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Finite element error analysis for a system coupling surface evolution to diffusion on the surface

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Abstract. We consider a numerical scheme for the approximation of a system that couples the evolution of a two-dimensional hypersurface to a reaction–diffusion equation on the surface. The surfaces are assumed to be graphs and evolve according to forced mean curvature flow. The method uses continuous, piecewise linear finite elements in space and a backward Euler scheme in time. Assuming the existence of a smooth solution, we prove optimal error bounds both in $L^\infty(L^2)$ and in $L^2(H^1)$. We present several numerical experiments that confirm our theoretical findings and apply the method in order to simulate diffusion induced grain boundary motion.

1. Introduction

In this paper, we analyse a finite element scheme for approximating a system which couples diffusion on a surface to an equation that determines the evolution of the surface. More precisely, we want to find a family of surfaces $(\Gamma(t))_{t\in[0,T]} \subset \mathbb{R}^3$ and a function $w : \bigcup_{t\in[0,T]}(\Gamma(t)\times\{t\}) \to \mathbb{R}$ such that

$$V = H + f(w) \qquad \text{on } \Gamma(t), \quad t \in (0,T], \tag{1.1a}$$

$$\partial^\bullet w = \Delta_\Gamma w + H\,V\,w + g(V,w) \qquad \text{on } \Gamma(t), \quad t \in (0,T]. \tag{1.1b}$$

Here, V and H are the normal velocity and the mean curvature of $\Gamma(t)$ corresponding to the choice ν of a unit normal, while Δ_Γ denotes the Laplace–Beltrami operator on $\Gamma(t)$. Furthermore, $\partial^\bullet w = w_t + V\,\frac{\partial w}{\partial \nu}$ is the material derivative of w and $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are given functions. We are interested in surfaces $\Gamma(t)$ which can be represented as the graph of a function $u : \overline{\Omega} \times [0,T] \to \mathbb{R}$, i.e.,

$$\Gamma(t) = \big\{(x, u(x,t)) \in \mathbb{R}^3 \mid x \in \overline{\Omega}\big\}, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary. Thus, $(\Gamma(t))_{t\in[0,T]}$ is a family of surfaces with boundary, which evolves according to forced mean curvature flow

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in the cylindrical set $A = \overline{\Omega} \times \mathbb{R}$. In what follows, we consider the following boundary conditions:

$$\nu \cdot \nu_{\partial A} = 0 \qquad \text{on } \partial\Gamma(t), \quad t \in (0, T], \tag{1.3a}$$
$$w = 0 \qquad \text{on } \partial\Gamma(t), \quad t \in (0, T]. \tag{1.3b}$$

Here, $\nu_{\partial A}$ is the unit outward normal to ∂A, so that we assume that the evolving surfaces meet the boundary of the cylinder at a right angle. Finally, we impose the initial conditions

$$\Gamma(0) = \Gamma^0, \qquad w(\cdot, 0) = w^0 \text{ on } \Gamma^0, \tag{1.4}$$

where $\Gamma^0 = \{(x, u^0(x)) \,|\, x \in \overline{\Omega}\}$, and $u^0 : \overline{\Omega} \to \mathbb{R}$, $w^0 : \Gamma^0 \to \mathbb{R}$ are given functions. The system (1.1) occurs, for example, in the modelling of diffusion induced grain boundary motion – see [8], [5], and Section 5.3. Further examples of systems that arise by coupling a geometric evolution equation to a PDE on the evolving surface can be found in [7, Section 10].

A semi-discrete finite element scheme for the approximation of (1.1) in the case that $\Gamma(t)$ is a closed curve was first analysed by Pozzi and Stinner in [12]. Using a tangentially modified parametrisation of the evolving curves, [1] obtains error bounds for a corresponding fully discrete scheme. In [13], this idea is applied to the case of open curves $\Gamma(t)$ meeting a given boundary orthogonally. In each of these papers, the error bounds are optimal in H^1. A first error analysis involving the evolution of two-dimensional closed (i.e., compact without boundary) surfaces was obtained in [11] for a regularized version of (1.1a). Extending ideas used in the error analysis for pure mean curvature flow in [9], Kovács, Li, and Lubich obtain in [10] a convergence proof for the system (1.1) in the case of closed surfaces. The scheme uses polynomials of degree at least two and is based on a system coupling the variable w in (1.1b) with the velocity, the normal, and the mean curvature of $\Gamma(t)$. The error estimates are optimal in H^1, while the restriction on the polynomial degree is essentially used to guarantee, via inverse estimates, that the discrete surfaces are non-degenerate.

The purpose of our paper is to derive and analyse a simple, fully discrete finite element scheme for the system (1.1) when the evolving surfaces are of the form (1.2). In order to translate (1.1) into problems which are posed on $\overline{\Omega} \times [0, T]$, we introduce

$$Q(u) = \sqrt{1 + |\nabla u|^2}.$$

Then, the upward pointing unit normal $\nu(u)$, the normal velocity V and the mean curvature H of $\Gamma(t)$ are given by

$$\nu(u) = \frac{1}{Q(u)}(-\nabla u, 1), \quad V = \frac{u_t}{Q(u)}, \quad H = \nabla \cdot \Big(\frac{\nabla u}{Q(u)}\Big), \tag{1.5}$$

respectively. Furthermore, if we denote by n the outward unit normal to $\partial\Omega$, then $\nu_{\partial A} = (n, 0)$ and hence

$$\nu(u) \cdot \nu_{\partial A} = -\frac{\nabla u \cdot n}{Q(u)}.$$

If we let $\widetilde{w} : \overline{\Omega} \times [0,T] \to \mathbb{R}$, $\widetilde{w}(x,t) := w(x,u(x,t),t)$, then we may write (1.1a), (1.3a) as

$$\frac{u_t}{Q(u)} - \nabla \cdot \Big(\frac{\nabla u}{Q(u)} \Big) + f(\widetilde{w}) = 0 \quad \text{in } \Omega \times (0,T]; \tag{1.6}$$

$$\frac{\nabla u \cdot n}{Q(u)} = 0 \quad \text{on } \partial\Omega \times (0,T]. \tag{1.7}$$

Let us next rewrite (1.1b) in terms of $\widetilde{w}$. To do so, we make use of the formulae (2.1) and (2.2) in [7], which yield (temporarily suppressing the dependence on t)

$$(\nabla_\Gamma w)(\Phi(x)) = \sum_{i,j=1}^{2} g^{ij}(x) \widetilde{w}_{x_j}(x) \Phi_{x_i}(x), \tag{1.8}$$

$$(\Delta_\Gamma w)(\Phi(x)) = \frac{1}{\sqrt{q(x)}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} \Big(g^{ij}(x) \sqrt{q(x)} \widetilde{w}_{x_i}(x) \Big). \tag{1.9}$$

In the above, $\Phi(x) = (x, u(x))$ and (g^{ij}) is the inverse matrix of (g_{ij}), where $g_{ij} = \Phi_{x_i} \cdot \Phi_{x_j} = \delta_{ij} + u_{x_i} u_{x_j}$, $i,j = 1,2$. Furthermore, $q = \det(g_{ij}) = 1 + |\nabla u|^2 = Q(u)^2$. A simple calculation shows that

$$(g^{ij}) = I - \frac{\nabla u \otimes \nabla u}{Q(u)^2}.$$

We can expand the velocity vector $(0, u_t)$ for the evolving family of graphs in terms of Φ_{x_1}, Φ_{x_2} and $\nu(u)$ as follows:

$$(0, u_t) = V \nu(u) + \sum_{k=1}^{2} \frac{u_t u_{x_k}}{Q(u)^2} \Phi_{x_k}.$$

Combining this relation with (1.8), we find

$$\begin{aligned} \widetilde{w}_t &= w_t + \nabla w \cdot (0, u_t) = w_t + V \frac{\partial w}{\partial \nu} + \nabla_\Gamma w \cdot \sum_{k=1}^{2} \frac{u_t u_{x_k}}{Q(u)^2} \Phi_{x_k} \\ &= \partial^\bullet w + \sum_{i,j,k=1}^{2} g^{ij} \widetilde{w}_{x_j} \frac{u_t u_{x_k}}{Q(u)^2} \Phi_{x_i} \cdot \Phi_{x_k} = \partial^\bullet w + \frac{u_t}{Q(u)^2} \nabla \widetilde{w} \cdot \nabla u. \end{aligned}$$

Recalling (1.5), we deduce that

$$\begin{aligned} \partial^\bullet w - H\, V\, w &= \widetilde{w}_t - \frac{u_t}{Q(u)} \frac{\nabla u}{Q(u)} \cdot \nabla \widetilde{w} - \frac{u_t}{Q(u)} \nabla \cdot \Big(\frac{\nabla u}{Q(u)} \Big) \widetilde{w} \\ &= \widetilde{w}_t - \frac{u_t}{Q(u)} \nabla \cdot \Big(\widetilde{w} \frac{\nabla u}{Q(u)} \Big). \end{aligned}$$

Hence, (1.1b), (1.3b) take the form

$$\widetilde{w}_t - \frac{1}{Q(u)} \sum_{i,j=1}^{2} \left(g^{ij} Q(u) \widetilde{w}_{x_i}\right)_{x_j} = \frac{u_t}{Q(u)} \nabla \cdot \left(\widetilde{w} \frac{\nabla u}{Q(u)}\right) + g(V, \widetilde{w}) \quad \text{in } \Omega \times (0, T], \tag{1.10}$$

$$\widetilde{w} = 0 \quad \text{on } \partial\Omega \times (0, T]. \tag{1.11}$$

For ease of notation, from now on we will write again w instead of $\widetilde{w}$. Our discretisation will be based on a weak formulation of the system (1.6, 1.10) and uses continuous, piecewise linear finite elements in space and a backward Euler scheme in time (see Section 2). A crucial point in the error analysis is the uniform control of the gradient of the discrete height function. This control is achieved with the help of a superconvergence estimate between the discrete height and a nonlinear projection previously employed in [3] for the numerical analysis of the mean curvature flow of graphs. The properties of this projection and a suitable projection for the function w are collected in Section 3. As our main results, we obtain an $O(\tau + h)$-error bound in H^1 and an $O(\tau + h^2 |\log h|^2)$-estimate in L^2 both for u and w, provided that the time step τ is appropriately related to the mesh size h. To the best of our knowledge, a quasioptimal L^2-bound is new for coupled systems of the form (1.1). The proof of the error bounds is presented in Section 4 and split into two parts: for the analysis of the graph part, we shall refer whenever possible to [3] in order to keep the presentation short. The analysis of the surface PDE requires much more work since the estimates have to be carried out in such a way as not to lose the optimal order. Finally, in Section 5 we present several numerical tests that confirm our error estimates and apply the method in order to simulate diffusion induced grain boundary motion. In some tests we will also consider boundary conditions that are not covered by the theory.

Let us finish the introduction with a few comments on our notation. We shall denote the norm of the Sobolev space $W^{m,p}(\Omega)$ ($m \in \mathbb{N}_0, 1 \le p \le \infty$) by $\|\cdot\|_{m,p}$. For $p = 2$, $W^{m,2}(\Omega)$ will be denoted by $H^m(\Omega)$ with norm $\|\cdot\|_m$, where we simply write $\|\cdot\| = \|\cdot\|_0$.

2. Weak formulation and finite element approximation

In what follows, we make the following assumptions on the data and the solution (u, w):

(A1) We have $f \in C^{0,1}_{\text{loc}}(\mathbb{R})$ and $g : \mathbb{R}^2 \to \mathbb{R}$ has the form

$$g(r, s) = \alpha(r)\, \beta(s) + \tilde{\beta}(s), \tag{2.1}$$

where $\beta, \tilde{\beta} \in C^{0,1}_{\text{loc}}(\mathbb{R})$ and

$$\alpha(r) = \begin{cases} \alpha_1 |r|, & r \ge 0, \\ \alpha_2 |r|, & r < 0 \end{cases}$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$.

(A2) The solution (u, w) solves (1.6), (1.7), (1.10), and (1.11) and satisfies

$$\begin{aligned} u &\in L^\infty((0,T); H^4(\Omega)) \cap L^2((0,T); H^5(\Omega)), \\ u_t &\in L^\infty((0,T); H^2(\Omega)) \cap L^2((0,T); H^3(\Omega)); \end{aligned} \tag{2.2}$$

$$\nabla u_t \in L^\infty(\Omega \times (0,T)),\ u_{tt} \in L^\infty((0,T); H^1(\Omega)); \tag{2.3}$$

$$\begin{aligned} w &\in C^0([0,T]; W^{2,\infty}(\Omega)), \\ w_t &\in C^0([0,T]; W^{1,\infty}(\Omega) \cap H^2(\Omega)), \\ w_{tt} &\in L^\infty((0,T); L^2(\Omega)). \end{aligned} \tag{2.4}$$

Multiplying (1.6) by $\varphi \in H^1(\Omega)$ and integrating by parts yields the weak formulation

$$\int_\Omega \frac{u_t \varphi}{Q(u)}\, dx + \int_\Omega \frac{\nabla u \cdot \nabla \varphi}{Q(u)}\, dx = \int_\Omega f(w)\varphi\, dx \quad \forall \varphi \in H^1(\Omega). \tag{2.5}$$

