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The multiphase Muskat problem with equal viscosities in two dimensions

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Abstract. We study the two-dimensional multiphase Muskat problem describing the motion of three immiscible fluids with equal viscosities in a vertical homogeneous porous medium identified with $\mathbb{R}^2$ under the effect of gravity. We first formulate the governing equations as a strongly coupled evolution problem for the functions that parameterize the sharp interfaces between the fluids. Afterwards we prove that the problem is of parabolic type and establish its well-posedness together with two parabolic smoothing properties. For solutions that are not global we exclude, in a certain regime, that the interfaces come into contact along a curve segment.

1. Introduction and the main results

The mathematical model

In this paper we study the two-dimensional multiphase Muskat problem describing the motion of three incompressible fluids with positive constant densities

$$\rho_3 > \rho_2 > \rho_1$$

in a vertical porous medium identified with $\mathbb{R}^2$. Such three-phase flows are of great interest not only from a mathematical point of view, but also in many other areas of science and technology, such as petroleum extraction and environmental engineering, cf., e.g., [4, 8]. In this paper, we restrict our attention to the particular case when the three fluids have equal viscosities, which we denote by $\mu > 0$. We further assume that the fluid phases are separated at each time instant $t \geq 0$ by sharp interfaces which we describe as being the graphs

$$\Gamma_f^{c_\infty}(t) := \{(x, c_\infty + f(t,x)) : x \in \mathbb{R}\} \quad \text{and} \quad \Gamma_h(t) := \{(x, h(t,x)) : x \in \mathbb{R}\}.$$

Here, c_∞ is a fixed positive constant and the functions $f(t), h(t) : \mathbb{R} \to \mathbb{R}$ are unknown and are assumed to satisfy $f(t) + c_\infty > h(t)$ during the motion. The fluid with density ρ_i occupies the domain $\Omega_i(t) \subset \mathbb{R}^2$ for $1 \leq i \leq 3$, where $\Omega_2(t) := \mathbb{R}^2 \setminus \overline{\Omega_1(t) \cup \Omega_3(t)}$ and

$$\Omega_1(t) := \{(x, y) : y > f(t,x) + c_\infty\}, \quad \Omega_3(t) := \{(x, y) : y < h(t,x)\}.$$

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In the fluid layers, the dynamic is governed by the equations

$$\left.\begin{aligned} v_i(t) &= -c\frac{k}{\mu}\big(\nabla p_i(t) + (0, \rho_i g)\big), \\ \operatorname{div} v_i(t) &= 0 \end{aligned}\right\} \quad \text{in } \Omega_i(t),\ 1 \le i \le 3, \tag{1.1a}$$

where $p_i(t)$ is the pressure and $v_i(t) := (v_i^1(t), v_i^2(t))$ denotes the velocity field of the fluid located at $\Omega_i(t)$. The positive constant k is the permeability of the homogeneous porous medium and g is the Earth's gravity. The equation $(1.1a)_1$ is Darcy's law which is often used to model flows in porous media, cf., e.g., [7], and $(1.1a)_2$ describes the conservation of mass in all phases.

We supplement (1.1a) with the following boundary conditions at the free interfaces:

$$\left.\begin{aligned} p_i(t) &= p_{i+1}(t), \\ \langle v_i(t)|\nu_i(t)\rangle &= \langle v_{i+1}(t)|\nu_i(t)\rangle \end{aligned}\right\} \quad \text{on } \partial\Omega_i(t) \cap \partial\Omega_{i+1}(t),\ i = 1, 2. \tag{1.1b}$$

Here, $\nu_i(t)$ denotes the unit normal at $\partial\Omega_i(t) \cap \partial\Omega_{i+1}(t)$ pointing into $\Omega_i(t)$ and $\langle\cdot|\cdot\rangle$ is the inner scalar product in $\mathbb{R}^2$. Additionally, we impose the far-field boundary conditions

$$\left.\begin{aligned} v_i(t,x,y) &\to 0 \quad \text{for } |(x,y)| \to \infty,\ 1 \le i \le 3, \\ f^2(t,x) + h^2(t,x) &\to 0 \quad \text{for } |x| \to \infty \end{aligned}\right\} \tag{1.1c}$$

which state that far away, the flow is nearly stationary.

Finally, in order to describe the motion of the free interfaces, we set their normal velocity equal to the normal component of the velocity field at the free boundary, that is,

$$\left.\begin{aligned} \partial_t f(t) &= \langle v_1(t)|(-\partial_x f(t), 1)\rangle \quad \text{on } \Gamma_f^{c_\infty}(t), \\ \partial_t h(t) &= \langle v_2(t)|(-\partial_x h(t), 1)\rangle \quad \text{on } \Gamma_h(t) \end{aligned}\right\} \tag{1.1d}$$

and we impose the initial condition

$$(f, h)(0, \cdot) = (f_0, h_0). \tag{1.1e}$$

We call the closed system (1.1) the multiphase Muskat problem.

Summary of results and outline of the paper

The classical Muskat problem describing the dynamics of two fluid phases under the influence of gravity has recently received much attention in the mathematics community. The numerous studies addressed the well-posedness issue [2, 3, 14, 18, 20, 31, 37, 38, 40], questions related to global existence of solutions [10, 15–17, 24, 32, 38, 41] and to singularity formation [11, 13], but also the modeling and dynamics of such flows in an inhomogeneous porous medium [9, 34, 42].

For the multiphase Muskat problem (1.1) considered herein, much less is known. This setting has been studied before in three dimensions in [21], where the authors established a local existence and uniqueness result in $H^k(\mathbb{R}^2)$ with $k \geq 4$. Moreover, it is shown in [21] that for solutions which are not global but bounded in $C^{1+\gamma}(\mathbb{R}^2)$, where $\gamma \in (0,1)$, the fluid interfaces cannot touch along a curve segment when the time approaches the maximal existence time, thus excluding the occurrence of so-called squirt singularities. Moreover, in the context of the two-dimensional multiphase Muskat problem (1.1), it has been shown in [33] that control of the curvature of the interfaces prevents also the formation of so-called splash singularities, that is, single point collisions of the interfaces.

It is worth pointing out that in the framework of the one-phase Muskat problem, splash singularities are one of the blow-up mechanisms, see [12, 23], where different initial geometries that lead to finite time splash singularities are presented, whereas self-intersection of the interface along a curve segment cannot occur in finite time, see [22]. A related scenario has been considered in two dimensions, but in a periodic setting, and with one of the fluids being air at uniform pressure. An example of this can be found in [27, 28], where the well-posedness and the stability of equilibria are investigated.

Similarly as in the three-dimensional case [21], we show herein that problem (1.1) can be expressed as a nonlinear, nonlocal, and strongly coupled evolution problem with nonlinearities described by contour integrals, cf. (1.10) below. The equivalence of the formulations (1.1) and (1.10) is rigorously established in Theorem 1.1 below, by making use of the results from Appendix A. The analysis in Appendix A, where in particular we extend Privalov's theorem to contour integrals over unbounded graphs (see Theorem A.3), also motivates the choice of homogeneous Sobolev spaces in the study of the Muskat problem [2]. Our second main result stated in Theorem 1.2 establishes the well-posedness of the problem in the subcritical Sobolev spaces $H^s(\mathbb{R})^2$ with $s \in (3/2, 2)$. It also provides two parabolic smoothing properties. Finally, in Proposition 1.3 we show for bounded solutions with finite existence time that the fluid interfaces intersect when the time approaches the maximal existence time at least in one point, but we exclude also in this two-dimensional scenario the formation of squirt singularities.

Compared to the two-phase Muskat problem, cf. [20], new difficulties arise from the fact that the coupling terms in (1.10) are of highest order. However, based on the mapping properties established in Section 2, we prove that the linearized operator, which is represented as a 2×2 matrix (see Section 3), has lower order off-diagonal entries. A similar feature has been evinced for the Muskat problem investigated in [28]. The benefit of this weak coupling at the level of the linearization is that only the diagonal terms need to be considered when establishing parabolicity for the problem. Once this is done, we can make use of the abstract parabolic theory from [36] in the study of this multiphase Muskat problem.

Notation

Given $k, n \in \mathbb{N}$ and an open set $\Omega \subset \mathbb{R}^n$, we denote by $C^k(\Omega)$ the space consisting of real-valued k-time continuously differentiable functions on Ω, and $UC^k(\Omega)$ is the subspace of $C^k(\Omega)$ having functions with uniformly continuous derivatives up to order k as elements. Moreover, $BUC^k(\Omega)$ is the Banach space of functions with bounded and uniformly continuous derivatives up to order k. Finally, given $\alpha \in (0,1)$, we set

$$\mathrm{BUC}^{k+\alpha}(\Omega) := \Big\{ f \in \mathrm{BUC}^k(\Omega) : [\partial^\beta f]_\alpha := \sup_{x \neq y} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^\alpha} < \infty \ \forall\, |\beta| = k \Big\}.$$

Given Banach spaces X and Y, the space $\mathrm{C}^{1-}(X,Y)$ consists of all locally Lipschitz maps from X to Y. Moreover, we write $A \in \mathcal{L}^k_{\mathrm{sym}}(X,Y)$ if $A:\ X^k \to Y$ is k-linear, bounded, and symmetric.

Solving the fixed time problem

A remarkable property of problem (1.1) is the fact that equations (1.1a)–(1.1c) are linear and have constant coefficients. This property enables us to identify the velocity field in terms of the a priori unknown functions f and h by means of contour integrals. Such an approach has been followed in the context of the Muskat problem already in the 1980s, at least at a formal level, cf. [25]. For the clarity of the exposition, we omit in this part the time-dependence and write $(\cdot)'$ for the x-derivative of functions that depend only on x. In Theorem 1.1 below we provide, under suitable regularity constraints, an explicit formula for the velocity field in terms of $X := (f,h)$. Our approach generalizes the one followed in [38] in the context of the two-phase Muskat problem and strongly relies on results from Appendix A.

Theorem 1.1. *Let $r \in (3/2, 2)$, $c_\infty > 0$, and $f, h \in H^r(\mathbb{R})$ with $c_\infty + f > h$ be given. The boundary value problem*

$$\left.\begin{aligned} v_i &= -\frac{k}{\mu}\big(\nabla p_i + (0, \rho_i g)\big) && \text{in } \Omega_i,\ 1 \le i \le 3,\\ \operatorname{div} v_i &= 0 && \text{in } \Omega_i,\ 1 \le i \le 3,\\ p_i &= p_{i+1} && \text{on } \partial\Omega_i \cap \partial\Omega_{i+1},\ i = 1,2,\\ \langle v_i | \nu_i \rangle &= \langle v_{i+1} | \nu_i \rangle && \text{on } \partial\Omega_i \cap \partial\Omega_{i+1},\ i = 1,2,\\ v_i(x,y) &\to 0 && \text{for } |(x,y)| \to \infty,\ 1 \le i \le 3 \end{aligned}\right\} \tag{1.2}$$

has a unique solution[1] $(v_1, v_2, v_3, p_1, p_2, p_3)$ *with*

$$v_i \in \mathrm{BUC}(\Omega_i) \cap \mathrm{C}^\infty(\Omega_i) \quad \text{and} \quad p_i \in \mathrm{UC}^1(\Omega_i) \cap \mathrm{C}^\infty(\Omega_i), \qquad 1 \le i \le 3.$$

Moreover, setting $v := v_1 \mathbf{1}_{\Omega_1} + v_2 \mathbf{1}_{\Omega_2} + v_3 \mathbf{1}_{\Omega_3}$, it holds for $z := (x,y) \in \mathbb{R}^2 \setminus (\Gamma_h \cup \Gamma_f^{c_\infty})$ that

$$\begin{aligned} v(z) &= \frac{\Theta_1}{\pi} \int_{\mathbb{R}} \frac{(c_\infty + f(s) - y, x - s)}{(x-s)^2 + (y - c_\infty - f(s))^2} f'(s)\, ds\\ &\quad + \frac{\Theta_2}{\pi} \int_{\mathbb{R}} \frac{(h(s) - y, x - s)}{(x-s)^2 + (y - h(s))^2} h'(s)\, ds, \end{aligned} \tag{1.3}$$

[1] The pressures (p_1, p_2, p_3) are unique only up to the same additive constant.

with constants

$$\Theta_1 := \frac{kg(\rho_1 - \rho_2)}{2\mu} \quad \text{and} \quad \Theta_2 := \frac{kg(\rho_2 - \rho_3)}{2\mu}. \tag{1.4}$$

Proof. We devise the proof into two steps.

Existence. Let $v : \mathbb{R}^2 \setminus (\Gamma_h \cup \Gamma_f^{c_\infty}) \to \mathbb{R}^2$ be given by (1.3) and set $v_i := v|_{\Omega_i}$, where $1 \leq i \leq 3$. In the notation from Appendix A, see (A.1), it holds that

$$v(z) = 2\Theta_1 v(f)[f'](z - (0, c_\infty)) + 2\Theta_2 v(h)[h'](z), \quad z \in \mathbb{R}^2 \setminus (\Gamma_h \cup \Gamma_f^{c_\infty}).$$

Then $v \in C^\infty(\mathbb{R}^2 \setminus (\Gamma_h \cup \Gamma_f^{c_\infty}))$ and, according to Theorem A.3, we also have that $v_i \in \mathrm{BUC}^{r-3/2}(\Omega_i)$ for $1 \leq i \leq 3$. Moreover, Lemma A.4 yields that $(1.2)_2$ and $(1.2)_5$ hold true. In view of Lemma A.1, we further get

$$\begin{aligned} v_i(x, c_\infty + f(x)) = {} & \frac{\Theta_1}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{(-\delta_{[x,s]} f, s)}{s^2 + (\delta_{[x,s]} f)^2} f'(x-s)\, ds \\ & + \frac{\Theta_2}{\pi} \int_{\mathbb{R}} \frac{(-\delta_{[x,s]} X, s)}{s^2 + (\delta_{[x,s]} X)^2} h'(x-s)\, ds \\ & + (-1)^i \Theta_1 \frac{f'(1, f')}{1 + f'^2}(x), \quad i = 1, 2, \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} v_i(x, h(x)) = {} & \frac{\Theta_1}{\pi} \int_{\mathbb{R}} \frac{(-\delta'_{[x,s]} X, s)}{s^2 + (\delta'_{[x,s]} X)^2} f'(x-s)\, ds \\ & + \frac{\Theta_2}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{(-\delta_{[x,s]} h, s)}{s^2 + (\delta_{[x,s]} h)^2} h'(x-s)\, ds \\ & + (-1)^{i+1} \Theta_2 \frac{h'(1, h')}{1 + h'^2}(x), \quad i = 2, 3, \end{aligned} \tag{1.6}$$

where PV is the principal value and, setting $X := (f, h)$, we defined

$$\begin{aligned} \delta_{[x,s]} f &:= f(x) - f(x-s), \\ \delta_{[x,s]} X &:= c_\infty + f(x) - h(x-s), \\ \delta'_{[x,s]} X &:= h(x) - c_\infty - f(x-s). \end{aligned} \tag{1.7}$$

The formulas (1.5) and (1.6) now show the validity of $(1.2)_4$.

