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# Bifurcations of spherically asymmetric solutions to an evolution equation for curves

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**Abstract.** We show that a certain non-local curvature flow for planar curves has non-trivial self-similar solutions with $n$-fold rotational symmetry, bifurcated from a trivial circular solution. Moreover, we show that the trivial solution is stable with respect to perturbations which keep the geometric center and the enclosed area, and that, for $n$ different from 3, the $n$-fold symmetric solution is stable with respect to perturbations which satisfy the same conditions as above and have the same symmetry as the solutions.

## 1. Introduction

The following evolution equation is known as the curve shortening flow:

$$\frac{\partial u}{\partial t} \cdot \nu = \kappa, \tag{1.1}$$

where $u = u(\theta, t)$ for $\theta \in [0, 2\pi)$ is a simple closed curve, $\nu$ is the outward unit normal vector field, and $\kappa$ is the curvature of $u$. Moreover, we choose the sign of $\kappa$ so that $\kappa < 0$ if $u$ is convex.

If $u$ is a solution to (1.1), it is well known that the area enclosed by $u$ decreases at the constant rate $2\pi$. Indeed, by the first variation formula,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^2(\Omega_t) = \int_{\Gamma_t} \frac{\partial u}{\partial t} \cdot \nu = \int_{\Gamma_t} \kappa = -2\pi,$$

where $\mathcal{L}^2$ denotes the two-dimensional Lebesgue measure, $\Gamma_t$ denotes the simple closed curve $u$ at time $t$, and $\Omega_t$ denotes the domain enclosed by $\Gamma_t$; see [14].

It is also well known that the solution $u$ of (1.1), after an appropriate scaling so that the enclosed area becomes $\pi$, approaches the unit circle. These results were originally obtained by Gage and Hamilton [8] and Grayson [9], and are comprehensively explained in [3].

In this paper, we consider the parameterized non-local curvature flow

$$\frac{\partial u}{\partial t} \cdot \nu = \kappa + q, \tag{1.2}$$

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where $q = \frac{2\pi-\beta}{\mathcal{H}^1(\Gamma_t)}$, $\beta > 0$ is a constant, and $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. The non-local term $q$ is introduced to control the rate of decrease in the enclosed area by $\Gamma_t$, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^2(\Omega_t) = \int_{\Gamma_t} \frac{\partial u}{\partial t} \cdot \nu = \int_{\Gamma_t} \left(\kappa + \frac{2\pi - \beta}{\mathcal{H}^1(\Gamma_t)}\right) = -\beta,$$

i.e., the area of $\Omega_t$ decreases at the prescribed constant rate $\beta$. In particular, (1.2) contains the curve shortening flow ($\beta = 2\pi$) and the area-preserving curvature flow ($\beta = 0$) studied by Chao, Ling and Wang [2] and Gage [7] as special cases. Regarding these flows, higher-dimensional cases are also studied, and it is known that sphere-shaped solutions are stable [6, 10].

Unlike these cases, it has been expected from numerical computations by Dallaston and McCue [5] that higher values of $\beta$ destabilize the circular shape and a variety of asymptotic shapes other than circles appear. In this paper, we rigorously justify this numerical observation as the bifurcation phenomena of non-circular solutions to (1.2) from circular solutions at

$$\beta_{n,0} := 2\pi(n^2 - 1), \quad n \geq 2.$$

Moreover, we clarify the local behavior of the bifurcating solution in terms of the bifurcating parameter $\beta$.

In order to state our main result, let us recall that the fractional Sobolev space on the unit circle $S^1$ is defined by

$$H^s(S^1) := \left\{ a_0 + \sum_{k=1}^{\infty}(a_k \cos(k\theta) + b_k \sin(k\theta)) \in L^2(S^1) \;\middle|\; a_0^2 \sum_{k=1}^{\infty}(1 + k^2)^s(a_k^2 + b_k^2) < \infty \right\}.$$

Here, we suppose that $s > \frac{5}{2}$, that is, we choose $s > 0$ so that every element in $H^s(S^1)$ becomes $C^2$. Then, the solution $u$ is described by $r \in H^s(S^1)$ as $u(\theta, t) = (1 + r(\theta, t))\xi$, where $\xi := (\cos\theta, \sin\theta) \in S^1$, as long as $\Omega_t$ is star-shaped with respect to the origin.

Using these notations, we have the following main results:

**Theorem 1.1** (Existence of non-trivial self-similar solutions). *For $n = 2$ or $n \geq 4$, there exists $\varepsilon_n > 0$ such that, for any $\beta_{n,0} < \beta < \beta_{n,0} + \varepsilon_n$, there is a non-trivial self-similar solution to (1.2) which has symmetry with respect to $\frac{2\pi}{n}$-rotation and symmetry with respect to the $x$-axis.*

*Similarly, there exists $\varepsilon_3 > 0$ such that, for any $\beta_{3,0} - \varepsilon_3 < \beta < \beta_{3,0}$, there is a non-trivial self-similar solution to (1.2) which has symmetry with respect to $\frac{2\pi}{3}$-rotation and symmetry with respect to the $x$-axis.*

*Moreover, the non-trivial self-similar solutions with respect to $\beta$'s near $\beta_n$ can be expressed by a real variable $\sigma$ as $(u_{n,\sigma}, \beta_n(\sigma))$ where $\sigma \neq 0$ is sufficiently small, and $\beta_n(\sigma)$ is analytic near 0 and satisfies $\beta_n(0) = \beta_{n,0}$.*

**Theorem 1.2** (Stability of self-similar solutions). *If $0 < \beta < 6\pi$, the trivial circular solution to (1.2) is stable with respect to small perturbations $r \in H^s(S^1)$ that do not change the geometric center of the curve and the enclosed area. In other words, if we suppose that sufficiently small $r_0 \in H^s(S^1)$ satisfies $\mathcal{L}^2(\Omega_0) = \pi$, $\int_{\Gamma_0} x = 0$, and $\int_{\Gamma_0} y = 0$, the solution*

$$u(\theta, t) = \sqrt{1 - \frac{\beta t}{\pi}}(1 + r(\theta, t))\xi$$

*with initial curve $(1 + r_0(\theta))\xi$ exists for $t \in (0, \frac{\beta}{\pi}]$ and satisfies*

$$r(\cdot, t) \to 0 \quad \text{in } H^s(S^1)$$

*as $t \to \frac{\pi}{\beta}$.*

*Similarly, if $n \neq 3$ and $\beta_{n,0} < \beta < \beta_{n,0} + \varepsilon_n$, a non-trivial self-similar solution $u_{n,\sigma}$ to (1.2) expressed as*

$$\sqrt{1 - \frac{\beta t}{\pi}}(1 + r_{n,\sigma}(\theta))\xi$$

*is stable with respect to small perturbations $r \in H^s(S^1)$ that have reflectional and $n$-fold symmetry and do not change the geometric center of the curve, nor the enclosed area. In other words, if $r_0 \in H^s(S^1)$ sufficiently close to $r_{n,\beta}$ satisfies $r_0(\theta) = r_0(2\pi - \theta)$, $r_0(\theta') = r_0(\theta)$ if $\theta' - \theta \equiv \frac{2\pi}{n} \bmod 2\pi$, $\mathcal{L}^2(\Omega_0) = \pi$, $\int_{\Gamma_0} x = 0$, and $\int_{\Gamma_0} y = 0$, the solution*

$$u(\theta, t) = \sqrt{1 - \frac{\beta t}{\pi}}(1 + r(\theta, t))\xi$$

*to (1.2) with initial curve $(1 + r_0(\theta))\xi$ exists for $t \in (0, \frac{\beta}{\pi}]$ and satisfies*

$$r(\cdot, t) \to r_{n,\sigma} \quad \text{in } H^s(S^1)$$

*as $t \to \frac{\pi}{\beta}$.*

## 2. Preliminaries

### 2.1. Sobolev space

All eigenvectors of linear operators which appear in this paper are trigonometric functions. For this reason, we define Sobolev spaces on a circle using these functions. In this subsection, we list some properties of Sobolev spaces that are necessary in this paper.

