

On differentiability of the membrane-mediated mechanical interaction energy of discrete–continuum membrane–particle models

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We consider a discrete–continuum model of a biomembrane with embedded particles. While the membrane is represented by a continuous surface, embedded particles are described by rigid discrete objects which are free to move and rotate in lateral direction. For the membrane we consider a linearized Canham–Helfrich energy functional and height and slope boundary conditions imposed on the particle boundaries resulting in a coupled minimization problem for the membrane shape and particle positions.

When considering the energetically optimal membrane shape for each particle position we obtain a reduced energy functional that models the implicitly given interaction potential for the membrane-mediated mechanical particle–particle interactions. We show that this interaction potential is differentiable with respect to the particle positions and orientations. Furthermore, we derive a fully practical representation of the derivative only in terms of well defined derivatives of the membrane. This opens the door for the application of minimization algorithms for the computation of minimizers of the coupled system and for further investigation of the interaction potential of membrane-mediated mechanical particle–particle interaction.

The results are illustrated with numerical examples comparing the explicit derivative formula with difference quotient approximations. We furthermore demonstrate the application of the derived formula to implement a gradient flow for the approximation of optimal particle configurations.

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1. Introduction

Particles embedded into membranes are commonly expected to be crucial for various biological processes involving the shaping of the membrane. Examples of such particles are transmembrane and bar-domain proteins. For example the latter are conjectured to play an important role in early stages of clathrin-mediated endocytosis [15]. The reason for the importance of proteins for membrane shaping is, that they may induce local membrane deformations in their vicinity. Since the membrane itself consists of a lipid bilayer, which – in the lateral direction – can be seen as a fluid, such particles are able to move easily within the membrane. As a consequence the local particle-induced membrane deformation implicitly induces a membrane-mediated mechanical

particle–particle interaction. Driven by the corresponding interaction potential, particles may cluster and form energetically preferable patterns.

This observation and the fact that membrane-shaping particles seem to be crucial for biological membrane functions stimulated various research directions on such particles and membrane-mediated particle–particle interactions. A common approach is based on atomistic or coarse-grained molecular dynamics (MD) models of the membrane. To overcome the severe length scale limitations of such approaches, alternative modeling techniques representing the membrane as continuous surface that minimizes an elastic energy have been developed [3, 16]. Using such an models it was shown that there are long-range interactions between particles that are predominantly membrane-mediated [10]. Since then further work has been done to investigate particle interactions within elasticity models. Typically a flatness assumption on the membrane is made and particles are modeled as circular disks or points and their coupling to the membrane is prescribed by radially symmetric boundary conditions for which the interaction energy can either be computed analytically or approximately by asymptotic expansion [6, 9, 19, 24, 26]. However, it turns out that the shape of particles has a significant impact on their interaction [20]. More recent work is also interested in numerical computations with pattern formation of many non-circular particles [14, 18], and attention was also given to situations where the flatness assumption is no longer fulfilled [7, 22, 23]. Also more elaborate models for proteins in continuum elastic models have recently been considered in [2].

To understand the pattern formation of particles it is desirable to quantify the forces exerted on the particles by the membrane in a framework that is as widely applicable as possible. General results in this direction have been obtained based on arguments from differential geometry [5]. The methods derived therein give insight into the qualitative behavior of particle interactions, but – to the best of the authors’ knowledge – they have not yet been made fully available for numerical computations.

In this paper we consider a discrete–continuum model where the membrane is modeled as a continuous graph minimizing a linearized Canham–Helfrich bending energy and where an arbitrary amount of particles are embedded into the membrane. These particles are modeled as discrete entities which are coupled to the membrane through certain boundary conditions. As particles are free to move in the membrane, those boundary conditions depend on each particle’s position. Consequently, the overall system’s energy given fixed boundary conditions and an optimal membrane shape can be written as a function of the particle positions, which we call the *interaction energy*.

In this setting we propose a method to prove differentiability of the interaction energy for arbitrary shapes and boundary conditions. Furthermore, we derive an expression for the derivative that can be evaluated numerically within a finite element scheme and where the evaluation error is bounded in terms of the discretization error of the finite element approximation. The proof is based on an application of the implicit function theorem and ideas from shape calculus [4, 17]. As such, the method is rather general and hence it naturally extends to a wider class of models that for example use nonlinear elastic energies or certain other membrane–particle couplings.

In the following we give an outline of this paper. In Section 2 we introduce the Canham–Helfrich energy in Monge–gauge as a model for the membrane and parametric boundary conditions for the coupling of the particles. Section 3 is then concerned with further mathematical notation that we use in order to define the interaction energy. There we also reformulate the parametric boundary conditions as linear constraints by using trace operators and appropriate projection operators. Afterwards, in Section 4, we prove differentiability of the interaction energy and

derive a numerically feasible expression for the gradient. Finally, Section 5 shows some example computations that illustrate that the derived formula can indeed be applied in a numerical scheme.

2. Membrane and particle model

The Canham–Helfrich model of a membrane given as two-dimensional surface $\mathfrak{M} \subset \mathbb{R}^3$ is based on the elastic bending energy

$$J_{\text{CH}}(\mathfrak{M}) := \int_{\mathfrak{M}} \frac{\kappa}{2} (H - c_0)^2 + \kappa_G K + \sigma \, d\mathfrak{M}$$

where H and K denote the mean and Gaussian curvature with corresponding bending rigidities κ and κ_G , σ is the surface tension, and c_0 is the so called spontaneous curvature. Considering membranes with fixed topology and fixed geodesic curvature the Gaussian curvature term is a constant and can be dropped. For simplicity we will also restrict our considerations to the case of vanishing spontaneous curvature $c_0 = 0$.

Assuming that the membrane is given by a graph $\mathfrak{M} = \{(x, u(x)) \mid x \in \Omega\}$ which is almost flat in the sense that $|\nabla u| \ll 1$ we consider the well-established *linearized Canham–Helfrich model in Monge-gauge*: Given a two-dimensional reference domain $\Omega \subseteq \mathbb{R}^2$ and a function $u \in H^2(\Omega)$, the membrane shape is described by the graph of u . The bending energy of this membrane is approximated by

$$J(\Omega, u) := \frac{1}{2} \int_{\Omega} \kappa (\Delta u(x))^2 + \sigma \|\nabla u(x)\|^2 \, dx$$

where $\kappa > 0$ and $\sigma \geq 0$ denote the *bending rigidity* and the *surface tension*, respectively. It is noted that this corresponds to the geometric linearization of the full nonlinear Canham–Helfrich energy J_{CH} near a flat membrane with $\nabla u = 0$. For more details on the model we refer to [8] and the references therein.

In absence of particles interacting with the membrane, this model determines the stationary shape of the membrane solely by minimizing the energy J . In the following we explain how the embedded particles are coupled to the membrane, before we state the model problem that is central to this paper.

For simplicity we first consider a single *transmembrane protein* that interacts with the membrane. Such a protein is not merely connected to one side of the membrane surface but rather is included in the membrane. The reason for this situation is that the protein has a *hydrophobic belt* which is shielded from the surrounding water by the membrane lipids. As a consequence the membrane preferably connects to the belt which in turn deforms the membrane according to its shape. We assume that the particle does not undergo any deformation and denote its rigid three-dimensional shape $\mathfrak{B} \subseteq \mathbb{R}^3$.

In the following we will suppose that the hydrophobic belt is approximated by a curve \mathfrak{G} . We assume that \mathfrak{G} is a simple closed curve that can be parameterized over the twodimensional Euclidean plane. This means that there exist a simple closed curve $\Gamma = \partial B \subseteq \mathbb{R}^2$ and a continuous function $g_0: \Gamma \rightarrow \mathbb{R}$ such that $\mathfrak{G} = \{(x, g_0(x)) \mid x \in \Gamma\}$. This gives rise to the boundary condition $u|_{\Gamma} = g_0$, which models that the membrane is connected to the particle at the interface \mathfrak{G} . The fact that the membrane is connected perpendicular to the hydrophobic region is modeled by additionally assuming that it is attached to \mathfrak{G} with a fixed slope. To this end, we impose the additional boundary

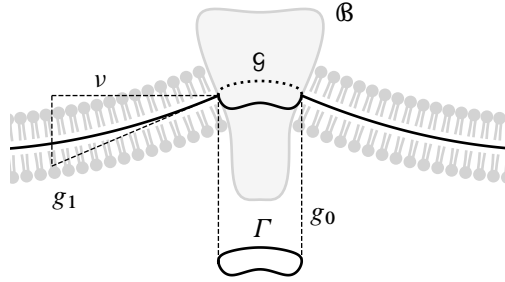


FIG. 2.1. Side view of a particle with hydrophobic belt \mathcal{G} , its projection to the plane Γ , and the height contour g_0 and slope g_1 prescribed at Γ

condition $\partial_\nu u|_\Gamma = g_1$ with a function $g_1: \Gamma \rightarrow \mathbb{R}$ describing the slope. Here ν is an oriented unit normal on Γ and $\partial_\nu u|_\Gamma$ denotes the normal derivative of u on Γ . The situation is illustrated in Figure 2.1 showing the hydrophobic belt \mathcal{G} , its projection to the plane Γ , and the boundary conditions on Γ .

Those constraints do not yet account for the fact that the particle is in principle free to move in space. To this end we parameterize the current position of the particle using translations z_j along the x_j -axes and rotations α_j around the x_j -axes. More precisely, let \mathcal{B}^0 be a reference state of the particle that is centered in the origin with hydrophobic belt $\mathcal{G}^0 = \{(\hat{x}, g_0^0(\hat{x})) \mid \hat{x} \in \Gamma^0\}$ for some Γ^0 and let $R_j(\alpha_j) \in \mathbb{R}^{3 \times 3}$ be the α_j -rotation matrix around the x_j -axis. Then for $\hat{y} \in \mathcal{B}^0$ the rigid body transformation $\Psi(z, \alpha) : \mathcal{B}^0 \rightarrow \mathbb{R}^3$ corresponding to (z, α) is given by

$$\Psi(z, \alpha)(\hat{y}) := \Psi(z, \alpha; \hat{y}) := R_1(\alpha_1)R_2(\alpha_2)R_3(\alpha_3)\hat{y} + z$$

and we define the parameterized particle as

$$\mathcal{B} = \mathcal{B}(z, \alpha) := \{\Psi(z, \alpha)(\hat{y}) \mid \hat{y} \in \mathcal{B}^0\}.$$

For the hydrophobic belt straight forward application of the parameterization would lead to

$$\mathcal{G} = \mathcal{G}(z, \alpha) := \{\Psi(z, \alpha)(\hat{y}) \mid \hat{y} \in \mathcal{G}^0\}$$

and the corresponding projection $\Gamma = \{(y_1, y_2) \mid y \in \mathcal{G}\}$ to the plane. Given the particle slope $g_1^0 : \Gamma^0 \rightarrow \mathbb{R}$ on the reference curve, the membrane–particle constraints introduced above could be imposed to the parametrized particle by first transforming the membrane given by the graph of u according to $\Psi(z, \alpha)^{-1}$ and then imposing constraints

$$\hat{u}|_{\Gamma^*} = g_0^0, \quad \partial_{\hat{\nu}} \hat{u}|_{\Gamma^0} = g_1^0$$

on the transformed graph of (if existing) \hat{u} over the reference curve Γ^0 with unit normal $\hat{\nu}$. However, this approach suffers from several drawbacks related to the rotations $R_1(\alpha_1)$ and $R_2(\alpha_2)$. First, they can easily exceed the regime where \mathcal{G} can be written as a graph over Γ and where the transformation of the graph of u is a graph of some \hat{u} . Second, the shape of the projected curve Γ differs from Γ^0 and normals of Γ^0 do not transform to normals of Γ . To avoid these complications, we will simplify the transformation assuming small rotational angles:

