On the entropy of parabolic Allen–Cahn equation

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We define a local (mean curvature flow) entropy for Radon measures in $\mathbb{R}^n$ or in a compact manifold. Moreover, we prove a monotonicity formula of the entropy of the measures associated with the parabolic Allen–Cahn equations. If the ambient manifold is a compact manifold with non-negative sectional curvature and parallel Ricci curvature, this is a consequence of a new monotonicity formula for the parabolic Allen–Cahn equation. As an application, we show that when the entropy of the initial data is small enough (less than twice of the energy of the one-dimensional standing wave), the limit measure of the parabolic Allen–Cahn equation has unit density for all future time.

1. Introduction

The parabolic Allen–Cahn equation

$$\frac{\partial}{\partial t} u^\epsilon = \Delta u^\epsilon - \frac{1}{\epsilon^2} f(u^\epsilon) \tag{1.1}$$

was introduced by Allen and Cahn in 1979. It is the gradient flow of the energy functional

$$M^\epsilon(u) = \int_{\mathbb{R}^n} \frac{\epsilon}{2} |Du|^2 + \frac{1}{\epsilon} F(u) \, dx \tag{1.2}$$

with a speed up factor $1/\epsilon$. Here $F(u)$ is the potential function known to be the "double well potential", and f is the derivative of the potential function, see Section 2.

The solutions to the Allen–Cahn equation are the models to the motion of phase boundaries by surface tension. When $\epsilon \to 0$, the term $-\frac{1}{\epsilon^2} f(u^\epsilon)$ makes the phase boundary sharp, and the limit should be the motion of surfaces by mean curvature. In [11], Ilmanen studied the measures μ_t^ϵ associated with the solution u^ϵ, which is defined by

$$d\mu_t^\epsilon = \left(\frac{\epsilon}{2} |Du^\epsilon(\cdot, t)|^2 + \frac{1}{\epsilon} F\big(u^\epsilon(\cdot, t)\big) \right) dx. \tag{1.3}$$

In this paper, we study the (mean curvature flow) entropy of this measure. Entropy in $\mathbb{R}^n$ was introduced by Colding–Minicozzi [6] in the study of mean curvature flow. Later the author [17]

generalized Colding–Minicozzi's idea to define entropy in a manifold. Entropy is a quantity that characterizes a submanifold/measure from all scales. Hypersurfaces with small entropy have been studied in many contexts, see Colding–Ilmanen–Minicozzi–White [5], Bernstein–Wang [2, 3] and Zhu [20].

The monotonicity formula of the entropy of mean curvature flow plays a very important role in the study of mean curvature flow, see [6, 17]. The main result of this paper is the following monotonicity formula of the entropy of Allen–Cahn equations. We use λ to denote the entropy (see Definition 4.1), and we use ξ^ϵ to denote the discrepancy measure (see (2.8)).

Theorem 1.1 *Suppose $M = \mathbb{R}^n$ or M is a compact manifold with non-negative sectional curvature and parallel Ricci curvature. Suppose μ_t^ϵ is the measure associated with the parabolic Allen–Cahn equation on M. Suppose $\xi_0^\epsilon \leq 0$. For every $0 \leq t_1 \leq t_2$, we have*

$$\lambda(\mu_{t_2}^\epsilon) \leq \lambda(\mu_{t_1}^\epsilon). \tag{1.4}$$

If we let $\epsilon \to 0$, we obtain a monotonicity formula of entropy for the limit measure.

Theorem 1.2 *Suppose $M = \mathbb{R}^n$ or M is a compact manifold with non-negative sectional curvature and parallel Ricci curvature. Suppose μ_t^ϵ is the measure associated with the parabolic Allen–Cahn equation on M, and μ_t is the limit measure as $\epsilon \to 0$. For every $0 < t_1 \leq t_2$, we have*

$$\lambda(\mu_{t_2}) \leq \lambda(\mu_{t_1}). \tag{1.5}$$

We remark that there are also monotonicity formulas for local entropy (see Definition 4.3), and Theorem 1.1 and Theorem 1.2 are just special cases. See Corollary 4.5 and Corollary 4.6.

The monotonicity of entropy provides information on the evolutions. In this paper, we focus on the problem of the density of the measures associated with the parabolic Allen–Cahn equations. In [11], Ilmanen proved that under certain reasonable initial conditions, as $\epsilon \to 0$, μ_t^ϵ converge to a rectifiable measure μ_t. See also [8, 15]. Moreover, μ_t is a Brakke flow, a geometric measure theoretic weak solution of mean curvature flow. In [11, Section 13], Ilmanen asked the following question: when does μ_t have unit density? Let α be the energy of the one-dimensional standing wave (see (2.6)). Then unit density means that μ_t has density α almost everywhere.

