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A free boundary problem for binary fluids

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A free boundary problem for the dynamics of a glasslike binary fluid naturally leads to a singular perturbation problem for a strongly degenerate parabolic partial differential equation in 1D. We present a conjecture for an asymptotic formula for the velocity of the free boundary and prove a weak version of the conjecture. The results are based on the analysis of a family of local travelling wave solutions.

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1. Introduction

In [3], Benzi, Sbragaglia, Bernaschi and Succi propose a new phase-field model for binary fluid mixtures. The fluid mixture is described by an order parameter ϕ which, in 1D, satisfies the partial differential equation (PDE)

$$\begin{aligned}\phi_t &= \big((D_0 + D_2\phi^2)\phi_x\big)_x - D_2\phi\phi_x^2 + \phi(1-\phi^2)\\ &= (D_0 + D_2\phi^2)\phi_{xx} + D_2\phi\phi_x^2 + \phi(1-\phi^2) \quad \text{in } Q = (a,b)\times(0,T],\end{aligned} \tag{1.1}$$

where D_0 and D_2 are positive constants and $a < b$. The parabolic PDE (1.1) corresponds to a gradient flow in L^2 for the energy functional

$$F[\phi] = \int_a^b \big(V(\phi) + \tfrac{1}{2}D(\phi)(\phi')^2\big)\mathrm{d}x, \qquad V(\phi) = -\tfrac{1}{2}\phi^2 + \tfrac{1}{4}\phi^4, \quad D(\phi) = D_0 + D_2\phi^2.$$

The double-well potential $V(\phi)$ is the bulk free-energy density and favours the two bulk phases, $\phi = \pm 1$. Identifying the two phases $\phi > 0$ and $\phi < 0$ with two fluids, the stiffness function $D(\phi)$ controls the cost of building and maintaining the interface, $\phi = 0$, between the two fluids. Since the interface is a priori unknown, the interface is the free boundary of the binary fluid problem.

The model is a variant of the standard Ginzburg–Landau equation, in which the stiffness function is constant. The stiffness function determines the value of the surface tension γ. Positive values of γ, corresponding to $D(\phi) > 0$, promote coarsening. Negative values of γ, corresponding to $D(\phi) < 0$, trigger an unstable growth of the interface.

The model is characterised by the choice of $D(\phi) = D_0 + D_2\phi^2$. Other variants of the Ginzburg–Landau model have been proposed, with different choices of $D(\phi)$. Piecewise constant stiffness functions have been used to study microemulsions [9]. A second degree polynomial $D(\phi)$ was proposed by Gompper et al. in [10], with a crucial difference compared to the model that we study in this paper. In [10] D_0 is negative, triggering local interface instabilities, and the higher order term $\kappa(\phi'')^2$, with $\kappa > 0$ known as rigidity, was added to the energy functional to guarantee thermodynamic stability. In [3], on the other hand, D_0 is strictly positive, and this guarantees the surface tension γ to be positive and, hence, thermodynamic stability is reached even with $\kappa = 0$.

The idea of Benzi et al. is to take the limit $D_0 \to 0$ since, for $D_0 = 0$, a local zero-cost condition is achieved, $D(\phi)$ being zero only at the interface ($\phi = 0$). Local zero-cost, in turn, is responsible for the peculiar features of this model, that exhibits long-time relaxation, ageing and long-term *dynamical arrest*, typical signatures of self-glassiness. This makes the model suitable for the description of coarsening in soft-glassy materials, in which domain growth is subject to long-term slow-down and, possibly, dynamical arrest. In fact, using suitable approximations, the model can also be derived from the continuous equation of the mesoscopic Lattice Boltzmann equation of soft-glassy system described in [2].

Computational evidence in [3] suggests that the two-fluid interface $\phi = 0$ becomes almost immobile for small values of $\varepsilon = D_0/D_2$. The main purpose of the present paper is to prove this phenomenon analytically and possibly quantify it in terms of the smallness of ε.

Rescaling x by a factor $\sqrt{2/D_2}$ (and changing the interval (a, b) accordingly) and t by a factor 2, equation (1.1) becomes

$$\phi_t = (\varepsilon + \phi^2)\phi_{xx} + \phi\phi_x^2 + \tfrac{1}{2}\phi(1 - \phi^2) \quad \text{in } Q. \tag{1.2}$$

To understand its mathematical structure for small values of ε it is convenient to replace ϕ by

$$u(x,t) = U_\varepsilon\big(\phi(x,t)\big), \quad U_\varepsilon(\phi) = 2\int_0^\phi \sqrt{\varepsilon + s^2}\,\mathrm{d}s. \tag{1.3}$$

Since the map U_ε is strictly increasing, odd and onto in $\mathbb{R}$, its inverse Φ_ε is well defined and odd. This change of variable transforms (1.2) into

$$u_t = \big(\varepsilon + \Phi_\varepsilon^2(u)\big)u_{xx} + \Phi_\varepsilon(u)\big(1 - \Phi_\varepsilon^2(u)\big)\sqrt{\varepsilon + \Phi_\varepsilon^2(u)} \quad \text{in } Q. \tag{1.4}$$

In the formal limit $\varepsilon \to 0$, $U_\varepsilon(\phi) \to |\phi|\phi$ and $\Phi_\varepsilon(u) \to u/\sqrt{|u|}$. The limit equation is

$$u_t = |u|u_{xx} + u(1 - |u|) \quad \text{in } Q. \tag{1.5}$$

The parabolicity of equation (1.5) is strongly degenerate. A few decades ago it was studied, mostly without reaction term, in the context of *nonnegative* solutions [4–7, 12]. Several singular phenomena were identified which indicate that its degeneracy is indeed much stronger than that of, for example, the well-known porous medium and p-laplacian equations [8, 13]. We mention a few of them: solutions are not always uniquely determined by their initial-boundary data and the support of nonnegative solutions is non-expanding in time and may also shrink. In higher spatial dimension solutions of a slightly different but similar equation may even be discontinuous.

As far as we are aware of, sign-changing solutions of (1.5) were never studied analytically. In this paper we consider a class of classical solutions u_ε of the uniformly parabolic equation (1.4) which initially have a finite number of interfaces, and we show that for vanishing ε they converge to a well-defined solution u of the degenerate parabolic limit equation (1.5) and the interfaces of u are constant in time. In particular, we prove the phenomenon which was numerically observed in [3], namely that for positive but small values of ε the interfaces of u_ε are almost immobile.

To quantify the latter result we analyse the existence of a family of local *travelling wave solutions* of (1.4). As we shall see, this naturally leads to the following conjecture for an asymptotic formula of the velocity of the interface $x = \zeta_\varepsilon(t)$ (here for simplicity we consider the case that $u_\varepsilon(x,t)$ is monotonic in x, so the interface is unique):

$$\zeta_\varepsilon'(t) = \frac{u_x(x_1^+,t) - u_x(x_1^-,t)}{2\log\varepsilon}\big(1 + o(1)\big) \quad \text{as } \varepsilon \to 0. \tag{1.6}$$

Here x_1 is the position of the constant interface of the limit solution u. The travelling wave solutions do satisfy (1.6), and in that case the wave velocity and the right-hand side are independent of time. We also prove that the one-sided spatial derivatives of u at x_1 exist and are continuous with respect to t (with the possible exception of at most two values of t, see Theorem 3.2); generically they do not coincide, as was already observed in [3] in the case of stationary solutions (in [3] they were referred to as *compactons*). For the moment being we are not able to prove the conjecture, but we do prove a weaker version of (1.6) (see Theorem 3.3 and the discussion in Section 7).

The paper is organised as follows. In Section 2 we collect some preliminary results on the limit problem which will be proved in Section 8. In Section 3 we present the main results and in Section 4 we analyse the family of travelling wave solutions. In Section 5 we characterise the limit of solutions of (1.4) for vanishing ε and show that asymptotically, for vanishing ε, interfaces do not move. In Section 6 we prove the regularity result for the one-sided spatial derivatives of the limit solution, and in Section 7 we prove the weak version of the interface condition (1.6).

2. The limit problem

We consider the problem for equation (1.5) in a bounded interval (a,b), and impose that u is in one of phases $u = \pm 1$ at a and b:

$$\begin{cases} u_t = |u|u_{xx} + u(1-|u|) & \text{in } Q = (a,b)\times(0,T], \\ u(a,t) = -1,\ u(b,t) = 1 & \text{for } t \in (0,T], \\ u(x,0) = u_0(x) & \text{for } x \in (a,b). \end{cases} \tag{2.1}$$

Here $u_0 : [a,b] \to \mathbb{R}$ is a given initial function which satisfies

$$\begin{cases} u_0 \in C([a,b]);\ u_0(a) = -1,\ u_0(b) = 1; \\ \text{the number of zeros of } u_0 \text{ in } (a,b) \text{ is finite;} \\ u_0 \text{ changes sign at each of its zeros.} \end{cases} \tag{2.2}$$

Definition 2.1. Let $u \in L^\infty(Q) \cap L^2(0,T;H^1(a,b))$; u is called a *weak solution* of problem (2.1) if $u(a,t) = -1$ and $u(b,t) = 1$ for a.e. $t \in (0,T)$, and

$$\int_a^b u_0(x)\psi(x,0)\mathrm{d}x + \iint_Q \big(u\psi_t - |u|u_x\psi_x - u_x^2 \operatorname{sgn}(u)\psi + u(1-|u|)\psi\big)\,\mathrm{d}x\mathrm{d}t = 0$$

for all $\psi \in C_c^{1,1}((a,b)\times[0,T))$.

The strong degeneracy of the parabolicity causes singular phenomena which, since a few decades, are known for nonnegative solutions. In particular, nonnegative weak solutions have non-expanding spatial supports and suffer various nonuniqueness phenomena ([5, 7, 12]). There do exist nonnegative weak solutions with spatial supports which are independent of time.

To understand the case of sign-changing solutions we consider initial data which satisfy (2.2). To be more specific, let $x_1 < x_2 < \cdots < x_k$ be the finite number of zeros of u_0 in (a,b):

$$\mathfrak{N}_0 := \{x_1, x_2, \ldots, x_k\}, \qquad \begin{cases} u_0 > 0 \text{ in } (x_i, x_{i+1}) \text{ if } i \text{ is odd,} \\ u_0 < 0 \text{ in } (x_i, x_{i+1}) \text{ if } i \text{ is even} \end{cases} \tag{2.3}$$

(obviously $u_0 < 0$ in $[a, x_1)$ and $u_0 > 0$ in $(x_k, b]$). The idea is that we can solve the problem for u independently in each interval (x_i, x_{i+1}) with homogeneous Dirichlet data at x_i and x_{i+1}, without creating zeros of u at the interior of the interval. This naturally leads to the concept of "classical" solution of the equation.

Definition 2.2. Let (2.2) and (2.3) be satisfied, and set

$$\mathfrak{N} := \mathfrak{N}_0 \times [0,T]. \tag{2.4}$$

A function $u \in C(\overline{Q})$ is called a *classical solution* of problem (2.1) if

- $u(x,t) = 0$ if and only if $(x,t) \in \mathfrak{N}$,
- $u \in C^{2,1}(Q \setminus \mathfrak{N})$,
- the second and third equation of (2.1) are satisfied,
- the first equation of (2.1) is satisfied in $Q \setminus \mathfrak{N}$.