Let $\varphi_{0,0} = 1$ on $S^1$ and for $k \geq 1$, set $\varphi_{k,0} = \cos(k\theta)$ and $\varphi_{k,1} = \sin(k\theta)$. Then, $\{\varphi_{k,j} \mid j = 0, \dots, \min\{k, 1\}\}$ forms an orthogonal basis of $H_k$, and

$$\bigcup_{k=0}^{\infty} \{\varphi_{k,j} \mid j = 0, \dots, \min\{k, 1\}\}$$

forms an orthogonal basis of $L^2(S^1)$.

**Definition 2.1** (Sobolev space). Suppose $s > 0$. We define the Sobolev space on $S^1$ as

$$H^s(S^1) := \left\{ f = \sum_{k=0}^{\infty} \sum_{j=0}^{\min\{k,1\}} a_{k,j}\varphi_{k,j} \in L^2(S^1) \;\middle|\; \sum_{k=0}^{\infty} \sum_{j=0}^{\min\{k,1\}} (1+k^2)^s a_{k,j}^2 < \infty \right\},$$

and for each $f = \sum_{k=0}^{\infty} \sum_{j=0}^{\min\{k,1\}} a_{k,j}\varphi_{k,j}$ and $g = \sum_{k=0}^{\infty} \sum_{j=0}^{\min\{k,1\}} b_{k,j}\varphi_{k,j} \in H^s(S^1)$, we define an inner product of $f$ and $g$ as

$$\langle f, g \rangle := \sum_{k=0}^{\infty} \sum_{j=0}^{\min\{k,1\}} (1+k^2)^s a_{k,j} b_{k,j}.$$

We regard $H^s(S^1)$ as a Hilbert space with respect to this inner product.

The next proposition follows from a straightforward calculation.

**Proposition 2.2.** *Suppose $s > 1$. Then, a linear map $H^s(S^1) \to H^{s-1}(S^1)$ defined by $\sum_k \sum_j a_{k,j}\varphi_{k,j} \mapsto \sum_k \sum_j k a_{k,j}\varphi_{k,j}$ is bounded.*

The above definition of the Sobolev space on a circle is different from that on an open subset of a Euclidean space. However, the Sobolev embedding theorem still holds. We use the following special case of this theorem in this paper:

**Proposition 2.3** ([1]). *Let $l \geq 0$ be an integer. Then, if $s > l + \frac{1}{2}$, an element of $H^s(S^1)$ is a $C^l$ function.*

In addition, $H^s(S^1)$ is closed under multiplication, and the proposition which follows also holds for $H^s(\mathbb{R})$. We omit the proof because it is analogous to that of $H^s(\mathbb{R})$.

**Proposition 2.4.** *Suppose $s > \frac{1}{2}$. For each $f, g \in H^s(S^1)$, $fg$ defined as $(fg)(\xi) := f(\xi)g(\xi)$ is in $H^s(S^1)$, and there exists a constant $C > 0$ independent of $f$ and $g$ such that*

$$\|fg\|_{H^s(S^1)} \leq C\|f\|_{H^s(S^1)}\|g\|_{H^s(S^1)}.$$

## 2.2. Analytic functions

**Definition 2.5.** Let $X, Y$ be Banach spaces, and let $U \subset X$ be open. Then, a function $F : U \to Y$ is called analytic at $x_0$ if there exist $\varepsilon > 0$ and a bounded $k$-linear map $F_k : X^k \to Y$ for each $k \geq 0$ such that the series $\sum_{k=0}^{\infty} \|F_k\|\varepsilon^k$ converges, and $F$ can be expressed as

$$F(x) = \sum_{k=0}^{\infty} F_k(x - x_0, \ldots, x - x_0), \quad \forall x \in B_{x_0}(\varepsilon).$$

If $F$ is analytic at every point of $U$, $F$ is called analytic on $U$.

Regarding analytic functions, the following holds:

**Proposition 2.6** ([13, Lemma 6]). *Suppose that* $F : X \to Y$ *is analytic at* $x_0 \in X$ *and that* $G : Y \to Z$ *is analytic at* $F(x_0) \in Y$*. Then,* $F \circ G : X \to Z$ *is analytic at* $x_0$*.*

**Proposition 2.7** ([13, Lemma 7]). *Let* $X$ *be a Banach algebra with the multiplicative identity. Suppose that* $x_0, u_0 \in X$ *is an invertible element satisfying* $x_0^2 = u_0$*. Then, there exist analytic functions* $u \mapsto \sqrt{u} : X \to X$ *and* $u \mapsto u^{-1} : X \to X$ *such that* $(\sqrt{u})^2 = u$ *and* $u^{-1}u = 1$.

## 3. Transformation of the equation

At first, we show that concentric shrinking circles give a solution to (1.2). If we suppose that an initial simple closed curve is a unit circle, then by (1),

$$\frac{\mathrm{d}(\pi R(t)^2)}{\mathrm{d}t} = -\beta, \quad R(0) = 1,$$

where $R(t)$ denotes the radius of the circle at time $t$. By solving this ODE, we get

$$\begin{aligned} \pi R(t)^2 &= \pi - \beta t, \\ R(t) &= \sqrt{1 - \frac{\beta t}{\pi}}. \end{aligned} \tag{3.1}$$

**Proposition 3.1.** *Suppose that* $R(t)$ *is defined as above. Then,* $u(\theta, t) = (R(t)\cos(\theta), R(t)\sin(\theta))$ *is a solution to* (1.2).

*Proof.* Since the curvature $\kappa$ and the unit normal vector field $\nu$ of the curve are

$$\kappa = -\frac{1}{R(t)}, \quad \nu = (\cos(\theta), \sin(\theta))$$

respectively, the right-hand side of (1.2) is

$$-\frac{1}{R(t)} + \frac{2\pi - \beta}{2\pi R(t)} = -\frac{\beta}{2\pi R(t)}.$$

On the other hand, the left-hand side is

$$\frac{\partial u}{\partial t} \cdot \nu = R'(t),$$

and for $R'(t)$,

$$\begin{aligned} 2\pi R(t) R'(t) &= -\beta, \\ R'(t) &= -\frac{\beta}{2\pi R(t)}, \end{aligned}$$

by (3.1). Therefore, $u(\theta, t)$ is a solution to (1.2). ∎

Next, we change the scale by using $R(t)$, that is, if $u$ is a solution to (1.2), we derive an equation which $v := R(t)^{-1}u$ satisfies. Since $Rv$ is a solution to (1.2), by substituting it for $u$, we obtain

$$\begin{aligned}\frac{\partial(Rv)}{\partial t}\cdot\nu &= \kappa+q,\\ \left(R'v+R\frac{\partial v}{\partial t}\right)\cdot\nu &= \kappa+q,\\ \left(-\frac{\beta}{2\pi R}v+R\frac{\partial v}{\partial t}\right)\cdot\nu &= \kappa+q. \end{aligned}\tag{3.2}$$

Defining $\tau$ as $\tau := -\log(1-\frac{\beta t}{\pi})$, we have

$$\frac{\mathrm{d}\tau}{\mathrm{d}t}=\frac{\beta}{\pi}\frac{1}{1-\frac{\beta t}{\pi}}=R^{-2}\frac{\beta}{\pi}.$$

Substituting it into (3.2) and using $\frac{\partial v}{\partial t}=\frac{\partial v}{\partial \tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}$, we have

$$\begin{aligned}\left(-\frac{\beta}{2\pi R}v+\frac{\beta}{\pi R}\frac{\partial v}{\partial \tau}\right)\cdot\nu &= \kappa+q,\\ 2\frac{\partial v}{\partial \tau}\cdot\nu &= v\cdot\nu+\frac{2\pi R}{\beta}(\kappa+q).\end{aligned}\tag{3.3}$$

Define $\widetilde{\kappa}$ as the curvature of $v$ and $\widetilde{q}(t) := \frac{2\pi-\beta}{L(v(t))}$. Then, they can be expressed as

$$\widetilde{\kappa}=R\kappa,\quad \widetilde{q}(t)=Rq.$$

Therefore, (3.3) can be expressed as

$$2\frac{\partial v}{\partial \tau}\cdot\nu=v\cdot\nu+\frac{2\pi}{\beta}(\widetilde{\kappa}+\widetilde{q}).\tag{3.4}$$