On the one hand we make the assumption that the reference set \mathfrak{B}^0 is oriented in such a way that the belt \mathcal{G}^0 is ‘almost flat’, by which we mean that $\max_{\hat{x} \in \Gamma^0} |g_0^0(\hat{x})| / \|\hat{x}\|$ is small. On the other hand we assume that the angles α_1 and α_2 are small, such that the parameterized belt is still ‘almost flat’. Based on these assumptions we first replace $R_1(\alpha_1)$ and $R_2(\alpha_2)$ by first order Taylor expansions $\tilde{R}_1(\alpha_1)$ and $\tilde{R}_2(\alpha_2)$ near $\alpha_1 = 0 = \alpha_2$ leading to

$$R_1(\alpha_1)R_2(\alpha_2)y \approx \tilde{R}_1(\alpha_1)\tilde{R}_2(\alpha_2)y = \begin{pmatrix} 1 & 0 & -\alpha_2 \\ -\alpha_1\alpha_2 & 1 & -\alpha_1 \\ \alpha_2 & \alpha_1 & 1 \end{pmatrix} y = \begin{pmatrix} y_1 - \alpha_2 y_3 \\ y_2 - \alpha_1 \alpha_2 y_1 - \alpha_1 y_3 \\ y_3 + \alpha_2 y_1 + \alpha_1 y_2 \end{pmatrix}.$$

Using $y = R_3(\alpha_3)\hat{y}$ for $\hat{y} = (\hat{x}, g_0^0(\hat{x})) \in \mathcal{G}^0$ and dropping higher order terms in α_1, α_2 , and $y_3 = \hat{y}_3 = g_0^0(\hat{x})$ finally leads to the approximate transformation

$$\tilde{\Psi}(z, \alpha)(\hat{y}) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_2 & \alpha_1 & 1 \end{pmatrix} R_3(\alpha_3)\hat{y} + z \approx \Psi(z, \alpha)(\hat{y})$$

and the corresponding parametrized particle belt

$$\mathcal{G}(z, \alpha) := \{\tilde{\Psi}(z, \alpha)(\hat{y}) \mid \hat{y} \in \mathcal{G}^0\}.$$

In the following we use the shortcut $p = (z_1, z_2, \alpha_3)$ for the in-plane translation and rotation. Introducing

$$R(\alpha) := \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \quad (2.1)$$

the in-plane transformation induced by p is given by the rigid body motion

$$\varphi(p)(\hat{x}) := \varphi(p; \hat{x}) := R(\alpha_3)\hat{x} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad (2.2)$$

with its inverse

$$\varphi^{-1}(p; x) := \varphi(p)^{-1}(x) := R(-\alpha_3) \left(x - \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right).$$

Utilizing φ , we can write $\tilde{\Psi}$ on \mathcal{G}^0 as

$$\tilde{\Psi}(z, \alpha)(\hat{x}, g_0^0(\hat{x})) = \begin{pmatrix} \varphi(p)(\hat{x}) \\ g_0^0(\hat{x}) + z_3 + (\alpha_2, \alpha_1)R(\alpha_3)(\hat{x}) \end{pmatrix}.$$

Thus $\mathcal{G}(z, \alpha)$ can be written as a graph over its projection to the plane

$$\Gamma(p) := \Gamma(z, \alpha) := \{(y_1, y_2) \mid y \in \mathcal{G}(z, \alpha)\} = \{\varphi(p)\hat{x} \mid \hat{x} \in \Gamma^0\} = \varphi(p; \Gamma^0).$$

Figure 2.2 illustrates the reference particle curve Γ^0 and the transformed particle curve $\Gamma(p) = \Gamma(z_1, z_2, \alpha_3)$. The figure shows a top view that complements the side view of Figure 2.1.

The zeroth and first order boundary condition for the parameterized particle are now obtained by pulling back the graph of u using the approximate transformation $\tilde{\Psi}(z, \alpha)^{-1}$ and then imposing

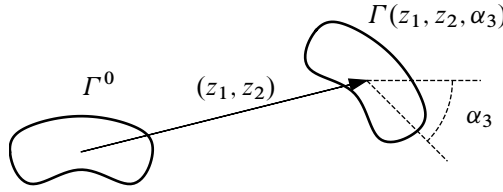


FIG. 2.2. Top view of reference particle Γ^0 and moved particle $\Gamma(z_1, z_2, \alpha_3)$

the conditions for g_0^0 and g_1^0 on Γ^0 . Noting that $\Gamma^0 = \varphi^{-1}(p; \Gamma(p))$ and defining $g_k(p; x) := g_k^0(\varphi^{-1}(p; x))$ the boundary conditions then read

$$\begin{aligned} u(x) &= g_0(p; x) + \alpha_2(x_1 - z_1) + \alpha_1(x_2 - z_2) + z_3 && \text{on } \Gamma(z_1, z_2, \alpha_3), \\ \partial_\nu u(x) &= g_1(p; x) + \alpha_2\nu_1 + \alpha_1\nu_2 && \text{on } \Gamma(z_1, z_2, \alpha_3), \end{aligned}$$

where ν_1 and ν_2 denote the components of an oriented unit normal on $\Gamma(p)$.

The dependence of these boundary conditions on the parameters (z, α) can be split naturally into the nonlinear dependence on the in-plane position $p = (z_1, z_2, \alpha_3)$ and the linear dependence on variations of the height and tilt given by $(z_3, \alpha_1, \alpha_2)$. We will exploit the latter by minimizing the membrane energy with respect to u and $(z_3, \alpha_1, \alpha_2)$ simultaneously. It can easily be seen, that this can be written equivalently by factoring out $(z_3, \alpha_1, \alpha_2)$ from the boundary condition in the sense that they are only enforced up to an arbitrary selection of $(z_3, \alpha_1, \alpha_2)$. Thus the boundary conditions become *parametric boundary conditions*:

$$\exists \gamma \in \mathbb{R}^3: \begin{cases} u|_{\Gamma(p)}(x) = g_0(p; x) + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3, \\ \partial_\nu u|_{\Gamma(p)}(x) = g_1(p; x) + \gamma_1 \nu_1 + \gamma_2 \nu_2, \end{cases} \quad \forall x \in \Gamma(p). \quad (2.3)$$

Notice that, z_3, α_1 , and α_2 are no longer relevant for describing the particle’s position and are now rather implicit to the boundary conditions. A particle in the model is then solely determined by its reference curve Γ^0 , its reference boundary conditions g_0^0, g_1^0 , and its position $p = (z_1, z_2, \alpha_3)$ in the Euclidean plane.

Parameterized boundary conditions for particle–membrane coupling with variable height and tilt have first been considered in [8]. They can be interpreted physically as the particles being tied only to the membrane such that they can freely change their height and tilt angle with the membrane. In contrast, the lateral motion of particles is an independent process. This is motivated by the fact, that the membrane has a bending rigidity in normal direction but behaves like a viscous fluid in tangential direction.

In the case where multiple particles are present we state the constraints analogously by imposing the above constraints for each particle separately.

3. Interaction energy

Before we can formulate the final model problem, we need to introduce some notation. We also augment the parametric boundary conditions by Dirichlet boundary conditions on the outer boundary $\partial\Omega$ which reflects the fact that we consider a small almost flat patch of the full membrane. For a discussion of other possible boundary conditions we refer to [8]. Particle–membrane coupling

and outer boundary conditions will both be formulated using simple linear operators in the following.

We consider N particles with reference curves Γ_i^0 , height profiles g_{i0}^0 and slopes g_{i1}^0 . Given a *particle configuration* $\mathbf{p} = (\mathbf{p}_i)_{i=1,\dots,N} \in \mathbb{R}^{N \times 3}$ we define the curves

$$\Gamma_i(\mathbf{p}_i) := \{\varphi(\mathbf{p}_i; y) \mid y \in \Gamma_i^0\}$$

and $\Gamma_0 := \Gamma_0^0 := \partial\Omega$. Their union is $\Gamma(\mathbf{p}) := \bigcup_{i=0}^N \Gamma_i(\mathbf{p}_i)$ where we use $\mathbf{p}_0 := 0$ and $\Gamma_0(\mathbf{p}_0) := \Gamma_0$ for the sake of a consistent notation. For $i > 0$ we denote the set enclosed by $\Gamma_i(\mathbf{p}_i)$ by $B_i(\mathbf{p}_i)$ and the union of these is denoted by $B(\mathbf{p}) := \bigcup_{i=1}^N B_i(\mathbf{p}_i)$. We define the \mathbf{p} -dependent reference domain as $\Omega(\mathbf{p}) := \Omega \setminus B(\mathbf{p})$.

Based on this we define the interior of the set of *feasible particle configurations* as

$$\Lambda^\circ := \left\{ \mathbf{p} \in \mathbb{R}^{N \times 3} \mid \forall i, j \in \{1, \dots, N\}, i \neq j: \Gamma_i(\mathbf{p}_i) \subseteq \Omega^\circ \wedge \overline{B_i(\mathbf{p}_i)} \cap \overline{B_j(\mathbf{p}_j)} = \emptyset \right\},$$

and set $\Lambda := \overline{\Lambda^\circ}$.

Now suppose $\mathbf{p} \in \Lambda$. Then we define the trace operators

$$\begin{aligned} T_i(\mathbf{p}): H^2(\Omega(\mathbf{p})) &\rightarrow H^{3/2}(\Gamma_i^0) \times H^{1/2}(\Gamma_i^0) \\ u &\mapsto \begin{pmatrix} u|_{\Gamma_i(\mathbf{p}_i)} \circ \varphi(\mathbf{p}_i) \\ (\partial_\nu u|_{\Gamma_i(\mathbf{p}_i)}) \circ \varphi(\mathbf{p}_i) \end{pmatrix}. \end{aligned} \quad (3.1)$$

Here ν is the unit outer normal on $\Gamma(\mathbf{p})$ with respect to the domain $\Omega(\mathbf{p})$. We define the joint trace operator by $T(\mathbf{p})u := (T_i(\mathbf{p})u)_{i=0,\dots,N}$. We also define $g_{0k}^0 := 0$, $g := ((g_{i0}^0, g_{i1}^0))_{i=0,\dots,N}$, and $g_i = (g_{i0}^0, g_{i1}^0)$.

In the next step we prove a useful reformulation of the parametric boundary conditions as linear constraints. The aim is to get rid of the parameters γ in conditions of the form (2.3) for the i -th particle by writing it as $P_i T_i(\mathbf{p})u = P_i g_i$ for a suitable projection operator

$$P_i: L^2(\Gamma_i^0) \times L^2(\Gamma_i^0) \rightarrow L^2(\Gamma_i^0) \times L^2(\Gamma_i^0).$$

For the domain boundary represented by $i = 0$ we can simply use $P_0(v_1, v_2) := (v_1, v_2)$. While all P_i are different, the construction works exactly the same for all $i \in \{1, \dots, N\}$. Thus, to simplify the notation, we will drop the index i in the following construction of $P_i = P$. Define for $\hat{x} \in \mathbb{R}^2$ the functions

$$\eta_1(\hat{x}) := \hat{x}_1, \quad \eta_2(\hat{x}) := \hat{x}_2, \quad \eta_3(\hat{x}) := 1,$$

spanning the space $V := \text{span}\{\eta_1, \eta_2, \eta_3\}$.

We will define P such that $\{(v, \partial_{\hat{\nu}} v) \mid v \in V\} \subset L^2(\Gamma^0) \times L^2(\Gamma^0)$ is the kernel of P where $\hat{\nu}$ again denotes the unit normal to Γ^0 . To this end let $\mathcal{P}: L^2(\Gamma^0) \rightarrow V$ be the $L^2(\Gamma^0)$ -orthogonal projection into V . Notice that \mathcal{P} can easily be computed using

$$\mathcal{P}(v) = \mathcal{O} \left(G^{-1} \begin{pmatrix} \langle v, \eta_1 \rangle_{L^2(\Gamma^0)} \\ \langle v, \eta_2 \rangle_{L^2(\Gamma^0)} \\ \langle v, \eta_3 \rangle_{L^2(\Gamma^0)} \end{pmatrix} \right)$$

where $G \in \mathbb{R}^{3 \times 3}$ is the Gramian matrix with $G_{jk} = \langle \eta_k, \eta_j \rangle_{L^2(\Gamma^0)}$ and $\Theta : \mathbb{R}^3 \rightarrow V$ the coordinate isomorphism with $\Theta(\xi) = \xi_1 \eta_1 + \xi_2 \eta_2 + \xi_3 \eta_3$. Now we define $P : L^2(\Gamma^0) \times L^2(\Gamma^0) \rightarrow L^2(\Gamma^0) \times L^2(\Gamma^0)$ by

$$P(v_1, v_2) = \begin{pmatrix} v_1 - \mathcal{P}(v_1)|_{\Gamma^0} \\ v_2 - \partial_{\hat{v}} \mathcal{P}(v_1)|_{\Gamma^0} \end{pmatrix}.$$

Since \mathcal{P} is an orthogonal projection into V , we have

$$0 = P(v_1, v_2) \iff v_1 \in V \text{ and } v_2 = \partial_{\hat{v}} v_1.$$

Hence $\{(v, \partial_{\hat{v}} v) \mid v \in V\} \subset L^2(\Gamma^0) \times L^2(\Gamma^0)$ is indeed the kernel of P .