Observe that the continuity of u and hypothesis (2.2) on u_0 imply that, for all $t \in (0,T]$, $u(\cdot,t) \neq 0$ in (x_i, x_{i+1}), and $u(\cdot,t)$ has the same sign as u_0 in (x_i, x_{i+1}). The same observation applies to (a, x_1) and (x_k, b).

Problem (2.1) is well-posed in the class of classical solutions, and the class of classical solutions is a uniqueness class in the set of weak solutions:

Theorem 2.3. *Let u_0 satisfy (2.2) and (2.3). Then problem (2.1) admits a unique classical solution u in the sense of Definition 2.2. In addition u is also a weak solution of problem (2.1).*

The proof of Theorem 2.3 is based on standard techniques (such as maximum principle, a priori estimates, regularity theory and integral estimates for parabolic equations) and in Section 8 we sketch its main lines. Below (Theorem 3.2 and Section 6) we shall establish additional regularity properties of u.

3. Main results

The main results of the paper concern the behaviour of the unique solution $u_\varepsilon \in C^{2,1}(Q) \cap C(\overline{Q})$ of the problem

$$\begin{cases} u_t = \big(\varepsilon + \Phi_\varepsilon^2(u)\big)u_{xx} + \Phi_\varepsilon(u)\big(1 - \Phi_\varepsilon^2(u)\big)\sqrt{\varepsilon + \Phi_\varepsilon^2(u)} & \text{in } Q, \\ u(a,t) = -u_{1\varepsilon},\ u(b,t) = u_{1\varepsilon} & \text{for } t \in (0,T], \\ u(x,0) = u_{0\varepsilon}(x) & \text{for } x \in (a,b) \end{cases} \tag{3.1}$$

for small values of $\varepsilon > 0$. Here Φ_ε is defined as in the Introduction, $a < b$, $T > 0$ and $u_{1\varepsilon} > 0$ are constants such that

$$\Phi_\varepsilon(u_{1\varepsilon}) = 1 \quad \big(\Rightarrow \Phi_\varepsilon(-u_{1\varepsilon}) = -1, \ \text{ and } \ u_{1\varepsilon} \to 1 \text{ as } \varepsilon \to 0\big), \tag{3.2}$$

and the initial functions $u_{0\varepsilon}$ satisfy

$$\begin{cases} u_{0\varepsilon} \in C^\infty([a,b]), \quad u_{0\varepsilon}(a) = -u_{1\varepsilon}, \quad u_{0\varepsilon}(b) = u_{1\varepsilon}; \\ u_{0\varepsilon}(x) = 0 \Leftrightarrow x \in \mathfrak{N}_0, \quad u_{0\varepsilon} \to u_0 \text{ in } C([a,b]) \text{ as } \varepsilon \to 0. \end{cases} \tag{3.3}$$

The first result shows that u_ε converges uniformly to the solution of the limit problem for vanishing ε, which implies that away from the set $\mathfrak{N}$, defined by (2.4), $u_\varepsilon(x,t)$ has the same sign as $u_0(x)$ if ε is small enough. More precisely we have:

Theorem 3.1. *Let u_0 satisfy (2.2) and (2.3), let $\varepsilon > 0$ and let $u_{0\varepsilon}$ satisfy (3.3). Let $u_\varepsilon \in C^{2,1}(Q) \cap C(\overline{Q})$ be the solution of problem (3.1), u the unique classical solution of the limit problem (2.1) defined by Theorem 2.3, and $\mathfrak{N}$ the set defined by (2.4). Then $u_\varepsilon \to u$ uniformly in Q and in $C^{2,1}_{\mathrm{loc}}(Q \setminus \mathfrak{N})$, and*

$$\sup_{\{(x,t)\in Q;\, u_\varepsilon(x,t)=0\}} \operatorname{dist}\big((x,t), \mathfrak{N}\big) \to 0 \quad \textit{as } \varepsilon \to 0. \tag{3.4}$$

To state the asymptotic result on the free boundary condition (1.6) we need the following regularity result for the one-sided spatial derivatives of the limit problem.

Theorem 3.2. *Let u_0 satisfy (2.2) and let u be the unique classical solution of the limit problem (2.1), defined by Theorem 2.3. Then*

(i) *$u_x \in L^\infty(((a,b) \setminus \mathfrak{N}_0) \times (t_0, T))$ for all $t_0 \in (0,T)$, and, for all $t \in (0,T]$, the function $x \mapsto u_x(x,t)$ has at most a jump discontinuity at $x_i \in \mathfrak{N}_0$ $(i = 1, \dots, k)$;*

(ii) *for all $i = 1, \dots, k$ there exist $\tau_i^\pm \in [0, \infty]$ (independent of T) such that*

$$u_x(x_i^\pm, t) \begin{cases} = 0 & \textit{if } 0 < t < \tau_i^\pm, \\ \neq 0 & \textit{if } t > \tau_i^\pm; \end{cases}$$

in addition the functions $t \mapsto u_x(x_i^\pm, t)$ are continuous in $(0,T] \setminus \{\tau_i^\pm\}$.

Concerning the asymptotic expansion (1.6) we limit ourselves, for the sake of simplicity (see Remark 7.2), to the case of a strictly increasing initial function:

$$u_0 \in C([a,b]), \quad u_0 \text{ is strictly increasing in } [a,b], \quad u_0(a) = -1,\ u_0(b) = 1. \tag{3.5}$$

The condition on the approximating initial data $u_{0\varepsilon}$ ($\varepsilon > 0$) is changed accordingly:

$$\begin{cases} u_{0\varepsilon} \in C^\infty([a,b]), \quad u'_{0\varepsilon} > 0 \text{ in } [a,b], \quad u_{0\varepsilon}(a) = -u_{1\varepsilon},\, u_{0\varepsilon}(b) = u_{1\varepsilon}, \\ u_0(x_1) = 0 \Rightarrow u_{0\varepsilon}(x_1) = 0, \quad u_{0\varepsilon} \to u_0 \text{ in } C([a,b]) \text{ as } \varepsilon \to 0. \end{cases} \tag{3.6}$$

Theorem 3.3. *Let u_0 satisfy (3.5), let $\varepsilon > 0$ and let $u_{0\varepsilon}$ satisfy (3.6). Let $u_\varepsilon \in C^{2,1}(Q) \cap C(\overline{Q})$ be the solution of problem (3.1) and let u be the unique classical solution of the limit problem (2.1), defined by Theorem 2.3. Then $u_{\varepsilon x} > 0$ in Q. Let $x = X_\varepsilon(u,t)$ be defined by $u_\varepsilon(X_\varepsilon(u,t),t) = u$. Then there exists $0 < \delta_\varepsilon \to 0$ as $\varepsilon \to 0$ such that for all $t \in [0,T]$*

$$\left(\int_{-\delta_\varepsilon}^{\delta_\varepsilon} \frac{du}{\varepsilon + \Phi_\varepsilon^2(u)}\right)^{-1} \int_{-\delta_\varepsilon}^{\delta_\varepsilon} \frac{X_{\varepsilon t}(u,t)}{\varepsilon + \Phi_\varepsilon^2(u)} du = \frac{u_x(x_1^+,t) - u_x(x_1^-,t)}{2\log\varepsilon}\big(1 + o(1)\big) \quad \text{as } \varepsilon \to 0. \tag{3.7}$$

The interpretation is immediate: let $x = \zeta_\varepsilon(t)$ be the interface of $u_\varepsilon(x,t)$; then $\zeta'_\varepsilon(t) = X_{\varepsilon t}(0,t)$, so (3.7) is nothing else than the interface condition (1.6) with the left-hand side, $\zeta'_\varepsilon(t)$, replaced by a weighted average of $X_{\varepsilon t}(u,t)$ in a neighbourhood $(-\delta_\varepsilon, \delta_\varepsilon)$ of $u = 0$, a neighbourhood which shrinks to a single point as $\varepsilon \to 0$.

In Section 7 (see Remark 7.1) we shall briefly discuss a different weak version of (1.6).

4. Travelling waves

In this section we analyse a family of travelling wave solutions (TWs) which play a key role in the proof of Theorem 3.1 and the formulation of the conjecture (1.6).

Let $u(x,t) = w(x - ct)$ be a travelling wave solution of (1.4) with velocity $c \in \mathbb{R}$:

$$-cw' = \big(\varepsilon + \Phi_\varepsilon^2(w)\big)w'' + \Phi_\varepsilon(w)\big(1 - \Phi_\varepsilon^2(w)\big)\sqrt{\varepsilon + \Phi_\varepsilon^2(w)}. \tag{4.1}$$

Assuming monotonicity of $w(z)$ in an interval, we use w as independent variable to reduce the order of this autonomous ODE: the function $p(w)$, defined by

$$p\big(w(z)\big) = w'(z),$$

satisfies

$$\frac{dp}{dw} = -\frac{c}{\varepsilon + \Phi_\varepsilon^2(w)} - \frac{\Phi_\varepsilon(w)\big(1 - \Phi_\varepsilon^2(w)\big)}{p\sqrt{\varepsilon + \Phi_\varepsilon^2(w)}}. \tag{4.2}$$

We are interested in solutions of equation (4.2) in a neighbourhood of $w = 0$ with $p > 0$, solutions which in the original variables can be viewed as "local TWs" $w(z)$, with $w'(z) > 0$, which are defined in a neighbourhood of the interface. If p is strictly positive, the second term on the right hand side of (4.2) is bounded. Therefore we focus on the first term on the right-hand side which becomes singular at $w = 0$ for vanishing ε. Given $\delta > 0$, we have that

$$I_\delta := -c\int_0^\delta \frac{1}{\varepsilon + \Phi_\varepsilon^2(w)}\, dw = -2c\int_0^{\Phi_\varepsilon(\delta)} \frac{1}{\sqrt{\varepsilon + \phi^2}}\, d\phi.$$

We set $\phi = \sqrt{\varepsilon}\sinh y$. Since $d\phi = \sqrt{\varepsilon}\cosh y\, dy$, we obtain that

$$\int \frac{1}{\sqrt{\varepsilon + \phi^2}}\, d\phi = \int \frac{\cosh y}{\sqrt{1 + \sinh^2 y}}\, dy = y = \log\left(\frac{1}{\sqrt{\varepsilon}}\left(\phi + \sqrt{\varepsilon + \phi^2}\right)\right),$$

and

$$I_\delta = -2c \log\left(\frac{1}{\sqrt{\varepsilon}}\left(\Phi_\varepsilon(\delta) + \sqrt{\varepsilon + \Phi_\varepsilon^2(\delta)}\right)\right). \tag{4.3}$$

If $\frac{\Phi_\varepsilon(\delta)}{\sqrt{\varepsilon}}$ is large, then

$$I_\delta \approx -2c \log\left(\frac{2\Phi_\varepsilon(\delta)}{\sqrt{\varepsilon}}\right) = -2c\left(\log(2\Phi_\varepsilon(\delta)) - \tfrac{1}{2}\log\varepsilon\right) \approx c\log\varepsilon.$$