In this paper, we use entropy and local entropy of Radon measures (see Definition 4.1) to answer this question in a special case.

Theorem 1.3 *Suppose $M = \mathbb{R}^n$ or M is a compact manifold with non-negative sectional curvature and parallel Ricci curvature. Suppose μ_t^ϵ are measures associated with the Allen–Cahn equations on M. Suppose there exists $\kappa > 0$ such that $\lambda^{(0,T)}(\mu_0^\epsilon) < 2\alpha - \kappa$, and the discrepancy measure $\xi_0^\epsilon \leq 0$ for a sequence of $\epsilon \to 0$. Then μ_t^ϵ has unit density for $t \in (0, T)$. Here $T \in (0, \infty)$ or $T = \infty$.*

In the above theorem, we need a technical assumption that the discrepancy measure is non-positive. This assumption is used by Ilmanen in [11]. Even we do not require the discrepancy measure to be non-positive, we can still use an argument by Soner [15] to show that the limit measure has unit density.

Theorem 1.4 *Suppose $M = \mathbb{R}^n$ or M is a compact manifold with non-negative sectional curvature and parallel Ricci curvature. Suppose μ_t^ϵ are measures associated with the Allen–Cahn equations on M, and μ_t is a limit measure as $\epsilon \to 0$. Suppose there exists $\kappa > 0, \epsilon_0 > 0$ and $\delta > 0$ such that*

the entropy $\lambda(\mu_\delta^\epsilon) < 2\alpha - \kappa$, or the local entropy $\lambda^{(0,T)}(\mu_\delta^\epsilon) < 2\alpha - \kappa$, for $\epsilon < \epsilon_0$. Then the limit measure μ_t has unit density for $t \geqslant \delta$, i.e. the density of μ_t is α almost everywhere.

Bronsard–Stoth [4] constructed examples such that the limits do not have unit density. The limit of the initial data constructed by Bronsard–Stoth has density 2α, this implies that the entropy of their initial data is at least 2α (see Lemma 4.4). Thus our theorem is sharp in some sense. We highlight that our theorems provide long time information, i.e., they hold for arbitrarily large time t.

There is a connection between the unit density problem of the limit of the Allen–Cahn equation and the unit density problem of Brakke flow. In [12, Appendix E], Ilmanen asked that can we show the unit density is preserved for the Brakke flow starting from the boundaries of sets. Ilmanen also constructed an example called "double spoon" to illustrate that in general unit density may fail under Brakke flow, even when the initial measure has unit density. Thus the assumption that the Brakke flow starts from the boundaries of sets is necessary.

It is not too hard to construct a motion of the boundaries of sets by the limit of the Allen–Cahn equation, see [11, Section 1.4]. In particular, the double spoon example can not be obtained by the Allen–Cahn equation. So the unit density conjecture on Brakke flow is true if the unit density problem for the limit of the Allen–Cahn equation is true. As a special case, our result implies that for a Brakke flow, if the entropy of the initial measure is small (less than 2), the unit density is preserved under the flow.

Idea of proof

The proof relies on monotonicity formulas of Allen–Cahn equation (see Theorem 3.3 and Theorem 3.4). If the ambient manifold is $\mathbb{R}^n$, then the monotonicity formula has been proved by Ilmanen [11] and Soner [15], and similar results for the non-scalar case (correspondence to higher codimensional mean curvature flow) was studied by Ambrosio–Soner in [1] and Jerrard–Soner in [13]. We refer the readers to [16] for an overview discussion.

In this paper, we generalize the monotonicity formula to a compact manifold M with non-negative sectional curvature and parallel Ricci curvature. The proof follows the calculations by Ilmanen and Soner, with an extra term appearing because the ambient space M is not $\mathbb{R}^n$. In order to handle this extra term, we need an idea by Hamilton. In [10], Hamilton proved a monotonicity formula for mean curvature flow in a compact manifold M. Similarly, because the ambient space M is not $\mathbb{R}^n$, there is an extra term, and Hamilton used a Harnack inequality [9] to bound this term. In our case, we can also use Hamilton's Harnack inequality to bound the extra term.

