Interfaces Free Bound. (Online first)
DOI 10.4171/IFB/466

Published by EMS Press

K-mean convex and K-outward minimizing sets

Annalisa Cesaroni and Matteo Novaga

Abstract. We consider the evolution of sets by nonlocal mean curvature and we discuss the preservation along the flow of two geometric properties, which are the mean convexity and the outward minimality. The main tools in our analysis are the level set formulation and the minimizing movement scheme for the nonlocal flow. When the initial set is outward minimizing, we also show the convergence of the (time integrated) nonlocal perimeters of the discrete evolutions to the nonlocal perimeter of the limit flow.

1. Introduction

Given an initial set $E \subseteq \mathbb{R}^n$, we consider its evolution E_t for $t > 0$ according to the nonlocal curvature flow

$$\partial_t x \cdot \nu = -H^K_{E_t}(x), \tag{1.1}$$

where ν is the outer normal at $x \in \partial E_t$. The quantity $H^K_E(x)$ is the K-curvature of E at x, which is defined in (1.3) below. More precisely, we take a kernel $K : \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$ such that

$$\min\{1, |x|\}\, K(x) \in L^1(\mathbb{R}^n) \quad \text{and} \quad K(x) = K(-x), \tag{1.2}$$

and we define the K-curvature of a set E of class $C^{1,1}$, at $x \in \partial E$, as

$$H^K_E(x) := \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \big(\chi_{\mathbb{R}^n \setminus E}(y) - \chi_E(y)\big)\, K(x-y)\, dy, \tag{1.3}$$

where, as usual,

$$\chi_E(y) := \begin{cases} 1, & \text{if } y \in E, \\ 0, & \text{if } y \notin E. \end{cases}$$

Notice that, under assumption (1.2), the singular integral in (1.3) is always a well-defined real number. For more general sets, the K-curvature will be understood in the viscosity sense (see Definition 2.1 below).

We point out that (1.2) is a very mild integrability assumption, which fits the requirements in [9, 24] in order to have existence and uniqueness for the level set flow associated to (1.1). Furthermore, when $K(x) = \frac{1}{|x|^{n+s}}$ for some $s \in (0,1)$, we will denote the

2020 Mathematics Subject Classification. 53E10, 35R11, 49Q20.
Keywords. Nonlocal curvature flows, mean convexity, nonlocal minimal surfaces, level set flow, minimizing movements.

K-curvature of a set E at a point x as $H^s_E(x)$, and we indicate it as the fractional mean curvature of E at x. We also observe that the K-curvature is the first variation of the following nonlocal perimeter functional (see [9]),

$$\mathrm{Per}_K(E) := \int_E \int_{\mathbb{R}^n \setminus E} K(x-y)\, dx\, dy, \tag{1.4}$$

and the geometric evolution law (1.1) can be interpreted as the L^2 gradient flow of this perimeter functional, as shown in [9].

The K-curvature flow has been recently studied from different perspectives, mainly in the case of the fractional mean curvature, taking into account several geometric features. In particular, we recall the results about small time existence of a classical solutions [25], existence and uniqueness of level set solutions [9, 24], preservation of convexity [11, 14], formation of singularities [13], classification of symmetric self-shrinkers [7], fattening phenomena [5] and stability results for nonlocal curvature flows [4, 8].

In this paper, we are interested in the analysis of the flows starting from K-mean convex sets, that is, sets with positive K-curvature, and from sets which are one-side minimizers of the nonlocal perimeter functional, the so called K-outward minimizing set. This second property can be interpreted as the variational analogue of the K-mean convexity, as we will see in Theorem 2.9. In the case of the fractional curvature, the preservation of the K-mean convexity for smooth sets has been studied in [30]. Here we consider more general flows, and also nonsmooth initial data. We show that K-mean convexity is a too weak condition to be conserved during the evolution; as a consequence, we introduce the notions of regular K-mean convexity and strong K-mean convexity (see Definition 2.2). We introduce the notion of K-outward minimality and strong K-outward minimality (see Definition 2.6). The main results are contained in Theorem 4.5, about the preservation of regular K-mean convexity and strong K-mean convexity, and Theorem 6.3 about preservation of K-outward minimality. Our main tools are the level set approach for geometric nonlocal curvature flows, developed in [9, 24], that we review in Section 3, and the variational scheme, called minimizing movements or Almgren–Taylor–Wang scheme, introduced in [1, 26] for the classical mean curvature flow, and extended to the nonlocal setting in [9].

We conclude by recalling that, in the local case, there is a vast literature on the analysis of the mean curvature flow starting from convex sets (see [2, 17–19, 31]) and more generally from mean-convex sets (see [10, 29, 32, 33] and the references therein). In particular, these geometric properties are preserved by the flow, both in the isotropic and in the anisotropic case, and the singularity formation is well understood (see for instance [20–23]).

The paper is organized as follows: Section 2 contains the definition of K-mean convexity and K-outward minimality, some examples, and the analysis of the relation between the two notions. Section 3 is essentially a review of the level set formulation of nonlocal curvature flows, and contains the comparison results between level set flows and classical strict subflows and superflows. Section 4 is devoted to the analysis of the flows starting from K-mean convex sets. Section 5 provides a review of the minimizing movement scheme in the nonlocal setting. Finally, Section 6 contains the analysis of the flows starting from K-outward minimizing sets.

2. Main definitions and properties

In this section, we introduce the notions of K-mean convexity and K-outward minimality, we give some examples and characterizations of these properties, and we analyze their relation.

We now recall the definition of constant K-mean curvature in the viscosity sense; for more details we refer to [9, 24] and [3, Section 5].

Definition 2.1. Let $c \in \mathbb{R}$, $E \subseteq \mathbb{R}^n$ and $x \in \partial E$. Then,

(1) $H_E^K(x) \leqslant c$ if for all sets F with compact boundary of class $C^{1,1}$ such that $E \subseteq F$ and $x \in \partial F$, one has $H_F^K(x) \leqslant c$;

(2) $H_E^K(x) \geqslant c$ if for all sets F with compact boundary of class $C^{1,1}$ such that $E \supseteq F$ and $x \in \partial F$, one has $H_F^K(x) \geqslant c$;

(3) $H_E^K(x) = c$ if both $H_E^K(x) \geqslant c$ and $H_E^K(x) \leqslant c$.

From (1.3), it follows that the K-mean curvature satisfies the following monotonicity property: if $E \subseteq F$ and $x \in \partial E \cap \partial F$ is a point where both $H_E^K(x)$ and $H_F^K(x)$ are defined, then $H_E^K(x) \geqslant H_F^K(x)$. As a consequence, the inequalities in Definition 2.1 are consistent with the definition of H_E^K in (1.3).

We observe that the viscosity inequality $H_E^K(x) \leqslant c$ can be checked only at points $x \in \partial E$ where E satisfies an exterior ball condition, that is, there exist y_0, r_0 such that $B(y_0, r_0) \subseteq \mathbb{R}^n \setminus E$ and $x \in \partial B(x_0, r_0)$. Analogously, the viscosity inequality $H_E^K(x) \geqslant c$ can be checked only at points $x \in \partial E$ where E satisfies an interior ball condition, that is, there exist y_0, r_0 such that $B(y_0, r_0) \subseteq E$ and $x \in \partial B(x_0, r_0)$. In particular, if E is a closed set with empty interior, then the viscosity inequality $H_E^K(x) \geqslant k$ is always verified for every $k \in \mathbb{R}$.

As usual, we will denote the distance between a point x and a set E by $d(x, E) := \inf_{y \in E} |y - x|$, and we define the signed distance from E as follows:

$$d_E(x) := d(x, \mathbb{R}^n \setminus E) - d(x, E).$$

For $\lambda > 0$, we define

$$E^\lambda := \{x \in \mathbb{R}^n : d_E(x) \geqslant -\lambda\} = \{x \in \mathbb{R}^n : d(x, E) \leqslant \lambda\}. \tag{2.1}$$

Observe that if E is a closed set, then $E = \bigcap_{\lambda > 0} E^\lambda$.

Finally, we define the distance between two sets $A, B \subseteq \mathbb{R}^n$ as follows:

$$d(A, B) := \inf_{a \in \partial A, b \in \partial B} |a - b|.$$

Definition 2.2 (K-mean convexity and regular/strong K-mean convexity).

(1) A closed set $E \subseteq \mathbb{R}^n$ is *K-mean convex* if $H^K_E(x) \geqslant 0$ for all $x \in \partial E$.

(2) A closed set $E \subseteq \mathbb{R}^n$ is *regularly K-mean convex* if there exist $\eta_E > 0$ and $c_E > 0$ such that for all $\lambda \in [0, \eta_E]$,

$$H^K_{E^\lambda}(x) \geqslant -c_E \lambda \quad \text{for any } x \in \partial E^\lambda,$$

where $E^0 = E$.

(3) A closed set $E \subseteq \mathbb{R}^n$ is *strongly K-mean convex* if there exist $\delta \geqslant 0$ and $\xi_E > 0$ such that

$$H^K_{E^\lambda}(x) \geqslant \delta \quad \text{for any } x \in \partial E^\lambda,$$

for every $\lambda \in [0, \xi_E]$.

To keep track of the constant δ, in the following we will say that $E \subseteq \mathbb{R}^n$ is a *strongly K-mean convex set* with associated constant δ.

Note that if E is strongly K-mean convex, then E is also regularly K-mean convex.

Remark 2.3 (Sets with $C^{1,1}$ boundary). Let E be a compact set with $C^{1,1}$ boundary.

If $H^K_E(x) \geqslant \delta$ for all $x \in \partial E$, then for all $\delta' < \delta$, there exists $\xi_E(\delta')$ such that

$$H^K_{E^\eta}(x) \geqslant \delta' \quad \text{for all } \eta \in [0, \xi_E(\delta')] \text{ and } x \in \partial E^\eta,$$

due to the continuity of H^K with respect to $C^{1,1}$ convergence of sets (see [9]), and therefore, E is strongly K-mean convex with constant δ'.

If $H^K_E(x) \geqslant 0$ for all $x \in \partial E$, and $K(x) = \frac{1}{|x|^{n+s}}$, then

$$E \text{ is regularly } K\text{-mean convex,}$$

due to the result about the variation of fractional curvature with respect to $C^{1,1}$ diffeomorphisms of sets proved in [15].

Remark 2.4 (Convex sets). Let C be a convex closed set. Then,

$$C \text{ is strongly } K\text{-mean convex with associated constant } 0,$$

since it is easy to show that $H^K_C(x) \geqslant 0$ for every $x \in \partial C$ in the viscosity sense and, moreover, C^λ are convex sets. Moreover, if C is compact and $\operatorname{supp} K$ is not compact, then there exists $\delta_C > 0$, depending on K and C, such that

$$C \text{ is strongly } K\text{-mean convex with associated constant } \delta_C.$$

Indeed, it is easy to check that if $C \subseteq \mathbb{R}^n$ is a convex set of diameter R, then

$$H^K_C(x) \geqslant \int_{\mathbb{R}^n \setminus B(0,R)} K(y)\, dy := \delta_C \quad \text{for every } x \in \partial C.$$

Remark 2.5. If the kernel $K \in L^1(\mathbb{R}^n)$, then H^K is continuous with respect to L^1 convergence of sets, see [27], that is, if $|E_n \Delta E| \to 0$ and $x_n \in \partial E_n \to x \in \partial E$, then $H^K_{E_n}(x_n) \to H^K_E(x)$. In this case, K-mean convexity and strong K-mean convexity are equivalent in the following sense. Let E be a compact set such that $H^K_E(x) \geqslant \delta$ in the viscosity sense for every $x \in \partial E$, then for all $\delta' < \delta$, there exists $\xi_E(\delta')$ such that

$$H^K_{E^\eta}(x) \geqslant \delta' \quad \text{for all } \eta \in [0, \xi_E(\delta')] \text{ and } x \in \partial E^\eta,$$

due to the continuity of H^K with respect to L^1 convergence. Therefore, E is strongly K-mean convex with constant δ'.