In order to derive a weak formulation for (1.10), we proceed as in [7, Section 5] and calculate for a test function $\eta \in H_0^1(\Omega)$,

$$\begin{aligned} \frac{d}{dt}\int_\Omega w\,\eta\, Q(u) dx &= \int_\Omega w_t\,\eta\, Q(u) dx + \int_\Omega w\,\eta\,[Q(u)]_t dx \\ &= \sum_{i,j=1}^{2} \int_\Omega \big(g^{ij} w_{x_i}\, Q(u)\big)_{x_j}\,\eta\, dx + \int_\Omega u_t \nabla\cdot\Big(w\,\frac{\nabla u}{Q(u)}\Big)\,\eta\, dx \\ &\quad + \int_\Omega w\,\eta\,\frac{\nabla u\cdot\nabla u_t}{Q(u)} dx + \int_\Omega g(V,w)\eta\, Q(u)\, dx \\ &= -\sum_{i,j=1}^{2} \int_\Omega g^{ij} w_{x_i}\,\eta_{x_j}\, Q(u) dx - \int_\Omega u_t \frac{\nabla u\cdot\nabla\eta}{Q(u)}\, w dx \\ &\quad + \int_\Omega g(V,w)\,\eta\, Q(u)\, dx \\ &= -\int_\Omega E(\nabla u)\nabla w\cdot\nabla\eta dx - \int_\Omega \nabla u\cdot\nabla\eta\, V w dx \\ &\quad + \int_\Omega g(V,w)\,\eta\, Q(u)\, dx, \end{aligned} \tag{2.6}$$

where V is given by (1.5) and

$$E(p) = \sqrt{1+|p|^2}\Big(I - \frac{p\otimes p}{1+|p|^2}\Big), \quad p \in \mathbb{R}^2. \tag{2.7}$$

Note that for all $p, \xi \in \mathbb{R}^2$,

$$\begin{aligned} E(p)\xi\cdot\xi &= \sqrt{1+|p|^2}\Big(|\xi|^2 - \frac{(\xi\cdot p)^2}{1+|p|^2}\Big) \\ &\geq \sqrt{1+|p|^2}\,|\xi|^2\Big(1 - \frac{|p|^2}{1+|p|^2}\Big) \\ &= \frac{|\xi|^2}{\sqrt{1+|p|^2}}. \end{aligned} \tag{2.8}$$

Next, let $(\mathcal{T}_h)_{0<h\le h_0}$ be a family of triangulations of Ω, where we allow boundary elements to have one curved face in order to avoid the analysis of domain approximation. We denote by $h := \max_{S\in\mathcal{T}_h} \operatorname{diam}(S)$ the maximum mesh size and assume that the triangulation is quasiuniform in the sense that there exists $\kappa > 0$ which is independent of h, such that each $S \in \mathcal{T}_h$ is contained in a ball of radius $\kappa^{-1}h$ and contains a ball of radius κh. Our finite element spaces are given by

$$X_h = \{\varphi_h \in C^0(\overline{\Omega}) \,|\, \varphi_h \text{ is a linear polynomial on each } S \in \mathcal{T}_h\}, \quad X_{h0} = X_h \cap H^1_0(\Omega),$$

where we note that in the curved elements φ_h is a composition of a linear polynomial with a suitably defined nonlinear mapping from S to the unit triangle. We refer to [14] for a detailed construction of X_h. The following well-known estimates will be useful:

$$\|\nabla\varphi_h\| \le ch^{-1}\|\varphi_h\| \qquad \forall \varphi_h \in X_h; \tag{2.9}$$
$$\|\nabla\varphi_h\|_{0,\infty} \le ch^{-1}\|\nabla\varphi_h\| \qquad \forall \varphi_h \in X_h; \tag{2.10}$$
$$\|\varphi_h\|_{0,\infty} \le c|\log h|^{\frac{1}{2}}\|\varphi_h\|_1 \quad \forall \varphi_h \in X_h. \tag{2.11}$$

Finally, let $\tau > 0$ be a time step and $t_m = m\tau,\ m = 0,\ldots,M$, where $M = \frac{T}{\tau}$. In what follows, an upper index m will refer to the time level m.

Our discretisation reads: Given $u_h^m \in X_h$, $w_h^m \in X_{h0}$, first find $u_h^{m+1} \in X_h$ such that

$$\frac{1}{\tau}\int_\Omega \frac{(u_h^{m+1}-u_h^m)\,\varphi_h}{Q(u_h^m)}\,dx + \int_\Omega \frac{\nabla u_h^{m+1}\cdot\nabla\varphi_h}{Q(u_h^m)}\,dx = \int_\Omega f(w_h^m)\,\varphi_h\,dx \tag{2.12}$$

for all $\varphi_h \in X_h$. Afterwards, find $w_h^{m+1} \in X_{h0}$ such that

$$\begin{aligned}
&\frac{1}{\tau}\Big(\int_\Omega w_h^{m+1}\,\eta_h\,Q(u_h^{m+1})dx - \int_\Omega w_h^m\,\eta_h Q(u_h^m)dx\Big) \\
&\qquad + \int_\Omega E(\nabla u_h^{m+1})\nabla w_h^{m+1}\cdot\nabla\eta_h dx \\
&= -\int_\Omega \nabla u_h^{m+1}\cdot\nabla\eta_h\,V_h^{m+1}\,w_h^m dx + \int_\Omega g(V_h^{m+1}, w_h^m)\,\eta_h\,Q(u_h^{m+1})\,dx
\end{aligned} \tag{2.13}$$

for all $\eta_h \in X_{h0}$. Here,

$$V_h^{m+1} = \frac{1}{\tau}\frac{u_h^{m+1}-u_h^m}{Q(u_h^{m+1})}.$$

We note that each time step requires the consecutive solution of two linear systems. In view of (2.8), it is easily seen that $u_h^{m+1} \in X_h$ and $w_h^{m+1} \in X_{h0}$ exist and are uniquely determined. The algorithm is initialised by $u_h^0 = \widehat{u}_h^0$, $w_h^0 = \widehat{w}_h^0$, given by (3.1) and (3.6) defined in the next section. Our main result reads as follows:

Theorem 2.1. *There exist $h_0 > 0$ and $\delta_0 > 0$ such that for all $0 < h \le h_0$ and all $\tau > 0$ satisfying $\tau \le \delta_0 h |\log h|^{-\frac{1}{2}}$, the following error bounds hold:*

$$\max_{0\le m\le M} \left[\|u^m - u_h^m\|^2 + \|w^m - w_h^m\|^2 \right] + \sum_{m=0}^{M-1} \tau \Big\| u_t^m - \frac{u_h^{m+1} - u_h^m}{\tau} \Big\|^2$$
$$\le c\big(\tau^2 + h^4 |\log h|^4\big),$$
$$\max_{0\le m\le M} \|\nabla(u^m - u_h^m)\|^2 + \sum_{m=0}^{M} \tau \|\nabla(w^m - w_h^m)\|^2 \le c\big(\tau^2 + h^2\big).$$

3. Projections

Our error analysis relies on the use of suitable Ritz projections of the solutions u and w. Omitting the time dependence for a moment, we define for a given function $u \in H^1(\Omega)$ the minimal surface type projection $\widehat{u}_h \in X_h$ by

$$\int_\Omega \frac{\nabla \widehat{u}_h \cdot \nabla \varphi_h}{Q(\widehat{u}_h)}\,dx + \int_\Omega \widehat{u}_h\,\varphi_h\,dx = \int_\Omega \frac{\nabla u \cdot \nabla \varphi_h}{Q(u)}\,dx + \int_\Omega u\,\varphi_h\,dx \quad \forall \varphi_h \in X_h. \tag{3.1}$$

Note that we have added the zero order term in order to ensure the $H^1(\Omega)$-coercivity of the problem. For functions that also depend on t we have the following error bounds:

Lemma 3.1. *Assume that u satisfies (2.2) and (2.3). Then,*

$$\sup_{0\le t\le T} \|(u - \widehat{u}_h)(t)\| + h \sup_{0\le t\le T} \|\nabla(u - \widehat{u}_h)(t)\| \le ch^2, \tag{3.2}$$
$$\sup_{0\le t\le T} \|(u - \widehat{u}_h)(t)\|_{0,\infty} + h \sup_{0\le t\le T} \|\nabla(u - \widehat{u}_h)(t)\|_{0,\infty} \le ch^2 |\log h|, \tag{3.3}$$
$$\sup_{0\le t\le T} \|(u_t - \widehat{u}_{h,t})(t)\| \le ch^2 |\log h|^2, \tag{3.4}$$
$$\sup_{0\le t\le T} \|\nabla(u_t - \widehat{u}_{h,t})(t)\| \le ch. \tag{3.5}$$

Proof. The proofs of (3.2) and (3.3) follow from [6, p. 160] using that $u(\cdot,t) \in H^4(\Omega) \subset W^{2,\infty}(\Omega)$ for every $t \in [0,T]$. The arguments required to show (3.4) and (3.5) can be found in [2, Section 4] for the case of homogeneous Dirichlet boundary conditions. In order to prove (3.5) for (3.1), one proceeds in the same way as in [2, p. 202] to obtain

$$\|\nabla(u_t - \widehat{u}_{h,t})\|^2 \le ch \|\nabla(u_t - \widehat{u}_{h,t})\| \big(\|\nabla u_t\|_{0,\infty} + \|u_t\|_2 \big) + ch^2 \|\nabla u_t\|_{0,\infty} \|u_t\|_2,$$

which yields (3.5) when taking (2.2) and (2.3) into account. The bound (3.4) can be shown for the Neumann case by modifying the dual problem in [2, p. 203] as follows:

$$-\nabla \cdot \big(F'(\nabla u)\nabla v\big) + v = u_t - \widehat{u}_{h,t} \quad \text{in } \Omega, \qquad F'(\nabla u)\nabla v \cdot n = 0 \quad \text{on } \partial\Omega,$$

where $F(p) = p/\sqrt{1+|p|^2}$, $p \in \mathbb{R}^2$. ∎

Let us next use $\widehat{u}_h$ in order to define a projection $\widehat{w}_h \in X_{h0}$ of w as follows:

$$\int_\Omega E(\nabla \widehat{u}_h)\nabla \widehat{w}_h \cdot \nabla \eta_h \, dx = \int_\Omega E(\nabla u)\nabla w \cdot \nabla \eta_h \, dx \quad \forall \eta_h \in X_{h0}. \tag{3.6}$$

Lemma 3.2. *Assume that w satisfies (2.4). Then,*

$$\sup_{0\le t\le T} \|\nabla(w - \widehat{w}_h)(t)\| \le ch, \tag{3.7}$$

$$\sup_{0\le t\le T} \|(w - \widehat{w}_h)(t)\| \le ch^2|\log h|, \tag{3.8}$$

$$\sup_{0\le t\le T} \|\nabla(w_t - \widehat{w}_{h,t})(t)\| \le ch, \tag{3.9}$$

$$\sup_{0\le t\le T} \|(w_t - \widehat{w}_{h,t})(t)\| \le ch^2|\log h|^2. \tag{3.10}$$

Proof. Using (3.2)–(3.5), these bounds have been obtained in [4, Appendix] for a slightly more complicated projection, see (2.22) in that paper. The same arguments can be applied to our case where we note that the matrix valued function $E(p)$ used in [4] differs from (2.7) by a factor of $1 + |p|^2$. However, since ∇u and $\nabla \widehat{u}_h$ vary in a bounded set that is independent of h, the analysis in [4] also applies to (3.6). ∎

We set

$$\rho_u = u - \widehat{u}_h, \tag{3.11}$$

and for later use, we record the following estimates which will be helpful in retaining the optimality of the error bounds:

Lemma 3.3. *Suppose that $F : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable and that $u \in W^{2,\infty}(\Omega)$. Then, we have for $f \in W_0^{1,1}(\Omega)$*

$$\Big|\int_\Omega \big(F(\nabla u) - F(\nabla \widehat{u}_h)\big)\, f \, dx\Big| \le ch^2|\log h|\, \|f\|_{1,1}.$$

Proof. Noting (3.11), we have

$$\int_\Omega \big(F(\nabla u) - F(\nabla \widehat{u}_h)\big)\, f\, dx = \int_\Omega F'(\nabla u)\cdot \nabla \rho_u\, f\, dx + R,$$

where

$$\begin{aligned} |R| &= \Big|\int_\Omega \int_0^1 \big(F'(\nabla u - s\nabla \rho_u) - F'(\nabla u)\big) ds \cdot \nabla \rho_u\, f\, dx\Big| \\ &\le c\|\nabla \rho_u\|_{0,\infty}\|\nabla \rho_u\|\|f\| \\ &\le ch^2|\log h|\|f\|_{1,1}, \end{aligned}$$

in view of (3.2), (3.3), and the embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$. Integration by parts together with (3.3) yields

$$\Big|\int_\Omega F'(\nabla u)\cdot\nabla\rho_u\, f\,dx\Big| = \Big|-\int_\Omega \nabla\cdot\big(F'(\nabla u)\, f\big)\rho_u\,dx\Big| \\ \le c\|\rho_u\|_{0,\infty}\|f\|_{1,1} \\ \le ch^2|\log h|\|f\|_{1,1},$$

and the result follows. ∎

Lemma 3.4. *Suppose that $f\in H^1_0(\Omega)\cap C^0(\overline{\Omega})$ with $f\in H^2(T)$ for all $T\in\mathcal{T}_h$. Then,*

$$\Big|\int_\Omega f\Big(\frac{\nabla u}{Q(u)}-\frac{\nabla\widehat{u}_h}{Q(\widehat{u}_h)}\Big)\cdot\nabla\varphi_h\,dx\Big| \le ch|\log h|\|\varphi_h\|\Big(\sum_{T\in\mathcal{T}_h}\|f\|^2_{H^2(T)}\Big)^{\frac12}\quad \forall\varphi_h\in X_h.$$

If in addition, $f\in H^2(\Omega)$, then

$$\Big|\int_\Omega f\Big(\frac{\nabla u}{Q(u)}-\frac{\nabla\widehat{u}_h}{Q(\widehat{u}_h)}\Big)\cdot\nabla\varphi_h\,dx\Big| \le ch^2|\log h|\|\varphi_h\|_1\|f\|_2\quad \forall\varphi_h\in X_h.$$

