Interfaces Free Bound. 24 (2022), 197–233
DOI 10.4171/IFB/471

Fully nonlinear free transmission problems with nonhomogeneous degeneracies

Cristiana De Filippis

Abstract. We prove existence and regularity results for free transmission problems governed by fully nonlinear elliptic equations with nonhomogeneous degeneracies.

1. Introduction

In this paper we provide existence and regularity results for the free transmission problem

$$\begin{aligned}\big[|Du|^{p^+\mathbb{1}_{\{u>0\}}+p_-\mathbb{1}_{\{u<0\}}} &+ a(x)\mathbb{1}_{\{u>0\}}|Du|^q \\ &+ b(x)\mathbb{1}_{\{u<0\}}|Du|^s\big]F(D^2u) = f(x) \quad \text{in } \Omega,\end{aligned} \tag{1.1}$$

where $F(\cdot)$ is a uniformly elliptic second order operator, $a(\cdot)$ and $b(\cdot)$ are continuous, nonnegative functions, $f(\cdot)$ is continuous and bounded and p_-, p^+, q, s are nonnegative numbers; we refer to Sections 2.2 and 2.3 for more details on the natural functional setting related to (1.1). Equation (1.1) models anisotropic diffusion processes characterized by multiple degeneracy phenomena such as material depending conductivity or electromagnetic processes in nonhomogeneous ferromagnetic media, cf. [2]. It is also connected to the analysis of composite, anisotropic materials, characterized by the coexistence of different media, whose viscosity features are linked to the exponents dictating the growth of the gradient variable in (1.1), and that are mixed according to the behavior of the modulating coefficients $a(\cdot)$ and $b(\cdot)$, see [30]. In fact, the degeneracy law displayed in (1.1) develops discontinuities along $\partial\{x \in \Omega : u(x) > 0\}$ and $\partial\{x \in \Omega : u(x) < 0\}$, and it is also influenced by the possible vanishing of the coefficients $a(\cdot)$, $b(\cdot)$. The various regions where each degeneracy regime is in force are in part unknown a priori as they vary according to the sign of solutions, the transmission interface can be interpreted as a free boundary, and a drastic degeneracy variation occurs in correspondence of the zero sets of the modulating coefficients $\{x \in \Omega : a(x) = 0\}$ and $\{x \in \Omega : b(x) = 0\}$. Transmission problems are essentially related to the analysis of models involving different constitutive laws holding in separate subregions of the domain. The systematic study of such problems started with [45] and since then it has undergone an intensive development, see

2020 Mathematics Subject Classification. 35A01, 35B65, 35J60, 35J70, 35R35.
Keywords. Fully nonlinear degenerate equations, Double Phase problems, free transmission problems.

[2, 14, 35, 36, 44, 46, 49] and references therein. We mention in particular [35], where the authors consider the degenerate free transmission problem

$$|Du|^{\theta_1 \mathbb{1}_{\{u>0\}}+\theta_2 \mathbb{1}_{\{u<0\}}} F(D^2 u) = f(x) \quad \text{in } \Omega, \tag{1.2}$$

where θ_1, θ_2 are nonnegative constants, and prove existence and optimal Hölder continuity for the gradient of solutions to the associated Dirichlet problem. Notice that the degeneracy law appearing in (1.2) is close to being homogeneous, in the sense that for any fixed point it behaves as a power. Another way of interpreting equation (1.2) is as an instance of a fully nonlinear elliptic equation with discontinuous variable exponent

$$|Du|^{p(x)} F(D^2 u) = f(x) \quad \text{in } \Omega. \tag{1.3}$$

The regularity theory for (1.3) has been developed in [12], under the assumption that the exponent is a continuous function; however, all the estimates obtained there do not depend on its modulus of continuity. In sharp contrast with the models described so far, equation (1.1) features a strongly anisotropic structure in which several nonhomogeneous phases coexist and switch according to the sign of solutions. More precisely, in correspondence of positive values of u (resp. negative values of u) we see the nonhomogeneous degeneracy $[|Du|^{p^+} + a(x)|Du|^q]$ (resp. $[|Du|^{p_-} + b(x)|Du|^s]$). Degeneracies of Double Phase type have been introduced in [26], where the Hölder continuity of the gradient of solutions to fully nonlinear elliptic equations such as

$$[|Du|^p + a(x)|Du|^q] F(D^2 u) = f(x) \quad \text{in } \Omega, \tag{1.4}$$

where $0 \le p \le q$, $0 \le a(\cdot) \in C(\Omega)$ and $f \in C(\Omega) \cap L^\infty(\Omega)$, was investigated. This new model received lots of attention recently in the setting of free boundary problems, nonhomogeneous ∞-laplacian equations or obstacle problems, cf. [20–22], while in [32] the authors carefully combine the approaches of [12, 26] to derive local C^{1,α_0}-regularity for the viscosity solution of the fully nonlinear equation with variable exponents and nonhomogeneous degeneracy

$$[|Du|^{p(x)} + a(x)|Du|^{q(x)}] F(D^2 u) = f(x) \quad \text{in } \Omega. \tag{1.5}$$

These results also cover Multi Phase equations with variable exponents, i.e., a generalization of (1.3), (1.4) and (1.5):

$$\left[|Du|^{p(x)} + \sum_{\iota=1}^{\kappa} a_\iota(x)|Du|^{p_\iota(x)}\right] F(D^2 u) = f(x) \quad \text{in } \Omega. \tag{1.6}$$

In this framework, we introduce a new model for anisotropic free transmission problems which is essentially based on the alternation (according to the positivity of solutions) of degeneracies of type (1.4), wherein we consider a Dirichlet problem governed by (1.1) and prove that at least a solution exists. This is the content of the following theorem:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain satisfying the uniform exterior sphere condition. Assume* (2.6)*,* (2.1)*,* (2.2)*,* (2.3)*,* (2.4) *and let $g \in C(\partial\Omega)$. Then there exists a viscosity solution $u \in C(\overline{\Omega})$ to Dirichlet problem*

$$\begin{cases} \left[|Du|^{p_u(x)} + a(x)\mathbb{1}_{\{u>0\}}|Du|^q + b(x)\mathbb{1}_{\{u<0\}}|Du|^s\right]F(D^2u) = f(x) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $p_u(x) := p^+\mathbb{1}_{\{u>0\}}(x) + p_-\mathbb{1}_{\{u<0\}}(x)$.

To prove Theorem 1.1 we first approximate a regularized variant of equation (1.7), obtained by relating the switch of degeneracies to the positivity of an arbitrary, globally continuous function, with a family of fully nonlinear Multi Phase equations with continuous variable exponents, cf. (1.6), and prove local Hölder continuity estimates that are uniform with respect to the parameter of approximation and to the moduli of continuity of the variable exponents and of the coefficients, see Appendix A. Then we establish a comparison principle for the approximating Dirichlet problems, construct continuous supersolutions/subsolutions and design a recursive procedure that will ultimately produce a solution to problem (1.7) via Perron theorem. Once the matter of existence of solutions to (1.7) has been settled, we turn to regularity. In this perspective, we have

Theorem 1.2. *Assume* (2.6)*,* (2.1)*,* (2.2)*,* (2.3)*,* (2.4) *and let $u \in C(\Omega)$ be a viscosity solution of equation* (1.1)*. Then there exists $\alpha_0 \equiv \alpha_0(n, \lambda, \Lambda, p^+, p_-) \in (0, 1)$ so that $u \in C^{1,\alpha_0}_{\mathrm{loc}}(\Omega)$. In particular, whenever $\Omega' \Subset \Omega$ is an open set it holds that*

$$[Du]_{0,\alpha_0;\Omega'} \le c(\mathtt{data}, \|u\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)}, \operatorname{dist}(\Omega', \partial\Omega)). \tag{1.8}$$

We refer to Section 2.2 for a description of the various quantities appearing in the above statement. The proof of Theorem 1.2 consists in three main steps. The first key observation to be made is that any viscosity solution $u \in C(\Omega)$ of (1.1), in the sense of Definition 2.4 below, turns out to be a viscosity subsolution of

$$\begin{aligned} &\min\left\{F(D^2u), [|Du|^{p^+} + a(x)|Du|^q]F(D^2u), [|Du|^{p_-} + b(x)|Du|^s]F(D^2u)\right\} \\ &\quad = \|f\|_{L^\infty(\Omega)} \end{aligned} \tag{1.9}$$

and a viscosity supersolution to

$$\begin{aligned} &\max\left\{F(D^2u), [|Du|^{p^+} + a(x)|Du|^q]F(D^2u), [|Du|^{p_-} + b(x)|Du|^s]F(D^2u)\right\} \\ &\quad = -\|f\|_{L^\infty(\Omega)}. \end{aligned} \tag{1.10}$$

Then we blow-up u and define a map with at most unitary oscillation that is a viscosity subsolution of an equation with the same structure of (1.9) and a viscosity supersolution of an equation similar to (1.10), both having arbitrarily small right-hand side datum. This blow-up procedure is particularly delicate due to the severe nonhomogeneity of (1.9)–(1.10) caused by the presence of different gradient powers. In contrast with the single power degeneracy case [38], the quantities involving the gradient variable on the left-hand side of (1.9)–(1.10) are not identically preserved after scaling; nonetheless, the Multi Phase structure can still be reproduced by introducing new modulating coefficients that incorporate the inhomogeneity excess. This way it is possible to reduce (1.9)–(1.10) to a smallness regime at the price of losing (uniform) control on both the L^∞-norm and the modulus of continuity of the coefficients during the blow-up procedure, see Section 5.1. For this reason, all the bounding constants appearing in the preliminary compactness estimates are carefully tracked in order to guarantee that none of them depend on the coefficients, which is coherent with the findings of [12, 22, 26, 32, 35]. Within the smallness framework, delicate perturbation arguments allow us to build a tangential path connecting viscosity subsolutions/supersolutions of (1.9)–(1.10) to a viscosity solution of the limiting profile—a homogeneous problem of the form

$$F(D^2h) = 0 \quad \text{in } B_1(0), \tag{1.11}$$

for which the Krylov–Safonov regularity theory is available. At this stage, the core of the proof becomes transferring such regularity from solutions of (1.11) to solutions of (1.1) via an iterative linearization scheme, eventually leading to (1.8). It is worth mentioning that the Hölder continuity exponent appearing in (1.8) depends on (n, λ, Λ), particularly through the exponent associated to the maximal regularity available for solutions to problem (1.11). We refer to Section 2.4 for more information on this matter. We can safely conjecture that the strategies exposed here and in [12, 26, 35] provide a solid blueprint for studying also in the setting of free transmission problems models that are more anisotropic than (1.3)–(1.6), such as

$$\big[|Du|\log(1 + |Du|) + a(x)|Du|^q\big]F(D^2u) = f(x) \quad \text{in } \Omega. \tag{1.12}$$

or, whenever $\varphi(\cdot)$ and $\psi(\cdot)$ are Orlicz functions,

$$\big[\varphi(|Du|) + a(x)\psi(|Du|)\big]F(D^2u) = f(x) \quad \text{in } \Omega. \tag{1.13}$$

In (1.12)–(1.13), $0 \le a(\cdot) \in C(\Omega)$ is expected and no restrictions on the size of q nor constraints linking $\varphi(\cdot)$ and $\psi(\cdot)$ should be imposed. Equations (1.3)–(1.6) as well as (1.12)–(1.13) are sophisticated examples of degenerate fully nonlinear elliptic equations, whose most celebrated prototype is

$$|Du|^p F(D^2u) = f \quad \text{in } \Omega, \tag{1.14}$$

see, e.g., [3, 11, 38]. Several aspects of this class of partial differential equations are very well known: comparison principle and Liouville type theorems [7], properties of eigenvalues and eigenfunctions [8], Alexandrov–Bakelman–Pucci estimates [24, 37], Harnack inequalities [25, 37], and regularity [9–11, 23, 38, 39].

1.1. Nonhomogeneous structures in the variational setting

As one could expect, equations (1.3)–(1.6) and (1.12)–(1.13) have a variational counterpart. Although the study of nonhomogeneous structures in the fully nonlinear framework started very recently with [12] for variable exponents and [26] in the Double Phase case, in the variational setting this field is the object of intense investigation and the first results date back to the pioneering papers [41,42], where the author introduced the so-called functionals with (p,q)-growth, aimed at treating in a unified fashion some regularity aspects of several anisotropic functionals or equations with unbalanced polynomial growth. Later on, considerable efforts were devoted to the analysis of specific nonautonomous models such as the $p(x)$-Laplacian [1,47,50]

$$W^{1,1}_{\mathrm{loc}}(\Omega) \ni w \mapsto \int_\Omega |Dw|^{p(x)}\,dx, \quad 1 < \inf_{x\in\Omega} p(x) \le p(\cdot) \in C^{0,\alpha}(\Omega) \tag{1.15}$$

or the Double Phase energy [6,17,27]

$$\begin{gathered} W^{1,1}_{\mathrm{loc}}(\Omega) \ni w \mapsto \int_\Omega \left[|Dw|^p + a(x)|Du|^q\right] dx, \\ 1 < p \le q, \quad \frac{q}{p} \le 1 + \frac{\alpha}{n}, \quad 0 \le a(\cdot) \in C^{0,\alpha}(\Omega). \end{gathered} \tag{1.16}$$

See also [29] for the analysis of nonhomogeneous equations related to (1.15)–(1.16) and obstacle problems. A nontrivial extension of (1.16) is the Multi Phase energy [4,30]

$$\begin{gathered} W^{1,1}_{\mathrm{loc}}(\Omega) \ni w \mapsto \int_\Omega \left[|Dw|^p + \sum_{\iota=1}^{\kappa} a_\iota(x)|Du|^{p_\iota}\right] dx, \\ 0 \le a_\iota(\cdot) \in C^{0,\alpha_\iota}(\Omega), \quad 1 \le \frac{p_\iota}{p} \le 1 + \frac{\alpha_\iota}{n}, \quad 1 < p \le p_1 \le \cdots \le p_\kappa, \end{gathered} \tag{1.17}$$

which features several phase transitions where the functional changes its ellipticity. This seems to be the right choice for modeling anisotropic free transmission problems and in fact, its fully nonlinear version (1.6) is fundamental for the formulation of (1.1). A borderline version of (1.16) is the following [28]:

$$\begin{gathered} W^{1,1}_{\mathrm{loc}}(\Omega) \ni w \mapsto \int_\Omega \left[|Dw|\log(1+|Dw|) + a(x)|Dw|^q\right] dx, \\ 0 \le a(\cdot) \in W^{1,d}(\Omega) \text{ with } d > n, \quad q < 1 + \frac{1}{n} - \frac{1}{d}, \end{gathered}$$

which in nondivergence form becomes (1.12). Other models inspired by (1.15)–(1.16) are the Double Phase energy with variable exponents [48,51], see (1.5)–(1.6), and the generalized Double Phase integral [13], cf. (1.13); we further refer to [43] for an account of the state of the art on this matter. The peculiarity of these functionals is that in the variational setting there is a strict interplay between the regularity of the x-depending coefficients and the regularity of minimizers, therefore each of them has to be treated in a very specific way that takes into account the structure of the operator involved. Only recently has a unified approach been proposed within the framework of Musielak–Orlicz spaces, see [34]. As already observed in [26, Section 1], there is a huge difference in the behavior of the nonhomogeneous structures listed above between the variational and the nonvariational setting, and this phenomenon is confirmed by Theorem 1.2 for anisotropic free transmission problems. In sharp contrast to what happens for instance with (1.16)–(1.17), where a sharp constraint linking the Hölder continuity exponent of $a_\iota(\cdot)$ with the growth exponents p, p_ι is needed to get regular minima [31, 33], here the plain continuity of $a(\cdot)$ and $b(\cdot)$ suffices, cf. (2.3). In fact, to prove our regularity results, we just ask that the coefficients $a(\cdot)$, $b(\cdot)$ are continuous and no restriction on the size of the differences $0 \le q - p^+$, $0 \le s - p_-$ is imposed, see (2.2). This makes Theorem 1.2 sharp from the viscosity theory viewpoint.

Organization of the paper. This paper is organized as follows: In Section 2 we display our notation, describe the main assumptions considered by Theorems 1.1 and 1.2 and recall some well-known results that will be needed later on. In Section 3 we prove Theorem 1.1, i.e., that there exists at least one solution to Dirichlet problem (1.7). In Section 4 we establish a uniform Hölder continuity result for solutions of suitable switched equations related to (1.1). Finally, Section 5 contains a description of the scaling properties of the viscosity differential inequalities (1.9)–(1.10), a "harmonic" approximation lemma, and the proof of Theorem 1.2.

2. Preliminaries

We shall split this section in three parts: first, we display our notation, then we collect the main assumptions governing problem (1.1), and finally we report some well-known results on the theory of viscosity solutions to uniformly elliptic operators.