On the other hand, we point out that if K is not L^1, and E is a compact set such that $H^K_E(x) \geqslant \delta > 0$ for all $x \in \partial E$, but $\partial E \notin C^{1,1}$, then in general it is not true that E is strongly K-mean convex or even regularly K-mean convex.

We recall the following example, studied in [5]. We consider the fractional kernel in dimension 2, that is, $K(x) = \frac{1}{|x|^{2+s}}$. We define the set E as follows:

$$E := \mathcal{G}_+ \cup \mathcal{G}_- \subseteq \mathbb{R}^2,$$

where $\mathcal{G}_+$ is the convex hull of $B((-1,1),1)$ with the origin, and $\mathcal{G}_-$ is the convex hull of $B((1,-1),1)$ with the origin.

Note that $\partial E \setminus (0,0)$ is $C^{1,1}$ and in $(0,0)$ the viscosity supersolution condition $H^K_E(0,0) \geqslant \delta$ is true for every δ since there is no interior ball in E containing $(0,0)$, that is, there are no regular sets F such that $F \subseteq E$ and $(0,0) \in \partial F$. It is an easy computation to check, using the radial symmetry of K, that for all $x \neq 0$, $x \in \partial E$,

$$H^s_E(x) \geqslant \int_{\mathbb{R}^2 \setminus B(0,1+\sqrt{2})} \frac{1}{|y|^{2+s}}\, dy = \frac{2\pi}{(1+\sqrt{2})^s}.$$

Let $Q_r = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 \in [-r,r],\ -|x_2| \leqslant x_1 \leqslant |x_2|\}$. It has been proven in [5, Lemma 7.1] that there exists a constant $c > 0$, depending on s, such that for all $r < c$,

$$H^s_{E\cup Q_r}(t,r),\ H^s_{E\cup Q_r}(t,-r) \leqslant -\frac{c}{r^s} \quad \text{for all } t \in (-r,r).$$

Note that for every point $(t,-r)$ and (t,r) with $t \in (-r,r)$, there exists a neighborhood where $\partial(E \cup Q_r)$ is $C^{1,1}$; therefore, the previous inequality holds in the classical sense. Consider now $E^r = \{x \in \mathbb{R}^n : d(x,E) \leqslant r\}$ and note that $(0,r) \in \partial E^r$. Let F be a set with boundary $C^{1,1}$ such that $F \subseteq E^r$, $(0,r) \in \partial F$ and such that there exists $\delta \ll r$ for which $\partial F \cap B((0,r),\delta) = \partial(E \cup q_r) \cap B((0,r),\delta)$. Then, $H^s_F(0,r) \leqslant -\frac{c}{r^s}$.

If E was regularly K-mean convex, there would exist $c_E > 0$ such that $H^s_F(0,r) \geqslant -c_E r$ for every $r \in [0,\eta_E]$. Therefore, we would get $-\frac{c}{r^s} \geqslant -\eta_E r$ for every $r \in [0,\eta_E]$, which is not possible. We conclude that E is not regularly K-mean convex.

Given a measurable set $E \subseteq \mathbb{R}^n$ and an open set $\Omega \subseteq \mathbb{R}^n$ we let

$$\mathrm{Per}_K(E,\Omega) := \int_E \int_{\Omega\setminus E} K(x-y)\, dx\, dy + \int_{E\cap\Omega} \int_{\mathbb{R}^n\setminus(\Omega\cup E)} K(x-y)\, dx\, dy.$$

Notice that if $E \subseteq \Omega$, then $\mathrm{Per}_K(E,\Omega) = \mathrm{Per}_K(E)$; in particular, $\mathrm{Per}_K(E,\mathbb{R}^n) = \mathrm{Per}_K(E)$ for all sets E.

Definition 2.6 (K-outward minimizing set and strongly K-outward minimizing set). Let $\Omega \subseteq \mathbb{R}^n$ be an open set. $E \subseteq \mathbb{R}^n$ is a *K-outward minimizing set* in Ω if for every $F \subseteq \mathbb{R}^n$ such that $E \subseteq F$ and $F \setminus E \Subset \Omega$,

$$\mathrm{Per}_K(E,\Omega) \leqslant \mathrm{Per}_K(F,\Omega).$$

$E \subseteq \mathbb{R}^n$ is a *strongly K-outward minimizing set* in Ω if there exists $\delta > 0$ for which for every $F \subseteq \mathbb{R}^n$ such that $E \subseteq F$ and $F \setminus E \Subset \Omega$,

$$\mathrm{Per}_K(E,\Omega) \leqslant \mathrm{Per}_K(F,\Omega) - \delta|F \setminus E|.$$

To keep track of the constant δ, we will say in the following that $E \subseteq \mathbb{R}^n$ is a strongly K-outward minimizing set with associated constant δ.

We now provide some equivalent characterizations of K-outward minimality and strong K-outward minimality, which imply in particular the stability under L^1 convergence of K-outward minimizing sets.

Proposition 2.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. The following assertions are equivalent:*

(1) *E is a K-outward minimizing set in Ω (resp. strongly K-outward minimizing set with associated constant $\delta > 0$).*

(2) *For every $G \subseteq \mathbb{R}^n$ such that $G \setminus E \Subset \Omega$,*

$$\mathrm{Per}_K(E \cap G,\Omega) \leqslant \mathrm{Per}_K(G,\Omega) \tag{2.2}$$
$$\big(\textit{resp. } \mathrm{Per}_K(E \cap G,\Omega) \leqslant \mathrm{Per}_K(G,\Omega) - \delta|G \setminus E|\big).$$

(3) *For all $A \subseteq \Omega \setminus E$, $A \Subset \Omega$,*

$$\int_A \int_E K(x-y)\,dx\,dy \leqslant \int_A \int_{\mathbb{R}^n \setminus (A \cup E)} K(x-y)\,dx\,dy \tag{2.3}$$
$$\left(\textit{resp. } \int_A \int_E K(x-y)\,dx\,dy \leqslant \int_A \int_{\mathbb{R}^n \setminus (A \cup E)} K(x-y)\,dx\,dy - \delta|A|\right).$$

In particular, if E_n is a sequence of K-outward minimizing sets (resp. strongly K-outward minimizing sets with associated constant δ) in Ω such that $E_n \to E$ in $L^1(\Omega)$, then E is a K-outward minimizing set (resp. strongly K-outward minimizing set with associated constant δ) in Ω.

Proof. We only prove the characterization for K-outward minimizers, since the case of strongly K-outward minimizers is completely analogous. We recall that for all $A, B \subseteq \mathbb{R}^n$, the following submodularity property holds:

$$\mathrm{Per}_K(A,\Omega) + \mathrm{Per}_K(B,\Omega) \geqslant \mathrm{Per}_K(A \cap B,\Omega) + \mathrm{Per}_K(A \cup B,\Omega); \tag{2.4}$$

see e.g. [6].

If (2.2) holds, then it is immediate to check Definition 2.6: we fix $F \supseteq E$, with $F \setminus E \Subset \Omega$ and we apply (2.2) to $G = F$. On the other hand, if E is a K-outward minimizing set in Ω and G is such that $G \setminus E \Subset \Omega$, letting $F = G \cup E$ and using the submodularity for the first inequality and Definition 2.6 for the second one, we get

$$\begin{aligned}\mathrm{Per}_K(E,\Omega) + \mathrm{Per}_K(G,\Omega) &\geqslant \mathrm{Per}_K(F,\Omega) + \mathrm{Per}_K(G \cap E,\Omega)\\ &\geqslant \mathrm{Per}_K(E,\Omega) + \mathrm{Per}_K(E \cap G,\Omega).\end{aligned}$$

We now assume that E is a K-outward minimizing set in Ω and we fix $A \subseteq \Omega \setminus E$, with $A \Subset \Omega$. Let $F := E \cup A$, so that $E \subseteq F$ and $F \setminus E \Subset \Omega$. By Definition 2.6, we know that

$$0 \leqslant \mathrm{Per}_K(F,\Omega) - \mathrm{Per}_K(E,\Omega) = \int_A \int_{\mathbb{R}^n \setminus F} K(x-y)\,dx\,dy - \int_A \int_E K(x-y)\,dx\,dy,$$

which gives (2.3). On the other hand, if we assume that (2.3) holds and fix F such that $E \subseteq F$ and $A := F \setminus E \Subset \Omega$, then (2.3) gives

$$\mathrm{Per}_K(F,\Omega) - \mathrm{Per}_K(E,\Omega) = \int_A \int_{\mathbb{R}^n \setminus F} K(x-y)\,dx\,dy - \int_A \int_E K(x-y)\,dx\,dy \geqslant 0,$$

which implies that E is K-outward minimizing.

Finally, the stability under L^1 convergence is a direct consequence of (2.2) and of the lower semicontinuity of Per_K. Indeed, fix F such that $E \subseteq F$ and $F \setminus E \Subset \Omega$. Since $E_n \to E$ in $L^1(\Omega)$, we get that for n sufficiently large, $F \setminus E_n \Subset \Omega$. Then, by the fact that E_n are K-outward minimizers in Ω, $\mathrm{Per}_K(E_n \cap F,\Omega) \leqslant \mathrm{Per}_K(F,\Omega)$, and we conclude that $\mathrm{Per}_K(E \cap F,\Omega) \leqslant \mathrm{Per}_K(F,\Omega)$ by the lower semicontinuity of $\mathrm{Per}_K(\cdot,\Omega)$. ∎

Remark 2.8 (Hyperplanes and convex sets). Let $\nu \in \mathbb{R}^n$ with $|\nu| = 1$ and define the hyperplane $H = \{x \in \mathbb{R}^n : x \cdot \nu \geqslant 0\}$. Then H is a K-outward minimizer in every ball $B(0,R)$ for $R > 0$, since H is a local minimizer of Per_K in every ball $B(0,R)$; see [28].

Moreover, every convex set C is a K-outward minimizer in every ball $B(0,R)$ for $R > 0$. Indeed, $C = \bigcap_{j \in J} H_j$ with H_j being hyperplanes. Let E such that $E \setminus C \Subset B(0,R)$. Then, $E \setminus H_i \Subset B(0,R)$ for every $i \in J$ and, by the minimality of H_i, we get

$$\mathrm{Per}_K\big(C \cap E, B(0,R)\big) = \mathrm{Per}_K\Big(\bigcap_j H_j \cap E, B(0,R)\Big) \leqslant \mathrm{Per}_K\Big(\bigcap_{j \neq i} H_j \cap E, B(0,R)\Big).$$

By repeating the same argument for every $j \in J$, we conclude that

$$\mathrm{Per}_K(C \cap E, B(0,R)) \leqslant \mathrm{Per}_K(E, B(0,R)).$$

We now analyze the relation between K-outward minimality and K-mean convexity for compact sets. In some sense, (strong) K-outward minimality is the variational analogue of (strong) K-mean convexity.

Theorem 2.9.