Finding a self-similar solution $u$ to the original equation (1.2) is the same as finding a solution $v(\tau)$ to (3.4) satisfying $\frac{\partial v}{\partial \tau}=0$. For this reason, we consider a solution $v: S^1\to\mathbb{R}^2$ to

$$v\cdot\nu+\frac{2\pi}{\beta}(\widetilde{\kappa}+\widetilde{q})=0.\tag{3.5}$$

If a curve $v$ is close to a circle, by using a function $r: S^1\to\mathbb{R}$ which represents the gap between $v$ and the circle, the curve can be expressed as

$$v(\xi)=\xi+r(\xi)\xi,\quad \xi\in S^1.\tag{3.6}$$

Note that if $r$ is sufficiently small in the $C^2$ sense, the curve expressed by (3.6) is a simple closed curve. In particular, we have the following proposition:

**Proposition 3.2** ([13, Lemma 5]). *Suppose $s>\frac{5}{2}$. Then, $\xi\mapsto\xi+r(\xi)\xi$ is a homeomorphism of $S^1\to\{\xi+r(\xi)\xi \mid \xi\in S^1\}$ for every $r\in H^s(S^1)$ satisfying $r>-1$.*

Substituting (3.6) for (3.5), we obtain

$$(1 + r(\xi))\xi \cdot \nu(r)(\xi) + \frac{2\pi}{\beta}(\widetilde{\kappa}(r)(\xi) + \widetilde{q}(r)) = 0, \tag{3.7}$$

where $\nu(r)(\xi)$ is the unit outward normal vector of $v(\xi)$ at $\xi$, $\widetilde{\kappa}(r)(\xi)$ is the curvature of $v(\xi)$ at $\xi$, and $\widetilde{q}(r)$ is a product of $2\pi - \beta$ and the reciprocal of the length of a curve defined as $\xi + r(\xi)\xi (\xi \in S^1)$.

## 4. Differentiation of the equation

We regard the left-hand side of (3.7) as a map $r : S^1 \to \mathbb{R}$ with a parameter $\beta > 0$, and we define

$$F(r(\xi), \beta) := (1 + r(\xi))\xi \cdot \nu(r)(\xi) + \frac{2\pi}{\beta}(\widetilde{\kappa}(r)(\xi) + \widetilde{q}(r)). \tag{4.1}$$

This map is smooth if the domain and the target set are specific spaces, which we now show.

At first, regarding $\nu$ and $\widetilde{\kappa}$ in (4.1), the following holds:

**Proposition 4.1** ([13, Lemma 16]). *Suppose $s > \frac{5}{2}$. Then,*

(1) *$r \mapsto \nu(r)$ is an analytic map from a neighborhood of 0 in $H^s(S^1)$ to $(H^{s-1}(S^1))^2$,*

(2) *$r \mapsto \widetilde{\kappa}(r)$ is an analytic map from a neighborhood of 0 in $H^s(S^1)$ to $H^{s-2}(S^1)$.*

Since a product of two analytic maps to $H^{s-1}(S^1)$ is analytic by Proposition 2.4, $r(\xi)\xi \cdot \nu(r)(\xi)$ is analytic from a neighborhood of 0 in $H^s(S^1)$ to $H^{s-1}(S^1)$. By considering this together with the smoothness of $\widetilde{\kappa}$, (4.1) is analytic from a neighborhood of 0 in $H^s(S^1)$ to $H^{s-2}(S^1)$, except for $\widetilde{q}$.

Regarding $\widetilde{q}$, we consider the length of $\xi + r(\xi)\xi$ (for $\xi \in S^1$). Let this be denoted by $L(r)$. We shall use the formula of the length of a curve expressed in polar coordinates, i.e.,

$$L(r) = \int_0^{2\pi} \sqrt{(1 + r(\theta))^2 + (r'(\theta))^2} d\theta. \tag{4.2}$$

The argument of the square root in this expression is an analytic function $H^s(S^1) \to H^{s-1}(S^1)$ by Proposition 2.2 which implies that $r \mapsto r'$ is a bounded linear operator of $H^s(S^1) \to H^{s-1}(S^1)$ and by Proposition 2.4 which implies that $(f, g) \mapsto fg$ is a bounded bilinear operator of the form $H^s(S^1) \times H^s(S^1) \to H^s(S^1)$. Moreover, the integral of the square root of this is analytic by Proposition 2.6 and Proposition 2.7. In other words, $L(r)$ is an analytic map from a neighborhood of 0 in $H^s(S^1)$ to $H^{s-1}(S^1)$. Furthermore, $L(r) \in \mathbb{R}$, and the product of $2\pi - \beta$ and the reciprocal of $L(r)$ is analytic as a map from a neighborhood of 0 in $H^s(S^1)$ to $H^{s'}(S^1)$ for all $s' > 0$ by Proposition 2.6. Consequently, we get the following:

**Theorem 4.2.** *Suppose that $s > \frac{5}{2}$. Then, $F$ in (4.1) is an analytic map from a neighborhood of* $0$ *in $H^s(S^1)$ to $H^{s-2}(S^1)$.*

We showed that $F$ is smooth in a neighborhood of the origin. Next, we consider the Fréchet derivative of $F$ at 0.

First, regarding the derivative of $\widetilde{\kappa}(r)$ at 0, the following is known:

**Proposition 4.3** ([13, Lemma 33]). *The function $\mathrm{D}\widetilde{\kappa}(0) : H^s(S^1) \to H^{s-2}(S^1)$ is given by $\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + I$. Here, $S^1$ is parameterized by $\theta \mapsto (\cos(\theta), \sin(\theta))$, and $I$ denotes the inclusion map $H^s(S^1) \to H^{s-2}(S^1)$.*

Next, we consider the Fréchet derivative of $\widetilde{q}$ at 0. The next proposition follows from a straightforward calculation.

**Proposition 4.4.** *Suppose that $s > \frac{3}{2}$. Then, regarding $L$ in (4.2), we have*

$$\left| L(r) - L(0) - \int_0^{2\pi} r(\theta)\mathrm{d}\theta \right| = o(\|r\|_{H^s(S^1)}).$$

*In other words, $\mathrm{D}L(0)$, which denotes the Fréchet derivative of $L : H^s(S^1) \to \mathbb{R}$ at $r = 0$, satisfies*

$$\mathrm{D}L(0)[h] = \int_0^{2\pi} h(\theta)\mathrm{d}\theta.$$

By this proposition, the Fréchet derivative of $\widetilde{q}(r)$ at 0 equals

$$h \mapsto -\frac{2\pi - \beta}{L(0)^2}\mathrm{D}L(0)[h] = -\frac{2\pi - \beta}{(2\pi)^2}\int_0^{2\pi} h(\theta)\mathrm{d}\theta.$$

Summarizing the above, we get the following proposition:

**Proposition 4.5.** *Suppose that $\beta > 0$. Then, $F(\cdot, \beta) : H^s(S^1) \to H^{s-2}(S^1)$ is analytic at 0, and*

$$\mathrm{D}F(0, \beta)[h] = h + \frac{2\pi}{\beta}\left\{\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}h(\theta) + h\right) - \frac{2\pi - \beta}{(2\pi)^2}\int_0^{2\pi} h(\theta)\mathrm{d}\theta\right\}$$

*is its Fréchet derivative at 0.*

## 5. Existence of solutions

First, we consider $\beta$ such that no solutions to $F(r, \beta) = 0$ other than 0 exist. If $\beta > 0$ cannot be expressed as $\beta = \beta_{n,0} = 2\pi(n^2 - 1)$ for any natural number $n$, then the Fréchet derivative $\mathrm{D}F(0,\beta) : H^s(S^1) \to H^{s-2}(S^1)$ is bijective. It is because $\mathrm{D}F(0,\beta)$ satisfies

$$\begin{aligned} 1 &\mapsto 2, \\ \cos(k\theta) &\mapsto \Big(1 + \frac{2\pi}{\beta}(-k^2+1)\Big)\cos(k\theta) \neq 0, \quad k \in \mathbb{N}, \\ \sin(k\theta) &\mapsto \Big(1 + \frac{2\pi}{\beta}(-k^2+1)\Big)\sin(k\theta) \neq 0, \quad k \in \mathbb{N}, \end{aligned}$$

and these form an orthonormal basis of $H^s(S^1)$. Then, by the implicit function theorem, there exist a neighborhood of $\beta$ denoted by $U$, a neighborhood $0 \in H^s(S^1)$ denoted by $V$, and a unique map $U \to V$ denoted by $r(\beta)$ such that $F(r,\beta) = 0$ implies $r = r(\beta)$. Since the constant map 0 satisfies the condition to be $r(\beta)$, $r(\beta) = 0$. Therefore, under this condition, no solutions other than 0 exist in a sufficiently small neighborhood of 0.