Proof. In view of definition (3.1) of $\widehat{u}_h$, we obtain

$$\begin{aligned}\int_\Omega f\Big(\frac{\nabla u}{Q(u)}-\frac{\nabla\widehat{u}_h}{Q(\widehat{u}_h)}\Big)\cdot\nabla\varphi_h\,dx &= \int_\Omega\Big(\frac{\nabla u}{Q(u)}-\frac{\nabla\widehat{u}_h}{Q(\widehat{u}_h)}\Big)\cdot\nabla(f\,\varphi_h)\,dx \\ &\quad -\int_\Omega\varphi_h\Big(\frac{\nabla u}{Q(u)}-\frac{\nabla\widehat{u}_h}{Q(\widehat{u}_h)}\Big)\cdot\nabla f\,dx \\ &= \int_\Omega\Big(\frac{\nabla u}{Q(u)}-\frac{\nabla\widehat{u}_h}{Q(\widehat{u}_h)}\Big)\cdot\nabla\big(f\varphi_h-I_h(f\varphi_h)\big)\,dx - \int_\Omega\rho_u I_h(f\varphi_h)\,dx \\ &\quad -\int_\Omega\varphi_h\Big(\frac{\nabla u}{Q(u)}-\frac{\nabla\widehat{u}_h}{Q(\widehat{u}_h)}\Big)\cdot\nabla f\,dx =: I+II+III.\end{aligned}$$

Here, I_h denotes the Lagrange interpolation operator. An interpolation estimate implies

$$\begin{aligned}|I| &\le \|\nabla\rho_u\|_{0,\infty}\|\nabla\big(f\varphi_h-I_h(f\varphi_h)\big)\|_{0,1} \\ &\le ch^2|\log h|\sum_{T\in\mathcal{T}_h}\|D^2(f\varphi_h)\|_{L^1(T)} \\ &\le ch^2|\log h|\|\varphi_h\|_1\Big(\sum_{T\in\mathcal{T}_h}\|f\|^2_{H^2(T)}\Big)^{\frac12}.\end{aligned}$$

Next,

$$\begin{aligned}|II| &\le \|\rho_u\|_{0,\infty}\|I_h(f\varphi_h)\|_{0,1} \\ &\le ch^2|\log h|\big(\|f\varphi_h\|_{0,1}+\|f\varphi_h-I_h(f\varphi_h)\|_{0,1}\big) \\ &\le ch^2|\log h|\|\varphi_h\|_1\Big(\sum_{T\in\mathcal{T}_h}\|f\|^2_{H^2(T)}\Big)^{\frac12}.\end{aligned}$$

Finally,

$$|III| \le \|\nabla \rho_u\|_{0,\infty} \|\varphi_h\| \, \|f\|_1 \le ch|\log h| \|\varphi_h\| \, \|f\|_1,$$

while applying Lemma 3.3 with $F_i(p) = \frac{p_i}{\sqrt{1+|p|^2}}$, yields

$$\begin{aligned} |III| &\le \sum_{i=1}^{2} \Big| \int_\Omega \big(F_i(\nabla u) - F_i(\nabla \widehat{u})\big) \varphi_h f_{x_i} \, dx \Big| \\ &\le ch^2 |\log h| \|\varphi_h \nabla f\|_{1,1} \le ch^2 |\log h| \|\varphi_h\|_1 \|f\|_2, \end{aligned}$$

in the case that $f \in H^2(\Omega)$. The result now follows from the above bounds together with (2.9). ■

4. Error analysis

Let us begin with two useful estimates involving the quantities Q and ν.

Lemma 4.1. *Let $u, v \in W^{1,\infty}(\Omega)$. Then, we have almost everywhere in Ω*

$$|\nabla(v-u)| \le \big(1 + \sup_\Omega |\nabla v|\big) Q(u) |\nu(v) - \nu(u)|, \tag{4.1}$$

$$Q(v) - Q(u) = \frac{\nabla u}{Q(u)} \cdot \nabla(v-u) + \frac{|\nabla(v-u)|^2}{2Q(u)} - \frac{(Q(v)-Q(u))^2}{2Q(u)}. \tag{4.2}$$

Proof. Estimate (4.1) is a consequence of the relation

$$\nabla v - \nabla u = Q(u)\Big(\frac{\nabla v}{Q(v)} - \frac{\nabla u}{Q(u)}\Big) + Q(u)\Big(\frac{1}{Q(u)} - \frac{1}{Q(v)}\Big)\nabla v$$

and the fact that $\nu(u) = \big(\frac{-\nabla u}{Q(u)}, \frac{1}{Q(u)}\big)$, while (4.2) follows from a straightforward calculation. ■

Let us decompose the errors $e_u^m = u^m - u_h^m$, $e_w^m = w^m - w_h^m$ as follows:

$$e_u^m = (u^m - \widehat{u}_h^m) + (\widehat{u}_h^m - u_h^m) =: \rho_u^m + e_{h,u}^m, \tag{4.3}$$

$$e_w^m = (w^m - \widehat{w}_h^m) + (\widehat{w}_h^m - w_h^m) =: \rho_w^m + e_{h,w}^m \tag{4.4}$$

and note that $e_{h,u}^m \in X_h$, $e_{h,w}^m \in X_{h0}$. It will be convenient to introduce the quantities

$$\mathcal{A}^m := \int_\Omega |\nu(u_h^m) - \nu(\widehat{u}_h^m)|^2 Q(u_h^m) \, dx, \tag{4.5}$$

$$\mathcal{B}^m := \frac{1}{2}\mathcal{A}^m - \int_\Omega d^m \cdot \nabla e_{h,u}^m \, \rho_u^m \, dx, \tag{4.6}$$

where

$$d^m = -\frac{u_t^m \nabla u^m}{\sqrt{1+|\nabla u^m|}^3}. \tag{4.7}$$

We shall use an induction argument and claim that

$$\mathcal{B}^m + \frac{\theta^2}{2}\int_\Omega (e_{h,w}^m)^2 Q(u_h^m)\,dx \le \big(\tau^2 + h^4|\log h|^4\big)\,e^{\mu t_m}, \quad m = 0, 1, \ldots, M, \tag{4.8}$$

provided that $\tau \le \delta_0 h|\log h|^{-\frac{1}{2}}$. The constants $\delta_0,\ 0 < \theta \le 1$, and $\mu > 0$ are independent of h and τ, and will be chosen a posteriori. To begin, choose $h_0 > 0$ small enough so that

$$h^2|\log h|^5 e^{\mu T} \le \frac{1}{2} \quad \text{and} \quad |\log h| \ge \frac{1}{\theta^2} \qquad \text{for all } 0 < h \le h_0. \tag{4.9}$$

Clearly, (4.8) holds for $m = 0$ since $e_{h,u}^0 = e_{h,w}^0 = 0$ by the choice of our initial data for the scheme. Let us assume that it is true for some $m \in \{0, \ldots, M-1\}$. Then, we have for $0 < h \le h_0$ that

$$\begin{aligned}\mathcal{B}^m + \frac{\theta^2}{2}\int_\Omega (e_{h,w}^m)^2 Q(u_h^m)\,dx &\le \big(\delta_0^2 h^2|\log h|^{-1} + h^4|\log h|^4\big)e^{\mu T}\\ &\le h^2|\log h|^{-1},\end{aligned} \tag{4.10}$$

provided that δ_0 and μ satisfy

$$\delta_0^2 e^{\mu T} \le \frac{1}{2}. \tag{4.11}$$

In what follows, we shall denote by c a generic constant that is independent of δ_0, θ, and μ. We infer from an inverse estimate, (4.10), the fact that $Q(u_h^m) \ge 1$ and from (4.9) that

$$\|w_h^m\|_{0,\infty} \le \|\widehat{w}_h^m\|_{0,\infty} + \|e_{h,w}^m\|_{0,\infty} \le c + ch^{-1}\|e_{h,w}^m\| \le c + \frac{c}{\theta}|\log h|^{-\frac{1}{2}} \le c. \tag{4.12}$$

Next, we deduce with the help of $\|\nabla\widehat{u}_h^m\|_{0,\infty} \le c$ and (2.10) that

$$\begin{aligned}\sup_{\overline{\Omega}} Q(u_h^m) &\le 1 + \sup_{\overline{\Omega}}|\nabla u_h^m|\\ &\le 1 + \|\nabla\widehat{u}_h^m\|_{0,\infty} + \|\nabla e_{h,u}^m\|_{0,\infty} \le c + ch^{-1}\|\nabla e_{h,u}^m\|.\end{aligned} \tag{4.13}$$

It follows from (4.1) that

$$\begin{aligned}|\nabla e_{h,u}^m| &= |\nabla(u_h^m - \widehat{u}_h^m)|\\ &\le (1 + \sup_{\overline{\Omega}}|\nabla\widehat{u}_h^m|)Q(u_h^m)|\nu(u_h^m) - \nu(\widehat{u}_h^m)| \le c|\nu(u_h^m) - \nu(\widehat{u}_h^m)|Q(u_h^m).\end{aligned}$$

Thus,

$$\begin{aligned}\|\nabla e_{h,u}^m\|^2 &\le c\int_\Omega |\nu(u_h^m) - \nu(\widehat{u}_h^m)|^2 Q(u_h^m)^2\,dx \le c\sup_{\overline{\Omega}} Q(u_h^m)\mathcal{A}^m\\ &\le c\sup_{\overline{\Omega}} Q(u_h^m)\big(\mathcal{B}^m + \|\rho_u^m\|\|\nabla e_{h,u}^m\|\big) \le c\sup_{\overline{\Omega}} Q(u_h^m)\big(\mathcal{B}^m + h^2\|\nabla e_{h,u}^m\|\big)\end{aligned}$$

and hence,

$$\|\nabla e_{h,u}^m\|^2 \le c \sup_{\overline{\Omega}} Q(u_h^m)\mathcal{B}^m + ch^4 (\sup_{\overline{\Omega}} Q(u_h^m))^2. \tag{4.14}$$

If we insert this bound into (4.13) and recall (4.10), we obtain

$$\sup_{\overline{\Omega}} Q(u_h^m) \le c + c\big(|\log h|^{-1} \sup_{\overline{\Omega}} Q(u_h^m)\big)^{\frac{1}{2}} + ch \sup_{\overline{\Omega}} Q(u_h^m)$$

and therefore,

$$\sup_{\overline{\Omega}} Q(u_h^m) \le c, \tag{4.15}$$

provided that $0 < h \le h_1$ for some sufficiently small $0 < h_1 \le h_0$. Furthermore, we infer from (4.10), (4.14), and (4.15) that

$$\|\nabla e_{h,u}^m\|^2 \le c\mathcal{B}^m + ch^4 \le ch^2 |\log h|^{-1}, \tag{4.16}$$

$$\frac{1}{2}\mathcal{A}^m \le \mathcal{B}^m + c\|\nabla e_{h,u}^m\| \|\rho_u^m\| \le ch^2 |\log h|^{-1}. \tag{4.17}$$

4.1. The graph equation

Evaluating (2.5) at $t = t_m$ and using the definition (3.1) of $\widehat{u}_h$, we derive for $\varphi_h \in X_h$

$$\int_\Omega \frac{u_t^m \varphi_h}{Q(u^m)} \, dx + \int_\Omega \frac{\nabla \widehat{u}_h^m \cdot \nabla \varphi_h}{Q(\widehat{u}_h^m)} \, dx = \int_\Omega f(w^m) \varphi_h \, dx + \int_\Omega \rho_u^m \varphi_h \, dx$$

and hence,

$$\begin{aligned} &\frac{1}{\tau} \int_\Omega \frac{(u^{m+1} - u^m)\varphi_h}{Q(u^m)} \, dx + \int_\Omega \frac{\nabla \widehat{u}_h^{m+1} \cdot \nabla \varphi_h}{Q(\widehat{u}_h^m)} \, dx \\ &\quad = \int_\Omega f(w^m) \varphi_h \, dx + \int_\Omega \frac{\nabla (\widehat{u}_h^{m+1} - \widehat{u}_h^m) \cdot \nabla \varphi_h}{Q(\widehat{u}_h^m)} \, dx + \int_\Omega R^m \varphi_h \, dx. \end{aligned} \tag{4.18}$$

Here, $R^m = \frac{1}{Q(u^m)}(\frac{u^{m+1}-u^m}{\tau} - u_t^m) + \rho_u^m$, so that in view of (3.2),

$$\|R^m\| \le \int_{t_m}^{t_{m+1}} \|u_{tt}\| \, dt + \|\rho_u^m\| \le c(\tau + h^2). \tag{4.19}$$

Combining (4.18) with (2.12) we obtain the error relation

$$\begin{aligned} &\frac{1}{\tau} \int_\Omega \frac{(e_u^{m+1} - e_u^m)\varphi_h}{Q(u_h^m)} \, dx + \int_\Omega \Big(\frac{\nabla \widehat{u}_h^{m+1}}{Q(\widehat{u}_h^m)} - \frac{\nabla u_h^{m+1}}{Q(u_h^m)} \Big) \cdot \nabla \varphi_h \, dx \\ &\quad = \int_\Omega (f(w^m) - f(w_h^m))\varphi_h \, dx + \frac{1}{\tau} \int_\Omega (u^{m+1} - u^m) \Big(\frac{1}{Q(u_h^m)} - \frac{1}{Q(u^m)} \Big) \varphi_h \, dx \\ &\qquad + \int_\Omega R^m \varphi_h \, dx + \int_\Omega \frac{\nabla (\widehat{u}_h^{m+1} - \widehat{u}_h^m) \cdot \nabla \varphi_h}{Q(\widehat{u}_h^m)} \, dx. \end{aligned} \tag{4.20}$$

