb{R}$ be a smooth function satisfying the inequality $\left\|\psi_0^\epsilon\right\|_{C^1(\Omega)} \leqslant C_{\psi_0}\epsilon^M$ for $\epsilon \in (0,\epsilon_0)$. Moreover, let $c_0^\epsilon(x) := c_A^\epsilon(x,0) + \psi_0^\epsilon(x)$ for all $x \in \Omega$. Then there is some $\tilde{\epsilon} \in (0, \epsilon_0]$ and a constant $C_0 > 0$ which only depends on $\tilde{\epsilon}$, C_{ψ_0} and $\sup_{\epsilon\in(0,\epsilon_0)} \left\|c_A^\epsilon(x,0)\right\|_{L^\infty(\Omega)}$, such that*

$$E^\epsilon\left(c_0^\epsilon\right) \leqslant C_0, \quad \|c_0^\epsilon\|_{L^\infty(\Omega)} \leqslant C_0 \qquad \text{for all } \epsilon \in (0, \tilde{\epsilon}),$$

where E^ϵ is given as in (1.29).

Proof. For simplicity we consider $c_0^\epsilon(x) = c_A^\epsilon(x,0)$ and highlight the situations where ψ_0^ϵ would play a role. The estimate for $\left\|c_0^\epsilon\right\|_{L^\infty(\Omega)}$ follows immediately by the construction of c_A^ϵ. Considering $\frac{\epsilon}{2}\int_\Omega \left|\nabla c_A^\epsilon(x,0)\right|^2 \mathrm{d}x$ we note that $\left\|\nabla c_A^\epsilon\right\|_{L^\infty(\Omega_{T_0}\setminus\Gamma(2\delta))} \leqslant C\epsilon$ and estimate

$$\begin{aligned}\frac{\epsilon}{2}\int_{\Gamma_0(2\delta)} \left|\nabla c_A^\epsilon(x,0)\right|^2 \mathrm{d}x &\leqslant \frac{\epsilon}{2}\int_{\Gamma_0(2\delta)} \left|\xi\left(d_\Gamma\right)\nabla c_I\left(x,0\right)\right|^2 \mathrm{d}x \\ &+ \frac{\epsilon}{2}\int_{\Gamma_0(2\delta)} \left|\left(1-\xi\left(d_\Gamma\right)\right)\nabla c_{O,\mathbf{B}}(x,0) + \nabla\left(\xi\left(d_\Gamma\right)\right)\left(c_I - c_{O,\mathbf{B}}\right)(x,0)\right|^2 \mathrm{d}x. \qquad (3.37)\end{aligned}$$

Now we have $\nabla c_{O,\mathbf{B}}(.,0) \in \mathcal{O}(\epsilon)$ in $L^\infty\left(\Omega^\pm(0)\right)$ and $c_I, c_{O,\mathbf{B}} \in \mathcal{O}(1)$ in $L^\infty\left(\Gamma_0(2\delta)\setminus\Gamma_0(\delta)\right)$. Moreover, $\rho(x,0) = \frac{d_\Gamma(x,0)}{\epsilon}$, as $h_A^\epsilon(x,0) = 0$, and thus

$$\nabla\big(c_0(\rho(x,0),x,0)\big) = \frac{1}{\epsilon}\theta_0'\big(\rho(x,0)\big)\cdot\mathbf{n}(x,0).$$

In particular

$$\frac{\epsilon}{2}\int_{\Gamma_0(2\delta)} \left|\xi\left(d_\Gamma\right)\nabla\Big(c_0\big(\rho(x,0),x,0\big)\Big)\right|^2 \mathrm{d}x \leqslant C\int_{\mathbb{T}^1}\int_{-\frac{2\delta}{\epsilon}}^{\frac{2\delta}{\epsilon}} \theta_0'(\rho)^2\,\mathrm{d}\rho\,\mathrm{d}s \leqslant C.$$

As $\epsilon^k\nabla\left(c_k\left(\rho\left(.,0\right),.,0\right)\right) \in \mathcal{O}(1)$ in $L^\infty\left(\Gamma_0(2\delta)\right)$ for $k \geqslant 1$, we find $\frac{\epsilon}{2}\int_{\Gamma_0(2\delta)}\left|\nabla c_A^\epsilon(x,0)\right|^2 \mathrm{d}x \leqslant C_1$ due to (3.37). Note that ψ_0^ϵ can be estimated uniformly in $C^1(\Omega)$ and is multiplied by ϵ^M, so would cause no troubles in these estimates. For the second term in $E^\epsilon\left(c_0^\epsilon\right)$, we compute

$$\frac{1}{\epsilon}\int_{\Omega^+(0)} f\left(c_0^\epsilon\right)\mathrm{d}x = \frac{1}{\epsilon}\int_{\Omega^+(0)} f'\big(\beta(x)\big)\big(c_A^\epsilon(x,0)-1\big)\,\mathrm{d}x \leqslant C$$

for some suitable $\beta(x) \in \left(1, c_A^\epsilon(x,0)\right)$, where we used a Taylor expansion and the explicit structure of c_A^ϵ. In particular, in $\Gamma_0^+(\delta) := \Omega^+(0) \cap \Gamma_0(\delta)$ a change of variables yields

$$\frac{1}{\epsilon}\int_{\Omega^+(0)} f'(\beta(x))\left(c_A^\epsilon(x,0)-1\right)\mathrm{d}x \leqslant C_1 + C_2\int_{\mathbb{T}^1}\int_0^{\frac{\delta}{\epsilon}} |\theta_0(\rho)-1|\,\mathrm{d}\rho\,\mathrm{d}s \leqslant C.$$

The appearance of ψ_0^ϵ would have changed nothing in this argumentation. This proves the claim. □

Lemma 3.8 *Let $\alpha, \kappa \in (0,1)$. There are some $C(K), C(K,\alpha)$ such that for all $\epsilon \in (0,\epsilon_1)$*

$$\begin{aligned}
\|R\|_{L^2(0,T_\epsilon;L^\infty(\Omega))} &\leqslant C(K,\alpha)\epsilon^{M-\frac{3}{2}}\epsilon^{-(M+2)\alpha},\\
\|\nabla R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} &\leqslant C(K)\epsilon^{-\frac{1}{2}},\\
\|\gamma R\|_{L^\infty(0,T_\epsilon;L^{2+\kappa}(\Omega))} &\leqslant C(K)\epsilon^{M-\frac{1}{2}-\frac{\kappa}{2+\kappa}M},\\
\|R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} &\leqslant C(K)\epsilon^{\frac{1}{2}\left(M-\frac{1}{2}\right)}.
\end{aligned} \tag{3.38}$$

Proof. For $\alpha \in (0,1)$ it holds

$$\|R\|_{L^\infty(\Omega)} \leqslant C(\alpha)\,\|R\|_{H^{1+\alpha}(\Omega)} \leqslant C(\alpha)\,\|R\|_{H^1(\Omega)}^{1-\alpha}\,\|R\|_{H^2(\Omega)}^{\alpha}. \tag{3.39}$$

Due to the construction and since h_A^ϵ is uniformly bounded in X_{T_ϵ} (cf. (3.34)). It can be easily verified by direct calculations and the properties of c_A^ϵ given in Section 3.1 that $\left\|\Delta c_A^\epsilon\right\|_{L^2(\Omega_{T_\epsilon})} \leqslant C(K)\frac{1}{\epsilon^2}$. Because of Lemma 1.3 and $R|_{\partial\Omega} = 0$, we get

$$\|R\|_{H^2(\Omega)} \leqslant C'\,\|\Delta R\|_{L^2(\Omega_{T_\epsilon})} \leqslant C(K)\epsilon^{-\frac{7}{2}}, \tag{3.40}$$

where $C(K)$ depends only on K, T_0, and C_0 (where C_0 is the constant from (1.28)). Using this and (1.27) in (3.39), we find

$$\|R\|_{L^2(0,T_\epsilon;L^\infty(\Omega))} \leqslant C(K)\left(\epsilon^{M-\frac{3}{2}}\right)^{1-\alpha}\left(\epsilon^{-\frac{7}{2}}\right)^{\alpha} = C(K)\epsilon^{M-\frac{3}{2}}\epsilon^{-(M+2)\alpha}.$$

In order to prove the second inequality, we employ Lemma 1.3, which yields

$$\epsilon^{\frac{1}{2}}\,\|\nabla R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \leqslant \epsilon^{\frac{1}{2}}\left(\|\nabla c^\epsilon\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} + \left\|\nabla c_A^\epsilon\right\|_{L^\infty(0,T_\epsilon;L^2(\Omega))}\right) \leqslant C(K).$$

Here we used $\epsilon^{\frac{1}{2}}\left\|\nabla c_A^\epsilon\right\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \leqslant C(K)$, which is a consequence of the uniform bound on $c_k, c_{O,\mathbf{B}}$ and their derivatives for $k \in \{0,\dots,M+1\}$ and the boundedness of $h_{M-\frac{1}{2}}^\epsilon$ in X_{T_ϵ}, combined with a change of variables.

For the proof of the third inequality we note that for $\kappa > 0$ we have for any $u \in H_0^1(\Omega)$

$$\|u\|_{L^{2+\kappa}(\Omega)} \leqslant C_1\,\|u\|_{L^2(\Omega)}^{1-\frac{\kappa}{2+\kappa}}\,\|\nabla u\|_{L^2(\Omega)}^{\frac{\kappa}{2+\kappa}} \tag{3.41}$$

for some $C_1 > 0$ due to the Gagliardo–Nirenberg interpolation inequality. Moreover, (3.41) together with by (3.38) and (1.27d) we obtain

$$\begin{aligned}\|\gamma R\|_{L^\infty(0,T_\epsilon;L^{2+\kappa}(\Omega))} &\leqslant C_1 \|\gamma R\|^{1-\frac{\kappa}{2+\kappa}}_{L^\infty(0,T_\epsilon;L^2(\Omega))} \|\nabla R\|^{\frac{\kappa}{2+\kappa}}_{L^\infty(0,T_\epsilon;L^2(\partial\Omega(\frac{\delta}{2})))} \\ &\leqslant C(K)\epsilon^{M-\frac{1}{2}-\frac{\kappa}{2+\kappa}M}. \end{aligned} \tag{3.42}$$

because of Poincaré's inequality, $\|R\|^2_{L^2(\Omega)} \leqslant \|R\|_{H^{-1}(\Omega)} \|\nabla R\|_{L^2(\Omega)}$, (1.27b) and (3.38). □

The following lemma is an adapted version of [4, Lemma 5.4].

Lemma 3.9 *Let $u \in H^1(\Omega)$. Then there is some constant $C > 0$ such that*

$$\|u\|^3_{L^3(\Gamma_t(\delta))} \leqslant C\left(\|u\|_{L^2(\Gamma_t(\delta))} + \left\|\nabla^\Gamma u\right\|_{L^2(\Gamma_t(\delta))}\right)^{\frac{1}{2}}\left(\|u\|_{L^2(\Gamma_t(\delta))} + \|\partial_{\mathbf{n}} u\|_{L^2(\Gamma_t(\delta))}\right)^{\frac{1}{2}} \cdot \left(\|u\|_{L^2(\Gamma_t(\delta))}\right)^2$$

holds for all $t \in [0, T_0]$.

Proof. Note

$$\|u\|^3_{L^3(\Gamma_t(\delta))} \leqslant C \int_{-\delta}^{\delta}\int_{\Gamma_t} |u\,(p,r)|^3 \, \mathrm{d}\mathcal{H}^1(p)\,\mathrm{d}r = C \left\| \|u\|_{L^3(\Gamma_t)} \right\|^3_{L^3(-\delta,\delta)}$$

and $\|u\|_{L^3(\Gamma_t)} \leqslant C \|u\|^{\frac{1}{6}}_{H^1(\Gamma_t)} \|u\|^{\frac{5}{6}}_{L^2(\Gamma_t)}$ as Γ_t is one-dimensional. Now Hölder's inequality leads to

$$\begin{aligned}\|u\|^3_{L^3(\Gamma_t(\delta))} &\leqslant C \left\| \|u\|^{\frac{1}{6}}_{H^1(\Gamma_t)} \|u\|^{\frac{5}{6}}_{L^2(\Gamma_t)} \right\|^3_{L^3(-\delta,\delta)} \\ &\leqslant C \left\| \|u\|_{H^1(\Gamma_t)} \right\|^{\frac{1}{2}}_{L^2(-\delta,\delta)} \left\| \|u\|_{L^{\frac{10}{3}}(-\delta,\delta)} \right\|^{\frac{5}{2}}_{L^2(\Gamma_t)} \\ &\leqslant C \left\| \|u\|_{H^1(\Gamma_t)} \right\|^{\frac{1}{2}}_{L^2(-\delta,\delta)} \left\| \|u\|_{H^1(-\delta,\delta)} \right\|^{\frac{1}{2}}_{L^2(\Gamma_t)} \left\| \|u\|_{L^2(-\delta,\delta)} \right\|^2_{L^2(\Gamma_t)},\end{aligned}$$

where we used $\|u\|_{L^{\frac{10}{3}}(-\delta,\delta)} \leqslant C \|u\|^{\frac{1}{5}}_{H^1(-\delta,\delta)} \|u\|^{\frac{4}{5}}_{L^2(-\delta,\delta)}$. □

3.2.1 *The error in the velocity.* For $\epsilon \in (0, \epsilon_0)$ we consider strong solutions $\overline{\mathbf{v}^\epsilon} : \Omega_{T_0} \to \mathbb{R}^2$ and $\overline{p^\epsilon} : \Omega_{T_0} \to \mathbb{R}$ of the system

$$-\Delta\overline{\mathbf{v}^\epsilon} + \nabla\overline{p^\epsilon} = \mu^\epsilon_A \nabla c^\epsilon_A \quad \text{in } \Omega_{T_0}, \tag{3.43}$$

$$\operatorname{div} \overline{\mathbf{v}^\epsilon} = 0 \quad \text{in } \Omega_{T_0}, \tag{3.44}$$

$$\left(-2D_s\overline{\mathbf{v}^\epsilon} + \overline{p^\epsilon}\mathbf{I}\right)\mathbf{n}_{\partial\Omega} = \alpha_0\overline{\mathbf{v}^\epsilon} \quad \text{on } \partial_{T_0}\Omega \tag{3.45}$$

(cf. Theorem 2.3) and weak solutions $\tilde{\mathbf{w}}_2^\epsilon : \Omega_{T_0} \to \mathbb{R}^2$ and $q_2^\epsilon : \Omega_{T_0} \to \mathbb{R}$ of

$$-\Delta \tilde{\mathbf{w}}_2^\epsilon + \nabla q_2^\epsilon = -\epsilon\big(\operatorname{div}(\mathbf{h} \otimes_s \nabla R) + \operatorname{div}(\nabla R \otimes \nabla R)\big) \quad \text{in } \Omega_{T_0}, \tag{3.46}$$

$$\operatorname{div} \tilde{\mathbf{w}}_2^\epsilon = 0 \quad \text{in } \Omega_{T_0}, \tag{3.47}$$

$$\left(-2D_s \tilde{\mathbf{w}}_2^\epsilon + q_2^\epsilon \mathbf{I}\right) \mathbf{n}_{\partial\Omega} = \alpha_0 \tilde{\mathbf{w}}_2^\epsilon \quad \text{in } \partial_{T_0}\Omega, \tag{3.48}$$

where $\mathbf{h}$ is defined as in (3.17). We consider the right-hand side of (3.46) as a functional in V_0' given by

$$\mathbf{g}^\epsilon(\psi) := \epsilon \int_\Omega \big((\mathbf{h} \otimes_s \nabla R) + (\nabla R \otimes \nabla R)\big) : \nabla \psi \, dx \quad \text{for all } \psi \in V_0. \tag{3.49}$$

Introducing

$$\mathbf{v}_{\text{err}}^\epsilon := \mathbf{v}^\epsilon - \left(\overline{\mathbf{v}^\epsilon} + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon\right) \tag{3.50}$$

we have $\mathbf{v}^\epsilon - \overline{\mathbf{v}^\epsilon} = \mathbf{v}_{\text{err}}^\epsilon + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon$. Hence, if we control $\mathbf{v}_{\text{err}}^\epsilon$, $\tilde{\mathbf{w}}_1^\epsilon$, and $\tilde{\mathbf{w}}_2^\epsilon$, we will control the error $\mathbf{v}^\epsilon - \overline{\mathbf{v}^\epsilon}$.

