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Existence of nonnegative solutions to stochastic thin-film equations in two space dimensions

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Abstract. We prove the existence of martingale solutions to stochastic thin-film equations in the physically relevant space dimension $d = 2$. Conceptually, we rely on a stochastic Faedo–Galerkin approach using tensor-product linear finite elements in space. Augmenting the physical energy on the approximate level by a curvature term weighted by positive powers of the spatial discretization parameter h, we combine Itô's formula with inverse estimates and appropriate stopping time arguments to derive stochastic counterparts of the energy and entropy estimates known from the deterministic setting. In the limit $h \searrow 0$, we prove our strictly positive finite element solutions to converge towards nonnegative martingale solutions—making use of compactness arguments based on Jakubowski's generalization of Skorokhod's theorem and subtle exhaustion arguments to identify third-order spatial derivatives in the flux terms.

1. Introduction

We are concerned with stochastic thin-film equations of the generic form

$$\mathrm{d}u = -\operatorname{div}\{m(u)\nabla(\Delta u - F'(u))\}\,\mathrm{d}t + \operatorname{div}\{\sqrt{m(u)}\mathrm{d}W\} \tag{1.1}$$

on a space-time cylinder $\mathcal{O} \times (0, T]$ where $\mathcal{O}$ is a bounded rectangular domain in $\mathbb{R}^2$. This kind of equations has been introduced to model dewetting of unstable liquid films under the influence of thermal fluctuations. Here, the mobility $m(\cdot)$ may be chosen as $m(u) = u^3 + \beta u^2$; a prototypical example for the effective interface potential $F(u)$ is $F(u) = u^{-8} - u^{-2} + 1$. It is based on a 6–12 Lennard–Jones pair potential and it models disjoining/conjoining van der Waals interactions. Numerical simulations in 1D have shown (see [21, 48]) that discrepancies with respect to time scales of dewetting between physical experiment and deterministic numerical simulation can be overcome if appropriately scaled noise terms are considered. The equations used for those models come along with a number of intrinsic difficulties. Firstly, the degeneracy of the mobility $m(\cdot)$ at $u = 0$; secondly, the singular behavior of the effective interface potential at $u = 0$;

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and thirdly, the fact that even in the deterministic case it is still an open problem whether solutions are continuous or bounded if the spatial dimension is at least $d = 2$. This is in sharp contrast to the one-dimensional case where Sobolev embedding results are the key to establish Hölder continuity in space and time.

The scope of the present work is threefold. First, we wish to establish the existence of martingale solutions for a model problem with quadratic mobility $m(u) = u^2$. Secondly, we shall address the case that the stochastic integral in (1.1) is to be understood in the sense of Stratonovich. Finally, as the third perspective of the techniques developed in the present paper, we recall that Cahn–Hilliard equations with degenerate mobility are intimately related to thin-film equations [22, 42]. Hence, we expect only slight modifications to be required to obtain existence results for Cahn–Hilliard equations with Stratonovich conservative noise—for results on stochastic degenerate Cahn–Hilliard equations with nonconservative noise, we refer to the recent paper [64].

With respect to the second goal, it turns out that the correction term necessary to rewrite the Stratonovich integral as an Itô integral can be included into the frame of the generic form (1.1) just as a modification of the singular potential $F(\cdot)$. In particular, this modification does not affect the structural conditions formulated on F in the Itô-case, provided some natural hypotheses on the decay parameters of the corresponding Wiener processes are met. Therefore, the arguments presented hereafter carry over to the case of Stratonovich noise via slight changes in the singular potential F.

Growing interest in stochastic thin-film equations with Stratonovich noise arose with the work of Gess and Gnann [32] who proved existence of solutions to stochastic versions of thin-film equations driven only by surface tension, i.e., with $F(\cdot) \equiv 0$. Using a novel approximation scheme, Dareiotis, Gess, Gnann, and the second author of this paper [16] extended this result to stochastic thin-film equations with conservative nonlinear multiplicative noise, covering in particular the case of a no-slip condition at the liquid-solid interface which corresponds to $m(u) = u^3$. In contrast to [16] which covers the case $m(u) = u^n$ with $n \in [8/3, 4)$, [32] allows for initial data with compact support—however, at the price of reduced regularity in space. Another approach to construct nonnegative solutions to initial data with compact support is presented in [47]. Here, the 1D-predecessor of the current work (see [29]) is used as the starting point of the analysis. So called α-entropy estimates are derived which guarantee almost surely that almost everywhere in time the solution is of class C^1 in space, this way providing more regularity than [32]. In this spirit, the present paper may similarly serve as a starting point to investigate the case $F(\cdot) \equiv 0$ and to obtain solutions to equations with compactly supported initial data in the 2D setting. Two months after the current work had been submitted, Sauerbrey [63] suggested another approach, combining α-entropy estimates with the method of [32] to establish existence of solutions in 2D for quadratic mobilities in the case of Stratonovich noise with $F \equiv 0$.

Note that Davidovitch et al. [17] derived stochastic thin-film equations for surface tension driven flow to study numerically the influence of thermal noise on the spreading behavior. For mathematically rigorous results on the noise impact on free boundary propagation, see [2,20,28,31,41] and the references therein, which are preliminary studies devoted to stochastic second-order degenerate parabolic equations of porous-media and of parabolic p-Laplace-type. However, it is worth mentioning that the techniques of [28,41] are expected to be universal in the sense that they are based on energy methods developed in [1,12,15,26,27,44], which work for general classes of degenerate parabolic equations of second and higher order.

In [29], it has already been argued that space-time white noise is not compatible with the finiteness of the physical energies encountered in thin-film flow. Therefore, we consider Q-Wiener processes. To guarantee conservation of mass, we work on rectangular physical domains $\mathcal{O} := \mathcal{O}^x \times \mathcal{O}^y := (0, L_x) \times (0, L_y)$, and we prescribe periodic boundary conditions. We may consider an ON-basis $(\mathfrak{g}_{kl})_{k,l\in\mathbb{N}}$ where the $\mathfrak{g}_{kl}$ are given as the product $\mathfrak{g}_{kl}(x, y) := g_k^x(x) g_l^y(y)$ of appropriately scaled eigenfunctions of the one-dimensional Laplace operator on $\mathcal{O}^x$ and $\mathcal{O}^y$, respectively (cf. Remark 3.2).

We consider driving noise $\mathbf{W}$ given by

$$\begin{pmatrix} \sum_{k,l\in\mathbb{Z}} \lambda_{kl}^x \mathfrak{g}_{kl} \beta_{kl}^x \\ \sum_{k,l\in\mathbb{Z}} \lambda_{kl}^y \mathfrak{g}_{kl} \beta_{kl}^y \end{pmatrix}, \tag{1.2}$$

where

- the β_{kl}^α, $\alpha \in \{x, y\}$, $k, l \in \mathbb{Z}$ constitute a family of i.i.d. Brownian motions,
- the λ_{kl}^α, $\alpha \in \{x, y\}$, $k, l \in \mathbb{Z}$ are a family of nonnegative real numbers converging sufficiently fast to zero—see Hypothesis (B3) for more details.

Therefore, we are interested in global existence of a.s. nonnegative martingale solutions to the stochastic thin-film equation

$$\mathrm{d}u = -\operatorname{div}\big\{m(u)\nabla(\Delta u - F'(u))\big\}\,\mathrm{d}t + \sum_{\alpha\in\{x,y\}} \sum_{k,l\in\mathbb{Z}} \partial_\alpha\big(\sqrt{m(u)}\lambda_{kl}^\alpha \mathfrak{g}_{kl}\big)\,\mathrm{d}\beta_{kl}^\alpha \tag{1.3}$$

on $\mathcal{O} \times [0, \infty)$ subject to periodic boundary conditions.

As the analysis of stochastic thin-film equations is influenced by the deterministic theory, we give a brief account on the literature, following here the exposition in [16].

A theory of existence of weak solutions for the deterministic thin-film equation in space dimension $d = 1$ has been developed in [3,5,6] and [8,59,61] for zero and nonzero contact angles at the intersection of the liquid-gas and liquid-solid interfaces, respectively, while the multi-dimensional version with $F \equiv 0$ in $\mathcal{O} \times (0, T]$ and zero contact angles has been the subject of [14,46]. For these solutions, a number of quantitative results has been obtained—including optimal estimates on spreading rates of free boundaries, i.e., the triple lines separating liquid, gas, and solid (see [7,26,43,51]), optimal conditions on the occurrence of waiting time phenomena [15], as well as scaling laws for the size of waiting times [27,34]. For the deterministic case with $F \neq 0$, we refer to [45] for an existence result based on numerical analysis.

A corresponding theory of classical solutions, giving the existence and uniqueness for initial data close to generic solutions or short times, has been developed in [33,35–39] for zero contact angles and in [24, 55–58] for nonzero contact angles in one space dimension, while the higher-dimensional version has been the subject of [40, 53, 65] and [18] for zero and nonzero contact angles, respectively.

It is worth recalling that the thin-film equation is one of the very few examples of (degenerate) parabolic fourth-order equations which allow for globally nonnegative solutions. To retain this property also in the stochastic case, neither additive noise nor multiplicative noise not degenerating for $u(x) = 0$ seems to be appropriate. The special structure with the noise term in (1.1) given as the square root of the mobility $m(\cdot)$ has been suggested by the derivation of stochastic thin-film equations, see [17, 48].

The outline of our paper is as follows: Conceptually, our existence result is based on stochastic counterparts of integral estimates known from the deterministic setting which we combine with Jakubowski/Skorokhod-type methods to construct martingale solutions; see [11, 50] for the basic ideas of this approach and [9, 49] for applications to other problems. More precisely, we will control the energy

$$\mathcal{E}(u) := \int_{\mathcal{O}} \frac{1}{2}|\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathcal{O}} F(u)\,\mathrm{d}x\,\mathrm{d}y \tag{1.4}$$

and the so-called mathematical entropy

$$\mathcal{S}(u) := \int_{\mathcal{O}} G(u)\,\mathrm{d}x\,\mathrm{d}y, \tag{1.5}$$

where

$$G(u) := \int_1^u \int_1^s \frac{1}{m(r)} \mathrm{d}r\,\mathrm{d}s \tag{1.6}$$

is a second primitive of the reciprocal mobility.

Considering the case $m(u) := u^2$, we mimic the 1D strategy used in [29], i.e., we perform discretization in space and apply Itô's formula to the resulting system of SDEs to derive those integral estimates. Taking a step towards the derivation of tractable, fully discrete finite element schemes, we use a basis of finite element functions to perform the spatial discretization. Although this choice introduces additional mathematical difficulties, we strongly believe that it will serve as a cornerstone for fully discrete schemes. Extending this 1D approach (in particular, the treatment of the additional terms arising from Itô's formula) to the two-dimensional setting requires the use of tensor product finite elements.