(1) *Let $E \Subset \Omega$ be a K-outward minimizing set in Ω. Then $H^K_E(x) \geqslant 0$ for all $x \in \partial E$. If, moreover, E is a strongly K-outward minimizing set with associated constant $\delta > 0$, then $H^K_E(x) \geqslant \delta > 0$ for all $x \in \partial E$.*

(2) *Let $E \subseteq \mathbb{R}^n$ be a bounded set with $H^K_E(x) \geqslant \delta > 0$ for all $x \in \partial E$, and assume that the boundary of E is of class $C^{1,1}$ or that $K \in L^1(\mathbb{R}^n)$. Then, for every $\delta' < \delta$, there exists an open set Ω such that $E \Subset \Omega$ and E is a K-outward minimizer in Ω with associated constant δ'.*

Proof. (1) For the case of fractional perimeters, this result has been proved in [3, Proposition 5.1]. Let $\delta \geqslant 0$. If E is a K-outward minimizer, we choose $\delta = 0$; if E is a strongly K-outward minimizer, we choose $\delta > 0$ to be the constant associated to E according to Definition 2.6. We proceed by contradiction and we assume there exist $x_0 \in \partial E$, $F \subseteq E$ with $\partial F \in C^{1,1}$, $x_0 \in \partial E \cap \partial F$, and $H^K_F(x_0) \leqslant \delta - 2\rho < \delta$ for some $\rho > 0$. Then, by the continuity of H^K, there exists $r > 0$ such that $H^K_F(x) \leqslant \delta - \rho$ for every $x \in \partial E \cap B(x_0, r)$. We construct a 1-parameter family Φ_ε of $C^{1,1}$ diffeomorphisms, such that $F = \Phi_0(F) \subseteq \Phi_\varepsilon(F) \subseteq \Omega$ and $\Phi_\varepsilon(F) \setminus F \Subset B(x_0, r) \Subset \Omega$ for every $\varepsilon \in (0, \varepsilon_0)$. Again, by continuity, $H^K_{\Phi_\varepsilon(F)}(x) \leqslant \delta - \rho/2$ for all $x \in \partial\Phi_\varepsilon(F) \setminus F$. Using the fact that H^K is the first variation of Per_K with respect to $C^{1,1}$ diffeomorphisms, we get

$$\mathrm{Per}_K(\Phi_\varepsilon(F)) = \mathrm{Per}_K(F) + \int_{\Phi_\varepsilon(F)\setminus F} H^K_{\Phi_{\varepsilon(x)}(F)}(x)\,dx, \tag{2.5}$$

where $\varepsilon(x) := \sup\{\lambda \in (0, \varepsilon) : x \in \Phi_\lambda(F)\}$ and

$$\mathrm{Per}_K(E \cap \Phi_\varepsilon(F)) \geqslant \mathrm{Per}_K(F) + \int_{(E\cap\Phi_\varepsilon(F))\setminus F} H^K_{\Phi_{\varepsilon(x)}(F)}(x)\,dx; \tag{2.6}$$

see [9, Proposition 5.2]. From (2.5) and (2.6), recalling that $H^K_{\Phi_\varepsilon(F)}(x) \leqslant \delta - \rho/2$ in $\Phi_\varepsilon(F) \setminus E \subseteq \Phi_\varepsilon(F) \setminus F$, we conclude that

$$\begin{aligned}\mathrm{Per}_K(E \cap \Phi_\varepsilon(F)) &\geqslant \mathrm{Per}_K(\Phi_\varepsilon(F)) - \int_{\Phi_\varepsilon(F)\setminus E} H^K_{\Phi_{\varepsilon(x)}(F)}(x)\,dx \\ &\geqslant \mathrm{Per}_K(\Phi_\varepsilon(F)) + \left(-\delta + \frac{\rho}{2}\right)|\Phi_\varepsilon(F) \setminus E| \\ &> \mathrm{Per}_K(\Phi_\varepsilon(F)) - \delta|\Phi_\varepsilon(F) \setminus E|,\end{aligned}$$

in contradiction with the fact that E is a K-outward minimizing set in Ω, if $\delta = 0$, or a strong K-outward minimizing set, if $\delta > 0$.

(2) First of all, we observe that for every $\delta' < \delta$, there exists $\xi_E(\delta')$ such that $H^K_{E^\eta}(x) \geqslant \delta'$ for all $x \in \partial E^\eta$ and all $\eta \in [0, \xi_E(\delta')]$; see Remarks 2.3 and 2.5. We let $\Omega := E^{\xi_E(\delta')}$, so $E \Subset \Omega$ and, by [9, Proposition 5.2], for every F with $\mathrm{Per}_K(F) < +\infty$ such that $E \subseteq F \subseteq E^{\xi_E(\delta')}$,

$$\mathrm{Per}_K(F) \geqslant \mathrm{Per}_K(E) + \int_{F\setminus E} H^K_{\{y : d_E(y) \geqslant d_E(x)\}}(x)\,dx.$$

Note that if $K \notin L^1(\mathbb{R}^n)$, this formula is meaningful only if $d_E \in C^{1,1}$ in a neighborhood of E. Now, using the fact that $-\xi_E(\delta') < d_E(x) < 0$ for $x \in F \setminus E$, and that $H^K_{E^\lambda}(x) \geqslant \delta$ for all $\lambda \in [0, \xi_E(\delta')]$, we conclude

$$\mathrm{Per}_K(F) \geqslant \mathrm{Per}_K(E) + \int_{F\setminus E} H^K_{\{y\,:\,d_E(y)\geqslant d_E(x)\}}(x)\,dx \geqslant \mathrm{Per}_K(E) + \delta'|F \setminus E|,$$

which implies that E is K-outward minimizing in Ω with associated constant δ'. ∎

Remark 2.10. We point out that, in general, K-mean convexity does not imply K-outward minimality. We consider the example described in Remark 2.5 of a compact set $E \in \mathbb{R}^2$ which satisfies $\partial E \setminus \{(0,0)\} \in C^{1,1}$ and $H^s_E(x) \geqslant \frac{2\pi}{(1+\sqrt{2})^2}$ for all $x \in \partial E$, and we show that E is not K-outward minimizing, for $K(x) = |x|^{-2-s}$, in any open set Ω such that $E \subseteq \Omega$. We recall that

$$H^s_{E\cup Q_r}(t,r),\, H^s_{E\cup Q_r}(t,-r) \leqslant -\frac{c(n)}{r^s} \quad \text{for all } t \in (-r,r),$$

where $Q_r = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 \in [-r,r],\ -|x_2| \leqslant x_1 \leqslant |x_2|\}$; see Remark 2.5. Then, arguing exactly as in [5, Proposition 1.8], it is possible to show that $\mathrm{Per}_s(E \cup Q_r) < \mathrm{Per}_s(E)$ for all $r > 0$ sufficiently small, which implies that E is not a K-outward minimizing set in any open set Ω which contains E.

3. Level set formulation

In this section, we recall the level set formulation of the geometric flow (1.1) in the setting of viscosity solutions for nonlocal equations, and we collect some results that will be useful in the sequel.

The viscosity theory for the classical mean curvature flow is contained in [12, 16]; see also [18] for a comprehensive presentation of the level set approach for classical geometric flows. The existence and uniqueness of solutions for the fractional curvature flow in (1.1) in the viscosity sense have been investigated in [24] by introducing the level set formulation of the geometric evolution problem (1.1) and a proper notion of viscosity solution. The paper [9] is the main reference where it is introduced a general framework for the analysis via the level set formulation of a wide class of local and nonlocal translation-invariant geometric flows.

Proposition 3.1. *Given a closed set $E \subseteq \mathbb{R}^n$ and a uniformly continuous function $u_E(x) : \mathbb{R}^n \to \mathbb{R}$ such that*

$$E = \{x \in \mathbb{R}^n : u_E(x) \geqslant 0\} \quad \text{and} \quad \partial E = \{x \in \mathbb{R}^n : u_E(x) = 0\}, \tag{3.1}$$

there exists a unique uniformly continuous function $u_E(x,t) : \mathbb{R}^n \times [0+\infty)$ which solves, in the viscosity sense, the nonlocal parabolic problem

$$\begin{cases} \partial_t u(x,t) + |Du(x,t)| H^K_{\{y\,:\,u(y,t)\geqslant u(x,t)\}}(x) = 0, \\ u(x,0) = u_E(x). \end{cases} \tag{3.2}$$

Moreover, if $u_E(x)$ is Lipschitz continuous, then $u_E(\cdot, t)$ is also Lipschitz continuous for all $t > 0$, with the same Lipschitz constant.

For the precise definition of viscosity solution, we refer to [9]; see also [24]. The existence and uniqueness of a uniformly continuous solution follows from the comparison principle proven in full generality in [9]. We observe that the inequality $H_E^K(x) \leqslant c$ (resp. $H_E^K(x) \geqslant c$) for $x \in \partial E$ can be shown to be equivalent to $H^K_{\{y:u_E(y)\geqslant 0\}}(x) \leqslant c$ (resp. $H^K_{\{y:u_E(y)\geqslant 0\}}(x) \geqslant c$) for x with $u_E(x) = 0$, in the viscosity sense.

Remark 3.2 (Outer and inner flows). We define the outer and inner flows as follows:

$$E^+(t) := \{x \in \mathbb{R}^n : u_E(x,t) \geqslant 0\} \quad \text{and} \quad E^-(t) := \{x \in \mathbb{R}^n : u_E(x,t) > 0\}, \tag{3.3}$$

where $u_E(x,t)$ is the unique viscosity solution to (3.2) with initial data u_E as defined in (3.1). The level set flow of ∂E is given by

$$\Sigma_E(t) := \{x \in \mathbb{R}^n : u_E(x,t) = 0\}. \tag{3.4}$$

We observe that since the equation in (3.2) is geometric, if we replace the initial condition u_E with any function u_0 with the same level sets $\{u_0 \geqslant 0\}$ and $\{u_0 > 0\}$; the evolutions $\Sigma_E(t)$, $E^+(t)$ and $E^-(t)$ remain the same. For more details, we refer to [9, 24].

Finally, we observe that if $\mathrm{int}(E) = \emptyset$, then $u_E(x) \leqslant 0$ for every $x \in \mathbb{R}^n$, by (3.1). Therefore, by the comparison principle proved in [9], we get that $u_E(x,t) \leqslant 0$ for every $t > 0$. In particular, this implies that

$$\text{if } E \text{ has empty interior, then } E^-(t) = \emptyset \text{ for all } t \geqslant 0. \tag{3.5}$$

Finally, we recall some results about the comparison between the level set flow and the geometric regular subsolutions and supersolutions to (1.1), which have been proven in [5, Appendix] (see also [9]).

We start with a geometric comparison principle proven in [5, Corollary A.8].

Proposition 3.3.

(1) *Let $F \subseteq E$ be two closed sets in $\mathbb{R}^n$ such that $d(F, E) = \delta > 0$. Then, $F^+(t) \subseteq E^-(t)$ for all $t > 0$, and the map $t \mapsto d(F^+(t), E^-(t))$ is nondecreasing.*

(2) *Let $v : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ be a bounded uniformly continuous viscosity supersolution to (3.2), and assume that $F \subseteq \{x \in \mathbb{R}^n : v(x, 0) \geqslant 0\}$. Then,*

$$F^+(t) \subseteq \{x \in \mathbb{R}^n : v(x,t) \geqslant 0\} \quad \textit{for all } t \in (0, T).$$

Moreover, if $d(F, \{x \in \mathbb{R}^n : v(x, 0) > 0\}) = \delta > 0$, then

$$F^+(t) \subseteq \{x \in \mathbb{R}^n : v(x,t) > 0\} \quad \textit{for all } t \in (0, T),$$

and

$$d\big(F^+(t), \{x \in \mathbb{R}^n : v(x,t) > 0\}\big) \geqslant \delta.$$

(3) *Let $w : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ be a bounded uniformly continuous viscosity subsolution to (3.2), and assume that $E \supseteq \{x \in \mathbb{R}^n : w(x, 0) \geqslant 0\})$. Then,*

$$E^+(t) \supseteq \big\{x \in \mathbb{R}^n : w(x, t) \geqslant 0\big\} \quad \textit{for all } t \in (0, T).$$

Moreover, if $d(E, \{x \in \mathbb{R}^n : w(x, 0) \geqslant 0\}) = \delta > 0$, then

$$E^-(t) \supseteq \big\{x \in \mathbb{R}^n : w(x, t) \geqslant 0\big\} \quad \textit{for all } t \in (0, T),$$

and

$$d\big(E^-(t), \big\{x \in \mathbb{R}^n : w(x, t) \geqslant 0\big\}\big) \geqslant \delta.$$

We now state a comparison result between the level set flow and the geometric subsolutions or supersolutions to (1.1). We omit its proof since it follows exactly as in [5, Proposition A.10].