Lemma 3.10 *Let $\tilde{\mathbf{w}}_2^\epsilon$ be the unique weak solution to* (3.46)–(3.48) *in Ω_{T_0} for $\epsilon \in (0, \epsilon_1)$. Then it holds for all $r \in [1, 2]$ and $q \in (1, 2)$*

$$\|\tilde{\mathbf{w}}_2^\epsilon\|_{L^r(0,T_\epsilon;L^q(\Omega))} \leq C(K, r, q)\epsilon^{\frac{2(M-1)}{r}} \tag{3.51}$$

for all $\epsilon \in (0, \epsilon_1)$.

Proof. Since $\Omega \subset \mathbb{R}^2$, we have $W_{q'}^1(\Omega) \hookrightarrow C^0(\Omega)$, where $\frac{1}{q'} + \frac{1}{q} = 1$. Thus Lemma 2.4 implies

$$\|\tilde{\mathbf{w}}_2^\epsilon\|_{L^r(0,T_\epsilon;L^q(\Omega))} \leq C(q)\epsilon\left(\|\nabla R \otimes \mathbf{h}\|_{L^r(0,T_\epsilon;L^1(\Omega))} + \|\nabla R \otimes \nabla R\|_{L^r(0,T_\epsilon;L^1(\Omega))}\right).$$

We use $X_{T_\epsilon} \hookrightarrow C^0([0, T_\epsilon]; C^1(\mathbb{T}^1))$ and $\partial_\rho c_k \in \mathcal{R}_\alpha$ for $k \in \{0, \ldots, M+1\}$ and get

$$\begin{aligned}
&\epsilon \|\nabla R \otimes \mathbf{h}\|_{L^r(0,T_\epsilon;L^1(\Omega))} \\
&\leq C\epsilon^{M-\frac{1}{2}}\epsilon^{\frac{1}{2}} \left\| \sum_{k=0}^{M+1} \epsilon^k \partial_\rho c_k \right\|_{L^\infty(\Gamma(2\delta;T_0);L^2(\mathbb{R}))} \left\| h_{M-\frac{1}{2}}^\epsilon \right\|_{C^0([0,T_\epsilon];C^1(\mathbb{T}^1))} \|\nabla R\|_{L^2(0,T_\epsilon;L^2(\Omega))} \\
&\leq C(K)\epsilon^{2M-\frac{3}{2}}
\end{aligned}$$

for all $\epsilon \in (0, \epsilon_1)$ due to (3.34) and (1.27). Moreover,

$$\epsilon \|\nabla R \otimes \nabla R\|_{L^r(0,T_\epsilon;L^1(\Omega))} \leq \epsilon \|\nabla R\|_{L^2(0,T_\epsilon;L^2(\Omega))}^{\frac{2}{r}} \|\nabla R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))}^{2-\frac{2}{r}} \leq C(K)\epsilon^{\frac{2M}{r}-\frac{2}{r}}$$

for $\epsilon \in (0, \epsilon_1)$, by (1.27) and (3.38). Combining the above estimates and using $r \geq 1$ the claim follows. □

Lemma 3.11 *Let $\varphi \in L^\infty\left(0, T_\epsilon; H^1(\Omega)\right)$ and let the assumptions of Lemma 3.10 hold. Then there is some $r' > 0$ such that*

$$\int_0^{T_\epsilon} \left| \int_\Omega \left(\tilde{\mathbf{w}}_2^\epsilon \cdot \nabla c_A^\epsilon\right) \varphi \, dx \right| dt \leq C(K) T_\epsilon^{r'} \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \quad \text{for all } \epsilon \in (0, \epsilon_1).$$

Proof. Let $r \in (1,2)$. As $\nabla c_A^\epsilon \in \mathcal{O}(\epsilon)$ in $L^\infty\left(\Omega_{T_0}\backslash\Gamma(2\delta)\right)$ it immediately follows

$$\int_0^{T_\epsilon}\left|\int_{\Omega\backslash\Gamma_t(2\delta)}\left(\tilde{\mathbf{w}}_2^\epsilon\cdot\nabla c_A^\epsilon\right)\varphi\,\mathrm{d}x\right|\mathrm{d}t \leqslant C\epsilon\,\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\int_0^{T_\epsilon}\|\tilde{\mathbf{w}}_2^\epsilon\|_{L^q(\Omega)}\,\mathrm{d}t$$
$$\leqslant C(K)T_\epsilon^{\frac{1}{r'}}\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\,\epsilon^{\frac{2(M-1)}{r}+1} \tag{3.52}$$

by (3.51) for $q \in (1,2)$ and due to $H^1(\Omega) \hookrightarrow L^s(\Omega)$ for all $s \geqslant 1$. The same estimate holds for $(\nabla\xi(d_\Gamma)(c_I - c_{O,\mathbf{B}}) + (1-\xi(d_\Gamma))\nabla c_{O,\mathbf{B}})$ in $\Gamma_t(2\delta)\backslash\Gamma_t(\delta)$ by (3.36).

In $\Gamma(2\delta; T_\epsilon)$ we consider $\nabla(c_0(\rho(x,t),x,t)) = \nabla(\theta_0(\rho(x,t)))$ and compute

$$\begin{aligned}&\int_0^{T_\epsilon}\left|\int_{\Gamma_t(2\delta)}\left(\tilde{\mathbf{w}}_2^\epsilon\cdot\nabla\Big(\theta_0\big(\rho(x,t)\big)\Big)\right)\xi(d_\Gamma)\,\varphi\,\mathrm{d}x\right|\mathrm{d}t\\ &\quad\leqslant \int_0^{T_\epsilon}\int_{\Gamma_t(2\delta)}\left|\left(\tilde{\mathbf{w}}_2^\epsilon\cdot\left(\mathbf{n}-\epsilon\nabla^\Gamma h_A^\epsilon(x,t)\right)\frac{1}{\epsilon}\theta_0'\big(\rho(x,t)\big)\right)\varphi\right|\mathrm{d}x\,\mathrm{d}t\\ &\quad\leqslant C(K)\,\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\,\epsilon^{-1}\int_0^{T_\epsilon}\|\tilde{\mathbf{w}}_2^\epsilon\|_{L^q(\Omega)}\,\mathrm{d}t\,\left\|\theta_0'\right\|_{L^\infty(\mathbb{R})}\\ &\quad\leqslant C(K)T_\epsilon^{\frac{1}{r'}}\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\,\epsilon^{\frac{2(M-1)}{r}-1}.\end{aligned}$$

Since $\nabla(c_I - c_0(\rho(.),.)) \in \mathcal{O}(1)$ in $L^\infty(\Gamma(2\delta;T_\epsilon))$, we immediately get

$$\int_0^{T_\epsilon}\left|\int_{\Gamma_t(2\delta)}\left(\tilde{\mathbf{w}}_2^\epsilon\cdot\nabla\Big(c_I - c_0\big(\rho(.),.\big)\Big)\right)\varphi\,\mathrm{d}x\right|\mathrm{d}t \leqslant C(K)T_\epsilon^{\frac{1}{r'}}\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\,\epsilon^{\frac{2(M-1)}{r}}$$

by similar arguments as in (3.52).

As $M \geqslant 4$ there always exists $r \in (1,2)$ (and with it $r' \in (2,\infty)$) such that $\epsilon^{\frac{2(M-1)}{r}-1} < \epsilon^M$ which concludes the proof. □

Theorem 3.12 (Error in the velocity) *Let* $\overline{\mathbf{v}^\epsilon}$ *be a strong solution to* (3.43)–(3.45)*, let the assumptions of Lemma* 3.10 *hold true and let* $\mathbf{v}_{\mathrm{err}}^\epsilon := \mathbf{v}^\epsilon - \left(\overline{\mathbf{v}^\epsilon} + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon\right)$.

1. *There is a constant* $C(K) > 0$ *such that*

$$\left\|\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right\|_{L^2(0,T_\epsilon;H^1(\Omega))} \leqslant C(K)\epsilon^M \quad \textit{for all } \epsilon \in (0,\epsilon_1).$$

2. *For every* $\beta \in \left(0,\frac{1}{2}\right)$ *there are constants* $C_1(\beta), C_2(\beta), C(K) > 0$ *such that*

$$\begin{aligned}\left\|\mathbf{v}_{\mathrm{err}}^\epsilon\right\|_{H^1(\Omega)} \leqslant C_1&\left(\left\|r_{\mathrm{CH2}}^\epsilon\nabla c_A^\epsilon\right\|_{(H_\sigma^1(\Omega))'} + \epsilon\,\|\nabla R\|_{L^2(\partial\Omega(\frac{\delta}{2}))}^{1-2\beta}\,\|\gamma\nabla R\|_{H^1(\partial\Omega(\frac{\delta}{2}))}^{1+2\beta}\right)\\ &+ C_2\epsilon^2\,\|\nabla R\|_{L^2(\partial\Omega(\frac{\delta}{2}))}^{\frac{1}{2}-\beta}\,\|\gamma\nabla R\|_{H^1(\partial\Omega(\frac{\delta}{2}))}^{\frac{1}{2}+\beta}\end{aligned} \tag{3.53}$$

for almost every $t \in (0,T_\epsilon)$ *and*

$$\left\|\mathbf{v}_{\mathrm{err}}^\epsilon\right\|_{L^1(0,T_\epsilon;H^1(\Omega))} \leqslant C(K)C\left(T_\epsilon,\epsilon\right)\epsilon^M \tag{3.54}$$

for all $\epsilon \in (0,\epsilon_1)$*, where* $C(T,\epsilon) \to 0$ *as* $(T,\epsilon) \to 0$.

Proof. Ad 1. By definition, $\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}$ satisfies

$$\begin{aligned} -\Delta(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}) + \nabla(p_A^\epsilon - \overline{p^\epsilon}) &= \mathbf{r}_S^\epsilon && \text{in } \Omega_{T_\epsilon}, \\ \operatorname{div}(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}) &= r_{\text{div}}^\epsilon && \text{in } \Omega_{T_\epsilon}, \\ \big(-2D_s(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}) + (p_A^\epsilon - \overline{p^\epsilon})\mathbf{I}\big)\mathbf{n}_{\partial\Omega} &= \alpha_0(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}) && \text{on } \partial_{T_\epsilon}\Omega. \end{aligned}$$

Thus, we have by Theorem 2.1 and since $r_{\text{div}}^\epsilon = 0$ on $\partial_{T_0}\Omega$

$$\|\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\|_{L^2(0,T_\epsilon;H^1(\Omega))} \leqslant C\left(\|\mathbf{r}_S^\epsilon\|_{L^2\left(0,T_\epsilon;(H^1(\Omega))'\right)} + \|r_{\text{div}}^\epsilon\|_{L^2(\Omega_{T_\epsilon})}\right)$$

and the claim follows from (3.8).

Ad 2. First of all we have for $\psi \in H_\sigma^1(\Omega)$

$$\int_\Omega 2D_s\left(\mathbf{v}^\epsilon - \overline{\mathbf{v}^\epsilon}\right) : D_s\psi \,\mathrm{d}x + \alpha_0 \int_{\partial\Omega} \left(\mathbf{v}^\epsilon - \overline{\mathbf{v}^\epsilon}\right)\cdot\psi \,\mathrm{d}\mathcal{H}^1(s) = \int_\Omega \left(\mu^\epsilon\nabla c^\epsilon - \mu_A^\epsilon\nabla c_A^\epsilon\right)\cdot\psi\,\mathrm{d}x. \tag{3.55}$$

Plugging in (1.4), (3.4) and using integration by parts we get

$$\begin{aligned} &\int_\Omega \left(\mu^\epsilon\nabla c^\epsilon - \mu_A^\epsilon\nabla c_A^\epsilon\right)\cdot\psi\,\mathrm{d}x \\ &= \epsilon\int_\Omega \left(\nabla c^\epsilon\otimes\nabla c^\epsilon - \nabla c_A^\epsilon\otimes\nabla c_A^\epsilon\right) : \nabla\psi\,\mathrm{d}x - \int_\Omega r_{\text{CH2}}^\epsilon\nabla c_A^\epsilon\cdot\psi\,\mathrm{d}x \\ &+ \epsilon\int_{\partial\Omega}\left(\left(\nabla c_A^\epsilon\otimes\nabla c_A^\epsilon - \nabla c^\epsilon\otimes\nabla c^\epsilon\right)\mathbf{n}_{\partial\Omega} + \frac{1}{2}\left(|\nabla c^\epsilon|^2 - \left|\nabla c_A^\epsilon\right|^2\right)\mathbf{n}_{\partial\Omega}\right)\cdot\psi\,\mathrm{d}\mathcal{H}^1(s). \end{aligned} \tag{3.56}$$

Here we used $c^\epsilon = c_A^\epsilon = -1$ on $\partial_T\Omega$ together with $f(-1) = 0$ and $\operatorname{div}(\nabla c\otimes\nabla c) = \Delta c\nabla c + \frac{1}{2}\nabla\left(|\nabla c|^2\right)$ for sufficiently smooth $c\colon\Omega\to\mathbb{R}$.