2.1. Notation

In this paper, $\Omega \subset \mathbb{R}^n$, $n \ge 2$ is an open and bounded domain and the open ball of $\mathbb{R}^n$ centered at x_0 with positive radius ϱ is denoted by $B_\varrho(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < \varrho\}$. When irrelevant or clear from the context, we will omit indicating the center, so that $B_\varrho \equiv B_\varrho(x_0)$. In particular, for $\varrho = 1$ and $x_0 = 0$, we shall simply denote $B_1 \equiv B_1(0)$. By $\mathcal{S}(n)$ we mean the space of $n \times n$ symmetric matrices. As usual, we denote by c a general constant larger than one. Different occurrences from line to line will be still indicated by c and relevant dependencies from certain parameters will be emphasized using brackets, i.e., $c(n, p)$ means that c depends on n and p. With $z, \xi \in \mathbb{R}^n$, $\mu \in [0, 1]$, $p, q, s \in [0, \infty)$ and $a(\cdot)$, $b(\cdot)$ being nonnegative functions, we define

$$\ell_\mu(z) := \sqrt{\mu^2 + |z|^2}, \quad p_v(x) := p^+ \mathbb{1}_{\{v>0\}} + p_- \mathbb{1}_{\{v<0\}}$$

and

$$\begin{aligned} H_q(x, z; \xi) &:= \left[|\xi + z|^{p^+} + a(x)|\xi + z|^q \right], \\ H_s(x, z; \xi) &:= \left[|\xi + z|^{p^-} + b(x)|\xi + z|^s \right], \\ H(x, v, z; \xi) &:= \left[|\xi + z|^{p_v(x)} + a(x)\mathbb{1}_{\{v>0\}}|\xi + z|^q + b(x)\mathbb{1}_{\{v<0\}}|\xi + z|^s \right]. \end{aligned}$$

When $\xi \equiv 0$, we shall simply write $H_q(x,z;0) \equiv H_q(x,z)$, $H_s(x,z;0) \equiv H_s(x,z)$ and $H(x,z;0) \equiv H(x,z)$. If $g\colon \Omega \to \mathbb{R}^k$ is any map, $U \subset \Omega$ is an open set and $\beta \in (0,1]$ is a given number, we shall write

$$[g]_{0,\beta;U} := \sup_{x,y\in U; x\neq y} \frac{|g(x)-g(y)|}{|x-y|^\beta}, \quad [g]_{0,\beta} := [g]_{0,\beta;\Omega}.$$

It is well known that the quantity defined above is a seminorm and when $[g]_{0,\beta;U} < \infty$, we will say that g belongs to the Hölder space $C^{0,\beta}(U,\mathbb{R}^k)$. We stress also that $g \in C^{1,\beta}(U,\mathbb{R}^k)$, provided that

$$[g]_{1+\beta;U} := \sup_{\varrho>0, x\in U} \inf_{\xi\in\mathbb{R}^n,\kappa\in\mathbb{R}} \sup_{y\in B_\varrho(x)\cap U} \varrho^{-(1+\beta)}|g(y)-\xi\cdot y-\kappa| < \infty.$$

Finally, $\mathbf{I}$ denotes the identity of $\mathbb{R}^{n\times n}$ and given any $n\times n$ matrix A, by $\operatorname{tr}(A)$ we mean the trace of A, i.e., the sum of all its eigenvalues, by $\operatorname{tr}(A^+)$ the sum of all positive eigenvalues of A, and by $\operatorname{tr}(A^-)$ the sum of all negative eigenvalues of A.

2.2. Main assumptions

When dealing with equation (1.1) or with Dirichlet problem (1.7), the assumptions which follow are enforced. As mentioned before, the set $\Omega \subset \mathbb{R}^n$ is an open and bounded domain with smooth boundary. The nonlinear operator $F(\cdot)$ is continuous and (λ,Λ)-elliptic in the sense of (2.6) below. Moreover,

$$F \in C(\mathcal{S}(n),\mathbb{R}), \quad F(0) = 0. \tag{2.1}$$

Concerning the nonhomogeneous degeneracy term appearing in (1.1), we shall ask that the exponents p^+, p_-, q, s satisfy

$$0 \le p^+ \le q \quad \text{and} \quad 0 \le p_- \le s, \tag{2.2}$$

the modulating coefficients $a(\cdot), b(\cdot)$ are such that

$$0 \le a(\cdot) \in C(\Omega) \quad \text{and} \quad 0 \le b(\cdot) \in C(\Omega), \tag{2.3}$$

and the forcing term f is such that

$$f \in C(\Omega) \cap L^\infty(\Omega). \tag{2.4}$$

To simplify the notation, we shall collect the main parameters related to the problems under investigation in the shorthand $\texttt{data} := (n,\lambda,\Lambda,p^+,p_-,q,s)$.

2.3. On uniformly elliptic operators

A map $G \in C(\Omega\times\mathbb{R}^n\times\mathcal{S}(n),\mathbb{R})$ is monotone if

$$G(x,z,M) \le G(x,z,N) \quad \text{for all } M,N \in \mathcal{S}(n) \text{ such that } M \ge N. \tag{2.5}$$

The (λ, Λ)-ellipticity condition for an operator $F: \mathcal{S}(n) \to \mathbb{R}$ prescribes that, whenever $A, B \in \mathcal{S}(n)$ are symmetric matrices with $B \geq 0$,

$$\lambda \operatorname{tr}(B) \leq F(A) - F(A+B) \leq \Lambda \operatorname{tr}(B) \tag{2.6}$$

for some fixed constants $0 < \lambda \leq \Lambda$. With this definition, $F(A) := -\operatorname{tr}(A)$ is uniformly elliptic with $\lambda = \Lambda = 1$ [38], so the usual Laplace operator "$-\Delta$" is uniformly elliptic. Moreover, it is easy to see that if L is any fixed, positive constant, then the operator $F_L(M) := LF(\frac{1}{L}M)$ satisfies (2.6) with the same constants $0 < \lambda \leq \Lambda$. Moreover, (2.6) is also satisfied by the operator $\widetilde{F}(M) := -F(-M)$, cf. [26, Section 2.2]. In this framework, it is important to introduce the Pucci extremal operators $\mathcal{P}^{\pm}_{\lambda,\Lambda}(\cdot)$, which are, respectively, the maximum and the minimum of all the uniformly elliptic functions $F(\cdot)$ with $F(0) = 0$. In particular, they admit the compact form

$$\mathcal{P}^{+}_{\lambda,\Lambda}(A) = -\Lambda \operatorname{tr}(A^{-}) - \lambda \operatorname{tr}(A^{+}) \quad \text{and} \quad \mathcal{P}^{-}_{\lambda,\Lambda}(A) = -\Lambda \operatorname{tr}(A^{+}) - \lambda \operatorname{tr}(A^{-}). \tag{2.7}$$

We can give an alternative formulation of (2.6) involving the Pucci extremal operators as follows:

$$\mathcal{P}^{-}_{\lambda,\Lambda}(B) \leq F(A+B) - F(A) \leq \mathcal{P}^{+}_{\lambda,\Lambda}(B), \tag{2.8}$$

that holds for all $A, B \in \mathcal{S}(n)$. Next, we turn our attention to equation

$$G_\xi(x, Du, D^2u) := G(x, \xi + Du, D^2u) = 0 \quad \text{in } \Omega, \tag{2.9}$$

with $G(\cdot)$ continuous and satisfying (2.5) and $\xi \in \mathbb{R}^n$ an arbitrary vector. The concept of viscosity solution for equation (2.9) can be explained as follows, cf. [5, 18]:

Definition 2.1. A lower semicontinuous function v is a viscosity supersolution of (2.9) if whenever $\varphi \in C^2(\Omega)$ and $x_0 \in \Omega$ is a local minimum point of $v - \varphi$, then

$$G_\xi(x_0, D\varphi(x_0), D^2\varphi(x_0)) \geq 0,$$

while an upper semicontinuous function w is a viscosity subsolution to (2.9) provided that if x_0 is a local maximum point of $w - \varphi$, it holds that

$$G_\xi(x_0, D\varphi(x_0), D^2\varphi(x_0)) \leq 0.$$

The map $u \in C(\Omega)$ is a viscosity solution of (2.9) if it is at the same time a viscosity subsolution and a viscosity supersolution.

Another important notion is the one of subjets and superjets [5].

Definition 2.2. Let $v: \Omega \to \mathbb{R}$ be an upper semicontinuous function and $w: \Omega \to \mathbb{R}$ be a lower semicontinuous function.

- A pair $(z, X) \in \mathbb{R}^n \times \mathcal{S}(n)$ is a superjet of v at $x \in \Omega$ if

$$v(x+y) \leq v(x) + z \cdot y + \frac{1}{2} Xy \cdot y + o(|y|^2).$$

- A pair $(z, X) \in \mathbb{R}^n \times \mathcal{S}(n)$ is a subjet of w at $x \in \Omega$ if

$$w(x+y) \geq w(x) + z \cdot y + \frac{1}{2} Xy \cdot y + o(|y|^2).$$

- A pair $(z, X) \in \mathbb{R}^n \times \mathcal{S}(n)$ is a limiting superjet of v at $x \in \Omega$ if there exists a sequence $\{x_j, z_j, X_j\} \to_{j\to\infty} \{x, z, X\}$ such that $\{z_j, X_j\}$ is a superjet of v at x_j and $v(x_j) \to_{j\to\infty} v(x)$.
- A pair $(z, X) \in \mathbb{R}^n \times \mathcal{S}(n)$ is a limiting subjet of w at $x \in \Omega$ if there exists a sequence $\{x_j, z_j, X_j\} \to_{j\to\infty} \{x, z, X\}$ such that $\{z_j, X_j\}$ is a subjet of w at the point x_j and $w(x_j) \to_{j\to\infty} w(x)$.

Now we are in position to present a variation to the maximum principle [5, Lemma 1 and Corollaries 1–2], [18, Theorem 3.2].

Proposition 2.3. *Let v be an upper semicontinuous viscosity subsolution of* (2.9)*, w a lower semicontinuous viscosity supersolution of* (2.9)*, $U \Subset \Omega$ an open set and $\psi \in C^2(U \times U)$. If $(\overline{x}, \overline{y}) \in U \times U$ is a local maximum point of $v(x) - w(y) - \psi(x, y)$, then there exists a positive threshold $\widehat{\delta} \equiv \widehat{\delta}(\|D^2\psi\|)$ so that for any $\delta \in (0, \widehat{\delta})$ there are matrices $X_\delta, Y_\delta \in \mathcal{S}(n)$ satisfying*

$$G_\xi(\overline{x}, v(\overline{x}), \partial_x \psi(\overline{x}, \overline{y}), X_\delta) \leq 0 \leq G_\xi(\overline{y}, w(\overline{y}), -\partial_y \psi(\overline{x}, \overline{y}), Y_\delta),$$

and the matrix inequality

$$-\frac{1}{\delta}\boldsymbol{I} \leq \begin{bmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{bmatrix} \leq D^2\psi(\overline{x}, \overline{y}) + \delta \boldsymbol{I}$$

holds true.

So far, we have described the main features of equations governed by a continuous map $G(\cdot)$, while in the forthcoming sections we shall deal with problems with discontinuous degeneracies of the type

$$\begin{aligned} &\big[|\xi + Du|^{p_v(x)} + a(x)\mathbb{1}_{\{v>0\}}|\xi + Du|^q + b(x)\mathbb{1}_{\{v<0\}}|\xi + Du|^s\big] F(D^2u) \\ &\quad = f(x) \quad \text{in } \Omega, \end{aligned} \tag{2.10}$$

where assumptions (2.6), (2.1), (2.2), (2.3), and (2.4) are in force, $\xi \in \mathbb{R}^n$ is any vector, $v \in C(\overline{\Omega})$, and $p_v(x) := p^+\mathbb{1}_{\{v>0\}}(x) + p_-\mathbb{1}_{\{v<0\}}(x)$. In light of the discussion in [35, Section 2.2], we define a viscosity solution to (2.10) as follows.

Definition 2.4. Let $v \in C(\Omega)$ be a function, $\xi \in \mathbb{R}^n$ be a vector and assumptions (2.6), (2.1), (2.2), (2.3), and (2.4) be in force. The map $u \in C(\Omega)$ is a viscosity solution to (2.10) if

- in the set $\{x \in \Omega : v(x) > 0\} \cap \{x \in \Omega : v(x) < 0\}$, u is a viscosity solution of (2.10);

- u is a viscosity subsolution of

$$\min\left\{F(D^2u), H_q(x, Du; \xi)F(D^2u), H_s(x, Du; \xi)F(D^2u)\right\} = f(x)$$

  in $\{x \in \Omega\colon v(x) = 0\}$;
- u is a viscosity supersolution of

$$\max\left\{F(D^2u), H_q(x, Du; \xi)F(D^2u), H_s(x, Du; \xi)F(D^2u)\right\} = f(x)$$

  in $\{x \in \Omega\colon v(x) = 0\}$.

Clearly, the above definition also applies to (1.1) by letting $v \in C(\Omega)$ be the unknown function u, so from now on, whenever we refer to a continuous viscosity solution of equations (1.1) or (2.10) or to Dirichlet problem (1.7), we shall mean it in the sense of Definition 2.4. The previous position can be justified by noticing that although the ingredients of (2.10) are discontinuous, they are everywhere defined and this allows us to work within the realm of C-viscosity solutions. If we set

$$\mathfrak{H}_\xi(x, v, z, X) := \left[H(x, v, z; \xi)F(X) - f(x)\right]$$

and introduce the lower and upper semicontinuous envelopes, denoted by $\mathfrak{H}_{\xi *}(\cdot)$ and $\mathfrak{H}_\xi^*(\cdot)$ respectively, then $u \in C(\Omega)$ is a viscosity solution of (2.10) if and only if it is a viscosity subsolution of $\mathfrak{H}_{\xi *}(x, v, Du, D^2u) = 0$ as well as a viscosity supersolution of $\mathfrak{H}_\xi^*(x, v, Du, D^2u) = 0$, [16, 35]. As a consequence, a viscosity solution of equation (2.10) is a viscosity subsolution to

$$\min\left\{F(D^2u), H_q(x, Du; \xi)F(D^2u), H_s(x, Du; \xi)F(D^2u)\right\} = \|f\|_{L^\infty(\Omega)} \quad \text{in } \Omega \tag{2.11}$$

and a viscosity supersolution of

$$\max\left\{F(D^2u), H_q(x, Du; \xi)F(D^2u), H_s(x, Du; \xi)F(D^2u)\right\} = -\|f\|_{L^\infty(\Omega)} \quad \text{in } \Omega, \tag{2.12}$$

as prescribed by Definition 2.1. Therefore, it is always the classical Definition 2.1 that is adapted in a slightly unusual way to deal with the singularities developed in (2.10) when the continuous function v changes sign. This is also coherent with the definition of viscosity solutions to singular problems given in [8, 11], where testing against C^2-maps with vanishing gradient is forbidden because of possible blow-up on small gradients. However, we do not need to exclude test functions with null gradients since by (2.2), equations (2.10), (2.11), and (2.12) are degenerate, so (2.1) and the continuity of v assure that the definition in [8, 11] is equivalent to Definition 2.1.

Remark 2.5. If $u \in C(\Omega)$ is a viscosity solution of (2.10) in the sense of Definition 2.4, then it is a viscosity subsolution/supersolution of variants of (2.11)/(2.12) having as a right-hand side term any constant larger than or equal to $\|f\|_{L^\infty(\Omega)}$. This observation will be useful when proving regularity, see Sections 4–5 below.

2.4. The homogeneous problem

Viscosity solutions of the homogeneous problem (1.11) will play a crucial role in the proof of the main results of this paper. In fact, viscosity solutions of problem (1.11) have good regularity properties, as the next proposition shows. For a proof of this proposition, we refer to [15, Corollary 5.7].

Proposition 2.6. *Assume that $F(\cdot)$ satisfies (2.6)–(2.1) and let $h \in C(B_1(0))$ be a viscosity solution of (1.11). Then, there exist $\alpha \equiv \alpha(n,\lambda,\Lambda) \in (0,1)$ and $c \equiv c(n,\lambda,\Lambda) > 0$ such that*

$$\|h\|_{C^{1,\alpha}(\overline{B}_{1/2}(0))} \le c\|h\|_{L^\infty(B_1(0))}. \tag{2.13}$$

Proposition 2.6 yields in particular that if $h \in C(B_1(0))$ is a viscosity solution to (1.11) then it is $C^{1,\alpha}$-regular around zero, which means that for all $\varrho \in (0,1)$, there exists $\xi_\varrho \in \mathbb{R}^n$ such that

$$\operatorname{osc}_{B_\varrho}(h - \xi_\varrho \cdot x) \le c(n,\lambda,\Lambda)\varrho^{1+\alpha}. \tag{2.14}$$

Now, fix $\sigma \in (0,1)$ so small that

$$c\sigma^\alpha < \frac{1}{4}, \tag{2.15}$$

where $c = c(n,\lambda,\Lambda)$ is the constant appearing in (2.13), and let $\xi_\sigma \in \mathbb{R}^n$ be the corresponding vector in (2.14). According to the choice made in (2.15), estimate (2.14) reads as

$$\operatorname{osc}_{B_\sigma(0)}(h - \xi_\sigma \cdot x) \le \frac{1}{4}\sigma \quad \text{with } \sigma \equiv \sigma(n,\lambda,\Lambda). \tag{2.16}$$

This will be useful later on.

Remark 2.7. It is well known, and it will be also evident from the proof of Theorem 1.2, that the Hölder continuity exponent appearing in (2.13) acts as a threshold value for the maximal regularity of solutions to degenerate equations. In case $F(\cdot)$ is concave or convex, higher regularity for solutions of (1.11) is available and $\alpha = 1$ by means of Evans–Krylov theory, see [15, Chapter 6].

3. Existence of solutions

In this section, we prove the existence of a continuous viscosity solution to Dirichlet problem (1.7). To do so, we need to introduce a family of approximating problems, prove a comparison principle and then conclude via Perron method. For $\varepsilon \in (0,1)$, let $\{\phi_\varepsilon\} \subset C^\infty(\mathbb{R}^n)$ be a sequence of radially symmetric, nonnegative mollifiers of $\mathbb{R}^n$, and $v \in C(\overline{\Omega})$ be a continuous function. In addition, we write $\{\chi^+_{\varepsilon;v}\} := \{\phi_\varepsilon * \mathbb{1}_{\{v>0\}}\} \subset C^\infty_{\text{loc}}(\Omega)$ and $\{\chi^-_{\varepsilon;v}\} := \{\phi_\varepsilon * \mathbb{1}_{\{v<0\}}\} \subset C^\infty_{\text{loc}}(\Omega)$. We define

$$\begin{cases} p_{\varepsilon;v}(x) := \varepsilon + p^+\chi^+_{\varepsilon;v}(x) + p_-\chi^-_{\varepsilon;v}(x), \\ a_{\varepsilon;v}(x) := (\varepsilon + a(x)\chi^+_\varepsilon(x)), \\ b_{\varepsilon;v}(x) := (\varepsilon + b(x)\chi^-_\varepsilon(x)), \end{cases} \tag{3.1}$$

for $p^+, p_-, a(\cdot), b(\cdot)$ as in (2.2)–(2.3). By very definition, both coefficients defined in (3.1) are positive and continuous in Ω. With these positions at hand, set

$$\Omega \times \mathbb{R}^n \ni (x,z) \mapsto G_\varepsilon(x,z) := \left[\ell_\varepsilon(z)^{p_{\varepsilon;v}(x)} + a_{\varepsilon;v}(x)\ell_\varepsilon(z)^q + b_{\varepsilon;v}(x)\ell_\varepsilon(z)^s\right]$$

and consider the equation

$$G_\varepsilon(x, Du_\varepsilon)\big(\varepsilon u_\varepsilon + F(D^2 u_\varepsilon)\big) = f(x) \quad \text{in } \Omega, \tag{3.2}$$

with $F(\cdot)$ as in (2.1) and $f(\cdot)$ described by (2.4). Let us prove a comparison principle for subsolutions and supersolutions of (3.2).