We adapt the stopping time approach of [29] to comply with the singularities in the effective interface potential $F(\cdot)$. In contrast to the spatially one-dimensional setting of [29], boundedness of the physical energy $\mathcal{E}(u)$ no longer implies strict positivity in our case. Therefore, in the discrete setting, $\mathcal{E}(u)$ is augmented by the square of the L^2-norm of the discrete Laplacian of discrete solutions u_h weighted by a factor which vanishes in the limit $h \to 0$. For the details, see (3.1) and (2.7).

Section 2 is devoted to notation and the large number of technical preliminaries which come along with our discretization and the energy regularization mentioned before. Moreover, the assumptions on initial data, growth behavior of the effective interface potential F and on the driving noise $\mathbf{W}$ are specified in Section 2, too.

Section 3 is devoted to the discussion of the semi-discrete scheme which is formulated in such a way that it may be used for practical numerical simulations of stochastic thin-film equations as well. Section 3 contains also the main existence result together with the applied solution concept. Moreover, we present a lemma which permits the control of the oscillation of discrete solutions on single finite elements.

In Section 4, we present the core result of the analysis in this paper—a discrete combined energy-entropy estimate. It is based on a combination of Itô's formula with inverse estimates for finite-element functions, and with error estimates for interpolation operators which are collected in a synopsis in Appendix A. Moreover, Section 4 contains results on compactness in time of discrete solutions which—in combination with the aforementioned energy-entropy estimate—are the key to applying the Skorokhod–Jakubowski method (cf. [11, 50, 52]) to pass to the limit $h \to 0$, which is the topic of Section 5.

It is worth mentioning that the passage to the limit in the deterministic terms poses new intricacies due to the lack of strict positivity results. We base our arguments on appropriate exhaustion arguments combined with generalizations of Egorov's theorem for Bochner-integrable functions.

Notation. Throughout the paper, we use the standard notation for Sobolev spaces, i.e., for a spatial domain $\mathcal{O} \subset \mathbb{R}^2$, we denote the space of k-times weakly differentiable functions with weak derivatives in $L^p(\mathcal{O})$ by $W^{k,p}(\mathcal{O})$. For $p = 2$, we denote the Hilbert spaces $W^{k,2}(\mathcal{O})$ by $H^k(\mathcal{O})$. The corresponding subspaces of $\mathcal{O}$-periodic functions will be denoted by the subscript 'per'. The subspace of $\mathcal{O}$-periodic $H^1(\mathcal{O})$-functions with mean-value zero will be denoted by $H^1_*(\mathcal{O})$. Furthermore, we denote the dual space of $H^1_{\text{per}}(\mathcal{O})$ by $(H^1_{\text{per}}(\mathcal{O}))'$. The space of continuous $\mathcal{O}$-periodic functions is denoted by the symbol $C_{\text{per}}(\overline{\mathcal{O}})$ and $C^\gamma_{\text{per}}(\overline{\mathcal{O}})$ is the space of $\mathcal{O}$-periodic, Hölder continuous functions with Hölder exponent γ.

For a time interval I and a Banach space X, the space of L^p-integrable functions with values in X is denoted by $L^p(I; X)$. Similarly, we denote the space of k-times weakly differentiable functions from I to X with weak derivatives in $L^p(I; X)$ by $W^{k,p}(I; X)$ and the Hölder continuous functions from I to X with Hölder exponent γ by $C^\gamma(I; X)$.

We shall also use some standard notation from stochastic analysis: The notation $a \wedge b$ stands for the minimum of a and b, and $L_2(X, Y)$ denotes the set of Hilbert–Schmidt operators from X to Y. For a stopping time T, we write χ_T to denote the (ω-dependent) characteristic function of the time interval $[0, T]$.

Further notation related to the semi-discrete scheme is introduced in Section 2.

2. Notation, technical preliminaries, and basic assumptions on the data

We consider the torus $\mathcal{O} := \mathcal{O}^x \times \mathcal{O}^y := (0, L_x) \times (0, L_y)$. We introduce partitions $\mathcal{T}_h^x$ and $\mathcal{T}_h^y$ of $\mathcal{O}^x$ and $\mathcal{O}^y$, respectively, satisfying the following assumption:

(S) $\{\mathcal{T}_h^x\}_{h>0}$ and $\{\mathcal{T}_h^y\}_{h>0}$ are families of equidistant partitions of $\mathcal{O}^x$ and $\mathcal{O}^y$, respectively, into disjoint, open intervals such that

$$\overline{\mathcal{O}^x} \equiv \bigcup_{K^x \in \mathcal{T}_h^x} \overline{K^x} \quad \text{and} \quad \overline{\mathcal{O}^y} \equiv \bigcup_{K^y \in \mathcal{T}_h^y} \overline{K^y}.$$

In particular, there exist positive constants $\hat{c}_1$, $\hat{C}_2$ such that

$$\hat{c}_1 h \le h_x, h_y \le \hat{C}_2 h$$

with $h_x := \operatorname{diam} K^x$ $(K^x \in \mathcal{T}_h^x)$, $h_y := \operatorname{diam} K^y$ $(K^y \in \mathcal{T}_h^y)$, and $h \in (0, 1)$.

Combining $\{\mathcal{T}_h^x\}_h$ and $\{\mathcal{T}_h^y\}_h$, we obtain a family of partitions $\{\mathcal{Q}_h\}_h$ of $\mathcal{O}$ which is defined via

$$\mathcal{Q}_h = \left\{Q = K^x \times K^y : K^x \in \mathcal{T}_h^x \text{ and } K^y \in \mathcal{T}_h^y\right\}. \tag{2.1}$$

Based on these partitions, we introduce the following spaces of continuous, piecewise linear finite element functions:

$$U_h^x := \left\{v \in C_{\text{per}}(\overline{\mathcal{O}^x}) : v|_{K^x} \in \mathcal{P}_1(K^x) \quad \forall K^x \in \mathcal{T}_h^x\right\}, \tag{2.2a}$$

$$U_h^y := \left\{v \in C_{\text{per}}(\overline{\mathcal{O}^y}) : v|_{K^y} \in \mathcal{P}_1(K^y) \quad \forall K^y \in \mathcal{T}_h^y\right\}, \tag{2.2b}$$

$$U_h := U_h^x \otimes U_h^y. \tag{2.2c}$$

Imposing periodic boundary conditions, we denote the vertices of $\mathcal{T}_h^x$ by

$$\left\{x_i\right\}_{i=1,\dots,\dim U_h^x+1} = \left\{(i-1)h_x\right\}_{i=1,\dots,\dim U_h^x+1}$$

and identify $x_{\dim U_h^x+1} = L_x$ with $x_1 = 0$. Furthermore, we denote the dual basis to these vertices by $\{e_i^x\}_{i=1,\dots,\dim U_h^x}$. We also denote the vertices of $\mathcal{T}_h^y$ by $\{y_j\}_{j=1,\dots,\dim U_h^y+1}$, identify

$$y_{\dim U_h^y+1} = L_y \text{ with } y_1 = 0,$$

and consider their dual basis $\{e_j^y\}_{j=1,\dots,\dim U_h^y}$. Throughout this work, we will also identify

$$x_0 \text{ with } x_{\dim U_h^x}, \quad y_0 \text{ with } y_{\dim U_h^y}.$$

In the same spirit, we shall identify

$$e^x_{\dim U_h^x+1} \text{ with } e_1^x, \quad e_0^x \text{ with } e^x_{\dim U_h^x}, \quad e^y_{\dim U_h^y+1} \text{ with } e_1^y, \quad e_0^y \text{ with } e^y_{\dim U_h^y}.$$

For the spaces introduced in (2.2), we define the interpolation operators

$$\mathcal{I}_h^x : C_{\text{per}}(\overline{\mathcal{O}^x}) \to U_h^x, \quad a \mapsto \sum_{i=1}^{\dim U_h^x} a(x_i) e_i^x, \tag{2.3}$$

$$\mathcal{I}_h^y : C_{\mathrm{per}}(\overline{\mathcal{O}^y}) \to U_h^y, \quad a \mapsto \sum_{j=1}^{\dim U_h^y} a(y_j) e_j^y, \tag{2.4}$$

$$\mathcal{I}_h^{xy} : C_{\mathrm{per}}(\overline{\mathcal{O}}) \to U_h, \quad a \mapsto \mathcal{I}_h^x\{\mathcal{I}_h^y\{a\}\} = \mathcal{I}_h^y\{\mathcal{I}_h^x\{a\}\}. \tag{2.5}$$

For future reference, we state the following norm equivalence for $p \in [1, \infty)$ and $u_h \in U_h$:

$$c\Big(\int_{\mathcal{O}} (u_h)^p \,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p} \le \Big(\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{(u_h)^p\} \,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p} \le C\Big(\int_{\mathcal{O}} (u_h)^p \,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p} \tag{2.6}$$

with $c, C > 0$ independent of h. Similar (lower-dimensional) results also hold true for $\mathcal{I}_h^x$ on U_h^x and $\mathcal{I}_h^y$ on U_h^y. These nodal interpolation operators satisfy error estimates similar to the ones established in [60] for simplicial elements. For the reader's convenience, we collect these estimates in Lemma A.1 in Appendix A.

With these interpolation operators, we define the discrete Laplacian $\Delta_h : U_h \to U_h \cap H_*^1(\mathcal{O})$ as follows:

$$\begin{aligned}\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{-\Delta_h u_h \psi_h\} \,\mathrm{d}x\,\mathrm{d}y &:= \int_{\mathcal{O}} \mathcal{I}_h^y\{\partial_x u_h \partial_x \psi_h\} \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathcal{O}} \mathcal{I}_h^x\{\partial_y u_h \partial_y \psi_h\} \,\mathrm{d}x\,\mathrm{d}y \\ &=: -\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{\Delta_h^x u_h \psi_h\} \,\mathrm{d}x\,\mathrm{d}y - \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{\Delta_h^y u_h \psi_h\} \,\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{2.7}$$

for all $\psi \in U_h$. Thereby, the operator Δ_h^x can be interpreted pointwise in y as the one-dimensional discrete Laplacian with respect to x mapping U_h^x onto $U_h^x \cap H_*^1(\mathcal{O}^x)$, and Δ_h^y can be interpreted pointwise in x as the one-dimensional discrete Laplacian with respect to y mapping U_h^y onto $U_h^y \cap H_*^1(\mathcal{O}^y)$.