Next, we consider the case that $\beta > 0$ can be expressed as $\beta = \beta_{n,0} = 2\pi(n^2 - 1)$ for a natural number $n \geq 2$. We use the following theorem to show the existence of the non-trivial solution to $F(r,\beta) = 0$:

**Proposition 5.1** ([4, Theorem 1.7]). *Let $X$, $Y$ be Banach spaces and $\Omega$ be a neighborhood of $0 \in X$. Let $(\lambda_1, \lambda_2)$ be an open interval in $\mathbb{R}$ and let $V = (\lambda_1, \lambda_2) \times \Omega$. Let $f : V \to Y$ be a $C^1$ map that satisfies $f(\lambda, 0) = 0$. In addition, suppose that $f$ satisfies the following conditions:*

(1) *$f_{\lambda x}$ exists and is continuous on $V$,*
(2) *the dimension of* $\mathrm{Ker}(f_x(\lambda_0, 0))$ *equals* 1,
(3) *the codimension of* $\mathrm{Range}(f_x(\lambda_0, 0))$ *equals* 1,
(4) *there exists $x_0 \in \mathrm{Ker}(f_x(\lambda_0, 0))$ such that $f_{\lambda x}(\lambda_0, 0)x_0 \notin \mathrm{Range}(f_x(\lambda_0, 0))$.*

*Let $Z$ be a complementary space of $\mathrm{Ker}(f_x(\lambda_0, 0))$. Then, there exist a neighborhood of $(\lambda_0, 0)$ in $V$ denoted by $V_0$, an open interval $I$ containing 0, and a continuous real-valued function $\lambda = \lambda(\sigma)$ ($\sigma \in I$) such that a set consisting of all $(\lambda, x) \in V_0$ satisfying $f(\lambda, x) = 0$ is a union of the curves*

$$\begin{aligned} \Gamma_1 &= \big\{(\lambda(\sigma), \sigma x_0 + z(\sigma) \mid \sigma \in I\big\}, \\ \Gamma_2 &= \big\{(\lambda, 0) \mid (\lambda, 0) \in V_0\big\}, \end{aligned}$$

*where $z = z(\sigma)$ is a $Z$-valued continuous function defined on $I$ satisfying $z(\sigma) = o(\sigma)$. Moreover, if $f$ is $C^p$ ($p \geq 3$) on $V$, we can choose $\lambda, z$ to be $C^{p-2}$.*

Since the dimension of the kernel of $\mathrm{D}F(0, \beta_{n,0})$ equals 2, we cannot apply this theorem to $F$ directly. However, restricting $F$ to a specific subspace, we become able to apply it. To prove this, we show the following:

**Proposition 5.2.** *The function $F$ in (4.1) keeps symmetry with respect to the $x$-axis. Moreover, $F$ keeps symmetry with respect to $\frac{2\pi}{n}$-rotation. In other words, if a function $r$ on $S^1$ satisfies $r(\theta) = r(-\theta)$, so does $F(r,\beta)$. In the same way, if $r$ satisfies $r(\theta) = r(\theta + \frac{2\pi}{n})$, so does $F(r,\beta)$.*

*Proof.* The function $F(r,\beta)$ can be expressed as

$$F(r,\beta)=\big((1+r)^2+(r')^2\big)^{-\frac{1}{2}}(1+r)^2 \\ +\frac{2\pi}{\beta}\Big\{-\big((1+r)^2+(r')^2\big)^{-\frac{3}{2}}\big((1+r)^2+2(r')^2-(1+r)r''\big) \\ +(2\pi-\beta)\Big(\int_{S^1}\sqrt{(1+r(\theta))^2+(r'(\theta))^2}\mathrm{d}\theta\Big)^{-1}\Big\}. \tag{5.1}$$

We can check this as follows: first, if a simple closed curve is parameterized as $S^1\to\mathbb{R}^2$, $\theta\mapsto(1+r(\theta))(\cos(\theta),\sin(\theta))$, the velocity vector can be expressed as $\theta\mapsto r'(\theta)(\cos(\theta),\sin(\theta))+(1+r(\theta))(-\sin(\theta),\cos(\theta))$. Therefore, the outward unit normal vector field can be expressed as

$$\theta\mapsto\big((1+r(\theta))^2+(r'(\theta))^2\big)^{-\frac{1}{2}}\big(r'(\theta)(\sin(\theta),-\cos(\theta))+(1+r(\theta))(\cos(\theta),\sin(\theta))\big).$$

Hence,

$$(1+r(\xi))\xi\cdot\nu(\xi)=\big((1+r(\xi))^2+(r'(\xi))^2\big)^{-\frac{1}{2}}(1+r(\xi))^2.$$

Next, regarding $\kappa(\theta)$, by using the expression of the velocity vector above, we get

$$\kappa(\theta)=-\frac{(1+r(\theta))^2+2(r'(\theta))^2-(1+r(\theta))r''(\theta)}{\big((1+r(\theta))^2+(r'(\theta))^2\big)^{\frac{3}{2}}}.$$

Moreover, since the norm of the velocity vector equals $((1+r(\theta))^2+(r'(\theta))^2)^{\frac{1}{2}}$, the length $L(r)$ of the curve can be expressed as

$$L(r)=\int_0^{2\pi}\big((1+r(\theta))^2+(r'(\theta))^2\big)^{\frac{1}{2}}\mathrm{d}\theta.$$

Substituting these expressions into (4.1), we get (5.1).

Because of this expression, if $r$ is an even function with respect to $\theta$, so is $F(r,\beta)$. In the same way, if $r$ is invariant under the change of variables $\theta\mapsto\theta+\frac{2\pi}{n}$, so is $F(r,\beta)$. ■

Defining a Banach space $\mathcal{X}_n^s$ as

$$\mathcal{X}_n^s:=\Big\{\sum_{k=0}^{\infty}a_k\cos(kn\theta)\in H^s(S^1)\Big\},$$

we can regard $F(\cdot,\beta)$ as a map $\mathcal{X}_n^s\to\mathcal{X}_n^{s-2}$. Then, we can check the four conditions necessary to apply Theorem 5.1 at $\beta=\beta_{n,0}$. First, condition (1) follows from the analyticity of $F$. Next, since $\mathrm{Ker}(\mathrm{D}F(0,\beta_{n,0}))=\mathrm{Range}(\mathrm{D}F(0,\beta_{n,0}))^{\perp}=\{a\cos(n\theta);a\in\mathbb{R}\}$, condition (2) and condition (3) hold. Finally, condition (4) holds since

$$F_{\beta r}(0,\beta_{n,0})(\cos(n\theta))=-\frac{2\pi}{\beta_{n,0}^2}(-n^2+1)\cos(n\theta)\notin\mathrm{Range}(\mathrm{D}F(0,\beta_{n,0}))$$

for $\cos(n\theta)$, which spans $\mathrm{Ker}(\mathrm{D}F(0,\beta_{n,0}))$.

Summarizing the above, we get the following:

**Theorem 5.3.** *Let $n \geq 2$ and $\mathcal{X}_n^s$ be defined as above. Then, there exist an open interval $I$; a real-valued function $\beta_n$ such that $\beta_n(0) = \beta_{n,0}$; and a $\mathcal{X}_n^{s-2}$-valued function $z_n$ such that $z_n(\sigma) = o(\sigma)$, $z_n$ is orthogonal to $\cos(n\theta)$, and*

$$F(r_n(\sigma), \beta_n(\sigma)) = 0, \quad \sigma \in I,$$

*where $r_n(\sigma) = \sigma\cos(n\theta) + z_n(\sigma)$.*

Note that the same argument holds even if we replace $\mathcal{X}_n^s$ with a space consisting of functions which are symmetric with respect to the $x$-axis. Moreover, such solutions have symmetry with respect to rotation because of uniqueness of the solutions.