We next define pressures $p_i : \Omega_i \to \mathbb{R}$, $1 \leq i \leq 3$, by the formula

$$p_i(x, y) := -\frac{\mu}{k}\Big(\int_0^x \langle v_i(s, d_i(s)) | (1, d_i'(s))\rangle\, ds + \int_{d_i(x)}^y v_i^2(x, s)\, ds\Big) - \rho_i g y + c_i, \tag{1.8}$$

where $v_i := (v_i^1, v_i^2)$ and $c_i \in \mathbb{R}$ are constants, and with

$$d_1 := \|f\|_\infty + c_\infty + 1, \quad d_2 := \frac{1}{2}(c_\infty + f + h), \quad d_3 := -\|h\|_\infty - 1.$$

Taking advantage of $\partial_y v^1 = \partial_x v^2$ in $\mathbb{R}^2 \setminus (\Gamma_h \cup \Gamma_f^{c_\infty})$, cf. Lemma A.4, we deduce that $p_i \in \mathrm{C}^1(\Omega_i)$ and that $(1.2)_1$ is satisfied. The regularity properties established for v_i now imply that indeed $p_i \in \mathrm{UC}^1(\Omega_i) \cap \mathrm{C}^\infty(\Omega_i)$, $1 \le i \le 3$. Then, $(1.2)_1$ and (1.5)–(1.6), it immediately follows that $(p_1 - p_2)|_{\Gamma_f^{c_\infty}}$ and $(p_2 - p_3)|_{\Gamma_h}$ are constants. Hence, for a suitable choice of c_i, we may achieve that $(1.2)_3$ is satisfied. Therewith, we established the existence of at least one solution to system (1.2).

Uniqueness. It remains to show that the system (1.2) has, when setting the gravity constant g equal to zero, only the trivial solutions defined by $v = (v^1, v^2) = 0$ and $p = c \in \mathbb{R}$. To begin, we note that $(1.2)_1$ implies $\partial_y v_i^1 - \partial_x v_i^2 = 0$ in Ω_i,, where $1 \le i \le 3$. Moreover, combining $(1.2)_1$, $(1.2)_3$ and $(1.2)_4$, we obtain that $v \in \mathrm{BUC}(\mathbb{R}^2)$. Stokes' theorem then yields

$$\partial_y v^1 - \partial_x v^2 = 0 \tag{1.9}$$

in $\mathscr{D}'(\mathbb{R}^2)$. We next set $\Psi := \psi_1 \mathbf{1}_{\overline{\Omega_1}} + \psi_2 \mathbf{1}_{\overline{\Omega_2}} + \psi_3 \mathbf{1}_{\overline{\Omega_3}}$, where $\psi_i : \overline{\Omega_i} \to \mathbb{R}$ are given by

$$\psi_i(x, y) := \int_{h(x)}^{y} v^1(x, s)\, ds - \int_0^x \langle v(s, h(s)) | (-h'(s), 1)\rangle\, ds, \quad i = 2, 3,$$

and

$$\psi_1(x, y) := \int_{c_\infty + f(x)}^{y} v^1(x, s)\, ds + \psi_2(x, c_\infty + f(x)).$$

It follows immediately that $\Psi \in \mathrm{C}(\mathbb{R}^2)$. Additionally, using Stokes's theorem and $(1.2)_2$, we may show that $\nabla \psi_i = (-v^2, v^1)$ in $\mathscr{D}'(\Omega_i)$, where $1 \le i \le 3$. As a direct consequence, we get $\psi_i \in \mathrm{UC}^1(\Omega_i)$ where $1 \le i \le 3$. Additionally, $\nabla \Psi \in \mathscr{D}'(\mathbb{R}^2)$ belongs to $\mathrm{BUC}(\mathbb{R}^2)$, hence $\Psi \in \mathrm{UC}^1(\mathbb{R}^2)$. Therefore, given $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^2)$, we have

$$\begin{aligned}\langle \Delta\Psi, \varphi\rangle = \int_{\mathbb{R}^2} \Psi \Delta\varphi\, dz = -\int_{\mathbb{R}^2} \langle \nabla\Psi | \nabla\varphi\rangle\, dz &= \int_{\mathbb{R}^2} \langle (v^2, -v^1) | \nabla\varphi\rangle\, dz \\ &= \langle \partial_y v^1 - \partial_x v^2, \varphi\rangle,\end{aligned}$$

and (1.9) then yields $\Delta\Psi = 0$ in $\mathscr{D}'(\mathbb{R}^2)$. Consequently, Ψ is the real part of a holomorphic function $u : \mathbb{C} \to \mathbb{C}$. Since u' is also holomorphic and $u' = (\partial_x \Psi, -\partial_y \Psi) = -(v^2, v^1)$ is bounded, cf. $(1.2)_5$, Liouville's theorem yields $u' = 0$, hence $v = 0$. Moreover, in view of $(1.2)_1$, we now obtain that $\nabla p = 0$ in $\mathbb{R}^2$, meaning that p is constant in $\mathbb{R}^2$. This completes our arguments. ∎

The contour integral formulation and the main results

Concerning the multiphase Muskat problem (1.1), Theorem 1.1 implies that if, at any given time $t \ge 0$, $f(t)$ and $h(t)$ belong to $H^r(\mathbb{R})$, with $r \in (3/2, 2)$, and $c_\infty + f(t) > h(t)$, then $v_1(t)|_{\Gamma_f^{c_\infty}(t)}$ and $v_2(t)|_{\Gamma_h(t)}$ are given by (1.5) and (1.6). Recalling also (1.1d), we can thus formulate the moving boundary problem (1.1) as an autonomous evolution problem for the pair $X := (f, h)$ which reads as

$$\frac{dX(t)}{dt} = \Phi(X(t)), \quad t \geq 0, \qquad X(0,\cdot) = X_0 := (f_0, h_0), \tag{1.10}$$

where the nonlinear operator $\Phi = (\Phi_1, \Phi_2)$ is defined by

$$\Phi_1(X) := \Theta_1 \mathbb{B}(f)[f'] + \frac{\Theta_2}{\pi}\big((c_\infty + f)f' C_1(X)[h'] - f' C_1(X)[hh'] + D_1(X)[h']\big) \tag{1.11}$$

and

$$\Phi_2(X) := \Theta_2 \mathbb{B}(h)[h'] + \frac{\Theta_1}{\pi}\big((h - c_\infty)h' C_1'(X)[f'] - h' C_1'(X)[ff'] + D_1'(X)[f']\big). \tag{1.12}$$

The constants Θ_i, $i = 1, 2$, are introduced in (1.4) and, given $u \in H^r(\mathbb{R})$, with $r \in (3/2, 2)$ which is fixed in the remaining part, we denote by $\mathbb{B}(u)$ the linear operator

$$\mathbb{B}(u) := \frac{1}{\pi}\big(B^0_{0,1}(u) + u' B^0_{1,1}(u)\big), \tag{1.13}$$

where the operators $B^0_{m,1}$, $m = 0,\ 1$, as well as C_1, C_1', D_1, and D_1' are defined in (2.1)–(2.2) and (2.3) below. We shall treat (1.10) as a fully nonlinear evolution problem in $H^{r-1}(\mathbb{R})^2$. To this end, we prove in Corollary 2.7 below that Φ is smooth, that is,

$$\Phi \in \mathrm{C}^\infty(\mathcal{O}_r, H^{r-1}(\mathbb{R})^2), \tag{1.14}$$

where

$$\mathcal{O}_r := \{(f, h) \in H^r(\mathbb{R})^2 \ :\ c_\infty + f > h\}. \tag{1.15}$$

Moreover, our analysis (see Proposition 3.2 below) will disclose that (1.10) is of parabolic type in the phase space $\mathcal{O}_r$; the Fréchet derivative $\partial\Phi(X)$ at any $X \in \mathcal{O}_r$, viewed as an unbounded operator in $H^{r-1}(\mathbb{R})^2$ with domain $H^r(\mathbb{R})^2$, is the generator of an analytic semigroup in $\mathcal{L}(H^{r-1}(\mathbb{R})^2)$. In the notation introduced in [5], the latter property writes as

$$-\partial\Phi(X) \in \mathcal{H}(H^r(\mathbb{R})^2, H^{r-1}(\mathbb{R})^2). \tag{1.16}$$

Properties (1.14) and (1.16) enable us to use the parabolic theory presented in [36] to establish the following results for (1.10), the proofs of which are postponed to the very end of Section 3.

Theorem 1.2. *Let $r \in (3/2, 2)$. Given $X_0 \in \mathcal{O}_r$, the multiphase Muskat problem* (1.10) *has a unique maximal solution $X := X(\cdot, X_0)$ such that*

$$X \in \mathrm{C}([0, T^+), \mathcal{O}_r) \cap \mathrm{C}^1([0, T^+), H^{r-1}(\mathbb{R})^2),$$

with $T^+ = T^+(X_0) \in (0, \infty]$ denoting the maximal time of existence. Moreover, we have:

(i) *The solution depends continuously on the initial data;*

(ii) *Given* $k \in \mathbb{N}$, *we have* $X \in \mathrm{C}^\infty((0,T^+)\times\mathbb{R},\mathbb{R}^2)\cap \mathrm{C}^\infty((0,T^+),H^k(\mathbb{R})^2)$;

(iii) *If* $T^+<\infty$, *then*

$$\sup_{t\in[0,T^+)}\|X(t)\|_{H^r(\mathbb{R})}=\infty \quad \textit{or} \quad \liminf_{t\to T^+}\operatorname{dist}(\Gamma_f^{c_\infty}(t),\Gamma_h(t))=0.$$

The next result shows, for bounded solutions with $T^+<\infty$, that the fluid interfaces intersect in at least one point along a sequence $t_n\to T^+$. Moreover, using the same strategy as in [19,21], we exclude for such solutions that the two fluid interfaces collapse along a curve segment.

Proposition 1.3. *Let* $X\in \mathrm{C}([0,T^+),\mathcal{O}_r)\cap \mathrm{C}^1([0,T^+),H^{r-1}(\mathbb{R})^2)$ *be a maximal solution to* (1.10) *with* $T^+<\infty$ *and such that, for some* $M>0$, $\|X(t)\|_{H^r}\le M$, *for all* $t\in[0,T^+)$. *Then there exists* $x_0\in\mathbb{R}$ *with the property that*

$$\liminf_{t\to T^+}(c_\infty+f(t,x_0)-h(t,x_0))=0. \tag{1.17}$$

Moreover, for each x_0 *satisfying* (1.17) *and for each* $\delta>0$, *we have*

$$\liminf_{t\to T^+}\sup_{|x-x_0|\le\delta}(c_\infty+f(t,x)-h(t,x))>0.$$

2. Mapping properties

In this section we introduce the operators $B^0_{m,1}$, $m=0,1$, and C_1, C_1', D_1, D_1' which appear in (1.11) and (1.12) in a more general context, and study the properties of these operators. The main goal is to establish the smoothness property (1.14), see Corollary 2.7 below.

Motivated by formulas (1.5) and (1.6), we introduce the family $B_{n,m}$ $(n,m\in\mathbb{N})$ of singular integral operators on the real line, where, given Lipschitz continuous maps $u_1,\ldots,u_m,v_1,\ldots,v_n:\mathbb{R}\to\mathbb{R}$ and $\overline{\omega}\in L_2(\mathbb{R})$, the operator $B_{n,m}$ is defined by

$$B_{n,m}(u_1,\ldots,u_m)[v_1,\ldots,v_n,\overline{\omega}](x):=\mathrm{PV}\int_{\mathbb{R}} c\,\frac{\prod_{i=1}^n\big(\delta_{[x,s]}u_i/s\big)}{\prod_{i=1}^m\Big[1+\big(\delta_{[x,s]}v_i/s\big)^2\Big]}\,\frac{\overline{\omega}(x-s)}{s}\,ds. \tag{2.1}$$

Here, we use the notation introduced in (1.7). Furthermore, we define

$$B^0_{n,m}(u)[\overline{\omega}]:=B_{n,m}(u,\ldots,u)[u,\ldots,u,\overline{\omega}]. \tag{2.2}$$

The operators $B_{n,m}$ were introduced in [38], but they are also important in the study of the Stokes problem, cf. [39]. It is important to point out that $B_{0,0}=\pi H$, where H denotes the Hilbert transform. We now recall some important properties of these operators.

Lemma 2.1. *Let $r \in (3/2, 2)$ be fixed.*

(i) *Given $u_1, \ldots, u_m \in H^r(\mathbb{R})$, there exists a constant C that depends only on n, m, r, and $\max_{1\le i\le m} \|u_i\|_{H^r}$ such that*

$$\|B_{n,m}(u_1, \ldots, u_m)[v_1, \ldots, v_n, \overline{\omega}]\|_{H^{r-1}} \le C\|\overline{\omega}\|_{H^{r-1}} \prod_{i=1}^{n} \|v_i\|_{H^r}$$

for all $v_1, \ldots, v_n \in H^r(\mathbb{R})$ and $\overline{\omega} \in H^{r-1}(\mathbb{R})$.

(ii) *The mapping $[u \mapsto B^0_{n,m}(u)] : H^r(\mathbb{R}) \to \mathcal{L}(H^{r-1}(\mathbb{R}))$ is smooth.*

Proof. Claim (i) is established in [1, Lemma 5], and claim (ii) is proven in [39, Corollary C.5]. ∎

The evolution equation (1.10) actually consists of one equation for f and one for h which are coupled. The coupling terms contain highest (first) order derivatives of both variables and they are expressed using the aforementioned operators C_1, C_1', D_1, D_1'. We now introduce these operators as elements of a larger family of operators enjoying similar properties.

Given $1 \le m \in \mathbb{N}$ and $X_i := (f_i, h_i) \in \mathcal{O}_r$, $1 \le i \le m$, we set

$$\begin{aligned}
C_m(X_1, \ldots, X_m)[\overline{\omega}](x) &:= \int_{\mathbb{R}} \frac{\overline{\omega}(x-s)}{\prod_{i=1}^m \left[s^2 + (\delta_{[x,s]}X_i)^2\right]}\, ds, \\
C_m'(X_1, \ldots, X_m)[\overline{\omega}](x) &:= \int_{\mathbb{R}} \frac{\overline{\omega}(x-s)}{\prod_{i=1}^m \left[s^2 + (\delta'_{[x,s]}X_i)^2\right]}\, ds, \\
D_m(X_1, \ldots, X_m)[\overline{\omega}](x) &:= \int_{\mathbb{R}} \frac{s\overline{\omega}(x-s)}{\prod_{i=1}^m \left[s^2 + (\delta_{[x,s]}X_i)^2\right]}\, ds, \\
D_m'(X_1, \ldots, X_m)[\overline{\omega}](x) &:= \int_{\mathbb{R}} \frac{s\overline{\omega}(x-s)}{\prod_{i=1}^m \left[s^2 + (\delta'_{[x,s]}X_i)^2\right]}\, ds,
\end{aligned} \tag{2.3}$$

for $\overline{\omega} \in L_2(\mathbb{R})$ and $x \in \mathbb{R}$, where we used the notation introduced in (1.7). Since $X_i \in \mathcal{O}_r$ for $1 \le i \le m$, these operators are no longer singular. However, for large values of the integration variable s, the kernels of D_1 and D_1' behave similarly to that of the truncated Hilbert transform $H_\delta : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, with $\delta > 0$, which is defined by

$$H_\delta[\overline{\omega}](x) := \frac{1}{\pi} \int_{\{|s|>\delta\}} \frac{\overline{\omega}(x-s)}{s}\, ds, \quad x \in \mathbb{R}. \tag{2.4}$$

We recall that, given $\delta > 0$, H_δ is a Fourier multiplier with symbol $[\xi \mapsto m_\delta(\xi)]$ given by

$$m_\delta(\xi) := -\frac{2}{\pi} i \,\mathrm{sign}(\xi) \int_{\delta|\xi|}^{\infty} \frac{\sin(t)}{t}\, dt.$$

Since $\|m_\delta\|_\infty \le 2$ for all $\delta > 0$, it follows that

$$\|H_\delta\|_{\mathcal{L}(L_2(\mathbb{R}))} \le 2. \tag{2.5}$$

We next study the mapping properties of the operators C_m, C_m', D_m, D_m'.