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, assumptions (2.6), (2.1), (2.2), (2.3), (2.4) be in force, $w_1 \in \mathrm{USC}(\overline{\Omega})$ be a subsolution of (3.2), and $w_2 \in \mathrm{LSC}(\overline{\Omega})$ be a supersolution of (3.2). Then*

$$w_1 \le w_2 \text{ on } \partial\Omega \implies w_1 \le w_2 \text{ in } \Omega.$$

Proof. By contradiction, assume that

$$\omega_0 := \max_{x\in\Omega}\big(w_1(x) - w_2(x)\big) > 0. \tag{3.3}$$

For $\sigma > 0$, set

$$\Phi_\sigma(x,y) := w_1(x) - w_2(y) - \frac{|x-y|^2}{2\sigma}$$

and notice that, if $(x_\sigma, y_\sigma) \in \overline{\Omega} \times \overline{\Omega}$ is a point of maximum for $\Phi_\sigma(\cdot)$, i.e.,

$$\max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \Phi_\sigma(x,y) = \Phi_\sigma(x_\sigma, y_\sigma) \ge \omega_0, \tag{3.4}$$

by [18, Lemma 3.1] we have

$$\lim_{\sigma\to 0} \frac{|x_\sigma - y_\sigma|^2}{\sigma} = 0 \implies \lim_{\sigma\to 0} |x_\sigma - y_\sigma| = 0. \tag{3.5}$$

Notice that x_σ, y_σ cannot both belong to $\partial\Omega$, otherwise $\Phi_\sigma(x_\sigma, y_\sigma) < 0$, in contradiction with (3.3)–(3.4). Then at least one of them, say x_σ, must be in the interior of Ω and (3.5) forces also y_σ to stay inside Ω. We can then apply [18, Theorem 3.2] to obtain that for all $\delta > 0$ we have two symmetric matrices $X_\delta, Y_\delta \in \mathcal{S}(n)$ so that $(\frac{x_\sigma - y_\sigma}{\sigma}, X_\delta)$ is a superjet of w_1 at x_σ, $(\frac{x_\sigma - y_\sigma}{\sigma}, Y_\delta)$ is a subjet of w_2 at y_σ, and the matrix inequality

$$\Big(-\frac{1}{\delta} + c(n,\sigma)\Big)\begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \le \begin{bmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{bmatrix} \le \frac{3(1+\delta)}{\sigma}\begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{bmatrix}$$

holds. Therefore, testing against the couple $(\xi, \xi) \in \mathbb{R}^{2n}$ we get

$$2\Big(-\frac{1}{\delta} + c(n,\sigma)\Big)|\xi|^2 \le \langle (X_\delta - Y_\delta)\xi, \xi\rangle \le 0 \implies Y_\delta \ge X_\delta. \tag{3.6}$$

We can then recover the viscosity inequalities

$$\begin{cases} G_\varepsilon\Big(x_\sigma, \dfrac{x_\sigma - y_\sigma}{\sigma}\Big)\big(\varepsilon w_1(x_\sigma) + F(X_\delta)\big) \le f(x_\sigma), \\ G_\varepsilon\Big(y_\sigma, \dfrac{x_\sigma - y_\sigma}{\sigma}\Big)\big(\varepsilon w_2(y_\sigma) + F(Y_\delta)\big) \ge f(y_\sigma), \end{cases}$$

and subtract the second from the first to get

$$\begin{aligned} \frac{f(x_\sigma)}{G_\varepsilon\big(x_\sigma, \frac{x_\sigma - y_\sigma}{\sigma}\big)} - \frac{f(x_\sigma)}{G_\varepsilon\big(y_\sigma, \frac{x_\sigma - y_\sigma}{\sigma}\big)} &\ge \varepsilon(w_1(x_\sigma) - w_2(y_\sigma)) + F(X_\delta) - F(Y_\delta) \\ &\overset{(2.6)}{\ge} \varepsilon(w_1(x_\sigma) - w_2(y_\sigma)) + \lambda \operatorname{tr}(Y_\delta - X_\delta) \\ &\overset{(3.6)}{\ge} \varepsilon(w_1(x_\sigma) - w_2(y_\sigma)) \overset{(3.4)}{\ge} \varepsilon\omega_0. \end{aligned} \tag{3.7}$$

At this point, recall that for any $\iota_0 > 0$ there exists a constant $c \equiv c(\iota_0, l, m)$ such that for all $t \ge 0$, $l, m \ge 0$ it holds that $|t^l - t^m| \le c|l - m|(1 + t^{(1+\iota_0)\max\{l,m\}})$, so choosing

$$\iota_0 := \frac{\varepsilon}{16(p^+ + p_- + 1)},$$

we see that

$$\begin{cases} \iota_0 \max\{p_{\varepsilon;v}(x_\sigma), p_{\varepsilon;v}(y_\sigma)\} - \min\{p_{\varepsilon;v}(x_\sigma), p_{\varepsilon;v}(y_\sigma)\} < -\frac{15\varepsilon}{16(p^+ + p_- + 1)}, \\ |\iota_0 \max\{p_{\varepsilon;v}(x_\sigma), p_{\varepsilon;v}(y_\sigma)\} - \min\{p_{\varepsilon;v}(x_\sigma), p_{\varepsilon;v}(y_\sigma)\}| \le 4(p^+ + p_- + 1), \end{cases}$$

and so

$$\begin{aligned} \mathcal{L}(\varepsilon, \sigma) &:= \ell_\varepsilon\Big(\frac{x_\sigma - y_\sigma}{\sigma}\Big)^{-(p_{\varepsilon;v}(x_\sigma) + p_{\varepsilon;v}(y_\sigma))} \\ &\quad \cdot \Big|\ell_\varepsilon\Big(\frac{x_\sigma - y_\sigma}{\sigma}\Big)^{p_{\varepsilon;v}(x_\sigma)} - \ell_\varepsilon\Big(\frac{x_\sigma - y_\sigma}{\sigma}\Big)^{p_{\varepsilon;v}(y_\sigma)}\Big| \\ &\le c|p_{\varepsilon;v}(x_\sigma) - p_{\varepsilon;v}(y_\sigma)|\ell_\varepsilon\Big(\frac{x_\sigma - y_\sigma}{\sigma}\Big)^{-(p_{\varepsilon;v}(x_\sigma) + p_{\varepsilon;v}(y_\sigma))} \\ &\quad \cdot \Big[1 + \ell_\varepsilon\Big(\frac{x_\sigma - y_\sigma}{\sigma}\Big)^{(1+\iota_0)\widehat{p}_{\varepsilon;\sigma}}\Big] \\ &\le c|p_{\varepsilon;v}(x_\sigma) - p_{\varepsilon;v}(y_\sigma)|\Big[\varepsilon^{-6\max\{p^+, p_-, 1\}} + \ell_\varepsilon\Big(\frac{x_\sigma - y_\sigma}{\sigma}\Big)^{\iota_0 \widehat{p}_{\varepsilon;\sigma} - \tilde{p}_{\varepsilon;\sigma}}\Big] \\ &\le \frac{c|p_{\varepsilon;v}(x_\sigma) - p_{\varepsilon;v}(y_\sigma)|}{\varepsilon^{6(p^+ + p_- + 1)}}, \end{aligned} \tag{3.8}$$

where we set $\widehat{p}_{\varepsilon;\sigma} := \max\{p_{\varepsilon;v}(x_\sigma), p_{\varepsilon;v}(y_\sigma)\}$, $\tilde{p}_{\varepsilon;\sigma} := \min\{p_{\varepsilon;v}(x_\sigma), p_{\varepsilon;v}(y_\sigma)\}$, and we used that $\iota_0 \widehat{p}_{\varepsilon;v} - (p_{\varepsilon;v}(x_\sigma) + p_{\varepsilon;v}(y_\sigma)) \le \iota_0 \widehat{p}_{\varepsilon;v} - \tilde{p}_{\varepsilon;v} < 0$ and $c \equiv c(\varepsilon, p^+, p_-)$. Via (2.1), (2.4), (3.8) and using that $a_{\varepsilon;v}(\cdot), b_{\varepsilon;v}(\cdot) \ge \varepsilon$ and $\ell_\varepsilon(\sigma^{-1}(x_\sigma - y_\sigma)) \ge \varepsilon$, we manipulate (3.7) to obtain

$$
\begin{aligned}
&\frac{c\|f\|_{L^\infty(\Omega)}|p_{\varepsilon;v}(x_\sigma)-p_{\varepsilon;v}(y_\sigma)|}{\varepsilon^{6(p^++p_-+1)}}+\frac{\|f\|_{L^\infty(\Omega)}|a_{\varepsilon;v}(x_\sigma)-a_{\varepsilon;v}(y_\sigma)|}{\varepsilon^{2q}}\\
&\qquad+\frac{\|f\|_{L^\infty(\Omega)}|b_{\varepsilon;v}(x_\sigma)-b_{\varepsilon;v}(y_\sigma)|}{\varepsilon^{2s}}+\frac{|f(x_\sigma)-f(y_\sigma)|}{\varepsilon^{2(p^++p_-+1)}}\\
&\quad\ge \|f\|_{L^\infty(\Omega)}\mathcal{L}(\varepsilon,\sigma)+\frac{\|f\|_{L^\infty(\Omega)}|a_{\varepsilon;v}(x_\sigma)-a_{\varepsilon;v}(y_\sigma)|}{a_{\varepsilon;v}(x_\sigma)a_{\varepsilon;v}(y_\sigma)\ell_\varepsilon\big(\frac{x_\sigma-y_\sigma}{\sigma}\big)^q}\\
&\qquad+\frac{\|f\|_{L^\infty(\Omega)}|b_{\varepsilon;v}(x_\sigma)-b_{\varepsilon;v}(y_\sigma)|}{b_{\varepsilon;v}(x_\sigma)b_{\varepsilon;v}(y_\sigma)\ell_\varepsilon\big(\frac{x_\sigma-y_\sigma}{\sigma}\big)^s}+\frac{|f(x_\sigma)-f(y_\sigma)|}{G_\varepsilon\big(y_\sigma,\frac{x_\sigma-y_\sigma}{\sigma}\big)}\\
&\quad\ge \frac{f(x_\sigma)}{G_\varepsilon\big(x_\sigma,\frac{x_\sigma-y_\sigma}{\sigma}\big)}-\frac{f(y_\sigma)}{G_\varepsilon\big(y_\sigma,\frac{x_\sigma-y_\sigma}{\sigma}\big)}\\
&\quad\ge \varepsilon\omega_0,
\end{aligned}
$$

therefore, we have

$$
\begin{aligned}
&\frac{c\|f\|_{L^\infty(\Omega)}|p_{\varepsilon;v}(x_\sigma)-p_{\varepsilon;v}(y_\sigma)|}{\varepsilon^{6(p^++p_-+1)+1}}+\frac{|f(x_\sigma)-f(y_\sigma)|}{\varepsilon^{2(p^++p_-+1)+1}}\\
&\quad+\|f\|_{L^\infty(\Omega)}\Big[\frac{|a_{\varepsilon;v}(x_\sigma)-a_{\varepsilon;v}(y_\sigma)|}{\varepsilon^{2q+1}}+\frac{|b_{\varepsilon;v}(x_\sigma)-b_{\varepsilon;v}(y_\sigma)|}{\varepsilon^{2s+1}}\Big]\ge\omega_0,
\end{aligned}
\tag{3.9}
$$

with $c\equiv c(\varepsilon,p^+,p_-)$. Recalling that $f(\cdot)$, $p_{\varepsilon;v}(\cdot)$, $a_{\varepsilon;v}(\cdot)$ and $b_{\varepsilon;v}(\cdot)$ are continuous and that $|x_\sigma-y_\sigma|\to 0$ by (3.5), we can send $\sigma\to 0$ in (3.9) to reach a contradiction with (3.3). The proof is complete. ■

At this stage, we need to construct continuous viscosity subsolutions and supersolutions of (3.2) with a fixed boundary datum.

Lemma 3.2. *Let $\Omega\subset\mathbb{R}^n$ be an open, bounded domain satisfying a uniform exterior sphere condition. Assume (2.1), (2.2), (2.3), (2.4), (2.6) and let $g\in C(\partial\Omega)$ be any function with modulus of continuity $\omega_g(\cdot)$. Then equation (3.2) admits a viscosity supersolution $\overline{w}\in C(\overline{\Omega})$ and a viscosity subsolution $\underline{w}\in C(\overline{\Omega})$ for all numbers $\varepsilon\in(0,1)$, maps $v\in C(\overline{\Omega})$, so that $\underline{w}|_{\partial\Omega}=\overline{w}|_{\partial\Omega}=g$.*

Proof. The proof closely follows that of [35, Lemma 2], see also [19, Proposition 3.2]. We construct a continuous viscosity supersolution $\overline{w}$ to (3.2) agreeing with g on $\partial\Omega$ for any $\varepsilon\in(0,1)$ and all functions $v\in C(\overline{\Omega})$. The construction of a subsolution $\underline{w}$ with analogous features can be obtained in a similar way. Let $x_0\in\mathbb{R}^n$ be any point with $\operatorname{dist}(x_0,\Omega)\ge 1$. Set

$$
\Gamma_1:=\max\{\|f\|_{L^\infty(\Omega)},\lambda n\},\qquad \Gamma_2:=\frac{16(\operatorname{dist}(x_0,\Omega)+\operatorname{diam}(\Omega))^2\Gamma_1}{\lambda n}+\|g\|_{L^\infty(\partial\Omega)}
$$

and define the function

$$
\widetilde{w}(x):=\Gamma_2-\Gamma_1|x-x_0|^2(2\lambda n)^{-1}.
$$

The choice of Γ_1 and Γ_2 yields that $|D\widetilde{w}| \geq 1$, $\widetilde{w} \geq 0$ in Ω and that $\widetilde{w}|_{\partial\Omega} \geq \|g\|_{L^\infty(\partial\Omega)}$. Now, notice that $F(D^2\widetilde{w}) \geq 0$, in fact, being the identity positive definite, we obtain

$$F(D^2\widetilde{w}) \overset{(2.1)_2}{=} F\Big(-\frac{\Gamma_1 \mathbf{I}}{\lambda n}\Big) - F(0) \overset{(2.6)}{\geq} \lambda \operatorname{tr}\Big(\frac{\Gamma_1 \mathbf{I}}{\lambda n}\Big) \geq \Gamma_1,$$

so for all $x \in \Omega$ we have

$$\begin{aligned}\big[\ell_\varepsilon(D\widetilde{w})^{p_{\varepsilon;v}(x)} + a_{\varepsilon;v}(x)\ell_\varepsilon(D\widetilde{w})^q + b_{\varepsilon;v}(x)\ell_\varepsilon(D\widetilde{w})^s\big]\big(\varepsilon\widetilde{w} + F(D^2\widetilde{w})\big) &\geq F(D^2\widetilde{w})\\ &\geq f(x),\end{aligned}$$

because of the very definition of Γ_1. Let $r_* \equiv r_*(\partial\Omega) > 0$ be the radius provided by the uniform exterior sphere condition, $y \in \partial\Omega$ be any point and $x_y \in \mathbb{R}^n$ be so that $|y - x_y| = r_*$ and $\overline{B}_{r_*}(x_y) \cap \overline{\Omega} = \{y\}$. Let $\widetilde{r} := r_* + \operatorname{diam}(\Omega)$, $\gamma > \max\{2, \frac{1}{\lambda} + n\frac{\Lambda}{\lambda}\}$, $L > 0$ and

$$w_y(x) := L(r_*^{-\gamma} - |x - x_y|^{-\gamma}).$$

By construction, we have $w_y(y) = 0$, $w_y(x) \geq 0$ for $x \in \Omega$ and

$$\begin{aligned} Dw_y(x) &:= L\gamma \frac{x - x_y}{|x - x_y|^{\gamma+2}},\\ D^2 w_y &= \frac{L\gamma}{|x - x_y|^{\gamma+2}}\Big[\mathbf{I} - (\gamma + 2)\frac{(x - x_y) \otimes (x - x_y)}{|x - x_y|^2}\Big],\end{aligned}$$

so we can control from below

$$|Dw_y| \geq L\gamma \widetilde{r}^{-(\gamma+1)} \quad \text{in } \Omega \tag{3.10}$$

and

$$\begin{aligned} F(D^2 w_y) &\overset{(2.1)_2}{=} \Big(F(D^2 w_y) - F\Big(\frac{L\gamma \mathbf{I}}{|x - x_y|^{\gamma+2}}\Big)\Big) + \Big(F\Big(\frac{L\gamma \mathbf{I}}{|x - x_y|^{\gamma+2}}\Big) - F(0)\Big)\\ &\overset{(2.6)}{\geq} \frac{L\gamma\lambda(\gamma + 2)}{|x - x_y|^{\gamma+4}} \operatorname{tr}\big((x - x_y) \otimes (x - x_y)\big) - \frac{L\Lambda\gamma}{|x - x_y|^{\gamma+2}} \operatorname{tr}(\mathbf{I})\\ &= \frac{L\gamma(\lambda(\gamma + 2) - n\Lambda)}{|x - x_y|^{\gamma+2}} \geq \frac{L\gamma}{|x - x_y|^{\gamma+2}},\end{aligned}$$

where we also used the lower bound imposed on γ. We stress that the restrictions imposed on the size of γ yield that $\gamma \equiv \gamma(n, \lambda, \Lambda)$. At this stage, we select $L > 0$ so that

$$\frac{L\gamma}{\widetilde{r}^{\gamma+1}} \geq 1 \quad \text{and} \quad \frac{L\gamma}{\widetilde{r}^{2+\gamma}} \geq \|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)}, \tag{3.11}$$

thus fixing the dependency $L \equiv L(n, \lambda, \lambda, \|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\partial\Omega)}, \partial\Omega, \operatorname{diam}(\Omega))$. Now, let $\tau \in (0, 1)$ be any number and define the function

$$w_{y;\tau}(x) := g(y) + \tau + \Gamma_\tau w_y(x),$$

where $\Gamma_\tau \geq 1$ is selected in such a way that $w_{y;\tau}(x) \geq g(x)$ for $x \in \partial\Omega$. This can be done by defining

$$\Gamma_\tau := 4\Big(\sup_{x\in\partial\Omega, x\neq y} \frac{(\omega_g(|x-y|)-\tau)_+}{w_y(x)}\Big) + 1.$$

The uniform sphere condition imposed on $\partial\Omega$ yields that Γ_τ does not depend on $y \in \partial\Omega$. We then estimate using the very definition of $w_{y;\tau}(\cdot)$, (3.10), and (3.11), which in particular render that $\varepsilon w_{y;\tau} + F(D^2 w_{y;\tau}) \geq 0$, to get

$$\begin{aligned}
&\big[\ell_\varepsilon(Dw_{y;\tau})^{p_{\varepsilon;v}(x)} + a_{\varepsilon;v}(x)\ell_\varepsilon(Dw_{y;\tau})^q + b_{\varepsilon;v}(x)\ell_\varepsilon(Dw_{y;\tau})^s\big]\big(\varepsilon w_{y;\tau} + F(D^2 w_{y;\tau})\big) \\
&\qquad \geq -\|g\|_{L^\infty(\partial\Omega)} + \frac{\Gamma_\tau L\gamma}{\tilde{r}^{\gamma+2}} \\
&\qquad \geq -\|g\|_{L^\infty(\partial\Omega)} + \Gamma_\tau\big(\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)}\big) \\
&\qquad \geq \|f\|_{L^\infty(\Omega)} \geq f(x),
\end{aligned}$$

thus $w_{y;\tau}$ is a viscosity supersolution of equation (3.2) for all $y \in \partial\Omega$ and all $\tau \in (0,1)$, and, as a consequence, the map $\widetilde{w}_{y;\tau} := \min\{\widetilde{w}, w_{y;\tau}\}$ is a viscosity supersolution of (3.2). Finally, setting

$$\bar{w}(x) := \inf\big\{\widetilde{w}_{y;\tau}(x) \colon y \in \partial\Omega, \tau \in (0,1)\big\},$$

we obtain the required viscosity supersolution to (3.2) agreeing with $g(\cdot)$ on $\partial\Omega$. ■

As a consequence of the two above lemmas, we obtain the existence of a continuous viscosity solution to equation (3.2).