Lemma 3.1 (Parametric boundary conditions as linear constraint). *Let $i \in \{1, \dots, N\}$. Then P_i is a well-defined linear projection operator such that for all $u \in H^2(\Omega(\mathbf{p}))$*

$$P_i T_i(\mathbf{p})u = P_i g_i \tag{3.2}$$

holds if and only if

$$\exists \gamma \in \mathbb{R}^3: \begin{cases} u|_{\Gamma_i}(x) = g_{i1}(\varphi(\mathbf{p}_i)^{-1}(x)) + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3, \\ \partial_{\nu} u|_{\Gamma_i}(x) = g_{i2}(\varphi(\mathbf{p}_i)^{-1}(x)) + \gamma_1 \nu_1 + \gamma_2 \nu_2, \end{cases} \quad \forall x \in \Gamma_i(\mathbf{p}_i). \tag{3.3}$$

For $i = 0$ equation (3.2) is equivalent to $u|_{\partial\Omega} = \partial_{\nu} u|_{\partial\Omega} = 0$.

Proof. The statement for $i = 0$ is trivial, hence we only consider $i \in \{1, \dots, N\}$ and again drop the index i for simplicity. Well-definedness of P directly follows from the fact that $\eta_1, \eta_2, \eta_3 \in L^2(\Gamma^0)$ are linearly independent such that \mathcal{P} is well-defined. P is linear, because it is the composition of linear maps.

To show that P is a projection, let $(v_1, v_2) \in L^2(\Gamma^0) \times L^2(\Gamma^0)$ and set $(w_1, w_2) = P(v_1, v_2)$. Since \mathcal{P} is an $L^2(\Gamma^0)$ -orthogonal projection we have

$$\mathcal{P}(w_1) = \mathcal{P}(v_1) - \mathcal{P}(\mathcal{P}(v_1)|_{\Gamma^0}) = \mathcal{P}(v_1) - \mathcal{P}(\mathcal{P}(v_1)) = 0$$

and thus $P^2(v_1, v_2) = P(w_1, w_2) = (w_1, w_2) = P(v_1, v_2)$.

Finally, we show equivalence of (3.2) and (3.3). To this end we first note that $\partial_{\nu} \eta_1 = \nu_1$, $\partial_{\nu} \eta_2 = \nu_2$, and $\partial_{\nu} \eta_3 = 0$. Furthermore, since $\varphi(\mathbf{p})$ is a rigid body motion in \mathbb{R}^2 , the normal vectors \hat{v} and ν of Γ^0 and $\Gamma(\mathbf{p})$ and corresponding normal derivatives transform according to

$$\nu(\varphi(\mathbf{p})(\hat{x})) = (D\varphi(\mathbf{p}))(\hat{x})\hat{v}(\hat{x}), \quad \partial_{\hat{v}}(v \circ \varphi(\mathbf{p})) = (\partial_{\nu} v) \circ \varphi(\mathbf{p}),$$

where $D\varphi(\mathbf{p})$ denotes the Jacobian of the map $\varphi(\mathbf{p})$. Using the trace operator and the formulas for $\partial_{\nu} \eta_k$ from above, we can write (3.3) compactly as

$$\exists \gamma \in \mathbb{R}^3: \quad ((T(\mathbf{p})u) - g) \circ \varphi(\mathbf{p})^{-1} = \sum_{k=1}^3 \gamma_k \begin{pmatrix} \eta_k \\ \partial_{\nu} \eta_k \end{pmatrix}.$$

By transformation with $\varphi(\mathbf{p})$ we find that this is equivalent to

$$\exists \gamma \in \mathbb{R}^3: \quad (T(\mathbf{p})u) - g = \sum_{k=1}^3 \gamma_k \begin{pmatrix} \eta_k \circ \varphi(\mathbf{p}) \\ (\partial_{\nu} \eta_k) \circ \varphi(\mathbf{p}) \end{pmatrix} = \sum_{k=1}^3 \gamma_k \begin{pmatrix} \eta_k \circ \varphi(\mathbf{p}) \\ \partial_{\hat{v}}(\eta_k \circ \varphi(\mathbf{p})) \end{pmatrix}$$

which can also be written as

$$(T(\mathbf{p})u) - g \in \{(v, \partial_{\hat{\nu}} v) \mid v = w \circ \varphi(\mathbf{p}), w \in V\} = \{(v, \partial_{\hat{\nu}} v) \mid v \in V\}.$$

Here we used the fact, that the transformation of the space V of affine linear functions by $\varphi(\mathbf{p})$ is V itself. Since the right-hand-side is the kernel of P , the last inclusion is equivalent to

$$P(T(\mathbf{p})u - g) = 0$$

and thus (3.2). \square

For notational convenience we define $Pv := (P_i v_i)_{i=0, \dots, N}$. The set of *feasible membranes given the particle configuration \mathbf{p}* is defined by

$$U(\mathbf{p}) := \{u \in H^2(\Omega(\mathbf{p})) \mid P(T(\mathbf{p})u - g) = 0\}.$$

We use the shorthand $J(\mathbf{p}, u) := J(\Omega(\mathbf{p}), u)$ to define the *interaction energy*

$$\mathfrak{J}(\mathbf{p}) := \min_{u \in U(\mathbf{p})} J(\mathbf{p}, u)$$

where we use the convention $\min(\emptyset) := +\infty$. Altogether, our model problem then reads

$$\min_{\mathbf{p} \in \Lambda} \mathfrak{J}(\mathbf{p}).$$

In order to ensure well-posedness of the minimization problems on $U(\mathbf{p})$ we will from now on assume that g is smooth enough. Then the following lemma allows to show well-posedness using Lax–Milgram’s theorem.

Lemma 3.2. *Let $\mathbf{p} \in \Lambda$. Then the affine subspace $U(\mathbf{p})$ can be written as $U(\mathbf{p}) = U_0(\mathbf{p}) + \hat{g}$ for a closed subspace $U_0(\mathbf{p})$ of $H^2(\Omega(\mathbf{p}))$ and some $\hat{g} \in H^2(\Omega(\mathbf{p}))$. Furthermore, the bilinear form*

$$a(u, v) = \int_{\Omega(\mathbf{p})} \kappa \Delta u \Delta v + \sigma \nabla u \cdot \nabla x \, dx$$

is $H^2(\Omega(\mathbf{p}))$ -elliptic on $U_0(\mathbf{p})$.

Proof. First let $\hat{g} \in H^2(\Omega(\mathbf{p}))$ a function such that $T(\mathbf{p})\hat{g} = g$. Then it is clear that

$$U_0(\mathbf{p}) := U(\mathbf{p}) - \hat{g} = \{u \in H^2(\Omega(\mathbf{p})) \mid PT(\mathbf{p})u = 0\}$$

is a subspace of $H^2(\Omega(\mathbf{p}))$. Continuity of the trace operator $T(\mathbf{p})$ and the L^2 -projection \mathcal{P} implies that $P \circ T(\mathbf{p})$ is continuous and hence $U_0(\mathbf{p})$ and $U(\mathbf{p})$ are closed.

Continuity of $a(\cdot, \cdot)$ on $U_0(\mathbf{p})$ and $U(\mathbf{p})$ is obvious. To show coercivity let $u \in U_0(\mathbf{p})$. First we note that $u = \partial_{\nu} u = 0$ on $\partial\Omega$. Using similar arguments as in the proof of Lemma 3.1 we find that for each $i \in \{1, \dots, N\}$ the trace of u on $\Gamma_i(\mathbf{p}_i)$ coincides with some affine linear function $w_i \in V$. Hence, in each $B_i(\mathbf{p}_i)$ we can extend u by the corresponding function w_i to obtain a function $\tilde{u} \in H_0^2(\Omega)$ which coincides with u in $\Omega(\mathbf{p})$ and is affine linear in each $B_i(\mathbf{p}_i)$.

Using the notation $\|\cdot\|_{0, M}$ and $|\cdot|_{2, M}$ for the L^2 -norm and the H^2 -half norm on the domain M , respectively, we now have

$$\|\Delta u\|_{0, \Omega(\mathbf{p})}^2 = \|\Delta \tilde{u}\|_{0, \Omega}^2 = |\tilde{u}|_{2, \Omega}^2 = |u|_{2, \Omega(\mathbf{p})}^2.$$

In the first and third step we used that all second order partial derivatives of $\tilde{u}|_{B_i(\mathbf{p}_i)} = w_i$ vanish on $B_i(\mathbf{p}_i)$. The second step follows from $\tilde{u} \in H_0^2(\Omega)$ (see, e.g., [11, Lemma 5]). Finally, we obtain for some constant $C > 0$ independent of u

$$a(u, u) \geq \kappa^{-1} \|\Delta u\|_{0, \Omega(\mathbf{p})}^2 = \kappa^{-1} \|u\|_{2, \Omega(\mathbf{p})}^2 \geq C \kappa^{-1} \|u\|_{H^2(\Omega(\mathbf{p}))}^2.$$

The last bound follows from the fact that $U_0(\mathbf{p})$ does not contain any nontrivial affine linear functions due to the boundary conditions on $\partial\Omega$ (see, e.g., [11, Corollary 2]). \square

4. Differentiation of the reduced interaction energy

In this section we investigate the differentiability of \mathfrak{J} on Λ° . First we derive a technical result which shows that the admissible membrane sets $U(\mathbf{p})$ are isomorphic and which allows us to pose our problem locally over a fixed reference domain $\Omega(\mathbf{p})$. Afterwards we apply the implicit function theorem to prove differentiability of the reduced interaction potential and use matrix calculus to derive an explicit and numerically feasible expression for the first order derivatives.

4.1 Trace-preserving diffeomorphisms between the reference domains

In this part we construct a local diffeomorphism between the domains $\Omega(\mathbf{p})$ that preserves the boundary conditions. The basic setting is, that given a particle configuration $\mathbf{p} \in \Lambda^\circ$ we want to investigate the problem under changes of \mathbf{p} along a given direction $\mathbf{q} \in \mathbb{R}^{N \times 3}$. To this end we will construct a diffeomorphism $\mathfrak{X}(\mathbf{q})$ from $\Omega(\mathbf{p})$ to $\Omega(\mathbf{p} + \mathbf{q})$ and show that it has the desired properties. The construction is based on ordinary differential equations (ODEs) and in particular requires the following result from ODE theory.

For a function \mathfrak{F} depending on multiple arguments, we denote by $\frac{\partial \mathfrak{F}}{\partial a}$ the derivative with respect to the argument denoted by a . If one of the arguments is a spatial coordinate in \mathbb{R}^2 we denote the m -th order derivative with respect to the spatial coordinate by D^m .

Lemma 4.1. *Let $\mathfrak{B} \subseteq \mathbb{R}^{N \times 3}$ be an open connected set, $m > 1$ and let $\mathfrak{V} \in C^m([0, 1] \times \mathfrak{B} \times \mathbb{R}^2, \mathbb{R}^2)$ be Lipschitz-continuous. For $\mathbf{q} \in \mathfrak{B}$ and $x \in \mathbb{R}^2$ let $\eta(\cdot, \mathbf{q}, x): [0, 1] \rightarrow \mathbb{R}^2$ be the unique solution of the ordinary differential equation*

$$\frac{\partial \eta}{\partial t}(t, \mathbf{q}, x) = \mathfrak{V}(t, \mathbf{q}, \eta(t, \mathbf{q}, x)), \quad \eta(0, \mathbf{q}, x) = x.$$

Then the map \mathfrak{X} defined by $\mathfrak{X}(\mathbf{q}, x) := \eta(1, \mathbf{q}, x)$ fulfills $\mathfrak{X} \in C^m(\mathfrak{B} \times \mathbb{R}^2, \mathbb{R}^2)$ and is an m -diffeomorphism onto its image for all $\mathbf{q} \in \mathfrak{B}$.