The curvature assumption on M is necessary for Hamilton's Harnack inequality. Without the curvature assumption, we can only obtain a Gronwall type inequality rather than a monotonicity formula, and the entropy may become very large as $t \to \infty$. Thus the curvature assumption is necessary if we want to study long time behaviour.

We want to remark that Pisante–Punzo [14] proved a Huisken's type of inequality for parabolic Allen–Cahn equations in a manifold, without a specific curvature assumption. Their inequality was motivated by Ecker's local monotonicity formula, see [7]. Nevertheless, their inequality is a Gronwall type inequality, so it can not be used to study long time behaviours.

2. Preliminaries on Allen–Cahn equation

Let u^ϵ be the unique smooth solutions of the equation

$$\frac{\partial}{\partial t}u^\epsilon = \Delta u^\epsilon - \frac{1}{\epsilon^2}f(u^\epsilon) \quad \text{on } \mathbb{R}^n \times [0, \infty),$$
$$u^\epsilon(\cdot, 0) = u_0^\epsilon(\cdot) \quad \text{on } \mathbb{R}^n \times \{0\}. \tag{2.1}$$

where u_0^ϵ is the initial data. The potential function $F : \mathbb{R} \to \mathbb{R}$ satisfies

$$f = F', \quad F = \frac{1}{2}g^2,$$

where

$$\begin{cases} f(-1) = f(0) = f(1) = 0, \\ f > 0 \text{ on } (-1, 0), \quad f < 0 \text{ on } (0, 1), \\ f'(-1) > 0, \quad f'(1) > 0, \quad f'(0) < 0, \\ g(-1) = g(1) = 0, \quad g > 0 \text{ on } (-1, 1). \end{cases} \tag{2.2}$$

In this paper, we will assume F, f and g are the functions in the standard model

$$F(u) = \frac{1}{2}(1 - u^2)^2, \quad f(u) = 2u(u^2 - 1), \quad g(u) = 1 - u^2. \tag{2.3}$$

For general F, f and g satisfying (2.2), all the discussions in this paper still hold if we assume one extra assumption on F (see (3.11)). We will discuss this assumption in Remark 3.5.

The one-dimensional standing wave q^ϵ is defined to be the solution to

$$q_{xx}^\epsilon - \frac{1}{\epsilon^2}f(q^\epsilon(x)) = 0, \quad x \in \mathbb{R} \tag{2.4}$$

with the assumption that $q_x^\epsilon > 0$, $q^\epsilon(\pm\infty) = \pm 1$ and $q^\epsilon(0) = 0$. We can solve this ODE by solving the first order ODE

$$q_x^\epsilon - \frac{1}{\epsilon}g(q^\epsilon(x)) = 0, \quad x \in \mathbb{R} \tag{2.5}$$

with the assumption that $q_x^\epsilon > 0$, $q^\epsilon(\pm\infty) = \pm 1$ and $q^\epsilon(0) = 0$.

By ODE theory we can solve q^ϵ for any potential function satisfying (2.2). We define α to be its energy

$$\alpha = \int_{-\infty}^{\infty} \frac{\epsilon}{2}(q_x^\epsilon(x))^2 + \frac{1}{\epsilon}F(q^\epsilon(x))dx. \tag{2.6}$$

By change of variable we know that α is independent of ϵ. One can also check that

$$\alpha = \int_{-1}^{1} \sqrt{F(s)/2}ds.$$

In particular, for the model case $F(u) = \frac{1}{2}(1 - u^2)^2$, we have $q^\epsilon(x) = \tanh(x/\epsilon)$, $\alpha = 4/3$.

We define the Radon measure μ_t^ϵ by

$$d\mu_t^\epsilon = \left(\frac{\epsilon}{2}|Du^\epsilon(\cdot, t)|^2 + \frac{1}{\epsilon}F(u^\epsilon(\cdot, t))\right)dx. \tag{2.7}$$

We say μ_t^ϵ is associated with the solution to the Allen–Cahn equation, or for the sake of brevity say μ_t^ϵ is associated with the Allen–Cahn equation.

Ilmanen [11] proved that, under certain technical requirement, there exists $\epsilon_i \to 0$ such that $\mu_t^{\epsilon_i}$ converge to a $(n-1)$-rectifiable Radon measure μ_t for a.e. $t > 0$. Moreover, μ_t is a mean curvature flow in the sense of Brakke.