Proposition 3.4. *Let $C(t) \subseteq \mathbb{R}^n$, for $t \in [0, T]$, be a continuous family of closed sets with compact boundaries, and let $E \subseteq \mathbb{R}^n$ be a closed set.*

(1) *Assume that $C(t)$ satisfies a uniform interior ball condition at every point of its boundary, and that there exists $\delta > 0$ such that at every $x \in \partial C(t)$,*

$$\partial_t x \cdot \nu(x) + H^K_{C(t)}(x) \geqslant \delta. \tag{3.6}$$

If $E \subseteq C(0)$, with $d(E, C(0)) = k \geqslant 0$, then $E^+(t) \subseteq C(t)$ for all $t \in [0, T]$, with $d(E^+(t), C(t)) \geqslant k$.

(2) *Assume that $C(t)$ satisfies a uniform exterior ball condition at every point of its boundary, and that there exists $\delta > 0$ such that at every $x \in \partial C(t)$,*

$$\partial_t x \cdot \nu(x) + H^K_{C(t)}(x) \leqslant -\delta. \tag{3.7}$$

If $E \supseteq C(0)$, then $E^+(t) \supseteq C(t)$ for all $t \in [0, T]$.
If $d(C(0), \{x \in \mathbb{R}^n : u_E(x) > 0\}) = k > 0$, then $E^-(t) \supseteq C(t)$ for all $t \in [0, T]$, with $d(E^-(t), C(t)) \geqslant k$.

4. K-flow of K-mean convex sets

In this section, we discuss some properties of the K-flow (1.1) starting from a regularly or strongly K-mean convex set. We first show that the flow is monotone in the following sense.

Proposition 4.1.

(1) *Let $E \subseteq \mathbb{R}^n$ be a strongly K-mean convex set with associated constant $\delta > 0$. If $\mathrm{int}(E) = \emptyset$, then $E^-(t) = \emptyset$ and $\mathrm{int}(E^+(t)) = \emptyset$ for every $t \geqslant 0$, whereas if $\mathrm{int}(E) \neq \emptyset$, then*

$$E^+(t+s) \subseteq E^-(t) \textit{ with } d(E^+(t+s), E^-(t)) \geqslant \delta s \qquad \textit{for every } t \geqslant 0,\ s \in [0, \xi_E/\delta), \tag{4.1}$$

where $E^-(0) = \mathrm{int}(E)$*. In particular,* $E^+(t) \setminus E^-(t)$ *has empty interior for all* $t > 0$*.*

(2) *Let* $E \subseteq \mathbb{R}^n$ *be a regularly* K*-mean convex set. Then,*

$$E^+(t) \subseteq E \quad \textit{and} \quad E^+(t+s) \subseteq E^+(t) \quad \textit{for every } t, s \geqslant 0. \tag{4.2}$$

Proof. (1) Let $\delta > 0$ and ξ_E be the constants associated to E, according to Definition 2.2. Let $\xi \leqslant \xi_E$. For $0 < h < \min(\delta, \xi)$ and $s \in [0, 1]$,

$$C(s) := E^{\xi - hs}.$$

We observe that $C(s)$ is a supersolution to (1.1), in the sense that it satisfies

$$\partial_s x \cdot \nu + H^K_{C(s)}(x) = -h + H^K_{C(s)}(x) > 0.$$

Since $E \subseteq E^\xi = C(0)$, by Proposition 3.4, we get that for all $s \in (0, 1]$, for every $\xi \leqslant \xi_E$,

$$E^+(s) \subseteq C(s) = E^{\xi - hs} \subseteq E^\xi \quad \text{and} \quad d(E^+(s), E^{\xi - hs}) \geqslant d(E, E^\xi) = \xi.$$

This implies that for all $s \in [0, 1]$,

$$E^+(s) \subseteq \bigcap_{0 < \xi \leqslant \xi_E} E^\xi = E.$$

Therefore, if $\mathrm{int}(E) = \emptyset$, then we conclude that $\mathrm{int}(E^+(t)) = \emptyset$ and we recall that $E^-(t) = \emptyset$ for all $t > 0$ by (3.5).

Assume now that E has nonempty interior. Arguing as above, we define $C(s) = E^{\xi_E - \delta s}$ and we get that $C(s)$ is a supersolution to (1.1) for every $s \in [0, \xi_E/\delta)$. Therefore, as above, by Proposition 3.4, we get that $E^+(s) \subseteq E^{\xi_E - \delta s}$ for every $s \in [0, \xi_E/\delta)$ and $d(E^+(s), E^{\xi_E - \delta s}) \geqslant d(E, E^{\xi_E}) = \xi_E$.

Let $x \in \partial E^+(s)$. Then, $d(x, \partial E^{\xi_E - \delta s}) \geqslant d(E^+(s), E^{\xi_E - \delta s}) \geqslant \xi_E$. Therefore, for every $y \in \partial E$, we get

$$\begin{aligned} \xi_E \leqslant d(x, \partial E^{\xi_E - \delta s}) = \min_{z \in \partial E^{\xi_E - \delta s}} |x - z| &\leqslant |x - y| + \min_{z \in \partial E^{\xi - \delta s}} |y - z| \\ &= |x - y| + \xi_E - \delta s, \end{aligned}$$

which, in turn, gives that for all $x \in \partial E^+(s)$ with $s \in [0, \xi_E/\delta)$ and all $y \in \partial E$,

$$|x - y| \geqslant \delta s.$$

This implies that for all $s \in [0, \xi_E/\delta)$

$$d(E^+(s), E) \geqslant \delta s > 0. \tag{4.3}$$

In particular, it follows that $E^+(s) \subseteq \mathrm{int}(E)$.

By the comparison principle in Proposition 3.3, we get that

$$E^+(t+s) \subseteq E^-(t) \text{ with } d(E^+(t+s), E^-(t)) \geqslant \delta s \quad \text{for all } t \geqslant 0,\ s \in (0, \xi_E/\delta).$$

Finally, we recall the following lower semicontinuity result for the outer evolution proved in [5, Proposition A.12]: $\liminf_{\eta\to 0} |E^+(t+\eta)| \geqslant |\mathrm{int}(E^+(t))|$.

Then, since $E^+(t+s) \subseteq E^-(t)$ for $s \in (0, \xi_E/\delta)$, we get

$$\left|\mathrm{int}(E^+(t) \setminus E^-(t))\right| \leqslant \limsup_{s\to 0^+} \left|\mathrm{int}(E^+(t))\right| - \left|E^+(t+s)\right| \leqslant 0,$$

which gives the conclusion.

(2) Now, we consider the case of a regularly K-mean convex set E. Fix $\lambda \leqslant \eta_E$ and $T < \frac{1}{c_E}$ and define the flow $C(t) = E^{c_E\lambda t}$ for $t \in [0, T]$. Note that since $c_E\lambda t \leqslant \eta_E$, $H^K_{C(t)}(x) \geqslant -c_E^2\lambda t \geqslant -c_E^2\lambda T > -c_E\lambda$ for all $t \in [0, T]$, which implies that $C(t)$ is a strict supersolution to (1.1). Therefore, by Proposition 3.4, we get that

$$E^+(t) \subseteq E^{c_E\lambda t} \quad \text{for all } 0 \leqslant t \leqslant T < \frac{1}{c_E} \text{ and every } \lambda \in (0, \eta_E].$$

This implies that for $t \in [0, \frac{1}{c_E})$, $E^+(t) \subseteq \bigcap_{\lambda\in(0,\eta_E]} E^{c_E\lambda t} = E$, since E is closed. Then, by the comparison principle in Proposition 3.3, we get that

$$E^+(t+s) \subseteq E^+(t) \quad \text{for all } t \geqslant 0,\ s \in \left[0, \frac{1}{c_E}\right). \qquad \blacksquare$$

Remark 4.2. Observe that if E is K-mean convex and $H^K_E(x) \geqslant \delta > 0$ for all $x \in \partial E$ in the viscosity sense, but E is not regularly or strongly K-mean convex, then, in general, it is not true that $E^+(t) \subseteq E$ for $t > 0$ and, moreover, in general, the flow may develop fattening. The fattening phenomenon is related to the non-uniqueness of the geometric flow; for an analysis of this phenomenon, mainly in dimension 2, for geometric equations as (1.1), we refer to [5].

As an example, we consider the set E described in Remark 2.5. In [5, Theorem 1.10], it is proved that there exists $t > 0$ and $c > 0$ such that $E^-(t) \subseteq B(0, r(\tau)) \subseteq E^+(\tau)$ for all $\tau \in [0, t)$, where $r(\tau) = c(n)\tau^{1/(1+s)}$, which implies that (4.2) cannot hold.

Moreover, we show that the monotonicity of the flow implies K-mean convexity.

Proposition 4.3. *Let E be a closed set. Assume that there exists $h > 0$ such that*

$$E^+(t) \subseteq E \quad \textit{for every } 0 \leqslant t \leqslant h, \tag{4.4}$$

then $H^K_E(x) \geqslant 0$ in the viscosity sense for every $x \in \partial E$.

If, moreover, there exists $\delta > 0$ such that

$$E^+(t) \subseteq E \textit{ with } d(E, E^+(t)) \geqslant \delta t \quad \textit{for every } 0 \leqslant t \leqslant h,$$

then $H^K_E(x) \geqslant \delta$ in the viscosity sense for every $x \in \partial E$.

Proof. We prove directly the second statement, since the first can be proved in a similar way, just putting $\delta = 0$. Assume that it is not true that $H^K_E(x) \geqslant \delta$ in the viscosity sense for every $x \in \partial E$. Therefore, there exists $x \in \partial E$ and a set F with $C^{1,1}$ boundary such that $F \subseteq E$, $x \in \partial F \cap \partial E$ and $H^K_F(x) \leqslant \delta - 4\rho < \delta$. By the continuity of the curvature on regular sets (see [9]), there exists $r > 0$ such that for all $y \in \partial F \cap B(x, 4r)$, $H^K_F(y) \leqslant \delta - 3\rho$.

Now, we construct a strict subsolution $C(t)$ to (1.1) with $C(0) = F$ as follows. Let $c = \max_{y\in\partial F} H^K_F(y) \geqslant 0$ and let $\psi_r, \phi_r : \mathbb{R}^n \to [0, 1]$ be two smooth functions such that $\psi_r(y) = 1$, for $y \in B(x, r)$, and $\psi_r(y) = 0$, for $y \in \mathbb{R}^n \setminus B(x, 2r)$, and on the other hand $\phi_r(y) = 0$, for $y \in B(x, 3r)$, and $\phi_r(y) = 1$, for $y \in \mathbb{R}^n \setminus B(x, 4r)$. We construct a family of regular sets as follows: $C(0) = F$ and $C(t)$ is the set whose boundary is

$$\partial C(t) = \big\{y + (-\delta + \rho)t\psi_r(y)\nu_{\partial F}(y) - (c + 2\rho)t\phi_r(y)\nu_{\partial F}(y) \colon y \in \partial F\big\},$$

where $\nu_F(y)$ is the outer normal of F at $x \in \partial F$. For $t > 0$ sufficiently small, $C(t)$ is of class $C^{1,1}$ and moreover, by the continuity of the curvature on regular sets,

$$H^K_{C(t)}(y) \leqslant \delta - 2\rho \quad \text{for } y \in \partial C(t) \cap B(x, 4r) \qquad \text{and} \qquad c + \rho \geqslant \max_{y\in\partial C(t)} H^K_{C(t)}(y). \tag{4.5}$$

Finally, observe that at every $y \in \partial C(t)$,

$$\partial_t y \cdot \nu(y) = (-\delta + \rho)\psi_r(y) - (c + 2\rho)\phi_r(y) \leqslant -H^K_{C(t)}(y) - \rho,$$

where the last inequality is obtained by recalling the definition of ϕ_r, ψ_r and (4.5). We conclude, by Proposition 3.4, that since $C(0) = F \subseteq E$, then $C(t) \subseteq E^+(t)$ for all $t > 0$ sufficiently small.