We denote the forward and backward difference quotients with respect to the spatial coordinates x and y by $\partial_x^{+h_x}$, $\partial_x^{-h_x}$, $\partial_y^{+h_y}$, and $\partial_y^{-h_y}$, i.e.,

$$\partial_x^{+h_x} f(x,y) := \big(f(x+h_x, y) - f(x,y)\big)/h_x, \tag{2.8a}$$

$$\partial_x^{-h_x} f(x,y) := \big(f(x,y) - f(x-h_x, y)\big)/h_x, \tag{2.8b}$$

$$\partial_y^{+h_y} f(x,y) := \big(f(x, y+h_y) - f(x,y)\big)/h_y, \tag{2.8c}$$

$$\partial_y^{-h_y} f(x,y) := \big(f(x,y) - f(x, y-h_y)\big)/h_y \tag{2.8d}$$

(with f extended outside of $\mathcal{O}$ by periodicity). Assuming equidistant partitions with respect to x and y, the identities

$$\Delta_h^x v_h = \partial_x^{+h_x}(\partial_x^{-h_x} v_h) = \partial_x^{-h_x}(\partial_x^{+h_x} v_h), \tag{2.9a}$$

$$\Delta_h^y v_h = \partial_y^{+h_y}(\partial_y^{-h_y} v_h) = \partial_y^{-h_y}(\partial_y^{+h_y} v_h) \tag{2.9b}$$

hold true for $v_h \in U_h$.

In addition, we introduce similar local interpolation operators as follows: We consider the spaces

$$C_{\mathrm{per},\mathcal{T}_h^x} := \big\{v \in L^\infty_{\mathrm{per}}(\mathcal{O}^x) : v|_{K^x} \in C(K^x) \quad \forall K^x \in \mathcal{T}_h^x\big\}, \tag{2.10}$$

$$C_{\mathrm{per},\mathcal{T}_h^y} := \big\{v \in L^\infty_{\mathrm{per}}(\mathcal{O}^y) : v|_{K^y} \in C(K^y) \quad \forall K^y \in \mathcal{T}_h^y\big\} \tag{2.11}$$

of bounded, piecewise continuous, periodic functions. As we can extend a continuous function on an open interval to a continuous function on the closure of this interval, we may apply $\mathcal{I}_h^x$ and $\mathcal{I}_h^y$ locally on each element to obtain

$$\mathcal{I}_{h,\mathrm{loc}}^x : C_{\mathrm{per},\mathcal{T}_h^x} \to \big\{v \in C_{\mathrm{per},\mathcal{T}_h^x} : v|_{K^x} \in \mathcal{P}_1(K^x) \quad \forall K^x \in \mathcal{T}_h^x\big\}, \tag{2.12a}$$

$$\mathcal{I}_{h,\mathrm{loc}}^y : C_{\mathrm{per},\mathcal{T}_h^y} \to \big\{v \in C_{\mathrm{per},\mathcal{T}_h^y} : v|_{K^y} \in \mathcal{P}_1(K^y) \quad \forall K^y \in \mathcal{T}_h^y\big\}. \tag{2.12b}$$

Obviously, these local interpolation operators satisfy the identities

$$\mathcal{I}_{h,\mathrm{loc}}^x\big\{\partial_x a_h^x v\big\} = \partial_x a_h^x \mathcal{I}_{h,\mathrm{loc}}^x\{v\}, \quad \text{and} \quad \mathcal{I}_{h,\mathrm{loc}}^y\big\{\partial_y a_h^y \widehat{v}\big\} = \partial_y a_h^y \mathcal{I}_{h,\mathrm{loc}}^y\{\widehat{v}\}, \tag{2.13a}$$

$$\mathcal{I}_h^x\{\widetilde{v}\} = \mathcal{I}_{h,\mathrm{loc}}^x\{\widetilde{v}\}, \quad \text{and} \quad \mathcal{I}_h^y\{\bar{v}\} = \mathcal{I}_{h,\mathrm{loc}}^y\{\bar{v}\} \tag{2.13b}$$

for $v \in C_{\mathrm{per},\mathcal{T}_h^x}$, $\widehat{v} \in C_{\mathrm{per},\mathcal{T}_h^y}$, $\widetilde{v} \in C_{\mathrm{per}}(\overline{\mathcal{O}^x})$, and $\bar{v} \in C_{\mathrm{per}}(\overline{\mathcal{O}^y})$. Here, the first identity follows directly from the definition of $\mathcal{I}_{h,\mathrm{loc}}^x$, as $\partial_x a_h$ is constant with respect to x on every element.

In order to allow for a discrete version of the chain rule, we introduce for a continuous function $f : \mathbb{R} \to \mathbb{R}$ and $u \in C_{\mathrm{per}}(\overline{\mathcal{O}})$ for every $y \in \mathcal{O}^y$ and $x \in (ih, (i+1)h) =: K^x \in \mathcal{T}_h^x$ the function

$$[f(u)]_x(x,y) := \fint_{u(ih,y)}^{u((i+1)h,y)} f(s)\,\mathrm{d}s. \tag{2.14}$$

Similarly, we define for $x \in \mathcal{O}^x$ and $y \in (jh, (j+1)h) =: K^y \in \mathcal{T}_h^y$

$$[f(u)]_y(x,y) := \fint_{u(x,jh)}^{u(x,(j+1)h)} f(s)\,\mathrm{d}s. \tag{2.15}$$

Obviously, these definitions provide for $u_h \in U_h$

$$\partial_x \mathcal{I}_h^{xy}\{f(u_h)\} = \mathcal{I}_h^y\big\{[f'(u_h)]_x \partial_x u_h\big\}, \quad \partial_y \mathcal{I}_h^{xy}\{f(u_h)\} = \mathcal{I}_h^x\big\{[f'(u_h)]_y \partial_y u_h\big\}. \tag{2.16}$$

On the periodic domain $\mathcal{O}$, we define the Ritz projection operator $\mathcal{R} : H^1_{\mathrm{per}}(\mathcal{O}) \to U_h$ via

$$\int_{\mathcal{O}} \nabla \mathcal{R}\{u\} \cdot \nabla v_h \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathcal{O}} \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y \quad \text{for all } v_h \in U_h \tag{2.17}$$

with the additional constraint $\int_{\mathcal{O}} \mathcal{R}\{u\}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathcal{O}} u\,\mathrm{d}x\,\mathrm{d}y$.

In this publication, we consider the case of a quadratic mobility, i.e., $m(u) := u^2$. Let us specify our assumptions on initial data, effective interface potential, and the noise.

(I) Let Λ be a probability measure on $H^2_{\text{per}}(\mathcal{O})$ equipped with the Borel σ-algebra which is supported on the subset of strictly positive functions such that there is a positive constant C with the property

$$\operatorname*{ess\,sup}_{v\in\operatorname{supp}\Lambda}\Big(\mathcal{E}_h(\mathcal{I}_h^{xy}\{v\})+\int_{\mathcal{O}}\mathcal{I}_h^{xy}\{v\}\,\mathrm{d}x\,\mathrm{d}y+\Big(\int_{\mathcal{O}}\mathcal{I}_h^{xy}\{v\}\,\mathrm{d}x\,\mathrm{d}y\Big)^{-1}\Big)\le C \quad (2.18)$$

for any $h>0$ with $\mathcal{E}_h$ being a discrete version of the energy (1.4), which we will define in (3.1).

(P) The effective interface potential F has continuous second-order derivatives on $\mathbb{R}^+$ and satisfies for some $p>2$ and $u>0$ the following estimates with appropriate positive constants:

$$F(u)\ge c_1u^{-p},$$
$$|F'(u)|\le \widehat{C}u^{-p-1}+\widehat{C},$$
$$\widetilde{c}_1u^{-p-2}-\widetilde{c}_2\le F''(u)\le \widetilde{C}u^{-p-2}+\widetilde{C}.$$

For nonpositive u, we define $F(u):=+\infty$.

(B) Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration such that

(B1) $\boldsymbol{W}$ is a Q-Wiener process on Ω adapted to $(\mathcal{F}_t)_{t\ge0}$ which admits a decomposition of the form

$$\boldsymbol{W}=\sum_{\alpha\in\{x,y\}}\sum_{k,l\in\mathbb{Z}}\lambda^\alpha_{kl}\mathfrak{g}_{kl}\boldsymbol{b}_\alpha\beta^\alpha_{kl}$$

for independent sequences of i.i.d. Brownian motions β^α_{kl} ($\alpha\in\{x,y\}$) and a sequence of sufficiently smooth basis functions $\mathfrak{g}_{kl}$. Here, $\boldsymbol{b}_x$ and $\boldsymbol{b}_y$ denote the standard Cartesian basis vectors in $\mathbb{R}^2$. Furthermore, we will denote its components by

$$W_\alpha:=\sum_{k,l\in\mathbb{Z}}\lambda^\alpha_{kl}\mathfrak{g}_{kl}\beta^\alpha_{kl}$$

for $\alpha\in\{x,y\}$. The corresponding components of Q will be denoted by Q_x and Q_y.

(B2) There exists a $\mathcal{F}_0$-measurable random variable u^0 such that $\Lambda=\mathbb{P}\circ(u^0)^{-1}$.

(B3) The noise $\boldsymbol{W}$ is colored in the sense that

$$\sum_{k,l=1}^{\infty}({\lambda^x_{kl}}^2+{\lambda^y_{kl}}^2)\|\mathfrak{g}_{kl}\|^2_{W^{2,\infty}(\mathcal{O})}\le C$$

for a positive constant C.