One motivation of Ilmanen to study the limit behavior of the Allen–Cahn equation was to understand the weak mean curvature flow, see [11, Section 12]. Thus Ilmanen had some technical requirement on the initial data μ_0^ϵ, see [11, p. 423]. However, uniformly bounded total energy $\int d\mu_t^\epsilon$ is enough to give a convergence subsequence, see [15]. Soner [15] even proved that a weaker assumption is enough to yield a convergence subsequence. In our case we only require the initial data have uniform small entropy. This can be viewed as a restriction on (iv) on [11, p. 423], cf. Lemma 4.2.

Next we discuss the *discrepancy measure* ξ_t, which is defined to be

$$d\xi_t^\epsilon = \left(\frac{\epsilon}{2}|Du^\epsilon(\cdot,t)|^2 - \frac{1}{\epsilon}F\big(u^\epsilon(\cdot,t)\big)\right)dx. \tag{2.8}$$

If we define r^ϵ to be the function satisfying

$$u^\epsilon = q^\epsilon(r^\epsilon), \tag{2.9}$$

then r^ϵ satisfies the equation

$$\frac{\partial}{\partial t}r^\epsilon = \Delta r_\epsilon + \frac{2g'}{\epsilon}\big(|Dr^\epsilon|^2 - 1\big). \tag{2.10}$$

We can check

$$|Dr^\epsilon|^2 = \frac{(\epsilon/2)|Du^\epsilon|^2}{(1/\epsilon)F(u^\epsilon)}. \tag{2.11}$$

This implies that $|Dr^\epsilon|^2 - 1$ carries information of ξ_t. In particular, $|Dr^\epsilon| \leqslant 1$ implies that $\xi_t \leqslant 0$.

By using (2.10) and the equation of $|Dr^\epsilon|^2$, Ilmanen and Soner obtained some estimates on $|Dr^\epsilon|^2$. As a result, they proved certain bounds on ξ_t. We refer the readers to [11, Section 4] and [15, Appendix] for detailed discussions. Here we only state their results which we will use later. We also remark that although their results were proved for the Allen–Cahn equations in $\mathbb{R}^n$, the proofs are also valid for the Allen–Cahn equations in a closed manifold (in fact the maximum principle is even easier in this case because the domain is compact).

Theorem 2.1 ([11, Section 4]) *If $\xi_0 \leqslant 0$, then $\xi_t \leqslant 0$ for all $t > 0$.*

Theorem 2.2 ([15, Proposition 4.1]) *Assume $|u^\epsilon(\cdot,0)| \leqslant 1$. Then there exists $0 < \epsilon_0 < 1$ such that for any $\epsilon \leqslant \epsilon_0$ we have*

$$|Dr^\epsilon(x,t)|^2 \leqslant 1 + \frac{2}{\log(1/\epsilon)}\frac{\big(\epsilon r^\epsilon(x,t)\big)^2 + 1}{t}. \tag{2.12}$$

for $(x,t) \in M \times (0,\infty)$.

We remark that in [15], Soner used z^ϵ to denote ϵr^ϵ.

3. Monotonicity formula

Based on Huisken's monotonicity formula [9], Ilmanen [11, Section 3] proved a monotonicity formula for the measures which are associated with the solutions to the Allen–Cahn equation, with a technical assumption on the initial data (see Ilmanen [11, Section 1.4 (i)]). Later Soner [15] removed this assumption. Let us first state the monotonicity formula for Allen–Cahn equations on $\mathbb{R}^n$. Let $\rho_{y,s}$ be the backward heat kernel in $\mathbb{R}^n$ multiplying by $\sqrt{4\pi(s-t)}$:

$$\rho_{y,s}(x,t) = \frac{1}{\left(4\pi(s-t)\right)^{(n-1)/2}} e^{-\frac{|x-y|^2}{4(s-t)}}.$$

Theorem 3.1 ([11, Section 3.3]) *Suppose μ_t^ϵ is the measure associated with an Allen–Cahn equation on $\mathbb{R}^n$. Suppose the discrepancy measure ξ_0 at time 0 is non-positive. Then for every $0 \leqslant t_1 \leqslant t_2 < s$, we have*

$$\int \rho_{y,s}(t_2, x) d\mu_{t_2}^\epsilon(x) \leqslant \int \rho_{y,s}(t_1, x) d\mu_{t_1}^\epsilon(x). \tag{3.1}$$

Theorem 3.2 ([15, Corollary 5.1]) *Suppose μ_t is the limit measure associated with Allen–Cahn equations on $\mathbb{R}^n$. For every $0 < t_1 \leqslant t_2 < s$, we have*

$$\int \rho_{y,s}(t_2, x) d\mu_{t_2}(x) \leqslant \int \rho_{y,s}(t_1, x) d\mu_{t_1}(x). \tag{3.2}$$