Lemma 2.2. *Given* $1 \le m \in \mathbb{N}$ *and* $X_i := (f_i, h_i) \in \mathcal{O}_r$, $1 \le i \le m$, *we set*

$$c_0 := \min_{1\le i\le m} \operatorname{dist}(\Gamma_{f_i}^{c\infty}, \Gamma_{h_i}). \tag{2.6}$$

Then, there exists a constant C *that depends only on* m, c_0, *and* $\max_{1\le i\le m} \|h_i'\|_\infty$ *such that*

$$\|C_m(X_1,\ldots,X_m)[\overline{\omega}]\|_2 + \|C_m'(X_1,\ldots,X_m)[\overline{\omega}]\|_2 \le C\|\overline{\omega}\|_2, \quad \overline{\omega} \in L_2(\mathbb{R}). \tag{2.7}$$

Proof. Let

$$\delta := \frac{c_0}{2(\max_{1\le i\le m} \|h_i'\|_\infty + 1)}. \tag{2.8}$$

Given $x \in \mathbb{R}$ and $|s| < \delta$, the following holds:

$$\min\{|\delta_{[x,s]}X_i|, |\delta'_{[x,s]}X_i|\} \ge \operatorname{dist}(\Gamma_{f_i}^{c\infty}, \Gamma_{h_i}) - |\delta_{[x,s]}h_i| \ge c_0/2, \quad 1 \le i \le m. \tag{2.9}$$

Therefore, when considering C_m (the case C_m' is similar), we can make use of Minkowski's integral inequality to obtain

$$\begin{aligned}
\|C_m(X_1,\ldots,X_m)[\overline{\omega}]\|_2 &= \Big(\int_{\mathbb{R}} \Big| \int_{\mathbb{R}} \frac{\overline{\omega}(x-s)}{\prod_{i=1}^m \big[s^2 + (\delta_{[x,s]}X_i)^2\big]}\, ds\Big|^2 dx\Big)^{1/2} \\
&\le \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \Big|\frac{\overline{\omega}(x-s)}{\prod_{i=1}^m \big[s^2 + (\delta_{[x,s]}X_i)^2\big]}\Big|^2 dx\Big)^{1/2} ds \\
&\le \Big(\frac{2}{c_0}\Big)^{2m} \int_{\{|s|<\delta\}} \|\overline{\omega}\|_2\, ds + \int_{\{|s|>\delta\}} \frac{\|\overline{\omega}\|_2}{s^{2m}}\, ds \\
&\le C\|\overline{\omega}\|_2,
\end{aligned}$$

and (2.7) follows. ∎

It is not difficult to extend the proof of Lemma 2.2 in the context of the operators D_m and D_m' with $m \ge 2$. The case $m = 1$ is more subtle, however, and requires a different strategy which uses the estimate (2.5).

Lemma 2.3. *Given* $1 \le m \in \mathbb{N}$ *and* $X_i := (f_i, h_i) \in \mathcal{O}_r$, $1 \le i \le m$, *let* $c_0 > 0$ *be the constant defined in* (2.6). *Then, there exists a constant* C *that depends only on* r, m, c_0, *and* $\max_{1\le i\le m} \|X_i\|_{H^r}$ *such that*

$$\|D_m(X_1,\ldots,X_m)[\overline{\omega}]\|_2 + \|D_m'(X_1,\ldots,X_m)[\overline{\omega}]\|_2 \le C\|\overline{\omega}\|_2, \quad \overline{\omega} \in L_2(\mathbb{R}), \tag{2.10}$$

Proof. The proof in the case $m \ge 2$ follows along the lines of the proof of Lemma 2.2. We now consider the operator D_1 (the estimate for D_1' follows similarly). Let $\delta > 0$ be as defined in (2.8) (with $m = 1$) and set

$$I(x,s) := \frac{s\overline{\omega}(x-s)}{s^2 + (\delta_{[x,s]}X_1)^2} = \Big(1 - \frac{(\delta_{[x,s]}X_1/s)^2}{1 + (\delta_{[x,s]}X_1/s)^2}\Big)\frac{\overline{\omega}(x-s)}{s}, \quad x, s \in \mathbb{R}, s \ne 0.$$

With H_δ denoting the truncated Hilbert transform, see (2.4), we have for $x \in \mathbb{R}$ that

$$|D_1(X_1)[\overline{\omega}](x)| \le \int_{\{|s|<\delta\}} |I(x,s)|\,ds + |H_\delta[\overline{\omega}](x)| \\ + \int_{\{\delta<|s|\}} \frac{(\delta_{[x,s]}X_1/s)^2}{1+(\delta_{[x,s]}X_1/s)^2}\Big|\frac{\overline{\omega}(x-s)}{s}\Big|\,ds.$$

Given $|s|<\delta$, (2.9) implies $|\delta_{[x,s]}X_1| \ge c_0/2$ and Minkowski's integral inequality then yields

$$\Big\|\int_{\{|s|<\delta\}} |I(\cdot,s)|\,ds\Big\|_2 \le \int_{\{|s|<\delta\}}\Big(\int_{\mathbb{R}} |I(x,s)|^2\,dx\Big)^{1/2} ds \le \frac{8\delta^2}{c_0^2}\|\overline{\omega}\|_2.$$

Moreover, taking into account that $|\delta_{[\cdot,s]}X_1| \le c_\infty + \|f_1\|_\infty + \|h_1\|_\infty$, Minkowski's integral inequality leads us to

$$\Big\|\int_{\{\delta<|s|\}} \frac{(\delta_{[\cdot,s]}X_1/s)^2}{1+(\delta_{[\cdot,s]}X_1/s)^2}\Big|\frac{\overline{\omega}(\cdot-s)}{s}\Big|\,ds\Big\|_2 \\ \le (c_\infty + \|f_1\|_\infty + \|h_1\|_\infty)\int_{\{\delta<|s|\}} \frac{\|\overline{\omega}\|_2}{s^2}\,ds \le C\|\overline{\omega}\|_2.$$

Recalling (2.5), we conclude that (2.10) is satisfied. ∎

As a consequence of Lemma 2.2 and Lemma 2.3 we obtain the following result.

Corollary 2.4. *Given $1 \le m \in \mathbb{N}$, it holds that*

$$C_m, D_m, C'_m, D'_m \in \mathrm{C}^{1-}(\mathcal{O}_r^m, \mathcal{L}(L_2(\mathbb{R}))). \tag{2.11}$$

Proof. Let $X_i = (f_i, h_i)$, $\widetilde{X}_i = (\widetilde{f}_i, \widetilde{h}_i) \in \mathcal{O}_r$, for $1 \le i \le m$, $\overline{\omega} \in L_2(\mathbb{R})$, and choose $E_m \in \{C_m, D_m\}$. It then follows that

$$E_m(X_1,\ldots,X_m)[\overline{\omega}] - E_m(\widetilde{X}_1,\ldots,\widetilde{X}_m)[\overline{\omega}] \\ = \sum_{j=1}^m \Big((2c_\infty + \widetilde{f}_j + f_j)(\widetilde{f}_j - f_j)E_{m+1}(\widetilde{X}_1,\ldots,\widetilde{X}_j, X_j,\ldots,X_m)[\overline{\omega}] \\ - (\widetilde{f}_j - f_j)E_{m+1}(\widetilde{X}_1,\ldots,\widetilde{X}_j, X_j,\ldots,X_m)[(\widetilde{h}_j + h_j)\overline{\omega}] \\ - (2c_\infty + \widetilde{f}_j + f_j)E_{m+1}(\widetilde{X}_1,\ldots,\widetilde{X}_j, X_j,\ldots,X_m)[(\widetilde{h}_j - h_j)\overline{\omega}] \\ + E_{m+1}(\widetilde{X}_1,\ldots,\widetilde{X}_j, X_j,\ldots,X_m)[(\widetilde{h}_j^2 - h_j^2)\overline{\omega}]\Big), \tag{2.12}$$

respectively

$$E'_m(X_1,\ldots,X_m)[\overline{\omega}] - E'_m(\widetilde{X}_1,\ldots,\widetilde{X}_m)[\overline{\omega}] \\ = \sum_{j=1}^m \Big((\widetilde{h}_j^2 - h_j^2)E'_{m+1}(\widetilde{X}_1,\ldots,\widetilde{X}_j, X_j,\ldots,X_m)[\overline{\omega}]$$

$$
\begin{aligned}
&- (\widetilde{h}_j - h_j)E'_{m+1}(\widetilde{X}_1, \dots, \widetilde{X}_j, X_j, \dots, X_m)[(2c_\infty + \widetilde{f}_j + f_j)\overline{\omega}] \\
&- (\widetilde{h}_j + h_j)E'_{m+1}(\widetilde{X}_1, \dots, \widetilde{X}_j, X_j, \dots, X_m)[(\widetilde{f}_j - f_j)\overline{\omega}] \\
&+ E'_{m+1}(\widetilde{X}_1, \dots, \widetilde{X}_j, X_j, \dots, X_m)[(2c_\infty + \widetilde{f}_j + f_j)(\widetilde{f}_j - f_j)\overline{\omega}]\Big). \qquad (2.13)
\end{aligned}
$$

Combining equations (2.12) and (2.13), Lemma 2.2, and Lemma 2.3, we conclude that (2.11) holds true. ∎

We prove next that the operators C_m, C'_m, D_m, D'_m map, for given $X_i \in \mathcal{O}_r$, $1 \le i \le m$, the definition domain $L_2(\mathbb{R})$ actually into $H^1(\mathbb{R})$.

Lemma 2.5. *Given $1 \le m \in \mathbb{N}$ and $X_i := (f_i, h_i) \in \mathcal{O}_r$, $1 \le i \le m$, let $c_0 > 0$ be the constant defined in (2.6). Given $E_m \in \{C_m, D_m\}$, there exists a positive constant C that depends only on r, m, c_0, and $\max_{1 \le i \le m} \|X_i\|_{H^r}$ such that*

$$
\|E_m(X_1, \dots, X_m)[\overline{\omega}]\|_{H^1} + \|E'_m(X_1, \dots, X_m)[\overline{\omega}]\|_{H^1} \le C\|\overline{\omega}\|_2, \quad \overline{\omega} \in L_2(\mathbb{R}). \qquad (2.14)
$$

Moreover, $C_m, D_m, C'_m, D'_m \in \mathrm{C}^{1-}(\mathcal{O}_r^m, \mathcal{L}(L_2(\mathbb{R}), H^1(\mathbb{R})))$.

Proof. Let $\{\tau_\xi\}_{\xi \in \mathbb{R}}$ denote the group of right translations, and assume that $\overline{\omega} \in \mathrm{C}_0^\infty(\mathbb{R})$. Given $0 \ne \xi \in \mathbb{R}$, the formula (2.12) leads us to

$$
\begin{aligned}
&\frac{E_m(X_1, \dots, X_m)[\overline{\omega}] - \tau_\xi(E_m(X_1, \dots, X_m)[\overline{\omega}])}{\xi} \\
&= E_m(X_1, \dots, X_m)\Big[\frac{\overline{\omega} - \tau_\xi\overline{\omega}}{\xi}\Big] \\
&\quad + \sum_{j=1}^m \Big((2c_\infty + \tau_\xi f_j + f_j)\frac{\tau_\xi f_j - f_j}{\xi} E_{m+1}(\tau_\xi X_1, \dots, \tau_\xi X_j, X_j, \dots, X_m)[\tau_\xi \overline{\omega}] \\
&\quad - \frac{\tau_\xi f_j - f_j}{\xi} E_{m+1}(\widetilde{X}_1, \dots, \widetilde{X}_j, X_j, \dots, X_m)[(\tau_\xi h_j + h_j)\tau_\xi\overline{\omega}] \\
&\quad - (2c_\infty + \tau_\xi f_j + f_j)E_{m+1}(\tau_\xi X_1, \dots, \tau_\xi X_j, X_j, \dots, X_m)\Big[\frac{\tau_\xi h_j - h_j}{\xi}\tau_\xi\overline{\omega}\Big] \\
&\quad + E_{m+1}(\tau_\xi X_1, \dots, \tau_\xi X_j, X_j, \dots, X_m)\Big[\frac{(\tau_\xi h_j)^2 - h_j^2}{\xi}\tau_\xi\overline{\omega}\Big]\Big).
\end{aligned}
$$

Recalling (2.11), we may pass to the limit $\xi \to 0$ on the right of the latter equation and conclude that $E_m(X_1, \dots, X_m)[\overline{\omega}] \in H^1(\mathbb{R})$, with

$$
\begin{aligned}
(E_m(X_1, \dots, X_m)[\overline{\omega}])' &= E_m(X_1, \dots, X_m)[\overline{\omega}'] \\
&\quad - 2\sum_{j=1}^m \Big((c_\infty + f_j)f'_j E_{m+1}(X_1, \dots, X_m, X_j)[\overline{\omega}] - f'_j E_{m+1}(X_1, \dots, X_m, X_j)[h_j\overline{\omega}] \\
&\quad - (c_\infty + f_j)E_{m+1}(X_1, \dots, X_m, X_j)[h'_j\overline{\omega}] + E_{m+1}(X_1, \dots, X_m, X_j)[h_j h'_j\overline{\omega}]\Big).
\end{aligned}
$$

Except for $E_m(X_1,\dots,X_m)[\overline{\omega}']$, all the terms in the latter expression are well-defined and belong to $L_2(\mathbb{R})$ as long as $\overline{\omega}\in L_2(\mathbb{R})$. However, using integration by parts, we can rewrite this term as

$$\begin{aligned} C_m(X_1,\dots,X_m)[\overline{\omega}'] = -2\sum_{j=1}^{m}\big(&D_{m+1}(X_1,\dots,X_m,X_j)[\overline{\omega}]\\ &+(c_\infty+f_j)C_{m+1}(X_1,\dots,X_m,X_j)[h_j'\overline{\omega}]\\ &-C_{m+1}(X_1,\dots,X_m,X_j)[h_jh_j'\overline{\omega}]\big), \end{aligned}$$

respectively

$$\begin{aligned} D_m(X_1,\dots,X_m)[\overline{\omega}'] &= (1-2m)C_m(X_1,\dots,X_m)[\overline{\omega}]\\ &-2\sum_{j=1}^{m}\big((c_\infty+f_j)D_{m+1}(X_1,\dots,X_m,X_j)[h_j'\overline{\omega}]-D_{m+1}(X_1,\dots,X_m,X_j)[h_jh_j'\overline{\omega}]\\ &-(c_\infty+f_j)^2C_{m+1}(X_1,\dots,X_m,X_j)[\overline{\omega}]-C_{m+1}(X_1,\dots,X_m,X_j)[h_j^2\overline{\omega}]\\ &+2(c_\infty+f_j)C_{m+1}(X_1,\dots,X_m,X_j)[h_j\overline{\omega}]\big). \end{aligned}$$