Rough sketches of self-similar solutions are given in Figure 1. They are smooth, symmetric with respect to the $x$-axis and with respect to $\frac{2\pi}{n}$-rotation for certain $n$. Moreover, if a self-similar solution to (1.2) is sufficiently close to a circle, it is convex since the sign of the curvature is constant. For more accurate figures of self-similar solutions to (1.2), see [5].

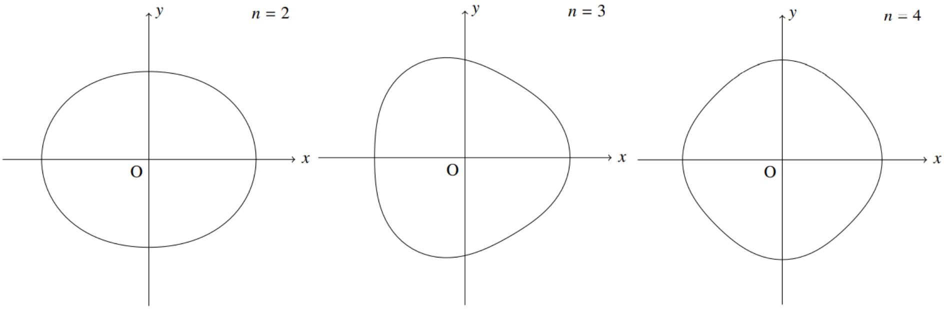


**Figure 1.** Rough sketches of $n$-fold symmetric self-similar solutions for $n = 2, 3, 4$.

## 6. Behavior near bifurcation points

In the above discussion, we showed that there exist $\beta_n$ sufficiently near $\beta_{n,0} = 2\pi(n^2 - 1)$ ($n \geq 2$) and $r_n$ sufficiently close to 0 such that $\beta_n$ and $r_n$ satisfy $F(r_n, \beta_n) = 0$. In this section, we derive the sign of $\beta_n(\sigma) - \beta_{n,0}$ for $\beta_n(\sigma)$ sufficiently near $\beta_{n,0}$.

First, we transform expression (5.1). By considering the Taylor expansion at 0 of each term in $F$, we divide $F$ into two parts: the part which is at most third order with respect to $\|r\|_{H^s(S^1)}$ and the remainder term $O(\|r\|^4_{H^s(S^1)})$.

Regarding the first term in the right-hand side of (5.1),

$$\left((1+r)^2 + (r')^2\right)^{-\frac{1}{2}}(1+r)^2 = \left(1 + 2r + (r^2 + (r')^2)\right)^{-\frac{1}{2}}(1 + 2r + r^2). \tag{6.1}$$

Since $(1+x)^{-\frac{1}{2}}$ can be expressed as $(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + O(|x|^4)$ for $|x| < 1$,

$$\begin{aligned}\text{RHS of (6.1)} &= \Big(1 - \frac{1}{2}\big(2r + r^2 + (r')^2\big) + \frac{3}{8}\big(2r + r^2 + (r')^2\big)^2 \\ &\quad - \frac{5}{16}\big(2r + r^2 + (r')^2\big)^3 + O\big(\|r\|^4_{H^s(S^1)}\big)\Big) \times (1 + 2r + r^2) \\ &= \Big(1 - r + \Big(r^2 - \frac{1}{2}(r')^2\Big) + \Big(-r^3 + \frac{3}{2}r(r')^2\Big) + O\big(\|r\|^4_{H^s(S^1)}\big)\Big) \\ &\quad \times (1 + 2r + r^2) \\ &= 1 + r - \frac{1}{2}(r')^2 + \frac{1}{2}r(r')^2 + O\big(\|r\|^4_{H^s(S^1)}\big) \end{aligned} \tag{6.2}$$

for functions $r$ for which the uniform norms of it and its first derivative are sufficiently small.

Next, regarding the term $((1+r)^2 + (r')^2)^{-\frac{3}{2}}((1+r)^2 + 2(r')^2 - (1+r)r'')$ in (5.1), by using $(1+x)^{-\frac{3}{2}} = 1 - \frac{3}{2}x + \frac{15}{8}x^2 - \frac{35}{16}x^3 + O(|x|^4)$, we get

$$\begin{aligned}&\big((1+r)^2 + (r')^2\big)^{-\frac{3}{2}}\big((1+r)^2 + 2(r')^2 - (1+r)r''\big) \\ &\quad = \big(1 + 2r + \big(r^2 + (r')^2\big)\big)^{-\frac{3}{2}}\big(1 + (2r - r'') + \big(r^2 - rr'' + 2(r')^2\big)\big) \\ &\quad = \Big(1 - \frac{3}{2}\big(2r + r^2 + (r')^2\big) + \frac{15}{8}\big(2r + r^2 + (r')^2\big)^2 - \frac{35}{16}\big(2r + r^2 + (r')^2\big)^3 \\ &\qquad + O\big(\|r\|^4_{H^s(S^1)}\big)\Big) \times \big(1 + (2r - r'') + \big(r^2 - rr'' + 2(r')^2\big)\big) \\ &\quad = \Big(1 - 3r + \Big(6r^2 - \frac{3}{2}(r')^2\Big) + \Big(-10r^3 + \frac{15}{2}r(r')^2\Big)\Big) \times \big(1 + (2r - r'') \\ &\qquad + (r^2 - rr'' + 2(r')^2)\big) \\ &\quad = 1 + (-r - r'') + \Big(r^2 + 2rr'' + \frac{1}{2}(r')^2\Big) + \Big(-r^3 - 3r^2r'' - \frac{3}{2}r(r')^2 \\ &\qquad + \frac{3}{2}(r')^2r''\Big) + O\big(\|r\|^4_{H^s(S^1)}\big). \end{aligned} \tag{6.3}$$

Moreover, we transform $(\int_{S^1}\sqrt{(1+r(\theta))^2 + (r'(\theta))^2}\mathrm{d}\theta)^{-1}$ in (5.1). In the following, for convenience, let $\int r$ denote $\int_{S^1} r(\theta)\mathrm{d}\theta$ for every function $r$. Then, transforming the integrand of $\int\sqrt{(1+r)^2 + (r')^2}$ by using $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + O(|x|^4)$, we get

$$\begin{aligned}\sqrt{(1+r)^2 + (r')^2} &= \sqrt{1 + 2r + (r^2 + (r')^2)} \\ &= 1 + \frac{1}{2}\big(2r + r^2 + (r')^2\big) - \frac{1}{8}\big(2r + r^2 + (r')^2\big)^2 \\ &\quad + \frac{1}{16}\big(2r + r^2 + (r')^2\big)^3 + O\big(\|r\|^4_{H^s(S^1)}\big) \\ &= 1 + r + \frac{1}{2}(r')^2 - \frac{1}{2}r(r')^2 + O\big(\|r\|^4_{H^s(S^1)}\big). \end{aligned}$$

Integrating this, we get

$$\int \sqrt{(1+r)^2+(r')^2} = 2\pi + \int r + \frac{1}{2}\int (r')^2 - \frac{1}{2}\int r(r')^2 + O\big(\|r\|^4_{H^s(S^1)}\big).$$

Taking the reciprocal of this, we have

$$\begin{aligned}\Big(\int \sqrt{(1+r)^2+(r')^2}\Big)^{-1} &= \frac{1}{2\pi}\Big\{1 - \frac{1}{2\pi}\Big(\int r + \frac{1}{2}\int (r')^2 - \frac{1}{2}\int r(r')^2\Big) \\ &\qquad + \frac{1}{(2\pi)^2}\Big(\int r + \frac{1}{2}\int (r')^2 - \frac{1}{2}\int r(r')^2\Big)^2 \\ &\qquad - \frac{1}{(2\pi)^3}\Big(\int r + \frac{1}{2}\int (r')^2 - \frac{1}{2}\int r(r')^2\Big)^3\Big\} \\ &\quad + O\big(\|r\|^4_{H^s(S^1)}\big) \\ &= \frac{1}{2\pi} - \frac{1}{4\pi^2}\int r - \frac{1}{8\pi^2}\int (r')^2 + \frac{1}{8\pi^3}\Big(\int r\Big)^2 \\ &\quad + \int O\big(\|r\|^3_{H^s(S^1)}\big) + O\big(\|r\|^4_{H^s(S^1)}\big). \end{aligned} \tag{6.4}$$