In this section, we are going to generalize this monotonicity formulas to the parabolic Allen–Cahn equation on a certain class of manifolds. Suppose M is a manifold. We use $\mathcal{H}(x, y, t)$ to denote the heat kernels on M and we call $\mathcal{H}(x, y, s - t)$ the backward heat kernel if we fix $s > 0$. We have

$$\partial_t \mathcal{H}(x, y, t) = \Delta_x \mathcal{H}(x, y, t).$$

Moreover, we define

$$\rho_{y,s}(t, x) = \sqrt{4\pi(s-t)} \mathcal{H}(x, y, s - t). \tag{3.3}$$

Note that if $M = \mathbb{R}^n$, then this definition coincides with the definition of $\rho_{y,s}$ in $\mathbb{R}^n$ as in the beginning of this section.

In the rest of this section we prove the following monotonicity formulas.

Theorem 3.3 *Suppose μ_t^ϵ is the measure associated with Allen–Cahn equations on M, where M is a compact manifold with parallel Ricci curvature and non-negative sectional curvature. Suppose the discrepancy measure ξ_0 at time 0 is non-positive. For every $0 \leqslant t_1 \leqslant t_2 < s$, we have*

$$\int \rho_{y,s}(t_2, x) d\mu_{t_2}^\epsilon(x) \leqslant \int \rho_{y,s}(t_1, x) d\mu_{t_1}^\epsilon(x). \tag{3.4}$$

Theorem 3.4 *Suppose μ_t is the limit measure associated to Allen–Cahn equations on M, where M is a compact manifold with parallel Ricci curvature and non-negative sectional curvature. For every $0 < t_1 \leqslant t_2 < s$, we have*

$$\int \rho_{y,s}(t_2, x) d\mu_{t_2}(x) \leqslant \int \rho_{y,s}(t_1, x) d\mu_{t_1}(x). \tag{3.5}$$

Proof of Theorem 3.3 *and Theorem* 3.4. The proof follows the calculations of [11, 15], and an idea of Hamilton in [10]. In the following, for two matrices A and B, $A : B$ denotes the inner product of them as matrices. We also define the unit normal vector to the level sets by $v = \frac{Du^\epsilon}{|Du^\epsilon|}$. Note that v is not well-defined at the points where $Du^\epsilon = 0$, but it does not matter the later calculations, and we can define v to be any unit vector at the point where $Du = 0$.

Recall the calculations in [11, Section 3.2]: for $\phi \in C^2(M, R^+)$,

$$\frac{d}{dt} \int \phi d\mu_t^\epsilon = -\epsilon \int \phi \left(-\Delta u + \frac{1}{\epsilon^2} f(u) - \frac{Du \cdot D\phi}{\phi} \right)^2 dx$$
$$+ \int \left(\Delta\phi + \frac{\partial}{\partial t}\phi \right) d\mu_t^\epsilon + \int \left(-v \otimes v : D^2\phi + \frac{(v \cdot D\phi)^2}{\phi} \right) (d\mu_t^\epsilon + d\xi_t^\epsilon)$$
$$\leqslant \int \left(\Delta\phi + \frac{\partial}{\partial t}\phi \right) d\mu_t^\epsilon + \int \left(-v \otimes v : D^2\phi + \frac{(v \cdot D\phi)^2}{\phi} \right) (d\mu_t^\epsilon + d\xi_t^\epsilon). \quad (3.6)$$

Now we insert $\rho_{y,s}(x, t)$ as the test function ϕ. Note that

$$\frac{\partial}{\partial t} \rho_{y,s}(x, t) = -\Delta\rho_{y,s}(x, t) - \frac{1}{2(s - t)} \rho_{y,s}(x, t).$$

Moreover, by Hamilton's Hanarck inequality (see [9, Corollary 4.4]),

$$v \otimes v : \left(D^2\mathcal{H} - \frac{D\mathcal{H} \otimes D\mathcal{H}}{\mathcal{H}} + \frac{1}{2(s - t)} \mathcal{H}g \right) \geqslant 0,$$

while $\mathcal{H} = \mathcal{H}(x, y, s - t)$ and g is the metric tensor on M. These imply

$$\frac{d}{dt} \int \rho_{y,s} d\mu_t^\epsilon \leqslant \int \left(\Delta\rho_{y,s} + \frac{\partial}{\partial t} \rho_{y,s} \right) d\mu_t^\epsilon + \int \frac{1}{2(s - t)} \rho_{y,s} (d\mu_t^\epsilon + d\xi_t^\epsilon)$$
$$\leqslant \int \frac{1}{2(s - t)} \rho_{y,s} d\xi_t^\epsilon. \quad (3.7)$$