If we insert $\varphi_h = \frac{1}{\tau}(e_{h,u}^{m+1} - e_{h,u}^m)$ into (4.20), we derive

$$\begin{aligned}
&\frac{1}{\tau^2}\int_\Omega \frac{(e_u^{m+1}-e_u^m)^2}{Q(u_h^m)}\,dx + \frac{1}{\tau}\int_\Omega \Big(\frac{\nabla\widehat{u}_h^{m+1}}{Q(\widehat{u}_h^m)} - \frac{\nabla u_h^{m+1}}{Q(u_h^m)}\Big)\cdot\nabla(e_{h,u}^{m+1}-e_{h,u}^m)\,dx \\
&\quad = \frac{1}{\tau^2}\int_\Omega \frac{(e_u^{m+1}-e_u^m)(\rho_u^{m+1}-\rho_u^m)}{Q(u_h^m)}\,dx \\
&\qquad + \frac{1}{\tau}\int_\Omega \frac{\nabla(\widehat{u}_h^{m+1}-\widehat{u}_h^m)\cdot\nabla(e_{h,u}^{m+1}-e_{h,u}^m)}{Q(\widehat{u}_h^m)}\,dx \\
&\qquad + \frac{1}{\tau^2}\int_\Omega (u^{m+1}-u^m)\,(e_{h,u}^{m+1}-e_{h,u}^m)\Big(\frac{1}{Q(u_h^m)} - \frac{1}{Q(u^m)}\Big)dx \\
&\qquad + \frac{1}{\tau}\int_\Omega R^m(e_{h,u}^{m+1}-e_{h,u}^m)\,dx + \frac{1}{\tau}\int_\Omega \big(f(w^m)-f(w_h^m)\big)(e_{h,u}^{m+1}-e_{h,u}^m)\,dx \\
&\quad =: \sum_{i=1}^{5} A_i\,.
\end{aligned} \tag{4.21}$$

In order to proceed, we make use of the analysis in [3] for the mean curvature flow of graphs subject to Dirichlet boundary conditions. The relation (4.21) corresponds to [3, (3.12)] where we use $e_u^m, e_{h,u}^m, \rho_u^m$ instead of $e^m, e_h^m, \varepsilon^m$, respectively. Furthermore, our remainder term R^m is defined in a different way and the term A_5 is not present in [3]. We shall refer to the calculations in [3] whenever possible and focus on the changes due to the differences mentioned above and the use of a Neumann boundary condition. To begin, it follows from [3, Lemma 2] that

$$\begin{aligned}
&\frac{1}{\tau}\int_\Omega \Big(\frac{\nabla\widehat{u}_h^{m+1}}{Q(\widehat{u}_h^m)} - \frac{\nabla u_h^{m+1}}{Q(u_h^m)}\Big)\cdot\nabla(e_{h,u}^{m+1}-e_{h,u}^m)\,dx \geq \frac{1}{2\tau}(\mathcal{A}^{m+1}-\mathcal{A}^m) \\
&\qquad + \frac{1}{4\tau}\int_\Omega \frac{|\nabla(e_{h,u}^{m+1}-e_{h,u}^m)|^2}{Q(\widehat{u}_h^m)}\,dx - c(\mathcal{A}^m+\mathcal{A}^{m+1}) - c\tau^2.
\end{aligned} \tag{4.22}$$

The lemma holds under the conditions that $h^{-2}\mathcal{A}^m \leq \gamma$ and $\gamma > 0$ is sufficiently small, which can be achieved in view of (4.17) if $0 < h \leq h_2$ and $h_2 \leq h_1$ is small enough.

Let us consider the terms on the right hand side of (4.21). The term S_1 is estimated in (i) at the bottom of [3, p. 352], so that

$$|A_1| \leq \frac{\delta}{\tau^2}\int_\Omega \frac{(e_u^{m+1}-e_u^m)^2}{Q(u_h^m)}\,dx + \frac{c}{\delta}h^4|\log h|^4. \tag{4.23}$$

The integral A_2 is treated in (ii) of [3, p. 353] and uses integration by parts for the term

$$\int_\Omega \frac{\nabla(u^{m+1}-u^m)\cdot\nabla(e_{h,u}^{m+1}-e_{h,u}^m)}{Q(u^m)}\,dx.$$

Since $\nabla(u^{m+1}-u^m)\cdot n = 0$ on $\partial\Omega$ in view of (1.7), the boundary integral vanishes and we obtain in the same way as in [3] that

$$|A_2| \le \frac{\delta}{\tau}\int_\Omega \frac{|\nabla(e_{h,u}^{m+1}-e_{h,u}^m)|^2}{Q(\widehat{u}_h^m)}\,dx + \frac{\delta}{\tau^2}\int_\Omega \frac{(e_u^{m+1}-e_u^m)^2}{Q(u_h^m)}\,dx + \frac{c}{\delta}\big(\tau^2 + h^4|\log h|^4\big). \tag{4.24}$$

The term A_3 is handled in (iii) in [3, pp. 353–356]. It again involves integration by parts, namely for the term

$$-\frac{1}{\tau^2}\int_\Omega (u^{m+1}-u^m)(e_{h,u}^{m+1}-e_{h,u}^m)b^m\cdot\nabla\rho_u^m\,dx,$$

which is II at the top of [3, p. 354]. Here, $b^m = B(\nabla u^m)$ with

$$B_i(p) = \frac{\partial}{\partial p_i}\Big(\frac{1}{\sqrt{1+|p|^2}}\Big) = -\frac{p_i}{\sqrt{1+|p|^2}^3}.$$

As a result, the boundary integral reads

$$-\frac{1}{\tau^2}\int_{\partial\Omega}(u^{m+1}-u^m)(e_{h,u}^{m+1}-e_{h,u}^m)\,b^m\cdot n\,\rho_u^m\,do = 0,$$

since

$$b^m\cdot n = -\frac{\nabla u^m\cdot n}{\sqrt{1+|\nabla u^m|^2}^3} = 0$$

on $\partial\Omega$ again by (1.7). Thus, we obtain from the top of [3, p. 356] that

$$\begin{aligned}|A_3| \le{}& \frac{1}{\tau}\int_\Omega d^{m+1}\cdot\nabla e_{h,u}^{m+1}\,\rho_u^{m+1}\,dx - \frac{1}{\tau}\int_\Omega d^m\cdot\nabla e_{h,u}^m\,\rho_u^m\,dx\\ &+ \frac{6\delta}{\tau^2}\int_\Omega\frac{(e_u^{m+1}-e_u^m)^2}{Q(u_h^m)}\,dx + \frac{2\delta}{\tau}\int_\Omega\frac{|\nabla(e_{h,u}^{m+1}-e_{h,u}^m)|^2}{Q(\widehat{u}_h^m)}\,dx\\ &+ \frac{c}{\delta}h^4|\log h|^4 + \frac{c}{\delta}\big(\mathcal{A}^m + \mathcal{A}^{m+1}\big)\end{aligned} \tag{4.25}$$

with d^m as in (4.7) (see top of [3, p. 355]). Next, (4.19) implies that

$$\begin{aligned}|A_4| &\le \frac{1}{\tau}\|R^m\|\|e_{h,u}^{m+1}-e_{h,u}^m\| \le \frac{c}{\tau}\big(\tau+h^2\big)\big(\|e_u^{m+1}-e_u^m\| + \|\rho_u^{m+1}-\rho_u^m\|\big)\\ &\le \frac{\delta}{\tau^2}\int_\Omega\frac{(e_u^{m+1}-e_u^m)^2}{Q(u_h^m)}\,dx + \frac{c}{\delta}\big(\tau^2+h^4|\log h|^4\big),\end{aligned} \tag{4.26}$$

since

$$\|\rho_u^{m+1}-\rho_u^m\| \le c\tau\sup_{t_m\le t\le t_{m+1}}\|u_t(\cdot,t)-\widehat{u}_{h,t}(\cdot,t)\| \le c\tau h^2|\log h|^2$$

by (3.4). Recalling (4.12) and the assumption that $f \in C^{0,1}_{\mathrm{loc}}(\mathbb{R})$, we obtain in a similar way

$$
\begin{aligned}
|A_5| &\le \frac{c}{\tau}\int_\Omega |e_w^m|\,|e_{h,u}^{m+1} - e_{h,u}^m|\,dx \\
&\le \frac{\delta}{\tau^2}\int_\Omega \frac{(e_u^{m+1}-e_u^m)^2}{Q(u_h^m)}\,dx + \frac{c}{\delta}\big(\|e_w^m\|^2 + h^4|\log h|^4\big).
\end{aligned} \tag{4.27}
$$

If we insert (4.22)–(4.27) into (4.21), we obtain after multiplying by τ and choosing $\delta > 0$ sufficiently small

$$
\begin{aligned}
&\frac{1}{2\tau}\int_\Omega \frac{(e_u^{m+1}-e_u^m)^2}{Q(u_h^m)}\,dx + \frac{1}{8}\int_\Omega \frac{|\nabla(e_{h,u}^{m+1}-e_{h,u}^m)|^2}{Q(\widehat{u}_h^m)}\,dx + (\frac{1}{2} - c\tau)\mathcal{A}^{m+1} \\
&\qquad - \int_\Omega d^{m+1}\cdot\nabla e_{h,u}^{m+1}\,\rho_u^{m+1}\,dx \\
&\le (\frac{1}{2} + c\tau)\mathcal{A}^m - \int_\Omega d^m\cdot\nabla e_{h,u}^m\,\rho_u^m\,dx + c\tau(\tau^2 + h^4|\log h|^4) + c\tau\|e_w^m\|^2.
\end{aligned}
$$

Recalling the definition of $\mathcal{B}^m$ (see (4.6)), and noting (4.17) and (3.2), we deduce that

$$
\begin{aligned}
&\frac{1}{2\tau}\int_\Omega \frac{(e_u^{m+1}-e_u^m)^2}{Q(u_h^m)}\,dx + \frac{1}{8}\int_\Omega \frac{|\nabla(e_{h,u}^{m+1}-e_{h,u}^m)|^2}{Q(\widehat{u}_h^m)}\,dx + (1 - c\tau)\mathcal{B}^{m+1} \\
&\quad \le (1+c\tau)\mathcal{B}^m + c\tau h^2\big(\|\nabla e_{h,u}^{m+1}\| + \|\nabla e_{h,u}^m\|\big) \\
&\qquad + c\tau(\tau^2 + h^4|\log h|^4) + c\tau\|e_w^m\|^2.
\end{aligned} \tag{4.28}
$$

The second term on the right hand side of (4.28) is estimated by

$$
\begin{aligned}
\tau h^2\big(\|\nabla e_{h,u}^{m+1}\| + \|\nabla e_{h,u}^m\|\big) &\le \tau h^2\big(\|\nabla(e_{h,u}^{m+1} - e_{h,u}^m)\| + 2\|\nabla e_{h,u}^m\|\big) \\
&\le \frac{1}{16}\int_\Omega \frac{|\nabla(e_{h,u}^{m+1}-e_{h,u}^m)|^2}{Q(\widehat{u}_h^m)}\,dx + c\tau\|\nabla e_{h,u}^m\|^2 + c\tau h^4 \\
&\le \frac{1}{16}\int_\Omega \frac{|\nabla(e_{h,u}^{m+1}-e_{h,u}^m)|^2}{Q(\widehat{u}_h^m)}\,dx + c\tau\mathcal{B}^m + c\tau h^4,
\end{aligned}
$$

where we have used (4.16) in the last step. Inserting this estimate into (4.28), we infer that

$$
\begin{aligned}
&\frac{1}{2\tau}\int_\Omega \frac{(e_u^{m+1}-e_u^m)^2}{Q(u_h^m)}\,dx + \mathcal{B}^{m+1} + \frac{1}{16}\int_\Omega \frac{|\nabla(e_{h,u}^{m+1}-e_{h,u}^m)|^2}{Q(\widehat{u}_h^m)}\,dx \\
&\quad \le (1+c\tau)\mathcal{B}^m + c\tau(\tau^2 + h^4|\log h|^4) + c\tau\int_\Omega (e_{h,w}^m)^2 Q(u_h^m)\,dx.
\end{aligned} \tag{4.29}
$$

We deduce from (4.29) and the induction hypothesis (4.8) together with (4.15) that

$$
\begin{aligned}
\frac{1}{2c}\frac{1}{\tau}\|e_u^{m+1} - e_u^m\|^2 + \mathcal{B}^{m+1} &\le (\tau^2 + h^4|\log h|^4)e^{\mu t_m}\Big(1 + c\tau\Big(1 + \frac{1}{\theta^2}\Big)\Big) \\
&\le (\tau^2 + h^4|\log h|^4)e^{\mu t_{m+1}} \le h^2|\log h|^{-1},
\end{aligned} \tag{4.30}
$$

provided that

$$\mu \geq c\Big(1 + \frac{1}{\theta^2}\Big). \tag{4.31}$$

Note that for the last inequality in (4.30), we have used again (4.9), (4.11), and the fact that $\tau \leq \delta_0 h|\log h|^{-\frac{1}{2}}$. In particular, we can repeat the arguments leading to (4.15) and (4.16) and obtain

$$\sup_{\overline{\Omega}} Q(u_h^{m+1}) \leq c \quad \text{and} \quad \|\nabla e_{h,u}^{m+1}\|^2 \leq ch^2|\log h|^{-1}. \tag{4.32}$$

4.2. The surface PDE

As already mentioned in the introduction, the error analysis of the surface equation is laborious. Much of this work is related to the handling of differences of the form $Q(u^{m+1}) - Q(u_h^{m+1})$, which are typically split into terms $Q(u^{m+1}) - Q(\widehat{u}_h^{m+1})$ and $Q(\widehat{u}_h^{m+1}) - Q(u_h^{m+1})$. The second term can be bounded in terms of $\nabla e_{h,u}^{m+1}$, which is naturally controlled within our induction. On the other hand, simply estimating the first term by $\nabla \rho_u^{m+1}$ will frequently lead to sub-optimal error bounds, which are not sufficient to control the gradient of the discrete height function uniformly. Instead, we will try to exploit the structure of $Q(u)$ and frequently apply integration by parts to take advantage of the quadratic convergence of ρ_u^{m+1}.