For all $\hat{\mathbf{q}} \in \mathbb{R}^{N \times 3}$ with $\eta_{\hat{\mathbf{q}}} \in C([0, 1] \times \mathfrak{B} \times \mathbb{R}^2, \mathbb{R}^2)$ as the unique solution of

$$\begin{aligned} \frac{\partial \eta_{\hat{\mathbf{q}}}}{\partial t}(t, \mathbf{q}, x) &= \frac{\partial \mathfrak{V}}{\partial \mathbf{q}}(t, \mathbf{q}, \eta(t, \mathbf{q}, x)) \hat{\mathbf{q}} + D\mathfrak{V}(t, \mathbf{q}, \eta(t, \mathbf{q}, x)) \eta_{\hat{\mathbf{q}}}(t, \mathbf{q}, x), \\ \eta_{\hat{\mathbf{q}}}(0, \mathbf{q}, x) &= 0 \end{aligned} \tag{4.1}$$

one has $\partial_{\hat{\mathbf{q}}} \mathfrak{X}(\mathbf{q}, x) = \eta_{\hat{\mathbf{q}}}(1, \mathbf{q}, x)$. Also, for all $y \in \mathbb{R}^2$ with $\eta_y \in C([0, 1] \times \mathfrak{B} \times \mathbb{R}^2, \mathbb{R}^2)$ as the unique solution of

$$\frac{\partial \eta_y}{\partial t}(t, \mathbf{q}, x) = D\mathfrak{V}(t, \mathbf{q}, \eta(t, \mathbf{q}, x)) \eta_y(t, \mathbf{q}, x), \quad \eta_y(0, \mathbf{q}, x) = y \tag{4.2}$$

one has $\partial_y \mathfrak{X}(\mathbf{q}, x) = \eta_y(1, \mathbf{q}, x)$.

Proof. The global existence and uniqueness of η is a consequence of the Lipschitz-continuity of \mathcal{U} and the well-known Picard–Lindelöf theorem. In particular, \mathfrak{X} is well-defined.

The smoothness of \mathfrak{X} and the characterization of its derivatives is a consequence of [13, Theorem 3.1, Theorem 4.1]

Concerning the inverse of $\mathfrak{X}(\mathbf{q})$, we note that $\tilde{\eta}(t, \mathbf{q}, x) := \eta(1 - t, \mathbf{q}, x)$ solves the equation

$$\frac{\partial \tilde{\eta}}{\partial t}(t, \mathbf{q}, x) = -\mathcal{U}(1 - t, \mathbf{q}, \tilde{\eta}(1, \mathbf{q}, x)), \quad \tilde{\eta}(0, \mathbf{q}, x) = \eta(1, \mathbf{q}, x) = \mathfrak{X}(\mathbf{q}, x).$$

Since $\tilde{\eta}(1, \mathbf{q}, x) = x$, the inverse of $\mathfrak{X}(\mathbf{q})$ is given by $\mathfrak{X}(\mathbf{q})^{-1} := \tilde{\eta}(1, \mathbf{q}, \cdot)$. Again, the smoothness of \mathcal{U} implies m -smoothness of $\tilde{\eta}$, and consequently \mathfrak{X} is an m -diffeomorphism. \square

In the following we restrict ourselves to a special class of vector fields that is described in the result below. We show afterwards that the diffeomorphisms induced by such vector fields have a certain trace preserving property that again can be used to construct an isomorphism between the admissible membrane sets.

Lemma 4.2. *Let $\mathbf{p} \in \Lambda^\circ$ and $m \geq 1$. Then there exist an open neighborhood $\mathfrak{B} \subseteq \mathbb{R}^{N \times 3}$ of $0 \in \mathbb{R}^{N \times 3}$ and a Lipschitz-continuous map $\mathcal{V} \in C^m([0, 1] \times \mathfrak{B} \times \mathbb{R}^2, \mathbb{R}^2)$ such that for all $t \in [0, 1]$, $\mathbf{q} \in \mathfrak{B}$, and $i \in \{0, \dots, N\}$ one has*

$$\begin{aligned} \mathcal{V}(t, \mathbf{q}, \cdot)|_{\Gamma_i(\mathbf{p}_i + t\mathbf{q}_i)} &= \begin{pmatrix} \mathbf{q}_{i1} \\ \mathbf{q}_{i2} \end{pmatrix} + \mathbf{q}_{i3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\cdot - \begin{pmatrix} \mathbf{p}_{i1} + t\mathbf{q}_{i1} \\ \mathbf{p}_{i2} + t\mathbf{q}_{i2} \end{pmatrix} \right), \\ D\mathcal{V}(t, \mathbf{q}, \cdot)|_{\Gamma_i(\mathbf{p}_i + t\mathbf{q}_i)} &= \mathbf{q}_{i3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{4.3}$$

Proof. This is a consequence of the Whitney extension theorem, see Appendix Theorem A.1. It uses the fact that the Γ_i are pairwise disjoint and that the right-hand-sides in (4.3) smoothly extend to \mathbb{R}^2 . \square

Next we show that $\mathfrak{X}(\mathbf{q})$ is indeed a trace preserving diffeomorphism from $\Omega(\mathbf{p})$ to $\Omega(\mathbf{p} + \mathbf{q})$.

Lemma 4.3. *Let \mathcal{V} be as in Lemma 4.2 for $m \geq 2$ and let \mathfrak{X} as in Lemma 4.1 be induced by \mathcal{V} . Then $\mathfrak{X}(0, \cdot) = \text{id}_{\mathbb{R}^2}$, $\mathfrak{X}(\mathbf{q}, \Omega(\mathbf{p})) = \Omega(\mathbf{p} + \mathbf{q})$, and for all $u \in H^2(\Omega(\mathbf{p}))$ one has*

$$T(\mathbf{p})u = T(\mathbf{p} + \mathbf{q})(u \circ \mathfrak{X}(\mathbf{q})^{-1}). \tag{4.4}$$

Proof. From $\mathcal{V}(t, 0, \cdot) = 0$ follows immediately that $\mathfrak{X}(0, x) = \eta(1, 0, x) = x$, i.e., $\mathfrak{X}(0) = \text{id}_{\mathbb{R}^2}$.

We now prove (4.4). First we show that \mathfrak{X} preserves the boundaries. Let $i \in \{1, \dots, N\}$ and $x \in \Gamma_i(\mathbf{p}_i)$ and define $\sigma(t) := \varphi(\mathbf{p}_i + t\mathbf{q}_i; \varphi^{-1}(\mathbf{p}_i; x))$. We will show that $\eta(t, \mathbf{q}, x) = \sigma(t)$ solves the ODE with right-hand-side \mathcal{V} from Lemma 4.2. To this end we will make use of the properties

$$R(\alpha + \beta) = R(\alpha)R(\beta), \quad R(\alpha)^{-1} = R(-\alpha), \quad R'(\alpha) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} R(\alpha)$$

of the rotation matrix R . Inserting the definition of φ and φ^{-1} into $\sigma(t)$ we get

$$\sigma(t) = R(t\mathbf{q}_{i3}) \left(x - \begin{pmatrix} \mathbf{p}_{i1} \\ \mathbf{p}_{i2} \end{pmatrix} \right) + \begin{pmatrix} \mathbf{p}_{i1} + t\mathbf{q}_{i1} \\ \mathbf{p}_{i2} + t\mathbf{q}_{i2} \end{pmatrix}.$$

Since we have $\sigma(t) \in \Gamma_i(\mathbf{p}_i + t\mathbf{q}_i)$ by construction, we can use the formula (4.3) for \mathcal{U} on $\Gamma_i(\mathbf{p}_i + t\mathbf{q}_i)$. Inserting $\sigma(t)$ then gives

$$\begin{aligned} \mathcal{V}(t, \mathbf{q}, \sigma(t)) &= \begin{pmatrix} \mathbf{q}_{i1} \\ \mathbf{q}_{i2} \end{pmatrix} + \mathbf{q}_{i3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\sigma(t) - \begin{pmatrix} \mathbf{p}_{i1} + t\mathbf{q}_{i1} \\ \mathbf{p}_{i2} + t\mathbf{q}_{i2} \end{pmatrix} \right) \\ &= \begin{pmatrix} \mathbf{q}_{i1} \\ \mathbf{q}_{i2} \end{pmatrix} + \mathbf{q}_{i3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} R(t\mathbf{q}_{i3}) \left(x - \begin{pmatrix} \mathbf{p}_{i1} \\ \mathbf{p}_{i2} \end{pmatrix} \right) = \frac{\partial \sigma}{\partial t}(t). \end{aligned}$$

Hence $\eta(t, \mathbf{q}, x) = \sigma(t)$ solves the ODE and we have

$$\mathfrak{X}(\mathbf{q}, x) = \sigma(1) = \varphi(\mathbf{p}_i + \mathbf{q}_i; \varphi^{-1}(\mathbf{p}_i; x)). \tag{4.5}$$

Recalling the convention $\mathbf{q}_0 := \mathbf{p}_0 := 0$, this is also true for $i = 0$.

In particular $\mathfrak{X}(\mathbf{q}, \Gamma_i(\mathbf{p}_i)) = \Gamma_i(\mathbf{p}_i + \mathbf{q}_i)$. Another immediate consequence of (4.5) is – now recalling the definition of the rotation matrix R in (2.1) – that

$$\nu|_{\Gamma_i(\mathbf{p}_i + \mathbf{q}_i)}(\mathfrak{X}(\mathbf{q}, x)) = R(\mathbf{q}_{i3})\nu|_{\Gamma_i(\mathbf{p}_i)}(x) \tag{4.6}$$

for $x \in \Gamma_i(\mathbf{p}_i)$. And, similarly, using the properties (4.3) of \mathcal{V} also allows us to compute $\frac{\partial \mathfrak{X}}{\partial x}(\mathbf{q}, x)$ on $\Gamma_i(\mathbf{p}_i)$ from solving the ODE (4.2). The result is

$$D\mathfrak{X}(\mathbf{q}, x) = R(\mathbf{q}_{i3}). \tag{4.7}$$

Now, let $\tilde{u} := u \circ \mathfrak{X}(\mathbf{q})^{-1}$. From (4.5) we infer for $x \in \Gamma_i(\mathbf{p}_i + \mathbf{q}_i)$ that

$$\tilde{u}(x) = u(\varphi(\mathbf{p}_i; \varphi^{-1}(\mathbf{p}_i + \mathbf{q}_i; x))). \tag{4.8}$$

Also, from (4.6) and (4.7) we infer for $x \in \Gamma_i(\mathbf{p}_i + \mathbf{q}_i)$ that

$$\begin{aligned} \partial_\nu \tilde{u}(x) &= Du(\mathfrak{X}^{-1}(\mathbf{q}; x)) \frac{\partial \mathfrak{X}^{-1}(\mathbf{q}; x)}{\partial x} \nu|_{\Gamma_i(\mathbf{p}_i + \mathbf{q}_i)}(x) \\ &= Du(\mathfrak{X}^{-1}(\mathbf{q}; x)) R(-\mathbf{q}_{i3}) R(\mathbf{q}_{i3}) \nu|_{\Gamma_i(\mathbf{p}_i)}(\mathfrak{X}^{-1}(\mathbf{q}; x)) \\ &= \partial_\nu u(\mathfrak{X}^{-1}(\mathbf{q}; x)) \\ &= \partial_\nu u(\varphi(\mathbf{p}_i; \varphi^{-1}(\mathbf{p}_i + \mathbf{q}_i; x))). \end{aligned} \tag{4.9}$$

Recalling the definition of the trace operators, (3.1), we have for almost-every $x \in \Gamma_i^0$ by (4.8)

$$T_{i1}(\mathbf{p} + \mathbf{q})\tilde{u}(x) = \tilde{u}(\varphi(\mathbf{p}_i + \mathbf{q}_i; x)) = u(\varphi(\mathbf{p}_i; x)) = T_{i1}(\mathbf{p})u(x)$$

and by (4.9)

$$T_{i2}(\mathbf{p} + \mathbf{q})\tilde{u}(x) = \partial_\nu \tilde{u}(\varphi(\mathbf{p}_i + \mathbf{q}_i; x)) = \partial_\nu u(\varphi(\mathbf{p}_i; x)) = T_{i2}(\mathbf{p})u(x).$$

Altogether this proves equation (4.4).