So if I_δ is bounded away from zero, c vanishes as $\varepsilon \to 0$, and since I_δ represents a variation in p, this simple calculation suggests the following result. Before stating it we observe that, given $B > 0$, the function $w_B \in C([0,\infty))$ defined by

$$w_B(x) = \max\{B\sinh x - \cosh x + 1, 0\} \quad \text{for } x \geqslant 0, \tag{4.4}$$

is a nonnegative steady state of the limit equation (1.5) in $[0,\infty)$ which satisfies $w_B(0) = 0$, $w_B'(0) = B$ and

$$w_B(x) > 0 \quad \text{if } \begin{cases} 0 < x < \log\frac{1+B}{1-B} & \text{if } 0 < B < 1, \\ x > 0 & \text{if } B \geqslant 1. \end{cases}$$

Lemma 4.1. *Let B and B_0 be positive constants. Let $w_B \in C([0,\infty))$ be the steady state defined by (4.4). Let $\varepsilon > 0$ and set*

$$c_\varepsilon = \frac{B - B_0}{\log\varepsilon}. \tag{4.5}$$

Let $w_{B,\varepsilon}$ be the local solution of the shooting problem

$$\begin{cases} -c_\varepsilon w' = \left(\varepsilon + \Phi_\varepsilon^2(w)\right)w'' + \Phi_\varepsilon(w)\left(1 - \Phi_\varepsilon^2(w)\right)\sqrt{\varepsilon + \Phi_\varepsilon^2(w)} & \text{for } x > 0, \\ w(0) = 0,\ w'(0) = B_0, \end{cases}$$

which can be continued as long as it stays bounded. Then

$$w_{B,\varepsilon} \to w_B \quad \text{in } \begin{cases} C^1_{\text{loc}}\left([0, \log\frac{1+B}{1-B})\right) & \text{if } 0 < B < 1, \\ C^1_{\text{loc}}\left([0,\infty)\right) & \text{if } B > 1. \end{cases} \tag{4.6}$$

Proof. Since $B_0 > 0$, $w_{B,\varepsilon}' > 0$ near $x = 0$. As long as $w_{B,\varepsilon}$ remains increasing and bounded, we argue as above and introduce $p_\varepsilon(w)$, which locally is a solution of

$$\begin{cases} \dfrac{dp}{dw} = -\dfrac{c_\varepsilon}{\varepsilon + \Phi_\varepsilon^2(w)} - \dfrac{\Phi_\varepsilon(w)\left(1 - \Phi_\varepsilon^2(w)\right)}{p\sqrt{\varepsilon + \Phi_\varepsilon^2(w)}} & \text{for } w > 0, \\ p(0) = B_0. \end{cases} \tag{4.7}$$

To understand the behaviour of p_ε near $w = 0$ we change variable and set

$$\begin{aligned} q_\varepsilon(w) &= p_\varepsilon(w) + \int_0^w \frac{c_\varepsilon}{\varepsilon + \Phi_\varepsilon^2(s)}\,ds \\ &= p_\varepsilon(w) + \frac{2(B - B_0)}{\log\varepsilon}\log\left[\frac{\Phi_\varepsilon(w) + \sqrt{\varepsilon + \Phi_\varepsilon^2(w)}}{\sqrt{\varepsilon}}\right], \end{aligned}$$

where we have used (4.3). The equation for q_ε is

$$\frac{dq_\varepsilon}{dw} = -\frac{\Phi_\varepsilon(w)\big(1-\Phi_\varepsilon^2(w)\big)}{\Big(q_\varepsilon(w) - \frac{2(B-B_0)}{\log\varepsilon}\log\Big[\frac{\Phi_\varepsilon(w)+\sqrt{\varepsilon+\Phi_\varepsilon^2(w)}}{\sqrt{\varepsilon}}\Big]\Big)\sqrt{\varepsilon+\Phi_\varepsilon^2(w)}}.$$

Let $w_0 > 0$ be a small number to be chosen below. If $0 \leqslant w \leqslant w_0$, then

$$1 \leqslant \frac{\Phi_\varepsilon(w)+\sqrt{\varepsilon+\Phi_\varepsilon^2(w)}}{\sqrt{\varepsilon}} \leqslant \frac{\Phi_\varepsilon(w_0)+\sqrt{\varepsilon+\Phi_\varepsilon^2(w_0)}}{\sqrt{\varepsilon}} = \frac{2\sqrt{w_0}}{\sqrt{\varepsilon}}\big(1+o(1)\big)$$

as $\varepsilon \to 0$, whence

$$-1+o(1) \leqslant \frac{2}{\log\varepsilon}\log\left[\frac{\Phi_\varepsilon(w)+\sqrt{\varepsilon+\Phi_\varepsilon^2(w)}}{\sqrt{\varepsilon}}\right] \leqslant 0 \quad \text{as } \varepsilon \to 0.$$

Since the factor $\Phi_\varepsilon(w)(1-\Phi_\varepsilon^2(w))/\sqrt{\varepsilon+\Phi_\varepsilon^2(w)}$ in the equation for q_ε is uniformly bounded, this implies that q_ε can be made arbitrarily near to $B_0 > 0$ in the interval $(0, w_0)$ by choosing w_0 and ε small enough.

If $B \geqslant B_0$ this means that $p_\varepsilon \geqslant q_\varepsilon$ is bounded away from 0 in $(0, w_0)$ if w_0 and ε are chosen small enough. If instead $B < B_0$, it is enough to slightly refine this argument and use that

$$(B-B_0)\big(1+o(1)\big) \leqslant -\frac{2(B-B_0)}{\log\varepsilon}\log\left[\frac{\Phi_\varepsilon(w)+\sqrt{\varepsilon+\Phi_\varepsilon^2(w)}}{\sqrt{\varepsilon}}\right] \leqslant 0 \quad \text{as } \varepsilon \to 0$$

uniformly in $(0, w_0)$, which implies that we can choose w_0 and ε so small that in $(0, w_0)$ the solution p_ε is contained in an interval which is only slightly larger than $[B, B_0]$ (and in particular bounded away from 0).

We check that p_ε changes by the "right" amount in the interval $(0, w_0)$, namely that $p_\varepsilon(w_0)$ can be made arbitrarily close to B by choosing w_0 and ε small enough. Since q_ε can be made arbitrarily close to B_0 in $(0, w_0)$, it is enough to observe that, as $\varepsilon \to 0$,

$$-\int_0^{w_0} \frac{c_\varepsilon}{\varepsilon+\Phi_\varepsilon^2(w)}\,dw = -\frac{2(B-B_0)}{\log\varepsilon}\log\left[\frac{\Phi_\varepsilon(w_0)+\sqrt{\varepsilon+\Phi_\varepsilon^2(w_0)}}{\sqrt{\varepsilon}}\right] \to B-B_0.$$

Turning to the original variables $w_{B,\varepsilon}(x)$, it follows that the slope $w'_{B,\varepsilon}(x)$ changes in a small right neighbourhood of the origin from B_0 to approximately B. For larger values of x we then use that $w_{B,\varepsilon}$ depends continuously on ε as long as it stays bounded and positive. This easily leads to (4.6). □

The above result has its natural counterpart for *nonpositive* steady states $\tilde{w}_A$ of the limit equation with support in $(-\infty, 0]$,

$$\tilde{w}_A(x) = \min\{A\sinh x + \cosh x - 1, 0\} \quad \text{for } x \leqslant 0. \tag{4.8}$$

Corollary 4.2. *Let A and A_0 be positive constants. Let $\tilde{w}_A \in C((-\infty, 0])$ be the steady state defined by* (4.8)*. Let $\varepsilon > 0$ and set*

$$\tilde{c}_\varepsilon = -\frac{A-A_0}{\log\varepsilon}. \tag{4.9}$$

Let $\tilde{w}_{A,\varepsilon}$ be the local solution of the shooting problem

$$\begin{cases} -\tilde{c}_\varepsilon w' = \big(\varepsilon + \Phi_\varepsilon^2(w)\big)w'' + \Phi_\varepsilon(w)\big(1 - \Phi_\varepsilon^2(w)\big)\sqrt{\varepsilon + \Phi_\varepsilon^2(w)} & \text{for } x < 0, \\ w(0) = 0, \ w'(0) = A_0, \end{cases}$$

which can be continued as long as it stays bounded. Then

$$\tilde{w}_{A,\varepsilon} \to \tilde{w}_A \quad \text{in} \begin{cases} C_{\text{loc}}\big((-\log \tfrac{1+A}{1-A}, 0]\big) & \text{if } 0 < A < 1, \\ C_{\text{loc}}\big((-\infty, 0]\big) & \text{if } A > 1. \end{cases} \tag{4.10}$$

We can easily merge the approximations of a nonnegative steady state in $[0, \infty)$ and a nonpositive one in $(-\infty, 0]$ to construct an approximation $w_{AB,\varepsilon}$ of the following steady state with changing sign:

$$w_{AB}(x) = \begin{cases} \tilde{w}_A(x) = \min\{A \sinh x + \cosh x - 1, 0\} & \text{if } x \leqslant 0, \\ w_B(x) = \max\{B \sinh x - \cosh x + 1, 0\} & \text{if } x > 0. \end{cases} \tag{4.11}$$

To do so we require that the two wave velocities coincide, $c_\varepsilon = \tilde{c}_\varepsilon$, as well as the two shooting parameters, $A_0 = B_0$. This means that

$$A_0 = B_0 = \tfrac{1}{2}(A + B), \qquad c_\varepsilon = \tilde{c}_\varepsilon = \frac{B - A}{2 \log \varepsilon}.$$

Combining Lemma 4.1 and Corollary 4.2, we obtain the following result.

Theorem 4.3. *Let A and B be positive constants and let $\varepsilon \in (0, 1]$. Let $w_{AB} \in C(\mathbb{R})$ be the steady state of the limit equation (1.5) defined by (4.11). Then there exists a travelling wave solution $w_{AB,\varepsilon}(x - c_\varepsilon t)$ of equation (1.4) with velocity*

$$c_\varepsilon = \frac{B - A}{2 \log \varepsilon} \tag{4.12}$$

such that $w_{AB,\varepsilon}(0) = 0$ and $w_{AB,\varepsilon} \to w_{AB}$ in $C_{\text{loc}}(J_{AB})$ as $\varepsilon \to 0$, where

$$J_{AB} = \begin{cases} \mathbb{R} & \text{if } A, B \geqslant 1, \\ \big(-\log \tfrac{1+A}{1-A}, \log \tfrac{1+B}{1-B}\big) & \text{if } A, B \in (0, 1), \\ \big(-\infty, \log \tfrac{1+B}{1-B}\big) & \text{if } A \geqslant 1, \ 0 < B < 1, \\ \big(-\log \tfrac{1+A}{1-A}, \infty\big) & \text{if } 0 < A < 1, \ B \geqslant 1. \end{cases}$$

For later use we observe that if $A > 1$, the steady state $\tilde{w}_A \in C^1((-\infty, 0])$, defined by (4.8), has an inflection point at $-\frac{1}{2} \log \frac{A+1}{A-1}$ and

$$\tilde{w}_A' \geqslant \tilde{w}_A'\big(-\tfrac{1}{2} \log \tfrac{A+1}{A-1}\big) = \sqrt{A^2 - 1} \quad \text{in } (-\infty, 0]. \tag{4.13}$$

5. Convergence to the limit problem

In this section we prove Theorem 3.1. Let u_0 satisfy (2.2) and (2.3), let $u_{0\varepsilon}$ satisfy (3.3), let $u_\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ be the unique smooth solution of problem (3.1) and let $u \in C(\overline{Q}) \cap C^{2,1}(Q \setminus \mathfrak{N})$ be the unique classical solution of the limit problem (2.1). We recall that $\mathfrak{N}_0 \subset (a,b)$ is the set containing the k zeros x_i of u_0 and that $\mathfrak{N} = \mathfrak{N}_0 \times [0,T]$.