Remark 2.1. (1) Under natural assumptions on the decay parameters λ_{kl}^{α} and the basis functions $\mathfrak{g}_{kl}$, $k,l \in \mathbb{Z}$, Assumption (P) covers in fact also the case that the stochastic integral is to be understood in the sense of Stratonovich. Assuming

- the basis functions to be given by $\mathfrak{g}_{kl}(x,y) = g_k^x(x) g_l^y(y)$ with $g_k^x(\cdot)$ and $g_l^y(\cdot)$ as in (3.4),
- the decay parameters λ_{kl}^{α}, $k,l \in \mathbb{Z}$, $\alpha \in \{x,y\}$ satisfy

$$\lambda_{kl}^{x} = \lambda_{kl}^{y}, \quad \lambda_{(-k)l}^{\alpha} = \lambda_{kl}^{\alpha}, \quad \lambda_{k(-l)}^{\alpha} = \lambda_{kl}^{\alpha} \tag{2.19}$$

  for all $k,l \in \mathbb{Z}$ and $\alpha \in \{x,y\}$,

the Itô correction of the Stratonovich term

$$\sum_{\alpha\in\{x,y\}} \sum_{k,l\in\mathbb{Z}} \partial_\alpha \big(u \lambda_{kl}^{\alpha} \mathfrak{g}_{kl}\big) \circ \mathrm{d}\beta_{kl}^{\alpha}$$

becomes

$$C_{\text{Strat}} \Delta u \,\mathrm{d}t + \sum_{\alpha\in\{x,y\}} \sum_{k,l\in\mathbb{Z}} \partial_\alpha \big(u \lambda_{kl}^{\alpha} \mathfrak{g}_{kl}\big) \,\mathrm{d}\beta_{kl}^{\alpha}. \tag{2.20}$$

Here, the positive constant C_{Strat} is given by

$$C_{\text{Strat}} := \frac{1}{L_x L_y}\Big(\lambda_{00}^2 + 4 \sum_{k,l\in\mathbb{Z}\setminus\{0\}} \lambda_{kl}^2 + 2 \sum_{k\in\mathbb{Z}\setminus\{0\}} (\lambda_{k0}^2 + \lambda_{0k}^2)\Big), \tag{2.21}$$

where we omitted the superscript α as the decay parameters were chosen to be independent of α. This allows us to write the stochastic thin-film equation with Stratonovich noise in the form

$$du = -\operatorname{div}\big(u^2 \nabla(\Delta u - F_{\text{Strat}}'(u))\big)\,\mathrm{d}t + \sum_{\alpha\in\{x,y\}} \sum_{k,l\in\mathbb{Z}} \partial_\alpha (u \lambda_{kl} \mathfrak{g}_{kl}) \,\mathrm{d}\beta_{kl}^{\alpha} \tag{2.22}$$

with the energy F_{Strat} given by

$$F_{\text{Strat}}(u) := \begin{cases} F(u) + C_{\text{Strat}}\,(u - \log u) + \text{const.} & \text{if } u > 0, \\ +\infty & \text{if } u \leq 0, \end{cases} \tag{2.23}$$

where the constant can be chosen in such a way that F_{Strat} satisfies Assumption (P) if Assumption (P) is satisfied by F itself. Hence, the analysis presented in this paper applies to the Stratonovich interpretation, too.

(2) The approximation of initial data is based on the nodal interpolation operator to cope with the requirement of strictly positive discrete initial data. Therefore, we need the space of initial data to be continuously embedded in $C(\overline{\mathcal{O}})$. The specification in Assumption (I) that initial data should have H^2-regularity is presumably not the optimal one. It is, however, consistent with our regularization procedure—see (3.6b)—of augmenting the pressure p_h by a discrete bi-Laplacian. As the energy estimate formulated for (1.3)

(see (3.9)) does not require more than H^1-regularity for initial data, it should have been possible to focus on H^1- initial data and to apply nodal interpolation operators to appropriate H^2-regularizations, e.g., by convolution. For the ease of presentation, we prefer to avoid those technicalities.

3. The semi-discrete scheme

In order to control the oscillation of the discrete solution u_h on each element, we regularize the energy under consideration. Introducing a regularization parameter $2 > \varepsilon > 0$, we define the regularized discrete energy and the discrete entropy as

$$\mathcal{E}_h(u_h) := \frac{1}{2}\int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x u_h|^2\} + \mathcal{I}_h^x\{|\partial_y u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{F(u_h)\}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|\Delta_h u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y, \tag{3.1}$$

$$\mathcal{S}_h(u_h) := \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{G(u_h)\}\,\mathrm{d}x\,\mathrm{d}y \quad \text{with} \quad G(s) := \int_1^s \int_1^r \frac{1}{m(\tau)}\mathrm{d}\tau\mathrm{d}r. \tag{3.2}$$

As it will be shown in Lemma 3.6, we assume that

(R) the regularization parameter ε is small enough such that there exists a constant $\rho > 0$ such that

$$1 > \frac{2}{p} + \frac{\varepsilon}{2} + \frac{\rho}{2p}, \tag{3.3}$$

where p is the exponent associated with the growth of F (cf. Assumption (P)).

Remark 3.1. For every exponent $p > 2$ in the effective interface potential, positive parameters ε and ρ exist, such that Assumption (R) holds true.

Given a positive time $T_{\max}$, we introduce a threshold energy $\mathcal{E}_{\max,h} := \widehat{C}h^{-\rho/(2+p)}$ for given $\widehat{C} > 0$ and $0 < \rho \ll 1$ satisfying Assumption (R). Similar to [29], we consider associated stopping times $T_h := T_{\max} \wedge \inf\{t \geq 0 : \mathcal{E}(u_h) \geq \mathcal{E}_{\max,h}\}$. We approximate the infinite- dimensional Wiener process by a finite-dimensional noise term. In particular, we introduce the sets $I_h^x \subset \mathbb{Z}$ and $I_h^y \subset \mathbb{Z}$ satisfying

(B3*) $\sum_{k\in I_h^x}\sum_{l\in I_h^y}({\lambda_{kl}^x}^2 + {\lambda_{kl}^y}^2)h^\varepsilon \|\mathfrak{g}_{kl}\|^2_{W^{3,\infty}(\mathcal{O})} \leq C$ for $h \searrow 0$,

(B4) $I_h^x \subseteq I_{\hat{h}}^x$, $I_h^y \subseteq I_{\hat{h}}^y$ for $h \geq \hat{h}$ and $\bigcup_{h>0} I_h^x = \mathbb{Z}$ and $\bigcup_{h>0} I_h^y = \mathbb{Z}$.

Remark 3.2. Often the basis functions $\mathfrak{g}_{kl}$ are assumed to be eigenfunctions of the negative Laplacian on $\mathcal{O}$ under periodic boundary conditions. In particular, the functions $\mathfrak{g}_{kl}$ are assumed to be the product of eigenfunctions g_k^x and g_l^y of the one-dimensional Laplacian on $\mathcal{O}^x$ and $\mathcal{O}^y$, respectively, i.e.,

$$
\begin{aligned}
g_k^x(x) &:= \sqrt{\frac{2}{L_x}} \begin{cases} \cos\Big(\frac{2\pi k x}{L_x}\Big) & \text{for } k \geq 1, \\ \frac{1}{\sqrt{2}} & \text{for } k = 0, \\ \sin\Big(\frac{2\pi k x}{L_x}\Big) & \text{for } k \leq -1, \end{cases} \\
g_l^y(y) &:= \sqrt{\frac{2}{L_y}} \begin{cases} \cos\Big(\frac{2\pi l y}{L_y}\Big) & \text{for } l \geq 1, \\ \frac{1}{\sqrt{2}} & \text{for } l = 0, \\ \sin\Big(\frac{2\pi l y}{L_y}\Big) & \text{for } l \leq -1. \end{cases}
\end{aligned} \tag{3.4}
$$

In this case, we have

$$
\|\mathfrak{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})} \sim (k^2 + l^2) \quad \text{and} \quad \|\mathfrak{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})} \leq C(k^3 + l^3).
$$

Therefore, one may choose

$$
I_h^x = I_h^y = \left\{z \in \mathbb{Z} : |z| \leq \widehat{C} h^{-\varepsilon/2}\right\}
$$

for given $\widehat{C} > 0$ to satisfy Assumptions (B3*) and (B4), i.e., the additional restrictions on the noise term imposed in (B3*) vanish when passing to the limit $h \searrow 0$.

With the goal to simplify the implementation in a practical numerical scheme, we approximate the basis functions $\mathfrak{g}_{kl}$ by $\widetilde{\mathfrak{g}}_{h,kl} := \mathcal{I}_h^{xy}\{\mathfrak{g}_{kl}\}$.

In this work, we consider solutions

$$
u_h \in L^2(\Omega; C([0, T_{\max}]; U_h)), \tag{3.5a}
$$

$$
p_h \in L^2(\Omega; L^\infty(0, T_{\max}; U_h)) \tag{3.5b}
$$

to the following regularized, semi-discrete version of (1.3):

$$
\begin{aligned}
&\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{u_h(T)\psi_h\}\,\mathrm{d}x\,\mathrm{d}y - \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{u_h(0)\psi_h\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_0^{T\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^y\left\{[G''(u_h)]_x^{-1}\partial_x p_h \partial_x \psi_h\right\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int_0^{T\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^x\left\{[G''(u_h)]_y^{-1}\partial_y p_h \partial_y \psi_h\right\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&= \sum_{k\in I_h^x}\sum_{k\in I_h^y} \lambda_{kl}^x \int_0^{T\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^y\left\{\mathcal{I}_{h,\mathrm{loc}}^x\{\partial_x(u_h\widetilde{\mathfrak{g}}_{h,kl})\psi_h\}\right\}\mathrm{d}x\,\mathrm{d}y\;\mathrm{d}\beta_{kl}^x \\
&\quad + \sum_{k\in I_h^x}\sum_{k\in I_h^y} \lambda_{kl}^y \int_0^{T\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^x\left\{\mathcal{I}_{h,\mathrm{loc}}^y\{\partial_y(u_h\widetilde{\mathfrak{g}}_{h,kl})\psi_h\}\right\}\mathrm{d}x\,\mathrm{d}y\;\mathrm{d}\beta_{kl}^y,
\end{aligned} \tag{3.6a}
$$

and

$$\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{p_h \psi_h\}\,\mathrm{d}x\,\mathrm{d}y = \chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^{y}\{\partial_x u_h \partial_x \psi_h\}\,\mathrm{d}x\,\mathrm{d}y + \chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^{x}\{\partial_y u_h \partial_y \psi_h\}\,\mathrm{d}x\,\mathrm{d}y + \chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{F'(u_h)\psi_h\}\,\mathrm{d}x\,\mathrm{d}y + \chi_{T_h} h^{\varepsilon} \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{\Delta_h u_h \Delta_h \psi_h\}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.6b}$$

Remark 3.3. Note that for discrete solutions of (3.6a), the mass of discrete solutions, i.e.,

$$\int_{\mathcal{O}} u_h(t)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{u_h(t)\}\,\mathrm{d}x\,\mathrm{d}y,$$

is constant in time. Of course, it is natural to choose $\psi_h \equiv 1$ as the test function in (3.6a). Obviously, the contribution from the elliptic terms vanishes. So, let us briefly prove that also the stochastic terms become zero. Using (2.13a) and the fact that both u_h and $\widetilde{\mathfrak{g}}_{h,kl}$ are contained in U_h, we find

$$\begin{aligned}&\int_{\mathcal{O}} \mathcal{I}_h^{y}\big\{\mathcal{I}_{h,\mathrm{loc}}^{x}\{\partial_x(u_h \widetilde{\mathfrak{g}}_{h,kl})\}\big\}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad= \int_{\mathcal{O}} \mathcal{I}_h^{y}\big\{\partial_x u_h \mathcal{I}_{h,\mathrm{loc}}^{x}\{\widetilde{\mathfrak{g}}_{h,kl}\} + \partial_x \widetilde{\mathfrak{g}}_{h,kl} \mathcal{I}_{h,\mathrm{loc}}^{x}\{u_h\}\big\}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad= \int_{\mathcal{O}} \mathcal{I}_h^{y}\{\partial_x(u_h \widetilde{\mathfrak{g}}_{h,kl})\}\,\mathrm{d}x\,\mathrm{d}y = 0,\end{aligned}$$

due to integration by parts.