Evaluating (2.6) at $t = t_{m+1}$ and using definition (3.6), we obtain for $\eta_h \in X_{h0}$

$$\int_\Omega (w\, Q(u))_t(\cdot, t_{m+1})\eta_h \, dx + \int_\Omega E(\nabla \widehat{u}_h^{m+1})\nabla \widehat{w}_h^{m+1} \cdot \nabla \eta_h \, dx$$
$$= -\int_\Omega \nabla u^{m+1} \cdot \nabla \eta_h \, V^{m+1} w^{m+1} \, dx + \int_\Omega g(V^{m+1}, w^{m+1})\, \eta_h \, Q(u^{m+1})\, dx.$$

If we combine this relation with (2.13), we deduce that

$$\begin{aligned}
&\int_\Omega e_{h,w}^{m+1} \eta_h Q(u_h^{m+1})\, dx - \int_\Omega e_{h,w}^m \eta_h Q(u_h^m)\, dx + \tau \int_\Omega E(\nabla u_h^{m+1})\nabla e_{h,w}^{m+1} \cdot \nabla \eta_h \, dx \\
&\quad = \int_\Omega \big(\widehat{w}_h^{m+1} Q(u_h^{m+1}) - \widehat{w}_h^m Q(u_h^m) - \tau (wQ(u))_t(\cdot, t_{m+1})\big)\eta_h \, dx \\
&\qquad + \tau \int_\Omega \big(E(\nabla u_h^{m+1}) - E(\nabla \widehat{u}_h^{m+1})\big)\nabla \widehat{w}_h^{m+1} \cdot \nabla \eta_h \, dx \\
&\qquad + \tau \int_\Omega \big(V_h^{m+1} w_h^{m+1} \nabla u_h^{m+1} - V^{m+1} w^{m+1} \nabla u^{m+1}\big) \cdot \nabla \eta_h \, dx \\
&\qquad + \tau \int_\Omega \big(g(V^{m+1}, w^{m+1}) Q(u^{m+1}) - g(V_h^{m+1}, w_h^m) Q(u_h^{m+1})\big)\eta_h \, dx.
\end{aligned} \tag{4.33}$$

Inserting $\eta_h = e_{h,w}^{m+1}$, we derive after some straightforward manipulations

$$\frac{1}{2}\int_\Omega (e_{h,w}^{m+1})^2 Q(u_h^{m+1})\, dx + \tau \int_\Omega E(\nabla u_h^{m+1})\nabla e_{h,w}^{m+1} \cdot \nabla e_{h,w}^{m+1}\, dx$$
$$+ \frac{1}{2}\int_\Omega (e_{h,w}^{m+1} - e_{h,w}^m)^2 Q(u_h^m)\, dx$$

$$
\begin{aligned}
&= \frac{1}{2}\int_\Omega (e_{h,w}^m)^2 Q(u_h^m)\,dx + \frac{1}{2}\int_\Omega (e_{h,w}^{m+1})^2\big(Q(u_h^m) - Q(u_h^{m+1})\big)\,dx \\
&\quad + \int_\Omega \big(\widehat{w}_h^{m+1} Q(u_h^{m+1}) - \widehat{w}_h^m Q(u_h^m) - \tau (wQ(u))_t(\cdot, t_{m+1})\big)e_{h,w}^{m+1}\,dx \\
&\quad + \tau \int_\Omega \big(E(\nabla u_h^{m+1}) - E(\nabla \widehat{u}_h^{m+1})\big)\nabla \widehat{w}_h^{m+1} \cdot \nabla e_{h,w}^{m+1}\,dx \\
&\quad + \tau \int_\Omega \big(V_h^{m+1} w_h^{m+1} \nabla u_h^{m+1} - V^{m+1} w^{m+1} \nabla u^{m+1}\big) \cdot \nabla e_{h,w}^{m+1}\,dx \\
&\quad + \tau \int_\Omega \big(g(V^{m+1}, w^{m+1}) Q(u^{m+1}) - g(V_h^{m+1}, w_h^m) Q(u_h^{m+1})\big)e_{h,w}^{m+1}\,dx \\
&=: \frac{1}{2}\int_\Omega (e_{h,w}^m)^2 Q(u_h^m)\,dx + \sum_{i=1}^{5} B_i. \qquad (4.34)
\end{aligned}
$$

(i) Rearranging the estimate

$$
\frac{\nabla u_h^m \cdot \nabla u_h^{m+1} + 1}{Q(u_h^m) Q(u_h^{m+1})} = \nu(u_h^m) \cdot \nu(u_h^{m+1}) \le 1
$$

implies that

$$
Q(u_h^m) - Q(u_h^{m+1}) \le \frac{\nabla u_h^m}{Q(u_h^m)} \cdot \nabla(u_h^m - u_h^{m+1}),
$$

so that

$$
\begin{aligned}
B_1 &\le \frac{1}{2}\int_\Omega (e_{h,w}^{m+1})^2 \frac{\nabla u_h^m}{Q(u_h^m)} \cdot \nabla(u_h^m - u_h^{m+1})\,dx \\
&= \frac{1}{2}\int_\Omega (e_{h,w}^{m+1})^2 \frac{\nabla u^m}{Q(u^m)} \cdot \nabla(u_h^m - u_h^{m+1})\,dx \\
&\quad + \frac{1}{2}\int_\Omega (e_{h,w}^{m+1})^2 \Big(\frac{\nabla u_h^m}{Q(u_h^m)} - \frac{\nabla u^m}{Q(u^m)}\Big) \cdot \nabla(u_h^m - u_h^{m+1})\,dx \\
&=: B_{1,1} + B_{1,2}.
\end{aligned}
$$

Integration by parts along with an inverse estimate yields

$$
\begin{aligned}
B_{1,1} &= \int_\Omega e_{h,w}^{m+1} \frac{\nabla e_{h,w}^{m+1} \cdot \nabla u^m}{Q(u^m)} (u_h^{m+1} - u_h^m)\,dx \\
&\quad + \frac{1}{2}\int_\Omega (e_{h,w}^{m+1})^2 \nabla \cdot \Big(\frac{\nabla u^m}{Q(u^m)}\Big)(u_h^{m+1} - u_h^m)\,dx \\
&\le c\int_\Omega \big(|e_{h,w}^{m+1}|\,|\nabla e_{h,w}^{m+1}| + |e_{h,w}^{m+1}|^2\big)\big(|e_u^{m+1} - e_u^m| + |u^{m+1} - u^m|\big)\,dx \\
&\le c\|e_{h,w}^{m+1}\|_{0,\infty}\|e_{h,w}^{m+1}\|_1\|e_u^{m+1} - e_u^m\| + c\tau \sup_{t_m \le t \le t_{m+1}} \|u_t\|_{0,\infty}\|e_{h,w}^{m+1}\|\,\|e_{h,w}^{m+1}\|_1 \\
&\le ch^{-1}\|e_{h,w}^{m+1}\|\,\|e_{h,w}^{m+1}\|_1\|e_u^{m+1} - e_u^m\| + c\tau\|e_{h,w}^{m+1}\|\,\|e_{h,w}^{m+1}\|_1.
\end{aligned}
$$

Next, we deduce from (2.10), (4.16), and (3.3) that

$$\|\nabla e_u^m\|_{0,\infty} \le \|\nabla e_{h,u}^m\|_{0,\infty} + \|\nabla \rho_u^m\|_{0,\infty} \le ch^{-1}\|\nabla e_{h,u}^m\| + ch|\log h| \le c|\log h|^{-\frac{1}{2}}$$

and therefore, by (2.11), (2.9), and (3.4),

$$\begin{aligned} B_{1,2} &\le c\|\nabla e_u^m\|_{0,\infty} \int_\Omega (e_{h,w}^{m+1})^2 \big(|\nabla(e_{h,u}^{m+1} - e_{h,u}^m)| + |\nabla(\widehat{u}_h^{m+1} - \widehat{u}_h^m)|\big)\,dx \\ &\le c|\log h|^{-\frac{1}{2}}\|e_{h,w}^{m+1}\|_{0,\infty}\|e_{h,w}^{m+1}\|\big(\|\nabla(e_{h,u}^{m+1} - e_{h,u}^m)\| + \tau\big) \\ &\le ch^{-1}\|e_{h,w}^{m+1}\|_1\|e_{h,w}^{m+1}\|\|e_{h,u}^{m+1} - e_{h,u}^m\| + c\tau\|e_{h,w}^{m+1}\|_1\|e_{h,w}^{m+1}\| \\ &\le ch^{-1}\|e_{h,w}^{m+1}\|_1\|e_{h,w}^{m+1}\|\|e_u^{m+1} - e_u^m\| + c\tau\|e_{h,w}^{m+1}\|_1\|e_{h,w}^{m+1}\|. \end{aligned}$$

Combining the above bounds, we find that

$$B_1 \le \varepsilon\tau\|e_{h,w}^{m+1}\|_1^2 + c_\varepsilon\tau\|e_{h,w}^{m+1}\|^2 + c_\varepsilon h^{-2}\|e_{h,w}^{m+1}\|^2\frac{1}{\tau}\|e_u^{m+1} - e_u^m\|^2. \tag{4.35}$$

(ii) Let us write

$$\begin{aligned} B_2 &= \int_\Omega \big(w^{m+1}Q(u^{m+1}) - w^m Q(u^m) - \tau(wQ(u))_t(\cdot,t_{m+1})\big)e_{h,w}^{m+1}\,dx \\ &\quad - \int_\Omega (\rho_w^{m+1} - \rho_w^m)Q(u_h^{m+1})e_{h,w}^{m+1}\,dx - \int_\Omega \rho_w^m\big(Q(u_h^{m+1}) - Q(u_h^m)\big)e_{h,w}^{m+1}\,dx \\ &\quad + \int_\Omega (w^{m+1} - w^m)\big(Q(u_h^{m+1}) - Q(u^{m+1})\big)e_{h,w}^{m+1}\,dx \\ &\quad + \int_\Omega w^m\Big(\big(Q(\widehat{u}_h^{m+1}) - Q(u^{m+1})\big) - \big(Q(\widehat{u}_h^m) - Q(u^m)\big)\Big)e_{h,w}^{m+1}\,dx \\ &\quad + \int_\Omega w^m\Big(\big(Q(u_h^{m+1}) - Q(\widehat{u}_h^{m+1})\big) - \big(Q(u_h^m) - Q(\widehat{u}_h^m)\big)\Big)e_{h,w}^{m+1}\,dx \\ &=: \sum_{j=1}^{6} B_{2,j}. \end{aligned} \tag{4.36}$$

Recalling (2.3), (2.4), (4.32), and (3.10), we have

$$|B_{2,1}| + |B_{2,2}| \le c\big(\tau^2 + \tau \sup_{t_m \le t \le t_{m+1}} \|\rho_{w,t}\|\big)\|e_{h,w}^{m+1}\| \le c\tau\big(\tau + h^2|\log h|^2\big)\|e_{h,w}^{m+1}\|.$$

Next, since $|Q(u_h^{m+1}) - Q(u_h^m)| \le |\nabla(u_h^{m+1} - u_h^m)|$, we obtain with the help of (3.8), (2.11), (3.2), (2.9), and (3.4) that

$$\begin{aligned} |B_{2,3}| &\le \|\rho_w^m\|\|\nabla(u_h^{m+1} - u_h^m)\|\|e_{h,w}^{m+1}\|_{0,\infty} \\ &\le ch^2|\log h|\big(\|\nabla(e_{h,u}^{m+1} - e_{h,u}^m)\| + \|\nabla(\widehat{u}_h^{m+1} - \widehat{u}_h^m)\|\big)|\log h|^{\frac{1}{2}}\|e_{h,w}^{m+1}\|_1 \\ &\le ch|\log h|^{\frac{3}{2}}\|e_{h,u}^{m+1} - e_{h,u}^m\|\|e_{h,w}^{m+1}\|_1 + c\tau h^2|\log h|^{\frac{3}{2}}\|e_{h,w}^{m+1}\|_1 \\ &\le c\|e_{h,w}^{m+1}\|_1\big(\|e_u^{m+1} - e_u^m\| + \tau h^2|\log h|^{\frac{3}{2}}\big). \end{aligned}$$

Applying Lemma 3.3 to $f = (w^{m+1} - w^m)e_{h,w}^{m+1}$ yields

$$\begin{aligned}
B_{2,4} &= \int_\Omega (w^{m+1} - w^m)\big(Q(u_h^{m+1}) - Q(\widehat{u}_h^{m+1})\big)e_{h,w}^{m+1}\,dx \\
&\quad + \int_\Omega (w^{m+1} - w^m)\big(Q(\widehat{u}_h^{m+1}) - Q(u^{m+1})\big)e_{h,w}^{m+1}\,dx \\
&\le c\|w^{m+1} - w^m\|_{0,\infty}\|\nabla e_{h,u}^{m+1}\|\|e_{h,w}^{m+1}\| + ch^2|\log h|\|(w^{m+1} - w^m)e_{h,w}^{m+1}\|_{1,1} \\
&\le c\tau\|e_{h,w}^{m+1}\|_1\big(\|\nabla e_{h,u}^{m+1}\| + h^2|\log h|\big).
\end{aligned}$$

Since $\big(Q(\widehat{u}_h) - Q(u)\big)_t = \dfrac{\nabla\widehat{u}_{h,t}\cdot\nabla\widehat{u}_h}{Q(\widehat{u}_h)} - \dfrac{\nabla u_t\cdot\nabla u}{Q(u)}$, we obtain

$$\begin{aligned}
B_{2,5} &= \int_{t_m}^{t_{m+1}}\int_\Omega w^m\Big(\frac{\nabla\widehat{u}_{h,t}\cdot\nabla\widehat{u}_h}{Q(\widehat{u}_h)} - \frac{\nabla u_t\cdot\nabla u}{Q(u)}\Big)e_{h,w}^{m+1}\,dx\,dt \\
&= \int_{t_m}^{t_{m+1}}\int_\Omega w^m\nabla u_t\cdot\Big(\frac{\nabla\widehat{u}_h}{Q(\widehat{u}_h)} - \frac{\nabla u}{Q(u)}\Big)e_{h,w}^{m+1}\,dx\,dt \\
&\quad + \int_{t_m}^{t_{m+1}}\int_\Omega w^m\frac{\nabla u}{Q(u)}\cdot\nabla(\widehat{u}_{h,t} - u_t)e_{h,w}^{m+1}\,dx\,dt \\
&\quad + \int_{t_m}^{t_{m+1}}\int_\Omega w^m\nabla(\widehat{u}_{h,t} - u_t)\cdot\Big(\frac{\nabla\widehat{u}_h}{Q(\widehat{u}_h)} - \frac{\nabla u}{Q(u)}\Big)e_{h,w}^{m+1}\,dx\,dt \\
&=: I + II + III. \qquad (4.37)
\end{aligned}$$