We choose one of the intervals (x_i, x_{i+1}). We assume that

$$u_0 > 0 \quad \text{in } (x_i, x_{i+1}).$$

The case that $u_0 < 0$ in (x_i, x_{i+1}) can be treated similarly, and also the intervals (a, x_1) and (x_k, b) are treated in a similar way.

We claim that it is enough to prove the following:

For all $n \in \mathbb{N}$, $n > 2(x_{i+1} - x_i)^{-1}$, there exists $\varepsilon_n \in (0, \varepsilon_{n-1})$ such that

$$u_\varepsilon(x_i + \tfrac{1}{n}, t) \geqslant 0 \text{ and } u_\varepsilon(x_{i+1} - \tfrac{1}{n}, t) \geqslant 0 \text{ for } 0 < t < T \text{ and for all } 0 < \varepsilon < \varepsilon_n. \tag{5.1}$$

Indeed, (5.1) and the Comparison Principle in $\big(x_i + \frac{1}{n}, x_{i+1} - \frac{1}{n}\big) \times (0,T)$ imply that

$$u_\varepsilon \geqslant \underline{v}_{\varepsilon,n} \quad \text{in } \big(x_i + \tfrac{1}{n}, x_{i+1} - \tfrac{1}{n}\big) \times (0,T) \text{ if } \varepsilon < \varepsilon_n,$$

where $\underline{v}_{\varepsilon,n}$ is the unique smooth solution of the problem

$$\begin{cases} v_t = \big(\varepsilon + \Phi_\varepsilon^2(v)\big)v_{xx} + \Phi_\varepsilon(v)\big(1 - \Phi_\varepsilon^2(v)\big)\sqrt{\varepsilon + \Phi_\varepsilon^2(v)} & \text{in } \big(x_i + \frac{1}{n}, x_{i+1} - \frac{1}{n}\big) \times (0,T], \\ v\big(x_i + \frac{1}{n}, t\big) = v(x_{i+1} - \frac{1}{n}, t) = 0 & \text{for } t \in (0,T], \\ v(x,0) = u_{0\varepsilon}(x) & \text{for } x \in \big(x_i + \frac{1}{n}, x_{i+1} - \frac{1}{n}\big), \\ v > 0 & \text{in } \big(x_i + \frac{1}{n}, x_{i+1} - \frac{1}{n}\big) \times (0,T]. \end{cases} \tag{5.2}$$

Since $\underline{v}_{\varepsilon,n+1} \geqslant \underline{v}_{\varepsilon,n} > 0$ in $(x_i + \frac{1}{n}, x_{i+1} - \frac{1}{n}) \times (0,T]$, this implies that there exists

$$\underline{v}_\varepsilon(x,t) = \lim_{n\to\infty} \underline{v}_{\varepsilon,n}(x,t) \quad \text{for } (x,t) \in (x_i, x_{i+1}) \times [0,T].$$

On the other hand we use the properties in the intervals (x_{i-1}, x_i) and (x_{i+1}, x_{i+2}) which are analogous to (5.1) and which, put together, imply that

for all sufficiently large $n \in \mathbb{N}$ there exists $\tilde{\varepsilon}_n \in (0, \varepsilon_{n-1})$ such that

$$u_\varepsilon(x_i - \tfrac{1}{n}, t) < 0 \text{ and } u_\varepsilon(x_{i+1} + \tfrac{1}{n}, t) < 0 \text{ for } 0 < t < T \text{ and for all } 0 < \varepsilon < \tilde{\varepsilon}_n.$$

Let $\overline{v}_{\varepsilon,n}$ be the unique smooth solution of problem (5.2) with $x_i + \frac{1}{n}$ and $x_{i+1} - \frac{1}{n}$ replaced by, respectively, $x_i - \frac{1}{n}$ and $x_{i+1} + \frac{1}{n}$. Then

$$u_\varepsilon \leqslant \overline{v}_{\varepsilon,n} \quad \text{in } \big(x_i - \tfrac{1}{n}, x_{i+1} + \tfrac{1}{n}\big) \times (0,T) \text{ if } \varepsilon < \tilde{\varepsilon}_n,$$

and since $\overline{v}_{\varepsilon,n+1} \leqslant \overline{v}_{\varepsilon,n}$ in $(x_i - \frac{1}{n+1}, x_{i+1} + \frac{1}{n+1}) \times (0,T]$ we may define

$$\overline{v}_\varepsilon(x,t) = \lim_{n\to\infty} \overline{v}_{\varepsilon,n}(x,t) \quad \text{for } (x,t) \in [x_i, x_{i+1}] \times [0,T].$$

Hence $0 < \underline{v}_\varepsilon \leqslant \overline{v}_\varepsilon$ in $(x_i, x_{i+1}) \times [0,T]$ and $\overline{v}_\varepsilon(x_i,t) = \overline{v}_\varepsilon(x_{i+1},t) = 0$. By local Schauder type estimates (see [11, Chapter 5, Theorem 5.4]), in $(x_i, x_{i+1}) \times (0,T]$ we may pass to the limit $n \to \infty$ in the equation for $\underline{v}_{\varepsilon,n}$ and $\overline{v}_{\varepsilon,n}$, whence both $\underline{v}_\varepsilon$ and $\overline{v}_\varepsilon$ coincide with the unique solution of the problem

$$\begin{cases} u_t = \big(\varepsilon + \Phi_\varepsilon^2(u)\big)u_{xx} + \Phi_\varepsilon(u)\big(1 - \Phi_\varepsilon^2(u)\big)\sqrt{\varepsilon + \Phi_\varepsilon^2(u)} & \text{in } (x_i, x_{i+1}) \times (0,T], \\ u(x_i,t) = u(x_{i+1},t) = 0 & \text{for } t \in (0,T], \\ u(x,0) = u_{0\varepsilon}(x) & \text{for } x \in (x_i, x_{i+1}), \\ u > 0 & \text{in } (x_i, x_{i+1}) \times (0,T]. \end{cases}$$

Now Theorem 3.1 would follow from a standard diagonal procedure if we could use again local Schauder type estimates in order to pass to the limit $\varepsilon \to 0$ and conclude that $\underline{v}_\varepsilon = \overline{v}_\varepsilon$ converges to the unique (by Theorem 2.3) solution of the problem

$$\begin{cases} u_t = uu_{xx} + u(1-u) & \text{in } (x_i, x_{i+1}) \times (0,T], \\ u(x_i,t) = u(x_{i+1},t) = 0 & \text{for } t \in (0,T], \\ u(x,0) = u_0(x) & \text{for } x \in (x_i, x_{i+1}), \\ u > 0 & \text{in } (x_i, x_{i+1}) \times (0,T]. \end{cases}$$

To justify this, we only need to prove that locally in $\big(x_i + \frac{1}{n}, x_{i+1} - \frac{1}{n}\big) \times [0,T]$ the solution $\underline{v}_{\varepsilon,n}$ is uniformly bounded away from 0, which makes the equation for $\underline{v}_{\varepsilon,n}$ locally uniformly parabolic. But establishing such uniform lower bound is easy: given $x_0 \in (x_i, x_{i+1})$ there exists a "small" steady state solution $\tilde{u}(x)$ of the limit equation which is strictly positive in a "small" neighbourhood $\mathfrak{U}$ of x_0, where "small" means that $\tilde{u} \leqslant \frac{1}{2}u_0$ in $\mathfrak{U}$ and $\operatorname{supp}\tilde{u} \subset (x_i, x_{i+1})$. Arguing as in Section 4 it easily follows that $\tilde{u}$ can be approximated by a family of steady state solutions of the equation with ε which can be used as subsolutions of the problem for $\underline{v}_{\varepsilon,n}$, and since x_0 is arbitrary we obtain the uniform lower bound for $\underline{v}_{\varepsilon,n}$, and thus for $\underline{v}_\varepsilon$. We leave the details to the reader.

It remains to prove (5.1).

Proof of (5.1). Since $u_{0\varepsilon} \to u_0$ uniformly in (a,b), for all $n \in \mathbb{N}$ there exist $m_n > 0$ and $\varepsilon_{1n} > 0$ such that $\Phi_\varepsilon^2(y) < 1$ if $y \in (0, m_n]$ and

$$u_{0\varepsilon} > m_n \quad \text{in } (x_i + \tfrac{1}{2n}, x_{i+1} - \tfrac{1}{2n}) \quad \text{for all } 0 < \varepsilon < \varepsilon_{1n}.$$

Let the steady state w_{AB} be defined by (4.11). Observe that u_ε is uniformly bounded in Q: it follows from the definition of $u_{1\varepsilon} \in \mathbb{R}$ in (3.2) that the constant $C_{1\varepsilon} = \max\{u_{1\varepsilon}, \sup_{(a,b)} u_{0\varepsilon}\}$ is a supersolution of problem (3.1) and, similarly, $C_{2\varepsilon} = \min\{-u_{1\varepsilon}, \inf_{(a,b)} u_{0\varepsilon}\}$ a subsolution, whence the result follows from the Comparison Principle for parabolic equations and the uniform boundedness of the constants $C_{1\varepsilon}$ and $C_{2\varepsilon}$. Hence there exists $M > 0$ such that $u_\varepsilon > -M$ in Q for all $\varepsilon \in (0,1]$. It follows from (4.13) that for all $n \in \mathbb{N}$ there exists $A_n > 1$ such that, independently of the choice of $B > 0$,

$$w_{A_nB}(y) < -M - 1 \quad \text{for all } y \leqslant -\tfrac{1}{2n}.$$

On the other hand there exist $y_n \in (\frac{1}{2n}, \frac{1}{n})$ and $B_n \in (0,1)$ such that

$$\tfrac{1}{n} - y_n < \tfrac{1}{2}\log\tfrac{1+B_n}{1-B_n} < \tfrac{1}{2}\big(\tfrac{1}{2}(x_i + x_{i+1}) - \tfrac{1}{n}\big), \qquad w_{A_nB_n}\big(\tfrac{1}{2}\log\tfrac{1+B_n}{1-B_n}\big) < m_n$$

(we recall that $\frac{1}{2}\log\frac{1+B_n}{1-B_n}$ is the point where $w_{A_nB_n}$ attains its maximal value). Finally, we choose a point $z_n \in (\frac{1}{2}\log\frac{1+B_n}{1-B_n}, \log\frac{1+B_n}{1-B_n})$, which implies that $w_{A_nB_n} > 0$ and $w'_{A_nB_n} < 0$ at z_n.