Definition 3.4. Let Λ be a probability measure on $H^2_{\mathrm{per}}(\mathcal{O})$ satisfying Assumption (I). A triple $((\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}}), \widetilde{u}, \widetilde{\boldsymbol{W}})$ is called a weak martingale solution to the stochastic thin-film equation (1.3) with initial data Λ on the time interval $[0, T_{\max}]$ provided

(1) $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$ is a stochastic basis with a complete, right-continuous filtration,

(2) $\widetilde{\boldsymbol{W}}$ satisfies Assumption (B) with respect to $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$,

(3) the solution $\widetilde{u}$ is element of

$$\begin{aligned}&L^q(\widetilde{\Omega}; L^\infty(0, T_{\max}; H^1_{\mathrm{per}}(\mathcal{O}))) \cap L^2(\widetilde{\Omega}; L^2(0, T_{\max}; H^2_{\mathrm{per}}(\mathcal{O}))) \\ &\qquad \cap L^\sigma(\widetilde{\Omega}; C^{1/4}([0, T_{\max}]; (H^1_{\mathrm{per}}(\mathcal{O}))'))\end{aligned} \tag{3.7}$$

for all $q < \infty$ and $\sigma < 8/5$ such that $\sqrt{m(\widetilde{u})}\nabla(\Delta\widetilde{u} - F'(\widetilde{u})) \in L^2([\widetilde{u} > 0])$,

(4) there exists an $\widetilde{F}_0$-measurable $H^2_{\mathrm{per}}(\mathcal{O};\mathbb{R}^+)$-valued random variable $\widetilde{u}^0$ such that $\Lambda = \widetilde{\mathbb{P}} \circ (\widetilde{u}^0)^{-1}$, and the equation

$$\int_{\mathcal{O}} \widetilde{u}(t)\phi \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathcal{O}} \widetilde{u}^0\phi \,\mathrm{d}x\,\mathrm{d}y + \iint_{[\widetilde{u}>0]} m(\widetilde{u})\nabla(\Delta\widetilde{u} - F'(\widetilde{u}))\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$- \sum_{\alpha\in\{x,y\}} \sum_{k,l\in\mathbb{Z}} \lambda^\alpha_{kl} \int_0^t \int_{\mathcal{O}} \sqrt{m(\widetilde{u})}\mathfrak{g}_{kl}\partial_\alpha\phi \,\mathrm{d}x\,\mathrm{d}y\;\mathrm{d}\widetilde{\beta}^\alpha_{kl} \tag{3.8}$$

holds true $\widetilde{\mathbb{P}}$-almost surely for all $t \in [0, T_{\max}]$ and all $\phi \in W^{1,q^*}_{\mathrm{per}}(\mathcal{O})$ with $q^* > 2$.

The aim of this work is to establish the existence of weak martingale solutions starting from semi-discrete solutions to (3.6). In particular, we shall prove the following theorem:

Theorem 3.5. *Let Assumptions (S), (I), (P), (B), (R), (B3*), and (B4) be satisfied and let $T_{\max} > 0$ be given. Furthermore, let $(u_h, p_h)_{h\searrow 0}$ be a sequence of solutions to the regularized Faedo–Galerkin scheme (3.6) for the stochastic thin-film equation (1.3) with $\mathcal{E}_{\max,h} = \widehat{C}h^{-\rho/(2+p)}$ for some given $\widehat{C} > 0$. Then, there exist a stochastic basis $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$ and processes $\widetilde{u}_h$, $\widetilde{J}^x_h$, $\widetilde{J}^y_h$, and $\widetilde{u}$ such that the following holds: The processes $\widetilde{u}_h$, $\widetilde{J}^x_h$, and $\widetilde{J}^y_h$ have the same law as the processes u_h,*

$$J^x_h := \mathcal{I}^y_h\{\sqrt{[G''(u_h)]^{-1}_x}\partial_x p_h\} \quad \text{and} \quad J^y_h := \mathcal{I}^x_h\{\sqrt{[G''(u_h)]^{-1}_y}\partial_y p_h\},$$

and for an appropriate subsequence, we $\widetilde{\mathbb{P}}$-almost surely have the convergences $\widetilde{u}_h \to \widetilde{u}$ strongly in $C([0,T_{\max}];L^q(\mathcal{O})) \cap L^2(0,T_{\max};W^{1,q}_{\mathrm{per}}(\mathcal{O}))$ $(1 \leq q < \infty)$; $\widetilde{J}^x_h \rightharpoonup \widetilde{J}^x$ weakly in $L^2(0,T_{\max};L^2(\mathcal{O}))$, which can be identified with $-\widetilde{u}\partial_x(\Delta\widetilde{u} - F'(\widetilde{u}))$ on $[\widetilde{u} > 0]$; and $\widetilde{J}^y_h \rightharpoonup \widetilde{J}^y$ weakly in $L^2(0,T_{\max};L^2(\mathcal{O}))$, which can be identified with $-\widetilde{u}\partial_y(\Delta\widetilde{u} - F'(\widetilde{u}))$ on $[\widetilde{u} > 0]$. Furthermore, $\widetilde{u}$ is a weak martingale solution to the stochastic thin-film equation in the sense of Definition 3.4 satisfying the additional bound

$$\widetilde{\mathbb{E}}\Big[\sup_{t\in[0,T_{\max}]} (\mathcal{E}(\widetilde{u}))^{\overline{p}}\Big] + \widetilde{\mathbb{E}}\Big[\iint_{[\widetilde{u}>0]} m(\widetilde{u})|\nabla(\Delta\widetilde{u} - F'(\widetilde{u}))|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big] \leq C(u^0, \overline{p}, T_{\max}) \tag{3.9}$$

with $1 \leq \overline{p} < \infty$. In particular, $\widetilde{\mathbb{P}}$-almost surely, $\widetilde{u}(\cdot,t)$ is strictly positive for almost all $t \in [0, T_{\max}]$.

Lemma 3.6. *Let $u_h \in U_h$ be strictly positive, let $1 > \gamma \geq \frac{2}{p} + \frac{\varepsilon}{2} + \frac{\rho}{2p}$ and let*

$$\mathcal{E}_h(u_h) \leq Ch^{-\rho/(2+p)}.$$

Then, there exists an h-independent constant $C_{\mathrm{osc}} > 0$ such that the estimate

$$\frac{u_h(x_i, y_j)}{u_h(x_{\widehat{i}}, y_{\widehat{j}})} \leq C_{\mathrm{osc}} \tag{3.10}$$

holds true for all $i \in \{1,\dots,\dim U^x_h\}$, $j \in \{1,\dots,\dim U^y_h\}$, $\widehat{i} \in \{i-1, i, i+1\}$, and $\widehat{j} \in \{j-1, j, j+1\}$.

Proof. Using the standard embedding theorems for Hölder continuous functions and the discrete embedding proven in Corollary A.4, we obtain

$$\|u_h\|_{C^\gamma(\overline{\mathcal{O}})} \le C\|\nabla u_h\|_{L^q(\mathcal{O})} \le C\big(\|u_h\|_{H^1(\mathcal{O})} + \|\Delta_h u_h\|_{L^2(\mathcal{O})}\big) \le C\sqrt{h^{-\varepsilon}\mathcal{E}_h(u_h)}$$

with $q < \infty$ large enough. Furthermore, we have

$$\sup_{(x,y)\in\mathcal{O}} u_h^{-1} = \big(\sup_{(x,y)\in\mathcal{O}} u_h^{-p}\big)^{1/p} \le C\Big(h^{-2}\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{F(u_h)\}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p}$$
$$\le Ch^{-2/p}\mathcal{E}_h(u_h)^{1/p}. \tag{3.11}$$

Since there exists an element $Q \in \mathcal{Q}_h$ including the vertices (x_i, y_j) and $(x_{\hat{i}}, y_{\hat{j}})$ by assumption, we combine the estimates above and obtain

$$\begin{aligned}\Big|\frac{u_h(x_i,y_j)}{u_h(x_{\hat{i}},y_{\hat{j}})} - 1\Big| &= \Big|\frac{u(x_i,y_j) - u_h(x_{\hat{i}},y_{\hat{j}})}{u_h(x_{\hat{i}},y_{\hat{j}})}\Big| \\ &\le C\big(\sup_{(x,y)\in\mathcal{O}} u_h^{-1}\big)h^\gamma\|u_h\|_{C^\gamma(\overline{\mathcal{O}})} \\ &\le Ch^{-2/p}\mathcal{E}_h(u_h)^{1/p}h^\gamma h^{-\varepsilon/2}\mathcal{E}_h(u_h)^{1/2} \\ &= Ch^{-2/p-\varepsilon/2+\gamma}\mathcal{E}_h(u_h)^{1/p+1/2} \le C =: C_{\mathrm{osc}},\end{aligned} \tag{3.12}$$

which completes the proof. ∎

We will start analyzing scheme (3.6) by showing that it admits a solution.

Lemma 3.7. *Let $T_{\max}$ be a positive real number and $\mathcal{E}_{\max,h} = \hat{C}h^{-\rho/(2+p)}$. Then there exist stochastic processes $u_h \in L^2(\Omega; C([0,T_{\max}];U_h))$ and $p_h \in L^2(\Omega; L^\infty(0,T_{\max};U_h))$ as well as associated stopping times T_h such that:*

- *Almost surely, we have $T_h = T_{\max} \wedge \inf\{t \in [0,\infty) : \mathcal{E}_h(u_h(\cdot,t)) \ge \mathcal{E}_{\max,h}\}$.*
- *Almost surely, the process p_h solves (3.6b) for $t \le T_{\max}$, and it is contained in $C([0,T_h];U_h)$.*
- *Almost surely, the process u_h solves (3.6a) for $t \le T_{\max}$ and it is constant for $t \in [T_h, T_{\max}]$.*

Proof. As the additional regularization term changes neither the Lipschitz continuity of $\mathcal{E}_h(u_h)$ with respect to u_h when $\mathcal{E}_h(u) \le 2\mathcal{E}_{\max,h}$ nor the Lipschitz continuous dependence of p_h on u_h when $\mathcal{E}_h(u_h) \le 3\mathcal{E}_{\max,h}$, the result follows along the lines of the proof of [29, Lemma 4.2]. ∎

As the solutions u_h are continuous in space and time for $h > 0$, the positivity of the initial data immediately provides the positivity of the semi-discrete solutions.