Substituting (6.2), (6.3), and (6.4) in (5.1), we obtain

$$\begin{aligned} F(r,\beta) &= r + \frac{2\pi}{\beta}\Big(r + r'' - \frac{2\pi-\beta}{4\pi^2}\int r\Big) - \frac{1}{2}(r')^2 \\ &\quad + \frac{2\pi}{\beta}\Big\{-r^2 - 2rr'' - \frac{1}{2}(r')^2 + (2\pi-\beta)\Big(-\frac{1}{8\pi^2}\int (r')^2 + \frac{1}{8\pi^3}\Big(\int r\Big)^2\Big)\Big\} \\ &\quad + \frac{1}{2}r(r')^2 + \frac{2\pi}{\beta}\Big(r^3 + 3r^2r'' + \frac{3}{2}r(r')^2 - \frac{3}{2}(r')^2r''\Big) \\ &\quad + \int O(\|r\|^3_{H^s(S^1)}) + O\big(\|r\|^4_{H^s(S^1)}\big). \end{aligned} \tag{6.5}$$

Solutions of $F(r,\beta)=0$ near $(0,\beta_{n,0})$ can be expressed as $(\sigma\rho_n + z_n(\sigma), \beta_n(\sigma))$ where $\rho_n(\theta) = \cos(n\theta)$, $\beta_n(0) = \beta_{n,0}$, $\sigma \in I$, $I$ contains 0, $z_n$ is orthogonal to $\rho_n$, and $z_n(\sigma) = o(\sigma)$. By substituting this into $F$, taking its inner product with specific functions and differentiating them with respect to $\sigma$, we obtain derivatives of $\beta_n$ with respect to $\sigma$ at 0.

In the following, we consider $\beta_n(\sigma)F(\sigma\rho_n + z_n(\sigma), \beta_n(\sigma)) = 0$ instead of $F(\sigma\rho_n + z_n(\sigma), \beta_n(\sigma)) = 0$ to make some calculations regarding differentiation slightly easier.

First, we take inner product with $\frac{1}{\pi}\rho_n$ of both sides of $\beta_n(\sigma)F(\sigma\rho_n + z_n(\sigma), \beta_n(\sigma)) = 0$. Then, second-order terms with respect to $\sigma$ vanish, and

$$\sigma\big(\beta_n(\sigma) + 2\pi(1-n^2)\big) + o(\sigma^2) = 0 \tag{6.6}$$

holds. From this expression, we can see that $\beta_n(0) = 2\pi(n^2 - 1)$, and we obtain

$$\dot{\beta_n}(0) = 0,$$

where $\dot{\beta_n}$ denotes derivatives with respect to $\sigma$.

Next, we calculate some derivatives of coefficients of the Fourier series of $z_n(\sigma)$ that are necessary later to calculate the second derivative of $\beta_n$.

By integrating both sides of $\beta_n(\sigma)F(\sigma\rho_n + z_n(\sigma), \beta_n(\sigma)) = 0$ on $S^1$ and dividing them by $2\pi$, we get

$$\begin{aligned}\Big\{\beta_n(\sigma) + 2\pi\Big(1 - \frac{2\pi - \beta_n(\sigma)}{2\pi}\Big)\Big\} z_n^{(0)}(\sigma) - \frac{n^2}{4}\beta_n(\sigma)\sigma^2 \\ + 2\pi\Big(-\frac{\sigma^2}{2} + n^2\sigma^2 - \frac{n^2}{4}\sigma^2 + (2\pi - \beta_n(\sigma))\frac{-n^2}{8\pi}\sigma^2\Big) + o(\sigma^2) = 0,\end{aligned}$$

where $z_n^{(0)}(\sigma) := \frac{1}{2\pi}\int z(\sigma)$. By differentiating twice with respect to $\sigma$ and substituting 0 into both sides, we have

$$2\beta_{n,0}\ddot{z}_n^{(0)}(0) - \frac{n^2}{2}\beta_{n,0} + 2\pi\Big(-1 + 2n^2 - \frac{n^2}{2} + (2\pi - \beta_{n,0})\frac{-n^2}{4\pi}\Big) = 0.$$

Substituting $\beta_{n,0} = 2\pi(n^2 - 1)$, we obtain

$$\ddot{z}_n^{(0)}(0) = -\frac{1}{2}.$$

Moreover, taking the inner product with $\frac{1}{\pi}\rho_{2n}$ of both sides of $\beta_n(\sigma)F(\sigma\rho_n + z_m(\sigma), \beta_n(\sigma)) = 0$, we have

$$\begin{aligned}\{\beta_n(\sigma) + 2\pi(1 - 4n^2)\} z_n^{(2n)}(\sigma) + \frac{n^2}{4}\beta_n(\sigma)\sigma^2 \\ + 2\pi\Big\{-\frac{\sigma^2}{2} + n^2\sigma^2 + \frac{n^2}{4}\sigma^2\Big\} + o(\sigma^2) = 0,\end{aligned}$$

where we define $z_n^{(2n)}(\sigma) := \frac{1}{\pi}\int(z(\sigma)\cos(2n\theta))$.

By differentiating twice as above and substituting in 0, we have

$$\{\beta_{n,0} + 2\pi(1 - 4n^2)\}\ddot{z}_n^{(2n)}(0) + \frac{n^2}{2}\beta_{n,0} + 2\pi\Big\{-1 + 2n^2 + \frac{n^2}{2}\Big\} = 0.$$

Substituting $\beta_{n,0} = 2\pi(n^2 - 1)$, we have

$$\ddot{z}_n^{(2n)}(0) = \frac{n^4 + 4n^2 - 2}{6n^2}.$$

Finally, we divide $o(\sigma^2)$ in (6.6) into the parts greater than $o(\sigma^3)$ and $o(\sigma^3)$. Taking the inner product with $\frac{1}{\pi}\rho_n$ of both sides of $\beta_n(\sigma)F(\sigma\rho_n + z_n(\sigma)) = 0$ again, we have

$$\begin{aligned} 0 &= \sigma\big(\beta_n(\sigma) + 2\pi(1-n^2)\big) - \frac{\beta_n(\sigma)}{2\pi}\int_{S^1}(r')^2\cos(n\theta)\mathrm{d}\theta \\ &\quad + 2\pi\Big\{-\frac{1}{\pi}\int_{S^1} r^2\cos(n\theta)\mathrm{d}\theta - \frac{2}{\pi}\int_{S^1} rr''\cos(n\theta)\mathrm{d}\theta - \frac{1}{2\pi}\int_{S^1}(r')^2\cos(n\theta)\mathrm{d}\theta\Big\} \\ &\quad + \frac{n^2\sigma^3}{8}\beta_n(\sigma) + 2\pi\Big\{\frac{3\sigma^3}{4} - \frac{9n^2\sigma^3}{4} + \frac{3n^2\sigma^3}{8} + \frac{3n^4\sigma^3}{8}\Big\} + o(\sigma^3). \end{aligned} \tag{6.7}$$

Then,

$$\begin{aligned} \frac{1}{\pi}\int_{S^1}(r')^2\cos(n\theta)\mathrm{d}\theta &= \int_{S^1}(-n\sigma\sin(n\theta) + (z_n(\sigma))')^2\cos(n\theta)\mathrm{d}\theta \\ &= -\frac{1}{\pi}\int_{S^1} 2n\sigma(z_n(\sigma))'\sin(n\theta)\cos(n\theta)\mathrm{d}\theta + O(s^4) \\ &= -\frac{1}{\pi}\int_{S^1} n\sigma(z_n(\sigma))'\sin(2n\theta)\mathrm{d}\theta + O(\sigma^4) \\ &= \frac{1}{\pi}\int_{S^1} 2n^2\sigma z_n(\sigma)\cos(2n\theta)\mathrm{d}\theta + O(\sigma^4) \\ &= 2n^2\sigma z_n^{(2n)}(\sigma) + O(\sigma^4). \end{aligned} \tag{6.8}$$