Corollary 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain satisfying the uniform sphere condition and assume* (2.1), (2.2), (2.3), (2.4), (2.6). *Then, for any $g \in C(\partial\Omega)$ and $v \in C(\overline{\Omega})$ there exists a viscosity solution $u_\varepsilon \in C(\overline{\Omega})$ to equation* (3.2) *so that $\underline{w} \leq u_\varepsilon \leq \bar{w}$, where $\underline{w}$ and $\bar{w}$ are respectively the subsolution and the supersolution constructed in Lemma* 3.2. *In particular, it holds that*

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq c(n, \lambda, \Lambda, \|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\partial\Omega)}, \partial\Omega, \operatorname{diam}(\Omega), \operatorname{dist}(\Omega', \partial\Omega)). \quad (3.12)$$

Proof. The proof immediately follows by combining [18, Theorem 4.1] with Lemmas 3.1 and 3.2. ■

Now we are ready to show the existence of a viscosity solution of Dirichlet problem

$$\begin{cases} \big[|Du_v|^{p_v(x)} + a(x)\mathbb{1}_{\{v>0\}}|Du_v|^q + b(x)\mathbb{1}_{\{v<0\}}|Du_v|^s\big]F(D^2u_v) = f(x) & \text{in } \Omega, \\ u_v = g & \text{on } \partial\Omega, \end{cases} \quad (3.13)$$

where $v \in C(\overline{\Omega})$, $g \in C(\partial\Omega)$, and assumptions (2.6), (2.1), (2.2), (2.3), (2.4) are in force.

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain satisfying the uniform sphere condition and assume* (2.6), (2.1), (2.2), (2.3), (2.4). *Then, for any $g \in C(\partial\Omega)$ and $v \in C(\overline{\Omega})$, Dirichlet problem* (3.13) *admits a viscosity solution $u_v \in C(\overline{\Omega})$ so that*

$u_v|_{\partial\Omega} = g|_{\partial\Omega}$ and $\underline{w} \le u_v \le \bar{w}$, where $\underline{w}$, $\bar{w}$ are respectively the subsolution and the supersolution constructed in Lemma 3.2. In particular, it holds that

$$\|u_v\|_{L^\infty(\Omega)} \le c(n, \lambda, \Lambda, \|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\partial\Omega)}, \partial\Omega, \operatorname{diam}(\Omega), \operatorname{dist}(\Omega', \partial\Omega)) \quad (3.14)$$

and, whenever $\Omega' \Subset \Omega$ is an open set, for all $\beta_0 \in (0,1)$ we have

$$[u]_{0,\beta_0;\Omega'} \le c(n, \lambda, \Lambda, \|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\partial\Omega)}, \partial\Omega, \operatorname{diam}(\Omega), \operatorname{dist}(\Omega', \partial\Omega), \beta_0). \quad (3.15)$$

Proof. By Corollary 3.3, there exists a viscosity solution $u_\varepsilon \in C(\overline{\Omega})$ to equation (3.2) so that

$$u_\varepsilon|_{\partial\Omega} = g|_{\partial\Omega} \quad \text{and} \quad \underline{w} \le u_\varepsilon \le \bar{w}, \quad (3.16)$$

where $\underline{w}$, $\bar{w}$ are respectively the viscosity subsolution and the viscosity supersolution to (3.2) determined by Lemma 3.2. We stress that $\underline{w}$ and $\bar{w}$ do not depend on ε. Notice that the bound in (3.12) is uniform in ε and that equation (3.2) falls in the class of those considered by Proposition A.1, with $\mu = \varepsilon$, $p(\cdot) \equiv p_{\varepsilon;v}(\cdot)$, $q(\cdot) \equiv q$, $s(\cdot) \equiv s$, $a(\cdot) \equiv a_{\varepsilon;v}(\cdot)$ and $b(\cdot) \equiv b_{\varepsilon;v}(\cdot)$, therefore, keeping in mind Remark A.2, we see that $\{u_\varepsilon\} \subset C^{0,\beta_0}_{\mathrm{loc}}(\Omega)$ for all $\beta_0 \in (0,1)$ with uniform estimates on the Hölder seminorm, cf. (A.3) and (3.12). This, together with (3.16), the compact embedding of the Hölder spaces $C^{0,\beta_1}(\Omega') \hookrightarrow C^{0,\beta_2}(\Omega')$ for $\beta_2 < \beta_1$, and (3.12), gives that $u_\varepsilon \to u_v$ uniformly on compact subsets of Ω, so we have

$$u_v \in C(\overline{\Omega}), \quad u_v|_{\partial\Omega} = g|_{\partial\Omega}, \quad \underline{w} \le u_v \le \bar{w}, \quad \|u_v\|_{L^\infty(\Omega')} + [u_v]_{0,\beta_0;\Omega'} \le c, \quad (3.17)$$

with $c \equiv c(n, \lambda, \Lambda, \|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\partial\Omega)}, \partial\Omega, \operatorname{diam}(\Omega), \operatorname{dist}(\Omega', \partial\Omega), \beta_0)$ for all $\beta_0 \in (0,1)$ (of course the dependency from β_0 occurs only when considering $[u_v]_{0,\beta_0;\Omega'}$). Finally, by very definition, we have that $p_{\varepsilon;v} \to p_v$, $a_{\varepsilon;v} \to a\mathbb{1}_{\{v>0\}}$ and $b_{\varepsilon;v} \to b\mathbb{1}_{\{v<0\}}$ in Ω, so by well-known stability properties of viscosity solutions, cf. [40, Chapter 3] and $(3.17)_2$, we have that $u_v \in C(\overline{\Omega})$ is a viscosity solution of equation (3.13). ■

3.1. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\tilde{u} \in C(\overline{\Omega})$ be any function. We recursively define the sequence of functions $\{u_\kappa\}_{\kappa\in\mathbb{N}\cup\{0\}}$ so that $u_0 = \tilde{u}$ and for $\kappa \ge 1$, u_κ is a solution of problem (3.13) with $v \equiv u_{\kappa-1}$, whose existence is assured by Corollary 3.4. Since the bounds in (3.14)–(3.15) do not depend on v and so in our case they are independent of κ, we have that the sequence $\{u_\kappa\}$ is uniformly bounded with respect to the full C^{0,β_0}-norm for all $\beta_0 \in (0,1)$, therefore $u_\kappa \to u_\infty$ uniformly on compact subsets of Ω, $u_\infty \in C(\overline{\Omega})$ and $u_\infty|_{\partial\Omega} = g|_{\partial\Omega}$. Standard stability results (see [40, Chapter 3]) eventually render that u_∞ is a viscosity solution of problem (1.7), and the proof is complete. ■

4. Compactness for switched differential inequalities

The main result of this section is uniform Hölder continuity for viscosity solutions of the switched equation (2.10). The uniformity is due to the fact that all the constants bounding the Hölder seminorm of solutions will not depend on ξ, nor on the moduli of continuity of coefficients $a(\cdot)$, $b(\cdot)$, nor on their L^{∞}-norm.

Proposition 4.1. *Under assumptions* (2.1)*,* (2.2)*,* (2.3)*,* (2.4) *and* (2.6)*, let* $u \in C(\Omega)$ *be a viscosity subsolution of* (2.11) *and a viscosity supersolution to* (2.12)*. Then* $u \in C^{0,\beta_0}_{\mathrm{loc}}(\Omega)$ *for all* $\beta_0 \in (0,1)$*. In particular, for any* $\beta_0 \in (0,1)$ *there exists a threshold radius* $r_* \equiv r_*(\beta_0) \in (0,1/4)$ *so that whenever* $B_{\varrho}(z_0) \Subset \Omega$ *is a ball with* $\varrho \in (0, r_*]$ *, it holds that*

$$|u(x) - u(y)| \le c|x-y|^{\beta_0} \quad \text{for all } x, y \in B_{\varrho/2}(z_0),$$

with $c \equiv c(n, \lambda, \Lambda, \|u\|_{L^{\infty}(B_{\varrho}(z_0))}, \|f\|_{L^{\infty}(B_{\varrho}(z_0))}, \varrho, \beta_0)$.

Proof. Let $u \in C(\Omega)$ be a viscosity subsolution to (2.11) and a viscosity supersolution of (2.12), $\beta_0 \in (0,1)$ be any number, and $B_{\varrho}(z_0) \Subset \Omega$ be a ball with radius $\varrho \in (0, r_*]$, where

$$r_* := \left(\beta_0/10\right)^{\frac{1}{1-\beta_0}}$$

is a threshold radius that will play an important role in a few lines. We aim to show that there are two positive constants $A_1 \equiv A_1(n, \lambda, \Lambda, \|u\|_{L^{\infty}(\Omega)}, \|f\|_{L^{\infty}(\Omega)}, \varrho, \beta_0)$ and $A_2 \equiv A_2(\|u\|_{L^{\infty}(\Omega)}, \varrho)$ such that

$$\mathcal{M}(x_0) := \sup_{x,y \in B_{\varrho}(z_0)} \left(u(x) - u(y) - A_1\omega(|x-y|) - A_2\left(|x-x_0|^2 + |y-x_0|^2\right)\right) \le 0 \tag{4.1}$$

holds for all $x_0 \in B_{\varrho/2}(z_0)$. In (4.1),

$$\omega(t) := t^{\beta_0} \quad \text{if } |\xi| \le \kappa_0^{-1}, \qquad \omega(t) := \begin{cases} t - \omega_0 t^{3/2} & \text{if } t \le t_0 \\ \omega(t_0) & \text{if } t > t_0 \end{cases} \quad \text{if } |\xi| > \kappa_0^{-1},$$

where $\kappa_0 := (2(A_1 + 2A_2))^{-1}$ is a limiting number, $\omega_0 = 1/3$ and $t_0 := \left(2/(3\omega_0)\right)^2 \ge 1$. By contradiction, we assume that

$$\text{there exists } x_0 \in B_{\varrho/2}(z_0) \text{ such that } \mathcal{M}(x_0) > 0 \text{ for all positive } A_1, A_2, \tag{4.2}$$

define quantities

$$\begin{cases} A_1 := \dfrac{40}{\beta_0(1-\beta_0)}\left[\dfrac{\|f\|_{L^{\infty}(B_{\varrho}(z_0))}}{\lambda} + (2A_2+1)\left(\dfrac{\Lambda}{\lambda}(n-1)+1\right)\right], \\ A_2 := 64\varrho^{-2}\max\left\{\|u\|_{L^{\infty}(B_{\varrho}(z_0))}, 1\right\}, \end{cases} \tag{4.3}$$

and consider the auxiliary functions

$$\begin{cases} \psi(x,y) := A_1\omega(|x-y|) + A_2\left(|x-x_0|^2 + |y-x_0|^2\right), \\ \phi(x,y) := u(x) - u(y) - \psi(x,y). \end{cases}$$

If $(\overline{x}, \overline{y}) \in \overline{B}_\varrho(z_0) \times \overline{B}_\varrho(z_0)$ is a maximum point of $\phi(\cdot)$, via (4.2) we have that $\phi(\overline{x}, \overline{y}) = \mathcal{M}(x_0) > 0$, so

$$A_1\omega(|\overline{x}-\overline{y}|) + A_2\big(|\overline{x}-x_0|^2 + |\overline{y}-x_0|^2\big) \leq u(\overline{x}) - u(\overline{y}) \leq 2\|u\|_{L^\infty(B_\varrho(z_0))}.$$

Inserting $(4.3)_2$ in the above inequality yields that $\overline{x}$, $\overline{y}$ both belong to the interior of $B_\varrho(z_0)$. In particular,

$$|\overline{x}-z_0| \leq |\overline{x}-x_0| + |x_0-z_0| \leq \frac{3\varrho}{4} \quad \text{and} \quad |\overline{y}-z_0| \leq |\overline{y}-x_0| + |x_0-z_0| \leq \frac{3\varrho}{4}.$$

Moreover, $\overline{x} \neq \overline{y}$, otherwise $\mathcal{M}(x_0) = \phi(\overline{x}, \overline{y}) = 0$ and (4.1) would be satisfied. This last remark implies that $\psi(\cdot)$ is smooth in a small neighborhood of $(\overline{x}, \overline{y})$, therefore we can determine its gradients

$$\xi_{\overline{x}} := \partial_x\psi(\overline{x}, \overline{y}) = A_1\omega'(|\overline{x}-\overline{y}|)\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|} + 2A_2(\overline{x}-x_0),$$
$$\xi_{\overline{y}} := -\partial_y\psi(\overline{x}, \overline{y}) = A_1\omega'(|\overline{x}-\overline{y}|)\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|} - 2A_2(\overline{y}-x_0).$$

To summarize, we have that $\phi(\cdot)$ attains its maximum at $(\overline{x}, \overline{y})$ inside $B_\varrho(z_0) \times B_\varrho(z_0)$ and $\phi(\cdot)$ is smooth around $(\overline{x}, \overline{y})$, thus Proposition 2.3 applies: we can find a threshold $\widehat{\delta} = \widehat{\delta}(\|D^2\psi\|)$ such that for all $\delta \in (0, \widehat{\delta})$ the couple $(\xi_{\overline{x}}, X_\delta)$ is a limiting subjet of u at $\overline{x}$ and the couple $(\xi_{\overline{y}}, Y_\delta)$ is a limiting superjet of u at $\overline{y}$ and the matrix inequality

$$\begin{bmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{bmatrix} \leq \begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix} + (2A_2 + \delta)\mathbf{I} \tag{4.4}$$

holds, where we set

$$\begin{aligned} Z :=& A_1(D^2\omega)(|\overline{x}-\overline{y}|) \\ =& A_1\Big[\frac{\omega'(|\overline{x}-\overline{y}|)}{|\overline{x}-\overline{y}|}\mathbf{I} + \Big(\omega''(|\overline{x}-\overline{y}|) - \frac{\omega'(|\overline{x}-\overline{y}|)}{|\overline{x}-\overline{y}|}\Big)\frac{(\overline{x}-\overline{y})\otimes(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|^2}\Big]. \end{aligned}$$

We fix

$$\delta \equiv \min\Big\{1, \frac{\widehat{\delta}}{4}\Big\}$$

and apply (4.4) to vectors of the form $(z, z) \in \mathbb{R}^{2n}$, to obtain

$$\langle (X_\delta - Y_\delta)z, z\rangle \leq (4A_2 + 2)|z|^2.$$

This means that

$$\text{all the eigenvalues of } X_\delta - Y_\delta \text{ are less than or equal to } 2(2A_2 + 1). \tag{4.5}$$

In particular, applying (4.4) to the vector

$$\overline{z} := \Big(\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|}, \frac{\overline{y}-\overline{x}}{|\overline{x}-\overline{y}|}\Big),$$

we get

$$\left\langle (X_\delta - Y_\delta)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}, \frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}\right\rangle \le 2(2A_2+1) + 4A_1\omega''(|\bar{x}-\bar{y}|).$$

This yields in particular that

$$\text{at least one eigenvalue of } X_\delta - Y_\delta \text{ is less than } 2(2A_2+1) + 4A_1\omega''(|\bar{x}-\bar{y}|). \tag{4.6}$$