If $\xi_0 \leqslant 0$, then Theorem 2.1 shows that $\xi_t \leqslant 0$, so we complete the proof of Theorem 3.3. In the following we are going to prove Theorem 3.4. Namely, even the discrepancy measure is not assumed to be non-positive, the inequality

$$\int \rho_{y,s}(t_2, x) d\mu_{t_2}(x) \leqslant \int \rho_{y,s}(t_1, x) d\mu_{t_1}(x) \quad (3.8)$$

holds for the limit measure μ_t.

By Theorem 2.2, (3.7) becomes

$$\frac{d}{dt} \int \rho_{y,s} d\mu_t^\epsilon \leqslant \int \frac{1}{2(s - t)} \frac{1}{\epsilon} \frac{2}{\log(1/\epsilon)} \frac{(\epsilon r^\epsilon)^2 + 1}{t} F(u^\epsilon) \rho_{y,s} dx. \quad (3.9)$$

We note that

$$\frac{1}{\epsilon} F(u^\epsilon) dx \leqslant d\mu_t^\epsilon \quad (3.10)$$

and

$$x^2 F\left(q^\epsilon(x) \right) \leqslant 4, \quad \forall x \in \mathbb{R}. \quad (3.11)$$

So we have

$$\frac{d}{dt}\int \rho_{y,s}d\mu_t^\epsilon \leqslant \frac{1}{(s-t)\log(1/\epsilon)t}\int \rho_{y,s}d\mu_t^\epsilon + \frac{4\epsilon}{(s-t)\log(1/\epsilon)t}\int \rho_{y,s}dx, \qquad (3.12)$$

and when $t \geqslant t_1$,

$$\frac{d}{dt}\int \rho_{y,s}d\mu_t^\epsilon \leqslant \frac{1}{(s-t)\log(1/\epsilon)t_1}\int \rho_{y,s}d\mu_t^\epsilon + \frac{4\epsilon}{(s-t)\log(1/\epsilon)t_1}\int \rho_{y,s}dx. \qquad (3.13)$$

We also note that

$$\int \rho_{y,s}(t,x)dx = \sqrt{4\pi(s-t)}\int \mathcal{H}(x,y,s-t)dx = \sqrt{4\pi(s-t)}.$$

In conclusion, we obtain the following differential inequality

$$\frac{d}{dt}\int \rho_{y,s}d\mu_t^\epsilon \leqslant \frac{1}{(s-t)\log(1/\epsilon)t_1}\left(\int \rho_{y,s}d\mu_t^\epsilon + 4\epsilon\sqrt{4\pi(s-t)}\right). \qquad (3.14)$$

Then we obtain a Gronwall type inequality for $t \geqslant t_1$

$$\frac{d}{dt}\left(\left(\frac{s-t}{s-t_1}\right)^K \int \rho_{y,s}d\mu_t^\epsilon - 8\epsilon\sqrt{\pi}K\int_{t_1}^t (s-b)^{-1/2}\left(\frac{s-b}{s-t_1}\right)^K db\right) \leqslant 0, \qquad (3.15)$$

where $K = (\log(1/\epsilon)t_1)^{-1}$. Therefore,

$$\int \rho_{y,s}d\mu_{t_2}^\epsilon \leqslant \left(\frac{s-t_1}{s-t_2}\right)^K \int \rho_{y,s}d\mu_{t_1}^\epsilon + 8\epsilon\sqrt{\pi}K\int_{t_1}^{t_2} (s-b)^{-1/2}\left(\frac{s-b}{s-t_2}\right)^K db. \qquad (3.16)$$

By letting $\epsilon \to 0$, we obtain the desired monotonicity formula in Theorem 3.4. □

REMARK 3.5 In the proof we assume

$$F(u) = \frac{1}{2}(1-u^2)^2, \quad f(u) = 2u(u^2-1), \quad g(u) = 1 - u^2.$$

For a general triple F, f and g satisfying (2.2), the proof still works if (3.11) holds (the right-hand side can be replaced by any fixed positive number).

4. Entropy of parabolic Allen–Cahn equation

We follow the idea of Colding–Minicozzi [6] to define the entropy of a Radon measure, cf. [17].