Combining the last three identities, Lemma 2.2, Lemma 2.3, and using a standard density argument we get that (2.14) holds for E_m. The claim for the operator E_m' follows similarly. Finally, the Lipschitz continuity property is obtained from (2.14) and (2.12)–(2.13). ∎

Since our goal is to establish the smoothness of Φ, cf. (1.14), we next prove that operators E_m and E_m' with $E_m\in\{C_m,D_m\}$ depend smoothly on the variable $X\in\mathcal{O}_r$. This requires some additional notation. Given $Y:=(u,v)\in H^r(\mathbb{R})^2$, we set

$$\overline{\delta}_{[x,s]}Y:=u(x)-v(x-s)\quad\text{and}\quad \overline{\delta}'_{[x,s]}Y:=v(x)-u(x-s),\qquad x,s\in\mathbb{R}. \tag{2.15}$$

Given $n,m,p\in\mathbb{N}$, $m\geq 1$, $X_i\in\mathcal{O}_r$, $1\leq i\leq m+p$, $Y_i\in H^r(\mathbb{R})^2$, $1\leq i\leq n$, $\overline{\omega}\in L_2(\mathbb{R})$, and $E\in\{C,D\}$, we set

$$\begin{aligned} &E_{n,m,p}(X_1,\dots,X_{m+p})[Y_1,\dots,Y_n,\overline{\omega}](x)\\ &\quad:=\int_{\mathbb{R}}\frac{s^j\overline{\omega}(x-s)\big(\prod_{i=m+1}^{m+p}\delta_{[x,s]}X_i\big)\prod_{i=1}^{n}\overline{\delta}_{[x,s]}Y_i}{\prod_{i=1}^{m}\big[s^2+(\delta_{[x,s]}X_i)^2\big]}\,ds, \end{aligned}$$

and

$$\begin{aligned} &E'_{n,m,p}(X_1,\dots,X_{m+p})[Y_1,\dots,Y_n,\overline{\omega}](x)\\ &\quad:=\int_{\mathbb{R}}\frac{s^j\overline{\omega}(x-s)\big(\prod_{i=m+1}^{m+p}\delta'_{[x,s]}X_i\big)\prod_{i=1}^{n}\overline{\delta}'_{[x,s]}Y_i}{\prod_{i=1}^{m}\big[s^2+(\delta'_{[x,s]}X_i)^2\big]}\,ds \end{aligned}$$

for $x\in\mathbb{R}$, where $j=0$ if $E=C$ and $j=1$ for $E=D$. It is worth pointing out that, given $E\in\{C,D\}$ and $m\geq 1$, we have $E_{0,m,0}(X,\dots,X)=E_m(X,\dots,X)$ and

$E'_{0,m,0}(X,\dots,X) = E'_m(X,\dots,X)$. Hence, the latter formulas extend our previous notation introduced in (2.3). Setting $Y_i = (u_i, v_i)$ for $1 \le i \le n$, it holds that

$$\begin{aligned}
&E_{n,m,p}(X_1,\dots,X_{m+p})[Y_1,\dots,Y_n,\overline{\omega}]\\
&\quad= \sum_{S\subset\{1,\dots,n\}} (-1)^{|S^c|}\Big(\prod_{j\in S} u_j\Big) E_{0,m,p}(X_1,\dots,X_{m+p})\Big[\overline{\omega}\prod_{j\in S^c} v_j\Big],\\
&E'_{n,m,p}(X_1,\dots,X_{m+p})[Y_1,\dots,Y_n,\overline{\omega}]\\
&\quad= \sum_{S\subset\{1,\dots,n\}} (-1)^{|S^c|}\Big(\prod_{j\in S} v_j\Big) E'_{0,m,p}(X_1,\dots,X_{m+p})\Big[\overline{\omega}\prod_{j\in S^c} u_j\Big],
\end{aligned}$$

where for each $S \subset \{1,\dots,n\}$ we set $S^c := \{1,\dots,n\}\setminus S$.

Moreover, letting $X_j := (f_j, h_j)$, $m+1 \le j \le m+p$, it holds that

$$\begin{aligned}
&E_{0,m,p}(X_1,\dots,X_{m+p})[\overline{\omega}]\\
&\quad= \sum_{S\subset\{m+1,\dots,m+p\}} (-1)^{|S^c|}\Big(\prod_{j\in S}(c_\infty + f_j)\Big) E_m(X_1,\dots,X_m)\Big[\overline{\omega}\prod_{j\in S^c} h_j\Big],\\
&E'_{0,m,p}(X_1,\dots,X_{m+p})[\overline{\omega}]\\
&\quad= \sum_{S\subset\{m+1,\dots,m+p\}} (-1)^{|S^c|}\Big(\prod_{j\in S}(h_j - c_\infty)\Big) E'_m(X_1,\dots,X_m)\Big[\overline{\omega}\prod_{j\in S^c} f_j\Big].
\end{aligned}$$

Recalling Lemma 2.5, we deduce for $E \in \{C, C', D, D'\}$, that

$$\|E_{n,m,p}(X_1,\dots,X_{m+p})[Y_1,\dots,Y_n,\cdot]\|_{\mathcal{L}(L_2(\mathbb{R}),H^1(\mathbb{R}))} \le C\prod_{i=1}^n \|Y_i\|_{H^r}, \tag{2.16}$$

where C is independent from Y_i, for $1 \le i \le n$.

Finally, given $E \in \{C, C', D, D'\}$, $n, m, p \in \mathbb{N}$, $m \ge 1$, $Y_i \in H^r(\mathbb{R})^2$, $1 \le i \le n$, and $X \in \mathcal{O}_r$, we define

$$E^n_{m,p}(X)[Y_1,\dots,Y_n] := E_{n,m,p}(X,\dots,X)[Y_1,\dots,Y_n,\cdot] \in \mathcal{L}(L_2(\mathbb{R}), H^1(\mathbb{R})). \tag{2.17}$$

The estimate (2.16) shows that $E^n_{m,p} : \mathcal{O}_r \to \mathcal{L}^n_{\mathrm{sym}}(H^r(\mathbb{R})^2, \mathcal{L}(L_2(\mathbb{R}), H^1(\mathbb{R})))$ (if $n = 0$ we identify $\mathcal{L}^n_{\mathrm{sym}}(H^r(\mathbb{R})^2, \mathcal{L}(L_2(\mathbb{R}), H^1(\mathbb{R})))$ with $\mathcal{L}(L_2(\mathbb{R}), H^1(\mathbb{R}))$). In the next lemma, we establish the Fréchet differentiability of $E^n_{m,p}$.

Lemma 2.6. *Given $n, m, p \in \mathbb{N}$, $m \ge 1$, and $X \in \mathcal{O}_r$, the operator $E^n_{m,p}$ is Fréchet differentiable in X and its Fréchet derivative is given by*

$$\begin{aligned}
\partial E^n_{m,p}(X)[Y][Y_1,\dots,Y_n] &= pE^{n+1}_{m,p-1}(X)[Y_1,\dots,Y_n,Y]\\
&\quad - 2mE^{n+1}_{m+1,p+1}(X)[Y_1,\dots,Y_n,Y]
\end{aligned}$$

for $Y, Y_1,\dots,Y_n \in H^r(\mathbb{R})^2$. Consequently, for each $n, m, p \in \mathbb{N}$ with $m \ge 1$, we have $E^n_{m,p} \in \mathrm{C}^\infty(\mathcal{O}_r, \mathcal{L}^n_{\mathrm{sym}}(H^r(\mathbb{R})^2, \mathcal{L}(L_2(\mathbb{R}), H^1(\mathbb{R}))))$.

Proof. Setting

$$
\begin{aligned}
R(X,Y)[Y_1,\dots,Y_n] := {} & E^n_{m,p}(X+Y)[Y_1,\dots,Y_n] - E^n_{m,p}(X)[Y_1,\dots,Y_n] \\
& - pE^{n+1}_{m,p-1}(X)[Y_1,\dots,Y_n,Y] \\
& + 2mE^{n+1}_{m+1,p+1}(X)[Y_1,\dots,Y_n,Y],
\end{aligned}
$$

elementary algebraic manipulations lead us to the identity

$$
\begin{aligned}
R(X,Y)[Y_1,\dots,Y_n] = {} & \sum_{j=0}^{p-1}(p-j-1)R_{1,j}[Y_1,\dots,Y_n,Y,Y] \\
& - \sum_{j=0}^{m-1} R^a_{2,j}[Y_1,\dots,Y_n,Y,Y] \\
& - \sum_{j=0}^{m-1} R^b_{2,j}[Y_1,\dots,Y_n,Y,Y,Y] \\
& + \sum_{j=0}^{m-1}\sum_{l=0}^{m-j-1} R^a_{3,j,l}[Y_1,\dots,Y_n,Y,Y] \\
& + \sum_{j=0}^{m-1}\sum_{l=0}^{m-j-1} R^b_{3,j,l}[Y_1,\dots,Y_n,Y,Y,Y],
\end{aligned}
$$

where

$$
\begin{aligned}
R_{1,j} &:= E_{n+2,m,p-2}(\underbrace{X+Y,\dots,X+Y}_{m},\underbrace{X+Y,\dots,X+Y}_{j},\underbrace{X,\dots,X}_{p-2-j}),\\
R^a_{2,j} &:= (1+2p)E_{n+2,m+1,p}(\underbrace{X+Y,\dots,X+Y}_{m-j},\underbrace{X,\dots,X}_{j+1},\underbrace{X,\dots,X}_{p}),\\
R^b_{2,j} &:= pE_{n+3,m+1,p-1}(\underbrace{X+Y,\dots,X+Y}_{m-j},\underbrace{X,\dots,X}_{j+1},\underbrace{X,\dots,X}_{p-1}),\\
R^a_{3,j,l} &:= 4E_{n+2,m+2,p+2}(\underbrace{X+Y,\dots,X+Y}_{m-j-l},\underbrace{X,\dots,X}_{j+l+2},\underbrace{X,\dots,X}_{p+2}),\\
R^b_{3,j,l} &:= 2E_{n+3,m+2,p+1}(\underbrace{X+Y,\dots,X+Y}_{m-j-l},\underbrace{X,\dots,X}_{j+l+2},\underbrace{X,\dots,X}_{p+1}).
\end{aligned}
$$

Hence, for all Y sufficiently close to X in $H^r(\mathbb{R})^2$, it follows from Lemma 2.5, by arguing as in the derivation of (2.16), that

$$
\|R(X,Y)[Y_1,\dots,Y_n]\|_{\mathcal{L}(L_2(\mathbb{R}),H^1(\mathbb{R}))} \le C\|Y\|^2_{H^r}\prod_{i=1}^n \|Y_i\|_{H^r},
$$

and the claim follows. ∎

We complete this section by establishing (1.14).

Corollary 2.7. *It holds that* $\Phi \in \mathrm{C}^\infty(\mathcal{O}_r, H^{r-1}(\mathbb{R})^2)$.

Proof. The claim follows from (1.11)–(1.12), Lemma 2.1 (ii) and Lemma 2.6, by using the algebra property of $H^{r-1}(\mathbb{R})$ and the embedding $H^1(\mathbb{R}) \hookrightarrow H^{r-1}(\mathbb{R})$. ∎

3. The generator property and the proof of the main results

In the first part of this section, we show that the evolution problem (1.10) is parabolic in $\mathcal{O}_r$ by establishing the generator property (1.16). In the second part, we prove our main results. With respect to the first task, let $X = (f, h) \in \mathcal{O}_r$ be fixed. We can represent the Fréchet derivative $\partial\Phi(X)$ as the matrix operator

$$\partial\Phi(X) = \begin{pmatrix} \partial_f \Phi_1(X) & \partial_h \Phi_1(X) \\ \partial_f \Phi_2(X) & \partial_h \Phi_2(X) \end{pmatrix} \in \mathcal{L}(H^r(\mathbb{R})^2, H^{r-1}(\mathbb{R})^2).$$

Although the coupling terms in (1.11)–(1.12) involve highest order derivatives of both unknowns, Lemma 2.6 and (2.16) show that the off-diagonal entry $\partial_h \Phi_1(X)$ can be treated as being a perturbation. Indeed, recalling Lemma 2.6, we obtain for $E \in \{C, D\}$ that

$$\partial E_1(X)[Y] = -2E^1_{2,1}(X)[Y] = 2E_{0,2,1}(X, X, X)[v\,\cdot] - 2uE_{0,2,1}(X, X, X) \tag{3.1}$$

for all $Y = (u, v) \in H^r(\mathbb{R})^2$. Using this formula, it follows from (1.11) that

$$\begin{aligned} \partial_h \Phi_1(X)[v] = \frac{\Theta_2}{\pi}\Big(&(c_\infty + f) f'\big(C_1(X)[v'] + 2C_{0,2,1}(X, X, X)[vh']\big) \\ &- f'\big(C_1(X)[hv' + vh'] + 2C_{0,2,1}(X, X, X)[vhh']\big) \\ &+ D_1(X)[v'] + 2D_{0,2,1}(X, X, X)[vh']\Big) \end{aligned}$$

and (2.16) yields

$$\|\partial_h \Phi_1(X)[v]\|_{H^{r-1}} \le C\|v\|_{H^1} \quad \text{for all } v \in H^r(\mathbb{R}).$$

In view of [5, Theorem I.1.6.1] and of the property $\|v\|_{H^1} \le \nu\|v\|_{H^r} + C(\nu)\|v\|_{H^{r-1}}$, $v \in H^r(\mathbb{R})$, which holds for any given arbitrary small $\nu > 0$, we conclude that (1.16) is satisfied provided the diagonal entries are analytic generators, that is,

$$\begin{aligned} -\partial_f \Phi_1(X) &\in \mathcal{H}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R})), \\ -\partial_h \Phi_2(X) &\in \mathcal{H}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R})). \end{aligned} \tag{3.2}$$

To establish the generator property for $\partial_f \Phi_1(X)$, we follow the strategy from [1, Section 4] (see also [26, 30], where similar arguments are used in other contexts). To this end, we first deduce from Lemma 2.1 (ii) that the mapping

$$[u \mapsto \mathbb{B}(u)] : H^r(\mathbb{R}) \to \mathcal{L}(H^{r-1}(\mathbb{R}))$$

is smooth. In view of this property, of (3.1), and of (2.16) we infer from (1.11) that

$$\partial_f \Phi_1(X)[u] = \Theta_1\big(\mathbb{B}(f)[u'] + \partial\mathbb{B}(f)[u][f']\big) + a(X)u' + T_{\mathrm{lot}}[u],$$

where, according to Lemma 2.5,

$$a(X) := \frac{\Theta_2}{\pi}\big((c_\infty + f)C_1(X)[h'] - C_1(X)[hh']\big) \in H^1(\mathbb{R}).$$