As by definition $\omega''(t) < 0$, we can majorize the quantity appearing in (4.6) as

$$2(2A_2+1) + 4A_1\omega''(|\bar{x}-\bar{y}|) \le 2(2A_2+1) - 4A_1|\omega''(1)| \overset{(4.3)_1}{<} 0,$$

where we also used that $|\bar{x}-\bar{y}| \le 1/2$. This means that at least one eigenvalue of $X_\delta - Y_\delta$ is negative, thus via $(2.7)_2$, (4.5) and (4.6), we obtain

$$\mathcal{P}^-_{\lambda,\Lambda}(X_\delta - Y_\delta) \ge -2(2A_2+1)\big[\Lambda(n-1)+\lambda\big] + 4\lambda A_1|\omega''(1)|, \tag{4.7}$$

therefore

$$F(X_\delta) - F(Y_\delta) \overset{(2.8)}{\ge} \mathcal{P}^-_{\lambda,\Lambda}(X_\delta - Y_\delta) \overset{(4.7)}{\ge} -2(2A_2+1)\big[\Lambda(n-1)+\lambda\big] + 4\lambda A_1|\omega''(1)|. \tag{4.8}$$

With $\xi_{\bar{x}}, \xi_{\bar{y}}$ computed before, we write the following viscosity inequalities deriving from (2.11)–(2.12):

$$\begin{cases} \min\big\{F(X_\delta), H_q(\bar{x},\xi_{\bar{x}};\xi)F(X_\delta), H_s(\bar{x},\xi_{\bar{x}};\xi)F(X_\delta)\big\} \le \|f\|_{L^\infty(\Omega)}, \\ \max\big\{F(Y_\delta), H_q(\bar{y},\xi_{\bar{y}};\xi)F(Y_\delta), H_s(\bar{y},\xi_{\bar{y}};\xi)F(Y_\delta)\big\} \ge -\|f\|_{L^\infty(\Omega)}. \end{cases} \tag{4.9}$$

For simplicity, define

$$\begin{aligned} \mathfrak{H}^-(x) &:= \min\big\{F(X_\delta), H_q(x,\xi_x;\xi)F(X_\delta), H_s(x,\xi_x;\xi)F(X_\delta)\big\}, \\ \mathfrak{H}^+(x) &:= \max\big\{F(Y_\delta), H_q(x,\xi_x;\xi)F(Y_\delta), H_s(x,\xi_x;\xi)F(Y_\delta)\big\}, \end{aligned}$$

and notice that

$$\begin{cases} \mathfrak{H}^-(\bar{x}) = \min\big\{1, H_q(\bar{x},\xi_{\bar{x}};\xi), H_s(\bar{x},\xi_{\bar{x}};\xi)\big\} & \text{if } F(X_\delta) \ge 0, \\ \mathfrak{H}^-(\bar{x}) = \max\big\{1, H_q(\bar{x},\xi_{\bar{x}};\xi), H_s(\bar{x},\xi_{\bar{x}};\xi)\big\} & \text{if } F(X_\delta) < 0, \\ \mathfrak{H}^+(\bar{y}) = \max\big\{1, H_q(\bar{y},\xi_{\bar{y}};Y_\delta), H_s(\bar{y},\xi_{\bar{y}};\xi)\big\} & \text{if } F(Y_\delta) \ge 0, \\ \mathfrak{H}^+(\bar{y}) = \min\big\{1, H_q(\bar{y},\xi_{\bar{y}};\xi), H_s(\bar{y},\xi_{\bar{y}};\xi)\big\} & \text{if } F(Y_\delta) < 0. \end{cases} \tag{4.10}$$

At this stage, we treat separately two cases: $|\xi| > \kappa_0^{-1}$ and $|\xi| \le \kappa_0^{-1}$.

Case $|\xi| > \kappa_0^{-1}$. We expand the expression of $\omega(\cdot)$ in (4.8) to get

$$F(X_\delta) - F(Y_\delta) \geq -2(2A_2+1)[\Lambda(n-1)+\lambda] + \lambda A_1. \tag{4.11}$$

Moreover, our choice of κ_0 assures that

$$\min\{|\xi + \xi_{\overline{x}}|, |\xi + \xi_{\overline{y}}|\} \geq \kappa_0^{-1} - \max\{|\xi_{\overline{x}}|, |\xi_{\overline{y}}|\} \geq A_1 + 2A_2 \geq 1,$$

which implies

$$\mathfrak{H}^-(\overline{x}) \geq 1 \quad \text{and} \quad \mathfrak{H}^+(\overline{y}) \geq 1. \tag{4.12}$$

Keeping in mind (4.10), we can manipulate the variational inequalities (4.9) to get

$$\begin{aligned} 4\|f\|_{L^\infty(B_\varrho(z_0))} &\overset{(4.12),(4.10)}{\geq} \frac{2\|f\|_{L^\infty(B_\varrho(z_0))}}{\mathfrak{H}^-(\overline{x})} + \frac{2\|f\|_{L^\infty(B_\varrho(z_0))}}{\mathfrak{H}^+(\overline{y})} \\ &\overset{(4.9)}{\geq} F(X_\delta) - F(Y_\delta) \\ &\overset{(4.8)}{\geq} -2(2A_2+1)\big[\Lambda(n-1)+\lambda\big] + \lambda A_1, \end{aligned}$$

which renders that

$$4\|f\|_{L^\infty(B_\varrho(z_0))} \geq -2(2A_2+1)\big[\Lambda(n-1)+\lambda\big] + \lambda A_1. \tag{4.13}$$

The content of (4.13) contradicts the choice made in $(4.3)_1$.

Case $|\xi| \leq \kappa_0^{-1}$. In this situation, (4.8) reads as

$$F(X_\delta) - F(Y_\delta) \geq -2(2A_2+1)[\Lambda(n-1)+\lambda] + 4\beta_0(1-\beta_0)\lambda A_1. \tag{4.14}$$

Now notice that (4.3) yields $A_1 - A_2 \geq A_1/4$ and $A_1/2 \geq 6A_2$, so keeping in mind the definitions of r_* and of $\omega(\cdot)$, via the Young inequality we have that

$$\begin{aligned} \min\{|\xi_{\overline{x}}|^2, |\xi_{\overline{y}}|^2\} &\geq \frac{1}{4}A_1^2\beta_0^2|\overline{x}-\overline{y}|^{2(\beta_0-1)} - \frac{52}{3}A_2^2\max\{|\overline{x}-x_0|^2, |\overline{y}-x_0|^2\} \\ &\geq \frac{1}{4}A_1^2\beta_0^2 r_*^{-2(1-\beta_0)} - 25A_2^2 \\ &\geq 25(A_1^2 - A_2^2) \geq \frac{25}{4}A_1^2. \end{aligned}$$

This allows us to conclude that

$$\begin{aligned} \min\{|\xi_{\overline{x}}|, |\xi_{\overline{y}}|\} - \kappa_0^{-1} &\geq \frac{5}{2}A_1 - 2(A_1 + 2A_2) \\ &= \frac{1}{2}A_1 - 4A_2 \geq 2A_2 \geq 1 \end{aligned}$$

and (4.12) holds in this case as well. Therefore, we can combine as before the variational inequalities (4.9) with (4.14) and (4.12) to deduce

$$4\|f\|_{L^\infty(\Omega)} \geq -2(2A_2+1)[\Lambda(n-1)+\lambda] + 4\beta_0(1-\beta_0)\lambda A_1,$$

which is again a contradiction of $(4.3)_1$.

Merging the two previous cases, we can conclude that if $u \in C(\Omega)$ is a viscosity solution to (2.10), then u is β_0-Hölder continuous on $B_{\varrho/2}(z_0)$ for all $\beta_0 \in (0,1)$, and the estimate

$$[u]_{0,\beta_0;B_{\varrho/2}(z_0)} \leq c(n,\lambda,\Lambda,\|u\|_{L^\infty(B_\varrho(z_0))},\|f\|_{L^\infty(B_\varrho(z_0))},\varrho,\beta_0)$$

holds true. The arbitrariness of $B_\varrho(z_0) \Subset \Omega$ and the fact that constants A_1, A_2 are increasing with respect to $\|u\|_{L^\infty(B_\varrho(z_0))}$ and to $\|f\|_{L^\infty(B_\varrho(z_0))}$ allow the use of a standard covering argument to deduce that given any open set $\Omega' \Subset \Omega$, $u \in C^{0,\beta_0}(\Omega')$ for all $\beta_0 \in (0,1)$ with

$$[u]_{0,\beta_0;\Omega'} \leq c(n,\lambda,\Lambda,\|u\|_{L^\infty(\Omega)},\|f\|_{L^\infty(\Omega)},\operatorname{dist}(\Omega',\partial\Omega),\beta_0),$$

and the proof is complete. ■

5. Gradient Hölder continuity

In this section we prove that viscosity solutions of equation (1.1) are locally C^{1,α_0}-regular for some $\alpha_0 \equiv \alpha_0(n,\lambda,\Lambda,p^+,p_-) \in (0,1)$. To do so, we shall first prove that in a suitable smallness regime, a continuous viscosity solution of the switched equation (2.10) is L^∞-close to a solution of a homogeneous problem of type (1.11). This closeness is assured by a "harmonic" approximation lemma, whose proof is based on [26, 38] and that strongly relies on the smallness of certain quantities and on the compactness earned via Proposition 4.1.

5.1. Smallness regime

We exploit the scaling properties of (2.11)–(2.12) for reducing the problem to a smallness regime. In other terms, if $\xi \in \mathbb{R}^n$ is an arbitrary vector and $u \in C(\Omega)$ is a viscosity subsolution/supersolution to (2.11)/(2.12), we blow-up and scale u in order to construct another map $\mathfrak{u}$, that is a viscosity subsolution of an equation having the same structure of (2.11), a viscosity supersolution of an equation similar to (2.12) and such that, for a given $\varepsilon \in (0,1)$ we have $\operatorname{osc}_{B_1(0)} \mathfrak{u} \leq 1$ and the right-hand side constant appearing in (2.11)–(2.12) can be controlled in modulus by ε. Under these conditions, $\mathfrak{u}$ is called an "ε-normalized viscosity solution". Let us show this construction. Let $\varepsilon \in (0,1)$ be any number and $B_\tau(x_0) \Subset \Omega$ be any ball with $\tau \in (0, \frac{1}{16}\min\{\operatorname{diam}(\Omega),1\})$ to be quantified later on, and define

$$\mathfrak{M} := 16\Big(1+\|u\|_{L^\infty(\Omega)}+\|f\|_{L^\infty(\Omega)}+\|f\|_{L^\infty(\Omega)}^{\frac{1}{p^++1}}+\|f\|_{L^\infty(\Omega)}^{\frac{1}{p_-+1}}\Big).$$

Now, if $u \in C(\Omega)$ is a viscosity solution to (2.10) on $B_\tau(x_0)$, then a straightforward computation shows that the map $\mathfrak{u}(x) := u(x_0+\tau x)\mathfrak{M}^{-1}$ is in particular an ε-normalized viscosity subsolution of

$$\min\{\mathfrak{F}(D^2\mathfrak{u}), \mathfrak{H}_q(x, D\mathfrak{u}; \bar{\xi})\mathfrak{F}(D^2\mathfrak{u}), \mathfrak{H}_s(x, D\mathfrak{u}; \bar{\xi})\mathfrak{F}(D^2\mathfrak{u})\} = \mathfrak{C} \quad \text{in } B_1(0) \tag{5.1}$$

and an ε-normalized viscosity supersolution to

$$\max\{\mathfrak{F}(D^2\mathfrak{u}), \mathfrak{H}_q(x, D\mathfrak{u}; \bar{\xi})\mathfrak{F}(D^2\mathfrak{u}), \mathfrak{H}_s(x, D\mathfrak{u}; \bar{\xi})\mathfrak{F}(D^2\mathfrak{u})\} = -\mathfrak{C} \quad \text{in } B_1(0), \tag{5.2}$$

where we set

$$\bar{\xi} := \frac{\tau}{\mathfrak{M}}\xi, \quad \mathfrak{a}(x) := \left(\frac{\mathfrak{M}}{\tau}\right)^{q-p^+} a(x_0 + \tau x), \quad \mathfrak{b}(x) := \left(\frac{\mathfrak{M}}{\tau}\right)^{s-p_-} b(x_0 + \tau x),$$

$$\mathfrak{H}_q(x, z; \bar{\xi}) := |\bar{\xi} + z|^{p^+} + \mathfrak{a}(x)|\bar{\xi} + z|^q, \quad \mathfrak{H}_s(x, z; \bar{\xi}) := |\bar{\xi} + z|^{p_-} + \mathfrak{b}(x)|\bar{\xi} + z|^s,$$

$$\mathfrak{F}(M) := \frac{\tau^2}{\mathfrak{M}} F\left(\frac{\mathfrak{M}}{\tau^2} M\right), \quad \mathfrak{C} := \max\left\{\frac{\tau^{p^+ + 2}}{\mathfrak{M}^{p^+ + 1}}, \frac{\tau^{p_- + 2}}{\mathfrak{M}^{p_- + 1}}, \frac{\tau^2}{\mathfrak{M}}\right\} \|f\|_{L^\infty(\Omega)},$$

as by (2.2) and since $\mathfrak{M} \geq 1$ and $\tau \leq 1$, we have

$$\max\left\{\frac{\tau^{p^+ + 2}}{\mathfrak{M}^{p^+ + 1}}, \frac{\tau^{p_- + 2}}{\mathfrak{M}^{p_- + 1}}, \frac{\tau^2}{\mathfrak{M}}\right\} = \frac{\tau^2}{\mathfrak{M}}.$$

A quick computation shows that if (2.1) is in force, then $\mathfrak{F}(\cdot)$ is (λ, Λ)-elliptic as well and, if $\varepsilon \in (0, 1)$ is the number introduced above, we fix $\tau = \varepsilon^{\frac{1}{2}}$. Therefore, by construction we have

$$\begin{cases} \|\mathfrak{a}\|_{L^\infty(B_1(0))} \leq \left(\frac{\mathfrak{M}}{\tau}\right)^{q-p^+} \|a\|_{L^\infty(B_\tau(x_0))}, \\ \|\mathfrak{b}\|_{L^\infty(B_1(0))} \leq \left(\frac{\mathfrak{M}}{\tau}\right)^{s-p_-} \|b\|_{L^\infty(B_\tau(x_0))}, \\ \|\mathfrak{u}\|_{L^\infty(B_1(0))} \leq 1, \qquad \operatorname{osc}_{B_1(0)} \mathfrak{u} \leq 1, \qquad \mathfrak{C} \leq \varepsilon. \end{cases} \tag{5.3}$$

Finally, notice that there is no loss of generality in assuming that $\mathfrak{u}(0) = 0$, since the function $(\mathfrak{u} - \mathfrak{u}(0))$ is still a ε-normalized viscosity subsolution/supersolution of (5.1)/(5.2) and satisfies all the conditions listed above. This is the announced smallness regime. Clearly, for $\xi \equiv 0$ we find a ε-normalized viscosity solution of equation (1.1). We refer to [35, Section 2.3] for the case in which no coefficients appear.

Remark 5.1. Due to the strong nonhomogeneity of (2.10) and (2.11)–(2.12), the scaling factor τ appears also in the definition of $\mathfrak{a}(\cdot)$ and $\mathfrak{b}(\cdot)$ and forces the (quite dangerous) bounds in $(5.3)_1$. Anyway, the L^∞-norms of $\mathfrak{a}(\cdot)$ and $\mathfrak{b}(\cdot)$ will never influence the constants appearing in the forthcoming estimates and it will ultimately be fixed as a function of `data`.

5.2. Harmonic approximation

In the next lemma we show that, in a suitable smallness regime, continuous viscosity solutions of (2.10) are close to solutions of the homogeneous problem (1.11).

Lemma 5.2. *Assume* (2.1), (2.2), (2.3), (2.4), (2.6) *and let* $\sigma \equiv \sigma(n, \lambda, \Lambda) \in (0, 1)$ *be as in* (2.16). *Then, there exists a positive* $\varepsilon_0 \equiv \varepsilon_0(\mathtt{data}) \in (0, 1)$ *such that if* $\mathfrak{u} \in C(B_1(0))$ *is an* ε_0*-normalized viscosity subsolution of* (5.1) *and an* ε_0*-normalized viscosity supersolution to* (5.2), *it is possible to find* $\xi_\sigma \in \mathbb{R}^n$ *such that*

$$\operatorname{osc}_{B_\sigma(0)}(\mathfrak{u} - \xi_\sigma \cdot x) < \frac{\sigma}{2}.$$

Proof. By contradiction, we find sequences of fully nonlinear operators $\{\mathfrak{F}_\kappa(\cdot)\}$ that are uniformly (λ, Λ)-elliptic, of vectors $\{\bar{\xi}_\kappa\} \subset \mathbb{R}^n$, of nonnegative functions $\{\mathfrak{a}_\kappa(\cdot)\}$, $\{\mathfrak{b}_\kappa(\cdot)\} \subset C(B_1(0))$, of numbers $\{\mathfrak{C}_\kappa\} \subset [0, \infty)$ so that $\mathfrak{C}_\kappa \le \kappa^{-1}$, and of maps $\{\mathfrak{u}_\kappa\} \in C(\Omega)$ that are κ^{-1}-normalized viscosity subsolutions to

$$\min\{\mathfrak{F}_\kappa(D^2\mathfrak{u}_\kappa), \mathfrak{H}_{q;\kappa}(x, D\mathfrak{u}_\kappa; \bar{\xi}_\kappa)\mathfrak{F}_\kappa(D^2\mathfrak{u}_\kappa), \mathfrak{H}_{s;\kappa}(x, D\mathfrak{u}_\kappa; \bar{\xi}_\kappa)\mathfrak{F}_\kappa(D^2\mathfrak{u}_\kappa)\} = \mathfrak{C}_\kappa \tag{5.4}$$

in $B_1(0)$ and κ^{-1}-normalized viscosity supersolutions of

$$\max\{\mathfrak{F}_\kappa(D^2\mathfrak{u}_\kappa), \mathfrak{H}_{q;\kappa}(x, D\mathfrak{u}_\kappa; \bar{\xi}_\kappa)\mathfrak{F}_\kappa(D^2\mathfrak{u}_\kappa), \mathfrak{H}_{s;\kappa}(x, D\mathfrak{u}_\kappa; \bar{\xi}_\kappa)\mathfrak{F}_\kappa(D^2\mathfrak{u}_\kappa)\} = -\mathfrak{C}_\kappa \tag{5.5}$$

in $B_1(0)$ for all $\kappa \in \mathbb{N}$, $\mathfrak{u}_\kappa(0) = 0$ and

$$\begin{aligned} \sup_{\kappa\in\mathbb{N}} \|\mathfrak{u}_\kappa\|_{L^\infty(B_1(0))} &\le 1, \\ \sup_{\kappa\in\mathbb{N}} (\operatorname{osc}_{B_1(0)} \mathfrak{u}_\kappa) &\le 1, \\ \operatorname{osc}_{B_\sigma(0)}(\mathfrak{u}_\kappa - \xi \cdot x) &\ge \frac{\sigma}{2} \quad \text{for all } \xi \in \mathbb{R}^n. \end{aligned} \tag{5.6}$$