Another application of Lemma 3.3 yields

$$I \le ch^2|\log h|\int_{t_m}^{t_{m+1}}\|w^m e_{h,w}^{m+1}\,\nabla u_t\|_{1,1}dt \le c\tau h^2|\log h|\|e_{h,w}^{m+1}\|_1.$$

After integration by parts, we obtain

$$\begin{aligned}
II &= \int_{t_m}^{t_{m+1}}\int_\Omega \nabla\cdot\Big(w^m\frac{\nabla u}{Q(u)}\Big)\cdot\rho_{u,t}\,e_{h,w}^{m+1}\,dx\,dt \\
&\quad + \int_{t_m}^{t_{m+1}}\int_\Omega w^m\rho_{u,t}\frac{\nabla u}{Q(u)}\cdot\nabla e_{h,w}^{m+1}\,dx\,dt \\
&\le c\tau\sup_{t_m\le t\le t_{m+1}}\|\rho_{u,t}\|\,\|e_{h,w}^{m+1}\|_1 \le c\tau h^2|\log h|^2\|e_{h,w}^{m+1}\|_1,
\end{aligned}$$

by (3.4). Next, (3.3) and (3.5) imply

$$III \le c\int_{t_m}^{t_{m+1}}\|\nabla\rho_{u,t}\|\,\|\nabla\rho_u\|_{0,\infty}\|e_{h,w}^{m+1}\|dt \le c\tau h^2|\log h|\|e_{h,w}^{m+1}\|.$$

If we insert the above estimates into (4.37) we obtain

$$B_{2,5} \le c\tau h^2|\log h|^2\|e_{h,w}^{m+1}\|_1.$$

In order to treat $B_{2,6}$, we write with the help of (4.2)

$$
\begin{aligned}
&\big(Q(u_h^{m+1}) - Q(\widehat{u}_h^{m+1})\big) - \big(Q(u_h^m) - Q(\widehat{u}_h^m)\big) \\
&\quad = -\frac{\nabla \widehat{u}_h^{m+1}}{Q(\widehat{u}_h^{m+1})} \cdot \nabla e_{h,u}^{m+1} + \frac{\nabla \widehat{u}_h^m}{Q(\widehat{u}_h^m)} \cdot \nabla e_{h,u}^m \\
&\qquad + \frac{|\nabla e_{h,u}^{m+1}|^2}{2Q(\widehat{u}_h^{m+1})} - \frac{|\nabla e_{h,u}^m|^2}{2Q(\widehat{u}_h^m)} - \frac{\big(Q(u_h^{m+1}) - Q(\widehat{u}_h^{m+1})\big)^2}{2Q(\widehat{u}_h^{m+1})} \\
&\qquad + \frac{\big(Q(u_h^m) - Q(\widehat{u}_h^m)\big)^2}{2Q(\widehat{u}_h^m)} \\
&\quad = -\frac{\nabla \widehat{u}_h^{m+1}}{Q(\nabla \widehat{u}_h^{m+1})} \cdot \nabla (e_{h,u}^{m+1} - e_{h,u}^m) - \big(\frac{\nabla \widehat{u}_h^{m+1}}{Q(\widehat{u}_h^{m+1})} - \frac{\nabla \widehat{u}_h^m}{Q(\widehat{u}_h^m)}\big) \cdot \nabla e_{h,u}^m \\
&\qquad + \frac{1}{2}\Big(\frac{1}{Q(\widehat{u}_h^{m+1})} - \frac{1}{Q(\widehat{u}_h^m)}\Big)\big(|\nabla e_{h,u}^m|^2 - \big(Q(u_h^m) - Q(\widehat{u}_h^m)\big)^2\big) \\
&\qquad + \frac{\nabla (e_{h,u}^{m+1} - e_{h,u}^m) \cdot \nabla (e_{h,u}^{m+1} + \nabla e_{h,u}^m)}{2Q(\widehat{u}_h^{m+1})} \\
&\qquad - \delta_h \{\big(Q(u_h^{m+1}) - Q(\widehat{u}_h^{m+1})\big) - \big(Q(u_h^m) - Q(\widehat{u}_h^m)\big)\},
\end{aligned} \tag{4.38}
$$

where

$$
\delta_h = \frac{\big(Q(u_h^{m+1}) - Q(\widehat{u}_h^{m+1})\big) + \big(Q(u_h^m) - Q(\widehat{u}_h^m)\big)}{2Q(\widehat{u}_h^{m+1})}.
$$

We remark that (2.10), (4.16), and (4.32) imply that

$$
|\delta_h| \le \frac{1}{2}\big(|\nabla e_{h,u}^{m+1}| + |\nabla e_{h,u}^m|\big) \le c h^{-1}\big(\|\nabla e_{h,u}^{m+1}\| + \|\nabla e_{h,u}^m\|\big) \le c|\log h|^{-\frac{1}{2}} \le \frac{1}{2}, \tag{4.39}
$$

provided that $0 < h \le h_3$ and $h_3 \le h_2$ is small enough. Thus, if we move the last term on the right hand side of (4.38) to the left hand side and divide by $1 + \delta_h \ge \frac{1}{2}$, we obtain

$$
\begin{aligned}
&\big(Q(u_h^{m+1}) - Q(\widehat{u}_h^{m+1})\big) - \big(Q(u_h^m) - Q(\widehat{u}_h^m)\big) \\
&\quad = -\frac{\nabla \widehat{u}_h^{m+1}}{Q(\nabla \widehat{u}_h^{m+1})} \cdot \nabla (e_{h,u}^{m+1} - e_{h,u}^m) \\
&\qquad + \frac{1}{1+\delta_h} \nabla (e_{h,u}^{m+1} - e_{h,u}^m) \cdot \Big(\delta_h \frac{\nabla \widehat{u}_h^{m+1}}{Q(\nabla \widehat{u}_h^{m+1})} + \frac{\nabla (e_{h,u}^{m+1} + \nabla e_{h,u}^m)}{2Q(\widehat{u}_h^{m+1})}\Big) \\
&\qquad + \frac{1}{1+\delta_h} \frac{Q(\widehat{u}_h^m) - Q(\widehat{u}_h^{m+1})}{2Q(\widehat{u}_h^{m+1})Q(\widehat{u}_h^m)}\big(|\nabla e_{h,u}^m|^2 - \big(Q(u_h^m) - Q(\widehat{u}_h^m)\big)^2\big) \\
&\qquad - \frac{1}{1+\delta_h}\Big(\frac{\nabla \widehat{u}_h^{m+1}}{Q(\widehat{u}_h^{m+1})} - \frac{\nabla \widehat{u}_h^m}{Q(\widehat{u}_h^m)}\Big) \cdot \nabla e_{h,u}^m \\
&\quad =: S_1 + S_2 + S_3 + S_4.
\end{aligned}
$$

Note that in order to derive the form of S_1 and S_2, we have split $-\frac{1}{1+\delta_h} = -1 + \frac{\delta_h}{1+\delta_h}$. The fact that S_1 does not contain δ_h will allow us to apply integration by parts to the integral involving this term. From the above calculations, we now have

$$B_{2,6} = \sum_{i=1}^{4} \int_\Omega w^m S_i e_{h,w}^{m+1}\, dx.$$

To begin, integration by parts together with (1.7) yields

$$\begin{aligned}
&\int_\Omega w^m S_1 e_{h,w}^{m+1}\, dx \\
&\quad = \int_\Omega w^m e_{h,w}^{m+1} \Big[\Big(\frac{\nabla u^{m+1}}{Q(u^{m+1})} - \frac{\nabla \widehat{u}_h^{m+1}}{Q(\widehat{u}_h^{m+1})} \Big) - \frac{\nabla u^{m+1}}{Q(u^{m+1})} \Big] \cdot \nabla (e_{h,u}^{m+1} - e_{h,u}^m)\, dx \\
&\quad = \int_\Omega w^m e_{h,w}^{m+1} \Big(\frac{\nabla u^{m+1}}{Q(u^{m+1})} - \frac{\nabla \widehat{u}_h^{m+1}}{Q(\widehat{u}_h^{m+1})} \Big) \cdot \nabla (e_{h,u}^{m+1} - e_{h,u}^m)\, dx \\
&\qquad + \int_\Omega \nabla \cdot \Big(\frac{w^m e_{h,w}^{m+1} \nabla u^{m+1}}{Q(u^{m+1})} \Big) (e_{h,u}^{m+1} - e_{h,u}^m)\, dx.
\end{aligned}$$

Using Lemma 3.4 and (3.4) for the first term, we obtain

$$\begin{aligned}
\int_\Omega w^m S_1 e_{h,w}^{m+1}\, dx &\le ch|\log h| \|e_{h,u}^{m+1} - e_{h,u}^m\| \Big(\sum_{T \in \mathcal{T}_h} \|w^m e_{h,w}^{m+1}\|_{H^2(T)}^2 \Big)^{\frac{1}{2}} \\
&\quad + c\|e_{h,u}^{m+1} - e_{h,u}^m\| \|e_{h,w}^{m+1}\|_1 \\
&\le ch|\log h| \|e_{h,w}^{m+1}\|_1 \|e_{h,u}^{m+1} - e_{h,u}^m\| \le c\|e_{h,w}^{m+1}\|_1 \|e_{h,u}^{m+1} - e_{h,u}^m\|,
\end{aligned}$$

where we also exploited the fact that the second derivatives of $e_{h,w}^{m+1}$ vanish. Since $1 + \delta_h \ge \frac{1}{2}$ and $|\delta_h| \le \frac{1}{2}(|\nabla e_{h,u}^{m+1}| + |\nabla e_{h,u}^m|)$, we derive with the help of (4.16), (4.32), (2.9), and (2.11)

$$\begin{aligned}
\int_\Omega w^m S_2 e_{h,w}^{m+1}\, dx &\le c\|\nabla(e_{h,u}^{m+1} - e_{h,u}^m)\| \big(\|\nabla e_{h,u}^{m+1}\| + \|\nabla e_{h,u}^m\| \big) \|e_{h,w}^{m+1}\|_{0,\infty} \\
&\le ch|\log h|^{-\frac{1}{2}} h^{-1} \|e_{h,u}^{m+1} - e_{h,u}^m\| \, |\log h|^{\frac{1}{2}} \|e_{h,w}^{m+1}\|_1 \\
&\le c\|e_{h,u}^{m+1} - e_{h,u}^m\| \|e_{h,w}^{m+1}\|_1.
\end{aligned}$$

Finally, we deduce with the help of (4.39), (2.11), and (4.16)

$$\begin{aligned}
&\int_\Omega w^m (S_3 + S_4) e_{h,w}^{m+1}\, dx \\
&\quad \le c \int_\Omega |\nabla(\widehat{u}_h^{m+1} - \widehat{u}_h^m)| \big(|\nabla e_{h,u}^m|^2 + |\nabla e_{h,u}^m| \big) |e_{h,w}^{m+1}|\, dx \\
&\quad \le c\|\nabla(\widehat{u}_h^{m+1} - \widehat{u}_h^m)\|_{0,\infty} \|\nabla e_{h,u}^m\| \big(\|\nabla e_{h,u}^m\| \|e_{h,w}^{m+1}\|_{0,\infty} + \|e_{h,w}^{m+1}\| \big) \\
&\quad \le c\tau \|\nabla e_{h,u}^m\| \big(h\|e_{h,w}^{m+1}\|_1 + \|e_{h,w}^{m+1}\| \big) \le c\tau \|\nabla e_{h,u}^m\| \|e_{h,w}^{m+1}\|_1.
\end{aligned}$$

Collecting the above estimates and recalling (3.4), we obtain

$$B_{2,6} \le c\|e_{h,w}^{m+1}\|_1\big(\|e_u^{m+1} - e_u^m\| + \tau h^2|\log h|^2 + \tau\|\nabla e_{h,u}^m\|\big).$$

If we insert the bounds for $B_{2,j}$, $j = 1, \dots, 6$ into (4.36), we obtain

$$\begin{aligned} B_2 &\le c\|e_{h,w}^{m+1}\|_1\big(\tau^2 + \tau h^2|\log h|^2 + \|e_u^{m+1} - e_u^m\| + \tau\|\nabla e_{h,u}^{m+1}\| + \tau\|\nabla e_{h,u}^m\|\big) \\ &\le \varepsilon\tau\|e_{h,w}^{m+1}\|_1^2 + c_\varepsilon\frac{1}{\tau}\|e_u^{m+1} - e_u^m\|^2 \\ &\quad + c_\varepsilon\tau\big(\tau^2 + h^4|\log h|^4 + \|\nabla e_{h,u}^{m+1}\|^2 + \|\nabla e_{h,u}^m\|^2\big). \end{aligned} \tag{4.40}$$

(iii) Recalling (2.7), it is not difficult to verify that $|E(p) - E(q)| \le c|p - q|$, and hence

$$B_3 \le C\tau\|\nabla e_{h,u}^{m+1}\|\|\nabla e_{h,w}^{m+1}\| \le \varepsilon\tau\|\nabla e_{h,w}^{m+1}\|^2 + c_\varepsilon\tau\|\nabla e_{h,u}^{m+1}\|^2. \tag{4.41}$$