In the same way,

$$\begin{aligned} \frac{1}{\pi}\int_{S^1} r^2\cos(n\theta)\mathrm{d}\theta &= \frac{1}{\pi}\int_{S^1}(\sigma\cos(n\theta) + z_n(\sigma))^2\cos(n\theta)\mathrm{d}\theta \\ &= \frac{1}{\pi}\int_{S^1} 2\sigma\cos^2(n\theta)z_n(\sigma)\mathrm{d}\theta + O(\sigma^4) \\ &= \frac{1}{\pi}\int_{S^1}\sigma(1+\cos(2n\theta))z_n(\sigma)\mathrm{d}\theta + O(\sigma^4) \\ &= \sigma(2z_n^{(0)}(\sigma) + z_n^{(2n)}(\sigma)) + O(\sigma^4) \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} \frac{1}{\pi}\int_{S^1} rr''\cos(n\theta)\mathrm{d}\theta &= \frac{1}{\pi}\int_{S^1}(\sigma\cos(n\theta) + z_n(\sigma))(-\sigma n^2\cos(n\theta) \\ &\qquad + (z_n(\sigma))'')\cos(n\theta)\mathrm{d}\theta \\ &= \frac{1}{\pi}\int_{S^1}\Big\{-n^2\sigma\cos^2(n\theta)z_n(\sigma) + \sigma\cos^2(n\theta)(z_n(\sigma))''\Big\}\mathrm{d}\theta \\ &\qquad + O(\sigma^4) \\ &= -n^2\sigma z_n^{(0)}(\sigma) - \frac{n^2\sigma}{2}z_n^{(2n)}(\sigma) - 2n^2\sigma z_n^{(2n)}(\sigma) + O(\sigma^4) \\ &= -n^2\sigma z_n^{(0)}(\sigma) - \frac{5n^2\sigma}{2}z_n^{(2n)}(\sigma) + O(\sigma^4). \end{aligned} \tag{6.10}$$

Substituting (6.8), (6.9), and (6.10) into (6.7), we have

$$\begin{aligned}0 &= \sigma\bigl(\beta_n(\sigma) + 2\pi(1-n^2)\bigr) - n^2\beta_n(\sigma)\sigma z_n^{(2n)}(\sigma) \\ &\quad + 2\pi\Bigl\{-\sigma(2z_n^{(0)}(\sigma) + z_n^{(2n)}(\sigma)) - 2\Bigl(-n^2\sigma z_n^{(0)}(\sigma) - \frac{5n^2\sigma}{2}z_n^{(2n)}(\sigma)\Bigr) \\ &\qquad - n^2\sigma z_n^{(2n)}(\sigma)\Bigr\} \\ &\quad + \frac{n^2\sigma^3}{8}\beta_n(\sigma) + 2\pi\Bigl\{\frac{3\sigma^3}{4} - \frac{9n^2\sigma^3}{4} + \frac{3n^2\sigma^3}{8} + \frac{3n^4\sigma^3}{8}\Bigr\} + o(\sigma^3).\end{aligned}$$

Moreover, by dividing this by $\sigma$, differentiating twice, and substituting 0 for $\sigma$, we get

$$\begin{aligned}0 &= \ddot{\beta}_n(0) - n^2\beta_{n,0}\ddot{z}_n^{(2n)}(0) \\ &\quad + 2\pi\Bigl\{-2\ddot{z}_n^{(0)}(0) - \ddot{z}_n^{(2n)}(0) - 2\Bigl(-n^2\ddot{z}_n^{(0)} - \frac{5n^2}{2}\ddot{z}_n^{(2n)}(0)\Bigr) - n^2\ddot{z}_n^{(2n)}(0)\Bigr\} \\ &\quad + \frac{n^2}{4}\beta_{n,0} + 2\pi\Bigl\{\frac{3}{2} - \frac{9n^2}{2} + \frac{3n^2}{4} + \frac{3n^4}{4}\Bigr\}.\end{aligned}$$

Substituting $\beta_{n,0} = 2\pi(n^2-1)$, $\ddot{z}_n^{(0)}(0) = -\frac{1}{2}$, and $\ddot{z}_n^{(2n)}(0) = \frac{n^4+4n^2-2}{6n^2}$, we obtain

$$\ddot{\beta}_n(0) = \pi\Bigl(\frac{n^6}{3} - 5n^4 + \frac{49n^2}{3} - 3 - \frac{2}{3n^2}\Bigr). \tag{6.11}$$

If $n = 2$, the right-hand side equals $\frac{7}{2}\pi$. If $n = 3$, the right-hand side equals $-\frac{488}{27}\pi$. Moreover, if $n > 3$, we can check that the right-hand side is positive. Therefore, $\beta_n$'s for which the original equation (equation (1.2)) has non-trivial self-similar solutions other than circles are as shown in Figure 2. For example, if $\beta_2$ is sufficiently close to $6\pi$, $\beta_2$ is greater than $6\pi$, and if $\beta_3$ is sufficiently close to $16\pi$, $\beta_3$ is smaller than $16\pi$.

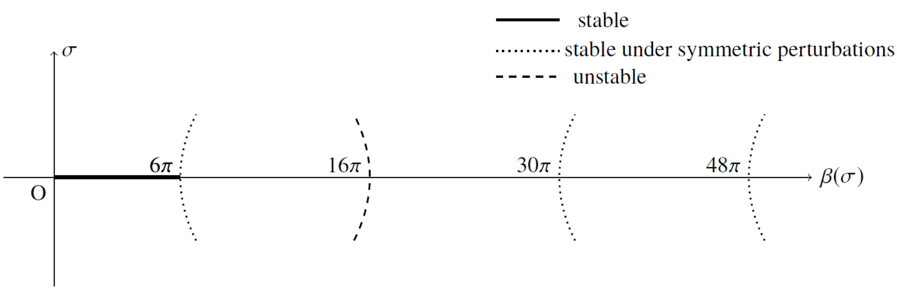


**Figure 2.** A graph depicting the values of $\beta$ for which non-trivial self-similar solutions exist and the stability of self-similar solutions.

## 7. Stability of solutions

This section is concerned with stability of stationary solutions to (3.4). In terms of $r$, (1.2) can be expressed as

$$2\frac{\partial r}{\partial \tau}\xi \cdot \nu(r(\xi)) = F(r,\beta). \tag{7.1}$$

Defining $G(r(\xi),\beta) := \frac{F(r(\xi),\beta)}{2\xi\cdot\nu(r)(\xi)}$, we have

$$\frac{\partial r}{\partial \tau} = G(r,\beta). \tag{7.2}$$

Note that the Fréchet derivative of $G$ at 0 equals a half of that of $F$.

At first, to show stability of non-trivial self-similar solutions with respect to symmetric perturbations, we need to exclude positive eigenvalues of D$G$. For this reason, we consider a set consisting of $r$ satisfying $\int \frac{(1+r)^2}{2} - \pi = 0$. Note that $r$ corresponds to a simple closed curve whose enclosed area equals $\pi$.

Define $\phi : \mathcal{X}_n^s \to \mathbb{R} \times P\mathcal{X}_n^s$ as

$$\phi(r) = \Big(\int \frac{(1+r)^2}{2} - \pi, Pr\Big),$$

where $P$ denotes a projection $\mathcal{X}_n^s \to \{r \in \mathcal{X}_n^s \mid \int r = 0\}$.

The inverse function theorem implies $\phi$ is a homeomorphism from a sufficiently small neighborhood of 0 to its image. Therefore, $\psi := \phi^{-1}$ is well-defined near 0.