In (5.4)–(5.5), $\mathfrak{H}_{q;\kappa}(\cdot)$, $\mathfrak{H}_{s;\kappa}(\cdot)$ are defined as in Section 5.1, with $\mathfrak{a}_\kappa(\cdot)$ and $\mathfrak{b}_\kappa(\cdot)$ replacing $\mathfrak{a}(\cdot)$ and $\mathfrak{b}(\cdot)$, respectively. As the sequence $\{\mathfrak{F}_\kappa(\cdot)\}$ is uniformly (λ, Λ)-elliptic, we have that

$$\mathfrak{F}_\kappa(\cdot) \to \mathfrak{F}_\infty(\cdot) \ \text{ for some } \mathfrak{F}_\infty \in C(\mathcal{S}(n), \mathbb{R}) \text{ uniformly } (\lambda, \Lambda)\text{-elliptic.} \tag{5.7}$$

Then, Proposition 4.1 applies to renormalized viscosity subsolutions/supersolutions of (5.4)/(5.5) as all the estimates made in its proof do not involve the coefficients in a quantitative way, and all the bounding constants are increasing with respect to the L^∞-norm of solutions and of the right-hand side datum. This means that $\{\mathfrak{u}_\kappa\} \subset C^{0,\beta_0}_{\text{loc}}(B_1(0))$ for all $\beta_0 \in (0, 1)$, so, recalling also $(5.6)_{1,2}$ and Arzelà–Ascoli theorem, we have that

$$\mathfrak{u}_\kappa \to \mathfrak{u}_\infty \quad \text{locally uniformly on } B_1(0) \tag{5.8}$$

and, by (5.6) and (5.8) it holds that $\mathfrak{u}_\infty \in C(B_1(0))$ with

$$\|\mathfrak{u}_\infty\|_{L^\infty(B_1(0))} \le 1 \qquad \text{and} \qquad \operatorname{osc}_{B_\sigma(0)}(\mathfrak{u}_\infty - \xi \cdot x) \ge \frac{\sigma}{2} \quad \text{for all } \xi \in \mathbb{R}^n. \tag{5.9}$$

We aim to prove that $\mathfrak{u}_\infty \in C(B_1(0))$ is a viscosity solution of

$$\mathfrak{F}_\infty(D^2\mathfrak{u}_\infty) = 0 \quad \text{in } B_1(0). \tag{5.10}$$

Let us show that $\mathfrak{u}_\infty$ is a viscosity supersolution of (5.10). Let $\varphi \in C^2(B_1(0))$ be so that $\mathfrak{u}_\infty - \varphi$ admits a local strict minimum at $x_0 \in B_1(0)$. There is no loss of generality in assuming that $\varphi(\cdot)$ is a quadratic polynomial, i.e.,

$$\varphi(x) := \frac{1}{2}A(x - x_0) \cdot (x - x_0) + b \cdot (x - x_0) + \mathfrak{u}_\infty(x_0).$$

By (5.8) and standard perturbations arguments [40, Lemma 5] we have that there exists a sequence of points $\{x_\kappa\} \subset B_1(0)$ so that $x_\kappa \to x_0$, $\mathfrak{u}_\kappa - \varphi$ attains a local minimum at x_κ, and $D\varphi(x_\kappa) \to b$. Suppose that

$$\mathfrak{F}_\infty(A) < 0 \implies \mathfrak{F}_\kappa(A) < 0 \quad \text{for } \kappa \in \mathbb{N} \text{ large enough.} \tag{5.11}$$

At this stage, we distinguish two cases according to the behavior of the sequence $\{\bar{\xi}_\kappa\}$.

Case 1: $\{\bar{\xi}_\kappa\}$ does not have a convergent subsequence. In this case, up to extracting a subsequence (which we do not relabel), we have that

$$|\bar{\xi}_\kappa| \to \infty. \tag{5.12}$$

In light of (5.7), (5.11) and (5.12), if we take $\kappa \in \mathbb{N}$ sufficiently large and then relabel, we can assume that

$$\sup_{\kappa \in \mathbb{N}} |D\varphi(x_\kappa)| \leq 2(|b| + 1), \; |\bar{\xi}_\kappa| > 4(|b| + 1) \implies |\bar{\xi}_\kappa + D\varphi(x_\kappa)| \geq 2(|b| + 1). \tag{5.13}$$

Notice that (5.11) yields

$$\begin{aligned} &\max\big\{\mathfrak{F}_\kappa(A), \mathfrak{H}_{q;\kappa}(x, D\varphi(x_\kappa); \bar{\xi}_\kappa)\mathfrak{F}_\kappa(A), \mathfrak{H}_{s;\kappa}(x, D\varphi(x_\kappa); \bar{\xi}_\kappa)\mathfrak{F}_\kappa(A)\big\} \\ &\quad = \min\big\{1, \mathfrak{H}_{q;\kappa}(x, D\varphi(x_\kappa); \bar{\xi}_\kappa), \mathfrak{H}_{s;\kappa}(x, D\varphi(x_\kappa); \bar{\xi}_\kappa)\big\}\mathfrak{F}_\kappa(A). \end{aligned} \tag{5.14}$$

As $\mathfrak{u}_\kappa$ is a κ^{-1}-normalized viscosity supersolution of (5.5), we get

$$\begin{aligned} \mathfrak{F}_\kappa(A) &\overset{(5.11)_2}{\geq} -\frac{\mathfrak{C}_\kappa}{\min\big\{1, \mathfrak{H}_{q;\kappa}(x, D\varphi(x_\kappa); \bar{\xi}_\kappa), \mathfrak{H}_{s;\kappa}(x, D\varphi(x_\kappa); \bar{\xi}_\kappa)\big\}} \\ &\overset{(5.13)}{\geq} -\frac{\kappa^{-1}}{\min\{1, 2^{p^+}(|b| + 1)^{p^+}, 2^{p_-}(|b| + 1)^{p_-}\}}. \end{aligned}$$

Passing to the limit as $\kappa \to \infty$ in the previous display and using (5.12), we obtain a contradiction to (5.11).

Case 2: $\{\bar{\xi}_\kappa\}$ admits a convergent subsequence. Up to extracting a subsequence (which we do not relabel), we may assume that $\bar{\xi}_\kappa \to \bar{\xi}_\infty$. We first consider the case $|\bar{\xi}_\infty + b| > 0$, which means that, up to selecting $\kappa \in \mathbb{N}$ large enough and then relabeling,

$$|\bar{\xi}_\kappa + D\varphi(x_\kappa)| \geq \frac{1}{4}|\bar{\xi}_\infty + b| > 0 \tag{5.15}$$

holds true, so from $(5.11)_2$ and (5.5) we obtain

$$\begin{aligned}\mathfrak{F}_\kappa(A) &\overset{(5.14),(5.15)}{\geq} -\frac{\mathfrak{C}_\kappa}{\min\{1, \mathfrak{H}_{q;\kappa}(x, 4^{-1}(\bar{\xi}_\infty + b); \bar{\xi}_\infty), \mathfrak{H}_{s;\kappa}(x, 4^{-1}(\bar{\xi}_\infty + b); \bar{\xi}_\infty)\}}\\ &\geq -\frac{4^{p^+ + p_- + 1}\kappa^{-1}}{\min\{1, |\bar{\xi}_\infty + b|^{p^+}, |\bar{\xi}_\infty + b|^{p_-}\}}.\end{aligned}$$

Sending $\kappa \to \infty$ in the above display, we contradict (5.11).

At this point, we only need to take care of the occurrence $|\bar{\xi}_\infty + b| = 0$. By (5.11) and ellipticity, we deduce that A has at least one positive eigenvalue. Let Σ_0 be the direct sum of all the eigensubspaces corresponding to nonnegative eigenvalues of A and $\Pi_0(\cdot)$ be the orthogonal projection over Σ_0. Since $\mathfrak{u}_\infty - \varphi$ has a local strict minimum at x_0, by (5.8) the function

$$\varphi_\delta(x) := \varphi(x) + \delta|\Pi_0(x - x_0)|$$

touches $\mathfrak{u}_\infty$ from below at a point $\hat{x}_0$ close to x_0 for $\delta > 0$ sufficiently small. We are then lead to consider two possible occurrences: $|\Pi_0(\hat{x}_0 - x_0)| = 0$ and $|\Pi_0(\hat{x}_0 - x_0)| > 0$. If $|\Pi_0(\hat{x}_0 - x_0)| = 0$, then

$$|\Pi_0(\hat{x}_0 - x_0)| = \max_{e \in \mathbb{S}^{n-1}} e \cdot \Pi_0(\hat{x}_0 - x_0) = \min_{e \in \mathbb{S}^{n-1}} e \cdot \Pi_0(\hat{x}_0 - x_0),$$

which means that the map

$$\hat{\varphi}_\delta(x) = \varphi(x) + \delta e \cdot \Pi_0(x - x_0)$$

touches $\mathfrak{u}_\infty$ from below at $\hat{x}_0$ for all $e \in \mathbb{S}^{n-1}$. This last fact, (5.8) and standard stability results, cf. [40, Lemma 5], yield that $\hat{\varphi}_\delta(\cdot)$ touches $\mathfrak{u}_\kappa$ from below at $\hat{x}_\kappa \to \hat{x}_0$. The uniformity prescribed by (5.8) guarantees that δ does not depend on κ. A direct computation shows that $D(e \cdot \Pi_0(x - x_0)) = \Pi_0(e)$ and $D^2(e \cdot \Pi_0(x - x_0)) = 0$. Moreover,

$$e \in \Sigma_0 \cap \mathbb{S}^{n-1} \to \Pi_0(e) = e \quad \text{and} \quad e \in \Sigma_0^\perp \cap \mathbb{S}^{n-1} \to \Pi_0(e) = 0, \tag{5.16}$$

where $\Sigma_0^\perp$ is the subspace orthogonal to Σ_0. We claim that

$$\text{there is } \hat{e} \in \mathbb{S}^{n-1} \text{ so that } |D\varphi(\hat{x}_0) + \bar{\xi}_\infty + \Pi_0(\hat{e})| > 0. \tag{5.17}$$

In fact, if $|D\varphi(\hat{x}_0) + \bar{\xi}_\infty| = 0$ we pick any $\hat{e} \in \mathbb{S}^{n-1} \cap \Sigma_0$ (which exists as $\Sigma_0 \neq \emptyset$ because of the previous considerations on the eigenvalues of A) and use $(5.16)_1$; while if $|D\varphi(\hat{x}_0) + \overline{\xi}_\infty| > 0$ and $\Sigma_0^\perp \neq \emptyset$, we fix $\hat{e} \in \Sigma_0^\perp \cap \mathbb{S}^{n-1}$ and exploit (5.16)$_2$; and if $|D\varphi(\hat{x}_0) + \overline{\xi}_\infty| > 0$ and $\Sigma_0^\perp = \emptyset$, i.e., $\Sigma_0 \equiv \mathbb{R}^n$ and $\Pi_0(\cdot) \equiv \mathbf{I}$, we let

$$\hat{e} := \frac{D\varphi(\hat{x}_0) + \overline{\xi}_\infty}{|D\varphi(\hat{x}_0) + \overline{\xi}_\infty|},$$

thus

$$|D\varphi(\hat{x}_0) + \overline{\xi}_\infty + \Pi_0(\hat{e})| = |D\varphi(\hat{x}_0) + \overline{\xi}_\infty| + 1 > 1.$$

Once (5.17) has been established, we can take $\kappa \in \mathbb{N}$ sufficiently large to assure that

$$|D\varphi(\hat{x}_\kappa) + \overline{\xi}_\kappa + \Pi_0(\hat{e})| \geq \frac{1}{4}|D\varphi(\hat{x}_0) + \overline{\xi}_\infty + \Pi_0(\hat{e})| \overset{(5.17)}{>} 0, \tag{5.18}$$

recall (5.11)$_2$ and use (5.5) to conclude that

$$\mathfrak{F}_\kappa(A) \overset{(5.14),(5.18)}{\geq} -\frac{4^{p^+ + p_- + 1}\kappa^{-1}}{\min\{1, |D\varphi(\hat{x}_0) + \overline{\xi}_\infty + \Pi_0(\hat{e})|^{p^+}, |D\varphi(\hat{x}_0) + \overline{\xi}_\infty + \Pi_0(\hat{e})|^{p_-}\}}.$$

As $\kappa \to \infty$ in the above display, we obtain a contradiction to (5.11).

On the other hand, if $|\Pi_0(\hat{x}_0 - x_0)| > 0$, we still have that $\varphi_\delta(\cdot)$ touches $\mathfrak{u}_\infty$ from below at $\hat{x}_0$ as above, so, by (5.8) and standard stability results [40, Lemma 5] we have that $\varphi_\delta(\cdot)$ touches from below $\mathfrak{u}_\kappa$ at $\hat{x}_\kappa$ for some points $\hat{x}_\kappa \to \hat{x}_0$. We remark that by (5.8), δ does not depend on κ. Since $|\Pi_0(\hat{x}_0 - x_0)| > 0$, it also holds that $|\Pi_0(\hat{x}_\kappa - x_0)| > 0$ for $\kappa \in \mathbb{N}$ sufficiently large, so the map $x \mapsto |\Pi_0(x - x_0)|$ is smooth and convex in a neighborhood of $\hat{x}_\kappa$. As $\Pi_0(\cdot - x_0)$ is a projector, it holds that

$$\Pi_0(x - x_0) D\Pi_0(x - x_0) = \Pi_0(x - x_0), \quad D^2|\Pi_0(x - x_0)| \text{ is nonnegative definite.} \tag{5.19}$$

Recall that we were assuming $|\overline{\xi}_\infty + b| = 0$, so using the very definition of Σ_0 we have

$$\begin{aligned}\left|\overline{\xi}_\infty + D\varphi(\hat{x}_0) + \delta\frac{\Pi_0(\hat{x}_0 - x_0)}{|\Pi_0(\hat{x}_0 - x_0)|}\right|^2 &= \left|A(\hat{x}_0 - x_0) + \delta\frac{\Pi_0(\hat{x}_0 - x_0)}{|\Pi_0(\hat{x}_0 - x_0)|}\right|^2 \\ &= |A(\hat{x}_0 - x_0)|^2 + \delta^2 \\ &\quad + 2\delta A(\hat{x}_0 - x_0)\cdot\frac{\Pi_0(\hat{x}_0 - x_0)}{|\Pi_0(\hat{x}_0 - x_0)|} \\ &= |A(\hat{x}_0 - x_0)|^2 + \delta^2 \\ &\quad + 2\delta A\Pi_0(\hat{x}_0 - x_0)\cdot\frac{\Pi_0(\hat{x}_0 - x_0)}{|\Pi_0(\hat{x}_0 - x_0)|} \\ &\geq |A(\hat{x}_0 - x_0)|^2 + \delta^2 \geq \delta^2,\end{aligned}$$

thus

$$\left|\overline{\xi}_\infty + D\varphi(\hat{x}_0) + \delta\frac{\Pi_0(\hat{x}_0 - x_0)}{|\Pi_0(\hat{x}_0 - x_0)|}\right| \geq \delta,$$

therefore, for $\kappa \in \mathbb{N}$ large enough we have

$$\left|\bar{\xi}_\kappa + D\varphi(\hat{x}_\kappa) + \delta \frac{\Pi_0(\hat{x}_\kappa - x_0)}{|\Pi_0(\hat{x}_\kappa - x_0)|}\right| \geq \frac{\delta}{4}. \tag{5.20}$$

We can then use (5.11)$_2$ and that $\mathfrak{u}_\kappa$ is a κ^{-1}-normalized viscosity supersolution of (5.5) to get

$$\begin{aligned}
\mathfrak{F}_\kappa(A) &\overset{(2.6)}{\geq} \mathfrak{F}_\kappa(A + D^2|\Pi_0(\hat{x}_\kappa - x_0)|) + \lambda\, \mathrm{tr}(D^2|\Pi_0(\hat{x}_\kappa - x_0)|) \\
&\overset{(5.19)_2}{\geq} -\frac{\mathfrak{C}_\kappa}{\min\left\{1, \left|\bar{\xi}_\kappa + D\varphi(\hat{x}_\kappa) + \frac{\Pi_0(\hat{x}_\kappa - x_0)}{|\Pi_0(\hat{x}_\kappa - x_0)|}\right|^{p^+}, \left|\bar{\xi}_\kappa + D\varphi(\hat{x}_\kappa) + \frac{\Pi_0(\hat{x}_\kappa - x_0)}{|\Pi_0(\hat{x}_\kappa - x_0)|}\right|^{p^-}\right\}} \\
&\overset{(5.20)}{\geq} -\frac{4^{p^+ + p_- + 1}\kappa^{-1}}{\min\{1, \delta^{p^+}, \delta^{p_-}\}}.
\end{aligned}$$

Sending $\kappa \to \infty$ in the above, we obtain a contradiction to (5.11). Combining Case 1 and Case 2, we can conclude that $\mathfrak{F}_\infty(A) \geq 0$, so $\mathfrak{u}_\infty$ is a supersolution of (5.10) in $B_1(0)$. To show that $\mathfrak{u}_\infty$ is also a subsolution to (5.10), we only observe that this is equivalent to proving that $\tilde{\mathfrak{u}}_\infty := -\mathfrak{u}_\infty$ is a supersolution of the equation

$$\tilde{\mathfrak{F}}_\infty(D^2\tilde{\mathfrak{u}}_\infty) = 0 \quad \text{in } B_1(0),$$

where we set $\tilde{\mathfrak{F}}_\infty(M) := -\mathfrak{F}_\infty(-M)$, which is uniformly (λ, Λ)-elliptic in the sense of (2.6). Hence, we can apply the whole procedure developed above to $\tilde{\mathfrak{u}}_\infty$ and conclude that $\mathfrak{u}_\infty$ is a viscosity solution of (5.10). Proposition 2.6 then applies, and $\mathfrak{u}_\infty \in C^{1,\alpha}(B_{1/2}(0))$. In particular, (2.16) is valid, which contradicts (5.9)$_2$, and the proof is complete. ∎

Lemma 5.2 essentially determines a certain parameter $\varepsilon_0 \equiv \varepsilon_0(\mathtt{data}) \in (0,1)$ so that it is possible to build a tangential path connecting an ε_0-normalized viscosity solution of (2.10) to viscosity solutions of a homogeneous limiting profile for which the Krylov–Safonov regularity theory is available. At this stage, we need to transfer such regularity from the limiting homogeneous problem to viscosity solutions of (1.1). In this perspective, we establish an oscillation control at discrete scales.