In order to show the equality $\mathfrak{X}(\mathbf{q}, \Omega(\mathbf{p})) = \Omega(\mathbf{p} + \mathbf{q})$ we define the set

$$\mathcal{Z} = \{(t, \eta(t, \mathbf{q}, x)) \mid x \in \partial\Omega(\mathbf{p})\}.$$

Now let $x_0 \in \Omega(\mathbf{p})^\circ$ and assume that $\mathfrak{X}(\mathbf{q}, x_0) = \eta(1, \mathbf{q}, x_0) \notin \Omega(\mathbf{p} + \mathbf{q})$. By continuity of \mathfrak{X} this would imply that there exists a $\hat{t} \in [0, 1]$ such that $\eta(\hat{t}, \mathbf{q}, x_0) \in \mathcal{Z}$ and therefore, by definition of \mathcal{Z} , there would exist a $x_1 \in \partial\Omega(\mathbf{p})$ such that $\eta(\hat{t}, \mathbf{q}, x_0) = \eta(\hat{t}, \mathbf{q}, x_1)$. As of $x_0 \neq x_1$ this would be a contradiction to the uniqueness of η . □

Lemma 4.4. *Let \mathfrak{X} be as in Lemma 4.3. Then for all $\mathbf{q} \in \mathfrak{B}$ the map*

$$\Phi(\mathbf{q}): U(\mathbf{p}) \rightarrow U(\mathbf{p} + \mathbf{q}), \quad u \mapsto u \circ \mathfrak{X}(\mathbf{q})^{-1}$$

is well-defined and an isomorphism.

Proof. From Lemma 4.3 we know for every $\mathbf{q} \in \mathfrak{B}$ that the restriction $\mathfrak{X}(\mathbf{q}, \cdot)|_{\Omega(\mathbf{p})}$ is a 2-diffeomorphism onto $\Omega(\mathbf{p} + \mathbf{q})$. Because $\overline{\Omega(\mathbf{p})}$ is compact we can assume without loss of generality that $\det \frac{\partial}{\partial x} \mathfrak{X}(\mathbf{q}, x)$ is uniformly bounded, i.e., that there exists a $c \in \mathbb{R}_{>0}$ such that for all $\mathbf{q} \in \mathfrak{B}$ and $x \in \Omega(\mathbf{p})$ we have $c \leq \left| \det \frac{\partial}{\partial x} \mathfrak{X}(\mathbf{q}, x) \right| \leq \frac{1}{c}$. Otherwise we may replace \mathfrak{B} by an appropriate sub-neighborhood. Hence, [1, Theorem 3.35] is applicable and the map

$$\tilde{\Phi}(\mathbf{q}): H^2(\Omega(\mathbf{p})) \rightarrow H^2(\Omega(\mathbf{p} + \mathbf{q})), \quad u \mapsto u \circ \mathfrak{X}(\mathbf{q})^{-1}$$

is well-defined and an isomorphism, and in particular also the restriction $\Phi(\mathbf{q}) = \tilde{\Phi}(\mathbf{q})|_{U(\mathbf{p})}$ is well-defined and an isomorphism onto its image.

It remains to show that $\text{range}(\Phi(\mathbf{q})) = U(\mathbf{p} + \mathbf{q})$. Suppose $u \in U(\mathbf{p})$ and $\tilde{u} \in U(\mathbf{p} + \mathbf{q})$. Because of the trace preserving property (4.4) and by definition of $U(\mathbf{p})$ and $U(\mathbf{p} + \mathbf{q})$ it follows that $\Phi(\mathbf{q})u = u \circ \mathfrak{X}(\mathbf{q}) \in U(\mathbf{p} + \mathbf{q})$ and $\Phi(\mathbf{q})^{-1}\tilde{u} = \tilde{u} \circ \mathfrak{X}(\mathbf{q}) \in U(\mathbf{p})$, and so $\text{range}(\Phi(\mathbf{q})) = U(\mathbf{p} + \mathbf{q})$. \square

4.2 Differentiability

In this part we use the maps \mathfrak{X} from Lemma 4.3 and Φ from Lemma 4.4 to transform the domain of definition for the functions $J(\mathbf{p} + \mathbf{q})$ from $U(\mathbf{p} + \mathbf{q})$ to $U(\mathbf{p})$. Afterwards we apply the implicit function theorem to derive a differentiability result.

For $\mathbf{q} \in \mathfrak{B}$ and $u \in U(\mathbf{p})$ the *transformed energy* is defined as

$$\hat{J}(\mathbf{q}, u) := J(\mathbf{p} + \mathbf{q}, \Phi(\mathbf{q}, u)), \tag{4.10}$$

and the *transformed reduced interaction energy* is

$$\hat{\mathfrak{J}}(\mathbf{q}) := \min_{v \in U(\mathbf{p})} \hat{J}(\mathbf{q}, v).$$

As in Lemma 3.2 we write the affine linear subspace $U(\mathbf{p})$ as $U(\mathbf{p}) = U_0 + \hat{g}$ where $U_0 \subseteq H^2(\Omega(\mathbf{p}))$ is a linear subspace and $\hat{g} \in H^2(\Omega(\mathbf{p}))$ is a function such that $T(\mathbf{p})\hat{g} = g$.

For notational convenience we define $D^k := \frac{\partial^k}{\partial x^k}$ to be the differential with respect to the spatial coordinates.

Lemma 4.5. *Let $\mathbf{q} \in \mathfrak{B}$ and define*

$$\mathfrak{Q}(\mathbf{q}, x) := |\det D\mathfrak{X}(\mathbf{q}, x)| (D\mathfrak{X}(\mathbf{q}, x))^{-1} (D\mathfrak{X}(\mathbf{q}, x))^{-T}.$$

One has

$$\hat{J}(\mathbf{q}, u) = \frac{1}{2} \int_{\Omega(\mathbf{p})} \kappa \frac{\text{div}(\mathfrak{Q}(\mathbf{q})\nabla u)^2}{|\det D\mathfrak{X}(\mathbf{q})|} + \sigma \|\nabla u\|_{\mathfrak{Q}(\mathbf{q})}^2 \, dx \tag{4.11}$$

and $\hat{J}_u \in C^{m-2}(\mathfrak{B} \times U(\mathbf{p}), U'_0)$.

Proof. Equation (4.11) is a direct application of Lemma A.3 applied to $X = \mathfrak{X}(\mathbf{q})$.

Furthermore, for all $v \in H^2(\Omega)$ we have

$$\hat{J}_u(\mathbf{q}, u; v) = \int_{\Omega(\mathbf{p})} \kappa \frac{\operatorname{div}(\mathfrak{R}(\mathbf{q})\nabla u) \operatorname{div}(\mathfrak{R}(\mathbf{q})\nabla v)}{|\det D\mathfrak{X}(\mathbf{q})|} + \sigma \mathfrak{R}(\mathbf{q})\nabla u \cdot \nabla v \, dx.$$

As of $\mathfrak{X} \in C^m$ we know that $D\mathfrak{X}(\mathbf{q})$ and $D^2\mathfrak{X}(\mathbf{q})$ are both $(m - 2)$ times continuously differentiable. Because the $\mathfrak{X}(\mathbf{q})$ are diffeomorphisms with $\mathfrak{X}(0)$ and because \mathfrak{X} is continuous we have $\det(D\mathfrak{X}(\mathbf{q})) > 0$. Because $\overline{\Omega(\mathbf{p})}$ is compact, we can assume without loss of generality that there exists a $c \in \mathbb{R}_{>0}$ such that $\det(D\mathfrak{X}(\mathbf{q})) > c$, else we replace \mathfrak{B} by a suitable sub-neighborhood. Consequently, the integrand of \hat{J}_u is $(m - 2)$ times differentiable with respect to \mathbf{q} . Moreover, the integrand is even smooth with respect to u and hence application of the dominated convergence theorem yields $\hat{J}_u \in C^{m-2}(\mathfrak{B} \times U(\mathbf{p}), U'_0)$. \square

Lemma 4.6. *There exists a neighborhood $\hat{\mathfrak{B}}$ of $0 \in \mathbb{R}^{N \times 3}$ such that $\mathfrak{J} \in C^{m-2}(\hat{\mathfrak{B}} + \mathbf{p})$ and for all $\mathbf{q} \in \hat{\mathfrak{B}}$ and multi-indices α with $|\alpha| \leq m - 2$ one has*

$$\frac{\partial^\alpha}{\partial \mathbf{p}^\alpha} \mathfrak{J}(\mathbf{p} + \mathbf{q}) = \frac{\partial^\alpha}{\partial \mathbf{q}^\alpha} \hat{\mathfrak{J}}(\mathbf{q}).$$

In particular, if $m \geq 3$ and $u = \arg \min_{v \in U(\mathbf{p})} J(\mathbf{p}, v)$ then

$$\frac{\partial}{\partial \mathbf{p}} \mathfrak{J}(\mathbf{p}) = \frac{\partial}{\partial \mathbf{q}} \hat{J}(0, u). \tag{4.12}$$

Proof. Let $\hat{J}_u := \frac{\partial}{\partial u} \hat{J}$ and $\hat{J}_{uu} := \frac{\partial^2}{\partial u^2} \hat{J}$. Define

$$F: \mathfrak{B} \times U_0 \rightarrow U'_0, \quad (\mathbf{q}, v) \mapsto \hat{J}_u(\mathbf{q}, v + \hat{g}).$$

Suppose that $u \in U(\mathbf{p})$ is the unique solution of $\min_{v \in U(\mathbf{p})} J(\mathbf{p}, v)$, and define $\hat{u} := u - \hat{g}$. Then by (4.10) and because of $\Phi(0)u = u$ it also follows that u is the unique minimizer of $\hat{J}(0, \cdot)$ over $U(\mathbf{p})$ and therefore and therefore

$$F(0, \hat{u}) = \hat{J}_u(0, \hat{u} + \hat{g}) = \hat{J}_u(0, u) = 0 \in U'_0.$$

Moreover, for all $v, w \in U_0$ we have

$$F_u(0, \hat{u}; v, w) = \hat{J}_{uu}(0, u; v, w) = \int_{\Omega(\mathbf{p})} \kappa \Delta v \Delta w + \sigma \nabla v \cdot \nabla w \, dx.$$

This defines an elliptic bilinear form over U_0 (cf. Lemma 3.2) and hence $F_u(0, \hat{u})$ is invertible in U_0 by virtue of Lax–Milgram’s theorem. Application of the implicit function theorem, Theorem A.2, yields a neighborhood $\hat{\mathfrak{B}} \subseteq \mathfrak{B}$ of 0 and a function $\hat{\mathbf{u}} \in C^{m-2}(\hat{\mathfrak{B}}, U_0)$ such that $\hat{\mathbf{u}}(0) = \hat{u}$ and $F(0, \hat{\mathbf{u}}(\mathbf{q})) = 0$ for all $\mathbf{q} \in \hat{\mathfrak{B}}$. In particular, $\hat{\mathfrak{J}}(\mathbf{q}) = \hat{J}(\mathbf{q}, \hat{\mathbf{u}}(\mathbf{q}) + \hat{g})$ for all $\mathbf{q} \in \hat{\mathfrak{B}}$. From $\mathfrak{J}(\mathbf{p} + \mathbf{q}) = \hat{\mathfrak{J}}(\mathbf{q})$ we infer

$$\frac{\partial^\alpha}{\partial \mathbf{p}^\alpha} \mathfrak{J}(\mathbf{p}) = \frac{\partial^\alpha}{\partial \mathbf{q}^\alpha} \hat{\mathfrak{J}}(\mathbf{q})$$

for all multi-indices α with $|\alpha| \leq m - 2$. For $m \geq 3$ this in particular implies

$$\frac{\partial}{\partial \mathbf{p}} \mathfrak{J}(\mathbf{p}) = \frac{\partial}{\partial \mathbf{q}} \hat{J}(0, \hat{\mathbf{u}}(0) + \hat{g}) + \hat{J}_u(0, \hat{\mathbf{u}}(0) + \hat{g}) \frac{\partial}{\partial \mathbf{q}} \hat{\mathbf{u}}(0) = \frac{\partial}{\partial \mathbf{q}} \hat{J}(0, u). \quad \square$$

4.3 A numerically feasible representation of the first derivative

In the following paragraphs we discuss a way to derive a numerically feasible expression for the first order derivative $\partial_e \mathfrak{J}(\mathbf{p})$ of the reduced interaction energy \mathfrak{J} in $\mathbf{p} \in \Lambda^\circ$ in direction of an $\mathbf{e} \in \mathbb{R}^{N \times 3}$.