So, defining $\mathbf{v}_{\text{err}}^\epsilon$ as in (3.50) and taking into account (3.55), (3.56), and the definitions of $\tilde{\mathbf{w}}_1^\epsilon$ and $\tilde{\mathbf{w}}_2^\epsilon$ (cf. (3.14), (3.46)) as weak solutions we find that $\mathbf{v}_{\text{err}}^\epsilon$ solves

$$\begin{aligned} &\int_\Omega 2D_s\mathbf{v}_{\text{err}}^\epsilon : D_s\psi\,\mathrm{d}x + \alpha_0\int_{\partial\Omega}\mathbf{v}_{\text{err}}^\epsilon\cdot\psi\,\mathrm{d}\mathcal{H}^1(s) \\ &\quad= \epsilon\int_{\partial\Omega}\left(\left(\nabla c_A^\epsilon\otimes\nabla c_A^\epsilon - \nabla c^\epsilon\otimes\nabla c^\epsilon\right)\mathbf{n}_{\partial\Omega} + \frac{1}{2}\left(|\nabla c^\epsilon|^2 - \left|\nabla c_A^\epsilon\right|^2\right)\mathbf{n}_{\partial\Omega}\right)\cdot\psi\,\mathrm{d}\mathcal{H}^1(s) \\ &\qquad - \int_\Omega r_{\text{CH2}}^\epsilon\nabla c_A^\epsilon\cdot\psi\,\mathrm{d}x =: \mathcal{F}^\epsilon(\psi) \end{aligned} \tag{3.57}$$

for all $\psi \in H_\sigma^1(\Omega)$. Due to (3.9) we have

$$\begin{aligned} \int_0^{T_\epsilon}\left|\int_\Omega r_{\text{CH2}}^\epsilon\nabla c_A^\epsilon\cdot\psi\,\mathrm{d}x\right|\mathrm{d}t &\leqslant \int_0^{T_\epsilon}\left\|r_{\text{CH2}}^\epsilon\nabla c_A^\epsilon\right\|_{(H_\sigma^1(\Omega))'}\mathrm{d}t\,\|\psi\|_{H^1(\Omega)} \\ &\leqslant C(K)C\left(T_\epsilon,\epsilon\right)\epsilon^M, \end{aligned} \tag{3.58}$$

where $C(T,\epsilon) \to 0$ as $(T,\epsilon) \to 0$. Thus, we only need to estimate the appearing boundary terms in (3.57). To this end, let $\beta \in \left(0, \frac{1}{2}\right)$ and we compute

$$\begin{aligned}
&\epsilon \int_0^{T_\epsilon} \int_{\partial\Omega} \left| \left(|\nabla c^\epsilon|^2 - \left|\nabla c_A^\epsilon\right|^2 \right) \psi \right| \, \mathrm{d}\mathcal{H}^1(s)\, \mathrm{d}t \\
&\quad \leqslant \epsilon \int_0^{T_\epsilon} \int_{\partial\Omega} \left(|\nabla R|^2 + 2 |\nabla R| \left|\nabla c_A^\epsilon\right| \right) |\psi| \, \mathrm{d}\mathcal{H}^1(s)\, \mathrm{d}t \\
&\quad \leqslant C \int_0^{T_\epsilon} \left(\epsilon \|\gamma \nabla R\|^2_{H^{\frac{1}{2}+\beta}(\Omega)} + \epsilon^2 \|\gamma \nabla R\|_{H^{\frac{1}{2}+\beta}(\Omega)} \right) \|\psi\|_{H^1(\Omega)} \, \mathrm{d}t \\
&\quad \leqslant C_1 \int_0^{T_\epsilon} \left(\epsilon \|\nabla R\|^{1-2\beta}_{L^2(\partial\Omega(\frac{\delta}{2}))} \|\gamma \nabla R\|^{1+2\beta}_{H^1(\Omega)} \right) \|\psi\|_{H^1(\Omega)} \, \mathrm{d}t \\
&\qquad + C_2 \int_0^{T_\epsilon} \left(\epsilon^2 \|\nabla R\|^{\frac{1}{2}-\beta}_{L^2(\partial\Omega(\frac{\delta}{2}))} \|\gamma \nabla R\|^{\frac{1}{2}+\beta}_{H^1(\Omega)} \right) \|\psi\|_{H^1(\Omega)} \, \mathrm{d}t \qquad (3.59) \\
&\quad \leqslant C_1 \left(\epsilon \|\nabla R\|^{1-2\beta}_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))} \|\gamma \nabla R\|^{1+2\beta}_{L^2(0,T_\epsilon;H^1(\Omega))} \right) \|\psi\|_{H^1(\Omega)} \\
&\qquad + C_2 T_\epsilon^{\frac{1}{2}} \left(\epsilon^2 \|\nabla R\|^{\frac{1}{2}-\beta}_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))} \|\gamma \nabla R\|^{\frac{1}{2}+\beta}_{L^2(0,T_\epsilon;H^1(\Omega))} \right) \|\psi\|_{H^1(\Omega)}, \qquad (3.60)
\end{aligned}$$

where we used in the second inequality that $\nabla c_A^\epsilon = \mathcal{O}(\epsilon)$ in $L^\infty\left(\overline{\partial_{T_0}\Omega\left(\frac{\delta}{2}\right)}\right)$ and that $H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^s(\partial\Omega)$ for all $s \in [1,\infty)$. and $H^\beta(\partial\Omega) \hookrightarrow L^{2+\beta}(\partial\Omega)$, since $\beta - \frac{1}{2} \geqslant -\frac{1}{2+\beta}$. Now we may estimate

$$\|\gamma \nabla R\|_{H^1(\Omega)} \leqslant C \, \|(\gamma \Delta R, |\nabla R|, R)\|_{L^2(\partial\Omega(\frac{\delta}{2}))} \qquad (3.61)$$

due to elliptic regularity theory and the definition of γ. Using this in (3.60) together with (1.27a) and (1.27d), we find

$$\begin{aligned}
\epsilon \int_0^{T_\epsilon} \int_{\partial\Omega} \left| \left(|\nabla c^\epsilon|^2 - \left|\nabla c_A^\epsilon\right|^2 \right) \psi \right| \, \mathrm{d}\mathcal{H}^1(s)\, \mathrm{d}t &\leqslant \|\psi\|_{H^1(\Omega)} C(K) \left(\epsilon^{2M-\frac{1}{2}-\beta} + T_\epsilon^{\frac{1}{2}} \epsilon^{M+\frac{5}{4}-\frac{1}{2}\beta} \right) \\
&\leqslant \|\psi\|_{H^1(\Omega)} C(K) \left(\epsilon^{\frac{1}{2}} + T_\epsilon^{\frac{1}{2}} \right) \epsilon^M
\end{aligned}$$

as $M \geqslant 4$ and $\beta > 0$ can be chosen sufficiently small.

For the remaining, not estimated term in (3.57), we note that

$$\begin{aligned}
\epsilon \int_0^{T_\epsilon} \int_{\partial\Omega} &\left| \left(-\nabla c^\epsilon \otimes \nabla c^\epsilon + \nabla c_A^\epsilon \otimes \nabla c_A^\epsilon \right) \mathbf{n}_{\partial\Omega} \cdot \psi \right| \, \mathrm{d}\mathcal{H}^1(s)\, \mathrm{d}t \\
&\leqslant \int_0^{T_\epsilon} \int_{\partial\Omega} \left(|\nabla R|^2 + 2 |\nabla R| \left|\nabla c_A^\epsilon\right| \right) |\psi| \, \mathrm{d}\mathcal{H}^1(s)\, \mathrm{d}t
\end{aligned}$$

and may then proceed as in (3.60). This proves (3.54) and also (3.53) if we use (3.58) and (3.59) without the integration in time. ◻

Corollary 3.13 *Let the assumptions of Theorem 3.12 hold true and let* $\varphi \in L^\infty(0, T_\epsilon; H^1(\Omega))$. *Then*

$$\int_0^{T_\epsilon} \left| \int_\Omega \left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^{\epsilon}} \right) \cdot \nabla c_A^\epsilon \varphi \, \mathrm{d}x \right| \mathrm{d}t \leqslant C(K) T_\epsilon^{\frac{1}{2}} \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}, \tag{3.62}$$

$$\int_0^{T_\epsilon} \left| \int_\Omega \mathbf{v}_{\mathrm{err}}^\epsilon \cdot \nabla c_A^\epsilon \varphi \, \mathrm{d}x \right| \mathrm{d}t \leqslant C(K) C(\epsilon, T_\epsilon) \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}, \tag{3.63}$$

$$\int_0^{T_\epsilon} \left| \int_\Omega R \mathbf{v}_{\mathrm{err}}^\epsilon \cdot \nabla R \gamma^2 \, \mathrm{d}x \right| \mathrm{d}t \leqslant C(K) C(\epsilon, T_\epsilon) \epsilon^{2M-1}, \tag{3.64}$$

$$\int_0^{T_\epsilon} \left| \int_\Omega \mathbf{v}_{\mathrm{err}}^\epsilon \cdot \nabla R \varphi \, \mathrm{d}x \right| \mathrm{d}t \leqslant C(K) C(\epsilon, T_\epsilon) \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \tag{3.65}$$

for all $\epsilon \in (0, \epsilon_1)$ *and* $C(\epsilon, T) \to 0$ *if* $(\epsilon, T) \to 0$.

Proof. Ad (3.62): We have $\nabla c_A^\epsilon \in \mathcal{O}(\epsilon)$ in $L^\infty(\Omega_{T_\epsilon} \backslash \Gamma(2\delta; T_\epsilon))$ and thus get the estimate in $\Omega_{T_\epsilon} \backslash \Gamma(2\delta; T_\epsilon)$ by simply using Hölder's inequality and Theorem 3.12.1. It remains to give an estimate inside $\Gamma(2\delta; T_\epsilon)$: We have $\nabla c_{O,\mathbf{B}} \in \mathcal{O}(\epsilon)$ in L^∞ and the term involving $(c_I - c_{O,\mathbf{B}})$ in ∇c_A^ϵ can be handled by using (3.36), Hölder's inequality and Theorem 3.12.1) as before. Moreover, we estimate

$$\begin{aligned} &\int_{\Gamma(2\delta;T_\epsilon)} \left| \left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon} \right) \xi \nabla (\theta_0 \circ \rho) \varphi \right| \mathrm{d}(x,t) \\ &\quad \leqslant C \int_0^{T_\epsilon} \int_{\mathbb{T}^1} \left\| \left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon} \right) \varphi \right\|_{L^\infty(-2\delta, 2\delta)} \int_{\mathbb{R}} \left| \theta_0' \left(\mathbf{n} + \nabla^\Gamma h_A^\epsilon \right) \right| \mathrm{d}\rho \, \mathrm{d}s \, \mathrm{d}t \\ &\quad \leqslant C(K) T_\epsilon^{\frac{1}{2}} \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}, \end{aligned}$$

where we used $H^1(\Gamma_t(2\delta)) \hookrightarrow L^{2,\infty}(\Gamma_t(2\delta))$ together with Theorem 3.12.1) in the last step. For $k \geqslant 1$ we can use $\epsilon^k \nabla (c_k(\rho(.), .)) \in L^\infty(\Gamma(2\delta; T_\epsilon))$ uniformly in ϵ. This proves (3.62).

Furthermore, (3.63) follows in the same way by using (3.54) and noting that we may not generate a term $T_\epsilon^{\frac{1}{2}}$ as we only control $\left\| \mathbf{v}_{\mathrm{err}}^\epsilon \right\|_{L^1(0,T_\epsilon;H^1(\Omega))}$.

Ad (3.64): Since $H^1(\Omega) \hookrightarrow L^s(\Omega)$ for all $s \in [1, \infty)$, we have

$$\int_0^{T_\epsilon} \left| \int_\Omega R \mathbf{v}_{\mathrm{err}}^\epsilon \cdot \nabla R \gamma^2 \, \mathrm{d}x \right| \mathrm{d}t \leqslant C(K) \int_0^{T_\epsilon} \left\| \mathbf{v}_{\mathrm{err}}^\epsilon \right\|_{H^1(\Omega)} \|\gamma R\|_{L^{2+\kappa}(\Omega)} \|\gamma \nabla R\|_{L^2(\Omega)} \, \mathrm{d}t \tag{3.66}$$

for $\kappa > 0$. Regarding (3.53), we need to show three estimates:

Firstly, we have

$$\begin{aligned} &\int_0^{T_\epsilon} \left\| r_{\mathrm{CH2}}^\epsilon \nabla c_A^\epsilon \right\|_{(H^1(\Omega))'} \|\gamma R\|_{L^{2+\kappa}(\Omega)} \|\gamma \nabla R\|_{L^2(\Omega)} \, \mathrm{d}t \\ &\quad \leqslant \left\| r_{\mathrm{CH2}}^\epsilon \nabla c_A^\epsilon \right\|_{L^2\left(0,T_\epsilon;(H^1_\sigma(\Omega))'\right)} \|\gamma R\|_{L^\infty(0,T_\epsilon;L^{2+\kappa}(\Omega))} \|\gamma \nabla R\|_{L^2(\Omega_{T_\epsilon})} \\ &\quad \leqslant C(K) C(T_\epsilon, \epsilon) \epsilon^{2M} \left(\epsilon^{M - \frac{1}{2} - \frac{\kappa}{2+\kappa} M} \right) \leqslant C(K) C(T_\epsilon, \epsilon) \epsilon^{2M-1}, \end{aligned} \tag{3.67}$$

where we used (3.9), (1.27d) and Lemma 3.8 3) and the fact that $M \geqslant 4$ and $\kappa > 0$ can be chosen arbitrarily.

Secondly, we estimate for $\beta \in \left(0, \frac{1}{2}\right)$

$$\int_0^{T_\epsilon} \epsilon \|\nabla R\|_{L^2(\partial\Omega(\frac{\delta}{2}))}^{1-2\beta} \|\gamma\nabla R\|_{H^1(\Omega)}^{1+2\beta} \|\gamma R\|_{L^{2+\kappa}(\Omega)} \|\gamma\nabla R\|_{L^2(\Omega)} \, dt$$
$$\leqslant C\epsilon \|\nabla R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))}^{1-2\beta} \|\gamma\nabla R\|_{L^2(0,T_\epsilon;H^1(\Omega))}^{1+2\beta} \|\gamma R\|_{L^\infty(0,T_\epsilon;L^{2+\kappa}(\Omega))} \|\nabla R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))}$$
$$\leqslant C(K)\left(\epsilon^{2M-\frac{1}{2}-\beta}\epsilon^{M-\frac{1}{2}-\frac{\kappa}{2+\kappa}M}\epsilon^{-\frac{1}{2}}\right) \leqslant C(K)\epsilon^{2M-\frac{1}{2}}, \tag{3.68}$$

where we used (3.61), (1.27a), (1.27d), Lemma 3.8 3) and (3.38), $M \geqslant 4$, and that $\beta > 0, \kappa > 0$ can be chosen arbitrarily small.

Similarly we obtain

$$\int_0^{T_\epsilon} \epsilon^2 \|\nabla R\|_{L^2(\partial\Omega(\frac{\delta}{2}))}^{\frac{1}{2}-\beta} \|\gamma\nabla R\|_{H^1(\Omega)}^{\frac{1}{2}+\beta} \|\gamma R\|_{L^{2+\kappa}(\Omega)} \|\gamma\nabla R\|_{L^2(\Omega)} \, dt$$
$$\leqslant C\epsilon^2 \|\nabla R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))}^{\frac{1}{2}-\beta} \|\gamma\nabla R\|_{L^2(0,T_\epsilon;H^1(\Omega))}^{\frac{1}{2}+\beta} \|\gamma R\|_{L^\infty(0,T_\epsilon;L^{2+\kappa}(\Omega))} \|\gamma\nabla R\|_{L^2(0,T_\epsilon;L^2(\Omega))}$$
$$\leqslant C(K)\left(\epsilon^{M+\frac{5}{4}-\frac{\beta}{2}}\epsilon^{M-\frac{1}{2}-\frac{\kappa}{2+\kappa}M}\epsilon^{M-\frac{1}{2}}\right) \leqslant C(K)\epsilon^{2M-\frac{1}{2}}. \tag{3.69}$$

Now (3.66)–(3.69) together with (3.53) yield (3.64).