Moreover, T_{lot} is a lower order operator. More precisely,

$$\|T_{\mathrm{lot}}[u]\|_{H^{r-1}} \le C\|u\|_{H^1} \quad \text{for all } u \in H^r(\mathbb{R}). \tag{3.3}$$

We next consider the continuous path $[\tau \mapsto \Psi(\tau)] : [0,1] \to \mathcal{L}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$ with

$$\Psi(\tau)[u] := \Theta_1\big(\mathbb{B}(\tau f)[u'] + \tau\partial\mathbb{B}(f)[u][f']\big) + \tau a(X)u' + \tau T_{\mathrm{lot}}[u].$$

Observe that $\Psi(1) = \partial_f \Phi_1(X)$, and $\Psi(0)$ is the Fourier multiplier

$$\Psi(0) = \Theta_1\mathbb{B}(0)\circ\frac{d}{dx} = \Theta_1 H\circ\frac{d}{dx} = \Theta_1\Big(-\frac{d^2}{dx^2}\Big)^{1/2}$$

with symbol $[\xi \mapsto \Theta_1|\xi|]$. The next step is to approximate $\Psi(\tau)$ locally by certain Fourier multipliers, see Lemma 3.1 below. Therefore, we choose for each $\varepsilon \in (0,1)$, a finite ε-localization family, that is, a family

$$\{\pi_j^\varepsilon : -N+1 \le j \le N\} \subset \mathrm{C}^\infty(\mathbb{R}, [0,1]),$$

with $N = N(\varepsilon) \in \mathbb{N}$ sufficiently large, such that

- $\operatorname{supp}\pi_j^\varepsilon$ is an interval of length ε for all $|j| \le N-1$, $\operatorname{supp}\pi_N^\varepsilon \subset \{|x| \ge 1/\varepsilon\}$;
- $\pi_j^\varepsilon \cdot \pi_l^\varepsilon = 0$ if $[|j-l| \ge 2, \max\{|j|,|l|\} \le N-1]$ or $[|l| \le N-2, j = N]$;
- $\sum_{j=-N+1}^{N}(\pi_j^\varepsilon)^2 = 1$; and
- $\|(\pi_j^\varepsilon)^{(k)}\|_\infty \le C\varepsilon^{-k}$ for all $k \in \mathbb{N}, -N+1 \le j \le N$.

To each finite ε-localization family we associate a family $\{\chi_j^\varepsilon : -N+1 \le j \le N\}$ with $\chi_j^\varepsilon \in \mathrm{C}^\infty(\mathbb{R}, [0,1])$, $-N+1 \le j \le N$, such that

- $\chi_j^\varepsilon = 1$ on $\operatorname{supp}\pi_j^\varepsilon$, $\operatorname{supp}\chi_N^\varepsilon \subset \{|x| \ge 1/\varepsilon - \varepsilon\}$; and
- $\operatorname{supp}\chi_j^\varepsilon$ is an interval of length 3ε and with the same midpoint as $\operatorname{supp}\pi_j^\varepsilon$ for $|j| \le N-1$.

Lemma 3.1. *Let $X \in \mathcal{O}_r$ be fixed and choose $r' \in (3/2, r)$. Given $\nu > 0$, there exist $\varepsilon \in (0,1)$, a finite ε-localization family $\{\pi_j^\varepsilon : -N+1 \le j \le N\}$, a constant $K = K(\varepsilon, X)$, and bounded operators $\mathbb{A}_{j,\tau} \in \mathcal{L}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$, $j \in \{-N+1,\ldots,N\}$ and $\tau \in [0,1]$, such that*

$$\|\pi_j^\varepsilon\Psi(\tau)[u] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon u]\|_{H^{r-1}} \le \nu\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}} \tag{3.4}$$

for all $-N+1 \leq j \leq N$, $\tau \in [0,1]$, *and* $u \in H^r(\mathbb{R})$. *The operators* $\mathbb{A}_{j,\tau}$ *are defined by*

$$\mathbb{A}_{j,\tau} := \alpha_\tau(x_j^\varepsilon)\Big(-\frac{d^2}{dx^2}\Big)^{1/2} + \beta_\tau(x_j^\varepsilon)\frac{d}{dx}, \quad |j| \leq N-1,$$

and

$$\mathbb{A}_{N,\tau} := \Theta_1\Big(-\frac{d^2}{dx^2}\Big)^{1/2},$$

where $x_j^\varepsilon \in \operatorname{supp}\pi_j^\varepsilon$, $|j| \leq N-1$, *and with functions* α_τ, β_τ *given by*

$$\alpha_\tau := \frac{1+(1-\tau)f'^2}{1+f'^2}\Theta_1, \quad \beta_\tau := \frac{\tau\Theta_1}{\pi}B^0_{1,1}(f)[f'] + \tau a(X).$$

Proof. As shown in the proof of [1, Theorem 7] (in a more general context), if ε is chosen sufficiently small, then for all $\tau \in [0,1]$ and $u \in H^r(\mathbb{R})$ we have

$$\begin{aligned}\Big\|\pi_j^\varepsilon\big(\mathbb{B}(\tau f)[u'] + \tau\partial\mathbb{B}(f)[u][f']\big) - \frac{\alpha_\tau(x_j^\varepsilon)}{\Theta_1}\Big(-\frac{d^2}{dx^2}\Big)^{1/2}[\pi_j^\varepsilon u]\\ - \frac{\tau}{\pi}B^0_{1,1}(f)[f'](x_j^\varepsilon)(\pi_j^\varepsilon u)'\Big\|_{H^{r-1}} \leq \frac{\nu}{2|\Theta_1|}\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}}\end{aligned}$$

for $|j| \leq N-1$, and

$$\begin{aligned}\Big\|\pi_N^\varepsilon\big(\mathbb{B}(\tau f)[u'] + \tau\partial\mathbb{B}(f)[u][f']\big) - \Big(-\frac{d^2}{dx^2}\Big)^{1/2}[\pi_N^\varepsilon u]\Big\|_{H^{r-1}}\\ \leq \frac{\nu}{2|\Theta_1|}\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}}.\end{aligned}$$

We next recall, see e.g., [1, Eq. 2.1], there exists a constant $C>0$ such that

$$\|ab\|_{H^{r-1}} \leq C(\|a\|_{H^{r-1}}\|b\|_\infty + \|a\|_\infty\|b\|_{H^{r-1}}) \quad \text{for all } a, b \in H^{r-1}(\mathbb{R}).$$

Using this estimate together with the identity $\chi_j^\varepsilon\pi_j^\varepsilon = \pi_j^\varepsilon$, $-N+1 \leq j \leq N$, we get in view of the relation $a(X) \in \mathrm{C}^{1/2}(\mathbb{R})$ that

$$\begin{aligned}&\|\pi_j^\varepsilon a(X)u' - a(X)(x_j^\varepsilon)(\pi_j^\varepsilon u)'\|_{H^{r-1}}\\ &\leq \|\chi_j^\varepsilon\big(a(X) - a(X)(x_j^\varepsilon)\big)(\pi_j^\varepsilon u)'\|_{H^{r-1}} + K\|u\|_{H^{r-1}}\\ &\leq C\|\chi_j^\varepsilon\big(a(X) - a(X)(x_j^\varepsilon)\big)\|_\infty\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}}\\ &\leq \frac{\nu}{4}\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}}, \quad |j| \leq N-1,\end{aligned}$$

if ε is sufficiently small, respectively, in view of the fact that $a(X)$ vanishes at infinity,

$$\begin{aligned}\|\pi_N^\varepsilon a(X)u'\|_{H^{r-1}} &\leq \|\chi_N^\varepsilon a(X)(\pi_N^\varepsilon u)'\|_{H^{r-1}} + K\|u\|_{H^{r-1}}\\ &\leq C\|\chi_N^\varepsilon a(X)\|_{H^{r-1}}\|\pi_N^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}}\\ &\leq \frac{\nu}{4}\|\pi_N^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}}\end{aligned}$$

for all $u \in H^r(\mathbb{R})$. These estimates together with (3.3) lead us to (3.4). ∎

Let us observe that there exists $\eta \in (0,1)$ such that the symbols of the Fourier multipliers identified in Lemma 3.1 satisfy

$$\eta \le -\alpha_\tau \le \frac{1}{\eta} \quad \text{and} \quad \|\beta_\tau\|_\infty \le \frac{1}{\eta} \qquad \text{for all } \tau \in [0,1].$$

Classical Fourier analysis arguments then show there exists $\kappa_0 = \kappa_0(\eta) \ge 1$ such that

$$\lambda - \mathbb{A}_{\alpha,\beta} \in \mathcal{L}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R})) \text{ is an isomorphism for all } \operatorname{Re}\lambda \ge 1, \text{ and} \tag{3.5}$$

$$\kappa_0\|(\lambda - \mathbb{A}_{\alpha,\beta})[u]\|_{H^{r-1}} \ge |\lambda| \cdot \|u\|_{H^{r-1}} + \|u\|_{H^r}, \quad \forall\, u \in H^r(\mathbb{R}),\ \operatorname{Re}\lambda \ge 1, \tag{3.6}$$

uniformly for $\mathbb{A}_{\alpha,\beta} := \alpha(-d^2/dx^2)^{1/2} + \beta(d/dx)$, where $-\alpha \in [\eta, 1/\eta]$, $|\beta| \le 1/\eta$. The properties (3.5)–(3.6) combined with Lemma 3.1 enable us to obtain the desired generator property for $\partial_f \Phi_1(X)$.

Proposition 3.2. *Given $X \in \mathcal{O}_r$, it holds that $-\partial\Phi(X) \in \mathcal{H}(H^r(\mathbb{R})^2, H^{r-1}(\mathbb{R})^2)$.*

Proof. According to our discussion above it remains to establish (3.2). To prove the generator property for $\partial_f \Phi_1(X)$, we may argue as in [1, Theorem 6] to find, in view of (3.5)–(3.6) and of Lemma 3.1, constants $\kappa = \kappa(X) \ge 1$ and $\omega = \omega(X) > 0$ such that

$$\kappa\|(\lambda - \Psi(\tau))[u]\|_{H^{r-1}} \ge |\lambda| \cdot \|u\|_{H^{r-1}} + \|u\|_{H^r} \tag{3.7}$$

for all $\tau \in [0,1]$, $\operatorname{Re}\lambda \ge \omega$, and $u \in H^r(\mathbb{R})$. Choosing $\omega \ge 1$, it follows from (3.5), in view of $\Psi(0) = \mathbb{A}_{\Theta_1,0}$, that $\omega - \Psi(0)$ is, as an element of $\mathcal{L}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$, an isomorphism. The method of continuity, cf. [5, Proposition I.1.1.1], and (3.7) then imply that $\omega - \partial_f \Phi_1(X) \in \mathcal{L}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$ is an isomorphism too. From this property and (3.7) (with $\tau = 1$) we deduce that indeed $-\partial_f \Phi_1(X)$ belongs to $\mathcal{H}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$. Since the generator property for $\partial_h \Phi_2(X)$ follows by using similar arguments (which we therefore omit), this proves our claim. ∎

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. The properties (1.14) and (1.16) enable us to use the abstract parabolic theory from [36, Chapter 8] in the context of the evolution problem (1.10). More precisely, given $X_0 \in \mathcal{O}$, [36, Theorem 8.1.1] implies there exists a time $T > 0$ and a solution $X = X(\cdot, X_0)$ to (1.10) such that[2]

$$X \in \mathrm{C}([0,T], \mathcal{O}_r) \cap \mathrm{C}^1([0,T], H^{r-1}(\mathbb{R})^2) \cap \mathrm{C}^{\alpha}_{\alpha}((0,T], H^r(\mathbb{R})^2)$$

[2]Given $\alpha \in (0,1)$, $T > 0$, and a Banach space X, let $B((0,T], X)$ denote the Banach space of all bounded functions from $(0,T]$ into X. The Banach space $\mathrm{C}^{\alpha}_{\alpha}((0,T], X)$ is then defined as

$$\mathrm{C}^{\alpha}_{\alpha}((0,T], X) := \Big\{ f \in B((0,T], X) : \|f\|_{\mathrm{C}^{\alpha}_{\alpha}} := \|f\|_\infty + \sup_{s \ne t} \frac{\|t^\alpha f(t) - s^\alpha f(s)\|_X}{|t-s|^\alpha} < \infty \Big\}.$$

for some $\alpha \in (0,1)$ (actually, since (1.10) is autonomous, for all $\alpha \in (0,1)$). Moreover, the solution is unique within the class

$$\bigcup_{\alpha\in(0,1)} \mathrm{C}^{\alpha}_{\alpha}((0,T], H^r(\mathbb{R})^2) \cap \mathrm{C}([0,T], \mathcal{O}_r) \cap \mathrm{C}^1([0,T], H^{r-1}(\mathbb{R})^2).$$

To prove that the solution is unique in $\mathrm{C}([0,T], \mathcal{O}_r) \cap \mathrm{C}^1([0,T], H^{r-1}(\mathbb{R})^2)$, we assume by contradiction there exist two solutions $X_i : [0,T] \to \mathcal{O}_r$, $i = 1, 2$, to (1.10) with the property that $X_1(0) = X_2(0)$ and $X_1(t) \neq X_2(t)$ for all $t \in (0,T]$. Let $r' \in (3/2, r)$ be arbitrary and set $\alpha := r - r' \in (0,1)$. The mean value theorem together with the inequality $\|a\|_{H^{r'}} \leq \|a\|^{\alpha}_{H^{r-1}} \|a\|^{1-\alpha}_{H^r}$, $a \in H^r(\mathbb{R})$, imply that there exists a constant $C > 0$ such that

$$\|X_i(t) - X_i(s)\|_{H^{r'}} \leq C|t_1 - t_2|^{\alpha}, \quad s,t \in [0,T],\ i = 1,2. \tag{3.8}$$

Hence, $X_i \in \mathrm{C}^{\alpha}([0,T], H^{r'}(\mathbb{R})^2) \hookrightarrow \mathrm{C}^{\alpha}_{\alpha}((0,T], H^{r'}(\mathbb{R})^2)$, and [36, Theorem 8.1.1] applied in the context of (1.10) with r replaced by r' ensures that $X_1 = X_2$ in $[0,T]$, which contradicts our assumption. This unique local solution can be extended up to a maximal existence time $T^+ = T^+(X_0)$, see [36, Section 8.2].

The continuous dependence of the solution on the initial data stated in part (i) of Theorem 1.2 follows from [36, Proposition 8.2.3]. The proof of part (ii) uses a parameter trick which was successfully applied also to other problems, cf., e.g., [1, 6, 29, 38, 43]. Since the details are very similar to those in [1, Theorem 2 (ii)], we omit them.