DEFINITION 4.1 Suppose $M = \mathbb{R}^n$. Given a Radon measure μ in M, we define the *entropy* $\lambda(\mu)$ to be

$$\lambda(\mu) = \sup_{(y,s)\in M\times(0,\infty)} \int \rho_{y,s}(x,0)d\mu(x).$$

Entropy is a quantity that characterizes the measure from all scales. For example, the following lemma indicates that the entropy is equivalent to the volume growth bound of a measure:

Lemma 4.2 *Suppose $M = \mathbb{R}^n$. There exists $C > 0$ only depending on the geometry of M such that for any Radon measure μ with bounded entropy,*

$$C^{-1} \sup_{x \in M, R > 0} \frac{\mu(B_R(x))}{R^{n-1}} \leq \lambda(\mu) \leq C \sup_{x \in M, R > 0} \frac{\mu(B_R(x))}{R^{n-1}}. \tag{4.1}$$

Proof of Lemma 4.2. On one hand, for any $x \in \mathbb{R}^n$ and $R > 0$, we have

$$\lambda(\mu) \geq \frac{1}{(4\pi t)^{n/2}} \int e^{\frac{-|y-x|^2}{4t}} \chi_{B_R(x)} d\mu(y) \geq \frac{1}{(4\pi t)^{n/2}} e^{-R^2/4t} \mu(B_R(x)).$$

Then by choosing $t = R^2$ we have $\frac{\mu(B_R(x))}{R^n} \leq C\lambda(\mu)$, where C is a constant.

On the other hand, we only need to prove $\int e^{-|x|^2} d\mu(x) \leq C \sup_{x \in \mathbb{R}^n, R > 0} \frac{\mu(B_R(x))}{R^{n-1}}$ to conclude the second inequality. Let $\chi_{B_r(x)}$ be the characteristic function of the set $B_r(x)$.

$$\int e^{-|x|^2} d\mu(x) \leq \sum_{y \in \mathbb{Z}^n} \int e^{-|x|^2} \chi_{B_2(y)} d\mu \leq C \sum_{y \in \mathbb{Z}^n} e^{-|y|^2} \mu(B_2(y))$$

$$\leq C \sup_{x \in \mathbb{R}^{n+1}, R > 0} \frac{\mu(B_R(x))}{R^{n-1}} \sum_{y \in \mathbb{Z}^n} e^{-|y|^2} \leq C \sup_{x \in \mathbb{R}^{n+1}, R > 0} \frac{\mu(B_R(x))}{R^{n-1}}. \tag{4.2}$$

Here C varies from line to line, but it is always a constant only depending on n. $\mathbb{Z}^n$ consists of all the integer points in $\mathbb{R}^n$. Then we conclude this Lemma. $\square$

If we only take the supremum over a subset of $\mathbb{R}^n \times (0, \infty)$, we get a localized version of entropy, see [18].

DEFINITION 4.3 Given a Radon measure μ and $U \subset M$, $I \subset (0, \infty)$, we define the *local entropy* $\lambda_U^I(\mu)$ to be

$$\lambda_U^I(\mu) = \sup_{(y,s) \in U \times I} \int \rho_{y,s}(x, 0) d\mu(x).$$

We will omit the subscription U if $U = M$.

If we only study the local property, the entropy/local entropy actually gives a bound on the density. We define the $(n - 1)$-dimensional density $\theta(x)$ by

$$\theta(x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{\omega_{n-1} r^{n-1}} \tag{4.3}$$

whenever the limit exists. Here ω_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$.

Lemma 4.4 *Suppose $M = \mathbb{R}^n$ or M is a compact manifold. Suppose μ is an integral $(n - 1)$-rectifiable Radon measure in M, then for $T \in (0, \infty)$ or $T = \infty$,*

$$\theta(x) \leq \lambda^{(0,T)}(\mu). \tag{4.4}$$

Proof. First let us consider the case that $M = \mathbb{R}^n$. Since μ is integral rectifiable, $\theta(x)$ just counts the multiplicity of the approximate tangent plane through x. Note the integral of the backward heat kernel on a hyperplane is 1. Thus

$$\theta(x) = \lim_{s \to 0} \int \frac{1}{(4\pi s)^{(n-1)/2}} e^{-\frac{|y-x|^2}{4s}} d\mu(y) \leq \lambda^{(0,T)}(\mu).$$

If M is a compact manifold, then the short time expansion of the heat kernels together with a similar argument as the case $M = \mathbb{R}^n$ show the same inequality. $\square$