Let $w_{A_nB_n,\varepsilon}$ be defined by Theorem 4.3. By the choice of A_n, B_n and z_n and by Theorem 4.3, there exists $\varepsilon_{2n} \leqslant \varepsilon_{1n}$ such that for all $\varepsilon \in (0, \varepsilon_{2n})$

$$w_{A_nB_n,\varepsilon}(y) < -M \text{ for } y \leqslant -\tfrac{1}{2n}, \quad 0 < w_{A_nB_n,\varepsilon} < m_n \text{ in } (0, z_n], \quad w'_{A_nB_n,\varepsilon}(z_n) < 0,$$

and

$$c_\varepsilon T = \frac{(B_n - A_n)T}{2\log\varepsilon} < \frac{1}{n} - y_n.$$

We define $\underline{u}_{\varepsilon,n} \in C([x_i, x_{i+1}] \times [0, T])$ by

$$\underline{u}_{\varepsilon,n}(x,t) = \begin{cases} w_{A_nB_n,\varepsilon}(x - x_i - y_n - c_\varepsilon t) & \text{if } x_i \leqslant x \leqslant x_i + y_n + c_\varepsilon t + z_n,\ 0 \leqslant t \leqslant T, \\ w_{A_nB_n,\varepsilon}(x_{i+1} - y_n - c_\varepsilon t - x) & \text{if } x_{i+1} - y_n - c_\varepsilon t - z_n \leqslant x \leqslant x_{i+1},\ 0 \leqslant t \leqslant T, \\ w_{A_nB_n,\varepsilon}(z_n) & \text{otherwise.} \end{cases}$$

By construction $\underline{u}_{\varepsilon,n}$ is a (weak) subsolution of the parabolic equation for u_ε in $(x_i, x_{i+1}) \times (0, T]$. In addition $\underline{u}_{\varepsilon,n} < -M \leqslant u_\varepsilon$ at $\{x_i, x_{i+1}\} \times (0, T)$ and $\underline{u}_{\varepsilon,n}(x, 0) \leqslant u_{0\varepsilon}(x)$ for $x \in (x_i, x_{i+1})$. Hence $u_\varepsilon \geqslant \underline{u}_{\varepsilon,n}$ in $[x_i, x_{i+1}] \times (0, T]$. Since

$$x_i + y_n + c_\varepsilon T < x_i + \tfrac{1}{n} \quad \text{and} \quad x_{i+1} - \tfrac{1}{n} < x_{i+1} - y_n - c_\varepsilon T$$

and $\underline{u}_{\varepsilon,n}(x,t) > 0$ if $x_i + y_n + c_\varepsilon T < x < x_{i+1} - y_n - c_\varepsilon T$, this implies that $u_\varepsilon(x,t) > 0$ if $x = x_i + \frac{1}{n}$ or $x = x_{i+1} - \frac{1}{n}$. □

6. Boundary regularity of the solution of the limit problem

In this section we prove Theorem 3.2, which exclusively concerns the unique solution u of the limit problem (2.1).

6.1 *Proof of Theorem* 3.2 (i)

We show first that it is sufficient to prove the following lemma, which reminds the well-known estimate by Aronson and Benilan [1] for nonnegative solutions of the porous medium equation $u_t = \Delta(u^m)$ if $m > 1$ (see also [4] in the case of nonnegative solutions of $u_t = u\Delta u - \gamma|\nabla u|^2$).

Lemma 6.1. *Let $x_i, x_{i+1} \in \mathfrak{N}_0$ and let u_i be the restriction of u to $\overline{Q}_i$, where we have set $Q_i = (x_i, x_{i+1}) \times (0, T]$. Then*

$$u_i > 0 \text{ in } Q_i \;\Rightarrow\; u_{it} \geqslant -\frac{u_i}{t} \quad \text{and} \quad u_{ixx} \geqslant -(1 - u_i) - \frac{1}{t} \quad \text{in } Q_i,$$
$$u_i < 0 \text{ in } Q_i \;\Rightarrow\; u_{it} \leqslant -\frac{u_i}{t} \quad \text{and} \quad u_{ixx} \leqslant -(1 - u_i) - \frac{1}{t} \quad \text{in } Q_i.$$

Consider for example the case that $u_i > 0$. We recall that u_{ix} is continuous in Q_i. By Lemma 6.1, u_{ixx} is bounded from below in $(x_i, x_{i+1}) \times [t_0, T]$ for all $t_0 \in (0, T]$. Since u_i is bounded, this implies that the following one-sided limits are well defined:

$$u_{ix}(x_i^+, t) \in [-\infty, \infty), \quad u_{ix}(x_{i+1}^-, t) \in (-\infty, +\infty].$$

Since the function $x \mapsto u_i(x, t)$ attains a minimum (=0) at x_i and x_{i+1}, we conclude that the limits $u_{ix}(x_i^+, t)$ and $u_{ix}(x_{i+1}^-, t)$ are bounded in $[t_0, T]$ for $t_0 \in (0, T)$. If $u_i < 0$ one argues similarly to arrive at the same conclusion. To complete the proof of part (i) it remains to prove Lemma 6.1.

Proof of Lemma 6.1. The inequalities for u_{it} and u_{ixx} in Lemma 6.1 are equivalent. Below we prove the inequality for u_{it} in the case that $u_i > 0$ in Q_i.

As we shall see in Section 8, u_i can be approximated from above by smooth solutions $u_{i,n} \geqslant \frac{1}{n}$ of the uniformly parabolic problem

$$\begin{cases} u_t = u(u_{xx} + 1 - u) & \text{in } Q_i, \\ u(x_i, t) = u(x_{i+1}, t) = \frac{1}{n} & \text{for } t \in (0, T], \\ u(x, 0) = u_0(x) + \frac{1}{n} & \text{for } x \in (x_i, x_{i+1}), \end{cases} \tag{6.1}$$

where $n \in \mathbb{N}$. We set

$$p = \frac{(u_{i,n})_t}{u_{i,n}} \quad \text{in } Q_i.$$

Then $(u_{i,n})_t = pu_{i,n}$ and $p = (u_{i,n})_{xx} + 1 - u_{i,n}$, whence

$$\begin{aligned} p_t &= (u_{i,n})_{txx} - (u_{i,n})_t = (pu_{i,n})_{xx} - pu_{i,n} = u_{i,n} p_{xx} + 2(u_{i,n})_x p_x + \big((u_{i,n})_{xx} - u_{i,n}\big) p \\ &= u_{i,n} p_{xx} + 2(u_{i,n})_x p_x + (p-1)p \qquad \text{in } Q_i. \end{aligned}$$

Let $t_0 \in (0, T)$ Since $p = 0$ at x_i and x_{i+1} and since $\underline{p} = -(t - t_0)^{-1}$ is a subsolution of the equation in $(x_i, x_{i+1}) \times (t_0, T)$ which tends to $-\infty$ as $t \to t_0^+$, it follows easily from the Comparison Principle that $p \geqslant \underline{p}$, i.e., $(t - t_0)(u_{i,n})_t \geqslant -u_{i,n}$, in $(x_i, x_{i+1}) \times (t_0, T)$. Since $u_{i,n} \to u_i$ in $C^{2,1}_{\text{loc}}(Q_i)$ as $n \to \infty$, this implies that $(t - t_0)u_{it} \geqslant -u_i$ in $(x_i, x_{i+1}) \times (t_0, T)$. Since $t_0 > 0$ is arbitrary, we have proved that $tu_{it} \geqslant -u_i$ in Q_i. □

6.2 *Proof of Theorem* 3.2 (ii)*(ii)*

Let u_i and Q_i be as above. We only consider the limit $u_i(x_i^+, t)$ in the case that $u_i > 0$ in Q_i. We define the difference quotient

$$q(x, t) = \frac{u_i(x, t) - u_i(x_i, t)}{x - x_i} = \frac{u_i(x, t)}{x - x_i} \quad \text{for } t \in (t_0, T].$$

Since, by part (i), $u_{it} \geqslant -u_i/t$ in Q_i, it follows at once that also $q_t \geqslant -q/t$ in Q_i. Integration with respect to t yields that

$$q(x, t) \geqslant \frac{t_0}{t} q(x, t_0) \quad \text{if } 0 < t_0 < t \leqslant T, \tag{6.2}$$

whence also

$$u_{ix}(x_i^+, t) \geqslant \frac{t_0}{t} u_{ix}(x_i^+, t_0) \quad \text{for } t \in (t_0, T].$$

In particular, $u_{ix}(x_i^+, t) > 0$ if $u_{ix}(x_i^+, t_0) > 0$ and $t > t_0$, and we have proved the existence of $\tau_i^+ \in [0, \infty]$.

The function $t \mapsto u_{ix}(x_i^+, t)$ is continuous in $(0, \tau_i^+)$ since $u_{ix}(x_i^+, t) = 0$ if $t \in (0, \tau_i^+)$. To prove the continuity in $(\tau_i^+, T]$ we shall show that

$$q : (x_i, x_{i+1}) \times (\tau_i^+, T] \to \mathbb{R} \text{ can be extended by continuity to } [x_i, x_{i+1}) \times (\tau_i^+, T]. \qquad (6.3)$$

The function q satisfies

$$q_t = \frac{u_{it}}{x - x_i} = q u_{ixx} + q(1 - u_i) = q\big((x - x_i)q_{xx} + 2q_x\big) + q(1 - (x - x_i)q) \ \text{ in } (x_i, x_{i+1}) \times (\tau_i^+, T].$$

Setting $y = \sqrt{x - x_i}$ and $h(y, t) = q(x_i + y^2, t)$, we obtain that

$$h_t = \frac{h}{4}\left(h_{yy} + 3\frac{h_y}{y}\right) + h(1 - y^2 h) \quad \text{in } (0, \sqrt{x_{i+1} - x_i}) \times (\tau_i^+, T].$$

Let B be the open ball in $\mathbb{R}^4$ centered at the origin with radius $\sqrt{x_{i+1} - x_i}$. Setting $y = |z|$ for $z \in B$ and $v(z, t) = h(|z|, t)$ for $(z, t) \in B \times (\tau_i^+, T]$, this means that

$$v_t = \frac{v}{4}\Delta v + v(1 - |z|^2 v) \quad \text{in } B \times (\tau_i^+, T].$$

It follows from (6.2) and the definition of τ_i^+ that, locally in $B \times (\tau_i^+, T]$, v is bounded away from zero, whence locally in $B \times (\tau_i^+, T]$ the equation for v is uniformly parabolic. This implies the continuity of v and we have proved (6.3). This completes the proof of Theorem 3.2.