Let $(r_{n,\sigma},\beta_{n,\sigma}) := (r_n(\sigma),\beta_n(\sigma))$ and $\widetilde{G}_{n,\sigma} := P \circ G(\cdot,\beta_{n,\sigma}) \circ \psi(0,\cdot) : P\mathcal{X}_n^s \to P\mathcal{X}_n^{s-2}$. Note that the spectrum of $\mathrm{D}\widetilde{G}_{n,0}(0)$ consists of elements of the form

$$\frac{1}{2} - \frac{\pi}{\beta_{n,0}}(n^2k^2 - 1), \quad k \geq 1.$$

Then, by virtue of (6.11), we can calculate signs of eigenvalues of $\mathrm{D}\widetilde{G}_{n,\sigma}(r_{n,\sigma})$ near 0, which is important for stability of the non-trivial stable solution $\widetilde{r}_{n,\sigma} := Pr_{n,\sigma}$. Actually, the following proposition holds by the Crandall–Rabinowitz theory:

**Proposition 7.1.** *Suppose that $n \geq 2$ is an integer satisfying $n \neq 3$. If $\sigma \neq 0$ is sufficiently small, there exists $\delta$ such that the intersection between the $\delta$-neighborhood of the origin and the spectrum of $\mathrm{D}\widetilde{G}_{n,\sigma}(\widetilde{r}_{n,\sigma})$ consists of a single point whose real part is negative.*

*Proof.* This proposition is a corollary of the Crandall–Rabinowitz theory (see [12, Lemma 3.6.1, Theorem 3.6.2]), that is, since 0 is a simple eigenvalue of $\mathrm{D}\widetilde{G}_0(0)$, the intersection of the spectrum of $\mathrm{D}\widetilde{G}_{n,\sigma}(\widetilde{r}_{n,\sigma})$ and a sufficiently small neighborhood of 0 consists of a single point $\mu_n(\sigma)$ for sufficiently small $\sigma$, by [12, Lemma 3.6.1].

Moreover, by [12, Theorem 3.6.2] and (6.11),

$$\frac{\sigma \dot{\beta}_n(\sigma)}{\mu_n(\sigma)} \frac{\pi}{\beta_{n,0}^2}(n^2 - 1) < 0$$

for sufficiently small $\sigma$. Hence, $\mu_n(\sigma) < 0$. ■

We remark that by replacing the word "negative" with the word "positive", the above proposition holds for $n = 3$.

Furthermore, applying [15, Lemma 4.1], we have the following:

**Proposition 7.2.** *Suppose $\sigma \neq 0$ is sufficiently small. Then, $\mathrm{D}\widetilde{G}_{n,\sigma}(\widetilde{r}_{n,\sigma})$ generates an analytic semigroup on $P\mathcal{X}_n^{s-2}$ with the domain of definition $P\mathcal{X}_n^{\sigma}$. Moreover, the supremum of the real part of the spectrum of $\mathrm{D}\widetilde{G}_{n,\sigma}(\widetilde{r}_{n,\sigma})$ is negative for $n \neq 3$.*

*Proof.* First, $\mathrm{D}\widetilde{G}_{n,0}(0)$ generates an analytic semigroup, since the resolvent of $\mathrm{D}\widetilde{G}_{n,0}(0)$ equals $\mathbb{C} \setminus \{\frac{1}{2} - \frac{\pi}{\beta_{n,0}}(n^2k^2 - 1) \mid k \geq 1\}$, and

$$\left| \frac{1}{\lambda - \left(\frac{1}{2} - \frac{\pi}{\beta_{n,0}}(n^2k^2 - 1)\right)} \right| \leq \frac{1}{\sin \alpha |\lambda|}$$

holds for every angle $\alpha \in (\frac{\pi}{2}, \pi)$ and every $\lambda \in \{z \mid -\alpha < \arg z < \alpha\}$. Moreover, $\mathrm{D}\widetilde{G}_{n,\sigma}(\widetilde{r}_{n,\sigma})$ also generates an analytic semigroup for sufficiently small $\sigma$ by continuity of $\sigma \mapsto \mathrm{D}\widetilde{G}_{n,\sigma}(\widetilde{r}_{n,\sigma})$ near 0.

Second, regarding the spectrum of $\mathrm{D}\widetilde{G}_{n,\sigma}(\widetilde{r}_{n,\sigma})$, we can apply the same argument as [15, Lemma 4.1]. ■

Consequently, we can apply a stability argument for evolution equations ([11, Theorem 9.1.2]) to $\widetilde{G}_{n,\sigma}$ at $\widetilde{r}_{n,\sigma}$, that is, the following holds:

**Proposition 7.3.** *Suppose that $\sigma \neq 0$ is sufficiently small. Then, the initial value problem of the equation*

$$\frac{\partial \widetilde{r}}{\partial \tau} = \widetilde{G}_{n,\sigma}(\widetilde{r}) \tag{7.3}$$

*is uniquely solvable on $\tau \in [0, \infty)$ if the initial value $\widetilde{r}_0 \in P\mathcal{X}_n^s$ is sufficiently close to $\widetilde{r}_{n,\sigma}$. Moreover, if $n \neq 3$, there exist $\lambda > 0$ and $M > 0$ such that*

$$\|\widetilde{r}(\tau) - \widetilde{r}_{n,\sigma}\|_{H^s(S^1)} \leq M e^{-\lambda\tau} \|\widetilde{r}_0 - \widetilde{r}_{n,\sigma}\|_{H^s(S^1)}. \tag{7.4}$$

If $\widetilde{r}(\tau)$ is a solution to (7.3), $r(\tau) := \psi(\widetilde{r}(\tau))$ is a solution to

$$\frac{\partial r}{\partial \tau} = G(r, \beta_{n,\sigma}). \tag{7.5}$$

Moreover, the same inequality as (7.4) holds for $r$, since the constant part of $r$ can be controlled by the $P\mathcal{X}_n^s$-component of $r$ because of the implicit function theorem. As a result, we obtain the following:

**Theorem 7.4.** *Suppose that* $\sigma \neq 0$ *is sufficiently small. Then, the initial value problem of* (7.5) *is uniquely solvable on* $\tau \in [0, \infty)$ *if the initial value* $r_0 \in \mathcal{X}_n^s$ *is sufficiently close to* $r_{n,\sigma}$ *and satisfies* $\int \frac{(1+r_0)^2}{2} - \pi = 0$. *Moreover, if* $n \neq 3$, *there exist* $\lambda > 0$ *and* $M > 0$ *such that*

$$\|r(\tau) - r_{n,\sigma}\|_{H^s(S^1)} \leq M e^{-\lambda\tau} \|r_0 - r_{n,\sigma}\|_{H^s(S^1)}.$$

At last, we show the stability of circular self-similar solutions (for $0 < \beta < 6\pi$) with respect to all perturbations in $H^s(S^1)$.

The discussion is almost the same as above.

Define $\mathcal{Y}^s \subset H^s(S^1)$ as

$$\mathcal{Y}^s := \Big\{ r \in H^s(S^1) \;\Big|\; \int r = 0, \int r(\theta)\cos(\theta)\mathrm{d}\theta = 0, \int r(\theta)\sin(\theta)\mathrm{d}\theta = 0 \Big\}.$$

Moreover, we redefine $P$ as the projection $H^s(S^1) \to \mathcal{Y}^s$ and redefine $\phi : H^s(S^1) \to \mathbb{R}^3 \times \mathcal{Y}^s$ as

$$\phi(r) = \Big( \int \frac{(1+r)^2}{2} - \pi, \int r(\theta)\cos(\theta)\mathrm{d}\theta, \int r(\theta)\sin(\theta)\mathrm{d}\theta, Pr \Big).$$

Then, the inverse function theorem implies that $\psi := \phi^{-1}$ is well-defined in a small neighborhood of 0 in $\mathbb{R}^3 \times \mathcal{Y}^s(S^1)$.

Suppose that $0 < \beta < 6\pi$ and define $\widetilde{G}_\beta := P \circ G(\cdot, \beta) \circ \psi(0,0,0,\cdot) : \mathcal{Y}^s \to \mathcal{Y}^{s-2}$.

Then, the spectrum of $\mathrm{D}\widetilde{G}_\beta(0)$ consists of elements of the form

$$\frac{1}{2} - \frac{\pi}{\beta}(k^2 - 1), \quad k \geq 2.$$

Since all of these are negative, we can apply the same argument as before, and obtain the stability of trivial circular solutions.

Notice that we showed the stability of the self-similar solutions only with respect to perturbations that do not move the geometric centers of the solutions. Therefore, whether the stability holds for perturbations that move the geometric centers of the solutions still remains unsolved. It is possible that the difficulty regarding stability with respect to such perturbations is not essential. However, it is difficult to overcome the problem as long as we scale curves with respect to a certain fixed point, such as the origin.

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