Lemma 5.3. *Assume* (2.1), (2.2), (2.3), (2.4), (2.6) *and let* $\varepsilon_0 \equiv \varepsilon_0(\mathtt{data}) \in (0,1)$ *be the smallness parameter determined in Lemma* 5.2. *There are* $\sigma \equiv \sigma(n, \lambda, \Lambda) \in (0,1)$ *and* $\alpha_0 \equiv \alpha_0(n, \lambda, \Lambda, p^+, p_-) \in (0,1)$ *so that if* $\mathfrak{u} \in C(B_1(0))$ *is an* ε_0*-normalized viscosity solution of equation* (1.1)*, then for any* $\kappa \in N$ *it is possible to find* $\bar{\xi}_\kappa \in \mathbb{R}^n$ *so that*

$$\operatorname{osc}_{B_{\sigma^\kappa}(0)} (\mathfrak{u} - \bar{\xi}_\kappa \cdot x) \leq \sigma^{\kappa(1+\alpha_0)}. \tag{5.21}$$

Proof. Let $\sigma \equiv \sigma(n, \lambda, \Lambda)$ be the one in (2.15) and let

$$\alpha_0 \in \left(0, \min\left\{\alpha, \frac{1}{\max\{p^+, p_-\} + 1}, \frac{\log(2)}{-\log(\sigma)}\right\}\right), \tag{5.22}$$

where $\alpha \equiv \alpha(n, \lambda, \Lambda) \in (0, 1)$ is the Hölder continuity exponent provided by Proposition 2.6. A direct consequence of the choice made in (5.22) is

$$\sigma^{\alpha_0} > \frac{1}{2}. \tag{5.23}$$

Now, we look back at the construction developed in Section 5.1 and fix a scaling parameter τ_0 equal to $\varepsilon_0^{1/2}$, where $\varepsilon_0 \equiv \varepsilon_0(\mathtt{data})$ is the one provided by Lemma 5.2. In this way we determine the dependency $\tau_0 \equiv \tau_0(\mathtt{data})$ and remove the ambiguity raised in Remark 5.1 described by the inequality

$$\|\mathfrak{a}\|_{L^\infty(B_1(0))} + \|\mathfrak{b}\|_{L^\infty(B_1(0))} \le c(\mathtt{data}, \|a\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)}).$$

Let $\mathfrak{u} \in C(B_1(0))$ be an ε_0-normalized viscosity solution of equation (1.1) in the sense of Definition 2.4 and of Section 5.1, which means that $\mathfrak{u}$ is an ε_0-normalized viscosity subsolution/supersolution of (5.1)/(5.2). With $\kappa \in \mathbb{N} \cup \{0\}$, we define $\sigma_\kappa := \sigma^\kappa$ and start an induction argument to show that (5.21) holds for all $\kappa \in \mathbb{N} \cup \{0\}$.

Basic step: $\kappa = 0$. By $(5.3)_2$ we see that (5.21) holds with $\bar{\xi}_0 = 0$. In fact, we have

$$\operatorname{osc}_{B_{\sigma_0(0)}}(\mathfrak{u} - \bar{\xi}_0 \cdot x) = \operatorname{osc}_{B_1(0)} \mathfrak{u} \overset{(5.3)_2}{\le} 1.$$

Induction step. Assume that there exists $\bar{\xi}_\kappa \in \mathbb{R}^n$ satisfying (5.21) and define

$$\mathfrak{u}_\kappa(x) := \sigma_\kappa^{-(1+\alpha_0)}\big[\mathfrak{u}(\sigma_\kappa x) - \sigma_\kappa \bar{\xi}_\kappa \cdot x\big].$$

Recalling Definition 2.4, a straightforward computation shows that $\mathfrak{u}_\kappa$ is a viscosity subsolution of

$$\begin{aligned} \min\{&\mathfrak{F}_\kappa(D^2\mathfrak{u}_\kappa), \mathfrak{H}_{q;\kappa}(x, D\mathfrak{u}_\kappa; \tilde{\xi}_\kappa)\mathfrak{F}_\kappa(D^2\mathfrak{u}_\kappa),\\ &\mathfrak{H}_{s;\kappa}(x, D\mathfrak{u}_\kappa; \tilde{\xi}_\kappa)\mathfrak{F}_\kappa(D^2\mathfrak{u}_\kappa)\} = \mathfrak{C}_\kappa \quad \text{in } B_1(0) \end{aligned} \tag{5.24}$$

and a viscosity supersolution to

$$\begin{aligned} \max\{&\mathfrak{F}_\kappa(D^2\mathfrak{u}_\kappa), \mathfrak{H}_{q;\kappa}(x, D\mathfrak{u}_\kappa; \tilde{\xi}_\kappa)\mathfrak{F}_\kappa(D^2\mathfrak{u}_\kappa),\\ &\mathfrak{H}_{s;\kappa}(x, D\mathfrak{u}_\kappa; \tilde{\xi}_\kappa)\mathfrak{F}_\kappa(D^2\mathfrak{u}_\kappa)\} = -\mathfrak{C}_\kappa \quad \text{in } B_1(0), \end{aligned} \tag{5.25}$$

where

$$\begin{aligned} \tilde{\xi}_\kappa &:= \sigma_\kappa^{-\alpha_0}\bar{\xi}_\kappa,\\ \mathfrak{a}_\kappa(x) &:= \sigma_\kappa^{\alpha_0(q-p^+)}\mathfrak{a}(\sigma_\kappa x),\\ \mathfrak{b}_\kappa(x) &:= \sigma_\kappa^{\alpha_0(s-p_-)}\mathfrak{b}(\sigma_\kappa x),\\ \mathfrak{F}_\kappa(M) &:= \sigma_\kappa^{1-\alpha_0}\mathfrak{F}(\sigma_\kappa^{\alpha_0-1}M),\\ \mathfrak{C}_\kappa &:= \sigma_\kappa^{1-\alpha_0(\max\{p^+,p_-\}+1)}\mathfrak{C}. \end{aligned}$$

Here, $\mathfrak{H}_{q;\kappa}(\cdot)$, $\mathfrak{H}_{s;\kappa}(\cdot)$ are defined in the proof of Lemma 5.2 and $\mathfrak{C}$ is the constant derived in Section 5.1 corresponding to the scaling parameter τ_0 fixed before. Notice that by construction, $\mathfrak{F}_\kappa(\cdot)$ satisfies (2.1) uniformly in κ and because of the choice of $\tau_0 \equiv \tau_0(\mathtt{data})$ made above, we have

$$\mathfrak{C}_\kappa \le \sigma_\kappa^{1-\alpha_0(\max\{p^+,p_-\}+1)}\mathfrak{C} \overset{(5.3)_2,(5.22)}{\le} \varepsilon_0.$$

Furthermore, the induction assumption assures that

$$\operatorname{osc}_{B_1(0)} \mathfrak{u}_\kappa = \sigma_\kappa^{-(1+\alpha_0)} \operatorname{osc}_{B_{\sigma_\kappa}(0)}(\mathfrak{u} - \bar{\xi}_\kappa \cdot x) \overset{(5.21)}{\le} 1, \tag{5.26}$$

and, as $\mathfrak{u}(0) = 0$, cf. Section 5.1, we also have $\mathfrak{u}_\kappa(0) = 0$. Therefore, by (5.26) we have $\|\mathfrak{u}_\kappa\|_{L^\infty(B_1(0))} \le 1$. Thus, we see that $\mathfrak{u}_\kappa$ is actually an ε_0-normalized viscosity subsolution/supersolution of (5.24)/(5.25), so all the assumptions of Lemma 5.2 are satisfied, and there is $\tilde{\xi}_{\kappa+1} \in \mathbb{R}^n$ so that

$$\operatorname{osc}_{B_\sigma(0)}(\mathfrak{u}_\kappa - \tilde{\xi}_{\kappa+1} \cdot x) \le \frac{\sigma}{2}.$$

Setting $\bar{\xi}_{\kappa+1} := \bar{\xi}_\kappa + \sigma_\kappa^{\alpha_0}\tilde{\xi}_{\kappa+1}$, we can rewrite the content of the previous display as

$$\sigma_\kappa^{-(1+\alpha_0)} \operatorname{osc}_{B_{\sigma_{\kappa+1}}(0)}(\mathfrak{u} - \bar{\xi}_{\kappa+1} \cdot x) \le \frac{\sigma}{2} \implies \operatorname{osc}_{B_{\sigma_{\kappa+1}}(0)}(\mathfrak{u} - \bar{\xi}_{\kappa+1} \cdot x) \overset{(5.23)}{\le} \sigma_{\kappa+1}^{1+\alpha_0},$$

and the proof is complete. ∎

Now we are ready to prove Theorem 1.2.

5.3. Proof of Theorem 1.2

Proof. Let $u \in C(\Omega)$ be a viscosity solution of equation (1.1). For the parameter $\varepsilon_0 \equiv \varepsilon_0(\mathtt{data}) \in (0, 1)$ provided by Lemma 5.2, we follow the scaling process outlined in Section 5.1 to turn u into an ε_0-normalized viscosity solution of (1.1). The choice of ε_0 assures that the assumptions of Lemma 5.3 are satisfied, so (5.21) is available to us. Given any $\varrho \in (0, 1]$, we can find $\kappa \in \mathbb{N} \cup \{0\}$ so that $\sigma^{\kappa+1} < \varrho \le \sigma^\kappa$. We then estimate

$$\operatorname{osc}_{B_\varrho(0)}(\mathfrak{u} - \xi_\kappa \cdot x) \le \operatorname{osc}_{B_{\sigma^\kappa}(0)}(\mathfrak{u} - \xi_\kappa \cdot x) \overset{(5.21)}{\le} \sigma^{\kappa(1+\alpha_0)} \le \sigma^{-(1+\alpha_0)}\varrho^{1+\alpha_0} \le c\varrho^{1+\alpha_0},$$

with $c \equiv c(n, \lambda, \Lambda, p^+, p_-)$, so $\mathfrak{u}$ is C^{1,α_0}-regular around zero. By standard translation arguments we can prove the same fact in a neighborhood of any $x_0 \in B_{1/2}(0)$. In particular, we have

$$[D\mathfrak{u}]_{0,\alpha_0;B_{1/2}(0)} \le c(n, \lambda, \Lambda, p^+, p_-).$$

Reversing the scaling procedure in Section 5.1 we get

$$[Du]_{0,\alpha_0;B_{1/2}(0)} \le c\tau^{-(1+\alpha_0)}(\mathtt{data}, \|u\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)})$$

and applying the usual covering argument we obtain (1.8), which immediately implies that $u \in C^{1,\alpha_0}_{\mathrm{loc}}(\Omega)$, and the proof is complete. ∎

A. Hölder estimates for Multi Phase equations with variable exponents

Let us derive uniform Hölder estimates for continuous viscosity solutions to fully nonlinear elliptic equations of Multi Phase type with variable exponents. Let $\mu \in [0,1]$ be any number, set for simplicity

$$\Omega \times \mathbb{R}^n \ni (x,z) \mapsto G_\mu(x,z) := \big[\ell_\mu(z)^{p(x)} + a(x)\ell_\mu(z)^{q(x)} + b(x)\ell_\mu(z)^{s(x)}\big],$$

and consider the equation

$$G_\mu(x, Du)\big(\mu u + F(D^2u)\big) = f(x) \quad \text{in } \Omega, \tag{A.1}$$

where

$$0 \le p(\cdot) \in C(\Omega), \quad 0 \le q(\cdot) \in C(\Omega), \quad 0 \le s(\cdot) \in C(\Omega), \tag{A.2}$$

and assume also (2.1), (2.3) and (2.4).

Proposition A.1. *Under assumptions (2.1), (2.3), (2.4) and (A.2), let $u \in C(\Omega)$ be a viscosity solution to the Multi Phase fully nonlinear equation with variable exponents (A.1). Then $u \in C^{0,\beta_0}_{\mathrm{loc}}(\Omega)$ for all $\beta_0 \in (0,1)$. In particular, if $B_\varrho(z_0) \Subset \Omega$ is any ball with radius $\varrho \in (0, \frac{1}{2})$, it holds that*

$$[u]_{0,\beta_0;B_{\varrho/2}(z_0)} \le c(n, \lambda, \Lambda, \|u\|_{L^\infty(B_\varrho(z_0))}, \|f\|_{L^\infty(B_\varrho(z_0))}, \varrho, \beta_0). \tag{A.3}$$

Proof. Let $u \in C(\Omega)$ be a viscosity solution to equation (A.1) and $B_\varrho(z_0) \Subset \Omega$ be any ball with radius $\varrho \in (0, \frac{1}{2})$. We prove that there are two constants $A_2 \equiv A_2(\varrho, \|u\|_{L^\infty(B_\varrho(z_0))})$ and $A_1 \equiv A_1(n, \lambda, \Lambda, p, \varrho, \beta_0, \|u\|_{L^\infty(B_\varrho(z_0))}, \|f\|_{L^\infty(B_\varrho(z_0))})$ so that

$$\mathcal{M}(x_0) := \sup_{x,y \in B_\varrho(z_0)} \big(u(x) - u(y) - A_1|x-y|^{\beta_0} - A_2\big(|x-x_0|^2 + |y-x_0|^2\big)\big) \le 0 \tag{A.4}$$

holds for all $x_0 \in B_{\varrho/2}(z_0)$. In (A.4), $\beta_0 \in (0,1)$ is any (fixed) number. By contradiction, we assume that

$$\text{there exists } x_0 \in B_{\varrho/2}(z_0) \text{ such that } \mathcal{M}(x_0) > 0 \text{ for all positive } A_1, A_2, \tag{A.5}$$

define quantities

$$\begin{cases} A_1 := \dfrac{40}{\beta_0(1-\beta_0)}\Big[\dfrac{\|f\|_{L^\infty(B_\varrho(z_0))} + \|u\|_{L^\infty(B_\varrho(z_0))}}{\lambda} + (2A_2+1)\Big(\dfrac{\Lambda(n-1)}{\lambda} + 1\Big)\Big], \\ A_2 := 64\varrho^{-2}\max\big\{\|u\|_{L^\infty(B_\varrho(z_0))}, 1\big\}, \end{cases} \tag{A.6}$$

and consider the auxiliary functions

$$\begin{cases} \psi(x,y) := A_1|x-y|^{\beta_0} + A_2\big(|x-x_0|^2 + |y-x_0|^2\big), \\ \phi(x,y) := u(x) - u(y) - \psi(x,y). \end{cases}$$

If $(\bar{x},\bar{y}) \in \overline{B}_{\varrho}(z_0) \times \overline{B}_{\varrho}(z_0)$ is a maximum point of $\phi(\cdot)$, via (A.5) we have $\phi(\bar{x},\bar{y}) = \mathcal{M}(x_0) > 0$, so

$$A_1|\bar{x}-\bar{y}|^{\beta_0} + A_2\big(|\bar{x}-x_0|^2 + |\bar{y}-x_0|^2\big) \le u(\bar{x}) - u(\bar{y}) \le 2\|u\|_{L^\infty(B_\varrho(z_0))}.$$

Plugging $(A.6)_2$ in the above inequality yields that $\bar{x}$, $\bar{y}$ both belong to the interior of $B_\varrho(z_0)$. In fact,

$$|\bar{x}-z_0| \le |\bar{x}-x_0| + |x_0-z_0| \le \frac{3\varrho}{4} \quad \text{and} \quad |\bar{y}-z_0| \le |\bar{y}-x_0| + |x_0-z_0| \le \frac{3\varrho}{4}.$$

Moreover, $\bar{x} \neq \bar{y}$, otherwise (A.4) would be satisfied. This last remark shows that $\psi(\cdot)$ is smooth in a small neighborhood of $(\bar{x},\bar{y})$, therefore we can determine vectors

$$\xi_{\bar{x}} := \partial_x\psi(\bar{x},\bar{y}) = A_1\beta_0|\bar{x}-\bar{y}|^{\beta_0-1}\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|} + 2A_2(\bar{x}-x_0),$$
$$\xi_{\bar{y}} := -\partial_y\psi(\bar{x},\bar{y}) = A_1\beta_0|\bar{x}-\bar{y}|^{\beta_0-1}\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|} - 2A_2(\bar{y}-x_0).$$

To summarize, we have that $\phi(\cdot)$ attains its maximum at $(\bar{x},\bar{y})$ inside $B_\varrho(z_0) \times B_\varrho(z_0)$ and $\phi(\cdot)$ is smooth around $(\bar{x},\bar{y})$, thus Proposition 2.3 applies: we can find a threshold $\hat{\delta} = \hat{\delta}(\|D^2\psi\|)$ such that for all $\delta \in (0,\hat{\delta})$ the couple $(\xi_{\bar{x}}, X_\delta)$ is a limiting subjet of u at $\bar{x}$ and the couple $(\xi_{\bar{y}}, Y_\delta)$ is a limiting superjet of u at $\bar{y}$ and the matrix inequality

$$\begin{bmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{bmatrix} \le \begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix} + (2A_2+\delta)\mathbf{I} \tag{A.7}$$

holds, where we set

$$Z := A_1 D^2(|x-y|^{\beta_0})|_{(\bar{x},\bar{y})} = \beta_0 A_1|\bar{x}-\bar{y}|^{\beta_0-2}\Big[\mathbf{I} - (2-\beta_0)\frac{(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|^2}\Big].$$