Note that $d(x + (-\delta + \rho)t\nu_F(x), x) = (\delta - \rho)t$ and then $d(C(t), E) \leqslant (\delta - \rho)t < \delta t$, in contradiction with the fact that $d(E^+(t), E) \geqslant \delta t$ and $C(t) \subseteq E^+(t) \subseteq E$. ∎

Remark 4.4. Note that, arguing exactly as in the proof of Proposition 4.3, we may prove the following result: if E is a closed set such that there exist $\delta > 0$ and $h > 0$ for which

$$\sup_{x\in E^+(t)} d(x, E) \leqslant \delta t \quad \text{for all } t \leqslant h,$$

then

$$H^K_E(x) \geqslant -\delta \quad \text{in the viscosity sense for all } x \in \partial E.$$

Indeed, we argue by contradiction and we choose F as in the proof of Proposition 4.3, with $C^{1,1}$ boundary such that $F \subseteq E$, $x \in \partial F \cap \partial E$ and $H^K_F(y) \leqslant -\delta - 2\rho$ for all $y \in \partial F \cap B(x, r)$. We now construct a strict subsolution to (1.1) as

$$\partial C(t) = \big\{y + (\delta + \rho)t\psi_r(y)\nu_{\partial F}(y) - (c + 2\rho)t\phi_r(y)\nu_{\partial F}(y) \colon y \in \partial F\big\},$$

where $c = \max_{y\in\partial F} H^K_F(y) \geqslant 0$ (since F is compact). Therefore, by comparison, $C(t) \subseteq E^+(t)$ and $\sup_{x\in C(t)} d(x, E) \geqslant (\delta + \rho)t$, which gives a contradiction.

We collect the previous results about flows of regularly and strongly K-mean convex sets.

Theorem 4.5.

(1) *Let E be a strongly K-mean convex set with associated constant $\delta \geqslant 0$. Then, for all $\eta \in [0, \xi_E)$, the outer flow $(E^\eta)^+(t)$ is monotone according to (4.1), if $\delta > 0$, or to (4.2), if $\delta = 0$, and, moreover,*

$$H^K_{(E^\eta)^+(t)}(x) \geqslant \delta \quad \textit{for all } t \geqslant 0.$$

(2) *Let E be a regularly K-mean convex set. Then, the outer flow $E^+(t)$ is monotone according to (4.2), and*

$$H^K_{E^+(t)}(x) \geqslant 0 \quad \textit{for all } t \geqslant 0.$$

Proof. (1) Note that by definition if K is strongly K-mean convex with associated constant $\delta \geqslant 0$, then for any $\eta \in (0, \xi_E)$, E^η is also strongly K-mean convex with associated constant $\delta \geqslant 0$, and $\xi_{E^\eta} = \xi_E - \eta$. Therefore, we may apply Proposition 4.1 to every E^η and deduce that if $\delta = 0$, then (4.2) holds for $(E^\eta)^+(t)$ for every $t \geqslant 0$ and if $\delta > 0$, then (4.1) holds for $s \in [0, \frac{\xi_E - \eta}{\delta}]$ and for every $t \geqslant 0$. Now, by Proposition 4.3, we get that

$$H^K_{(E^\eta)^+(t)}(x) \geqslant \delta \quad \text{for all } t \geqslant 0.$$

(2) The fact that $H^K_{E^+(t)}(x) \geqslant 0$ is a consequence of (4.1) and Proposition 4.3. ∎

5. Minimizing movements

We now recall the variational scheme, sometimes called minimizing movements, introduced in [1] for the classical mean curvature flow, and later extended to the nonlocal setting in [9].

Given a nonempty set $E \subseteq \mathbb{R}^n$ with compact boundary and a time step $h > 0$, if E is bounded, we define the set $T_h(E)$ as a solution of the minimization problem

$$\min_{F \subseteq \mathbb{R}^n} \operatorname{Per}_K(F) - \frac{1}{h} \int_F d_E(x)\, dx. \tag{5.1}$$

If E is unbounded, then we define $T_h(E) := \mathbb{R}^n \setminus T_h(\mathbb{R}^n \setminus E)$. We also let $T_h(\emptyset) := \emptyset$.

We iterate the scheme to obtain $T_h^{(k)}(E) = T_h(T_h^{(k-1)}(E))$, where we put $T_h^{(1)}(E) = T_h(E)$, and we define the following piecewise constant flows as follows:

$$E_h(t) = T_h^{(k)}(E) \quad \text{for } t \in \big[kh, (k+1)h\big). \tag{5.2}$$

In the sequel, we will identify a minimizer $T_h(E)$, and a time discrete flow $E_h(t)$, with the representative given by the set of Lebesgue points of the characteristic function.

We recall some results about this scheme from [9].

Theorem 5.1.

(1) *For any set E, the minimization problem* (5.1) *admits a maximal solution $T_h^+(E)$ and a minimal solution $T_h^-(E)$ (with respect to inclusion). We will denote the flow obtained in* (5.2) *by interpolating the minimal and the maximal solution as $E_h^-(t)$ and $E_h^+(t)$, respectively. Every flow constructed as in* (5.2) *satisfies $E_h^-(t) \subseteq E_h(t) \subseteq E^+(t)$.*

(2) *If $E \subseteq F$, then $T_h^{\pm}(E) \subseteq T_h^{\pm}(F)$. Moreover, if $d(E, F) \geqslant r$, then $d(T_h(E), T_h(F)) \geqslant r$.*

(3) *There exists a constant $C > 1$ depending only on the dimension, such that for every fixed $R > 0$ and every $h > 0$ such that*

$$R - h \min_{x \in \partial B(0,CR)} H^K_{B(0,CR)}(x) > 0,$$

one has

$$T_h^{\pm}(B(0,R)) \subseteq B\Big(0, R - h \min_{x \in \partial B(0,CR)} H^K_{B(0,CR)}(x)\Big).$$

(4) *For every $R_0 > 0$ and $\sigma > 1$, there exists $h_0 > 0$ depending on R_0, σ and C such that if $h \leqslant h_0$, then for any $R \geqslant R_0$ and $h \leqslant h_0$,*

$$B\Big(0, R - h \max_{x \in \partial B(0,R/\sigma)} H^K_{B(0,R/\sigma)}(x)\Big) \subseteq T_h^{\pm}(B(0,R)).$$

(5) *Let $E \subseteq F$ be a nonempty bounded set with $r = d(E, F) > 0$. Then, there exists $h_0 > 0$ depending on r and the dimension such that for all $h \leqslant h_0$, $T_h^{\pm}(E) \subseteq F$ and, moreover,*

$$d(T_h^+(E), F) \geqslant r - h \max_{x \in \partial B(0,r/2)} H^K_{B(0,r/2)}(x) > 0.$$

Proof. For the proof of items (1)–(4), we refer to Proposition 7.1, Lemma 7.2, Lemma 7.4, Lemma 7.5, Lemma 7.6 and Lemma 7.10 in [9].

We now show item (5). We fix $x \in \partial F$ and observe that by assumption, for every $r' < r$, $E \subseteq \mathbb{R}^n \setminus B(x, r')$ and then, by monotonicity,

$$T^{\pm}(E) \subseteq T^{\pm}(\mathbb{R}^n \setminus B(x, r')) = \mathbb{R}^n \setminus T^{\mp}(B(x, r')). \tag{5.3}$$

Now, we apply item (4), choosing $R_0 = r/2$ and $\sigma = 2$: there exists h_0 depending on r such that for all $r' > r/2$ and $h \leqslant h_0$,

$$B\Big(x, r' - h \max_{y \in \partial B(0,r'/2)} H^K_{B(0,r'/2)}(y)\Big) \subseteq T_h^{\pm}(B(x, r')).$$

Substituting in (5.3), we get that for all $x \in \partial F$,

$$T_h^{\pm}(E) \subseteq \mathbb{R}^n \setminus B\Big(x, r' - h \max_{y \in \partial B(0,r'/2)} H^K_{B(0,r'/2)}(y)\Big) \quad \text{for all } r' \in (r/2, r),\ h \leqslant h_0.$$

This implies that for all $h \leqslant h_0$, either $T_h^{\pm}(E) \subseteq F$ or $T_h^{\pm}(E) \subseteq \mathbb{R}^n \setminus F$, and in both cases

$$d(F, T_h^{\pm}(E)) \geqslant r - h \max_{y \in \partial B(0,r/2)} H^K_{B(0,r/2)}(y) > 0. \tag{5.4}$$

Finally, we observe that necessarily $T_h^{\pm}(E) \subseteq F$. Assume by contradiction that $T_h^{\pm}(E) \subseteq \mathbb{R}^n \setminus F$. Then, recalling that $E \subseteq F$ with $d(E,F) = r$, from (5.4) we would get that $d_E(x) \leqslant -2r + h \max_{y \in \partial B(0,r/2)} H^K_{B(0,r/2)}(y) < 0$ for every $x \in T_h^+(E)$. So, it would be possible, just by translating $T_h^+(E)$, to construct a competitor with strictly less energy, and so to prove that $T_h^+(E)$ could not be a solution to the minimization problem (5.1). ∎

Finally, we recall the convergence of the scheme to the K-mean curvature flow, as proved in [9, Proposition 7.12 and Theorem 7.16].

Theorem 5.2. *Let u_0 be a Lipschitz continuous function. We define*

$$T_h u_0(x) := \sup \big\{\lambda : x \in T_h\big(\{x : u_0(x) > \lambda\}\big)\big\},$$

and iteratively, for $k \in \mathbb{N}$,

$$T_h^{(k)} u_0(x) := T_h\big(T_h^{(k-1)} u_0(x)\big). \tag{5.5}$$

Let

$$u_h(x,t) := T_h^{[t/h]} u_0(x),$$

then

$$\begin{cases} T_h^-\big(\{x : u_h(x,(k-1)h) > \lambda\}\big) = \{x : u_h(x,kh) > \lambda\}, \\ T_h^+\big(\{x : u_h(x,(k-1)h) \geqslant \lambda\}\big) = \{x : u_h(x,kh) \geqslant \lambda\}, \end{cases}$$

where the second equality holds up to a negligible set and, moreover,

$$u_h(x,t) \to u(x,t) \quad \text{as } h \to 0, \text{ locally uniformly in } \mathbb{R}^n \times [0,+\infty),$$

where $u(x,t)$ is the unique solution to (3.2) with initial datum u_0.

6. K-flow of K-outward minimizing sets

In this section, we show that the level set flow preserves the K-outward minimality. In the case of the classical mean curvature flow, we refer to [29] for an analysis of the outward minimizing sets. In particular, in that paper, it is shown that these sets provide a class of initial data for which the minimizing movement scheme converges to the level set flow. For the generalization of this result to the anisotropic case and the crystalline case, we refer to [10].

First of all, we show that the minimizing movement scheme (5.2) starting from a K-outward minimizer is monotone (see [29, Lemma 2.7] for the case of the classical perimeter, and [10, Lemma 2.3] for the anisotropic perimeter).

Proposition 6.1. *Let Ω be an open set and let E be a nonempty bounded set with $E \Subset \Omega$. If E is a K-outward minimizing set in Ω, then there exists h_0 depending on $r = d(E, \Omega) > 0$ such that for all $h \leqslant h_0$, every piecewise constant flow $E_h(t) = T_h^{(k)}(E)$, for $t \in [kh, (k+1)h)$ defined in (5.2) satisfies*

$$E_h(t) \subseteq \overline{E_h(s)} \quad \textit{and} \quad \mathrm{Per}_K(E_h(t)) \leqslant \mathrm{Per}_K(E_h(s)) \quad \textit{for all } t \geqslant s \geqslant 0,$$

where $E_h(0) = E$. Moreover, $E_h(t)$ is a K-outward minimizing set in Ω, so that $H^K_{E_h(t)}(x) \geqslant 0$ in the viscosity sense at every $x \in \partial E_h(t)$.