Remark 6.2. It is not difficult to show that not only the functions $t \mapsto u_x(x_i^{\pm}, t)$ are continuous in $(0, T] \setminus \{\tau_i^{\pm}\}$, but also the restriction of u_x to $(x_i, x_{i+1}) \times (0, T]$ can be extended with continuity to the set $[x_i, x_{i+1}] \times (0, T] \setminus \{(x_i, \tau_i^+), (x_{i+1}, \tau_{i+1}^-)\}$.

Remark 6.3. The number $\tau_i^{\pm}$ reminds the concept of *waiting time* for the interfaces of the porous medium equation, which is always a finite number (see Theorem 15.15, equation (15.61) and Corollary 15.23 in [13]). It is natural to ask whether also in our case $\tau_i^{\pm}$ is always finite. The answer is negative, as the following example shows.

Let u_0 be strictly increasing in (a, b), let $u_0(x_1) = 0$ and let $\log u_0 \notin L^1(x_1, b)$. Let $\psi \in C_c^1([x_1, b))$ be such that $\psi(x_1) = 1$ and $\psi' \leqslant 0$ in (x_1, b) and set

$$\chi_n(x) = \begin{cases} 0 & \text{if } x_1 \leqslant x \leqslant x_1 + \frac{1}{n}, \\ n\big(x - x_i - \frac{1}{n}\big) & \text{if } x_1 + \frac{1}{n} < x < x_1 + \frac{2}{n}, \\ 1 & \text{if } x_1 + \frac{2}{n} \leqslant x \leqslant b. \end{cases}$$

Then $u_x > 0$ in $(x_1, b) \times (0, T)$ and

$$\begin{aligned} &\int_{x_1}^b \log u(x, t)\psi(x)\chi_n(x)\mathrm{d}x \\ &\quad = \int_{x_1}^b \log u_0(x)\psi(x)\chi_n(x)\mathrm{d}x + \iint_{(x_1,b)\times(0,t)} \big(-u_x(\psi'\chi_n + \psi\chi_n') + (1 - u)\psi\chi_n\big) \\ &\quad \leqslant \int_{x_i}^b \log u_0(x)\psi(x)\chi_n(x)\mathrm{d}x + \iint_{(x_1,b)\times(0,t)} \big(-u_x\psi'\chi_n + (1 - u)\psi\chi_n\big). \end{aligned}$$

Letting $n \to \infty$ we obtain that

$$\int_{x_1}^{b} \log u(x,t)\psi(x)\mathrm{d}x = -\infty \quad \text{for all } t \in (0,T],$$

and since T is arbitrary this implies that $\tau_1^+ = \infty$.

7. The interface condition

This section is devoted to the proof of Theorem 3.3 concerning the asymptotic formula for the interface condition (1.6).

The fact that $u_{\varepsilon x} > 0$ in Q follows from the strong maximum principle applied to the equation for $u_{\varepsilon x}$. Hence the interface $x = \zeta_\varepsilon(t)$ is well-defined by $u_\varepsilon(\zeta_\varepsilon(t),t) = 0$ and, by the implicit function theorem, $\zeta_\varepsilon \in C([0,T]) \cap C^1((0,T])$. By Theorem 3.1, in particular by (3.4),

$$\zeta_\varepsilon \to x_1 \quad \text{in } C([0,T]) \text{ as } \varepsilon \to 0.$$

For the limit function u we have a similar positivity result of u_x, but we must exclude the point x_1 where the parabolic equation degenerates:

$$u_x > 0 \quad \text{in } Q \setminus \big\{(x_1,t);\ t \in (0,T]\big\}. \tag{7.1}$$

We begin with some preliminary calculations. Let $X_\varepsilon : [-u_{1\varepsilon}, u_{1\varepsilon}] \times [0,T] \to [a,b]$ be defined by

$$u_\varepsilon\big(X_\varepsilon(u,t),t\big) = u \quad \big(\text{and so } \zeta_\varepsilon(t) = X_\varepsilon(0,t)\big).$$

Differentiating $x = X_\varepsilon(u,t)$ with respect to x and t we find that

$$X_{\varepsilon u}(u,t) = \frac{1}{u_{\varepsilon x}\big(X_\varepsilon(u,t),t\big)}, \quad X_{\varepsilon t}(u,t) = -\frac{u_{\varepsilon t}\big(X_\varepsilon(u,t),t\big)}{u_{\varepsilon x}\big(X_\varepsilon(u,t),t\big)},$$

and it follows from the equation for u_ε that X_ε satisfies the parabolic equation

$$X_t = -\big(\varepsilon + \Phi_\varepsilon^2(u)\big)\left(\frac{1}{X_u}\right)_u - \Phi_\varepsilon(u)\big(1 - \Phi_\varepsilon^2(u)\big)\sqrt{\varepsilon + \Phi_\varepsilon^2(u)}\,X_u. \tag{7.2}$$

We define $X_0 : [-1,1] \times [0,T] \to [a,b]$ by

$$u\big(X_0(u,t),t\big) = u \quad \big(\text{and so } X_0(0,t) = x_1\big),$$

and one shows in a similar way that X_0 satisfies

$$X_t = -|u|\left(\frac{1}{X_u}\right)_u - u(1-|u|)X_u \quad \text{in } \big([-1,0) \cup (0,1]\big) \times (0,T]. \tag{7.3}$$

It easily follows from Theorem 3.1 and (7.1) that

$$X_\varepsilon \to X_0 \quad \text{in } C([-1,1] \times [0,T]) \cap C^{2,1}_{\mathrm{loc}}\big(([-1,0) \cup (0,1]) \times (0,T]\big).$$

Since, by Theorem 3.1, $u_{\varepsilon x} \to u_x$ locally in $([a,b] \setminus \{x_1\}) \times (0,T]$, for all $\varepsilon \in (0,1]$ there exists $0 < y_\varepsilon \to 0$ such that

$$\sup_{y \in [y_\varepsilon, 1]} \left(\frac{1}{X_{\varepsilon u}(|y|,\cdot)} - \frac{1}{X_{0u}(|y|,\cdot)} \right) \to 0 \quad \text{in } C_{\text{loc}}\big((0,T]\big) \text{ as } \varepsilon \to 0.$$

In particular, since $y_\varepsilon \to 0$, there exists $0 < y_\varepsilon \leqslant \delta_\varepsilon \to 0$ such that $\varepsilon/\delta_\varepsilon \to 0$ as $\varepsilon \to 0$ and

$$\frac{1}{X_{\varepsilon u}(\pm\delta_\varepsilon,\cdot)} - \frac{1}{X_{0u}(\pm\delta_\varepsilon,\cdot)} \to 0 \quad \text{in } C_{\text{loc}}\big((0,T]\big) \text{ as } \varepsilon \to 0.$$

Hence it follows from the continuity properties of u_x (see Remark 6.2) that

$$\frac{1}{X_{\varepsilon u}(\pm\delta_\varepsilon,\cdot)} \to u_x(x_1^\pm,\cdot) \quad \text{pointwise in } (0,T] \text{ and in } C_{\text{loc}}\big((0,T] \setminus \{\tau_1^\pm\}\big) \text{ as } \varepsilon \to 0. \tag{7.4}$$

We fix $t \in (0,T]$. The idea of the proof is to integrate the equation for X_ε with respect to u in a neighbourhood of $u = 0$, but to do so we change variable and set

$$p = A_\varepsilon(u) := \int_0^u \frac{1}{\varepsilon + \Phi_\varepsilon^2(s)} \mathrm{d}s.$$

Then the second order term in (7.2) becomes a partial derivative with respect to p and integrating (7.2) with respect to p from $-A_\varepsilon(\delta_\varepsilon)$ to $A_\varepsilon(\delta_\varepsilon)$ we obtain that

$$\begin{aligned} \int_{-A_\varepsilon(\delta_\varepsilon)}^{A_\varepsilon(\delta_\varepsilon)} X_{\varepsilon t}(A_\varepsilon^{-1}(p),t)\,\mathrm{d}p + \frac{1}{X_{\varepsilon u}(\delta_\varepsilon,t)} - \frac{1}{X_{\varepsilon u}(-\delta_\varepsilon,t)} \\ = -B_{\varepsilon,\delta_\varepsilon}(t) := -\int_{-\delta_\varepsilon}^{\delta_\varepsilon} \frac{\Phi_\varepsilon(u)\big(1-\Phi_\varepsilon^2(u)\big)}{\sqrt{\varepsilon + \Phi_\varepsilon^2(u)}} X_{\varepsilon u}(u,t)\,\mathrm{d}u. \end{aligned} \tag{7.5}$$

Observe that, as $\varepsilon \to 0$,

$$|B_{\varepsilon,\delta_\varepsilon}(t)| \leqslant \int_{-\delta_\varepsilon}^{\delta_\varepsilon} X_{\varepsilon u}(u,t)\,\mathrm{d}u = X_\varepsilon(\delta_\varepsilon,t) - X_\varepsilon(-\delta_\varepsilon,t) \to 0$$

uniformly with respect to t; here we have used the (uniform) continuity of the map $[0,1] \times [-\frac{1}{2},\frac{1}{2}] \times [0,T] \ni (\varepsilon,u,t) \mapsto X_\varepsilon(u,t)$. In view of (7.4) and (7.5) this implies that

$$-\int_{-A_\varepsilon(\delta_\varepsilon)}^{A_\varepsilon(\delta_\varepsilon)} X_{\varepsilon t}\big(A_\varepsilon^{-1}(p),t\big)\,\mathrm{d}p \to u_x(x_1^+,t) - u_x(x_1^-,t) \quad \text{as } \varepsilon \to 0. \tag{7.6}$$

We claim that

$$A_\varepsilon(\delta_\varepsilon) = -\log\varepsilon\big(1+o(1)\big) \quad \text{if } \delta_\varepsilon \to 0 \text{ and } \varepsilon = o(\delta_\varepsilon) \text{ as } \varepsilon \to 0. \tag{7.7}$$

By the definition of Φ_ε,

$$u = 2\int_0^{\Phi_\varepsilon(u)} \sqrt{\varepsilon + s^2}\,\mathrm{d}s \;\Rightarrow\; 1 = 2\sqrt{\varepsilon + \Phi_\varepsilon^2(u)}\,\Phi_\varepsilon'(u),$$

whence, setting $\phi = \Phi_\varepsilon(s)$,

$$A_\varepsilon(\delta_\varepsilon) = \int_0^{\delta_\varepsilon} \frac{1}{\varepsilon+\Phi_\varepsilon^2(s)}\mathrm{d}s = 2\int_0^{\Phi_\varepsilon(\delta_\varepsilon)} \frac{1}{\sqrt{\varepsilon+\phi^2}}\mathrm{d}\phi = 2\log\left(\frac{\Phi_\varepsilon(\delta_\varepsilon)}{\sqrt{\varepsilon}} + \sqrt{1+\left(\frac{\Phi_\varepsilon(\delta_\varepsilon)}{\sqrt{\varepsilon}}\right)^2}\right).$$