Concerning (3.65) we note that

$$\left|\int_\Omega \mathbf{v}_{\mathrm{err}}^\epsilon \cdot \nabla R\varphi \, dx\right| \leqslant \left\|\mathbf{v}_{\mathrm{err}}^\epsilon\right\|_{H^1(\Omega)} \|\nabla R\|_{L^2(\Omega)} \|\varphi\|_{L^4(\Omega)} .$$

Regarding (3.53), we again consider three different terms: Firstly,

$$\int_0^{T_\epsilon} \left\|r_{\mathrm{CH2}}^\epsilon \nabla c_A^\epsilon\right\|_{(H^1(\Omega))'} \|\nabla R\|_{L^2(\Omega)} \|\varphi\|_{L^4(\Omega)} \, dt$$
$$\leqslant \left\|r_{\mathrm{CH2}}^\epsilon \nabla c_A^\epsilon\right\|_{L^2(0,T_\epsilon;H^1(\Omega)')} \|\nabla R\|_{L^2(\Omega_{T_\epsilon})} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$

where we may now use (3.9) and (1.27) and $M \geqslant 4$ to gain the estimate by the right-hand side of (3.65). Secondly,

$$\int_0^{T_\epsilon} \epsilon \|\nabla R\|_{L^2(\partial\Omega(\frac{\delta}{2}))}^{1-2\beta} \|\gamma\nabla R\|_{H^1(\Omega)}^{1+2\beta} \|\nabla R\|_{L^2(\Omega)} \|\varphi\|_{L^4(\Omega)} \, dt$$
$$\leqslant C\epsilon \|\nabla R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))}^{1-2\beta} \|\gamma\nabla R\|_{L^2(0,T_\epsilon;H^1(\Omega))}^{1+2\beta} \|\nabla R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$

for $\beta \in \left(0, \frac{1}{2}\right)$, where (1.27) and (3.38) together with $M \geqslant 4$ imply the desired estimate. Thirdly,

$$\int_0^{T_\epsilon} \epsilon^2 \|\nabla R\|_{L^2(\partial\Omega(\frac{\delta}{2}))}^{\frac{1}{2}-\beta} \|\gamma\nabla R\|_{H^1(\Omega)}^{\frac{1}{2}+\beta} \|\nabla R\|_{L^2(\Omega)} \|\varphi\|_{L^4(\Omega)} \, dt$$
$$\leqslant C\epsilon^2 \|\nabla R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))}^{\frac{1}{2}-\beta} \|\gamma\nabla R\|_{L^2(0,T_\epsilon;H^1(\Omega))}^{\frac{1}{2}+\beta} \|\nabla R\|_{L^2(\Omega_{T_\epsilon})} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$

for $\beta \in \left(0, \frac{1}{2}\right)$, where finally (1.27) and $M \geqslant 4$ imply the claim. □

Lemma 3.14 *Let $\varphi \in L^\infty\big(0, T_\epsilon; H^1(\Omega)\big)$ and $(\tilde{\mathbf{w}}_1^\epsilon)^\Gamma = \tilde{\mathbf{w}}_1^\epsilon - \tilde{\mathbf{w}}_1^\epsilon\big|_\Gamma$. Then*

$$\int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} \frac{1}{\epsilon} \left(\tilde{\mathbf{w}}_1^\epsilon\right)^\Gamma \cdot \mathbf{n}\theta_0'(\rho)\varphi \,\mathrm{d}x \right| \mathrm{d}t \leqslant C(K)(T_\epsilon)^{\frac{1}{2}}\epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}, \quad (3.70)$$

$$\int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} \left(\tilde{\mathbf{w}}_1^\epsilon\right)^\Gamma \cdot \nabla^\Gamma h_A^\epsilon \theta_0'(\rho)\varphi \,\mathrm{d}x \right| \mathrm{d}t \leqslant C(K)(T_\epsilon)^{\frac{1}{2}}\epsilon^{M+1} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}, \quad (3.71)$$

$$\int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} \left(\tilde{\mathbf{w}}_1^\epsilon\right)^\Gamma \cdot \left(\frac{\mathbf{n}}{\epsilon} - \nabla^\Gamma h_A^\epsilon\right) \epsilon \partial_\rho c_1 \varphi \,\mathrm{d}x \right| \mathrm{d}t \leqslant C(K)(T_\epsilon)^{\frac{1}{2}}\epsilon^{M+1} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \quad (3.72)$$

for all $\epsilon \in (0, \epsilon_1)$.

Proof. Proceeding as in [4, proof of Lemma 5.1] we find, using $\partial_{\mathbf{n}}\tilde{\mathbf{w}}_1^\epsilon = -\operatorname{div}_{\boldsymbol{\tau}} \tilde{\mathbf{w}}_1^\epsilon$,

$$\begin{aligned}
&\int_{\Gamma_t(\delta)} \frac{1}{\epsilon} \left(\tilde{\mathbf{w}}_1^\epsilon - \tilde{\mathbf{w}}_1^\epsilon|_\Gamma\right) \cdot \mathbf{n}\theta_0'(\rho)\varphi \,\mathrm{d}x \\
&= \int_{-\delta}^{\delta} \int_0^r \int_{\Gamma_t} \frac{1}{\epsilon} \tilde{\mathbf{w}}_1^\epsilon(\sigma, p, t) \cdot \nabla_{\boldsymbol{\tau}}\big(\rho(r, p, t)\big)\theta_0''\big(\rho(r, p, t)\big)\varphi(r, p, t)\, J(r, p, t)\, \mathrm{d}\mathcal{H}^1(p)\,\mathrm{d}\sigma\,\mathrm{d}r \\
&\quad + \int_{-\delta}^{\delta} \int_0^r \int_{\Gamma_t} \frac{1}{\epsilon} \tilde{\mathbf{w}}_1^\epsilon(\sigma, p, t) \cdot \nabla_{\boldsymbol{\tau}}\big(\varphi(r, p, t)\, J(r, p, t)\big)\theta_0'\big(\rho(r, p, t)\big)\,\mathrm{d}\mathcal{H}^1(p)\,\mathrm{d}\sigma\,\mathrm{d}r \\
&\quad + \int_{-\delta}^{\delta} \int_0^r \int_{\Gamma_t} \frac{1}{\epsilon} \tilde{\mathbf{w}}_1^\epsilon(\sigma, p, t) \cdot \mathbf{n}_{\Gamma_t}(p)\kappa(p)\theta_0'\big(\rho(r, p, t)\big)\varphi(r, p, t)\, J(r, p, t)\,\mathrm{d}\mathcal{H}^1(p)\,\mathrm{d}\sigma\,\mathrm{d}r \\
&=: I_1 + I_2 + I_3,
\end{aligned}$$

because of Lemma 2.8. To estimate the occurring integrals, we note that

$$\left| \int_0^r \tilde{\mathbf{w}}_1^\epsilon(\sigma, p, t)\,\mathrm{d}\sigma \right| \leqslant r\, \|\tilde{\mathbf{w}}_1^\epsilon(., p, t)\|_{L^\infty(-\delta,\delta)} \leqslant C r\, \|\tilde{\mathbf{w}}_1^\epsilon(., p, t)\|_{H^1(-\delta,\delta)} \quad (3.73)$$

holds for all $p \in \Gamma_t$ and $r \in (-\delta, \delta)$. After a change of variables, we get

$$\begin{aligned}
|I_2| &\leqslant C\epsilon \int_{\Gamma_t} \|\tilde{\mathbf{w}}_1^\epsilon(., p, t)\|_{H^1(-\delta,\delta)} \\
&\qquad \int_{-\frac{\delta}{\epsilon}-h_A^\epsilon}^{\frac{\delta}{\epsilon}-h_A^\epsilon} \left| \Big(\nabla_{\boldsymbol{\tau}}(\varphi J)\big(\epsilon(\rho + h_A^\epsilon), p, t\big)\Big)(\rho + h_A^\epsilon)\theta_0'(\rho) \right| \mathrm{d}\rho\,\mathrm{d}\mathcal{H}^1(p) \\
&\leqslant C(K)\epsilon^{\frac{1}{2}} \|\tilde{\mathbf{w}}_1^\epsilon(., t)\|_{L^2(\Gamma_t;H^1(-\delta,\delta))} \left\|(\rho + 1)\,\theta_0'\right\|_{L^2(\mathbb{R})} \left(\|\varphi\|_{H^1(\Omega)} + \epsilon^{\frac{1}{2}} \|\varphi\|_{L^{2,\infty}(\Gamma_t(\delta))}\right)
\end{aligned}$$

where we used (3.73), $\left\|h_A^\epsilon\right\|_{C^0([0,T];C^1(\mathbb{T}^1))} \leqslant C(K)$ as in (3.35). Employing Lemma 2.9 and the exponential decay of θ_0', we find

$$\int_0^{T_\epsilon} |I_2|\,\mathrm{d}t \leqslant C(K)\,(T_\epsilon)^{\frac{1}{2}}\,\epsilon^{\frac{1}{2}}\, \|\tilde{\mathbf{w}}_1^\epsilon\|_{L^2(0,T_\epsilon;H^1(\Omega))} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\,.$$

As $\nabla_{\boldsymbol{\tau}} (\rho (r, p, t)) = \nabla_{\boldsymbol{\tau}} \left(h_A^\epsilon (S (p, t), t)\right)$ we may estimate I_1 in a similar manner, and $|\kappa(p)| \leqslant C$ for all $p \in \Gamma_t$, implies the equivalent estimate for I_3. Lemma 3.4 together with the estimates on I_1, I_2 and I_3 completes the proof for (3.70).

To show (3.71), we calculate

$$\begin{aligned}
&\left| \int_{\Gamma_t(\delta)} ((\tilde{\mathbf{w}}_1^\epsilon - \tilde{\mathbf{w}}_1^\epsilon|_{\Gamma_t})) \cdot \nabla^\Gamma h_A^\epsilon\big(S(x,t),t\big)\theta_0'(\rho)\varphi \,\mathrm{d}x \right| \\
&\quad \leqslant C \int_{\mathbb{T}^1} \int_{-\delta}^{\delta} \int_0^r \left|(\partial_{\mathbf{n}} \tilde{\mathbf{w}}_1^\epsilon)\big(X(\sigma,s,t)\big)\right| \mathrm{d}\sigma \left|\nabla^\Gamma h_A^\epsilon(s,t)\theta_0'\Big(\rho\big(X(r,s,t)\big)\Big)\varphi\right| \mathrm{d}r\,\mathrm{d}s. \\
&\quad \leqslant C(K) \int_{\mathbb{T}^1} \left\|\tilde{\mathbf{w}}_1^\epsilon\big(X(.,s,t)\big)\right\|_{H^1(-\delta,\delta)} \|\varphi\|_{L^\infty(-\delta,\delta)} \int_{-\frac{\delta}{\epsilon}-h_A^\epsilon}^{\frac{\delta}{\epsilon}-h_A^\epsilon} \epsilon^{\frac{3}{2}} |\rho+1| \left|\theta_0'(\rho)\right| \mathrm{d}\rho\,\mathrm{d}s
\end{aligned}$$

since

$$\left| \int_0^r \left(\partial_{\mathbf{n}} \tilde{\mathbf{w}}_1^\epsilon\right) \left(X(r,s,t)\right) \mathrm{d}\sigma \right| \leqslant \left\|\tilde{\mathbf{w}}_1^\epsilon\big(X(.,s,t)\big)\right\|_{H^1(-\delta,\delta)} \sqrt{|r|} \quad \forall r \in (-\delta,\delta)$$

and $s \in \mathbb{T}^1$, $t \in [0, T_\epsilon]$. Integration from 0 to T_ϵ and Lemma 3.4 yield the assertion. The proof of (3.72) follows analogously to the proof of (3.71) since $\partial_\rho c_1 \in \mathcal{R}_\alpha$. □

Lemma 3.15 *Let $\varphi \in L^\infty\left(0, T_\epsilon; H^1(\Omega)\right)$ and $\mathbf{w}_1^\epsilon = \frac{\tilde{\mathbf{w}}_1^\epsilon}{\epsilon^{M-\frac{1}{2}}}$. Then it holds*

$$\int_0^{T_\epsilon} \left| \int_\Omega \epsilon^{M-\frac{1}{2}} \left(\mathbf{w}_1^\epsilon - \mathbf{w}_1^\epsilon|_\Gamma \,\xi\right) \cdot \nabla c_A^\epsilon \varphi \,\mathrm{d}x \right| \mathrm{d}t \leqslant C(K) C\left(T_\epsilon, \epsilon\right) \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))},$$

for all $\epsilon \in (0, \epsilon_1)$, where $C(T,\epsilon) \to 0$ as $(T,\epsilon) \to 0$.

Proof. In $\Omega_{T_0} \backslash \Gamma(2\delta)$ we have $\nabla c_A^\epsilon \in \mathcal{O}(\epsilon)$ in L^∞, thus the estimate in this region is a direct consequence of Lemma 3.4. Inside $\Gamma(2\delta; T_\epsilon)$ we have $\nabla c_A^\epsilon = \xi \nabla c_I + \xi' \mathbf{n} (c_I - c_{O,\mathbf{B}}) + (1-\xi) \nabla c_{O,\mathbf{B}}$. The term involving $\nabla c_{O,\mathbf{B}}$ can be treated as in the outer region and the estimate for the term $(c_I - c_{O,\mathbf{B}})$ is a consequence of (3.36). Now by definition

$$\nabla c_I(x,t) = \sum_{i=0}^{M+1} \epsilon^i \left(\partial_\rho c_i\big(\rho(x,t),x,t\big) \left(\frac{\mathbf{n}\big(S(x,t),t\big)}{\epsilon} - \nabla^\Gamma h_A^\epsilon(x,t) \right) + \nabla_x c_i\big(\rho(x,t),x,t\big) \right)$$

for $(x,t) \in \Gamma(2\delta; T_\epsilon)$. Since $\nabla_x c_0 \equiv 0$, we have $\sum_{i=0}^{M+1} \epsilon^i \nabla_x c_i \in \mathcal{O}(\epsilon)$ in $L^\infty(\mathbb{R} \times \Gamma(2\delta))$, allowing for a suitable estimate with the help of Lemma 3.4. Choosing $\epsilon > 0$ small enough, we have $\left|\frac{d_\Gamma}{\epsilon} - h_A^\epsilon\right| \geqslant \frac{\delta}{2\epsilon}$ in $\Gamma(2\delta; T_\epsilon) \backslash \Gamma(\delta; T_\epsilon)$ and as $\partial_\rho c_i \in \mathcal{R}_\alpha$, this leads to

$$\begin{aligned}
&\int_{\Gamma(2\delta;T_\epsilon)\backslash\Gamma(\delta;T_\epsilon)} \epsilon^{M-\frac{1}{2}} \left|\mathbf{w}_1^\epsilon - \mathbf{w}_1^\epsilon|_\Gamma \,\xi\right| \left|\partial_\rho c_i\big(\rho(x,t),x,t\big) \left(\frac{\mathbf{n}}{\epsilon} - \nabla^\Gamma h_A^\epsilon\right)\right| |\varphi| \,\mathrm{d}(x,t) \\
&\qquad \leqslant C(K) \|\tilde{\mathbf{w}}_1^\epsilon\|_{L^2(0,T_\epsilon;H^1(\Omega))} \|\varphi\|_{L^2(0,T_\epsilon;H^1(\Omega))} \frac{1}{\epsilon} C_1 e^{-C_2 \frac{\delta}{2\epsilon}}
\end{aligned}$$

for all $i \in \{0,\dots,M+1\}$, where we have used $\|h^\epsilon_{M-\frac{1}{2}}\|_{C^0(0,T_\epsilon;C^1(\mathbb{T}^1))} \leqslant C(K)$ due to (3.35). So we only need to show

$$\int_{\Gamma(\delta;T_\epsilon)} \epsilon^{M-\frac{1}{2}} \left|\left(\mathbf{w}^\epsilon_1 - \mathbf{w}^\epsilon_1|_\Gamma\right)\cdot\left(\epsilon^i\partial_\rho c_i\left(\rho(x,t),x,t\right)\left(\frac{\mathbf{n}}{\epsilon} - \nabla^\Gamma h^\epsilon_A\right)\right)\varphi\right| \mathrm{d}(x,t) \leqslant C\,(T,\epsilon)\,C(K)\epsilon^M \,\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$

for $i \in \{0,\dots,M+1\}$, where $C\,(T,\epsilon) \to 0$ as $(T,\epsilon) \to 0$. For $i \in \{0,1\}$ this is a consequence of Lemma 3.14 and for $i \geqslant 2$ this is a consequence of $\partial_\rho c_i \in L^\infty\,(\mathbb{R}\times\Gamma(2\delta))$. This shows the claim. □