To prove part (iii), we assume there exists a maximal solution $X = X(\cdot, X_0)$ to (1.10) with $T^+ < \infty$ and such that

$$\sup_{t\in[0,T^+)} \|X(t)\|_{H^r} < \infty \quad \text{and} \quad \liminf_{t\to T_+} \mathrm{dist}(\Gamma^{c_\infty}_f(t), \Gamma_h(t)) = c_0 > 0.$$

Arguing as above, we deduce for some fixed $r' \in (3/2, r)$, that $X : [0, T^+) \to \mathcal{O}_{r'}$ is Hölder continuous. Applying [36, Theorem 8.1.1] to (1.10) (with r replaced by r') we may extend the solution X to an existence interval $[0, T')$ with $T^+ < T'$ and such that $X \in \mathrm{C}([0,T'), \mathcal{O}_{r'}) \cap \mathrm{C}^1([0,T'), H^{r'-1}(\mathbb{R})^2)$. Moreover, the parabolic smoothing property established in part (ii) (with r replaced by r') implies that $X \in \mathrm{C}^1((0,T'), H^r(\mathbb{R})^2)$, and this contradicts the maximality property of X. This completes the proof. ∎

We conclude this section with the proof of Proposition 1.3.

Proof of Proposition 1.3. Since $\|X(t)\|_{H^r} \leq M$ for all $t \in [0, T^+)$, (1.1d) and Lemma A.2 imply there exists $C > 0$ such that

$$\Big\|\frac{dX(t)}{dt}\Big\|_{\infty} \leq C(1 + M^4), \quad t \in [0, T^+),$$

therefore $X^2 := (f^2, h^2) \in \mathrm{C}^1([0,T^+), L_2(\mathbb{R})^2)$ has a bounded derivative. Moreover, $X^2 : [0,T) \to H^r(\mathbb{R})^2$ is bounded. Since $\|a\|_{H^{r'}} \leq \|a\|_2^{1-r'/r} \|a\|^{r'/r}_{H^r}$, $a \in H^r(\mathbb{R})$, the mean value theorem yields $X^2 \in \mathrm{BUC}^{1-r'/r}([0,T^+), \mathcal{O}_{r'})$, where $r' \in (3/2, r)$ is fixed.

Hence, there exists $X_* \in H^{r'}(\mathbb{R})^2$ such that $X^2(t) \to X_* = (f_*, h_*)$ in $H^{r'}(\mathbb{R})^2$ when letting $t \to T^+$.

Since

$$\liminf_{t \to T^+} \operatorname{dist}(\Gamma_f^{c_\infty}(t), \Gamma_h(t)) = 0,$$

cf. Theorem 1.2 (iii), there exist sequences $t_n \nearrow T^+$ and $(x_n) \subset \mathbb{R}$ with

$$(c_\infty + f)(t_n, x_n) - h(t_n, x_n) \to 0 \quad \text{for } n \to \infty. \tag{3.9}$$

We next show that (x_n) is bounded. To this end, we infer from the convergence $X^2(t) \to X_*$ in $H^{r'}(\mathbb{R})^2$ that there exists $n_0 \in \mathbb{N}$ such that $|f(t_n, x)| + |h(t_n, x)| < c_\infty/2$ for all $n \geq n_0$ and $|x| \geq n_0$. The latter inequality together with (3.9) imply that (x_n) is indeed bounded.

We may thus assume (after eventually subtracting a subsequence), that $x_n \to x_0$ in $\mathbb{R}$. Since X_* is a continuous function and $X^2(t_n) \to X_*$ in $H^{r'}(\mathbb{R})^2$ when letting $n \to \infty$, we obtain that $X(t_n, x_n) \to (\sqrt{f_*(x_0), \sqrt{h_*}(x_0)})$. The relation (3.9) now leads us to the equation $c_\infty + \sqrt{f_*}(x_0) = \sqrt{h_*}(x_0)$. Finally, since $X^2(t_n, x_0) \to X_*(x_0)$, together with the latter identity we conclude that $c_\infty + f(t_n, x_0) - h(t_n, x_0) \to 0$, and therefore that

$$\liminf_{t \to T^+} (c_\infty + f(t, x_0) - h(t, x_0)) = 0.$$

In order to prove the second claim, we argue by contradiction and assume there exists $x_0 \in \mathbb{R}$ and $\delta > 0$ such that

$$\liminf_{t \to T^+} \sup_{\{|x - x_0| \leq \delta\}} (c_\infty + f(t, x) - h(t, x)) = 0.$$

Since (1.10) is invariant under horizontal translations, we may assume without loss of generality that $x_0 = 0$. Hence, there exists a sequence (t_n) with $t_n \nearrow T^+$ and

$$c_\infty + f(t_n) - h(t_n) \to 0 \quad \text{in } L_\infty([-\delta, \delta]). \tag{3.10}$$

Recalling Lemma A.2, we find a constant $c_1 = c_1(M)$ such that the velocity $v_2(t) = (v_2^1(t), v_2^2(t))$ satisfies

$$\|v_2(t)\|_{L_\infty(\Omega_2(t))} \leq c_1, \quad t \in [0, T^+). \tag{3.11}$$

Given $t \in (T^+ - \delta/c_1, T^+)$, let $R(t) := \delta + c_1(t - T^+)$. Then R is a positive increasing function with $R(t) \to \delta$ for $t \to T^+$. We further define the surface area

$$S(t) := \int_{-R(t)}^{R(t)} (c_\infty + f(t, x) - h(t, x))\, dx, \quad t \in (T^+ - \delta/c_1, T^+).$$

Let $n_0 \in \mathbb{N}$ be fixed such that $t_n > T^+ - \delta/c_1$ for all $n \geq n_0$. On the one hand, $S(t_n) > 0$ for all $n \geq n_0$. Moreover, the dominated convergence theorem together with (3.10) immediately imply that $S(t_n) \to 0$ for $n \to \infty$.

On the other hand, given $t \in (T^+ - \delta/c_1, T^+)$, Stokes' theorem together with $\mathrm{div} v_2(t) = 0$ in $\Omega_2(t)$ yields

$$S'(t) = \int_{h(t,-R(t))}^{c_\infty + f(t,-R(t))} \big(c_1 + v_2^1(t, -R(t), y)\big)\, dy + \int_{h(t,R(t))}^{c_\infty + f(t,R(t))} \big(c_1 - v_2^1(t, -R(t), y)\big)\, dy.$$

Hence, in view of the bound (3.11), we have $S'(t) \geq 0$ for all $t \in (T^+ - \delta/c_1, T^+)$. Consequently, $S(t_n) \geq S(t_{n_0}) > 0$ for $n \geq n_0$, in contradiction to $S(t_n) \to 0$ for $n \to \infty$. Therefore, our assumption was false and the argument is complete. ∎

A. An extension of Privalov's theorem

In this section, we fix $p \in (1, \infty)$, $\alpha \in (0, 1)$, $\overline{\omega} \in \mathrm{BUC}^\alpha(\mathbb{R}) \cap L_p(\mathbb{R})$, and a differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f' \in \mathrm{BUC}^\alpha(\mathbb{R})$. We study the map $v := v(f)[\overline{\omega}] : \mathbb{R}^2 \setminus \Gamma_f \to \mathbb{R}^2$ given by the formula

$$v(x, y) := v(f)[\overline{\omega}](x, y) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{(f(s) - y, x - s)}{(x - s)^2 + (y - f(s))^2} \overline{\omega}(s)\, ds, \tag{A.1}$$

where

$$\Gamma_f := \{(x, f(x)) : x \in \mathbb{R}\}.$$

Let us first note that v is the complex conjugate of a holomorphic function, see (A.3), so that v is smooth, that is, $v := (v^1, v^2) \in \mathrm{C}^\infty(\mathbb{R}^2 \setminus \Gamma_f)$. For this function we establish several additional properties below. In particular, we extend Privalov's theorem, cf., e.g., [35], and prove that v is α-Hölder continuous in the domains above and below the graph Γ_f, cf. Theorem A.3.

As a first step, we show in Lemma A.1 that the one-sided limits of v when approaching a point on Γ_f from below or from above exist. This is a consequence of the classical Plemelj formula, cf., e.g., [35], and of the observation that

$$v(x, y) = \overline{\frac{1}{2\pi i} \int_{\Gamma_f} \frac{\varphi(\xi)}{\xi - z}\, d\xi}, \qquad z = (x, y) \in \mathbb{R}^2 \setminus \Gamma_f, \tag{A.2}$$

where the function $\varphi : \Gamma_f \to \mathbb{C}$ in the contour integral (A.2) is defined by

$$\varphi(\xi) = -\frac{\overline{\omega}(1, -f')}{1 + f'^2}(s), \qquad \xi = (s, f(s)) \in \Gamma_f.$$

We note that $\varphi \in \mathrm{BUC}^\alpha(\Gamma_f)$, that is, φ is bounded and

$$[\varphi]_\alpha := \sup_{\zeta \neq \xi \in \Gamma_f} \frac{|\varphi(\zeta) - \varphi(\xi)|}{|\zeta - \xi|^\alpha} < \infty.$$

It is suitable to introduce the function $F : c \setminus \Gamma_f \to c$ defined by

$$F(z) := \overline{v(z)} = \frac{1}{2\pi i} \int_{\Gamma_f} \frac{\varphi(\xi)}{\xi - z} \, d\xi. \tag{A.3}$$

It is not difficult to prove that this function is holomorphic.

Lemma A.1. *Let*

$$\Omega_\pm := \{(x, y) \in \mathbb{R}^2 : \pm(y - f(x)) > 0\}.$$

The restrictions $v_\pm := v|_{\Omega_\pm} : \Omega_\pm \to \mathbb{R}^2$ *of the function* v *defined in* (A.1) *extend continuously up to* Γ_f *and, given* $x \in \mathbb{R}$*, we have*

$$v_\pm(x, f(x)) = \frac{1}{2\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{(f(s) - f(x), x - s)}{(x - s)^2 + (f(x) - f(s))^2} \overline{\omega}(s) \, ds \mp \frac{1}{2} \frac{\overline{\omega}(1, f')}{1 + f'^2}(x). \tag{A.4}$$

Proof. Let $z_0 := (x_0, f(x_0)) \in \Gamma_f$. In order to prove that v_+ can be extended continuously in z_0, we consider the polygonal path $\Gamma_1 \subset \overline{\Omega_+}$ defined by the segments

$$[(x_0 + 1, f(x_0 + 1)), (x_0 + 1, D)], \quad [(x_0 + 1, D), (x_0 - 1, D)],$$
$$[(x_0 - 1, D), (x_0 - 1, f(x_0 - 1))]$$

and oriented counterclockwise. Here we set

$$D := 1 + 2\|f'\|_\infty + \max\{f(x_0 - 1), f(x_0 + 1)\}.$$

Moreover, we let

$$\Gamma_0 := \{(x, f(x)) : |x - x_0| \le 1\}$$

and define the closed curve $\Gamma := \Gamma_0 + \Gamma_1$ which is again oriented counterclockwise. Additionally, we define the function $\widetilde{\varphi} \in \mathrm{BUC}^\alpha(\Gamma)$ by setting

$$\widetilde{\varphi}(\xi) := \begin{cases} \varphi(\xi), & \xi \in \Gamma_0, \\ \varphi_+, & \xi = (x_0 + 1, y), \ f(x_0 + 1) \le y \le D, \\ c\,\frac{(1+x_0-x)\varphi_- + (1+x-x_0)\varphi_+}{2}, & \xi = (x, D), \ |x - x_0| \le 1, \\ \varphi_-, & \xi = (x_0 - 1, y), \ f(x_0 - 1) \le y \le D, \end{cases}$$

where $\varphi_\pm := \varphi(x_0 \pm 1, f(x_0 \pm 1))$. It is not difficult to check that

$$\|\widetilde{\varphi}\|_\infty \le \|\varphi\|_\infty, \quad [\widetilde{\varphi}]_\alpha \le 2\|\varphi\|_\infty + [\varphi]_\alpha, \quad |\Gamma| \le 7(\|f'\|_\infty + 1). \tag{A.5}$$

Given $z \in \Omega_+$ which is sufficiently close to z_0, it then holds that

$$\int_{\Gamma_f} \frac{\varphi(\xi)}{\xi - z} \, d\xi = \int_{\Gamma_f - \Gamma_0} \frac{\varphi(\xi)}{\xi - z} \, d\xi + \int_{\Gamma} \frac{\widetilde{\varphi}(\xi)}{\xi - z} \, d\xi - \int_{\Gamma_1} \frac{\widetilde{\varphi}(\xi)}{\xi - z} \, d\xi.$$

Since $z_0 \in \Gamma_0$, Lebesgue's dominated convergence shows that

$$\int_{\Gamma_f - \Gamma_0} \frac{\varphi(\xi)}{\xi - z} \, d\xi - \int_{\Gamma_1} \frac{\widetilde{\varphi}(\xi)}{\xi - z} \, d\xi \xrightarrow[z \to z_0]{} \int_{\Gamma_f - \Gamma_0} \frac{\varphi(\xi)}{\xi - z_0} \, d\xi - \int_{\Gamma_1} \frac{\widetilde{\varphi}(\xi)}{\xi - z_0} \, d\xi.$$

Additionally, according to the Plemelj formula, cf., e.g., [35], it holds that

$$\frac{1}{2\pi i}\int_\Gamma \frac{\widetilde{\varphi}(\xi)}{\xi - z}\,d\xi \underset{z\to z_0}{\longrightarrow} \frac{1}{2\pi i}\,\mathrm{PV}\int_\Gamma \frac{\widetilde{\varphi}(\xi)}{\xi - z_0}\,d\xi + \frac{1}{2}\varphi(z_0).$$

These two convergences imply that v_+ can indeed be extended continuously in z_0, with the value of the extension in z_0 being as given in formula (A.4). Finally, the corresponding claim for v_- follows from a similar argument. ∎

We next prove in Lemma A.2 that v is bounded in $\mathbb{R}^2 \setminus \Gamma_f$. In fact, in Lemma A.2, we bound the L_∞-norm of v by a constant that depends explicitly on the norms of the functions f and $\overline{\omega}$.

Lemma A.2. *There exists a constant C, which is independent of f and $\overline{\omega}$, such that*

$$\|v\|_\infty \le C(\|\overline{\omega}\|_p + \|\overline{\omega}\|_{\mathrm{BUC}^\alpha})(1 + \|f'\|_{\mathrm{BUC}^\alpha})^2. \tag{A.6}$$

Proof. We devise the proof in several steps.