Corollary 3.8. *The solutions constructed in Lemma 3.7 are strictly positive for all $h > 0$.*

4. A priori estimates

In this section, we shall establish uniform a priori estimates for the semi-discrete solution established in Lemma 3.7. These results will be used in the next section to pass to the limit $h \searrow 0$ in order to to prove Theorem 3.5.

4.1. The combined energy-entropy estimate

We start this section by demonstrating that our spatial semi-discretization (3.6) satisfies a combined energy-entropy estimate as long as the energy remains below the critical threshold energy $\mathcal{E}_{\max,h}$ which becomes infinite for $h \searrow 0$. Due to the cut-off mechanism implemented in (3.6), it is possible to extend the results to $[0, T_{\max}]$.

Writing $u_h(x, y, t)$ as

$$\sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} u_{ij}(t) e_i^x(x) e_j^y(y)$$

and choosing $\psi_h(x, y) = e_i^x(x) e_j^y(y)$ in (3.6a) gives

$$\begin{aligned}
& \mathrm{d}u_{ij} + \chi_{T_h} M_{ij}^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \big\{ [G''(u_h)]_x^{-1} \partial_x p_h \partial_x (e_i^x(x) e_j^y(y)) \big\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \chi_{T_h} M_{ij}^{-1} \int_{\mathcal{O}} \mathcal{I}_h^x \big\{ [G''(u_h)]_y^{-1} \partial_y p_h \partial_y (e_i^x(x) e_j^y(y)) \big\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad - \chi_{T_h} M_{ij}^{-1} \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \big\{ \mathcal{I}_{h,\mathrm{loc}}^x \{ \partial_x (u_h \widetilde{\mathfrak{g}}_{h,kl}) e_i^x(x) e_j^y(y) \} \big\} \,\mathrm{d}x\,\mathrm{d}y \;\mathrm{d}\beta_{kl}^x \\
&\quad - \chi_{T_h} M_{ij}^{-1} \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^y \int_{\mathcal{O}} \mathcal{I}_h^x \big\{ \mathcal{I}_{h,\mathrm{loc}}^y \{ \partial_y (u_h \widetilde{\mathfrak{g}}_{h,kl}) e_i^x(x) e_j^y(y) \} \big\} \,\mathrm{d}x\,\mathrm{d}y \;\mathrm{d}\beta_{kl}^y \;= 0
\end{aligned} \tag{4.1}$$

with $M_{ij} = \int_{\mathcal{O}} e_i^x(x) e_j^y(y) \,\mathrm{d}x\,\mathrm{d}y$. As we assume the subdivision to be equidistant, we have $M_{ij} = h_x h_y$ for all $i \in \{1, \dots, \dim U_h^x\}$ and $j \in \{1, \dots, \dim U_h^y\}$.

Furthermore, we define for $i \in \{1, \dots, \dim U_h^x\}$ and $j \in \{1, \dots, \dim U_h^y\}$

$$\begin{aligned}
L_{ij}(t) := & -\chi_{T_h} M_{ij}^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \big\{ [G''(u_h)]_x^{-1} \partial_x p_h \partial_x (e_i^x(x) e_j^y(y)) \big\} \,\mathrm{d}x\,\mathrm{d}y \\
& - \chi_{T_h} M_{ij}^{-1} \int_{\mathcal{O}} \mathcal{I}_h^x \big\{ [G''(u_h)]_y^{-1} \partial_y p_h \partial_y (e_i^x(x) e_j^y(y)) \big\} \,\mathrm{d}x\,\mathrm{d}y,
\end{aligned} \tag{4.2}$$

$$Z_{ij}^x(\omega) := \chi_{T_h} M_{ij}^{-1} \sum_{k,l \in \mathbb{Z}} \int_{\mathcal{O}} \mathcal{I}_h^y \big\{ \mathcal{I}_{h,\mathrm{loc}}^x \{ \partial_x ((\mathfrak{g}_{kl}, \omega)_{L^2} u_h \widetilde{\mathfrak{g}}_{h,kl}) e_i^x(x) e_j^y(y) \} \big\} \,\mathrm{d}x\,\mathrm{d}y, \tag{4.3}$$

$$Z_{ij}^y(\omega) := \chi_{T_h} M_{ij}^{-1} \sum_{k,l\in\mathbb{Z}} \int_{\mathcal{O}} \mathcal{I}_h^x\{\mathcal{I}_{h,\mathrm{loc}}^y\{\partial_y(\langle \mathfrak{g}_{kl}, \omega\rangle_{L^2} u_h \widetilde{\mathfrak{g}}_{h,kl}) \mathrm{e}_i^x(x)\mathrm{e}_j^y(y)\}\}\,\mathrm{d}x\,\mathrm{d}y. \tag{4.4}$$

Here, $\langle\cdot,\cdot\rangle_{L^2}$ denotes the standard $L^2(\mathcal{O})$ inner product. With this notation we may rewrite (4.1) as

$$\mathrm{d}u_{ij} = L_{ij}(t)\,\mathrm{d}t + \sum_{k\in I_h^x, l\in I_h^y} \big(Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl})\,\mathrm{d}\beta_{kl}^x + Z_{ij}^y(\lambda_{kl}^y \mathfrak{g}_{kl})\,\mathrm{d}\beta_{kl}^y\big). \tag{4.5}$$

For given positive parameters α and κ, we consider the integral quantity

$$R_{\alpha,\kappa,h}(u_h(t)) := \alpha + \mathcal{E}_h(u_h(t)) + \kappa \mathcal{S}_h(u_h(t)). \tag{4.6}$$

For the ease of presentation, we will often drop the explicit dependence on u_h and use the abbreviation

$$R(t) := R(u_h(t)) := R_{\alpha,\kappa,h}(u_h(t)).$$

Lemma 4.1. *Let $\overline{p} \geq 1$ be given. The first and second variations of $R(s)^{\overline{p}}$ with respect to u_h are given by*

$$D\big(R(u_h(s))^{\overline{p}}\big) = \overline{p} R(s)^{\overline{p}-1}\big(D\mathcal{E}_h(u_h(s)) + \kappa D\mathcal{S}_h(u_h(s))\big) \tag{4.7a}$$

and

$$\begin{aligned} D^2(R(u_h(s))^{\overline{p}}) = {} & \overline{p} R(s)^{\overline{p}-1}\big(D^2\mathcal{E}_h(u_h(s)) + \kappa D^2\mathcal{S}_h(u_h(s))\big) \\ & + \overline{p}(\overline{p}-1) R(s)^{\overline{p}-2}\big(D\mathcal{E}_h(u_h(s)) + \kappa D\mathcal{S}_h(u_h(s))\big) \\ & \otimes \big(D\mathcal{E}_h(u_h(s)) + \kappa D\mathcal{S}_h(u_h(s))\big) \end{aligned} \tag{4.7b}$$

with

$$\begin{aligned} D\mathcal{E}_h(u_h(s))\psi_h = {} & \int_{\mathcal{O}} \mathcal{I}_h^y\{\partial_x u_h \partial_x \psi_h\} + \mathcal{I}_h^x\{\partial_y u_h \partial_y \psi_h\}\,\mathrm{d}x\,\mathrm{d}y \\ & + \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{F'(u_h)\psi_h\}\,\mathrm{d}x\,\mathrm{d}y \\ & + h^{\varepsilon} \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{\Delta_h u_h \Delta_h \psi_h\}\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{4.8}$$

$$\begin{aligned} D^2\mathcal{E}_h(u_h(s))(\phi_h,\psi_h) = {} & \int_{\mathcal{O}} \mathcal{I}_h^y\{\partial_x \phi_h \partial_x \psi_h\} + \mathcal{I}_h^x\{\partial_y \phi_h \partial_y \psi_h\}\,\mathrm{d}x\,\mathrm{d}y \\ & + \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{F''(u_h)\phi_h\psi_h\}\,\mathrm{d}x\,\mathrm{d}y \\ & + h^{\varepsilon} \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{\Delta_h \phi_h \Delta_h \psi_h\}\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{4.9}$$

$$D\mathcal{S}_h(u_h(s))\psi_h = \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{G'(u_h)\psi_h\}\,\mathrm{d}x\,\mathrm{d}y, \tag{4.10}$$

$$D^2\mathcal{S}_h(u_h(s))(\phi_h,\psi_h) = \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{G''(u_h)\phi_h\psi_h\}\,\mathrm{d}x\,\mathrm{d}y. \tag{4.11}$$

Applying Itô's formula, we are able to show the following combined energy-entropy estimate:

Proposition 4.2. *Let $\overline{p} \geq 1$ be arbitrary and let (u_h, p_h) be a solution to (3.6) for a parameter $h \in (0,1)$. Furthermore, let Assumptions (B), (B3*), (I), (P), (R), and (S) hold true. Then, for sufficiently large α and κ depending only on $(\lambda^x_{kl})_{kl}$, $(\lambda^y_{kl})_{kl}$, $\overline{p}$, and $T_{\max}$, there exists a positive, h-independent constant C such that*

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T_{\max}]} R(t)^{\overline{p}}\Big] + \mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}\int_{\mathcal{O}} \mathcal{I}_h^y\big\{[G''(u_h)]_x^{-1}|\partial_x p_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]\\
&\quad+\mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}\int_{\mathcal{O}} \mathcal{I}_h^x\big\{[G''(u_h)]_y^{-1}|\partial_y p_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]\\
&\quad+\mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}\|\Delta_h u_h\|_h^2\,\mathrm{d}s\Big]\\
&\quad+\mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}h^{\varepsilon}\int_{\mathcal{O}} \mathcal{I}_h^y\big\{|\partial_x \Delta_h u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]\\
&\quad+\mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}h^{\varepsilon}\int_{\mathcal{O}} \mathcal{I}_h^x\big\{|\partial_y \Delta_h u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]\\
&\quad+\mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}\int_{\mathcal{O}} \mathcal{I}_h^y\big\{[|u_h|^{-p-2}]_x|\partial_x u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]\\
&\quad+\mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}\int_{\mathcal{O}} \mathcal{I}_h^x\big\{[|u_h|^{-p-2}]_y|\partial_y u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big] \leq C.
\end{aligned}
\tag{4.12}
$$