We fix $\delta \equiv \min\{1, \frac{\hat{\delta}}{4}\}$ and apply (A.7) to vectors of the form $(z,z) \in \mathbb{R}^{2n}$ to obtain

$$\langle (X_\delta - Y_\delta)z, z\rangle \le (4A_2+2)|z|^2.$$

This means that

$$\text{all the eigenvalues of } X_\delta - Y_\delta \text{ are less than or equal to } 2(2A_2+1). \tag{A.8}$$

In particular, applying (A.7) to the vector

$$\bar{z} := \Big(\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}, \frac{\bar{y}-\bar{x}}{|\bar{x}-\bar{y}|}\Big),$$

we get

$$\left\langle (X_\delta - Y_\delta)\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|}, \frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|}\right\rangle \le 2(2A_2+1) - 4\beta_0(1-\beta_0)A_1|\overline{x}-\overline{y}|^{\beta_0-2}.$$

This yields in particular that

$$\begin{aligned}&\text{at least one eigenvalue of } X_\delta - Y_\delta \text{ is less than}\\ &\qquad 2(2A_2+1) - 4A_1\beta_0(1-\beta_0)|\overline{x}-\overline{y}|^{\beta_0-2}.\end{aligned} \tag{A.9}$$

Recalling (A.6)$_1$ we can conclude that

$$2(2A_2+1) - 4A_1\beta_0(1-\beta_0)|\overline{x}-\overline{y}|^{\beta_0-2} < 0,$$

where we also used that $|\overline{x}-\overline{y}| \le 1$. This means that at least one eigenvalue of $X_\delta - Y_\delta$ is negative, thus combining (2.7)$_2$, (A.8), and (A.9) we obtain

$$\mathcal{P}^-_{\lambda,\Lambda}(X_\delta - Y_\delta) \ge -2(2A_2+1)\big[\Lambda(n-1)+\lambda\big] + 4\lambda A_1\beta_0(1-\beta_0). \tag{A.10}$$

With $\xi_{\overline{x}}$, $\xi_{\overline{y}}$ computed before, we recover the viscosity inequalities

$$\begin{cases} G_\mu(\overline{x},\xi_{\overline{x}})\big(\mu u(\overline{x}) + F(X_\delta)\big) \le f(\overline{x}),\\ G_\mu(\overline{y},\xi_{\overline{y}})\big(\mu u(\overline{y}) + F(Y_\delta)\big) \ge f(\overline{y}). \end{cases} \tag{A.11}$$

Moreover, a quick computation based on the Young inequality shows that

$$\begin{aligned}\min\big\{\ell_\mu(\xi_{\overline{x}}), \ell_\mu(\xi_{\overline{y}})\big\} &\ge \min\big\{|\xi_{\overline{x}}|, |\xi_{\overline{y}}|\big\} \ge \sqrt{\frac{1}{4}A_1^2\beta_0^2 - 25A_2^2}\\ &\ge \sqrt{\Big(\frac{1}{2}A_1\beta_0 - 5A_2\Big)\Big(\frac{1}{2}A_1\beta_0 + 5A_2\Big)} \overset{(A.6)_1}{\ge} 1\end{aligned} \tag{A.12}$$

and, via ellipticity,

$$F(X_\delta) \overset{(2.8)}{\ge} F(Y_\delta) + \mathcal{P}^-_{\lambda,\Lambda}(X_\delta - Y_\delta). \tag{A.13}$$

Merging all the previous inequalities, we obtain

$$\begin{aligned}\frac{f(\overline{x})}{G_\mu(\overline{x},\xi_{\overline{x}})} &\overset{(A.11)_1}{\ge} \mu u(\overline{x}) + F(X_\delta) \overset{(A.13)}{\ge} \mu u(\overline{x}) + F(Y_\delta) + \mathcal{P}^-_{\lambda,\Lambda}(X_\delta - Y_\delta)\\ &\overset{(A.10),(A.11)_2}{\ge} \mu(u(\overline{x}) - u(\overline{y}))\\ &\qquad - 2(2A_2+1)\big[\Lambda(n-1)+\lambda\big] + 4\lambda A_1\beta_0(1-\beta_0) + \frac{f(\overline{y})}{G_\mu(\overline{y},\xi_{\overline{y}})},\end{aligned}$$

so with (A.12) we can complete the estimate in the above display as follows:

$$\begin{aligned}2\big(\|f\|_{L^\infty(B_\varrho(z_0))} + \|u\|_{L^\infty(B_\varrho(z_0))}\big) &+ 2(2A_2+1)\big[\Lambda(n-1)+\lambda\big]\\ &\ge \mu(u(\overline{y}) - u(\overline{x})) + \frac{f(\overline{x})}{G_\mu(\overline{x},\xi_{\overline{x}})} - \frac{f(\overline{y})}{G_\mu(\overline{y},\xi_{\overline{y}})}\end{aligned}$$

$$+ 2(2A_2 + 1)\big[\Lambda(n-1) + \lambda\big]$$
$$\geq 4\lambda A_1 \beta_0 (1 - \beta_0),$$

which contradicts the position in (A.6)$_1$. This means that there are two positive constants A_1, A_2 with the dependencies outlined before so that for all $x_0 \in B_{\varrho/2}(z_0)$, inequality (A.4) is satisfied, which in particular yields that $u \in C^{0,\beta_0}(B_{\varrho/2}(z_0))$ for all $\beta_0 \in (0,1)$. The arbitrariness of $B_\varrho(z_0)$ and a standard covering argument render that $u \in C^{0,\beta_0}_{\mathrm{loc}}(\Omega)$ for all $\beta_0 \in (0,1)$, and the proof is complete. ■

Remark A.2. Notice that the constant appearing in (A.3) does not depend on $\mu \in [0,1]$ nor on the moduli of continuity of $a(\cdot)$, $b(\cdot)$, $p(\cdot)$, $q(\cdot)$, $s(\cdot)$.

Acknowledgments. This work is supported by the University of Turin via the project "Regolaritá e proprietá qualitative delle soluzioni di equazioni alle derivate parziali".

References

[1] E. Acerbi and G. Mingione, Regularity results for a class of functionals with non-standard growth. *Arch. Ration. Mech. Anal.* **156** (2001), no. 2, 121–140 Zbl 0984.49020 MR 1814973

[2] M. D. Amaral and E. V. Teixeira, Free transmission problems. *Commun. Math. Phys.* **337** (2015), no. 3, 1465–1489 Zbl 1321.35260 MR 3339182

[3] D. J. Araújo, G. Ricarte, and E. V. Teixeira, Geometric gradient estimates for solutions to degenerate elliptic equations. *Calc. Var. Partial Differ. Equ.* **53** (2015), no. 3-4, 605–625 Zbl 1326.35152 MR 3347473

[4] A. Baasandorj, S.-S. Byun, and J. Oh, Gradient estimates for multi-phase problems. *Calc. Var. Partial Differ. Equ.* **60** (2021), no. 3, paper no. 104 Zbl 07355282 MR 4256270

[5] G. Barles and C. Imbert, Second-order elliptic integro-differential equations: viscosity solutions' theory revisited. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **25** (2008), no. 3, 567–585 Zbl 1155.45004 MR 2422079

[6] P. Baroni, M. Colombo, and G. Mingione, Regularity for general functionals with double phase. *Calc. Var. Partial Differ. Equ.* **57** (2018), no. 2, paper no. 62 Zbl 1394.49034 MR 3775180

[7] I. Birindelli and F. Demengel, Comparison principle and Liouville type results for singular fully nonlinear operators. *Ann. Fac. Sci. Toulouse, Math. (6)* **13** (2004), no. 2, 261–287 Zbl 1129.35369 MR 2126744

[8] I. Birindelli and F. Demengel, Eigenvalue, maximum principle and regularity for fully non linear homogeneous operators. *Commun. Pure Appl. Anal.* **6** (2007), no. 2, 335–366 Zbl 1132.35032 MR 2289825

[9] I. Birindelli and F. Demengel, Regularity for radial solutions of degenerate fully nonlinear equations. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods* **75** (2012), no. 17, 6237–6249 Zbl 1254.35099 MR 2959802

[10] I. Birindelli, and F. Demengel, $\mathcal{C}^{1,\beta}$ regularity for Dirichlet problems associated to fully nonlinear degenerate elliptic equations. *ESAIM, Control Optim. Calc. Var.* **20** (2014), no. 4, 1009–1024 Zbl 1315.35108 MR 3264232

[11] I. Birindelli, F. Demengel, and F. Leoni, $\mathcal{C}^{1,\gamma}$ regularity for singular or degenerate fully nonlinear equations and applications. *NoDEA, Nonlinear Differ. Equ. Appl.* **26** (2019), no. 5, paper no. 40 Zbl 1428.35132 MR 4010655

[12] A. C. Bronzi, E. A. Pimentel, G. C. Rampasso, and E. V. Teixeira, Regularity of solutions to a class of variable-exponent fully nonlinear elliptic equations. *J. Funct. Anal.* **279** (2020), no. 12, 108781 Zbl 1450.35092 MR 4156124

[13] S.-S. Byun and J. Oh, Regularity results for generalized double phase functionals. *Anal. PDE* **13** (2020), no. 5, 1269–1300 Zbl 07271830 MR 4149062

[14] L. A. Caffarelli, M. Soria-Carro, and P. R. Stinga, Regularity for $C^{1,\alpha}$ interface transmission problems. *Arch. Ration. Mech. Anal.* **240** (2021), no. 1, 265–294 Zbl 1467.35169 MR 4228861

[15] L. A. Caffarelli and X. Cabré, *Fully nonlinear elliptic equations*. Colloq. Publ., Am. Math. Soc. 43, American Mathematical Society, Providence, RI, 1995 Zbl 0834.35002 MR 1351007

[16] L. Caffarelli, M. G Crandall, M. Kocan, and A. Święch, On viscosity solutions of fully nonlinear equations with measurable ingredients. *Commun. Pure Appl. Math.* **49** (1996), no. 4, 365–397 Zbl 0854.35032 MR 1376656

[17] M. Colombo and G. Mingione, Regularity for double phase variational problems. *Arch. Ration. Mech. Anal.* **215** (2015), no. 2, 443–496 Zbl 1322.49065 MR 3294408

[18] M. G. Crandall, H. Ishii, and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc., New Ser.* **27** (1992), no. 1, 1–67 Zbl 0755.35015 MR 1118699

[19] M. G. Crandall, M. Kocan, P.-L. Lions, and A. Święch, Existence results for boundary problems for uniformly elliptic and parabolic fully nonlinear equations. *Electron. J. Differ. Equ.* (1999), no. 24 Zbl 0927.35029 MR 1696765

[20] J. V. da Silva, E. C. Júnior, and G. C. Ricarte, Fully nonlinear singularly perturbed models with non-homogeneous degeneracy. *Rev. Mat. Iberoam.* (2022), DOI 10.4171/RMI/1319

[21] J. V. da Silva, G. C. Rampasso, G. C. Ricarte, C. Gleydson, and H. A. Vivas, Free boundary regularity for a class of one-phase problems with non-homogeneous degeneracy. 2021, arXiv:2103.11028. To appear in *Isr. J. Math.*

[22] J. V. da Silva and G. C. Ricarte, Geometric gradient estimates for fully nonlinear models with non-homogeneous degeneracy and applications. *Calc. Var. Partial Differ. Equ.* **59** (2020), no. 5, paper no. 161 Zbl 1448.35058 MR 4148407

[23] P. Daskalopoulos, T. Kuusi, and G. Mingione, Borderline estimates for fully nonlinear elliptic equations. *Commun. Partial Differ. Equations* **39** (2014), no. 3, 574–590 Zbl 1290.35092 MR 3169795

[24] G. Dávila, P. Felmer, and A. Quaas, Alexandroff-Bakelman-Pucci estimate for singular or degenerate fully nonlinear elliptic equations. *C. R., Math., Acad. Sci. Paris* **347** (2009), no. 19-20, 1165–1168 Zbl 1180.35230 MR 2566996

[25] G. Dávila, P. Felmer, and A. Quaas, Harnack inequality for singular fully nonlinear operators and some existence results. *Calc. Var. Partial Differ. Equ.* **39** (2010), no. 3-4, 557–578 Zbl 1204.35088 MR 2729313

[26] C. De Filippis, Regularity for solutions of fully nonlinear elliptic equations with nonhomogeneous degeneracy. *Proc. Roy. Soc. Edinb., Sect. A, Math.* **151** (2021), no. 1, 110–132 Zbl 1459.35154 MR 4202634

[27] C. De Filippis and G. Mingione, Manifold constrained non-uniformly elliptic problems. *J. Geom. Anal.* **30** (2020), no. 2, 1661–1723 Zbl 1437.49008 MR 4081328

[28] C. De Filippis and G. Mingione, On the regularity of minima of non-autonomous functionals. *J. Geom. Anal.* **30** (2020), no. 2, 1584–1626 Zbl 1437.35292 MR 4081325

[29] C. De Filippis and G. Mingione, Lipschitz Bounds and Nonautonomous Integrals. *Arch. Ration. Mech. Anal.* **242** (2021), no. 2, 973–1057 Zbl 07388811 MR 4331020

[30] C. De Filippis and J. Oh, Regularity for multi-phase variational problems. *J. Differ. Equations* **267** (2019), no. 3, 1631–1670 Zbl 1422.49037 MR 3945612

[31] L. Esposito, F. Leonetti, and G. Mingione, Sharp regularity for functionals with (p,q) growth. *J. Differ. Equations* **204** (2004), no. 1, 5–55 Zbl 1072.49024 MR 2076158

[32] Y. Fang, V. D. Rădulescu, and C. Zhang, Regularity of solutions to degenerate fully nonlinear elliptic equations with variable exponent. *Bull. Lond. Math. Soc.* (2021), DOI 10.1112/blms.12550

[33] I. Fonseca, J. Malý, and G. Mingione, Scalar minimizers with fractal singular sets. *Arch. Ration. Mech. Anal.* **172** (2004), no. 2, 295–307 Zbl 1049.49015 MR 2058167

[34] P. Hästö and J. Ok, Maximal regularity for local minimizers of non-autonomous functionals. *J. Europ. Math. Soc.* (2021), DOI 10.4171/JEMS/1118.

[35] G. Huaroto, E. A. Pimentel, G. C. Rampasso, and A. Święch, A fully nonlinear degenerate free transmission problem. 2020, arXiv:2008.06917.

[36] V. A. Il'in and I. A. Šišmarev, The method of potentials for the problems of Dirichlet and Neumann in the case of equations with discontinuous coefficients (in Russian). *Sibirsk. Mat. Ž.*, 46-58, (1961). Zbl 0065.08702 MR 0133474

[37] C. Imbert, Alexandroff-Bakelman-Pucci estimate and Harnack inequality for degenerate/singular fully non-linear elliptic equations. *J. Differ. Equations* **250** (2011), no. 3, 1553–1574 Zbl 1205.35124 MR 2737217

[38] C. Imbert and L. Silvestre, $C^{1,\alpha}$ regularity of solutions of some degenerate fully non-linear elliptic equations. *Adv. Math.* **233** (2013), 196–206 Zbl 1262.35065 MR 2995669

[39] C. Imbert and L. Silvestre, Estimates on elliptic equations that hold only where the gradient is large. *J. Eur. Math. Soc. (JEMS)* **18** (2016), no. 6, 1321–1338 Zbl 1344.35049 MR 3500837

[40] N. Katzourakis, *An introduction to viscosity solutions for fully nonlinear PDE with applications to calculus of variations in L^∞*. SpringerBriefs Math. Springer, Cham, 2015 MR 3289084

[41] P. Marcellini, On the definition and the lower semicontinuity of certain quasiconvex integrals. *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **3** (1986), no. 5, 391–409 Zbl 0609.49009 MR 868523

[42] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions. *Arch. Ration. Mech. Anal.* **105** (1989), no. 3, 267–284 Zbl 0667.49032 MR 969900

[43] G. Mingione and V. Rădulescu, Recent developments in problems with nonstandard growth and nonuniform ellipticity. *J. Math. Anal. Appl.* **501** (2021), no. 1, article ID 125197 Zbl 1467.49003 MR 4258810

[44] O. A. Oleĭnik, Boundary-value problems for linear equations of elliptic parabolic type with discontinuous coefficients (in Russian). *Izv. Akad. Nauk SSSR Ser. Mat.* **25** (1961), 3–20 MR 0123814

[45] M. Picone, Sur un probléme nouveau pour l'équation linéaire aux dérivées partielles de la théorie mathématique classique de l'élasticité. *CBRM Bruxelles*, 9–11, (1954).

[46] E. A. Pimentel and A. Święch, Existence of solutions to a fully nonlinear free transmission problem. 2021, arXiv:2103.08974.

[47] M. A. Ragusa, A. Tachikawa, and H. Takabayashi, Partial regularity of $p(x)$-harmonic maps. *Trans. Am. Math. Soc.* **365** (2013), no. 6, 3329–3353 Zbl 1277.35092 MR 3034468

[48] M. A. Ragusa and A. Tachikawa, Regularity for minimizers for functionals of double phase with variable exponents. *Adv. Nonlinear Anal.* **9** (2020), no. 1, 710–728 Zbl 1420.35145 MR 3985000

[49] M. Schechter, A generalization of the problem of transmission. *Ann. Sc. Norm. Sup. Pisa, Sci. Fis. Mat., III. Ser.* **14** (1960), 207–236 Zbl 0094.29603 MR 131063

[50] A. Tachikawa, On the singular set of minimizers of $p(x)$-energies. *Calc. Var. Partial Differ. Equ.* **50** (2014), no. 1–2, 145–169 Zbl 1305.49049 MR 3194679

[51] A. Tachikawa, Boundary regularity of minimizers of double phase functionals. *J. Math. Anal. Appl.* **501** (2021), no. 1, paper no. 123946 Zbl 1467.35080 MR 4258788

Received 22 March 2021; revised 4 September 2021.

Cristiana De Filippis
Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Università di Parma, Parco Area delle Scienze 53/A, 43124 Parma, Italy; cristiana.defilippis@unipr.it