An important component of the integrand’s derivative is $\partial_e \mathfrak{X}(0)$ and its spatial derivatives. From (4.1) we know that this derivative can be evaluated by solving an ODE. This is not practically feasible, however, because those computations would be too expensive. Besides, it also requires knowledge of the vector field \mathfrak{U} , which may be hard to construct explicitly. Instead we restrict ourselves to a subclass of vector fields in the sense of Lemma 4.2 for which $\partial_e \mathfrak{X}(0)$ is can be computed easily from information that is available a-priori.

To this end, suppose a vector field $V: \Omega(\mathbf{p}) \rightarrow \mathbb{R}^2$ such that for $i \in \{0, \dots, N\}$

$$\begin{aligned} V|_{\Gamma_i(\mathbf{p}_i)} &= \begin{pmatrix} \mathbf{e}_{i1} \\ \mathbf{e}_{i2} \end{pmatrix} + \mathbf{e}_{i3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\cdot - \begin{pmatrix} \mathbf{p}_{i1} \\ \mathbf{p}_{i2} \end{pmatrix} \right), \\ DV|_{\Gamma_i(\mathbf{p}_i)} &= \mathbf{e}_{i3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{4.13}$$

where we again use the convention $\mathbf{e}_0 := 0$. Usually it is easy to construct such a V in a way that it is also numerically accessible. Next we extend this to a vector field \mathfrak{U} such that (4.3) is fulfilled, where we can make the simplifying assumption that \mathfrak{B} is a ball of radius $r \in \mathbb{R}_{>0}$, and such that for all $t \in [0, 1]$ and $\lambda \in (0, r)$ the scaling properties

$$\begin{aligned} \mathfrak{U}(t, \lambda \mathbf{e}, x) &= \frac{\lambda}{r} \mathfrak{U} \left(\frac{\lambda}{r} t, r \mathbf{e}, x \right), \\ \mathfrak{U}(0, r \mathbf{e}, x) &= r V(x) \end{aligned} \tag{4.14}$$

hold.

In view of (4.1) and given $x \in \Omega(\mathbf{p})$, we have $\partial_e \mathfrak{X}(0, x) = \eta_e(1, x)$ where $\eta_e(\cdot, x)$ solves the ODE

$$\frac{\partial \eta_e}{\partial t}(t, x) = \partial_e \mathfrak{U}(t, 0, \eta(t, 0, x)) + D \mathfrak{U}(t, 0, \eta(t, 0, x)) \eta_e(t, x), \quad \eta_e(0, x) = 0.$$

As of $\mathfrak{U}(t, 0, \cdot) \equiv 0$ we have $\eta(t, 0, x) = x$ and $D \mathfrak{U}(t, 0, \eta(t, 0, x)) = 0$. Furthermore, from (4.14) we are able to conclude

$$\partial_e \mathfrak{U}(t, 0, x) = \lim_{\lambda \searrow 0} \frac{\mathfrak{U}(t, \lambda \mathbf{e}, x) - \mathfrak{U}(t, 0, x)}{\lambda} = \lim_{\lambda \searrow 0} \frac{\mathfrak{U}(\frac{\lambda}{r} t, r \mathbf{e}, x)}{r} = V(x).$$

Therefore, η_e is the solution of the ODE

$$\frac{\partial \eta_e}{\partial t}(t, x) = V(x), \quad \eta_e(0, x) = 0,$$

which implies $\eta_e(t, x) = t V(x)$ and hence also $\partial_e \mathfrak{X}(0) = V$.

When computing the derivative, we will make use of the following identities from matrix calculus.

Lemma 4.7. *Suppose $M \in C^1(\mathbb{R}^d, \mathbb{R}^{n \times n})$ and that $M(x)$ is invertible for all $x \in \mathbb{R}^d$. Then*

$$\frac{\partial}{\partial x_i} \det(M(x)) = \det(M) \operatorname{Tr}\left(M(x)^{-1} \frac{\partial}{\partial x_i} M(x)\right), \tag{4.15}$$

$$\frac{\partial}{\partial x_i} M(x)^{-1} = -M(x)^{-1} \frac{\partial M(x)}{\partial x_i} M(x)^{-1}, \tag{4.16}$$

$$\frac{\partial}{\partial x_i} \operatorname{Tr}(M(x)) = \operatorname{Tr}\left(\frac{\partial}{\partial x_i} M(x)\right). \tag{4.17}$$

Proof. See literature on matrix calculus, e.g., [21, Chapter 9]. □

Lemma 4.8. *Let $V := \partial_e \mathfrak{X}(0)$, $u := \arg \min_{v \in U(p)} J(p, u)$, and*

$$\mathfrak{Q}'(0) := \operatorname{div}(V)I - DV - DV^T.$$

Then

$$\begin{aligned} \partial_e \mathfrak{J}(p) &= \int_{\Omega(p)} \kappa \Delta u \left(\mathfrak{Q}'(0) : D^2 u - \Delta V \cdot \nabla u - \frac{1}{2} \operatorname{div}(V) \Delta u \right) dx \\ &\quad + \int_{\Omega(p)} \frac{\sigma}{2} \|\nabla u\|_{\mathfrak{Q}'(0)}^2 dx. \end{aligned} \tag{4.18}$$

Proof. From (4.12) we know that $\partial_e \mathfrak{J}(p) = \partial_e \hat{J}(0, u)$ and hence it suffices to compute the latter. In the following we use without further emphasis the identities $D\mathfrak{X}(0, \cdot) \equiv \operatorname{id}_{\mathbb{R}^2}$ and $\det D\mathfrak{X}(0) \equiv 1$. Based on this and on the identities (4.15) and (4.16) we have

$$\begin{aligned} \left. \frac{\partial}{\partial \mathbf{e}} \det(D\mathfrak{X}(\mathbf{q})) \right|_{\mathbf{q}=0} &= \det(D\mathfrak{X}(\mathbf{q})) \operatorname{Tr}\left(D\mathfrak{X}(\mathbf{q})^{-1} \frac{\partial}{\partial \mathbf{e}} D\mathfrak{X}(\mathbf{q})\right) \Big|_{\mathbf{q}=0} \\ &= \operatorname{Tr}\left(\frac{\partial}{\partial \mathbf{e}} D\mathfrak{X}(\mathbf{q})\right) = \operatorname{div}(V), \end{aligned} \tag{4.19}$$

$$\begin{aligned} \left. \frac{\partial}{\partial \mathbf{e}} D\mathfrak{X}^{-1}(\mathbf{q}) \right|_{\mathbf{q}=0} &= -D\mathfrak{X}(\mathbf{q})^{-1} \frac{\partial D\mathfrak{X}(\mathbf{q})}{\partial \mathbf{e}} D\mathfrak{X}(\mathbf{q})^{-1} \Big|_{\mathbf{q}=0} \\ &= -\frac{\partial D\mathfrak{X}(0)}{\partial \mathbf{e}} = -DV. \end{aligned} \tag{4.20}$$

By definition of \mathfrak{Q} we have

$$\mathfrak{Q}(\mathbf{q}) = \det(D\mathfrak{X}(\mathbf{q})) D\mathfrak{X}(\mathbf{q})^{-1} D\mathfrak{X}(\mathbf{q})^{-T},$$

where we again used $\det(\mathfrak{X}(\mathbf{q})) > 0$. The product rule together with the identities (4.19) and (4.20)

then leads us to

$$\begin{aligned}
 \frac{\partial}{\partial \mathbf{e}} \mathcal{Q}(\mathbf{p}) \Big|_{\mathbf{q}=\mathbf{0}} &= \frac{\partial \det(D\mathcal{X}(\mathbf{q}))}{\partial \mathbf{e}} \Big|_{\mathbf{q}=\mathbf{0}} D\mathcal{X}(\mathbf{q})^{-1} D\mathcal{X}(\mathbf{q})^{-T} \\
 &\quad + \det(D\mathcal{X}(\mathbf{q})) \frac{\partial D\mathcal{X}(\mathbf{q})^{-1}}{\partial \mathbf{e}} \Big|_{\mathbf{q}=\mathbf{0}} D\mathcal{X}(\mathbf{q})^{-T} \\
 &\quad + \det(D\mathcal{X}(\mathbf{q})) D\mathcal{X}(\mathbf{q})^{-1} \frac{\partial D\mathcal{X}(\mathbf{q})^{-T}}{\partial \mathbf{e}} \Big|_{\mathbf{q}=\mathbf{0}} \\
 &= \operatorname{div}(V)I - DV - DV^T \\
 &= \mathcal{Q}'(0).
 \end{aligned} \tag{4.21}$$

Equation (4.21) gives us, using $\partial_i := \frac{\partial}{\partial x_i}$,

$$\partial_i \mathcal{Q}'(0) = \left(\sum_{k=1}^2 \partial_{ik} V_k \right) I - \partial_i DV - \partial_i DV^T$$

and so

$$\begin{aligned}
 \sum_{i=1}^2 \partial_i \mathcal{Q}'_{ij}(0) &= \sum_{i,k=1}^2 \partial_{ik} V_k \delta_{ij} - \sum_{i=1}^2 (\partial_{ij} V_i + \partial_{ii} V_j) \\
 &= \sum_{k=1}^2 \partial_{jk} V_k - \sum_{i=1}^2 (\partial_{ij} V_i + \partial_{ii} V_j) \\
 &= \sum_{i=1}^2 \left(\partial_{ji} V_i - \partial_{ij} V_i - \sum_{i=1}^2 \partial_{ii} V_j \right) \\
 &= -\Delta V_j
 \end{aligned} \tag{4.22}$$

for $j \in \{1, 2\}$. Noting

$$\begin{aligned}
 \operatorname{div}(\mathcal{Q}(\mathbf{q})\nabla u) &= \sum_{i=1}^2 \partial_i \left(\sum_{j=1}^2 \mathcal{Q}(\mathbf{q})_{ij} \partial_j u \right) \\
 &= \sum_{i,j=1}^2 (\mathcal{Q}(\mathbf{q})_{ij} \partial_{ij} u + \partial_i \mathcal{Q}(\mathbf{q})_{ij} \partial_j u) \\
 &= \mathcal{Q}(\mathbf{q}) : D^2 u + \sum_{j=1}^2 \left(\sum_{i=1}^2 \partial_i \mathcal{Q}(\mathbf{q})_{ij} \right) \partial_j u,
 \end{aligned}$$

we therefore conclude

$$\frac{\partial}{\partial \mathbf{e}} \operatorname{div}(\mathcal{Q}(\mathbf{q})\nabla u) \Big|_{\mathbf{q}=\mathbf{0}} = \mathcal{Q}'(0) : D^2 u - \Delta V \cdot \nabla u \tag{4.23}$$

where $\Delta V := (\Delta V_i)_{i=1,\dots,n} \in \mathbb{R}^2$.

We recall

$$\hat{J}(\mathbf{q}, u) = \frac{1}{2} \int_{\Omega(\mathbf{p})} \kappa \frac{\operatorname{div}(\mathcal{R}(\mathbf{q})\nabla u)^2}{\det D\mathcal{X}(\mathbf{q})} + \sigma \mathcal{R}(\mathbf{q})\nabla u \cdot \nabla u \, dx.$$

Combining (4.19), (4.21) and (4.23) yields

$$\begin{aligned} \left. \frac{\partial}{\partial \mathbf{e}} \hat{J}(\mathbf{q}, u) \right|_{\mathbf{q}=0} &= \frac{1}{2} \int_{\Omega(\mathbf{p})} 2\kappa \operatorname{div}(\mathcal{R}(\mathbf{q})\nabla u) \left. \frac{\partial}{\partial \mathbf{e}} \operatorname{div}(\mathcal{R}(\mathbf{q})\nabla u) \right|_{\mathbf{q}=0} dx \\ &\quad - \frac{1}{2} \int_{\Omega(\mathbf{p})} \kappa \frac{\operatorname{div}(\mathcal{R}(\mathbf{q})\nabla u)^2}{(\det D\mathcal{X}(\mathbf{q}))^2} \left. \frac{\partial}{\partial \mathbf{e}} \det(D\mathcal{X}(\mathbf{q})) \right|_{\mathbf{q}=0} dx \\ &\quad + \frac{1}{2} \int_{\Omega(\mathbf{p})} \sigma \left. \frac{\partial}{\partial \mathbf{e}} \mathcal{R}(\mathbf{q}) \right|_{\mathbf{q}=0} \nabla u \cdot \nabla u \, dx \\ &= \int_{\Omega(\mathbf{p})} \kappa \Delta u \left(\mathcal{R}'(0) : D^2 u - \Delta V \cdot \nabla u - \frac{1}{2} \operatorname{div}(V)\Delta u \right) \\ &\quad + \frac{\sigma}{2} \|\nabla u\|_{\mathcal{R}'(0)}^2 \, dx, \end{aligned}$$

which proves the claim as stated. □

In general, the exact minimizer for $\mathfrak{J}(\mathbf{p})$ is unknown and can only be approximated. The following result gives an upper bound on the approximation error.