3.3 *The proof of the main result*

Let the assumptions of Theorem 1.1 hold true. Moreover, let $c^\epsilon_A, \mu^\epsilon_A, \mathbf{v}^\epsilon_A, p^\epsilon_A, h^\epsilon_A$ be given as in [5, Definition 4.1], which implies in particular that the properties discussed in Section 3.1 hold. Let $\tilde{\mathbf{w}}^\epsilon_1$ and $\tilde{\mathbf{w}}^\epsilon_2$ be weak solutions to (3.14)–(3.16) and (3.46)–(3.48), resp., and let $\overline{\mathbf{v}^\epsilon}$ be a strong solution to (3.43)–(3.45). We denote $\mathbf{w}^\epsilon_1 = \frac{\tilde{\mathbf{w}}^\epsilon_1}{\epsilon^{M-\frac{1}{2}}}$. Additionally, let $(\mathbf{v}^\epsilon, p^\epsilon, c^\epsilon, \mu^\epsilon)$ be smooth solutions to (1.1)–(1.6) such that (1.22) is satisfied. Note that Proposition 3.7 implies that Lemma 1.3 is applicable in this situation. We define $R := c^\epsilon - c^\epsilon_A$ in Ω_{T_0} and let $\varphi\,(.,t) \in H^2(\Omega)\cap H^1_0(\Omega)$ for $t \in [0,T_0]$ be the unique solution of the problem

$$\begin{aligned} -\Delta\varphi\,(.,t) &= R\,(.,t) && \text{in } \Omega,\\ \varphi\,(.,t) &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Then φ is smooth and we have $\|\varphi\,(.,0)\|_{H^1(\Omega)} \leqslant C\|R\,(.,0)\,\|_{L^2(\Omega)} \leqslant C_{\psi_0}\epsilon^M$ for all $\epsilon \in (0,1)$. This implies the existence of some family $(\tau_\epsilon)_{\epsilon\in(0,1)} \subset (0,T_0]$ and $K \geqslant 1$ such that Assumption 1.2 is satisfied (and in particular (1.27) holds for τ_ϵ) and such that

$$\|\varphi\,(.,0)\|_{H^1(\Omega)} \leqslant \|R\,(.,0)\|_{L^2(\Omega)} \leqslant \frac{K}{2}\epsilon^M. \tag{3.74}$$

Moreover, we may choose $\epsilon_0 \in (0,1)$ small enough, such that (3.6)–(3.10), Lemma 3.5, (3.28) and (3.36) hold. This implies in particular that Assumption 3.6 is satisfied and that we may use all the results shown in Section 3.2. Now let $T \in (0,T_0]$ and for $\epsilon \in (0,\epsilon_0)$ we set

$$T_\epsilon := \sup\big\{t \in (0,T] \mid (1.27) \text{ holds true for } t\big\}. \tag{3.75}$$

We will show in the following that we may choose $T \in (0,T_0]$ (independent of ϵ) and ϵ_0 small enough, such that $T_\epsilon = T$ for all $\epsilon \in (0,\epsilon_0)$.

Now let $T' \in (0,T_0]$ be fixed. Multiplying the difference of the differential equations (1.3) and (3.3) by φ and integrating the result over Ω yields

$$0 = \int_\Omega \varphi\partial_t\,(-\Delta\varphi) + \varphi\left((\mathbf{v}^\epsilon\cdot\nabla R) - \left(\mathbf{v}^\epsilon_A - \overline{\mathbf{v}^\epsilon}\right)\cdot\nabla c^\epsilon_A + \left(\tilde{\mathbf{w}}^\epsilon_1 - \tilde{\mathbf{w}}^\epsilon_1|_\Gamma\xi\,(d_\Gamma)\right)\cdot\nabla c^\epsilon_A\right)\mathrm{d}x + \int_\Omega \varphi\left(\mathbf{v}^\epsilon_{\mathrm{err}}\cdot\nabla c^\epsilon_A + \tilde{\mathbf{w}}^\epsilon_2\cdot\nabla c^\epsilon_A - \Delta(\mu^\epsilon - \mu^\epsilon_A)\right) + \varphi r^\epsilon_{\mathrm{CH1}}\,\mathrm{d}x \tag{3.76}$$

for all $t \in (0, T)$. Here we used the definition of φ and the identity

$$\mathbf{v}^\epsilon \cdot \nabla c^\epsilon - \mathbf{v}_A^\epsilon \cdot \nabla c_A^\epsilon = \mathbf{v}^\epsilon \cdot \nabla R + \left(\tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon\right) \cdot \nabla c_A^\epsilon - \left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right) \cdot \nabla c_A^\epsilon + \mathbf{v}_{\text{err}}^\epsilon \cdot \nabla c_A^\epsilon, \tag{3.77}$$

which is a consequence of the definition of $\mathbf{v}_{\text{err}}^\epsilon$ (cf. (3.50)). In order to shorten the notation, we now write

$$\begin{aligned} \mathcal{E}(R, T') &:= \int_{\Omega_{T'}} \epsilon \,|\nabla R|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right) R^2 \,\mathrm{d}(x,t), \\ \mathcal{N}(c_A^\epsilon, R) &:= f'\left(c_A^\epsilon + R\right) - f'\left(c_A^\epsilon\right) - f''\left(c_A^\epsilon\right) R = f'''(c_A)\frac{R^2}{2} + f^{(4)}(c_A)\frac{R^3}{6}, \\ \mathcal{R}^\epsilon &:= \left(\epsilon^{M-\frac{1}{2}}\left(-\mathbf{w}_1^\epsilon + \mathbf{w}_1^\epsilon|_\Gamma \xi\left(d_\Gamma\right)\right) \cdot \nabla c_A^\epsilon\right) \end{aligned} \tag{3.78}$$

which leads us to

$$\begin{aligned} 0 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x + \mathcal{E}(R, T') + \int_\Omega \varphi\left(\mathbf{v}^\epsilon \cdot \nabla R\right) + \epsilon^{-1}\mathcal{N}(c_A^\epsilon, R) R \,\mathrm{d}x \\ - \int_\Omega \varphi\Big(\left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^{\mathrm{ffl}}}\right) \cdot \nabla c_A^\epsilon - \tilde{\mathbf{w}}_2^\epsilon \cdot \nabla c_A^\epsilon - \mathbf{v}_{\text{err}}^\epsilon \cdot \nabla c_A^\epsilon - r_{\text{CH1}}^\epsilon + \mathcal{R}^\epsilon\Big) + R r_{\text{CH2}}^\epsilon \,\mathrm{d}x \end{aligned} \tag{3.79}$$

for all $t \in (0, T')$ because of (1.4) and (3.4). We obtained this equality by using integration by parts in (3.76) and noting that the boundary integrals vanish due to the Dirichlet boundary conditions satisfied by φ, μ_A^ϵ and μ^ϵ.

Using Theorem 2.13, we obtain

$$\begin{aligned} &\int_\Omega \epsilon \,|\nabla R|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right) R^2 \,\mathrm{d}x \\ &\quad \geqslant C_1\left(\epsilon \,\|R\|_{L^2(\Omega)}^2 + \epsilon^{-1}\,\|R\|_{L^2(\Omega\setminus\Gamma_t(\delta))} + \epsilon\left\|\nabla^\Gamma R\right\|_{L^2(\Gamma_t(\delta))}^2\right) \\ &\qquad + C_2\left(\epsilon^3\,\|\nabla R\|_{L^2(\Omega)}^2 + \epsilon\,\|\nabla R\|_{L^2(\Omega\setminus\Gamma_t(\delta))}^2\right) - C_3\,\|\nabla\varphi\|_{L^2(\Omega)}^2 \end{aligned} \tag{3.80}$$

and due to the assumptions on f, [9, Lemma 2.2] yields

$$\frac{1}{\epsilon}\int_\Omega \mathcal{N}\left(c_A^\epsilon, R\right) R \,\mathrm{d}x \geqslant -\frac{C}{\epsilon}\int_\Omega |R|^3 \,\mathrm{d}x.$$

Plugging these observations into (3.79) enables us to get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x \\ &\quad + C_1\left(\left\|\left(\epsilon R, \epsilon^3\nabla R\right)\right\|_{L^2(\Omega)}^2 + \left\|\left(\epsilon^{-1} R, \epsilon\nabla R\right)\right\|_{L^2(\Omega\setminus\Gamma_t(\delta))} + \epsilon\left\|\nabla^\Gamma R\right\|_{L^2(\Gamma_t(\delta))}^2\right) \\ &\qquad \leqslant C_2\,\|\nabla\varphi\|_{L^2(\Omega)}^2 + \mathcal{RS}, \end{aligned} \tag{3.81}$$

where

$$\mathcal{RS} := \left|\int_\Omega \Big(\big(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^{\text{ffl}}}\big)\cdot\nabla c_A^\epsilon + r_{\text{CH1}}^\epsilon - \tilde{\mathbf{w}}_2^\epsilon\cdot\nabla c_A^\epsilon + \mathcal{R}^\epsilon - \mathbf{v}^\epsilon\cdot\nabla R - \mathbf{v}_{\text{err}}^\epsilon\cdot\nabla c_A^\epsilon\Big)\varphi\,\mathrm{d}x\right|. \\ + \frac{C_3}{\epsilon}\int_\Omega |R|^3\,\mathrm{d}x + \left|\int_\Omega R r_{\text{CH2}}^\epsilon\,\mathrm{d}x\right|.$$

Integrating (3.81) over $(0, T')$ and using Gronwall's inequality, we get

$$\sup_{0\leqslant\tau\leqslant T'}\|\nabla\varphi\|_{L^2(\Omega)}^2 + \left\|\left(\epsilon R, \epsilon^3|\nabla R|\right)\right\|_{L^2(\Omega_{T'})}^2 + \left\|\left(\epsilon^{-1}R, \epsilon|\nabla R|\right)\right\|_{L^2(\Omega\setminus\Gamma(\delta;T'))}^2 \\ + \epsilon\left\|\nabla^\Gamma R\right\|_{L^2(\Gamma(\delta;T'))}^2 \leqslant C(T_0)\left(\|\nabla\varphi(.,0)\|_{L^2(\Omega)}^2 + \int_0^{T'}\mathcal{RS}\,\mathrm{d}t\right) \tag{3.82}$$

for some positive constant $C(T_0) > 0$. On the other hand, (3.79) together with Gronwall's inequality and (3.74) also implies

$$\mathcal{E}\left(R, T'\right) \leqslant C(T_0)\left(\|\nabla\varphi(.,0)\|_{L^2(\Omega)}^2 + \int_0^{T'}\mathcal{RS}\,\mathrm{d}t\right). \tag{3.83}$$

The idea now is to show that we may choose $\epsilon_0 > 0$ and $T \in (0, T_0]$ in the definition of T_ϵ so small, that

$$C(T_0)\left(\|\nabla\varphi(.,0)\|_{L^2(\Omega)}^2 + \int_0^{T_\epsilon}\mathcal{RS}\,\mathrm{d}t\right) < K^2\epsilon^{2M}.$$

holds for all $\epsilon \in (0, \epsilon_0)$. To this end we have to estimate $\mathcal{RS}$ in the following.

Due to (3.6)–(3.7) and since (1.27) holds true for T_ϵ, we get

$$\int_0^{T_\epsilon}\left|\int_\Omega R r_{\text{CH2}}^\epsilon\,\mathrm{d}x\right|\mathrm{d}t + \int_0^{T_\epsilon}\left|\int_\Omega r_{\text{CH1}}^\epsilon\varphi\,\mathrm{d}x\right|\mathrm{d}t \leqslant C(K)C(T,\epsilon)\,\epsilon^{2M}.$$

Moreover, we immediately get

$$\int_0^{T_\epsilon}\left|\int_\Omega\Big(\big(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^{\text{ffl}}}\big)\cdot\nabla c_A^\epsilon + \mathbf{v}_{\text{err}}^\epsilon\cdot\nabla c_A^\epsilon + \mathcal{R}^\epsilon + \tilde{\mathbf{w}}_2^\epsilon\cdot\nabla c_A^\epsilon\Big)\varphi\,\mathrm{d}x\right|\mathrm{d}t \leqslant C(K)C(T;\epsilon)\,\epsilon^{2M},$$

as a consequence of Corollary 3.13 and Lemmata 3.15 and 3.11. Here $C(T,\epsilon) \to 0$ as $(T,\epsilon) \to 0$.

Moreover, as a consequence of Lemma 3.9 and Hölder's inequality we have

$$\int_0^{T_\epsilon}\|R\|_{L^3(\Gamma_t(\delta))}^3\,\mathrm{d}t \leqslant C\left(\|R\|_{L^2(0,T_\epsilon;L^2(\Gamma_t(\delta)))} + \left\|\nabla^\Gamma R\right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t(\delta)))}\right)^{\frac12} \\ \cdot\left(\|R\|_{L^2(0,T_\epsilon;L^2(\Gamma_t(\delta)))} + \|\partial_{\mathbf{n}}R\|_{L^2(0,T_\epsilon;L^2(\Gamma_t(\delta)))}\right)^{\frac12}\|R\|_{L^4(0,T_\epsilon;L^2(\Gamma_t(\delta)))}^2. \tag{3.84}$$

Since $\|R\|_{L^2(\Omega)}^2 \leqslant \|\nabla\varphi\|_{L^2(\Omega)} \|\nabla R\|_{L^2(\Omega)}$, we deduce

$$\|R\|_{L^4(0,T_\epsilon;L^2(\Omega))}^2 \leqslant \sup_{\tau\in(0,T_\epsilon)} \|\nabla\varphi\|_{L^2(\Omega)} \|\nabla R\|_{L^2(\Omega_{T_\epsilon})}. \tag{3.85}$$

Because of (1.27) and the definition of T_ϵ, this implies

$$\frac{1}{\epsilon}\int_0^{T_\epsilon} \|R\|_{L^3(\Gamma_t(\delta))}^3 \,\mathrm{d}t < \frac{1}{\epsilon} CK^3 \epsilon^{\frac{1}{2}M-\frac{1}{4}} \epsilon^{\frac{1}{2}M-\frac{3}{4}} \epsilon^M \epsilon^{M-\frac{3}{2}} = CK^3\epsilon^{3M-\frac{7}{2}} \leqslant CK^3\epsilon^{2M+\frac{1}{2}}$$

since $M \geqslant 4$. On the other hand, we have, for $\epsilon > 0$ small enough,

$$\begin{aligned}\frac{1}{\epsilon}\int_0^{T_\epsilon} \|R\|_{L^3(\Omega\backslash\Gamma_t(\delta))}^3 \,\mathrm{d}t &\leqslant \frac{1}{\epsilon} C \|R\|_{L^2(0,T_\epsilon;H^1(\Omega\backslash\Gamma_t(\delta)))} \|R\|_{L^4(0,T_\epsilon;L^2(\Omega\backslash\Gamma_t(\delta)))}^2 \\ &\leqslant \frac{1}{\epsilon} CK^3 \epsilon^{M-\frac{1}{2}} \epsilon^{2M-\frac{3}{2}},\end{aligned} \tag{3.86}$$

where we used the Gagliardo–Nirenberg interpolation theorem, (3.85) and (1.27). As $M \geqslant 4$, the estimate follows.