Step 1. In this step, we provide bounds for the restrictions of $v_\pm$ to Γ_f. Given $x \in \mathbb{R}$, it follows from (A.4) and Hölder's inequality that

$$\begin{aligned}|v_\pm(x, f(x))| &\le \|\overline{\omega}\|_\infty + \int_{\{|s|>1\}} \Big|\frac{\overline{\omega}(x-s)}{s}\Big|\,ds \\ &\quad + \Big|\mathrm{PV}\int_{-1}^1 \frac{(f(x-s)-f(x), s)}{s^2 + (f(x)-f(x-s))^2}\overline{\omega}(x-s)\,ds\Big| \\ &\le \|\overline{\omega}\|_\infty + C\|\overline{\omega}\|_p + I_1 + I_2,\end{aligned}$$

where

$$\begin{aligned}I_1 &:= \Big|\int_{-1}^1 \frac{(f(x-s)-f(x), s)}{s^2 + (f(x-s)-f(s))^2}(\overline{\omega}(x-s) - \overline{\omega}(x)\,ds\Big| \\ &\le [\overline{\omega}]_\alpha \int_{-1}^1 |s|^{\alpha-1}\,ds \le C[\overline{\omega}]_\alpha, \\ I_2 &:= \|\overline{\omega}\|_\infty\Big|\mathrm{PV}\int_{-1}^1 \frac{(f(x-s)-f(x), s)}{s^2 + (f(x)-f(x-s))^2}\,ds\Big|.\end{aligned}$$

Concerning I_2, we have

$$\begin{aligned}I_2 &\le \|\overline{\omega}\|_\infty \int_0^1 \Big|\frac{(f(x-s)-f(x), s)}{s^2 + (f(x)-f(x-s))^2} + \frac{(f(x+s)-f(x), -s)}{s^2 + (f(x)-f(x+s))^2}\Big|\,ds \\ &\le 3\|\overline{\omega}\|_\infty \int_0^1 \frac{|f(x+s)-2f(x)-f(x-s)|}{s^2}\,ds \\ &\le 6\|\overline{\omega}\|_\infty [f']_\alpha \int_0^1 |s|^{\alpha-1}\,ds \\ &\le C\|\overline{\omega}\|_\infty [f']_\alpha.\end{aligned}$$

Gathering these estimates, we conclude that

$$\|v_\pm|_{\Gamma_f}\|_\infty \le C(\|\overline{\omega}\|_p + \|\overline{\omega}\|_{\mathrm{BUC}^\alpha})(1 + \|f'\|_{\mathrm{BUC}^\alpha}). \tag{A.7}$$

Step 2. Given $z = (x, y) \in \mathbb{R}^2$, we set $d(z) := \mathrm{dist}(z, \Gamma_f)$. We next prove that

$$\sup_{\{1/4\le d(z)\}} |v_\pm(z)| \le C\|\overline{\omega}\|_p. \tag{A.8}$$

Indeed, if $1/4 \le d(z)$, then $\sqrt{s^2 + (y - f(x - s))^2} \ge \max\{1/4, |s|\}$ for all $s \in \mathbb{R}$, and together with Hölder's inequality we conclude from (A.1) that

$$|v(z)| \le \int_{\mathbb{R}} \frac{1}{\max\{1/4, |s|\}} |\overline{\omega}(x - s)|\, ds \le C\|\overline{\omega}\|_p.$$

Step 3. In this final step we prove that

$$\sup_{\{0<d(z)<1/4\}} |v_\pm(z)| \le C(\|\overline{\omega}\|_p + \|\overline{\omega}\|_{\mathrm{BUC}^\alpha})(1 + \|f'\|_{\mathrm{BUC}^\alpha})^2. \tag{A.9}$$

We first consider the case when $z \in \Omega_+$. We associate to z a point $z_\Gamma = (x_0, f(x_0)) \in \Gamma_f$ such that

$$d(z) = |z - z_\Gamma| \in (0, 1/4).$$

Let $\Gamma = \Gamma_0 + \Gamma_1$ and $\widetilde{\varphi}$ be as defined in the proof of Lemma A.1 (with z_Γ instead of z_0). Recalling (A.7), we have

$$\begin{aligned}|v_+(z)| &\le |v_+(z) - v_+(z_\Gamma)| + |v_+(z_\Gamma)|\\ &\le T_1 + T_2 + T_3 + C(\|\overline{\omega}\|_p + \|\overline{\omega}\|_{\mathrm{BUC}^\alpha})(1 + \|f'\|_{\mathrm{BUC}^\alpha}),\end{aligned}$$

where

$$T_1 := \Big|\int_{\Gamma_f-\Gamma_0} \Big(\frac{\varphi(\xi)}{\xi - z} - \frac{\varphi(\xi)}{\xi - z_\Gamma}\Big)\, d\xi\Big|, \quad T_2 := \Big|\int_{\Gamma_1} \Big(\frac{\widetilde{\varphi}(\xi)}{\xi - z} - \frac{\widetilde{\varphi}(\xi)}{\xi - z_\Gamma}\Big)\, d\xi\Big|,$$
$$T_3 := \Big|\frac{1}{2\pi i}\int_\Gamma \frac{\widetilde{\varphi}(\xi)}{\xi - z}\, d\xi - \frac{1}{2\pi i}\,\mathrm{PV}\int_\Gamma \frac{\widetilde{\varphi}(\xi)}{\xi - z_\Gamma}\, d\xi - \frac{1}{2}\varphi(z_\Gamma)\Big|.$$

Given $\xi \in \Gamma_1$, we have $\min\{|\xi - z|, |\xi - z_\Gamma|\} \ge |\xi - z_\Gamma| - 1/4 \ge 3/4$ and (A.5) yields

$$T_2 \le 2\|\varphi\|_\infty |\Gamma_1| \cdot |z - z_\Gamma| \le C\|\overline{\omega}\|_\infty (1 + \|f'\|_\infty).$$

Moreover, since $\min\{|\xi - z|, |\xi - z_\Gamma|\} \ge \max\{3/4,\ |s - x_0|/2\}$ for all $\xi = (s, f(s)) \in \Gamma_f - \Gamma_0$, Hölder's inequality leads us to

$$T_1 \le \frac{8}{3}\|\overline{\omega}\|_p |z - z_\Gamma| \Big(\int_{\mathbb{R}} \frac{1}{\max\{3/2, |s|\}^{p'}}\, ds\Big)^{1/p'} \le C\|\overline{\omega}\|_p,$$

where $p' \in (1, \infty)$ is the adjoint exponent to p, that is $p^{-1} + p'^{-1} = 1$. In order to estimate T_3 we first note that

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{\xi - z}\, d\xi = 1 \quad \text{and} \quad \frac{1}{2\pi i}\,\mathrm{PV}\int_\Gamma \frac{1}{\xi - z_\Gamma}\, d\xi = \frac{1}{2}. \tag{A.10}$$

The first relation follows from Cauchy's integral formula. The second identity is a direct consequence of Plemelj's formula, cf., e.g., [35]. Using these two identities we get

$$T_3 = \Big|\frac{1}{2\pi i}\int_\Gamma \frac{\widetilde{\varphi}(\xi)-\widetilde{\varphi}(z_\Gamma)}{\xi - z}\,d\xi - \frac{1}{2\pi i}\int_\Gamma \frac{\widetilde{\varphi}(\xi)-\widetilde{\varphi}(z_\Gamma)}{\xi - z_\Gamma}\,d\xi\Big| \le T_{3a} + T_{3b},$$

where

$$T_{3a} := \Big|\int_{\Gamma_1} \frac{(z-z_\Gamma)(\widetilde{\varphi}(\xi)-\widetilde{\varphi}(z_\Gamma))}{(\xi-z)(\xi-z_\Gamma)}\,d\xi\Big|,$$
$$T_{3b} := \Big|\int_{\Gamma_0} \frac{(z-z_\Gamma)(\varphi(\xi)-\varphi(z_\Gamma))}{(\xi-z)(\xi-z_\Gamma)}\,d\xi\Big|.$$

The arguments used to estimate T_2 lead us to

$$T_{3a} \le \|\varphi\|_\infty |\Gamma_1| \le C\|\overline{\omega}\|_\infty(1+\|f'\|_\infty).$$

In order to estimate T_{3b}, we note that $|\xi - z| \ge d(z) = |z - z_\Gamma|$ and $|\xi - z_\Gamma| \ge |s - x_0|$ for all $\xi = (s, f(s)) \in \Gamma_0$, and therefore

$$T_{3b} \le (1+\|f'\|_\infty)[\varphi]_\alpha \int_{\{|s|<1\}} s^{\alpha-1}\,ds \le C(1+\|f'\|_\infty)[\varphi]_\alpha.$$

Observing that

$$[\varphi]_\alpha \le C\|\overline{\omega}\|_{\mathrm{BUC}^\alpha}(1+\|f'\|_{\mathrm{BUC}^\alpha}),$$

the latter arguments show that (A.9) holds for $z \in \Omega_+$. Arguing along the same lines, it is easy to see that (A.9) is satisfied also for $z \in \Omega_-$. The claim (A.6) follows now from (A.7)–(A.9). ■

Now, in Theorem A.3, we extend Privalov's theorem to the setting considered herein, where the contour integral in (A.1) is defined over an unbounded graph.

Theorem A.3. *The restrictions $v_\pm := v|_{\Omega_\pm}$ of the function v defined in* (A.1) *satisfy $v_\pm \in \mathrm{BUC}^\alpha(\Omega_\pm)$.*

Proof. We only establish the Hölder continuity of $v_+ =: (v_+^1, v_+^2)$ (that of v_- follows from similar arguments). We devise the proof in several steps.

Step 1. Let $z, z' \in \overline{\Omega_+}$ satisfy $|z - z'| > 1/8$. Then, according to Lemma A.1 and Lemma A.2, we have

$$|v_+(z) - v_+(z')| \le 2\|v\|_\infty \le 16\|v\|_\infty |z-z'|^\alpha \le C|z-z'|^\alpha.$$

Step 2. Given $z \in \mathbb{R}^2$, we set again $d(z) := \mathrm{dist}(z, \Gamma_f)$. Assume now that $z, z' \in \Omega_+$ are chosen such that $|z - z'| \le 1/8$. Then, letting $S_{zz'} := \{(1-t)z + tz' : t \in [0,1]\}$ denote the segment that connects z and z', there exists at least a point $\overline{\zeta} \in S_{zz'}$ such that

$$d(\overline{\zeta}) = |\overline{\zeta} - \overline{\zeta}_\Gamma| = \mathrm{dist}(S_{zz'}, \Gamma_f).$$

We distinguish two cases.

Step 2a. If $|z-z'| < |\overline{\zeta}-\overline{\zeta}_\Gamma|$, then $S_{zz'} \subset \Omega_+$. Then we have

$$\begin{aligned}|v_+(z)-v_+(z')| &= |F(z)-F(z')|\\ &= \Big|\int_{S_{zz'}} F'(\zeta)\,d\zeta\Big|\\ &= \Big|\int_{S_{zz'}} \Big(\frac{1}{2\pi i}\int_{\Gamma_f}\frac{\varphi(\xi)}{(\xi-\zeta)^2}\,d\xi\Big)\,d\zeta\Big|,\end{aligned}$$

where F is the holomorphic function defined in (A.3). Given $\zeta \in S_{zz'}$, it holds that

$$\int_{\Gamma_f}\frac{1}{(\xi-\zeta)^2}\,d\xi = 0,$$

and therewith we get

$$\begin{aligned}|v_+(z)-v_+(z')| &= \Big|\int_{S_{zz'}} \Big(\frac{1}{2\pi i}\int_{\Gamma_f}\frac{\varphi(\xi)-\varphi(\zeta_\Gamma)}{(\xi-\zeta)^2}\,d\xi\Big)\,d\zeta\Big|\\ &\le |z-z'| \sup_{\zeta\in S_{zz'}}\Big|\int_{\Gamma_f}\frac{\varphi(\xi)-\varphi(\zeta_\Gamma)}{(\xi-\zeta)^2}\,d\xi\Big|\\ &\le |z-z'|\cdot[\varphi]_\alpha \sup_{\zeta\in S_{zz'}}\int_{\Gamma_f}\frac{|\xi-\zeta_\Gamma|^\alpha}{|\xi-\zeta|^2}\,|d\xi|.\end{aligned}$$

Recalling the definition of ζ_Γ, we have $|\xi-\zeta_\Gamma| \le |\xi-\zeta|+|\zeta-\zeta_\Gamma| \le 2|\xi-\zeta|$ for all $\zeta \in S_{zz'}$ and $\xi \in \Gamma_f$, hence $|\xi-\zeta_\Gamma|+|\zeta-\zeta_\Gamma| \le 3|\xi-\zeta|$. Noticing that $|z-z'| < |\zeta-\zeta_\Gamma|$ for $\zeta \in S_{zz'}$, we obtain in view of these inequalities that

$$\begin{aligned}\int_{\Gamma_f}\frac{|\xi-\zeta_\Gamma|^\alpha}{|\xi-\zeta|^2}\,|d\xi| &\le 9\int_{\Gamma_f}(|\xi-\zeta_\Gamma|+|\zeta-\zeta_\Gamma|)^{\alpha-2}\,|d\xi|\\ &\le 9(1+\|f'\|_\infty)\int_{\mathbb{R}}(|s|+|\zeta-\zeta_\Gamma|)^{\alpha-2}\,ds\\ &\le C|\zeta-\zeta_\Gamma|^{\alpha-1} \le C|z-z'|^{\alpha-1},\end{aligned}$$

and therefore $|v_+(z)-v_+(z')| \le C|z-z'|^\alpha$.

Step 2b. We now consider the second case when $|z-z'| \ge |\overline{\zeta}-\overline{\zeta}_\Gamma|$. Since $\overline{\zeta} \in S_{zz'}$, we have

$$\max\{|z-\overline{\zeta}_\Gamma|, |z'-\overline{\zeta}_\Gamma|\} \le \max\{|z-\overline{\zeta}|, |z'-\overline{\zeta}|\} + |\overline{\zeta}-\overline{\zeta}_\Gamma| \le 2|z-z'| \le 1/4.$$

Assuming there exists a constant $C > 0$ such that

$$|v_+(z)-v_+(z_0)| \le C|z-z_0|^\alpha \quad \forall\, z_0 \in \Gamma_f \text{ and } z \in \overline{\Omega_+} \text{ with } |z-z_0| \le 1/4, \quad \text{(A.11)}$$

we then have

$$
\begin{aligned}
|v_+(z)-v_+(z')| &\leq |v_+(z)-v_+(\overline{\xi}_\Gamma)|+|v_+(z')-v_+(\overline{\xi}_\Gamma)|\\
&\leq C\big(|z-\overline{\xi}_\Gamma|^\alpha+|z'-\overline{\xi}_\Gamma|^\alpha\big)\\
&\leq C|z-z'|^\alpha,
\end{aligned}
$$

and the claim then follows.