(iv) In view of the definition of V^{m+1} and V_h^{m+1}, we have

$$\begin{aligned} B_4 &= \int_\Omega \Big((u_h^{m+1} - u_h^m)w_h^m\frac{\nabla u_h^{m+1}}{Q(u_h^{m+1})} - \tau u_t^{m+1}w^{m+1}\frac{\nabla u^{m+1}}{Q(u^{m+1})}\Big)\cdot\nabla e_{h,w}^{m+1}\,dx \\ &= -\int_\Omega (e_u^{m+1} - e_u^m)w_h^m\frac{\nabla u_h^{m+1}}{Q(u_h^{m+1})}\cdot\nabla e_{h,w}^{m+1}\,dx \\ &\quad - \int_\Omega (u^{m+1} - u^m)e_w^m\frac{\nabla u_h^{m+1}}{Q(u_h^{m+1})}\cdot\nabla e_{h,w}^{m+1}\,dx \\ &\quad + \int_\Omega (u^{m+1} - u^m)w^m\Big(\frac{\nabla u_h^{m+1}}{Q(u_h^{m+1})} - \frac{\nabla\widehat{u}_h^{m+1}}{Q(\widehat{u}_h^{m+1})}\Big)\cdot\nabla e_{h,w}^{m+1}\,dx \\ &\quad + \int_\Omega (u^{m+1} - u^m)w^m\Big(\frac{\nabla\widehat{u}_h^{m+1}}{Q(\widehat{u}_h^{m+1})} - \frac{\nabla u^{m+1}}{Q(u^{m+1})}\Big)\cdot\nabla e_{h,w}^{m+1}\,dx \\ &\quad + \int_\Omega (u^{m+1} - u^m - \tau u_t^{m+1})w^m\frac{\nabla u^{m+1}}{Q(u^{m+1})}\cdot\nabla e_{h,w}^{m+1}\,dx \\ &\quad + \int_\Omega \tau u_t^{m+1}(w^m - w^{m+1})\frac{\nabla u^{m+1}}{Q(u^{m+1})}\cdot\nabla e_{h,w}^{m+1}\,dx \\ &=: \sum_{i=1}^6 B_{4,i}. \end{aligned}$$

It follows from (4.12) and (4.19) that

$$\sum_{i\ne4} B_{4,i} \le c\|\nabla e_{h,w}^{m+1}\|\big(\|e_u^{m+1} - e_u^m\| + \tau\|e_w^m\| + \tau\|\nabla e_{h,u}^{m+1}\| + \tau^2\big),$$

while Lemma 3.4 implies

$$B_{4,4} \le ch^2|\log h|\|e_{h,w}^{m+1}\|_1\|(u^{m+1} - u^m)w^m\|_2 \le c\tau h^2|\log h|\|e_{h,w}^{m+1}\|_1.$$

In conclusion,

$$B_4 \le \varepsilon\tau\|e_{h,w}^{m+1}\|_1^2 + c_\varepsilon\frac{1}{\tau}\|e_u^{m+1}-e_u^m\|^2 + c_\varepsilon\tau\big(\|e_{h,w}^m\|^2 + \|\nabla e_{h,u}^{m+1}\|^2 + \tau^2 + h^4|\log h|^2\big). \tag{4.42}$$

(v) Finally, in order to treat B_5 we recall (2.1) and note that $\alpha(\lambda r) = \lambda\alpha(r)$, $r \in \mathbb{R}$, $\lambda > 0$. As a consequence,

$$\begin{aligned} &g(V^{m+1}, w^{m+1})Q(u^{m+1}) - g(V_h^{m+1}, w_h^m)Q(u_h^{m+1}) \\ &\quad = \alpha(u_t^{m+1})\big(\beta(w^{m+1}) - \beta(w_h^m)\big) + \Big(\alpha(u_t^{m+1}) - \alpha\Big(\frac{u_h^{m+1}-u_h^m}{\tau}\Big)\Big)\beta(w_h^m) \\ &\qquad + \tilde\beta(w^{m+1})\big(Q(u^{m+1}) - Q(u_h^{m+1})\big) + \big(\tilde\beta(w^{m+1}) - \tilde\beta(w_h^m)\big)Q(u_h^{m+1}) \\ &\quad = \sum_{i=1}^4 S_i, \end{aligned}$$

so that

$$B_5 = \sum_{i=1}^4 \tau\int_\Omega S_i\, e_{h,w}^{m+1}\, dx =: \sum_{i=1}^4 B_{5,i}.$$

Since $\beta, \tilde\beta \in C^{0,1}_{\mathrm{loc}}(\mathbb{R})$ we obtain from (4.12), (4.32), and (3.8)

$$\begin{aligned} |B_{5,1}| + |B_{5,4}| &\le c\tau\|w^{m+1} - w_h^m\|\|e_{h,w}^{m+1}\| \\ &\le c\tau\big(\|w^{m+1} - w^m\| + \|\rho_w^m\| + \|e_{h,w}^m\|\big)\|e_{h,w}^{m+1}\| \\ &\le c\tau\big(\tau + h^2|\log h| + \|e_{h,w}^m\|\big)\|e_{h,w}^{m+1}\|. \end{aligned}$$

Next, we deduce with the help of the global Lipschitz continuity of $r \mapsto \alpha(r)$ and (4.19) that

$$\begin{aligned} |B_{5,2}| &\le c\tau\Big\|u_t^{m+1} - \frac{u_h^{m+1}-u_h^m}{\tau}\Big\|\|e_{h,w}^{m+1}\| \\ &\le c\tau^2\|e_{h,w}^{m+1}\| + c\|e_u^{m+1} - e_u^m\|\|e_{h,w}^{m+1}\|. \end{aligned}$$

Applying Lemma 3.3 with $f = \tilde\beta(w^{m+1})e_{h,w}^{m+1}$, we infer that

$$|B_{5,3}| \le c\tau h^2|\log h|\|\tilde\beta(w^{m+1})e_{h,w}^{m+1}\|_{1,1} \le c\tau h^2|\log h|\|e_{h,w}^{m+1}\|_1.$$

After collecting the above estimates, we obtain

$$B_5 \le \varepsilon\tau\|e_{h,w}^{m+1}\|_1^2 + c_\varepsilon\frac{1}{\tau}\|e_u^{m+1}-e_u^m\|^2 + c_\varepsilon\tau\big(\|e_{h,w}^m\|^2 + \tau^2 + h^4|\log h|^2\big). \tag{4.43}$$

If we insert (4.35), (4.40), (4.41), (4.42), and (4.43) into (4.34), use Poincaré's inequality, and observe (2.8) together with (4.32), we derive

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (e_{h,w}^{m+1})^2 Q(u_h^{m+1})\,dx + \tau c_0\|\nabla e_{h,w}^{m+1}\|^2 + \frac{1}{2}\|e_{h,w}^{m+1} - e_{h,w}^m\|^2 \\
&\quad\le \frac{1}{2}\int_\Omega (e_{h,w}^m)^2 Q(u_h^m)\,dx + \varepsilon\tau\|\nabla e_{h,w}^{m+1}\|^2 \\
&\qquad + c_\varepsilon\tau\big(\tau^2 + h^4|\log h|^4 + \|\nabla e_{h,u}^{m+1}\|^2 + \|\nabla e_{h,u}^m\|^2\big) \\
&\qquad + c_\varepsilon\tau\|e_{h,w}^m\|^2 + c_\varepsilon h^{-2}\frac{1}{\tau}\|e_{h,w}^{m+1}\|^2\|e_u^{m+1} - e_u^m\|^2 + c_\varepsilon\frac{1}{\tau}\|e_u^{m+1} - e_u^m\|^2.
\end{aligned} \tag{4.44}$$

In view of (4.10), (4.30), and (4.9), we have

$$\begin{aligned}
h^{-2}\|e_{h,w}^{m+1}\|^2\frac{1}{\tau}\|e_u^{m+1} - e_u^m\|^2 &\le 2h^{-2}\big(\|e_{h,w}^m\|^2 + \|e_{h,w}^{m+1} - e_{h,w}^m\|^2\big)\frac{1}{\tau}\|e_u^{m+1} - e_u^m\|^2 \\
&\le 4\frac{|\log h|^{-1}}{\theta^2}\frac{1}{\tau}\|e_u^{m+1} - e_u^m\|^2 + c|\log h|^{-1}\|e_{h,w}^{m+1} - e_{h,w}^m\|^2 \\
&\le c\frac{1}{\tau}\|e_u^{m+1} - e_u^m\|^2 + c|\log h|^{-1}\|e_{h,w}^{m+1} - e_{h,w}^m\|^2.
\end{aligned}$$

Using this bound in (4.44) and choosing ε and h_0 sufficiently small, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (e_{h,w}^{m+1})^2 Q(u_h^{m+1})\,dx + \tau\frac{c_0}{2}\|\nabla e_{h,w}^{m+1}\|^2 \le \frac{1}{2}\int_\Omega (e_{h,w}^m)^2 Q(u_h^m)\,dx + c\tau\|e_{h,w}^m\|^2 \\
&\qquad + c\tau\big(\tau^2 + h^4|\log h|^4\big) + c\tau\|\nabla(e_{h,u}^{m+1} - e_{h,u}^m)\|^2 + c\tau\|\nabla e_{h,u}^m\|^2 \\
&\qquad + \frac{c}{\tau}\|e_u^{m+1} - e_u^m\|^2 \\
&\quad\le (1 + c\tau)\frac{1}{2}\int_\Omega (e_{h,w}^m)^2 Q(u_h^m)\,dx + c\tau\big(\tau^2 + h^4|\log h|^4\big) + c\tau\mathcal{B}^m \\
&\qquad + c\tau\int_\Omega \frac{|\nabla(e_{h,u}^{m+1} - e_{h,u}^m)|^2}{Q(\widehat{u}_h^m)}\,dx + c\frac{1}{2\tau}\int_\Omega \frac{(e_u^{m+1} - e_u^m)^2}{Q(u_h^m)}\,dx,
\end{aligned} \tag{4.45}$$

where we used (4.16) and the fact that $Q(\widehat{u}_h^m), Q(u_h^m) \le c$ in order to derive the last estimate. Multiplying (4.45) by θ^2 ($0 < \theta \le 1$) and adding the result to (4.29), we obtain with the help of our induction hypothesis (4.8)

$$\begin{aligned}
&\mathcal{B}^{m+1} + \frac{\theta^2}{2}\int_\Omega (e_{h,w}^{m+1})^2 Q(u_h^{m+1})\,dx + \tau\frac{\theta^2 c_0}{2}\|\nabla e_{h,w}^{m+1}\|^2 \\
&\qquad + (1 - c\theta^2)\frac{1}{2\tau}\int_\Omega \frac{(e_u^{m+1} - e_u^m)^2}{Q(u_h^m)}\,dx \\
&\qquad + \Big(\frac{1}{16} - c\tau\Big)\int_\Omega \frac{|\nabla(e_{h,u}^{m+1} - e_{h,u}^m)|^2}{Q(\widehat{u}_h^m)}\,dx \\
&\quad\le \Big(1 + \frac{c}{\theta^2}\tau\Big)\Big(\mathcal{B}^m + \frac{\theta^2}{2}\int_\Omega (e_{h,w}^m)^2 Q(u_h^m)\,dx\Big) + c\tau\big(\tau^2 + \tau h^4|\log h|^4\big)
\end{aligned}$$

$$\leq \Big(1 + c\tau\Big(1 + \frac{1}{\theta^2}\Big)\Big)(\tau^2 + h^4|\log h|^4)e^{\mu t_m}$$
$$\leq (\tau^2 + h^4|\log h|^4)e^{\mu t_{m+1}}, \tag{4.46}$$

provided that

$$\mu \geq c\Big(1 + \frac{1}{\theta^2}\Big). \tag{4.47}$$

We are now in position to specify the choice of the constants θ, μ, δ_0, and h_0. To begin, choose $0 < \theta \leq 1$ such that $1 - c\theta^2 \geq \frac{1}{2}$ in the second line of (4.46). Next, choose $\mu > 0$ to satisfy (4.31) and (4.47) and then $\delta_0 > 0$ to satisfy (4.11). Finally, $h_0 > 0$ is fixed by (4.9) and additional smallness conditions on h that were required in the course of the calculations.

5. Numerical results

We begin this section by investigating the experimental order of convergence (EOC) of our scheme and then display some simulations of diffusion induced grain boundary motion. Throughout the computations in this section, we choose a uniform time step $\tau = h^2$.

5.1. Experimental order of convergence

We set $\Omega := \{x \in \mathbb{R}^2 \,|\, |x| < 1\}$, $T = 0.1$ and choose $f(w) = w^2$ as well as $g(V, w) = Vw$. We consider $u, w : \overline{\Omega} \times [0, T] \to \mathbb{R}$ given by

Example 1. $u(x,t) = 5\sin(t)(1 - |x|^2)$, $w(x,t) = e^{-t}(1 + |x|^2)$;
Example 2. $u(x,t) = 5\sin(t)\big(1 + (1 - |x|^2)^2\big)$, $w(x,t) = e^{-t}(1 + |x|^2)$,

and include additional right hand sides in order for (u, w) to be solutions of the corresponding PDEs, while the boundary conditions are $u(x,t) = 0$, $w(x,t) = 2e^{-t}$ in Example 1 and $\frac{\partial u}{\partial n}(x,t) = 0$, $w(x,t) = 2e^{-t}$ in Example 2. Let us point out that the Dirichlet condition for u in Example 1 is not covered by our theory. We commence our numerical results with Figure 1, in which we display the solution w_h^m plotted on the surface $\Gamma_h^m = \{(x, u_h^m(x)) \,|\, x \in \Omega\}$, at $t^m = 0$ and $t^m = 0.1$, for Example 2. When investigating the experimental order of convergence, we monitor the following errors:

$$\mathcal{E}_1 := \max_{0 \leq m \leq M} \|e_w^m\|^2, \quad \mathcal{E}_2 := \sum_{m=1}^{M} \tau\,\|\nabla e_w^m\|^2, \quad \mathcal{E}_3 := \max_{0 \leq m \leq M} \|e_u^m\|^2,$$
$$\mathcal{E}_4 := \max_{0 \leq m \leq M} \|\nabla e_u^m\|^2, \quad \mathcal{E}_5 := \sum_{m=0}^{M-1} \tau\Big\|\frac{e_u^{m+1} - e_u^m}{\tau}\Big\|^2.$$

In Tables 1 and 2, we display the values of $\mathcal{E}_i$, $i = 1, \ldots, 5$, evaluated using a quadrature rule of degree 4, for Example 1 and Example 2, respectively. For both examples we see

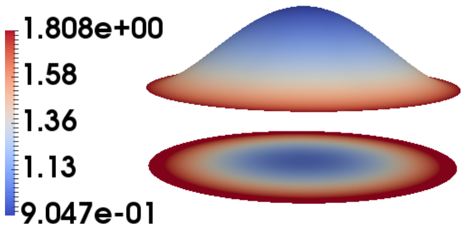


Figure 1. Example 2, w_h^m plotted on Γ_h^m at $t^m = 0.0$ and $t^m = 0.1$.

the expected order of convergence, with EOCs close to four for $\mathcal{E}_1$, $\mathcal{E}_3$ and $\mathcal{E}_5$, and EOCs close to two for $\mathcal{E}_2$ and $\mathcal{E}_3$. In particular, the results of Example 2 confirm the bounds obtained in Theorem 2.1.