For the last term in $\mathcal{RS}$ we have

$$\begin{aligned}\int_0^{T_\epsilon} \left|\int_\Omega \mathbf{v}^\epsilon \cdot \nabla R\varphi \,\mathrm{d}x\right| \mathrm{d}t &= \int_0^{T_\epsilon} \left|\int_\Omega \mathbf{v}^\epsilon \cdot \nabla\varphi R \,\mathrm{d}x\right| \mathrm{d}t \\ &\leqslant \int_0^{T_\epsilon} \left|\int_\Omega \mathbf{v}_A^\epsilon \cdot \nabla\varphi R \,\mathrm{d}x\right| + \left|\int_\Omega \left(\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon\right) \cdot \nabla\varphi R \,\mathrm{d}x\right| \mathrm{d}t.\end{aligned} \tag{3.87}$$

Before we continue with the estimates, we introduce $\hat{\mathbf{v}}_A^\epsilon := \mathbf{v}_A^\epsilon - \epsilon^{M-\frac{1}{2}} \mathbf{v}_{A,M-\frac{1}{2}}^\epsilon \in L^\infty\left(\Omega_{T_0}\right)$. First of all we have

$$\begin{aligned}\int_0^{T_\epsilon} \left|\int_\Omega \hat{\mathbf{v}}_A^\epsilon \cdot \nabla\varphi R \,\mathrm{d}x\right| \mathrm{d}t &\leqslant \int_0^{T_\epsilon} \left|\int_\Omega \gamma\hat{\mathbf{v}}_A^\epsilon \cdot \nabla\varphi R \,\mathrm{d}x\right| \mathrm{d}t \\ &+ \int_0^{T_\epsilon}\int_\Omega \left|\nabla\left((1-\gamma)\hat{\mathbf{v}}_A^\epsilon\right) : (\nabla\varphi\otimes\nabla\varphi)\right| + \left|(1-\gamma)\hat{\mathbf{v}}_A^\epsilon \cdot \nabla\left(\frac{|\nabla\varphi|^2}{2}\right)\right| \mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{3.88}$$

where we used $-\Delta\varphi = R$. We note that we introduced γ since $\hat{\mathbf{v}}_A^\epsilon$ does not satisfy Dirichlet boundary conditions (nor does φ satisfy Neumann boundary conditions).

Now $\left|\nabla\hat{\mathbf{v}}_A^\epsilon(x,t)\right| \leqslant \left|\xi(d_\Gamma(x,t))\partial_\rho\mathbf{v}_0\left(\rho(x,t),x,t\right)\frac{1}{\epsilon}\right| + C(K)$, which is a consequence of the uniform boundedness of the terms $\mathbf{v}_k$, $\mathbf{v}_{O,\mathbf{B}}$ and of $\left\|h_A^\epsilon\right\|_{C^0(0,T_\epsilon;C^1(\Gamma_t(2\delta)))} \leqslant C(K)$ (see (3.35)). Moreover, by (3.12), and since $d_\Gamma(x,t) = \epsilon\left(\rho(x,t) + h_A^\epsilon(x,t)\right)$ for $(x,t) \in \Gamma(2\delta)$, we have

$$\left|\partial_\rho\mathbf{v}_0\left(\rho(x,t),x,t\right)\right| \leqslant \epsilon\left|\eta'\left(\rho(x,t)\right)\right| \left|\rho(x,t) + h_A^\epsilon(x,t)\right| \left|\frac{\mathbf{v}^+(x,t) - \mathbf{v}^-(x,t)}{d_\Gamma(x,t)}\right| \tag{3.89}$$

for all $(x,t) \in \Gamma(2\delta)$.

Due to $\|\eta'(\rho)\rho| < C$ for all $\rho \in \mathbb{R}$ and $\mathbf{v}^+ = \mathbf{v}^-$ on Γ this results in

$$\int_0^{T_\epsilon} \left| \int_\Omega \nabla \left((1-\gamma)\, \hat{\mathbf{v}}_A^\epsilon \right) : (\nabla\varphi \otimes \nabla\varphi)\, \mathrm{d}x \right| \mathrm{d}t \leqslant C(K) T_\epsilon \|\nabla\varphi\|^2_{L^\infty(0,T_\epsilon;L^2(\Omega))} \leqslant C(K) T_\epsilon \epsilon^{2M}$$

by (1.27b) and the facts that $\hat{\mathbf{v}}_A^\epsilon \in L^\infty\left(\Omega_{T_0}\right)$ and γ, γ' are bounded.

Concerning the second term on the right-hand side of (3.88), we note that $\left\|\operatorname{div}\left(\hat{\mathbf{v}}_A^\epsilon\right)\right\|_{L^\infty(\Omega_{T_\epsilon})} \leqslant C(K)$ as a consequence of (3.89) and (3.35). Thus

$$\int_0^{T_\epsilon} \left| \int_\Omega \left((1-\gamma)\, \hat{\mathbf{v}}_A^\epsilon\right) \cdot \nabla \left(\frac{1}{2} |\nabla\varphi|^2 \right) \mathrm{d}x \right| \mathrm{d}t \leqslant C(K) T_\epsilon \|\nabla\varphi\|^2_{L^\infty(0,T_\epsilon;L^2(\Omega))} \leqslant C(K) T_\epsilon \epsilon^{2M}.$$

For the third term on the right-hand side of (3.88), we calculate

$$\int_0^{T_\epsilon} \left| \int_\Omega \gamma \hat{\mathbf{v}}_A^\epsilon \cdot \nabla\varphi R \,\mathrm{d}x \right| \mathrm{d}t \leqslant C T_\epsilon^{\frac{1}{2}} \|\nabla\varphi\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \|R\|_{L^2(\Omega_{T_\epsilon} \setminus \Gamma(2\delta;T_\epsilon))},$$

so (1.27) implies a suitable estimate. Regarding (3.87), we have

$$\begin{aligned} &\int_0^{T_\epsilon} \left| \int_\Omega \epsilon^{M-\frac{1}{2}} \mathbf{v}^\epsilon_{A,M-\frac{1}{2}} \cdot \nabla\varphi R \,\mathrm{d}x \right| \mathrm{d}t \\ &\leqslant \epsilon^{M-\frac{1}{2}} \|\mathbf{v}^\epsilon_{A,M-\frac{1}{2}}\|_{L^2(0,T_\epsilon;L^\infty(\Omega))} \|\nabla\varphi\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \cdot \|R\|_{L^2(0,T_\epsilon;L^2(\Omega))} < C(K)\epsilon^{2M+\frac{1}{2}} \end{aligned}$$

as $M \geqslant 4$. Here we used that $\|\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}\|_{L^2(0,T_\epsilon;L^\infty(\Omega^\pm(t)\cup\Gamma_t(2\delta)))} \leqslant C(K)$ due to (3.34) and $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. Hence

$$\begin{aligned} &\int_0^{T_\epsilon} \left| \int_\Omega \left(\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon\right) \cdot \nabla\varphi R\,\mathrm{d}x \right| \mathrm{d}t \\ &\leqslant \left(\|\tilde{\mathbf{w}}_1^\epsilon\|_{L^2(0,T_\epsilon;L^4(\Omega))} + \|\overline{\mathbf{v}^\epsilon} - \mathbf{v}_A^\epsilon\|_{L^2(0,T_\epsilon;L^4(\Omega))} \right) \|\nabla\varphi\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \|R\|_{L^2(0,T_\epsilon;L^4(\Omega))} \\ &\qquad + \int_0^{T_\epsilon} \left| \int_\Omega \tilde{\mathbf{w}}_2^\epsilon \cdot \nabla R\varphi \,\mathrm{d}x \right| \mathrm{d}t + \int_0^{T_\epsilon} \left| \int_\Omega \mathbf{v}^\epsilon_{\mathrm{err}} \cdot \nabla R\varphi\,\mathrm{d}x \right| \mathrm{d}t \\ &\leqslant C(K) \left(\epsilon^{2M+\frac{1}{2}} + \epsilon^{2M+\frac{1}{2}} + C(T,\epsilon)\,\epsilon^{2M} \right) + \int_0^{T_\epsilon} \left| \int_\Omega \tilde{\mathbf{w}}_2^\epsilon \cdot \nabla R\varphi\,\mathrm{d}x \right| \mathrm{d}t \end{aligned} \tag{3.90}$$

because of $\mathbf{v}^\epsilon_{\mathrm{err}} = \mathbf{v}^\epsilon - \left(\overline{\mathbf{v}^\epsilon} + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon\right)$, Theorem 3.12.1), (3.28), (1.27), (3.65), and $M \geqslant 4$. Regarding the $\tilde{\mathbf{w}}_2^\epsilon$ term we first note that for $\kappa > 0$ we have

$$\begin{aligned} \|\nabla R\|_{L^2(0,T_\epsilon;L^{2+\kappa}(\Omega))} &\leqslant C \left(\|\nabla R\|^{1-\frac{\kappa}{2+\kappa}}_{L^2(\Omega_{T_\epsilon})} \|\Delta R\|^{\frac{\kappa}{2+\kappa}}_{L^2(\Omega_{T_\epsilon})} + \|\nabla R\|_{L^2(\Omega_{T_\epsilon})} \right) \\ &\leqslant C(K) \left(\epsilon^{M-\frac{3}{2}} \epsilon^{-(M+2)\frac{\kappa}{2+\kappa}} \right) \end{aligned} \tag{3.91}$$

where we used $\|R\|_{H^2(\Omega)} \leqslant C\|\Delta R\|_{L^2(\Omega)}$ and $\|\Delta R\|_{L^2(\Omega_{T_\epsilon})} \leqslant C(K)\epsilon^{-\frac{7}{2}}$ as in (3.40). Thus, we may estimate for $\kappa > 0$ and $q \in (\frac{2+\kappa}{(2+\kappa)-1}, 2)$

$$\begin{aligned} \int_0^{T_\epsilon} \left| \int_\Omega \tilde{\mathbf{w}}_2^\epsilon \cdot \nabla R\varphi\,\mathrm{d}x \right| \mathrm{d}t &\leqslant \|\tilde{\mathbf{w}}_2^\epsilon\|_{L^2(0,T_\epsilon;L^q(\Omega))} \|\nabla R\|_{L^2(0,T_\epsilon;L^{2+\kappa}(\Omega))} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \\ &\leqslant C(K)\epsilon^{3M-\frac{5}{2}} \epsilon^{-(M+2)\frac{\kappa}{2+\kappa}} \leqslant C(K)\epsilon^{2M+\alpha} \end{aligned}$$

for some $\alpha > 0$, where we used (3.51), (1.27b), (3.91), $M \geqslant 4$ and that $\kappa > 0$ can be chosen arbitrarily small.

Because of (3.90), we get $\int_0^{T_\epsilon} \left| \int_\Omega \left(\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon\right) \cdot \nabla\varphi R \, dx \right| dt \leqslant C(K) C(T, \epsilon) \epsilon^{2M}$, which concludes the estimates for $\mathcal{RS}$. Since (3.82) and (3.83) do not imply estimates of the kind (1.23e) and (1.23f), we need to apply another strategy and test with $\gamma^2 R$ in the following.

Let again $T' \in (0, T_0]$. Multiplying the difference of the differential equations (1.3) and (3.3) by $\gamma^2 R$ and integrating the result over Ω yields

$$0 = \frac{1}{2} \int_\Omega \frac{d}{dt} \left(R^2\right) \gamma^2 \, dx + \int_{\Omega_T} \gamma^2 R \big(\mathbf{v}^\epsilon \cdot \nabla R + \mathbf{v}_{\text{err}}^\epsilon \cdot \nabla c_A^\epsilon + \big(\overline{\mathbf{v}^\epsilon} - \mathbf{v}_A^\epsilon + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon\big) \cdot \nabla c_A^\epsilon\big) \, dx$$
$$+ \int_\Omega \gamma^2 R r_{\text{CH1}}^\epsilon - \Delta\left(\gamma^2 R\right) \left(-\epsilon \Delta R + \frac{1}{\epsilon}\Big(f''\left(c_A^\epsilon\right) R + \mathcal{N}\left(c_A^\epsilon, R\right)\Big) - r_{\text{CH2}}^\epsilon\right) dx, \quad (3.92)$$

where we used $\operatorname{supp}\gamma \cap \operatorname{supp}\xi \circ d_{\Gamma_t} = \emptyset$ for all $t \in [0, T_0]$, (3.77), integration by parts and $R = \mu^\epsilon = \mu_A^\epsilon = 0$ on $\partial_{T_0}\Omega$.

As $c_A^\epsilon = -1 + \mathcal{O}(\epsilon)$ in $L^\infty\big(\partial_{T_0}\Omega\big(\frac{\delta}{2}\big)\big)$, we have $f''\big(c_A^\epsilon(x,t)\big) = f''(-1) + \epsilon \tilde{f}(x,t)$ for $(x,t) \in \partial_{T_0}\Omega\big(\frac{\delta}{2}\big)$ by a Taylor expansion, where $\tilde{f} \in L^\infty\big(\partial_{T_0}\Omega\big(\frac{\delta}{2}\big)\big)$. Moreover,

$$\nabla\left(\gamma^2 R\right) = 2\gamma R \nabla\gamma + \gamma^2 \nabla R, \quad \Delta\left(\gamma^2 R\right) = \Delta\left(\gamma^2\right) R + 4\gamma\nabla\gamma \cdot \nabla R + \gamma^2 \Delta R$$

and we find

$$\frac{1}{\epsilon} \int_\Omega -\Delta\left(\gamma^2 R\right) f''\left(c_A^\epsilon\right) R \, dx$$
$$= \frac{1}{\epsilon} f''(-1) \|\gamma \nabla R\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \int_\Omega f''(-1) R \nabla\left(\gamma^2\right) \cdot \nabla R \, dx - \int_\Omega \Delta\left(\gamma^2 R\right) \tilde{f} R \, dx, \quad (3.93)$$

where we used $R = 0$ on $\partial_{T_0}\Omega$. Moreover, we have

$$\nabla\mathcal{N}\left(c_A^\epsilon, R\right) = k_f \nabla c_A^\epsilon R^2 + \left(f^{(3)}\left(c_A^\epsilon\right) R + k_f R^2\right) \nabla R,$$

due to (3.78) and $k_f = \frac{f^{(4)}(c_A)}{2}$. This yields

$$\int_\Omega -\Delta\left(\gamma^2 R\right) \frac{1}{\epsilon} \mathcal{N}\left(c_A^\epsilon, R\right) d(x,t) = \frac{1}{\epsilon} \int_\Omega k_f \left(|\gamma(\nabla R) R|^2 + \nabla\left(\gamma^2\right) R^3 \cdot \nabla R\right) dx$$
$$+ \frac{1}{\epsilon} \int_\Omega \nabla\left(\gamma^2 R\right) \cdot \left(k_f \nabla c_A^\epsilon R^2 + f^{(3)}\left(c_A^\epsilon\right) R \nabla R\right) dx$$
$$= \frac{k_f}{\epsilon} \|\gamma |\nabla R| R\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \int_\Omega \mathcal{N}^\nabla\left(c_A^\epsilon, R\right) dx, \quad (3.94)$$

where the boundary terms due to integration by parts vanish since $f'(-1) = R(x,t) = 0$ and $c_A^\epsilon(x,t) = -1$ for $(x,t) \in \partial_{T_0}\Omega$. Here we used the notation

$$\mathcal{N}^\nabla\left(c_A^\epsilon, R\right) := k_f \nabla\left(\gamma^2\right) R^3 \cdot \nabla R + \nabla\left(\gamma^2 R\right) \cdot \left(k_f \nabla c_A^\epsilon R^2 + f^{(3)}\left(c_A^\epsilon\right) R \nabla R\right). \quad (3.95)$$

Additionally, we compute

$$\int_{\Omega} -\Delta\left(\gamma^2 R\right)(-\epsilon\Delta R)\,\mathrm{d}x = \epsilon\,\|\gamma\Delta R\|_{L^2(\Omega)}^2 + \epsilon\int_{\Omega} 4\gamma\nabla\gamma\cdot\nabla R\Delta R + \Delta\left(\gamma^2\right) R\Delta R\,\mathrm{d}x. \quad (3.96)$$

Plugging (3.93), (3.94) and (3.96) (noting that $k_f, f''(-1) > 0$) into (3.92) and integrating in time yields

$$\begin{aligned}
&\sup_{t\in(0,T')}\|\gamma R(.,t)\|_{L^2(\Omega)}^2 + \epsilon\|\gamma\Delta R\|_{L^2(\Omega_{T'})}^2 + \frac{k_f}{\epsilon}\|(\gamma\nabla R,\gamma R\nabla R)\|_{L^2(\Omega_{T'})}^2\\
&\leqslant \|\gamma R(.,0)\|_{L^2(\Omega)}^2 + C_1\int_0^{T'}\left|\int_{\Omega}\gamma^2 R\big(\mathbf{v}^\epsilon\cdot\nabla R + (\mathbf{v}_{\mathrm{err}}^\epsilon + \overline{\mathbf{v}^\epsilon} - \mathbf{v}_A^\epsilon + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon)\cdot\nabla c_A^\epsilon\big)\,\mathrm{d}x\right|\mathrm{d}t\\
&\quad + C_2\int_0^{T'}\left|\int_{\Omega}\gamma^2 R r_{\mathrm{CH1}}^\epsilon + \epsilon\big(\Delta(\gamma^2)R + 4\gamma\nabla\gamma\cdot\nabla R\big)\Delta R + \frac{1}{\epsilon}\mathcal{N}^\nabla\left(c_A^\epsilon, R\right)\mathrm{d}x\right|\mathrm{d}t\\
&\quad + C_3\int_0^{T'}\left|\int_{\Omega}\Delta(\gamma^2 R)\big(\tilde{f}R - r_{\mathrm{CH2}}^\epsilon\big) + R\nabla(\gamma^2)\cdot\nabla R\frac{1}{\epsilon}f''(-1)\,\mathrm{d}x\right|\mathrm{d}t. \quad (3.97)
\end{aligned}$$

If we may now give suitable estimates for the right-hand side of (3.97), replacing T' by T_ϵ, we get (1.23e) and (1.23f).