Lemma 4.9. *Let $u = \arg \min_{v \in U(\mathbf{p})} J(\mathbf{p}, v)$ and $\tilde{u} \in H^2(\Omega(\mathbf{p}))$. Then there exists a constant $C > 0$ such that*

$$\left| \partial_{\mathbf{e}} \hat{J}(0, u) - \partial_{\mathbf{e}} \hat{J}(0, \tilde{u}) \right| \leq C \|V\|_{C^2(\Omega(\mathbf{p}))} \|u + \tilde{u}\|_{H^2(\Omega(\mathbf{p}))} \|u - \tilde{u}\|_{H^2(\Omega(\mathbf{p}))}.$$

Proof. Note from (4.18) that \hat{J}_u is induced by a non-symmetric bilinear form, i.e., there exists a bilinear form $a: H^2(\Omega(\mathbf{p})) \times H^2(\Omega(\mathbf{p})) \rightarrow \mathbb{R}$ such that $\hat{J}_u(v) = a(v, v)$. Upon investigation of the coefficients of a it is readily seen that there exists a $C \in \mathbb{R}_{>0}$ such that for all $v, w \in H^2(\Omega(\mathbf{p}))$

$$|a(v, w)| \leq C \|V\|_{C^2(\Omega(\mathbf{p}))} \|v\|_{H^2(\Omega(\mathbf{p}))} \|w\|_{H^2(\Omega(\mathbf{p}))}.$$

Now, consider

$$\begin{aligned} \left| \partial_{\mathbf{e}} \hat{J}(0, u) - \partial_{\mathbf{e}} \hat{J}(0, \tilde{u}) \right| &= |a(u, u) - a(\tilde{u}, \tilde{u})| \\ &= \left| \frac{1}{2} a(u + \tilde{u}, u - \tilde{u}) + \frac{1}{2} a(u - \tilde{u}, u + \tilde{u}) \right| \\ &\leq C \|V\|_{C^2(\Omega(\mathbf{p}))} \|u + \tilde{u}\|_{H^2(\Omega(\mathbf{p}))} \|u - \tilde{u}\|_{H^2(\Omega(\mathbf{p}))}. \quad \square \end{aligned}$$

It is important to note that while the derivative $\frac{\partial}{\partial \mathbf{e}} \mathfrak{J}(\mathbf{p})$ itself is independent of V , the actual choice of V very well enters the approximation error. Therefore it is desirable to construct a V with a bounded C^2 -norm.

5. Numerical examples

In this section we illustrate our formula for the derivative by numerical computations for various particle configurations. Here we always defined the bending rigidity $\kappa = 1$ and specified no surface tension $\sigma = 0$. The optimal membrane shapes $u(\mathbf{p})$ for fixed particle configurations \mathbf{p} were approximated by finite element discretizations $u_h(\mathbf{p}) \approx u(\mathbf{p})$. A possible discretization using a penalty approach is discussed in [12] where we also gave a proof of convergence. The vector fields V that occur in the derivative were explicitly constructed in such a way that they both fulfill (4.13) and can be represented as finite element functions within the used discretization. The expressions for the discretized derivatives were evaluated exactly by using standard quadrature methods.

5.1 Two circular particles

Let $\Omega = [-10, 10]^2$ and consider two circular particles of radius one B_1, B_2 that each induce on $\Gamma_i := \partial B_i$ the boundary conditions

$$u|_{\Gamma_i}(y) = 0 + \gamma_{i1}y_1 + \gamma_{i2}y_2 + \gamma_{i3}, \quad \partial_\nu u|_{\Gamma_i}(y) = 1 + \gamma_{i1}\nu_1(y) + \gamma_{i2}\nu_2(y).$$

We are interested in the interaction energy as a function of the distance of these two particles and therefore we define

$$\hat{\mathbf{q}} := \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{q} := \frac{\hat{\mathbf{q}}}{\|\hat{\mathbf{q}}\|}, \quad f(r) := \mathfrak{J}(r\mathbf{q}).$$

In this formulation r is the distance between the particle centers and the particles touch for $r = 2$.

On the left picture of Figure 5.1 we depict the approximate values of $f(r)$ for $2.06 \leq r \leq 7.94$ that we obtained from our discretization, and on right we show two approximations of $f'(r)$. One approximation was obtained by computing the difference quotients from the function values and the other one was obtained from the derivative formula (4.18).

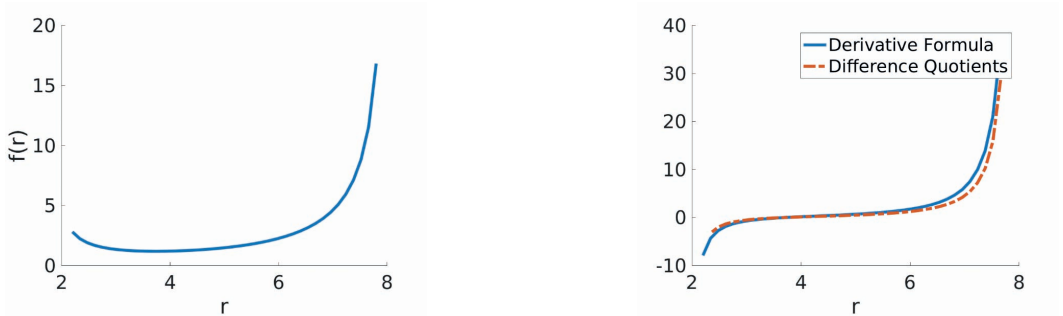


FIG. 5.1. Left: Elastic energy for two circular inclusions as a function of their distance. Right: Comparison of difference quotients with the derivative formula (4.18) for this function.

5.2 Two peanut shaped particles

Let $\Omega = [-5, 5]^2$ and consider particles whose shape is defined by the zero level set of

$$\frac{1}{20} - x^4 + \frac{19}{20}x^2 - 2x^2y^2 - \frac{19}{20}y^2 - y^4.$$

We assume that each particle induces the following boundary conditions:

$$u|_{\Gamma_i}(y) = 0 + \gamma_{i1}y_1 + \gamma_{i2}y_2 + \gamma_{i3}, \quad \partial_\nu u|_{\Gamma_i}(y) = \partial_\nu g(y) + \gamma_{i1}v_1(y) + \gamma_{i2}v_2(y)$$

where $g(y) := \frac{1}{2}(y_1^2 + y_2^2)$. Since the particles are not rotationally symmetric, the interaction energy might not just depend on the particle distance, but on the distances in x - and y -direction and the relative rotation angle. To investigate all three directions we define

$$\mathbf{p} = \begin{pmatrix} -2.5 & 0 & 0 \\ 2.5 & 0 & 0 \end{pmatrix}, \quad \mathbf{q}^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{q}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{q}^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and $f_i(t) := \mathfrak{g}(\mathbf{p} + t\mathbf{q}^i)$. Then f_1 , f_2 , and f_3 represent the interaction energy along changes of the relative distance in x - and y -direction and of the relative rotation, respectively.

In Figure 5.2 we show an example of an optimal membrane shape given the particle configuration \mathbf{p} as obtained from our discretization. In Figures 5.3 to 5.5 we evaluate the functions $f_i(t)$ for $-1.4 \leq t \leq 1.4$ and compare the approximation of $f_i'(t)$ by difference quotients with the approximation obtained from evaluating the derivative formula (4.18).

Also in this setting we observe that our formula is generally in good agreement with the approximation by difference quotients.

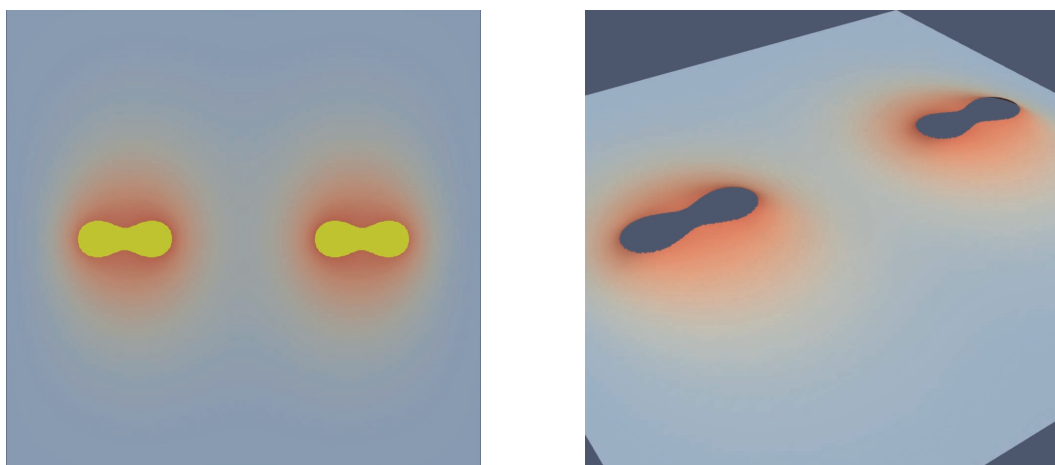
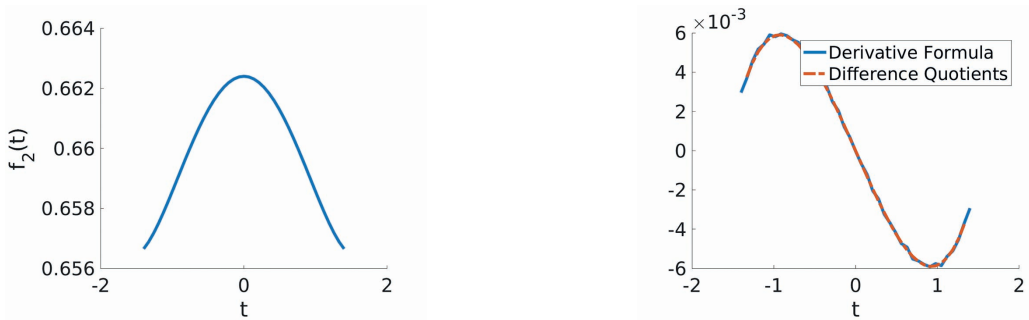
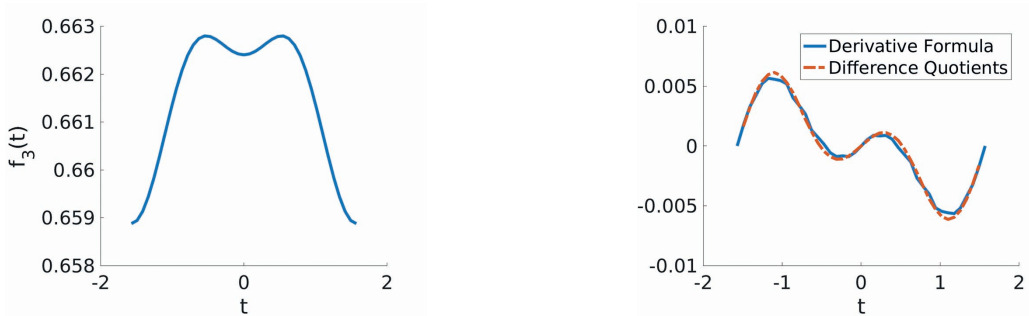


FIG. 5.2. Left: Level set view of the optimal membrane shape for the particle configuration \mathbf{p} . Right: Rendered 3D view.

FIG. 5.3. Left: Plot of $f_1(t)$. Right: Comparison of difference quotients with the derivative formula (4.18).FIG. 5.4. Left: Plot of $f_2(t)$. Right: Comparison of difference quotients with the derivative formula (4.18).FIG. 5.5. Left: Plot of $f_3(t)$. Right: Comparison of difference quotients with the derivative formula (4.18).

5.3 Gradient flow

An immediate application of our findings is to employ a gradient flow

$$\mathbf{p}'(t) = -\nabla \mathfrak{J}(\mathbf{p}(t)), \quad \mathbf{p}(0) = \mathbf{p}_0$$

in order to investigate stable particle configurations.