Proof. Using the notation

$$
\varphi_h(t) := \varphi_h(x,y,t) := \sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y} L_{ij}(t)\mathrm{e}_i^x(x)\mathrm{e}_j^y(y) \tag{4.13}
$$

and

$$
\begin{aligned}
\Phi_h(t)((\omega_x,\omega_y)^T) &:= \Phi_h(x,y,t)((\omega_x,\omega_y)^T)\\
&:= \sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}\big(Z_{ij}^x(\omega_x)+Z_{ij}^y(\omega_y)\big)\mathrm{e}_i^x(x)\mathrm{e}_j^y(y),
\end{aligned}
\tag{4.14}
$$

we may rewrite (3.6) as

$$
\mathrm{d}u_h = \varphi_h(t)\,\mathrm{d}t + \Phi_h(t)(\mathrm{d}\boldsymbol{W}_{Q,h}) \tag{4.15}
$$

with

$$
\boldsymbol{W}_{Q,h} := \sum_{\alpha\in\{x,y\}}\sum_{k\in I_h^x,\, l\in I_h^y} \lambda_{kl}^{\alpha}\mathfrak{g}_{kl}\beta_{kl}^{\alpha}\boldsymbol{b}_{\alpha}, \tag{4.16}
$$

where $\boldsymbol{b}_x, \boldsymbol{b}_y$ denote the standard Cartesian basis vectors of $\mathbb{R}^2$. Applying Itô's formula, we compute

$$
\begin{aligned}
R(t \wedge T_h)^{\overline{p}} = {} & R(0)^{\overline{p}} + \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} (D\mathcal{E}_h + \kappa D\mathcal{S}_h)\varphi_h(s)\,\mathrm{d}s \\
& + \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} (D\mathcal{E}_h + \kappa D\mathcal{S}_h)\Phi_h(s)\mathrm{d}\boldsymbol{W}_{Q,h} \\
& + \frac{1}{2} \sum_{k\in I_h^x, l\in I_h^y} \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} (D^2\mathcal{E}_h + \kappa D^2\mathcal{S}_h)(\Phi^x_{h,kl}, \Phi^x_{h,kl})\,\mathrm{d}s \\
& + \frac{1}{2} \sum_{k\in I_h^x, l\in I_h^y} \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} (D^2\mathcal{E}_h + \kappa D^2\mathcal{S}_h)(\Phi^y_{h,kl}, \Phi^y_{h,kl})\,\mathrm{d}s \\
& + \frac{1}{2} \sum_{k\in I_h^x, l\in I_h^y} \int_0^{t\wedge T_h} \overline{p}(\overline{p}-1) R(s)^{\overline{p}-2} (D\mathcal{E}_h + \kappa D\mathcal{S}_h) \\
& \qquad \otimes (D\mathcal{E}_h + \kappa D\mathcal{S}_h)(\Phi^x_{h,kl}, \Phi^x_{h,kl})\,\mathrm{d}s \\
& + \frac{1}{2} \sum_{k\in I_h^x, l\in I_h^y} \int_0^{t\wedge T_h} \overline{p}(\overline{p}-1) R(s)^{\overline{p}-2} (D\mathcal{E}_h + \kappa D\mathcal{S}_h) \\
& \qquad \otimes (D\mathcal{E}_h + \kappa D\mathcal{S}_h)(\Phi^y_{h,kl}, \Phi^y_{h,kl})\,\mathrm{d}s \\
=: {} & R(0)^{\overline{p}} + I + II + III + IV + V + VI,
\end{aligned}
\tag{4.17}
$$

where we used the abbreviations

$$
\Phi^x_{h,kl}(s) := \Phi_h(s)((\lambda^x_{kl}\mathfrak{g}_{kl}, 0)^T) = \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z^x_{ij}(\lambda^x_{kl}\mathfrak{g}_{kl}) e^x_i(x) e^y_j(y) \tag{4.18a}
$$

and

$$
\Phi^y_{h,kl}(s) := \Phi_h(s)((0, \lambda^y_{kl}\mathfrak{g}_{kl})^T) = \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z^y_{ij}(\lambda^y_{kl}\mathfrak{g}_{kl}) e^x_i(x) e^y_j(y). \tag{4.18b}
$$

Using Lemma 4.1, we compute

$$
\begin{aligned}
I = {} & \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}} \mathcal{I}_h^y \{\partial_x u_h \partial_x e^x_i(x) e^y_j(y)\}\,\mathrm{d}x\,\mathrm{d}y L_{ij}(s)\,\mathrm{d}s \\
& + \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}} \mathcal{I}_h^x \{\partial_y u_h \partial_y e^y_j(y) e^x_i(x)\}\,\mathrm{d}x\,\mathrm{d}y L_{ij}(s)\,\mathrm{d}s \\
& + \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{F'(u_h) e^x_i(x) e^y_j(y)\}\,\mathrm{d}x\,\mathrm{d}y L_{ij}(s)\,\mathrm{d}s
\end{aligned}
$$

$$+ h^\varepsilon \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1}$$

$$\cdot \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}} \mathcal{I}_h^{xy}\left\{\Delta_h u_h \Delta_h (e_i^x(x) e_j^y(y))\right\} \mathrm{d}x\, \mathrm{d}y L_{ij}(s)\, \mathrm{d}s$$

$$+ \kappa \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}} \mathcal{I}_h^{xy}\left\{G'(u_h) e_i^x(x) e_j^y(y)\right\} \mathrm{d}x\, \mathrm{d}y L_{ij}(s)\, \mathrm{d}s$$

$$=: I_a + I_b + I_c + I_d + I_e. \tag{4.19}$$

From (4.2), we obtain by straightforward computations

$$\int_{\mathcal{O}} \mathcal{I}_h^{xy}\left\{w \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} L_{ij}(s) e_i^x(x) e_j^y(y)\right\} \mathrm{d}x\, \mathrm{d}y$$

$$= \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} M_{ij} L_{ij} w\big((i-1)h_x, (j-1)h_y\big)$$

$$= -\chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^y\left\{[G''(u_h)]_x^{-1} \partial_x p_h \partial_x \mathcal{I}_h^{xy}\{w\}\right\}$$

$$+ \mathcal{I}_h^x\left\{[G''(u_h)]_y^{-1} \partial_y p_h \partial_y \mathcal{I}_h^{xy}\{w\}\right\} \mathrm{d}x\, \mathrm{d}y \tag{4.20}$$

for $w \in C_{\text{per}}(\overline{\mathcal{O}})$ and therefore, after integration by parts and using (3.6b),

$$I_a + I_b + I_c + I_d$$

$$= \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{(-\Delta_h u_h + F'(u_h) + h^\varepsilon \Delta_h \Delta_h u_h)$$

$$\times L_{ij}(s) e_i^x(x) e_j^y(y)\} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s$$

$$= \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}} \mathcal{I}_h^{xy}\left\{p_h L_{ij}(s) e_i^x(x) e_j^y(y)\right\} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s$$

$$= -\overline{p} \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y\left\{[G''(u_h)]_x^{-1} |\partial_x p_h|^2\right\}$$

$$+ \mathcal{I}_h^x\left\{[G''(u_h)]_y^{-1} |\partial_y p_h|^2\right\} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s. \tag{4.21}$$

Similarly, we use (4.20) with $w = G'(u_h)$ and (2.16) to compute

$$\int_{\mathcal{O}} \mathcal{I}_h^{xy}\left\{G'(u_h) \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} L_{ij}(s) e_i^x(x) e_j^y(y)\right\} \mathrm{d}x\, \mathrm{d}y$$

$$
\begin{aligned}
&= -\chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \big\{[G''(u_h)]_x^{-1} \partial_x p_h \partial_x \mathcal{I}_h^{xy}\{G'(u_h)\}\big\} \\
&\qquad + \mathcal{I}_h^x \big\{[G''(u_h)]_y^{-1} \partial_y u_h \partial_y \mathcal{I}_h^{xy}\{G'(u_h)\}\big\} \,\mathrm{d}x\,\mathrm{d}y \\
&= -\chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \{\partial_x u_h \partial_x p_h\} + \mathcal{I}_h^x \{\partial_y u_h \partial_y p_h\} \,\mathrm{d}x\,\mathrm{d}y \\
&= \chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{\Delta_h u_h p_h\} \,\mathrm{d}x\,\mathrm{d}y \\
&= -\chi_{T_h} \|\Delta_h u_h\|_h^2 - \chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \big\{\partial_x u_h \partial_x \mathcal{I}_h^{xy}\{F'(u_h)\}\big\} \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + \chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \big\{\partial_y u_h \partial_y \mathcal{I}_h^{xy}\{F'(u_h)\}\big\} \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + \chi_{T_h} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{\Delta_h u_h \Delta_h \Delta_h u_h\} \,\mathrm{d}x\,\mathrm{d}y \\
&= -\chi_{T_h} \|\Delta_h u_h\|_h^2 - \chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \big\{[F''(u_h)]_x |\partial_x u_h|^2\big\} \\
&\qquad + \mathcal{I}_h^x \big\{[F''(u_h)]_y |\partial_y u_h|^2\big\} \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad - \chi_{T_h} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \big\{|\partial_x \Delta_h u_h|^2\big\} + \mathcal{I}_h^x \big\{|\partial_y \Delta_h u_h|^2\big\} \,\mathrm{d}x\,\mathrm{d}y. \qquad (4.22)
\end{aligned}
$$

From Assumption (P), we obtain the estimates $[F''(u_h)]_x \geq \widetilde{c}_1[|u_h|^{-p-2}]_x - \widetilde{c}_2$ and $[F''(u_h)]_y \geq \widetilde{c}_1[|u_h|^{-p-2}]_y - \widetilde{c}_2$. Therefore, combining the above estimates we conclude

$$
\begin{aligned}
I_e \leq -\kappa \overline{p} \int_0^{t\wedge T_h} & R(s)^{\overline{p}-1} \|\Delta_h u_h\|_h^2 \,\mathrm{d}s \\
& - \kappa \overline{p} h^\varepsilon \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \big\{|\partial_x \Delta_h u_h|^2\big\} + \mathcal{I}_h^x \big\{|\partial_y \Delta_h u_h|^2\big\} \,\mathrm{d}x\,\mathrm{d}y \\
& - \widetilde{c}_1 \kappa \overline{p} \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \big\{[|u_h|^{-p-2}]_x |\partial_x u_h|^2\big\} \\
& \qquad + \mathcal{I}_h^x \big\{[|u_h|^{-p-2}]_y |\partial_y u_h|^2\big\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
& + \widetilde{c}_2 \kappa \overline{p} \int_0^{t\wedge T_h} R(s)^{\overline{p}} \,\mathrm{d}s. \qquad (4.23)
\end{aligned}
$$