Now we focus on the measures associated with the parabolic Allen–Cahn equations. Taking supremum of the monotonicity formula among all (y, s) leads to the monotonicity of entropy and local entropy:

Corollary 4.5 *Suppose $M = \mathbb{R}^n$ or M is a compact manifold with non-negative sectional curvature and parallel Ricci curvature. Suppose μ_t^ϵ is the measure associated with the parabolic Allen–Cahn equation on M. Suppose $\xi^\epsilon 0 \leqslant 0$. For every $0 \leqslant t_1 \leqslant t_2$, we have*

$$\lambda\left(\mu_{t_2}^\epsilon\right) \leqslant \lambda\left(\mu_{t_1}^\epsilon\right). \tag{4.5}$$

More generally, given $T > 0$, we have

$$\lambda^{(0,T)}\left(\mu_{t_2}^\epsilon\right) \leqslant \lambda^{(0,T+(t_2-t_1))}\left(\mu_{t_1}^\epsilon\right). \tag{4.6}$$

Corollary 4.6 *Suppose $M = \mathbb{R}^n$. Suppose μ_t^ϵ is the measure associated with the parabolic Allen–Cahn equation on M, and μ_t is the limit as $\epsilon \to 0$. For every $0 < t_1 \leqslant t_2$, we have*

$$\lambda(\mu_{t_2}) \leqslant \lambda(\mu_{t_1}). \tag{4.7}$$

More generally, given $T > 0$, we have

$$\lambda^{(0,T)}(\mu_{t_2}) \leqslant \lambda^{(0,T+(t_2-t_1))}(\mu_{t_1}). \tag{4.8}$$

Proof. Note $\rho_{y,s}(t, x) = \rho_{y,s-t}(0, x)$. Taking supremum among all $(y, s) \in \mathbb{R}^n \times (t_2, \infty)$ on the left-hand side of (3.1) and (3.2) gives us the monotonicity of entropy, and taking supremum among all $(y, s) \in \mathbb{R}^n \times (t_2, t_2 + T)$ on the left-hand side of (3.1) and (3.2) gives us the monotonicity of local entropy. $\square$

Together with Lemma 4.2, Corollary 4.5 and Corollary 4.6 imply the following uniform volume growth bound.

Corollary 4.7 *Suppose $M = \mathbb{R}^n$. Suppose μ_t^ϵ is the measure associated with the parabolic Allen–Cahn equation on M, and μ_t is the limit as $\epsilon \to 0$. Then*

$$\sup_{x \in M, R > 0} \frac{\mu_t\left(B_R(x)\right)}{R^{n-1}} \leqslant C \sup_{x \in M, R > 0} \frac{\mu_\delta\left(B_R(x)\right)}{R^{n-1}}, \quad \forall t \geqslant \delta > 0.$$

If we further assume $\xi_0^\epsilon \leqslant 0$, then

$$\sup_{x \in M, R > 0} \frac{\mu_t^\epsilon\left(B_R(x)\right)}{R^{n-1}} \leqslant C \sup_{x \in M, R > 0} \frac{\mu_0^\epsilon\left(B_R(x)\right)}{R^{n-1}}.$$

Here C is a constant depending on M but does not depend on time t.

5. Unit density of limit measure

Let μ_t be the limit of the measures which is associated with the parabolic Allen–Cahn equation. Recall that

$$\alpha = \int_{-1}^{1} \sqrt{F(s)/2}\,ds,$$

and we say μ_t has unit density if $\theta(x) = \alpha$ for almost all time and almost all x in the support of μ_t. Note α appears because of the nature of the Allen–Cahn equation. In this section we prove the main theorem.

Proof of Theorem 1.3. By Corollary 4.6, $\lambda^{(0,T-t)}(\mu_t) \leqslant \lambda^{(0,T)}(\mu_0) < 2\alpha$. Then Lemma 4.4 implies that the density of μ_t is strictly less than 2α. Tonegawa [19] proved that $\alpha^{-1}\mu_t$ is integral, thus μ_t has density α, i.e., μ_t has unit density. □

Proof of Theorem 1.4. By the lower semi-continuity of entropy and local entropy, $\lambda^{(0,T)}(\mu_0^\epsilon) < 2\alpha - \kappa$ implies that $\lambda^{(0,T)}(\mu_0) \leqslant 2\alpha - \kappa < 2\alpha$. Then Theorem 1.3 implies that μ_t has unit density when $t \in [0, T)$. □

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