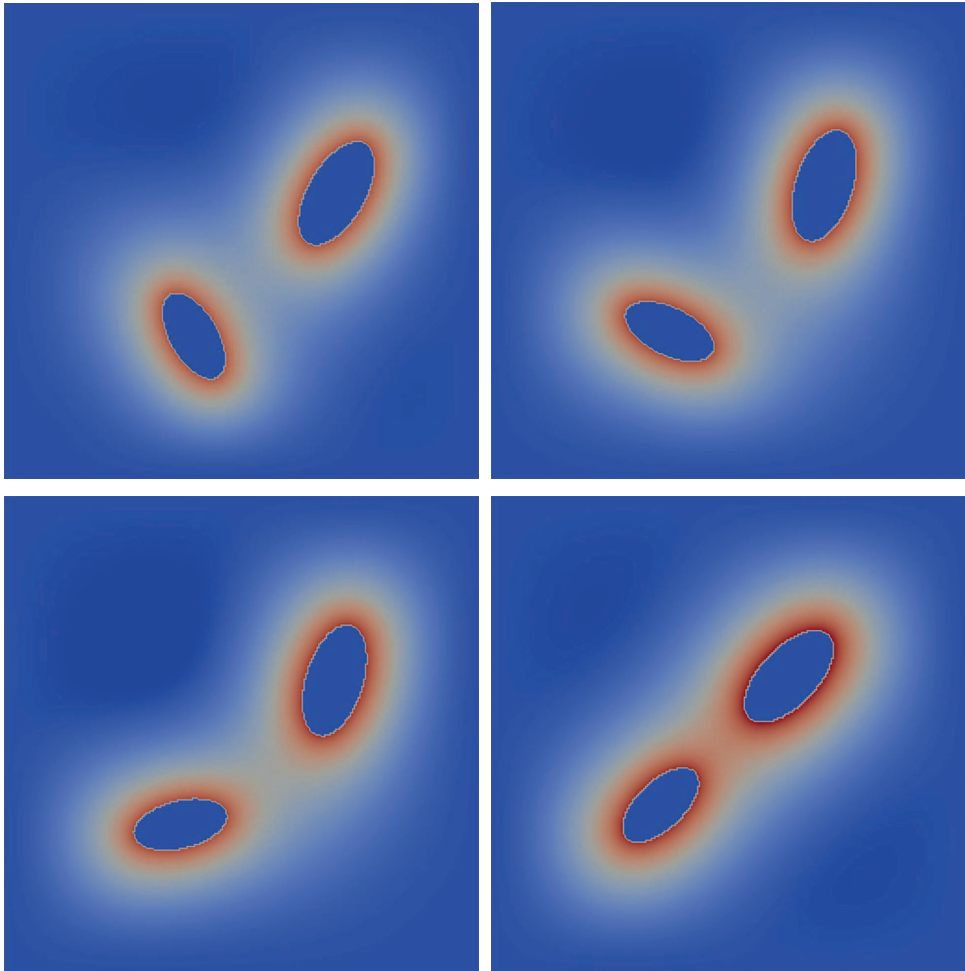


FIG. 5.6. Time steps \mathbf{p}_0 , \mathbf{p}_5 , \mathbf{p}_{10} , and \mathbf{p}_{75} of a discretized gradient flow for \mathfrak{J} with two elliptic particles (from left to right, top to bottom).

In Figure 5.6 we illustrate some time steps of the flow for two elliptic particles of different size on a square domain Ω . Here we assume the boundary conditions

$$u|_{\Gamma_i}(y) = \gamma_i, \quad \partial_\nu u|_{\Gamma_i}(y) = 1$$

for each particle. The computations use a discretization of the gradient flow by an explicit Euler scheme

$$\mathbf{p}_{k+1} := \mathbf{p}_k - \tau \nabla \mathfrak{J}(\mathbf{p}_k)$$

with a fixed time step size $\tau > 0$. The gradient $\nabla \mathfrak{J}(\mathbf{p}_k)$ is approximated using the derivative formula (4.18) for a finite element approximation of u . In fact, the time discrete gradient flow can be viewed as gradient descent method with fixed step size for the computation of minimizers of \mathfrak{J} .

Notice, that the simple gradient flow approach was used to simplify the presentation and that more sophisticated iterative methods based on first order derivatives could be used.

For the given setting with initially unaligned particles, the gradient flow leads to a configuration where the long axes of the elliptic particles are aligned. Furthermore, the distance of the particles is initially reduced and remains unchanged in a later stage indicating that the implicit particle–particle interaction is attractive and that this configuration is (close to) a local minimizer of \mathcal{J} .

6. Conclusion

This paper considered a typical model for membrane-mediated particle interactions where the membrane is described as a continuous surface and where the particles are treated as discrete entities that couple to the membrane through certain constraints. Based on methods from shape calculus and the implicit function theorem we were able to give a proof for the differentiability of the interaction energy. Matrix calculus then enabled us to derive a formula for the first derivative that is numerically feasible in the sense that it can be evaluated from a finite element approximation of the optimal membrane shape for a fixed particle configuration and that it is possible to bound the approximation error of the derivative in terms of the discretization error of the finite element method. Numerical examples suggest the correctness of our results.

We emphasize that the approach chosen in this paper is rather general and we expect that it can be used to prove similar results for other model formulations, too. Furthermore, as the differentiability proof is based on the implicit function theorem this readily gives constructive instructions on how to derive analogous formulas for higher order derivatives.

Our results allow the efficient differentiation of the interaction potential and may therefore be applied in order to develop new algorithms for investigating stable particle configurations.

A. Appendix

Theorem A.1 (Whitney extension theorem). *Let $A \subseteq \mathbb{R}^n$ be closed, $m \in \mathbb{N} \cup \{\infty\}$, and $f_\alpha: A \rightarrow \mathbb{R}$ for all multi-indices $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m$. Suppose that for all multi-indices α with $|\alpha| \leq m$ and all $x, x' \in A$ the equality*

$$f_\alpha(x') = \sum_{|\beta| \leq m - |\alpha|} \frac{f_{\alpha+\beta}(x)}{\beta!} (x' - x)^\beta + R_\alpha(x'; x)$$

holds, where $R_\alpha: A \times A \rightarrow \mathbb{R}$ is such that for all $x_0 \in A$ and all $\varepsilon \in \mathbb{R}_{>0}$ there exists $\delta \in \mathbb{R}_{>0}$ such that

$$\forall x, x' \in A: \|x - x_0\| < \delta \wedge \|x' - x_0\| < \delta \implies |R_\alpha(x'; x)| \leq \|x - x'\|^{m-|\alpha|} \varepsilon.$$

Then there exists a function $F \in C^m(\mathbb{R}^n)$ such that $\partial^\alpha F(x) = f_\alpha(x)$ for all $x \in A$ and all multi-indices α with $|\alpha| \leq m$.

Proof. See [25, Theorem I]. □

Theorem A.2 (Implicit function theorem). *Let X, Y, Z be real Banach spaces, $A \subseteq X \times Y$ open, $F: A \rightarrow Z$ and $(x_0, y_0) \in A$ such that $F(x_0, y_0) = 0$. Suppose that the partial Fréchet-derivative F_y exists on A , and F and F_y are continuous in (x_0, y_0) .*

If $F_y(x_0, y_0)$ is invertible, then there exist an open neighborhood $\mathfrak{B}(x_0)$ and a unique function $y: \mathfrak{B}(x_0) \rightarrow Y$ such that $(x, y(x)) \in A$ and $F(x, y(x)) = 0$ for all $x \in \mathfrak{B}(x_0)$. Furthermore, if $F \in C^m(A, Z)$ for some $m \in \mathbb{N}$, then also $y \in C^m(\mathfrak{B}, Y)$.

Proof. See [17]. □

Lemma A.3 (Transformation of derivatives). *Suppose $X: \Omega_1 \rightarrow \Omega_2$ is a diffeomorphism and let $u \in H^2(\Omega_1)$. Then*

$$\int_{\Omega_2} \kappa(\Delta(u \circ X^{-1}))^2 + \sigma \|\nabla(u \circ X^{-1})\|^2 \, dx = \int_{\Omega_1} \kappa \frac{\operatorname{div}(A\nabla u)^2}{|\det DX|} + \sigma \|\nabla u\|_A^2 \, dx$$

where

$$A(x) := |\det DX(x)| (DX(x))^{-1} (DX(x))^{-T}$$

and

$$\|\nabla u(x)\|_{A(x)}^2 := \nabla u(x)^T A(x) \nabla u(x).$$

Proof. First note that for $x \in \Omega_2$ and $v \in H^1(\Omega)$

$$\nabla(v \circ X^{-1})(x) = (D(X^{-1})(x))^T \nabla v(X^{-1}(x)) = (DX(X^{-1}(x)))^{-T} \nabla v(X^{-1}(x)) \tag{A1}$$

holds almost-everywhere. Equation (A1) together with the transformation formula applied to the diffeomorphism X we obtain for all $v, w \in H^1(\Omega)$

$$\begin{aligned} & \int_{\Omega_2} (\nabla(v \circ X^{-1})(x))^T \nabla(w \circ X^{-1})(x) \, dx \\ &= \int_{\Omega_2} \nabla v(X^{-1}(x))^T (DX(X^{-1}(x)))^{-1} (DX(X^{-1}(x)))^{-T} \nabla w(X^{-1}(x)) \, dx \\ &= \int_{\Omega_1} \nabla v(x)^T DX(x)^{-1} DX(x)^{-T} \nabla w(x) |\det DX(x)| \, dx \\ &= \int_{\Omega_1} \langle \nabla v(x), \nabla w(x) \rangle_{A(x)} \, dx. \end{aligned} \tag{A2}$$

By integration by parts and (A2) we know that for all $\phi \in C_0^\infty(\Omega_1)$ it holds that

$$\begin{aligned} & \int_{\Omega_2} \Delta(u \circ X^{-1})(x) (\phi \circ X^{-1})(x) \, dx \\ &= - \int_{\Omega_2} \nabla(u \circ X^{-1})(x) \cdot \nabla(\phi \circ X^{-1})(x) \, dx + \int_{\partial\Omega_2} \partial_\nu(u \circ X^{-1})(x) (\phi \circ X^{-1})(x) \, dx \\ &= - \int_{\Omega_1} (A(x)\nabla u(x)) \cdot \nabla\phi(x) \, dx \\ &= \int_{\Omega_1} \operatorname{div}(A(x)\nabla u(x)) \phi(x) \, dx - \int_{\partial\Omega_1} (A(x)\nabla u(x)) \partial_\nu \phi(x) \, dx \\ &= \int_{\Omega_1} \operatorname{div}(A(x)\nabla u(x)) \phi(x) \, dx \end{aligned} \tag{A3}$$

where the boundary terms vanish as of $\phi \circ X^{-1} \in C_0^\infty(\Omega_2)$ and $\phi \in C_0^\infty(\Omega_1)$, respectively. On the other hand, application of the transformation formula also yields

$$\int_{\Omega_2} \Delta(u \circ X^{-1})(x) (\phi \circ X^{-1})(x) dx = \int_{\Omega_1} \Delta(u \circ X^{-1})(X(x)) \phi(x) |\det DX(x)| dx. \quad (\text{A4})$$

Combining (A3) and (A4) leads to

$$\int_{\Omega_1} |\det DX(x)| \Delta(u \circ X^{-1})(X(x)) \phi(x) dx = \int_{\Omega_1} \operatorname{div}(A(x)\nabla u(x)) \phi(x) dx$$

for all $\phi \in C_0^\infty(\Omega_1)$. The fundamental theorem of calculus of variations then readily implies that

$$\Delta(u \circ X^{-1})(X(x)) = \frac{\operatorname{div}(A(x)\nabla u(x))}{|\det DX(x)|} \quad (\text{A5})$$

holds for almost-every $x \in \Omega_1$. Because X is a diffeomorphism, this expression is well-defined as of $|\det DX(x)| \neq 0$ for all $x \in \Omega_1$. Hence, by virtue of the transformation formula and (A5) we obtain

$$\begin{aligned} \int_{\Omega_2} (\Delta(u \circ X^{-1})(x))^2 dx &= \int_{\Omega_1} \left(\Delta(u \circ X^{-1})(X(x)) \right)^2 |\det DX(x)| dx \\ &= \int_{\Omega_1} \frac{\operatorname{div}(A(x)\nabla u(x))^2}{|\det DX(x)|} dx. \end{aligned} \quad (\text{A6})$$

Finally, the desired assertion is a direct consequence of (A2) and (A6). \square

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