Proof. First of all, we observe that by Theorem 5.1, since $E \Subset \Omega$, then there exists h_0 such that $T_h^+(E) \Subset \Omega$ for all $h \leqslant h_0$. Now, we proceed by induction on $k \geqslant 0$ and, to avoid long notation, we define $E_k := T_h^{(k)}(E)$. Since $T_h(E_k)$ is a minimizer of (5.1), choosing $E_k \cap T_h(E_k)$ as a competitor, we get

$$\begin{aligned}
&\mathrm{Per}_K(T_h(E_k)) - \mathrm{Per}_K(E_k \cap T_h(E_k)) \\
&\quad \leqslant \frac{1}{h}\int_{T_h(E_k)} d_{E_k}(x)\,dx - \frac{1}{h}\int_{E_k \cap T_h(E_k)} d_{E_k}(x)\,dx \\
&\quad = \frac{1}{h}\int_{T_h(E_k)\setminus E_k} d_{E_k}(x)\,dx \leqslant 0,
\end{aligned}$$

since $d_{E_k} \leqslant 0$ on $\mathbb{R}^n \setminus E_k$. Since E_k is a K-outward minimizer, we get that

$$\mathrm{Per}_K(E_k \cap T_h(E_k)) \leqslant \mathrm{Per}_K(T_h(E_k)) \quad \text{and} \quad \frac{1}{h}\int_{T_h(E_k)\setminus E_k} d_{E_k}(x)\,dx = 0,$$

which implies that $T_h(E_k) \subseteq \overline{E_k}$, up to a negligible set, recalling that $d_{E_k} < 0$ on $\mathbb{R}^n \setminus \overline{E_k}$. Now, using E_k as a competitor, we observe that, since $T_h(E_k) \subseteq \overline{E_k}$ and $d_{E_k} \geqslant 0$ in $\overline{E_k}$,

$$\mathrm{Per}_K(T_h(E_k)) \leqslant \mathrm{Per}_K(E_k) - \frac{1}{h}\int_{E_k} d_{E_k}(x)\,dx + \frac{1}{h}\int_{T_h(E_k)} d_{E_k}(x)\,dx \leqslant \mathrm{Per}_K(E_k).$$

Let $G \supseteq T_h(E_k)$ be such that $G \setminus T_h(E_k) \Subset \Omega$. Our aim is to prove that $\mathrm{Per}_K(T_h(E_k)) \leqslant \mathrm{Per}_K(G)$. Using the minimality of $T_h(E_k)$ and $G \cap E_k$ as competitor we get, recalling that $T_h(E_k) \subseteq \overline{E_k} \cap G$ and that $d_{E_k} = 0$ on $\overline{E_k} \setminus E_k$,

$$\begin{aligned}
\mathrm{Per}_K(T_h(E_k)) &\leqslant \mathrm{Per}_K(G \cap E_k) - \frac{1}{h}\int_{G\cap E_k} d_{E_k}(x)\,dx + \frac{1}{h}\int_{T_h(E_k)} d_{E_k}(x)\,dx \\
&\leqslant \mathrm{Per}_K(G \cap E_k).
\end{aligned}$$

We conclude by recalling that E_k is a K-outward minimizer, so that $\mathrm{Per}_K(G \cap E_k) \leqslant \mathrm{Per}_K(G)$. ■

Proposition 6.2. *Under the same assumptions of Proposition 6.1, if E is also strongly K-outward minimizing set in Ω with constant $\delta > 0$, for all $h \leqslant \min(h_0, \frac{d(E,\Omega)}{\delta})$, one has*

- *if E has empty interior, then $T_h(E) = \emptyset$;*
- *if E has nonempty interior, then the discrete flow $E_h(t)$ satisfies*

$$d(E_h(t), E_h(t+h)) \geqslant \delta h \quad \textit{and} \quad H^K_{E_h(t)}(x) \geqslant \delta \quad \textit{for all } t \geqslant 0 \textit{ and } x \in \partial E_h(t).$$

Proof. Observe that, by the definition of the piecewise constant flow $E_h(t)$, it is sufficient to prove the second statement for $E_k := T_h^{(k)}(E)$ for every $k \geqslant 1$. We start by considering the case $k = 1$. In this case, $E_1 = T_h(E)$. By Proposition 6.1, we know that $E_1 \subseteq \overline{E}$. We fix $z \in \mathbb{R}^n$ with $|z| < h\delta$ and observe that $E_1 + z \subseteq \overline{E} + z \subseteq \Omega$ since $h\delta \leqslant d(E, \Omega)$. Now, $E_1 + z$ is a solution to the minimization problem

$$\min_F \Big(\mathrm{Per}_K(F) - \frac{1}{h} \int_F d_E(x-z)\, dx \Big).$$

We choose $E \cap (E_1 + z)$ as a competitor and we get

$$\begin{aligned} &\mathrm{Per}_K(E_1 + z) - \frac{1}{h} \int_{E_1+z} d_E(x-z)\, dx \\ &\qquad \leqslant \mathrm{Per}_K(E \cap (E_1 + z)) - \frac{1}{h} \int_{E\cap(E_1+z)} d_E(x-z)\, dx. \end{aligned}$$

Since E is a strongly K-outward minimizer, we get

$$\mathrm{Per}_K(E \cap (E_1 + z)) \leqslant \mathrm{Per}_K(E_1 + z) - \delta \big|(E_1 + z) \setminus E\big|.$$

Substituting in the previous inequality, we get

$$\delta \big|(E_1 + z) \setminus E\big| \leqslant \frac{1}{h} \int_{(E_1+z)\setminus E} d_E(x-z)\, dx.$$

Finally, for $x \notin E$, by definition,

$$d_E(x-z) = d(x-z, \mathbb{R}^n \setminus E) - d(x-z, E) \leqslant d(x-z, \mathbb{R}^n \setminus E) \leqslant d(x-z, x) = |z|.$$

Therefore, in the previous inequality, we get

$$\delta \big|(E_1 + z) \setminus E\big| \leqslant \frac{1}{h}|z| \big|(E_1 + z) \setminus E\big| < \delta \big|(E_1 + z) \setminus E\big|,$$

which implies that $|(E_1 + z) \setminus E| = 0$ for every z with $|z| < \delta h$, that is,

$$E_1 + B(0, \delta h) \subseteq E.$$

Note that if $\mathrm{int}(E) = \emptyset$, then by the previous inclusion, we get that necessarily $E_1 = \emptyset$.

If $\mathrm{int}(E) \neq \emptyset$, one has that

$$E_1 \subseteq E \quad \text{and} \quad d(E_1, E) \geqslant h\delta.$$

By Theorem 5.1, we then get

$$E_2 = T_h(E_1) \subseteq T_h(E) = E_1 \quad \text{and} \quad d(E_2, E_1) \geqslant h\delta.$$

So, by iteration, we obtain

$$E_k \subseteq E_{k-1} \quad \text{and} \quad d(E_k, E_{k-1}) \geqslant h\delta.$$

Finally, we fix $k \geqslant 1$ and we claim that for any $\lambda \in (0,1)$,

$$H^K_{E_k}(x) \geqslant \delta(1-\lambda) \text{ in the viscosity sense for all } x \in \partial E_k.$$

So, sending $\lambda \to 0$, we get the statement.

The minimality of $E_k = T_h(E_{k-1})$ and the submodularity of the perimeter (2.4) give that for all G,

$$\begin{aligned} \mathrm{Per}_K(G \cap E_k) &\leqslant \mathrm{Per}_K(G) + \mathrm{Per}_K(E_k) - \mathrm{Per}_K(E_k \cup G) \\ &\leqslant \mathrm{Per}_K(G) - \frac{1}{h}\int_{G\setminus E_k} d_{E_{k-1}}(x)\,dx. \end{aligned} \tag{6.1}$$

We proceed as in the proof of Theorem 2.9 (1). We fix $\lambda \in (0,1)$ and we assume, by contradiction, that there exists $F \subseteq E_k$ with $\partial F \in C^{1,1}$, $x_0 \in \partial E_k \cap \partial F$, such that $H^K_F(x_0) \leqslant \delta(1-\lambda) - 2\rho$ for some $\rho > 0$ small. Then, by the continuity of H^K, there exists $r_0 > 0$ such that $H^K_F(x) \leqslant \delta(1-\lambda) - \rho$ for every $x \in \partial F \cap B(x_0, r_0)$. We fix

$$r < \min\Big(r_0, \frac{h\delta\lambda}{2}\Big), \tag{6.2}$$

so that $B(x_0, r) \Subset E_{k-1}$ (since $d(E_k, E_{k-1}) \geqslant \delta h$) and we construct a 1-parameter family Φ_ε of $C^{1,1}$ diffeomorphisms, such that $F = \Phi_0(F) \subseteq \Phi_\varepsilon(F) \subseteq E$, $|\Phi_\varepsilon(F) \setminus E_k| > 0$ and $\Phi_\varepsilon(F) \setminus E_k \subseteq \Phi_\varepsilon(F) \setminus F \Subset B(x_0, r) \Subset E_{k-1}$ for every $\varepsilon \in (0, \varepsilon_0)$. Again, by continuity, $H^K_{\Phi_\varepsilon(F)}(x) \leqslant \delta(1-\lambda) - \rho/2$ for all $x \in \partial\Phi_\varepsilon(F) \setminus F$. Using the fact that H^K is the first variation of Per_K with respect to $C^{1,1}$ diffeomorphisms, as in the proof of Theorem 2.9 (see (2.5) and (2.6)), we get

$$\mathrm{Per}_K(E_k \cap \Phi_\varepsilon(F)) \geqslant \mathrm{Per}_K(\Phi_\varepsilon(F)) + \Big(-\delta(1-\lambda) + \frac{\rho}{2}\Big)\big|\Phi_\varepsilon(F) \setminus E_k\big|.$$

By the previous inequality and (6.1) applied to $G = \Phi_\varepsilon(F)$, we get

$$\begin{aligned} &\mathrm{Per}_K(\Phi_\varepsilon(F)) - \frac{1}{h}\int_{\Phi_\varepsilon(F)\setminus E_k} d_{E_{k-1}}(x)\,dx \\ &\qquad \geqslant \mathrm{Per}_K(\Phi_\varepsilon(F)) + \Big(-\delta(1-\lambda) + \frac{\rho}{2}\Big)\big|\Phi_\varepsilon(F) \setminus E_k\big|, \end{aligned}$$

from which we deduce

$$\frac{1}{h}\int_{\Phi_\varepsilon(F)\setminus E_k} d_{E_{k-1}}(x)\,dx \leqslant \Big(\delta(1-\lambda) - \frac{\rho}{2}\Big)\big|\Phi_\varepsilon(F) \setminus E_k\big|.$$

Observe that $\Phi_\varepsilon(F) \setminus E_k \subseteq B(x_0, r)$ and then, $d_{E_{k-1}}(x) \geqslant h\delta - r > h(\delta - \frac{\delta\lambda}{2})$ for all $x \in \Phi_\varepsilon(F) \setminus E_k$, by (6.2). Therefore, we get

$$\Big(\delta - \frac{\delta\lambda}{2}\Big)\big|\Phi_\varepsilon(F) \setminus E_k\big| < \Big(\delta - \delta\lambda - \frac{\rho}{2}\Big)\,\big|\Phi_\varepsilon(F) \setminus E_k\big|,$$

which implies $\Phi_\varepsilon(F) \subseteq E_k$, in contradiction with our construction. ■

We now prove the main result of this section, about the flow of K-outward minimizing sets.