Now we estimate the right-hand side of (3.97). Starting from the last term in (3.97), we have

$$\int_0^{T_\epsilon}\left\|\nabla(\gamma^2)R\nabla R\frac{f''(-1)}{\epsilon}\right\|_{L^1(\Omega)}\mathrm{d}t \leqslant \frac{C}{\epsilon}\|\gamma\nabla R\|_{L^2(\Omega_{T_\epsilon})}\|\nabla\gamma R\|_{L^2(\Omega_{T_\epsilon})} \leqslant C(K)\epsilon^{2M-\frac12}$$

due to (1.27a) and (1.27d). For the next term, we note that $r_{\mathrm{CH2}}^\epsilon = r_{\mathrm{CH2,B}}^\epsilon$ in $\partial_{T_0}\Omega\left(\frac{\delta}{2}\right)$ and use (3.13) to conclude

$$\begin{aligned}
\int_0^{T_\epsilon}\left|\int_{\Omega}\Delta\left(\gamma^2 R\right) r_{\mathrm{CH2}}^\epsilon\,\mathrm{d}x\right|\mathrm{d}t &\leqslant C\|(\gamma\Delta R,\nabla R,R)\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))}\epsilon^{M+1}\\
&\quad + C_2\epsilon^{M-\frac12}\|\nabla\mu_{M-\frac12}^-\|_{L^2\left(\Omega_{T_\epsilon}^-\right)}\|(\gamma\nabla R,R)\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))} \leqslant C(K)\epsilon^{2M-\frac12},
\end{aligned}$$

where we used integration by parts, $\mu_{M-\frac12}^- = 0$ on $\partial_{T_\epsilon}\Omega$, (1.27a), (1.27d) and (3.34). Moreover,

$$\begin{aligned}
\int_0^{T_\epsilon}\left|\int_{\Omega}\Delta(\gamma^2 R)\tilde{f}R\,\mathrm{d}x\right|\mathrm{d}t &\leqslant C\|(\gamma\Delta R,\nabla R,R)\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))}\|R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))}\\
&\leqslant C(K)\epsilon^{2M-\frac12}.
\end{aligned}$$

Skipping $\mathcal{N}^\nabla\left(c_A^\epsilon, R\right)$ for now, we next estimate

$$\int_0^{T_\epsilon}\left|\int_{\Omega}\epsilon 4\left(\nabla\gamma\cdot\nabla R\right)\gamma\Delta R\,\mathrm{d}x\right|\mathrm{d}t \leqslant C\epsilon\|\gamma\Delta R\|_{L^2(\Omega_{T_\epsilon})}\|\nabla R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))} \leqslant C_1\epsilon^{2M-\frac12}$$

due to (1.27a) and (1.27d). Additionally,

$$\begin{aligned}
\int_0^{T_\epsilon}\left|\int_{\Omega}\epsilon\Delta(\gamma^2)R\Delta R\,\mathrm{d}x\right|\mathrm{d}t &= \int_0^{T_\epsilon}\left|\int_{\Omega}\epsilon\big(\nabla\Delta(\gamma^2)R + \Delta(\gamma^2)\nabla R\big)\cdot\nabla R\,\mathrm{d}x\right|\mathrm{d}t\\
&\leqslant C\epsilon\left(\|R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))}\|\nabla R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))} + \|\nabla R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))}^2\right) = O(\epsilon^{2M})
\end{aligned}$$

because of $R|_{\partial\Omega} = 0$ and (1.27b). Now

$$\int_0^{T_\epsilon} \left| \int_\Omega \gamma^2 R r_{CH1}^\epsilon \, dx \right| dt \leqslant C \, \|R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))} \left\| r_{\text{CH1}}^\epsilon \right\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))} \leqslant C(K)\epsilon^{2M+\frac{1}{2}}$$

due to (1.27a) and (3.10), and

$$\begin{aligned} &\int_0^{T_\epsilon} \left| \int_\Omega \gamma^2 R \left(\overline{\mathbf{v}^\epsilon} - \mathbf{v}_A^\epsilon + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon \right) \cdot \nabla c_A^\epsilon \, dx \right| dt \\ &\qquad \leqslant C\epsilon \, \|R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))} \left(\left\| \overline{\mathbf{v}^\epsilon} - \mathbf{v}_A^\epsilon \right\|_{L^2(0,T_\epsilon;H^1(\Omega))} + \left\| \tilde{\mathbf{w}}_1^\epsilon \right\|_{L^2(0,T_\epsilon;H^1(\Omega))} \right) \\ &\qquad\qquad + C\epsilon \, \|R\|_{L^2(0,T_\epsilon;L^{q'}(\partial\Omega(\frac{\delta}{2})))} \left\| \tilde{\mathbf{w}}_2^\epsilon \right\|_{L^2(0,T_\epsilon;L^q(\Omega))} \end{aligned}$$

where $q \in (1, 2)$, $\frac{1}{q'} + \frac{1}{q} = 1$ and we used $\nabla c_A^\epsilon = \mathcal{O}(\epsilon)$ in $L^\infty\big(\partial_{T_0}\Omega\big(\frac{\delta}{2}\big)\big)$. Now (3.28), Theorem 3.12.1 and (3.51) together with $H^1\big(\partial\Omega\big(\frac{\delta}{2}\big)\big) \hookrightarrow L^{q'}\big(\partial\Omega\big(\frac{\delta}{2}\big)\big)$ and (1.27a) imply that the term is of order $O(\epsilon^{2M+\frac{1}{2}})$. Next,

$$\int_0^{T_\epsilon} \left| \int_\Omega \gamma^2 R \mathbf{v}_{\text{err}}^\epsilon \cdot \nabla c_A^\epsilon \, dx \right| dt \leqslant \epsilon \, \|\gamma R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \left\| \mathbf{v}_{\text{err}}^\epsilon \right\|_{L^1(0,T_\epsilon;H^1(\Omega))} \leqslant C(K)\epsilon^{2M+\frac{1}{2}},$$

where we again used $\nabla c_A^\epsilon = \mathcal{O}(\epsilon)$ in $L^\infty\big(\partial_{T_0}\Omega\big(\frac{\delta}{2}\big)\big)$ in the first line and (3.54), (1.27d) in the second line. In view of the above considerations, $\|\gamma R\,(.,0)\|_{L^2(\Omega_{T_\epsilon})}^2 \leqslant \frac{K^2}{4}\epsilon^{2M}$ (cf. (3.74)) and (3.97), we have two more estimates to show:

Using the explicit form of $\mathcal{N}^\nabla$ given in (3.95), we calculate

$$\begin{aligned} &\frac{1}{\epsilon}\int_0^{T_\epsilon} \left| \int_\Omega \mathcal{N}^\nabla \left(c_A^\epsilon, R \right) dx \right| dt \\ &\qquad \leqslant \frac{1}{\epsilon}\int_0^{T_\epsilon} \left| \int_\Omega k_f \nabla(\gamma^2) R^3 \cdot \nabla R \, dx \right| dt + C_1 \, \|R\|_{L^3(\Omega_{T_\epsilon}\setminus\Gamma(2\delta;T_\epsilon))}^3 \\ &\qquad\qquad + C_2 \int_0^{T_\epsilon}\int_\Omega |\gamma^2 \nabla R R^2| \, dx \, dt \\ &\qquad\qquad\qquad + C_3 \frac{1}{\epsilon}\int_0^{T_\epsilon}\int_\Omega |\nabla(\gamma^2 R) R \nabla R| \, dx \, dt, \end{aligned} \tag{3.98}$$

where we again used $\nabla c_A^\epsilon = \mathcal{O}(\epsilon)$ in $L^\infty\big(\partial_{T_0}\Omega\big(\frac{\delta}{2}\big)\big)$ in the last step. Now we have

$$\begin{aligned} &\frac{1}{\epsilon}\int_0^{T_\epsilon}\int_\Omega \left| k_f \nabla(\gamma^2) R^3 \cdot \nabla R \right| dx \, dt \\ &\qquad \leqslant \frac{1}{\epsilon} C \, \|\gamma R \, |\nabla R|\|_{L^2(\Omega_{T_\epsilon})} \, \|R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \, \|R\|_{L^2(0,T_\epsilon;H^1(\partial\Omega(\frac{\delta}{2})))} \\ &\qquad \leqslant C(K)\epsilon^{-1}\epsilon^M \epsilon^{\frac{M}{2}-\frac{1}{4}}\epsilon^{M-\frac{1}{2}} \end{aligned}$$

where we used $\|u\|_{L^4(\Omega)} \leqslant C\|u\|_{L^2(\Omega)}^{\frac{1}{2}}\|u\|_{H^1(\Omega)}^{\frac{1}{2}}$ and (1.27) together with Lemma 3.8 4). The estimate follows since $M \geqslant 4$. Next we have $\|R\|_{L^3(\Omega_{T_\epsilon}\setminus\Gamma(2\delta;T_\epsilon))} \leqslant C(K)\epsilon^{2M+1}$ due to (3.86) and

$$\int_0^{T_\epsilon}\int_\Omega \left|\gamma^2\nabla R R^2\right| \mathrm{d}x\,\mathrm{d}t \leqslant C\,\|\gamma R\nabla R\|_{L^2(\Omega_{T_\epsilon})}\|R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))} \leqslant C(K)\epsilon^{2M+\frac{1}{2}}$$

due to (1.27). Regarding the last term in (3.98) we have on the one hand

$$\frac{1}{\epsilon}\int_0^{T_\epsilon}\int_\Omega \left|\left(\nabla\gamma^2\right) R^2\nabla R\right| \mathrm{d}x\,\mathrm{d}t \leqslant C\frac{1}{\epsilon}\,\|\gamma R\nabla R\|_{L^2(\Omega_{T_\epsilon})}\|R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))} \leqslant C(K)\epsilon^{2M-\frac{1}{2}}$$

as before and on the other hand

$$\begin{aligned}\frac{1}{\epsilon}\int_0^{T_\epsilon}\int_\Omega \left|\gamma^2\left(\nabla R\right)^2 R\right| \mathrm{d}x\,\mathrm{d}t &\leqslant C\frac{1}{\epsilon}\,\|R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))}\|\gamma\nabla R\|_{L^2(\Omega_{T_\epsilon})}\|\gamma\nabla R\|_{L^2(0,T_\epsilon;H^1(\Omega))}\\ &\leqslant C(K)\epsilon^{\frac{M}{2}-\frac{1}{4}}\epsilon^{M}\epsilon^{M-1}\epsilon^{-1} = C(K)\epsilon^{\frac{5}{2}M-2-\frac{1}{4}}\end{aligned}$$

where we use Lemma 3.8 4) and (1.27). Altogether we have $\frac{1}{\epsilon}\int_0^{T_\epsilon}\left|\int_\Omega \mathcal{N}^\nabla(c_A^\epsilon, R)\,\mathrm{d}x\right|\mathrm{d}t \leqslant \epsilon^{2M-\frac{1}{2}}$.

Finally, we estimate

$$\begin{aligned}&\int_0^{T_\epsilon}\left|\int_\Omega \gamma^2 R\left(\mathbf{v}^\epsilon\cdot\nabla R\right)\mathrm{d}x\right|\mathrm{d}t\\ &\leqslant \int_0^{T_\epsilon}\left|\int_\Omega \gamma^2 R\left(\mathbf{v}_{\mathrm{err}}^\epsilon\cdot\nabla R\right)\mathrm{d}x\right| + \int_0^{T_\epsilon}\left|\int_\Omega \left(\mathbf{v}^\epsilon-\mathbf{v}_{\mathrm{err}}^\epsilon\right)\cdot\frac{1}{2}\nabla(\gamma^2)R^2\,\mathrm{d}x\right|\mathrm{d}t\\ &\leqslant C(K)C\left(\epsilon,T_\epsilon\right)\epsilon^{2M-1} + \frac{1}{2}\int_{\Omega_{T_\epsilon}}\left|\left(\overline{\mathbf{v}^\epsilon}-\mathbf{v}_A^\epsilon+\tilde{\mathbf{w}}_1^\epsilon+\tilde{\mathbf{w}}_2^\epsilon\right)\cdot\nabla(\gamma^2)R^2\right|\\ &\qquad + \left|\mathbf{v}_A^\epsilon\cdot\nabla(\gamma^2)R^2\right|\mathrm{d}(x,t)\\ &\leqslant C(K)C\left(\epsilon,T_\epsilon\right)\epsilon^{2M-1} + \frac{1}{2}\int_{\Omega_{T_\epsilon}}\left|\left(\overline{\mathbf{v}^\epsilon}-\mathbf{v}_A^\epsilon+\tilde{\mathbf{w}}_1^\epsilon+\tilde{\mathbf{w}}_2^\epsilon\right)\cdot\nabla(\gamma^2)R^2\right|\mathrm{d}(x,t)\\ &\qquad + \epsilon^{M-\frac{1}{2}}\frac{1}{2}\int_0^{T_\epsilon}\int_\Omega\left|\mathbf{v}_{M-\frac{1}{2}}^{-,\epsilon}\cdot\nabla(\gamma^2)R^2\right|\mathrm{d}t,\end{aligned}$$