Recall that $\Phi_\varepsilon = U_\varepsilon^{-1}$ with U_ε defined in (1.3):

$$U_\varepsilon(\phi) = 2\int_0^{\phi} \sqrt{\varepsilon + s^2}\,\mathrm{d}s = \phi\sqrt{\varepsilon+\phi^2} + \varepsilon\log\left(\frac{\phi}{\sqrt{\varepsilon}} + \sqrt{1+\frac{\phi^2}{\varepsilon}}\right),$$

and observe that $U_\varepsilon(\sqrt{\varepsilon}) = (\sqrt{2} + \log(1+\sqrt{2}))\varepsilon$. Setting $\phi_\varepsilon = \Phi_\varepsilon(\delta_\varepsilon)$ and $\xi_\varepsilon = \frac{\phi_\varepsilon}{\sqrt{\varepsilon}}$,

$$\frac{\delta_\varepsilon}{\varepsilon} = \frac{U_\varepsilon(\phi_\varepsilon)}{\varepsilon} = \frac{U_\varepsilon(\xi_\varepsilon\sqrt{\varepsilon})}{\varepsilon} = \xi_\varepsilon\sqrt{1+\xi_\varepsilon^2} + \log\left(\xi_\varepsilon + \sqrt{1+\xi_\varepsilon^2}\right),$$

and since $\delta_\varepsilon/\varepsilon \to \infty$ as $\varepsilon \to 0$ this implies that $\xi_\varepsilon \to \infty$ as $\varepsilon \to 0$. Hence

$$\frac{\Phi_\varepsilon(\delta_\varepsilon)}{\sqrt{\delta_\varepsilon}} = \frac{\phi_\varepsilon}{\sqrt{U_\varepsilon(\phi_\varepsilon)}} = \frac{\xi_\varepsilon}{\sqrt{\xi_\varepsilon\sqrt{1+\xi_\varepsilon^2} + \log\left(\xi_\varepsilon + \sqrt{1+\xi_\varepsilon^2}\right)}} \to 1 \text{ as } \varepsilon \to 0$$

and we obtain (7.7): as $\varepsilon \to 0$

$$A_\varepsilon(\delta_\varepsilon) = 2\log\left(2\frac{\sqrt{\delta_\varepsilon}}{\sqrt{\varepsilon}}(1+o(1))\right) = 2\log(2\sqrt{\delta_\varepsilon}) - \log\varepsilon + o(1) = -\log\varepsilon\big(1+o(1)\big).$$

It follows from (7.6) and (7.7) that

$$\left(\int_{-\delta_\varepsilon}^{\delta_\varepsilon} \frac{\mathrm{d}u}{\varepsilon+\Phi_\varepsilon^2(u)}\right)^{-1} \int_{-\delta_\varepsilon}^{\delta_\varepsilon} \frac{2\log\varepsilon X_{\varepsilon t}(u,t)}{\varepsilon+\Phi_\varepsilon^2(u)}\mathrm{d}u = \frac{\log\varepsilon}{A_\varepsilon(\delta_\varepsilon)} \int_{-A_\varepsilon(\delta_\varepsilon)}^{A_\varepsilon(\delta_\varepsilon)} X_{\varepsilon t}\big(A_\varepsilon^{-1}(p),t\big)\,\mathrm{d}p$$
$$\to u_x(x_1^+,t) - u_x(x_1^-,t) \quad \text{as } \varepsilon \to 0.$$

This completes the proof of Theorem 3.3.

Remark 7.1. The weak formulation of (1.6), which is given by Theorem 3.3 and proved in this Section, is not the only possible weak version of the asymptotic formula for the velocity of the interface, $\zeta_\varepsilon'(t)$. For example, if

$$\tau_1 = \max\{\tau_1^+, \tau_1^-\} < \infty,$$

one can show that

$$2\log\varepsilon\,\zeta_\varepsilon' \text{ converges weakly in } L^2_{\mathrm{loc}}(\tau_1, T) \text{ to } u_x(x_1^+,\cdot) - u_x(x_1^-,\cdot) \text{ as } \varepsilon \to 0. \tag{7.8}$$

Indeed, in view of Theorem 3.3 we obtain (7.8) if we prove that, given $\varphi \in C_c^\infty((\tau,T))$,

$$\lim_{\varepsilon\to 0} 2\log\varepsilon\left(\int_{-\delta_\varepsilon}^{\delta_\varepsilon} \frac{\mathrm{d}u}{\varepsilon+\Phi_\varepsilon^2(u)}\right)^{-1} \int_0^T \int_{-\delta_\varepsilon}^{\delta_\varepsilon} \frac{X_{\varepsilon t}(u,t)\varphi(t)}{\varepsilon+\Phi_\varepsilon^2(u)}\mathrm{d}u\mathrm{d}t$$
$$= \lim_{\varepsilon\to 0} 2\log\varepsilon \int_0^T X_{\varepsilon t}(0,t)\varphi(t)\mathrm{d}t.$$

Due to (7.7), this is equivalent to proving that

$$\lim_{\varepsilon\to 0}\int_{-\delta_\varepsilon}^{\delta_\varepsilon}\frac{1}{\varepsilon+\Phi_\varepsilon^2(u)}\left(\int_0^T\big(X_{\varepsilon t}(u,t)-X_{\varepsilon t}(0,t)\big)\varphi(t)\mathrm{d}t\right)\mathrm{d}u=0.$$

By the definition of $\tau_1^\pm$, there exists $C>0$ which does not depend on ε such that $|X_{\varepsilon u}(x,t)|<C$ for $x\in(a,b)$ and $t\in\operatorname{supp}\varphi$. Hence

$$\begin{aligned}&\left|\int_{-\delta_\varepsilon}^{\delta_\varepsilon}\frac{1}{\varepsilon+\Phi_\varepsilon^2(u)}\left(\int_0^T\big(X_{\varepsilon t}(u,t)-X_{\varepsilon t}(0,t)\big)\varphi(t)\mathrm{d}t\right)\mathrm{d}u\right|\\ &\qquad=\left|\int_{-\delta_\varepsilon}^{\delta_\varepsilon}\frac{1}{\varepsilon+\Phi_\varepsilon^2(u)}\left(\int_0^T\left(\int_0^u X_{\varepsilon u}(s,t)\mathrm{d}s\right)\varphi'(t)\mathrm{d}t\right)\mathrm{d}u\right|\\ &\qquad\leqslant CT\|\varphi'\|_\infty\int_{-\delta_\varepsilon}^{\delta_\varepsilon}\frac{|u|}{\varepsilon+\Phi_\varepsilon^2(u)}\mathrm{d}u\to 0\quad\text{as }\varepsilon\to 0,\end{aligned}$$

since

$$\frac{|u|}{\varepsilon+\Phi_\varepsilon^2(u)}\leqslant\frac{|u|}{\Phi_\varepsilon^2(u)}\to 1\quad\text{as }\varepsilon\to 0.$$

Remark 7.2. We have imposed condition (3.5) to simplify the proof of Theorem 3.3 and it can be relaxed considerably. For example, if u_0 satisfies the weaker condition (2.2), it is enough to require that u_0 (and hence $u_{0\varepsilon}$) is strictly monotonic in a neighbourhood of each of its zero's. More precisely, it is possible to use the maximum principle to show that, for small enough ε, this monotonicity property persists for positive times $t\in(0,T]$, which makes it possible to "localize" the proof of Theorem 3.3. To keep the proof as transparent as possible, we have preferred to avoid technical complications and require the more restrictive condition (3.5).

8. The limit problem: Proof of Theorem 2.3

Consider the Dirichlet problem in $(x_i,x_{i+1})\times(0,T]$:

$$\begin{cases}u_t=|u|u_{xx}+u(1-|u|) & \text{in } Q_i:=(x_i,x_{i+1})\times(0,T],\\ u(x_i,t)=u(x_{i+1},t)=0 & \text{for } t\in(0,T],\\ u(x,0)=u_0(x) & \text{for } x\in(x_i,x_{i+1}).\end{cases}\tag{8.1}$$

We assume that $u_0>0$ in (x_i,x_{i+1}) (the case that $u_0<0$ is completely similar) and approximate problem (8.1) by

$$\begin{cases}u_t=u(u_{xx}+1-u) & \text{in } Q_i,\\ u(x_i,t)=u(x_{i+1},t)=\frac{1}{n} & \text{for } t\in(0,T],\\ u(x,0)=u_0(x)+\frac{1}{n} & \text{for } x\in(x_i,x_{i+1}),\end{cases}\tag{8.2}$$

where $n\in\mathbb{N}$. We collect various basic results on problems (8.1) and (8.2) in:

Lemma 8.1. *Let u_0 satisfy (2.2) and let $u_0 > 0$ in (x_i, x_{i+1}). Then problem (8.2) has, for all $n \in \mathbb{N}$, a unique solution $u_{i,n} \in C(\overline{Q}_i) \cap C^{2,1}([x_i, x_{i+1}] \times (0,T])$, $u_{i,n}$ is pointwise decreasing with respect to n, and its pointwise limit u_i is the unique solution in $C(\overline{Q}_i) \cap C^{2,1}(Q_i)$ of problem (8.1) which is positive in Q_i.*

Proof. By the Comparison Principle for uniformly parabolic equations, each smooth solution u of (8.2) satisfies $\frac{1}{n} \leqslant u \leqslant \max\{1, \|u_0\|_\infty + \frac{1}{n}\}$, whence, by standard theory of quasilinear parabolic equations, problem (8.2) has a unique solution $u_{i,n}$ in $C(\overline{Q}_i) \cap C^{2,1}([x_i, x_{i+1}] \times (0,T])$. In addition $u_{i,n}$ is pointwise decreasing with respect to n and there exists

$$u_i(x,t) = \lim_{n\to\infty} u_{i,n}(x,t) \geqslant 0 \quad \text{for } (x,t) \in \overline{Q}_i.$$

It follows easily from the construction of explicit positive subsolutions (not depending on t and n) in subintervals of (x_i, x_{i+1}) that $u_{i,n}(x,t) \geqslant g_i(x) > 0$ in Q_i for some continuous function g, whence also $u_i(x,t) \geqslant g(x) > 0$. Hence, by standard a priori local Schauder type bounds ([11, Chapter 5, Theorem 5.4]) for solutions of quasilinear parabolic equations,

$$u_{i,n} \to u_i \quad \text{in } C^{2,1}_{\text{loc}}(Q_i), \tag{8.3}$$

and u_i satisfies the PDE pointwise in Q_i. Since $u_{i,n} = \frac{1}{n}$ at x_i and x_{i+1} and since $u_{i,n}$ is decreasing in n, u_i vanishes and is continuous on the lateral boundary $\{x_i, x_{i+1}\} \times [0,T]$ of Q_i. In addition, by local Hölder estimates ([11, Chapter 5, Theorem 1.1]) in $(x_i, x_{i+1}) \times [0,T]$ and the Lipschitz continuity of u_0, u_i is continuous in $(x_i, x_{i+1}) \times \{0\}$. Hence $u_i \in C(\overline{Q}_i) \cap C^{2,1}(Q_i)$ is a solution of problem (8.1) which is strictly positive in Q_i.