Theorem 6.3. *Let Ω be an open set, E a bounded set with $E \Subset \Omega$. Assume that E is a strongly K-outward minimizing set in Ω with constant $\delta > 0$. Then, for every $t > 0$ up to a countable set, one has*

$$E_h(t) \to E^-(t) \quad \text{in } L^1(\Omega), \text{ as } h \to 0.$$

Moreover, $E^-(t)$ is a K-outward minimizing set in Ω for every $t > 0$, and

$$E^-(t+s) \subseteq E^-(t) \quad \text{with} \quad d\big(E^-(t+s), E^-(t)\big) \geqslant \delta s \quad \text{for every } t, s > 0.$$

Moreover, $\bigcap_{s<t} E^-(s) \setminus E^-(t)$ has empty interior for all $t > 0$, $|\bigcap_{s<t} E^-(s) \setminus E^-(t)|$ $= 0$ for every $t > 0$, up to a countable set, and

$$H^K_{E^-(t)}(x) \geqslant \delta \quad \text{for all } x \in \partial E^-(t) \text{ and } t > 0.$$

Finally, if E has boundary of class $C^{1,1}$ or if $K \in L^1(\mathbb{R}^n)$, the same results hold also for the outer flow $E^+(t)$, $E^+(t) \setminus E^-(t)$ has empty interior for all $t > 0$, and

$$H^K_{E^+(t)}(x) \geqslant \delta \quad \text{for all } x \in \partial E^+(t) \text{ and } t \geqslant 0.$$

Proof. Note that by Proposition 6.2 and Remark 3.2, we may assume $\mathrm{int}(E) \neq \emptyset$; otherwise the statement is trivial. We divide the proof into six steps.

Step 1: Definition of a continuous minimal time function u. We recall that $E_h(t) = T_h^{(k)}(E)$ for $t \in [kh, (k+1)h)$. We define the discrete arrival time function as follows:

$$u_h(x) := \begin{cases} h \displaystyle\sum_{k \geqslant 0} \chi_{E_k}(x) = \int_0^{+\infty} \chi_{E_h(t)}\, dt, & \text{if } x \in E, \\ 0, & \text{if } x \in \mathbb{R}^n \setminus E. \end{cases} \tag{6.3}$$

Note that by Proposition 6.2, u_h is well defined and

$$\big\{x : u_h(x) > t\big\} = E_h(t).$$

By its very definition, we get that

$$T_h^{(k)} u_h(x) = u_h(x) - hk, \tag{6.4}$$

where $T_h^{(k)} u_h(x)$ is defined as in (5.5).

Moreover, by Proposition 6.2, we get that $d(T_h^{(k)}(E), T_h^{(k')}(E)) \geqslant \delta h(|k-k'|-1)$. Let $x \in T_h^{(k)}(E)$ and $y \in T_h^{(k')}(E)$,

$$|u_h(x) - u_h(y)| = h|k'-k| \leqslant \frac{d\big(T_h^{(k)}(E), T_h^{(k')}(E)\big)}{\delta} + h \leqslant \frac{|x-y|}{\delta} + h.$$

This implies that, up to a subsequence, $u_h \to u$ uniformly as $h \to 0$, where $u : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz continuous function such that $u = 0$ in $\mathbb{R}^n \setminus E$ and $|u(x) - u(y)| \leqslant \frac{|x-y|}{\delta}$.

Step 2: For all $t > 0$, $E^-(t) = \{x : u(x) > t\}$. Note that since $u_h \to u$ uniformly, then it is also true that $\|T_h u_h - T_h u\|_\infty \to 0$ as $h \to +\infty$ and then also $\|T_h^{(k)} u - T_h^{(k)} u_h\|_\infty \to 0$ as $h \to 0$ for all $k \geqslant 1$, where $T_h u$, $T_h^{(k)} u$ are defined as in (5.5). Therefore, by Theorem 5.2, we conclude that

$$T_h^{[\frac{t}{h}]} u_h(x) \to u(x,t)$$

locally uniformly in $\mathbb{R}^n \times [0, +\infty)$ as $h \to 0$, where $u(x,t)$ is the unique viscosity solution to (3.2) with initial datum u.

On the other hand, by (6.4), we get that

$$T_h^{[\frac{t}{h}]} u_h(x) \to u(x) - t$$

locally uniformly. This implies that $u(x) - t$ is the unique viscosity solution to (3.2) with initial datum u and, in particular, since the operator is geometric and the level set $\{u(x) > 0\}$ coincide with the level set $\{d_E(x) > 0\}$, we conclude that

$$E^-(t) = \big\{x : u(x) > t\big\} \quad \text{for all } t > 0.$$

Note that by this equality, we also deduce that

$$\bigcap_{s<t} E^-(s) = \big\{x : u(x) \geqslant t\big\},$$

and that the limit u of u_h is unique, so the whole family u_h converges to u uniformly as $h \to 0$. By its characterization, we get also that $E^-(t+s) \subseteq \{x : u(x) \geqslant t+s\} \subseteq E^-(t)$ for all $s > 0$.

Step 3: L^1 convergence and K-outward minimality property of $E^-(t)$. By the uniform convergence of $u_h \to u$, we get that for all $t > 0$, one has

$$E^-(t) = \big\{x : u(x) > t\big\} \subseteq \lim_{h\to 0} E_h(t) \subseteq \big\{x : u(x) \geqslant t\big\} = \bigcap_{s<t} E^-(s),$$

where the limit is taken in the L^1 sense. Since u is Lipschitz continuous, we know that $|\{x : u(x) = t\}| = 0$ for almost every $t > 0$, which implies that $E_h(t) \to E^-(t)$ in $L^1(\mathbb{R}^n)$ for almost every $t > 0$. Moreover, by the stability with respect to L^1 convergence of the K-outward minimizing sets (see Proposition 2.7), since $E_h(t)$ are K-outward minimizers in Ω, by Proposition 6.1, we conclude that $E^-(t)$ and $\bigcap_{s<t} E^-(s)$ are also K-outward minimizer sets in Ω for almost every $t > 0$.

Now, we observe that $E^-(t)$ is a K-outward minimizer set in Ω for every $t > 0$, again, by the stability under L^1 convergence, since $E^-(t) = \bigcup_{s>0} E^-(t+s) = \lim_{s\to 0^+} E^-(t+s)$. Then, $\bigcap_{s<t} E^-(s) = \lim_{s\to t^-} E^-(s)$ is also a K-outward minimizer set in Ω for every $t > 0$.

Step 4: K-curvature of $E^-(t)$. Since E is strongly K-outward minimizer with $\delta > 0$, then, by Proposition 6.2, we get that

$$d\big(E_h(t), E_h(t+s)\big) \geqslant \delta\Big(h\Big[\frac{t+s}{h}\Big] - h\Big[\frac{t}{h}\Big] - h\Big) \geqslant s\delta - 2h\delta.$$

Then,

$$E_h(t+s) + B(0, \delta s - 2h\delta) \subseteq E_h(t).$$

Passing to the limit as $h \to 0$, we get that for almost every $t, s > 0$,

$$d\big(\{u(x) \geqslant t+s\}, \{u(x) > t\}\big) \geqslant \delta s. \tag{6.5}$$

Arguing as before, we get that this inequality holds for all $s, t > 0$.

Now, we apply Theorem 4.3, choosing as initial set $\{u(x) > t\}$ and observing that the outer flow at time $s > 0$ of $\{u(x) > t\}$ is given by $\{u(x) \geqslant t + s\}$. So we get that

$$H^K_{\{u(y)>t\}}(x) \geqslant \delta \quad \text{in the viscosity sense for all } x \in \partial\{u(y) > t\} \text{ and for all } t > 0. \tag{6.6}$$

Step 5: The set $\{x : u(x) = t\}$. We show that for all $t > 0$,

$$\operatorname{int}\big(\{x : u(x) \geqslant t\} \setminus \{x : u(x) > t\}\big) = \operatorname{int}\big(\{x : u(x) = t\}\big) = \emptyset.$$

We assume by contradiction that there exist z and $r > 0$ such that $B(z, r) \Subset \{x : u(x) = t\} \subseteq \{x : u(x) \geqslant t\}$. Let

$$\alpha := \max_{k\in[r/2,r]}\Big(\max_{y\in\partial B(0,k)} H^K_{B(0,k)}(y)\Big).$$

Note that by the definition of curvature, then $\alpha = \max_{y\in\partial B(0,r/2)} H^K_{B(0,r/2)}(y) > 0$. Let $s_0 > 0$ be such that $r - \alpha s_0 > r/2$, and define the flow $B(s) = B(z, r - \alpha s)$ for $s \in [0, s_0]$. Then, we get that $B(s)$ is a strict subsolution to (1.1) since $H^K_{B(s)}(y) \leqslant 2\alpha$ for every $s \in [0, s_0]$. Recalling that $u(x) - t$ is a viscosity solution to (3.2), we conclude by Proposition 3.4, that $B(z, r - \alpha s) = B(s) \subseteq \{x : u(x) \geqslant t + s\}$. This implies that $t = u(z) \geqslant t + s$ for all $s \in [0, s_0]$, which is not possible.

Moreover, observe that by (6.5), the set of $t > 0$ where $|\{x : u(x) = t\}| > 0$ coincides with the set of jumps of the strictly decreasing function $t \mapsto |E^-(t)|$. Therefore, such set is countable.

Step 6: Case of E with $C^{1,1}$ boundary or $K \in L^1(\mathbb{R}^n)$. Note that if E is strongly K-outward minimizing, then by Theorem 2.9, $H^K_E(x) \geqslant \delta$ for all $x \in \partial E$, so that, by Remarks 2.3 and 2.5, for any $\eta \in (0, \delta)$, there exists an open set $\Omega_\eta \subseteq \Omega$, with $E \Subset \Omega_\eta$, such that E it is strongly K-mean convex in Ω_η with associated constant η. Therefore, by Proposition 4.1 (1), we get that $E^+(t) \subseteq E^-(t-s)$ for every $t > s > 0$, and so

$$E^+(t) \subseteq \bigcap_{0<s<t} E^-(t-s) = \bigcap_{0<s<t} \{x : u(x) > t-s\} = \{x : u(x) \geqslant t\} \subseteq E^+(t),$$

which implies that for all $t > 0$,

$$\{x : u(x) \geqslant t\} = E^+(t) \quad \text{and} \quad \{x : u(x) = t\} = E^+(t) \setminus E^-(t).$$

Therefore, by Step 5, we get that $E^+(t) \setminus E^-(t)$ has empty interior for all $t > 0$ and has measure zero for all $t > 0$, except from a countable set. By Step 3, we get that for almost every t, $E_h(t) \to E^+(t)$ in L^1. Then, by the stability with respect to L^1 convergence, $E^+(t)$ is a K-outward minimizing set for almost every t, and then for all t, observing that $E^+(t) = \lim_{s\to 0^+} E^-(t-s)$. Finally, the same argument of Step 4 gives that

$$H^K_{E^+(t)} \geqslant \delta \quad \text{for all } t \geqslant 0.$$

■

Remark 6.4. If the outer flow satisfies $E^+(t) \Subset E$ for $t > 0$, then the same results as in Theorem 6.3 hold also for the outer flow $E^+(t)$, since we may prove that $\{x : u(x) \geqslant t\} = E^+(t)$, arguing exactly as in Step 5 of the proof. In particular, we would get that $E^+(t) \setminus E^-(t)$ has empty interior for all t.

We expect this monotonicity property to hold true for the flow starting from a strongly K-outward minimizer.

Remark 6.5. In Theorem 6.3, we show that the volume function

$$t \mapsto |E^-(t)|$$

is strictly decreasing. We expect that this function is also continuous, as it happens in the local case.

We conclude with a corollary about the convergence of the K-perimeter of the discrete flow to the K-perimeter of the limit level set flow (we refer to [10,29] for analogous results in the local case).