5.2. Non-orthogonal boundary contact

Even though we have restricted our error analysis to the case where the evolving surface meets the boundary of the cylinder at a right angle, it is not difficult to apply our approach to the case where it meets the boundary of the cylinder at a given angle α. In order to do this, we replace boundary condition (1.3a) with

$$\nu \cdot \nu_{\partial A} = \cos(\alpha) \quad \text{on } \partial\Gamma(t), \qquad t \in (0, T],$$

leading to the following boundary condition for the height function u:

$$\frac{\nabla u \cdot n}{Q(u)} = -\cos(\alpha) \quad \text{on } \partial\Omega \times (0, T].$$

The weak formulation for u then takes the form

$$\int_\Omega \frac{u_t \varphi}{Q(u)}\, dx + \int_\Omega \frac{\nabla u \cdot \nabla\varphi}{Q(u)}\, dx = \int_\Omega f(w)\, \varphi\, dx - \int_{\partial\Omega} \cos(\alpha)\varphi\, dx \quad \forall \varphi \in H^1(\Omega),$$

from which we derive the corresponding finite element approximation replacing (2.12).

We set $\Omega := \{x \in \mathbb{R}^2 \mid |x| < 1\}$, $f(w) = w$, and $g(V, w) = |V|w$ and specify the following boundary conditions for u and w:

$$\frac{\nabla u \cdot n}{Q(u)} = -\cos(2\pi t - \pi/2) \quad \text{and} \quad w = 1 \qquad \text{on } \partial\Omega \times (0, T].$$

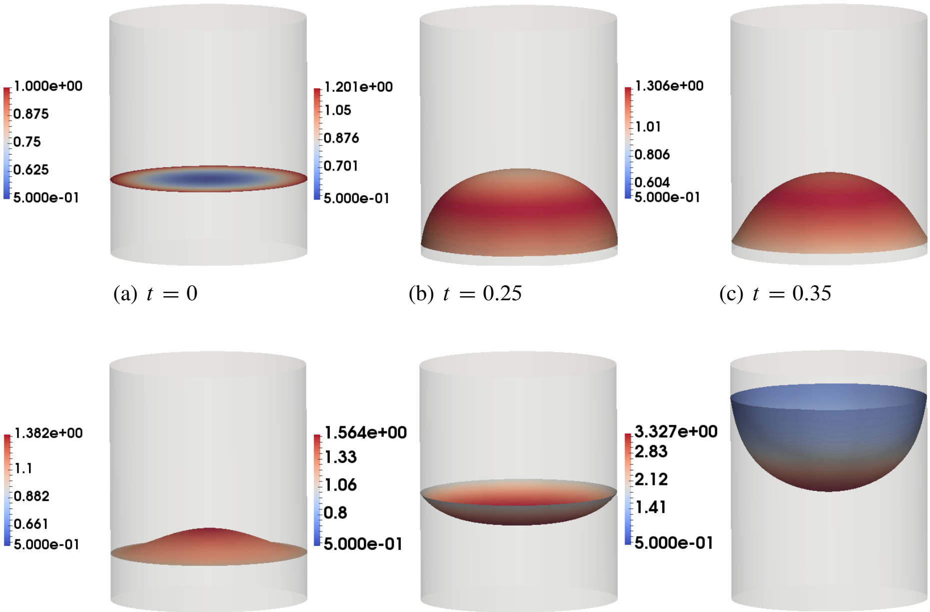
As initial data we choose $u^0(x) = 0$ and $w^0(x) = \frac{1}{2}(1 + |x|^2)$. In Figure 2, we display w_h^m on the surface $\Gamma_h^m = \{(x, u_h^m(x)) \mid x \in \Omega\}$ at $t^m = 0, 0.25, 0.35, 0.5, 0.65, 0.75$. As $|\cos(2\pi t - \pi/2)| = 1$ for $t = 0.25, 0.75$, the gradient of u will blow up on the boundary. However, for the mesh sizes we chose, the discrete solution was able to flow through these singularities without problems.

h	$\mathcal{E}_1 \times 10^2$	EOC_1	$\mathcal{E}_2 \times 10^2$	EOC_2	$\mathcal{E}_4 \times 10^4$	EOC_3	$\mathcal{E}_4 \times 10^2$	EOC_4	$\mathcal{E}_5 \times 10^4$	EOC_5
0.1961	29.33177	–	64.7340	–	81.45108	–	35.3598	–	9.88350	–
0.0996	0.06181	9.10	0.5881	6.94	0.16291	9.18	1.1420	5.07	0.04050	8.12
0.0538	0.00514	4.04	0.1006	2.87	0.00668	5.19	0.2347	2.57	0.00279	4.34
0.0269	0.00032	3.99	0.0217	2.21	0.00037	4.16	0.0574	2.03	0.00018	4.00
0.0135	0.00002	4.00	0.0052	2.06	0.00002	4.07	0.0142	2.02	0.00001	4.01

Table 1. Example 1, the errors, $\mathcal{E}_i$, together with their associated estimated order of convergence, EOC_i, for $i = 1, 2, 3, 4, 5$.

h	$\mathcal{E}_1 \times 10^2$	EOC_1	$\mathcal{E}_2 \times 10^2$	EOC_2	$\mathcal{E}_4 \times 10^4$	EOC_3	$\mathcal{E}_4 \times 10^2$	EOC_4	$\mathcal{E}_5 \times 10^4$	EOC_5
0.1961	38.25517	–	152.3575	–	99.40560	–	101.7628	–	38.94582	–
0.0996	0.27386	7.29	1.0056	7.41	1.72052	5.99	4.4905	4.61	0.23208	7.56
0.0538	0.02398	3.96	0.1325	3.29	0.10679	4.51	0.8869	2.63	0.01604	4.34
0.0269	0.00153	3.97	0.0237	2.48	0.00652	4.03	0.2105	2.08	0.00099	4.01
0.0135	0.00011	3.82	0.0054	2.15	0.00041	3.98	0.0515	2.03	0.00006	4.04

Table 2. Example 2, the errors, $\mathcal{E}_i$, together with their associated estimated order of convergence, EOC_i, for $i = 1, 2, 3, 4, 5$.



(a) $t = 0$ (b) $t = 0.25$ (c) $t = 0.35$

(d) $t = 0.5$ (e) $t = 0.65$ (f) $t = 0.75$

Figure 2. w_h^m plotted on the surface Γ_h^m at $t^m = 0, 0.25, 0.35, 0.5, 0.65, 0.75$.

5.3. Simulations of diffusion induced grain boundary motion

We conclude our numerical results with two simulations of diffusion induced grain boundary motion. We consider the physical set up of a film of metal, containing a single grain boundary. We denote the film by $A = \Omega \times [0, 5] \subset \mathbb{R}^3$, with $\Omega = (-2, 2)^2$, and we model the grain boundary by the surface $\Gamma(t) = \{(x, u(x, t)) \,|\, x \in \Omega\}$. We impose the boundary condition

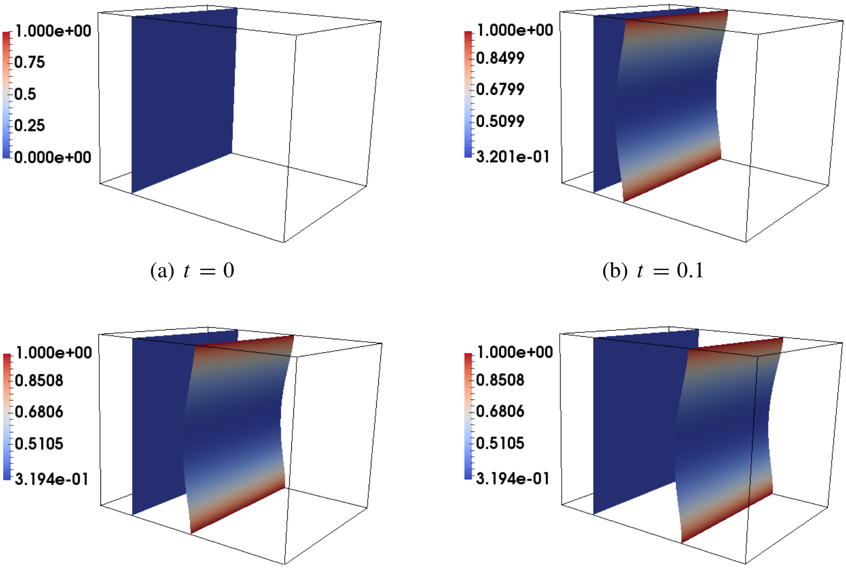
$$\frac{\partial u}{\partial n}(x, t) = 0 \quad \forall\, (x, t) \in \partial\Omega \times (0, T]$$

such that the grain boundary meets the boundaries of the film orthogonally.

The film is immersed in a solute that diffuses into the grain boundary at the surfaces $x_1 = \pm 2$. We denote the concentration of the solute on the grain boundary by $w(x, t) \in [0, 1]$, for $x \in \Omega$, and we assume that the solute concentration is set to one on the surfaces $x_1 = \pm 2$ and satisfies zero flux boundary conditions at the surfaces $x_2 = \pm 2$, i.e.,

$$w(x, t) = 1 \quad \text{for } x_1 = \pm 2, \qquad \frac{\partial w}{\partial n}(x, t) = 0 \quad \text{for } x_2 = \pm 2.$$

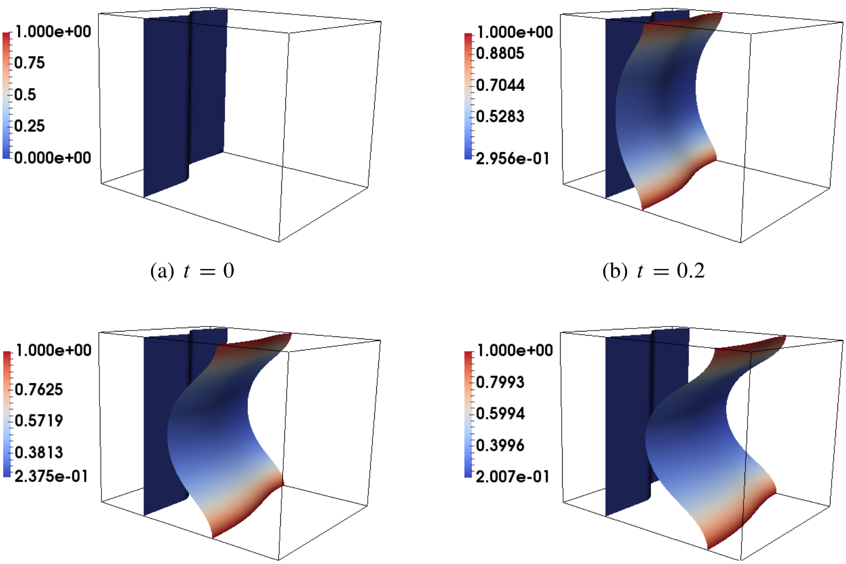
We consider two initial configurations for the grain boundary: in the first configuration we take the grain boundary to be the planar surface $x_3 = 1$ such that $u^0(x) = 1$, while in the



(a) $t = 0$ (b) $t = 0.1$

(c) $t = 0.2$ (d) $t = 0.3$

Figure 3. Travelling wave solution showing the grain boundary with the solute concentration at $t^m = 0, 0.1, 0.2, 0.3$, with $u_h^0 \equiv 1$ and $w_h^0 \equiv 0$.



(a) $t = 0$ (b) $t = 0.2$

(c) $t = 0.4$ (d) $t = 0.6$

Figure 4. Evolving grain boundary with the solute concentration at $t^m = 0, 0.2, 0.4, 0.6$, with the initial surface defined by (5.1) and $w_h^0 \equiv 0$.

second we take

$$u^0(x_1, x_2) = \begin{cases} 1+\varepsilon & \text{if } x_1 > \frac{\pi\varepsilon}{2}, \\ \varepsilon \sin\left(\frac{x_1}{\varepsilon}\right) & \text{if } |x_1| \le \frac{\pi\varepsilon}{2}, \\ 1-\varepsilon & \text{if } x_1 < -\frac{\pi\varepsilon}{2} \end{cases} \tag{5.1}$$

with $\varepsilon = 0.4$.

For both configurations, we assume that the concentration of solute on the grain boundary is initially zero, such that $w^0(x) = 0$ for $x \in \Omega$. In this set up, physically meaningful choices for $f(w)$ and $g(V, w)$ are $f(w) = w^2$ and $g(V, w) = |V|\, w$. Figure 3 displays the solute concentration, $w_h^m(x)$, plotted on the grain boundary, $\Gamma_h^m = \{(x, u_h^m(x)) \mid x \in \Omega\}$, at times $t^m = 0, 0.1, 0.2, 0.3$. Additionally, in each plot we display the initial grain boundary, depicted by the blue surface, and the outline of the metallic film $A = \Omega \times [0, 5]$. The symmetry of this set up makes it equatable to the two-dimensional configurations studied in [5] and [13]. In particular, we see a travelling wave solution comparable to the ones displayed in [5, Figures 9 and 10] and [13, Figure 4.4]. In Figure 4 the initial surface is defined by (5.1), which gives rise to a fully three-dimensional simulation. We display the solute concentration, $w_h^m(x)$, plotted on the grain boundary, at times $t^m = 0, 0.2, 0.4, 0.6$, together with the initial grain boundary and the outline of the film.

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