It remains to prove the uniqueness claim of Lemma 8.1. Let v be another classical solution such that $v > 0$ in Q_i. First we show that

$$0 < v \leqslant u_i \quad \text{in } Q_i. \tag{8.4}$$

Let $n \in \mathbb{N}$ and let $\delta_n > 0$ be so small that $v(x_i + \delta_n, t) < \frac{1}{n}$ and $v(x_{i+1} - \delta_n, t) < \frac{1}{n}$ for $t \in [0,T]$. Then it follows from the Comparison Principle for uniformly parabolic equations that $v < u_{i,n}$ in $(x_i + \delta_n, x_{i+1} - \delta_n) \times [0,T]$. Since n is arbitrary and $\delta_n \to 0$ as $n \to \infty$, this implies (8.4).

To show that $v = u_i$ in Q_i we observe that, by the equations for u_i and v,

$$(\log u_i - \log v)_t = (u_i - v)_{xx} - (u_i - v) \text{ in } Q_i. \tag{8.5}$$

We would like to test this equation with the first eigenfunction φ_0 of the laplacian in (x_i, x_{i+1}) with homogeneous Dirichlet data such that $\max \varphi_0 = 1$. But since (8.5) is singular at the lateral boundaries we slightly shrink the interval: let $\delta > 0$ be sufficiently small and let φ_δ be the first eigenfunction of the laplacian in $(x_i + \delta, x_{i+1} - \delta)$ with homogeneous Dirichlet data such that $\max \varphi_\delta = 1$. We denote the first eigenvalue by λ_δ:

$$\lambda_\delta = -\frac{\pi^2}{(x_{i+1} - x_i - 2\delta)^2} \to \lambda_0 := -\frac{\pi^2}{(x_{i+1} - x_i)^2} \quad \text{as } \delta \downarrow 0.$$

In addition $\varphi_\delta \uparrow \varphi$ locally in (x_i, x_{i+1}) as $\delta \downarrow 0$.

Let $t \in (0, T]$ and $\delta > 0$. Multiplying (8.5) by φ_δ and integrating, we obtain

$$\int_{x_i+\delta}^{x_{i+1}-\delta} \big(\log u_i(t) - \log v(t)\big)\varphi_\delta \mathrm{d}x + (1-\lambda_\delta)\int_0^t \int_{x_i+\delta}^{x_{i+1}-\delta} (u_i - v)\varphi_\delta \,\mathrm{d}x\mathrm{d}t$$
$$= \sqrt{|\lambda_\delta|}\left(\int_0^t \big(u_i(x_{i+1}-\delta,t) - v(x_{i+1}-\delta,t)\big)\mathrm{d}t + \int_0^t \big(u_i(x_i+\delta,t) - v(x_i+\delta,t)\big)\mathrm{d}t\right).$$

Since the integrals on the right-hand side vanish as $\delta \to 0$, it follows from (8.4) and the monotone convergence theorem that

$$0 \leqslant \int_{x_i}^{x_{i+1}} \big(\log u_i(t) - \log v(t)\big)\varphi_0 \mathrm{d}x = -(1-\lambda_0)\int_0^t \int_{x_i}^{x_{i+1}} (u_i - v)\varphi_0 \,\mathrm{d}x\mathrm{d}t \leqslant 0.$$

Hence, by (8.4), the continuity of u_i and v and the arbitrariness of t, we conclude that $u_i = v$ in Q_i. □

Lemma 8.1 concerns the sets Q_i for $i = 1\ldots,k-1$, but a similar result can be easily proved in $(a, x_1) \times (0, T]$ and $(x_k, b) \times (0, T]$. At this point the solutions in the $(k+1)$ single intervals can be "merged together" to define a function u in all of Q, i.e., the restriction of u to one of the sets Q_i coincides with the smooth solution of problem (8.1) in Q_i. By construction, u is a classical solution of problem (2.1).

Vice versa, the restriction of any classical solution of problem (2.1) to one of the sets Q_i is a smooth solution of problem (8.1) in Q_i, so the uniqueness statement in Lemma 8.1 implies the uniqueness of the classical solution of problem (8.1).

To complete the proof of Theorem 2.3 we must show that u is a weak solution of problem (2.1). This is an immediate consequence of the integral equality (8.8) in the following result.

Lemma 8.2. *Let $u_{i,n}$ and u_i be defined by Lemma 8.1 and let $\alpha > -1$. Then there exists a positive constant K which does not depend on n such that*

$$\tfrac{4(\alpha+1)}{(\alpha+2)^2}\iint_{Q_i}\left(u_{i,n}^{\frac{\alpha+2}{2}}\right)_x^2 \mathrm{d}x\mathrm{d}t + \int_0^T n^{-(\alpha+1)}\left(|(u_{i,n})_x(x_{i+1},t)| + |(u_{i,n})_x(x_i,t)|\right)\mathrm{d}t \leqslant K. \quad (8.6)$$

In addition $u_i^{\frac{\alpha+2}{2}} \in L^2(0,T; H^1((x_i,x_{i+1})))$,

$$\frac{\partial}{\partial x}\left(u_{i,n}^{\frac{\alpha+2}{2}}\right) \rightharpoonup \frac{\partial}{\partial x}\left(u_i^{\frac{\alpha+2}{2}}\right) \quad \text{in } L^2(Q_i) \text{ as } n \to \infty, \quad (8.7)$$

and, for all $\psi \in C_c^{1,1}([x_i,x_{i+1}] \times [0,T))$,

$$\int_{x_i}^{x_{i+1}} u_0(x)\psi(x,0)\mathrm{d}x + \iint_{Q_i}\left(u_i\psi_t - u_i u_{ix}\psi_x - u_{ix}^2\psi + u_i(1-u_i)\psi\right)\mathrm{d}x\mathrm{d}t = 0. \quad (8.8)$$

Proof. Let $\tau \in (0,T)$. Integration by parts over $Q_i^\tau := (x_i, x_{i+1}) \times (\tau, T)$ yields

$$\tfrac{4(\alpha+1)}{(\alpha+2)^2}\iint_{Q_i^\tau}\left(u_{i,n}^{\frac{\alpha+2}{2}}\right)_x^2 \mathrm{d}x\mathrm{d}t - \int_\tau^T n^{-(\alpha+1)}\big((u_{i,n})_x(x_{i+1},t) - (u_{i,n})_x(x_i,t)\big)\mathrm{d}t$$
$$= \frac{1}{\alpha+1}\int_{x_i}^{x_{i+1}}\big(u_{i,n}^{\alpha+1}(x,\tau) - u_{i,n}^{\alpha+1}(x,T)\big)\mathrm{d}x + \iint_{Q_{\tau,T}} u_{i,n}^{\alpha+1}(1-u_{i,n})\mathrm{d}x\mathrm{d}t.$$

Since $u_{i,n} = \frac{1}{n}$ at x_i and x_{i+1} and $u_{i,n} \geqslant \frac{1}{n}$ in Q_i, we have that $(u_{i,n})_x(x_{i+1}, t) \leqslant 0$ and $(u_{i,n})_x(x_i, t) \geqslant 0$ for $t \in (0, T]$. Letting $\tau \to 0$, the existence of the constant K follows from the uniform boundedness of $u_{i,n}$ in Q_i.

It remains to prove (8.8). Let $\psi \in C_c^{1,1}([x_i, x_{i+1}] \times [0, T))$ and let $\tau \in (0, T)$. Integration by parts over Q_i^τ yields

$$\int_{x_i}^{x_{i+1}} u_{i,n}(x,\tau)\psi(x,\tau)\mathrm{d}x + \frac{1}{n}\int_\tau^T (u_{i,n})_x\psi\Big|_{(x_i,t)}^{(x_{i+1},t)}\mathrm{d}t \\ + \iint_{Q_i^\tau}\left(u_{i,n}\psi_t - \tfrac{1}{2}(u_{i,n}^2)_x\psi_x - (u_{i,n})_x^2\psi + u_{i,n}(1-u_{i,n})\psi\right)\mathrm{d}x\mathrm{d}t = 0. \tag{8.9}$$

We first pass to the limit $n \to \infty$ inside the integrals, for fixed τ. By the Dominated Convergence Theorem, the only nontrivial terms are those containing $(u_{i,n})_x$. In the term with $(u_{i,n}^2)_x$ it is enough to use (8.7) with $\alpha = 2$ to pass to the limit. In addition, we obtain from (8.6) with $\alpha \in (-1, 0)$ that the integral

$$\frac{1}{n}\left|\int_\tau^T (u_{i,n})_x\psi\Big|_{(x_i,t)}^{(x_{i+1},t)}\mathrm{d}t\right| \leqslant Cn^\alpha\int_\tau^T n^{-(\alpha+1)}\left(|(u_{i,n})_x(x_{i+1},t)| + |(u_{i,n})_x(x_i,t)|\right)$$

vanishes as $n \to \infty$. It remains to consider the term containing $(u_{i,n})_x^2$. By (8.3), for each $\delta > 0$

$$\iint_{(x_i+\delta, x_{i+1}-\delta)\times(\tau,T)} (u_{i,n})_x^2\psi\,\mathrm{d}x\mathrm{d}t \to \iint_{(x_i+\delta, x_{i+1}-\delta)\times(\tau,T)} (u_i)_x^2\psi\,\mathrm{d}x\mathrm{d}t \quad \text{as } n \to \infty.$$

On the other hand, by the pointwise monotonicity of $u_{i,n}$ with respect to n and the lateral boundary condition $u_{i,n} = \frac{1}{n}$, for all $\eta > 0$ there exist $\delta_\eta > 0$ and $n_\eta \in \mathbb{N}$ such that $u_{i,n}(x,t) < \eta$ if $n > n_\eta$ and if $x < x_i + \delta_\eta$ or $x > x_{i+1} - \delta_\eta$. Hence, by (8.6) with $\alpha = -1/2$,

$$\begin{aligned}
&\left|\iint_{(x_i, x_i+\delta_\eta)\times(\tau,T)} (u_{i,n})_x^2\psi\,\mathrm{d}x\mathrm{d}t\right| \\
&\qquad \leqslant \left(\sup_{(x_i,x_i+\delta_\eta)\times(\tau,T)}\sqrt{u_{i,n}}\right)\iint_{(x_i,x_i+\delta_\eta)\times(\tau,T)} u_{i,n}^{-1/2}(u_{i,n})_x^2|\psi|\,\mathrm{d}x\mathrm{d}t \\
&\qquad \leqslant C_1 \sup_{(x_i,x_i+\delta_\eta)\times(\tau,T)}\sqrt{u_{i,n}} \leqslant C_1\sqrt{\eta} \quad \text{for all } n > n_\eta.
\end{aligned}$$

Since η is arbitrary this implies that we can let $n \to \infty$ in (8.9):

$$\int_{x_i}^{x_{i+1}} u_i(x,\tau)\psi(x,\tau)\mathrm{d}x + \iint_{Q_i^\tau}\left(u_i\psi_t - u_iu_{ix}\psi_x - u_{ix}^2\psi + u_i(1-u_i)\psi\right)\mathrm{d}x\mathrm{d}t = 0.$$

Since $u_{ix} \in L^2(Q_i)$ we can let $\tau \to 0$ and obtain (8.8). □

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