Corollary 6.6. *Let Ω be a domain and $E \Subset \Omega$ be a strongly K-outward minimizing set in Ω with constant $\delta > 0$. Then, for every $T > 0$,*

$$\int_0^T \mathrm{Per}_K(E_h(t))\, dt \to \int_0^T \mathrm{Per}_K(E^-(t))\, dt \quad \textit{as } h \to 0,$$

where $E_h(t)$ is any piecewise constant flow as defined in (5.2) and $E^-(t)$ is the viscosity inner flow as defined in (3.3).

Proof. By Theorem 6.3, $E_h(t) \to E^-(t)$ in $L^1(\Omega)$ for almost every t; therefore, by the lower semicontinuity of Per_K with respect to the L^1 convergence and Fatou's lemma, we get that for every $T > 0$,

$$\liminf_{h\to 0} \int_0^T \mathrm{Per}_K(E_h(t))\, dt \geqslant \int_0^T \mathrm{Per}_K(E^-(t))\, dt. \tag{6.7}$$

We now introduce the functional

$$J_K(v) := \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big|v(x)-v(y)\big| K(x-y)\,dx\,dy \quad \text{for } v \in L^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{6.8}$$

Note that $J_K(\chi_E) = \mathrm{Per}_K(E)$ for all measurable $E \subseteq \mathbb{R}^n$. The coarea formula [6, Proposition 2.3] states that

$$J_K(v) = \int_{-\infty}^{+\infty} \mathrm{Per}_K\big(\{v > s\}\big)\,ds \tag{6.9}$$

for all $v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Let u_h be as defined in (6.3). We claim that

$$J_K(u_h) \leqslant J_K(v) \quad \text{for all } v \in L^1_{\mathrm{loc}}(\mathbb{R}^n),\ v \geqslant u_h \text{ and } \operatorname{supp} v \Subset \Omega. \tag{6.10}$$

The proof of this claim is a direct consequence of the coarea formula and the fact that $E_h(t)$ is K-outward minimizer for every t, by Proposition 6.1. Indeed, since $u_h \leqslant v$, for every $s > 0$,

$$E_h(s) = \big\{x : u_h(x) > s\big\} \subseteq \big\{x : v(x) > s\big\} \Subset \Omega,$$

which implies, since $E_h(t)$ is a K-outward minimizing set, that

$$\mathrm{Per}_K(E_h(s)) = \mathrm{Per}_K\big(\{x : u_h > s\}\big) \leqslant \mathrm{Per}_K\big(\{x : v > s\}\big).$$

Integrating for $s \in (0, +\infty)$, and recalling (6.9), we get the conclusion.

Now, we use the same argument as in [29, Proposition 5.1]. We recall that by Theorem 6.3, $u_h \to u$ uniformly as $h \to 0$, where u is Lipschitz continuous and $u = 0$ in $\mathbb{R}^n \setminus E$. By the uniform convergence, we get that for any $\varepsilon > 0$, there exists h_0 such that $u_h \leqslant u + \varepsilon$ for all $h < h_0$. Let $v(x) := (u(x) + \varepsilon)\chi_E(x)$, so $v(x) \geqslant u_h(x)$ by construction and, moreover, $\operatorname{supp} v = E \Subset \Omega$.

Therefore, by (6.10),

$$\begin{aligned} J_K(u_h) &\leqslant J_K(v) = J_K((u+\varepsilon)\chi_E) \\ &\leqslant J_K(u) + J_K(\varepsilon\chi_E) = J_K(u) + \varepsilon\,\mathrm{Per}_K(E). \end{aligned}$$

Sending $\varepsilon \to 0$, we conclude that

$$J_K(u_h) \leqslant J_K(u). \tag{6.11}$$

Recalling that $E_h(t) = \{x : u_h(x) > t\}$ and $E^-(t) = \{x : u(x) > t\}$, (6.11), by the coarea formula, coincides with

$$\int_0^{+\infty} \mathrm{Per}_K(E_h(t))\,dt \leqslant \int_0^{+\infty} \mathrm{Per}_K(E^-(t))\,dt \quad \text{for all } h \leqslant h_0.$$

This inequality, together with (6.7), gives the claim. ■

Funding. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). M. N. acknowledges partial support by the PRIN 2017 Project *Variational methods for stationary and evolution problems with singularities and interfaces*.

References

[1] F. Almgren, J. E. Taylor, and L. Wang, Curvature-driven flows: a variational approach. *SIAM J. Control Optim.* **31** (1993), no. 2, 387–438 Zbl 0783.35002 MR 1205983

[2] G. Bellettini, V. Caselles, A. Chambolle, and M. Novaga, Crystalline mean curvature flow of convex sets. *Arch. Ration. Mech. Anal.* **179** (2006), no. 1, 109–152 Zbl 1148.53049 MR 2208291

[3] L. Caffarelli, J.-M. Roquejoffre, and O. Savin, Nonlocal minimal surfaces. *Comm. Pure Appl. Math.* **63** (2010), no. 9, 1111–1144 Zbl 1248.53009 MR 2675483

[4] A. Cesaroni, L. de Luca, M. Novaga, and M. Ponsiglione, Stability results for nonlocal geometric evolutions and limit cases for fractional mean curvature flows. *Comm. Partial Differential Equations* **46** (2021), no. 7, 1344–1371 MR 4279969

[5] A. Cesaroni, S. Dipierro, M. Novaga, and E. Valdinoci, Fattening and nonfattening phenomena for planar nonlocal curvature flows. *Math. Ann.* **375** (2019), no. 1-2, 687–736 Zbl 1426.53095 MR 4000255

[6] A. Cesaroni and M. Novaga, The isoperimetric problem for nonlocal perimeters. *Discrete Contin. Dyn. Syst. Ser. S* **11** (2018), no. 3, 425–440 Zbl 1379.53013 MR 3732175

[7] A. Cesaroni and M. Novaga, Symmetric self-shrinkers for the fractional mean curvature flow. *J. Geom. Anal.* **30** (2020), no. 4, 3698–3715 Zbl 1462.53083 MR 4167262

[8] A. Cesaroni and V. Pagliari, Convergence of nonlocal geometric flows to anisotropic mean curvature motion, Discrete Contin. Dyn. Syst. **41** (2021), 4987–5008 Zbl 07373646 MR 4284646

[9] A. Chambolle, M. Morini, and M. Ponsiglione, Nonlocal curvature flows. *Arch. Ration. Mech. Anal.* **218** (2015), no. 3, 1263–1329 Zbl 1328.35166 MR 3401008

[10] A. Chambolle and M. Novaga, Anisotropic and crystalline mean curvature flow of mean-convex sets, to appear in Ann. Sc. Norm. Super. Pisa Cl. Sci., DOI 10.2422/2036-2145.202005_009

[11] A. Chambolle, M. Novaga, and B. Ruffini, Some results on anisotropic fractional mean curvature flows. *Interfaces Free Bound.* **19** (2017), no. 3, 393–415 Zbl 1380.53070 MR 3713894

[12] Y. G. Chen, Y. Giga, and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.* **33** (1991), no. 3, 749–786 Zbl 0696.35087 MR 1100211

[13] E. Cinti, C. Sinestrari, and E. Valdinoci, Neckpinch singularities in fractional mean curvature flows. *Proc. Amer. Math. Soc.* **146** (2018), no. 6, 2637–2646 Zbl 1390.53068 MR 3778164

[14] E. Cinti, C. Sinestrari, and E. Valdinoci, Convex sets evolving by volume-preserving fractional mean curvature flows. *Anal. PDE* **13** (2020), no. 7, 2149–2171 Zbl 1462.53084 MR 4175821

[15] M. Cozzi, On the variation of the fractional mean curvature under the effect of $C^{1,\alpha}$ perturbations. *Discrete Contin. Dyn. Syst.* **35** (2015), no. 12, 5769–5786 Zbl 1335.35274 MR 3393254

[16] L. C. Evans and J. Spruck, Motion of level sets by mean curvature. I. *J. Differential Geom.* **33** (1991), no. 3, 635–681 Zbl 0726.53029 MR 1100206

[17] M. Gage and R. S. Hamilton, The heat equation shrinking convex plane curves. *J. Differential Geom.* **23** (1986), no. 1, 69–96 Zbl 0621.53001 MR 840401

[18] Y. Giga, *Surface evolution equations*. Monogr. Math. 99, Birkhäuser, Basel, 2006 Zbl 1096.53039 MR 2238463

[19] Y. Giga, S. Goto, H. Ishii, and M.-H. Sato, Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains. *Indiana Univ. Math. J.* **40** (1991), no. 2, 443–470 Zbl 0836.35009 MR 1119185

[20] R. Haslhofer and B. Kleiner, Mean curvature flow of mean convex hypersurfaces. *Comm. Pure Appl. Math.* **70** (2017), no. 3, 511–546 Zbl 1360.53069 MR 3602529

[21] O. Hershkovits and B. White, Nonfattening of mean curvature flow at singularities of mean convex type. *Comm. Pure Appl. Math.* **73** (2020), no. 3, 558–580 Zbl 1441.53077 MR 4057901

[22] G. Huisken, Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom.* **31** (1990), no. 1, 285–299 Zbl 0694.53005 MR 1030675

[23] G. Huisken and C. Sinestrari, Mean curvature flow singularities for mean convex surfaces. *Calc. Var. Partial Differential Equations* **8** (1999), no. 1, 1–14 Zbl 0992.53052 MR 1666878

[24] C. Imbert, Level set approach for fractional mean curvature flows. *Interfaces Free Bound.* **11** (2009), no. 1, 153–176 Zbl 1173.35533 MR 2487027

[25] V. Julin and D. A. La Manna, Short time existence of the classical solution to the fractional mean curvature flow. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **37** (2020), no. 4, 983–1016 Zbl 07208003 MR 4104832

[26] S. Luckhaus and T. Sturzenhecker, Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differential Equations* **3** (1995), no. 2, 253–271 Zbl 0821.35003 MR 1386964

[27] J. M. Mazón, J. D. Rossi, and J. J. Toledo, *Nonlocal perimeter, curvature and minimal surfaces for measurable sets*. Front. Math., Birkhäuser/Springer, Cham, 2019 Zbl 1446.49001 MR 3930619

[28] V. Pagliari, Halfspaces minimise nonlocal perimeter: a proof *via* calibrations. *Ann. Mat. Pura Appl. (4)* **199** (2020), no. 4, 1685–1696 Zbl 1452.49025 MR 4117514

[29] G. De Philippis and T. Laux, Implicit time discretization for the mean curvature flow of mean convex sets, Ann. Sc. Norm. Super. Pisa, Cl. Sci. **21**, Spec. Iss., (2020), 911–930 Zbl 07373235 MR 4288621

[30] M. Sáez and E. Valdinoci, On the evolution by fractional mean curvature. *Comm. Anal. Geom.* **27** (2019), no. 1, 211–249 Zbl 1411.35274 MR 3951024

[31] H. M. Soner, Motion of a set by the curvature of its boundary. *J. Differential Equations* **101** (1993), no. 2, 313–372 Zbl 0769.35070 MR 1204331

[32] E. Spadaro, Mean-convex sets and minimal barriers. *Matematiche (Catania)* **75** (2020), no. 1, 353–375 Zbl 1434.53011 MR 4069612

[33] B. White, The size of the singular set in mean curvature flow of mean-convex sets. *J. Amer. Math. Soc.* **13** (2000), no. 3, 665–695 Zbl 0961.53039 MR 1758759

Received 25 November 2020; revised 18 June 2021.

Annalisa Cesaroni
Dipartimento di Scienze Statistiche, Università di Padova, Via Cesare Battisti 141, 35121 Padova, Italy; annalisa.cesaroni@unipd.it

Matteo Novaga
Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy; matteo.novaga@unipi.it