Step 3. It remains to establish (A.11). Let $z_0\in\Gamma_f$ and $z\in\Omega_+$ satisfy $|z-z_0|\leq 1/4$. Also let $\Gamma=\Gamma_0+\Gamma_1$, and $\widetilde{\varphi}$ be as defined in the proof of Lemma A.1. Recalling Lemma A.1, it follows similarly as in Step 3 of the proof of Lemma A.2 that

$$
\begin{aligned}
|v_+(z)-v_+(z_0)| \leq& \Big|\int_{\Gamma_f-\Gamma_0}\Big(\frac{\varphi(\xi)}{\xi-z}-\frac{\varphi(\xi)}{\xi-z_0}\Big)\,d\xi\Big|+\Big|\int_{\Gamma_1}\Big(\frac{\widetilde{\varphi}(\xi)}{\xi-z}-\frac{\widetilde{\varphi}(\xi)}{\xi-z_0}\Big)\,d\xi\Big|\\
&+\Big|\frac{1}{2\pi i}\int_\Gamma\frac{\widetilde{\varphi}(\xi)}{\xi-z}\,d\xi-\frac{1}{2\pi i}\,\mathrm{PV}\int_\Gamma\frac{\widetilde{\varphi}(\xi)}{\xi-z_0}\,d\xi-\frac{1}{2}\varphi(z_0)\Big|\\
=:& \sum_{i=1}^{3}T_i,
\end{aligned}
$$

with

$$
T_1+T_2\leq C|z-z_0|\leq C|z-z_0|^\alpha.
$$

It remains to estimate the term T_3 which, in view of (A.10), can be written as

$$
T_3(z):=\Big|\frac{1}{2\pi i}\int_\Gamma\frac{\widetilde{\varphi}(\xi)}{\xi-z}\,d\xi-\frac{1}{2\pi i}\,\mathrm{PV}\int_\Gamma\frac{\widetilde{\varphi}(\xi)}{\xi-z_0}\,d\xi-\frac{1}{2}\varphi(z_0)\Big|\leq\sum_{i=1}^{3}S_i,
$$

where, letting z_Γ be defined by the relation $d(z)=|z-z_\Gamma|$, we set

$$
\begin{aligned}
S_1 &:= |\varphi(z_\Gamma)-\varphi(z_0)|,\quad S_2:=\Big|\int_{\Gamma_1}\Big(\frac{\widetilde{\varphi}(\xi)-\widetilde{\varphi}(z_\Gamma)}{\xi-z}-\frac{\widetilde{\varphi}(\xi)-\widetilde{\varphi}(z_0)}{\xi-z_0}\Big)\,d\xi\Big|,\\
S_3 &:= \Big|\int_{\Gamma_0}\Big(\frac{\varphi(\xi)-\varphi(z_\Gamma)}{\xi-z}-\frac{\varphi(\xi)-\varphi(z_0)}{\xi-z_0}\Big)\,d\xi\Big|.
\end{aligned}
$$

Noticing that $|z_\Gamma-z_0|\leq|z_\Gamma-z|+|z-z_0|\leq 2|z-z_0|$, we obtain

$$
S_1\leq[\varphi]_\alpha|z_\Gamma-z_0|^\alpha\leq C|z-z_0|^\alpha.
$$

Moreover, given $\xi\in\Gamma_1$, we have $\min\{|\xi-z|,\,|\xi-z_0|\}\geq 3/4$ and together with (A.5) we get

$$
\begin{aligned}
S_2 &\leq \frac{16}{9}\int_{\Gamma_1}\big(|\widetilde{\varphi}(\xi)-\widetilde{\varphi}(z_\Gamma)|\cdot|z-z_0|+|\varphi(z_0)-\varphi(z_\Gamma)|\big)\,|d\xi|\\
&\leq C|\Gamma_1|(\|\varphi\|_\infty|z-z_0|+[\varphi]_\alpha|z_0-z_\Gamma|^\alpha)\leq C|z-z_0|^\alpha.
\end{aligned}
$$

In order to estimate S_3, we let $\eta := |z - z_0| \in (0, 1/4]$, we set $z_0 =: (x_0, f(x_0))$, and we introduce the curve $\Gamma_\eta := \{(s, f(s)) : |s - x_0| \le 2\eta\}$. It then holds that

$$S_3 \le S_{3a} + S_{3b} + S_{3c},$$

where

$$\begin{aligned}
S_{3a} &:= \Big| \int_{\Gamma_\eta} \Big(\frac{\varphi(\xi) - \varphi(z_\Gamma)}{\xi - z} - \frac{\varphi(\xi) - \varphi(z_0)}{\xi - z_0} \Big) \, d\xi \Big|, \\
S_{3b} &:= |z - z_0| \cdot \Big| \int_{\Gamma_0 - \Gamma_\eta} \frac{\varphi(\xi) - \varphi(z_\Gamma)}{(\xi - z)(\xi - z_0)} \, d\xi \Big|, \\
S_{3c} &:= \Big| \int_{\Gamma_0 - \Gamma_\eta} \frac{\varphi(z_0) - \varphi(z_\Gamma)}{\xi - z_0} \, d\xi \Big|.
\end{aligned}$$

The relation $|z - z_\Gamma| \le |z - z_0| = \eta$ implies that $z_\Gamma \in \Gamma_\eta$. Taking also into account the inequality $|\xi - z_\Gamma| \le |\xi - z| + |z - z_\Gamma| \le 2|\xi - z|$ for $\xi \in \Gamma_f$, we have

$$\begin{aligned}
S_{3a} &\le 2[\varphi]_\alpha \int_{\Gamma_\eta} (|\xi - z_\Gamma|^{\alpha-1} + |\xi - z_0|^{\alpha-1}) \, |d\xi| \\
&\le C[\varphi]_\alpha (1 + \|f'\|_\infty) \eta^\alpha \le C|z - z_0|^\alpha.
\end{aligned}$$

Given $\xi \in \Gamma_0 - \Gamma_\eta$, the relation $|\xi - z_0| \ge 2\eta = 2|z - z_0|$ leads us to

$$\begin{aligned}
|\xi - z| &\ge |\xi - z_0| - |z - z_0| \ge \eta = |z - z_0|, \\
2|\xi - z_0| &\ge |\xi - z_0| + |z_0 - z| \ge |\xi - z|, \\
3|\xi - z| &\ge |\xi - z_0| - |z - z_0| + 2|z - z_0| = |\xi - z_0| + |z - z_0|.
\end{aligned}$$

Recalling also that $|\xi - z_\Gamma| \le 2|\xi - z|$, we then obtain

$$\begin{aligned}
S_{3b} &\le 4|z - z_0| \cdot [\varphi]_\alpha \int_{\Gamma_0 - \Gamma_\eta} |\xi - z|^{\alpha-2} \, |d\xi| \\
&\le C|z - z_0| \cdot [\varphi]_\alpha \int_{\Gamma_0 - \Gamma_\eta} (|\xi - z_0| + |z - z_0|)^{\alpha-2} \, |d\xi| \\
&\le C[\varphi]_\alpha (1 + \|f'\|_\infty) |z - z_0| (2\eta + |z - z_0|)^{\alpha-1} \le C|z - z_0|^\alpha.
\end{aligned}$$

Finally, since $|z_0 - z_\Gamma| \le 2|z - z_0|$, we have

$$S_{3c} \le [\varphi]_\alpha |z_0 - z_\Gamma|^\alpha \Big| \int_{\Gamma_0 - \Gamma_\eta} \frac{1}{\xi - z_0} \, d\xi \Big| \le C|z_0 - z|^\alpha \Big| \int_{\Gamma_0 - \Gamma_\eta} \frac{1}{\xi - z_0} \, d\xi \Big|,$$

and, after identifying the real and imaginary parts of the integral, we obtain the following:

$$\begin{aligned}
\Big| \int_{\Gamma_0 - \Gamma_\eta} \frac{1}{\xi - z_0} \, d\xi \Big| \le \Big| &\int_{\{2\eta \le |s| \le 1\}} \frac{s + f'(x_0 - s)(f(x_0) - f(x_0 - s))}{s^2 + (f(x_0) - f(x_0 - s)^2} \, ds \Big| \\
&+ \Big| \int_{\{2\eta \le |s| \le 1\}} \frac{sf'(x_0 - s) - (f(x_0) - f(x_0 - s))}{s^2 + (f(x_0) - f(x_0 - s)^2} \, ds \Big|
\end{aligned}$$

$$\le \Big|\frac{1}{2}\ln\Big(\frac{(2\eta)^2+(f(x_0)-f(x_0+2\eta))^2}{(2\eta)^2+(f(x_0)-f(x_0-2\eta))^2}\cdot\frac{1+(f(x_0)-f(x_0-1))^2}{1+(f(x_0)-f(x_0+1))^2}\Big)\Big|$$
$$+[f']_\alpha\int_{\{2\eta\le|s|\le 1\}}|s|^{\alpha-1}\,ds\le \ln(1+\|f'\|_\infty^2)+C[f']_\alpha.$$

Hence, we have shown that $S_{3c}\le C|z_0-z|^\alpha$ and the proof is completed. ∎

We conclude this section with the following result.

Lemma A.4. *It holds that*

$$\partial_x v^1+\partial_y v^2=0=\partial_y v^1-\partial_x v^2\quad \textit{in } \mathbb{R}^2\setminus\Gamma_f \tag{A.12}$$

and

$$v_\pm(z)\to 0\quad \textit{for } z\in\Omega_\pm \textit{ with } |z|\to\infty. \tag{A.13}$$

Proof. The relations (A.12) follow by direct computation. We next prove that v_+ vanishes at infinity (the claim for v_- follows by arguing along the same lines). We divide the proof in two steps.

Step 1. We first show that $v_+(x,f(x))\to 0$ for $|x|\to\infty$. Recalling Lemma A.1 and (2.2), we write

$$v_+(x,f(x))=\frac{1}{2\pi}\big(-B^0_{1,1}(f)[\overline{\omega}],B^0_{0,1}(f)[\overline{\omega}]\big)(x)-\frac{1}{2}\frac{\overline{\omega}(1,f')}{1+f'^2}(x),\qquad x\in\mathbb{R}.$$

Because $\overline{\omega}\in \mathrm{BUC}^\alpha(\mathbb{R})\cap L_p(\mathbb{R})$, the last term on the right vanishes at infinity. We next prove that, given $n,m\in\mathbb{N}$, we also have

$$B^0_{n,m}(f)[\overline{\omega}](x)\to 0\quad \text{for } |x|\to\infty. \tag{A.14}$$

Thus, let $\varepsilon>0$ be given and choose $N>0$ such that

$$\|f'\|_\infty^n\|\overline{\omega}\|_p\Big(\frac{2}{(p'-1)N^{p'-1}}\Big)^{1/p'}\le\frac{\varepsilon}{2},$$

where p' is the adjoint exponent to p. This choice together with Hölder's inequality then yields

$$|B^0_{n,m}(f)[\overline{\omega}](x)|\le T(x)+\|f'\|_\infty^n\|\overline{\omega}\|_p\Big(\frac{2}{(p'-1)N^{p'-1}}\Big)^{1/p'}\le T(x)+\frac{\varepsilon}{2},$$

where

$$T(x):=\Big|\mathrm{PV}\int_{\{|s|\le N\}}c\frac{\big(\delta_{[x,s]}f/s\big)^n}{\big[1+\big(\delta_{[x,s]}f/s\big)^2\big]^m}\frac{\overline{\omega}(x-s)}{s}\,ds\Big|,\qquad x\in\mathbb{R}.$$

In order to estimate $T(x)$ we note that

$$T(x) \leq C \int_0^N \Big|\frac{\overline{\omega}(x-s) - \overline{\omega}(x+s)}{s}\Big| + |\overline{\omega}(x+s)| \cdot \Big|\frac{f(x+s) - 2f(x) + f(x-s)}{s^2}\Big|\, ds,$$

with C depending only on n, m, and $\|f'\|_\infty$. Taking into account that $\overline{\omega}$ vanishes at infinity, we obtain for $|x| > M$, where $M > N$ is chosen to be sufficiently large, that

$$T(x) \leq C\big([\overline{\omega}]_\alpha^{1/2} \|\overline{\omega}\|^{1/2}_{L_\infty(\{|x|>M-N\})} + \|\overline{\omega}\|_{L_\infty(\{|x|>M-N\})}\big) \leq \frac{\varepsilon}{2}.$$

This establishes (A.14).

Step 2. We now prove that $v_+(z) \to 0$ for $|z| \to \infty$. Let $\varepsilon > 0$ be given. From Step 1 we find $x_0 > 0$ such that $|v_+(x, f(x))| \leq \varepsilon/2$ for all $|x| \geq x_0$. Given $z = (x, y) \in \Omega_+$, let again $d(z) := \mathrm{dist}(z, \Gamma_f) = |z - z_\Gamma|$ with $z_\Gamma \in \Gamma$.

Assume first that $d(z) \leq \delta$, where $\delta := \min\{1, \varepsilon/(2(1 + [v_+]_\alpha))\}$. If $z = (x, y) \in \Omega_+$ satisfies $d(z) \leq \delta$, and $|x| \geq x_1$, we deduce for the corresponding point $z_\Gamma := (x_\Gamma, f(x_\Gamma))$ that $|x_\Gamma| \geq x_0$. Hence, for all such $z \in \Omega_+$, Theorem A.3 leads us to

$$|v_+(z)| = |v_+(z) - v_+(z_\Gamma)| + |v_+(z_\Gamma)| \leq [v_+]_\alpha d(z) + \varepsilon/2 \leq \varepsilon.$$

Assume now that $d(z) \geq \delta$. Let $s_0 > 0$ be chosen such that

$$\|\overline{\omega}\|_p \Big(\frac{2}{(p'-1)s_0^{p'-1}}\Big)^{1/p'} \leq \frac{\varepsilon}{2}.$$

It then holds that

$$\begin{aligned} |v_+(z)| &\leq \int_{\mathbb{R}} \frac{|\overline{\omega}(x-s)|}{\sqrt{s^2 + (y - f(x-s))^2}}\, ds \\ &\leq T(z) + \int_{\{|s|>s_0\}} \frac{|\overline{\omega}(x-s)|}{|s|}\, ds \\ &\leq T(z) + \|\overline{\omega}\|_p \Big(\frac{2}{(p'-1)N^{p'-1}}\Big)^{1/p'} \\ &\leq T(z) + \frac{\varepsilon}{2}, \end{aligned}$$

where

$$T(z) := \int_{\{|s|<s_0\}} \frac{|\overline{\omega}(x-s)|}{\sqrt{s^2 + (y - f(x-s))^2}}\, ds, \quad z = (x, y) \in \Omega_+,\ d(z) \geq \delta.$$

Let $N > 0$ be chosen such that

$$\frac{4s_0\|\overline{\omega}\|_\infty}{N} + \frac{2s_0\|\overline{\omega}\|_{L_\infty(\{|x|\geq N\})}}{\delta} \leq \frac{\varepsilon}{2}$$

and set $M_1 := N + s_0$, $M_2 := N + 2\|f\|_{L_\infty(\{|x|\le M_1+s_0\})}$, and $M := 2\max\{M_1, M_2\}$. Given $|z| \ge M$, we distinguish two cases.

(1) If $|x| \ge M_1$, then

$$T(z) \le \frac{2s_0}{\delta}\|\overline{\omega}\|_{L_\infty(\{|x|\ge M_1-s_0\})} = \frac{2s_0}{\delta}\|\overline{\omega}\|_{L_\infty(\{|x|\ge N\})} \le \frac{\varepsilon}{2}.$$

(2) If $|x_1| \le M_1$ and $|y| \ge M_2$, then $|y - f(x-s)| \ge |y/2|$, and therefore

$$T(z) \le \frac{4s_0}{|y|}\|\overline{\omega}\|_\infty \le \frac{4s_0}{N}\|\overline{\omega}\|_\infty \le \frac{\varepsilon}{2}.$$

Hence $|v_+(z)| \le \varepsilon$ for all $z \in \Omega_+$ that satisfy $d(z) \ge \delta$ and $|z| \ge M$.

To summarize, for all $z \in \Omega_+$ with $|z| \ge \max\{M, x_1 + \|f\|_{L_\infty(\{|x|\le x_1+1\})} + 1\}$, we have established that $|v_+(z)| \le \varepsilon$, and this completes the proof. ∎

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