where we used that $\mathbf{v}^\epsilon - \mathbf{v}_{\mathrm{err}}^\epsilon$ is divergence free and $R|_{\partial\Omega} = 0$, as well as (3.64) and the definition of $\mathbf{v}_{\mathrm{err}}^\epsilon$ in (3.50). Furthermore, we used $\mathbf{v}_A^\epsilon - \epsilon^{M-\frac{1}{2}}\mathbf{v}_{A,M-\frac{1}{2}}^\epsilon \in L^\infty\left(\Omega_{T_0}\right)$ and (1.27a). Note that $\mathbf{v}_{A,M-\frac{1}{2}}^\epsilon = \mathbf{v}_{M-\frac{1}{2}}^{-,\epsilon}$ in $\partial_{T_0}\Omega\left(\frac{\delta}{2}\right)$. We may continue estimating

$$\begin{aligned}&\int_{\Omega_{T_\epsilon}}\left|\left(\overline{\mathbf{v}^\epsilon}-\mathbf{v}_A^\epsilon+\tilde{\mathbf{w}}_1^\epsilon\right)\cdot\nabla(\gamma^2)R^2\right|\mathrm{d}(x,t)\\ &\leqslant \left(\left\|\overline{\mathbf{v}^\epsilon}-\mathbf{v}_A^\epsilon\right\|_{L^2(0,T_\epsilon;H^1(\Omega))}+\|\tilde{\mathbf{w}}_1^\epsilon\|_{L^2(0,T_\epsilon;H^1(\Omega))}\right)\cdot\|\gamma R\|_{L^\infty(0,T_\epsilon;L^{2+\kappa}(\Omega))}\|R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))}\\ &\leqslant C(K)\left(\epsilon^M+\epsilon^{M-\frac{1}{2}}\right)\epsilon^{M-\frac{1}{2}-\frac{\kappa}{2+\kappa}M}\epsilon^{M+\frac{1}{2}}\end{aligned}$$

where we used $H^1(\Omega) \hookrightarrow L^s(\Omega)$ for all $1 \leqslant s < \infty$ in the first inequality, Theorem 3.12.1, Lemma 3.4 (in particular (3.28)), Lemma 3.8 3) and (1.27a) in the second inequality. A suitable estimate follows since $M \geqslant 4$ and we may choose $\kappa > 0$ arbitrarily small. Regarding $\tilde{\mathbf{w}}_2^\epsilon$, we choose $\kappa > 0$ and $q = \frac{2+\kappa}{(2+\kappa)-1}$ and estimate

$$\begin{aligned}\int_{\Omega_{T_\epsilon}} \left|\tilde{\mathbf{w}}_2^\epsilon \cdot \nabla(\gamma^2) R^2\right| \mathrm{d}(x,t) &\leqslant \|\tilde{\mathbf{w}}_2^\epsilon\|_{L^2(0,T_\epsilon;L^q(\Omega))} \|\gamma R\|_{L^\infty(0,T_\epsilon;L^{2+\kappa}(\Omega))} \|R\|_{L^2(0,T_\epsilon;L^\infty(\Omega))} \\ &\leqslant C(K,\alpha)\,\epsilon^{M-1}\epsilon^{M-\frac12}\epsilon^{-\frac{\kappa}{2+\kappa}M}\epsilon^{M-\frac32}\epsilon^{-(M+2)\alpha}\end{aligned}$$

for $\alpha > 0$, where we used (3.51), (1.27d) and Lemma 3.8 1). Again $M \geqslant 4$ and a suitable choice of $\alpha > 0$ and $\kappa > 0$ yield the final estimate by $C(T,\epsilon)\epsilon^{2M-1}$. For the term involving $\mathbf{v}_{M-\frac12}^{-,\epsilon}$ we obtain

$$\begin{aligned}&\epsilon^{M-\frac12}\int_{\Omega_{T_\epsilon}} \left|\mathbf{v}_{M-\frac12}^{-,\epsilon}\cdot\nabla\left(\gamma^2\right)R^2\right| \mathrm{d}(x,t) \\ &\leqslant C\epsilon^{M-\frac12}\left\|\mathbf{v}_{M-\frac12}^{-,\epsilon}\right\|_{L^2(0,T_\epsilon;L^\infty(\Omega^-(t)))}\|\gamma R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))}\cdot\|R\|_{L^2(\partial_{T_\epsilon}\Omega(\frac{\delta}{2}))} \leqslant C(K)\epsilon^{3M-\frac12}\end{aligned}$$

where we used (3.34) (together with $H^2(\Omega^-(t)) \hookrightarrow L^\infty(\Omega^-(t))$) and (1.27) in the second estimate.

Thus, we have shown

$$\int_0^{T_\epsilon}\left|\int_\Omega \gamma^2 R\left(\mathbf{v}^\epsilon\cdot\nabla R\right)\mathrm{d}x\right|\mathrm{d}t \leqslant C(K)C\left(T_\epsilon,\epsilon\right)\epsilon^{2M-1}$$

and with that may conclude using (3.97) that

$$\begin{aligned}\sup_{t\in(0,T_\epsilon)}\|\gamma R(.,t)\|_{L^2(\Omega)}^2 + \epsilon\|\gamma\Delta R\|_{L^2(\Omega_{T_\epsilon})}^2 \\ + \frac{1}{\epsilon}\|(\gamma\nabla R,\gamma R\nabla R)\|_{L^2(\Omega_{T_\epsilon})}^2 \leqslant C(K)C(T,\epsilon)\,\epsilon^{2M-1}. \quad (3.99)\end{aligned}$$

where $C(T,\epsilon) \to_{(T,\epsilon)\to 0} 0$.

Altogether we may now choose $\epsilon_0 > 0$ and $T \in (0,T_0]$ so small that (1.27a)–(1.27c) follow for $T_\epsilon = T$ from (3.82) and (3.83) as a consequence of the estimates for $\mathcal{RS}$. (1.27d) follows for $T_\epsilon = T$ from (3.99). This shows (1.23). Regarding (1.24), we have by the definition of $\mathbf{v}_{\text{err}}^\epsilon$ in (3.50) for $q \in (1,2)$

$$\begin{aligned}\left\|\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon\right\|_{L^1(0,T;L^q(\Omega))} &\leqslant \left\|\mathbf{v}_{\text{err}}^\epsilon + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon\right\|_{L^1(0,T;L^q(\Omega))} + C\left\|\overline{\mathbf{v}^\epsilon} - \mathbf{v}_A^\epsilon\right\|_{L^1(0,T;L^2(\Omega))} \\ &\leqslant C(K,q)\,\epsilon^{M-\frac12}\end{aligned}$$

by (3.28), (3.51) and Theorem 3.12. The convergence results (1.25) and (1.26) are then due to the construction of c_A^ϵ and $\mathbf{v}_A^\epsilon$, more precisely to the discussed form of the zero-th order terms, where it is important to note (3.34) for $\mathbf{v}_{M-\frac12}^{\pm,\epsilon}$. This finishes the proof of Theorem 1.1.

REMARK 3.16 In this final remark, we want to discuss the consequences of considering Neumann boundary conditions $\partial_{\mathbf{n}_{\partial\Omega}}\mu^\epsilon = 0$ on $\partial_{T_0}\Omega$ instead of $\mu^\epsilon = 0$. Of course, in this case we would construct μ_A^ϵ such that $\partial_{\mathbf{n}_{\partial\Omega}}\mu_A^\epsilon = 0$ is satisfied on $\partial_{T_0}\Omega$. To gain (3.79), which is a vital point of the proof, we need to ensure that

$$\int_\Omega \varphi\Delta\left(\mu^\epsilon - \mu_A^\epsilon\right)\mathrm{d}x = \int_\Omega \Delta\varphi\left(\mu^\epsilon - \mu_A^\epsilon\right)\mathrm{d}x$$

holds, which is satisfied if we choose Neumann boundary conditions for φ. In particular, φ should be the solution to

$$-\Delta\varphi(.,t) = R(.,t) \text{ in } \Omega, \ \partial_{\mathbf{n}_{\partial\Omega}}\varphi = 0 \text{ on } \partial\Omega, \tag{3.100}$$

together with $\int_\Omega \varphi(.,t)\,\mathrm{d}x = 0$. However, in order for (3.100) to be well-posed, $\int_\Omega R(.,t)\,\mathrm{d}x = 0$ needs to be satisfied, where

$$\begin{aligned}\int_\Omega R(x,t)\,\mathrm{d}x &= \int_0^t\int_\Omega \partial_t\left(c^\epsilon - c_A^\epsilon\right)\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega c_0^\epsilon - c_A^\epsilon|_{t=0}\,\mathrm{d}x\\ &= \int_0^t\int_\Omega \operatorname{div}\left(\mathbf{v}_A^\epsilon\right)c_A^\epsilon + \tilde{\mathbf{w}}_1^\epsilon|_\Gamma\cdot\nabla c_A^\epsilon\xi - r_{\mathrm{CH}1}^\epsilon\,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega c_0^\epsilon - c_A^\epsilon|_{t=0}\,\mathrm{d}x\end{aligned}$$

in the case of no-slip boundary conditions for $\mathbf{v}^\epsilon$. This expression does not vanish and we are not able to estimate it to a high enough power of ϵ. A similar problem arises in the case of periodic boundary conditions. To circumvent this difficulty, we decided to stick to Dirichlet boundary values for μ.

List of notation

D	Jacobian matrix
∇	Gradient
D_s	symmetrized gradient
∂_t^Γ,∇^Γ,Δ^Γ	cf. (2.20) and (2.25)
$\operatorname{div}^\Gamma$	(2.21)
$D_{t,\Gamma}$, ∇_Γ, Δ_Γ	Remark 2.6, (2.24)
$\left[\partial_{\mathbf{n}}, \nabla^\Gamma\right]$	(2.32)
$\mathbf{a}$ (bold letter)	Element in $\mathbb{R}^2$ or $\mathbb{R}^2$-valued function
$\cdot$	Euclidean scalar product on $\mathbb{R}^2$, e.g., $\mathbf{a}\cdot\mathbf{b}$
$\otimes$	$\mathbf{a}\otimes\mathbf{b} = \mathbf{a}^T\mathbf{b} = (a_i b_j)_{i,j=1,2}$
$\otimes_s$	$\mathbf{a}\otimes_s\mathbf{b} := \mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a}$
Ω	smooth domain in $\mathbb{R}^2$
Ω_T, $\partial_T\Omega$	$\Omega_T := \Omega\times(0,T)$, $\partial_T\Omega := \partial\Omega\times(0,T)$
$\Omega^\pm(t)$,$\Omega_T^\pm$	domains for the different phases, cf. Section 1
Γ_t,Γ	Interface separating the different phases, cf. Section 1
$\Gamma_t(\alpha)$, $\Gamma(\alpha;T)$, $\Gamma(2\delta)$	neighborhoods of the interface, cf. Section 1
$\partial\Omega(\alpha)$, $\partial_T\Omega(\alpha)$	neighborhoods of the boundary $\partial\Omega$, cf. Section 1
$\mathbf{n}_{\Gamma_t}$, $\mathbf{n}_{\partial\Omega}$	normals of $\Gamma(t)$ and $\partial\Omega$, resp.
$\mathbf{n}$, τ	parametrized normal and tangential vector, cf. (2.14)
V_{Γ_t}, H_{Γ_t}	normal velocity and (mean) curvature of Γ_t
Pr_{Γ_t}, $Pr_{\partial\Omega}$	orthogonal projections onto $\Gamma(t)$ and $\partial\Omega$, resp.
d_Γ, $d_{\mathbf{B}}$	signed distance functions to Γ_t and $\partial\Omega$, resp.
δ	small positive constant such that $d_\Gamma\colon\Gamma(3\delta)\to\mathbb{R}$ is smooth
$S\colon\Gamma(2\delta)\to\mathbb{T}^1$	cf. (2.19)

X_0	parametrization of Γ_t, cf. (2.13)
$X_0^*, X_0^{*,-1}$	pull-back and push-forward with X_0, cf. (2.15),(2.16)
X	diffeomorphism onto neighborhood of $\Gamma(2\delta)$, cf. (2.17)
$\mathcal{R}_\alpha$	function space for remainders, cf. Definition 2.10
$L^{p,\infty}(\Gamma_t(\alpha))$	cf. Section 2.3
$L^q(0,T;L^p(\Gamma_t(\alpha)))$	cf. Section 2.3
X_T	function space for "height-functions" h, cf. (3.21)
$H^s, s>0$	$H^s := W_2^s$, Sobolev–Slobodeckij space
H^{-1}	$H^{-1} = \left(H_0^1\right)'$
$(X_0, X_1)_{\theta,p}$	Real interpolation space of (X_0, X_1) with exponents θ, p
$\cdot\vert_\Gamma$	$g\vert_\Gamma(x,t) = g\left(Pr_{\Gamma_t}(x), t\right)$
f	double-well potential, cf. (1.19)
θ_0	optimal profile determined by (1.18)
σ	surface tension constant, cf. (1.17)
$\mu, \mathbf{v}, p$	solutions of the sharp interface limit (1.7)–(1.16)
$c^\epsilon, \mu^\epsilon, \mathbf{v}^\epsilon, p^\epsilon$	solutions of the diffuse interface model (1.1)–(1.6)
$c_A^\epsilon, \mu_A^\epsilon, \mathbf{v}_A^\epsilon, p_A^\epsilon$	approximate solution, cf. Theorem 3.1
$\mu^\epsilon_{A,M-\frac{1}{2}}, \mathbf{v}^\epsilon_{A,M-\frac{1}{2}}$	highest order terms of approx. sol., cf. Section 3.1
$\rho^\epsilon, \rho(x,t)$	stretched coordinate, cf. (2.27), (3.11)
$c_O, \mu_O, \mathbf{v}_O, p_O$	outer expansion, cf. Section 3.1
$c_I, \mu_I, \mathbf{v}_I, p_I$	inner expansion, cf. Section 3.1
$c_k, \mu_k, \mathbf{v}_k, p_k$	integer order coeff. of inner exp., cf. Section 3.1
$c^\epsilon_{M-\frac{1}{2}}, \mu^\epsilon_{M-\frac{1}{2}}.\mathbf{v}^\epsilon_{M-\frac{1}{2}}, p^\epsilon_{M-\frac{1}{2}}$	highest order coeff. of inner exp., cf. Section 3.1
$c_{O,\mathbf{B}}, \mu_{O,\mathbf{B}}, \mathbf{v}_{O,\mathbf{B}}, p_{O,\mathbf{B}}$	boundary expansion, cf. Section 3.1
h_A^ϵ	height function of approx. sol., cf. Section 3.1
$h_k, h^\epsilon_{M-\frac{1}{2}}$	coefficients of height function, cf. Section 3.1
$\mathbf{v}^\epsilon_{\mathrm{err}}$	error of velocity, cf. (3.50)
$\tilde{\mathbf{w}}_1^\epsilon, q_1^\epsilon$	leading errors, cf. (3.14)–(3.16)
$\mathbf{w}_1^\epsilon$	$\mathbf{w}_1^\epsilon := \frac{\tilde{\mathbf{w}}_1^\epsilon}{\epsilon^{M-\frac{1}{2}}}$
$\tilde{\mathbf{w}}_2^\epsilon, q_2^\epsilon$	lower order errors (3.48)
$\overline{\mathbf{v}^\epsilon}, \overline{p^\epsilon}$	auxiliary solutions, cf. (3.43)–(3.45)
$r^\epsilon_{\mathrm{CH1}}, r^\epsilon_{\mathrm{CH2}}, \mathbf{r}^\epsilon_{\mathrm{S}}, r^\epsilon_{\mathrm{div}}$	remainders from approx. sol., cf. (3.1)–(3.4)
$\mathbf{h}$	auxilliary function, cf. (3.17)

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