Noting that

$$
\begin{aligned}
D^2 \mathcal{E}_h(\Phi_{h,kl}^x, \Phi_{h,kl}^x) &= \int_{\mathcal{O}} \mathcal{I}_h^y \Big\{\Big| \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) \partial_x e_i^x(x) e_j^y(y) \Big|^2\Big\} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathcal{O}} \mathcal{I}_h^x \Big\{\Big| \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) e_i^x(x) \partial_y e_j^y(y) \Big|^2\Big\} \,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

$$+\int_{\mathcal{O}} \mathcal{I}_h^{xy}\Big\{F''(u_h)\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}|Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl})|^2 e_i^x(x)e_j^y(y)\Big\}\,\mathrm{d}x\,\mathrm{d}y$$

$$+h^\varepsilon\int_{\mathcal{O}} \mathcal{I}_h^{xy}\Big\{\Big|\Delta_h\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl})e_i^x(x)e_j^y(y)\Big|^2\Big\}\,\mathrm{d}x\,\mathrm{d}y, \tag{4.24}$$

$$D^2\mathcal{S}_h(\Phi_{h,kl}^x,\Phi_{h,kl}^x) = \int_{\mathcal{O}} \mathcal{I}_h^{xy}\Big\{G''(u_h)\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}|Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl})|^2 e_i^x(x)e_j^y(y)\Big\}\,\mathrm{d}x\,\mathrm{d}y, \tag{4.25}$$

and similar identities for $D^2\mathcal{E}_h(\Phi_{h,kl}^y,\Phi_{h,kl}^y)$ and $D^2\mathcal{S}_h(\Phi_{h,kl}^y,\Phi_{h,kl}^y)$, we combine *III* and *IV* and obtain after reordering

$$\frac{1}{2}\sum_{k\in I_h^x,\,l\in I_h^y}\int_0^{t\wedge T_h}\overline{p}R(s)^{\overline{p}-1}\int_{\mathcal{O}}\mathcal{I}_h^y\Big\{\Big|\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl})\partial_x e_i^x(x)e_j^y(y)\Big|^2\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$

$$+\frac{1}{2}\sum_{k\in I_h^x,\,l\in I_h^y}\int_0^{t\wedge T_h}\overline{p}R(s)^{\overline{p}-1}$$

$$\int_{\mathcal{O}}\mathcal{I}_h^x\Big\{\Big|\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl})\partial_y e_j^y(y)e_i^x(x)\Big|^2\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$

$$+\frac{1}{2}\sum_{k\in I_h^x,\,l\in I_h^y}\int_0^{t\wedge T_h}\overline{p}R(s)^{\overline{p}-1}$$

$$\int_{\mathcal{O}}\mathcal{I}_h^y\Big\{\Big|\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}Z_{ij}^y(\lambda_{kl}^y \mathfrak{g}_{kl})\partial_x e_i^x(x)e_j^y(y)\Big|^2\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$

$$+\frac{1}{2}\sum_{k\in I_h^x,\,l\in I_h^y}\int_0^{t\wedge T_h}\overline{p}R(s)^{\overline{p}-1}$$

$$\int_{\mathcal{O}}\mathcal{I}_h^x\Big\{\Big|\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}Z_{ij}^y(\lambda_{kl}^y \mathfrak{g}_{kl})\partial_y e_j^y(y)e_i^x(x)\Big|^2\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$

$$+\frac{1}{2}\sum_{k\in I_h^x,\,l\in I_h^y}\int_0^{t\wedge T_h}\overline{p}R(s)^{\overline{p}-1}$$

$$\int_{\mathcal{O}}\mathcal{I}_h^{xy}\Big\{F''(u_h)\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}|Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl})|^2 e_i^x(x)e_j^y(y)\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$

$$+\frac{1}{2}\sum_{k\in I_h^x,\,l\in I_h^y}\int_0^{t\wedge T_h}\overline{p}R(s)^{\overline{p}-1}$$

$$\int_{\mathcal{O}}\mathcal{I}_h^{xy}\Big\{F''(u_h)\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}|Z_{ij}^y(\lambda_{kl}^y\mathfrak{g}_{kl})|^2 e_i^x(x)e_j^y(y)\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$

$$+\frac{1}{2}h^{\varepsilon}\sum_{k\in I_h^x,\,l\in I_h^y}\int_0^{t\wedge T_h}\overline{p}R(s)^{\overline{p}-1}$$

$$\int_{\mathcal{O}}\mathcal{I}_h^{xy}\Big\{\Big|\Delta_h\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}Z_{ij}^x(\lambda_{kl}^x\mathfrak{g}_{kl})e_i^x(x)e_j^y(y)\Big|^2\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$

$$+\frac{1}{2}h^{\varepsilon}\sum_{k\in I_h^x,\,l\in I_h^y}\int_0^{t\wedge T_h}\overline{p}R(s)^{\overline{p}-1}$$

$$\int_{\mathcal{O}}\mathcal{I}_h^{xy}\Big\{\Big|\Delta_h\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}Z_{ij}^y(\lambda_{kl}^y\mathfrak{g}_{kl})e_i^x(x)e_j^y(y)\Big|^2\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$

$$+\frac{1}{2}\kappa\sum_{k\in I_h^x,\,l\in I_h^y}\int_0^{t\wedge T_h}\overline{p}R(s)^{\overline{p}-1}$$

$$\int_{\mathcal{O}}\mathcal{I}_h^{xy}\Big\{G''(u_h)\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}|Z_{ij}^x(\lambda_{kl}^x\mathfrak{g}_{kl})|^2 e_i^x(x)e_j^y(y)\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$

$$+\frac{1}{2}\kappa\sum_{k\in I_h^x,\,l\in I_h^y}\int_0^{t\wedge T_h}\overline{p}R(s)^{\overline{p}-1}$$

$$\int_{\mathcal{O}}\mathcal{I}_h^{xy}\Big\{G''(u_h)\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}|Z_{ij}^y(\lambda_{kl}^y\mathfrak{g}_{kl})|^2 e_i^x(x)e_j^y(y)\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$

$$= III_a + III_b + III_c + III_d + III_e + III_f + III_g + III_h + III_i + III_j\,. \tag{4.26}$$

To derive an estimate for III_a, we adapt the ideas of [29]. Using the periodicity, the specific form of the one-dimensional stiffness matrix on equidistant meshes, and the fact that $M_{ij} = h_x h_y$ for all $i = 1,\ldots,\dim U_h^x$ and $j = 1,\ldots,\dim U_h^y$, we compute using (2.13a) and (2.13b)

$$\int_{\mathcal{O}}\mathcal{I}_h^y\Big\{\Big|\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y}Z_{ij}^x(\lambda_{kl}^x\mathfrak{g}_{kl})\partial_x e_i^x(x)e_j^y(y)\Big|^2\Big\}\,\mathrm{d}x\,\mathrm{d}y$$

$$=\int_{\mathcal{O}^y}\sum_{j=1}^{\dim U_h^y}\int_{\mathcal{O}^x}\Big|\sum_{i=1}^{\dim U_h^x}Z_{ij}^x(\lambda_{kl}^x\mathfrak{g}_{kl})\partial_x e_i^x(x)\Big|^2\,\mathrm{d}x\;e_j^y(y)\,\mathrm{d}y$$

$$
\begin{aligned}
&= \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}^y} \big(2(Z_{ij}^x)^2 - Z_{i-1,j}^x Z_{ij}^x - Z_{ij}^x Z_{i+1,j}^x\big)(\lambda_{kl}^x \mathfrak{g}_{kl}) e_j^y(y)\,\mathrm{d}y h_x^{-1} \\
&= \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}^y} (Z_{i+1,j}^x - Z_{ij}^x)^2 (\lambda_{kl}^x \mathfrak{g}_{kl}) e_j^y(y)\,\mathrm{d}y\, h_x^{-1} \\
&= \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \Big\{ \mathcal{I}_{h,\mathrm{loc}}^x \Big\{ \partial_x (\lambda_{kl}^x u_h \widetilde{\mathfrak{g}}_{h,kl}) \frac{e_{i+1}^x(x) - e_i^x(x)}{h_x} \Big\} \\
&\qquad \cdot e_j^y(y) \Big\}\,\mathrm{d}x\,\mathrm{d}y \Big)^2 h_x^{-1} h_y^{-1} \\
&\leq 2\chi_{T_h} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \Big\{ \lambda_{kl}^x \partial_x u_h \mathcal{I}_h^x \Big\{ \widetilde{\mathfrak{g}}_{h,kl} \frac{e_{i+1}^x(x) - e_i^x(x)}{h_x} \Big\} \\
&\qquad \cdot e_j^y(y) \Big\}\,\mathrm{d}x\,\mathrm{d}y \Big)^2 h_x^{-1} h_y^{-1} \\
&\quad + 2\chi_{T_h} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \Big\{ \lambda_{kl}^x \partial_x \widetilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \Big\{ u_h \frac{e_{i+1}^x(x) - e_i^x(x)}{h_x} \Big\} \\
&\qquad \cdot e_j^y(y) \Big\}\,\mathrm{d}x\,\mathrm{d}y \Big)^2 h_x^{-1} h_y^{-1} \\
&= (*).
\end{aligned}
\tag{4.27}
$$

Recalling $e_{i+1}^x(x) = e_i^x(x - h_x)$ and performing a discrete integration by parts (cf. Lemma A.6 in Appendix A), we continue with the estimate

$$
\begin{aligned}
(*) \leq{}& \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \Big(\lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \mathcal{I}_h^x \{ \partial_x^{-h_x} \widetilde{\mathfrak{g}}_{h,kl} e_{i+1}^x(x) \} e_j^y(y) \}\,\mathrm{d}x\,\mathrm{d}y \Big)^2 h_x^{-1} h_y^{-1} \\
&+ \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \Big(\lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x^{+h_x} \partial_x u_h \mathcal{I}_h^x \{ \widetilde{\mathfrak{g}}_{h,kl} e_i^x(x) \} \\
&\qquad \cdot e_j^y(y) \}\,\mathrm{d}x\,\mathrm{d}y \Big)^2 h_x^{-1} h_y^{-1} \\
&+ \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \Big(\lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x \widetilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \{ \partial_x^{-h_x} u_h e_{i+1}^x(x) \} \\
&\qquad \cdot e_j^y(y) \}\,\mathrm{d}x\,\mathrm{d}y \Big)^2 h_x^{-1} h_y^{-1} \\
&+ \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \Big(\lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x^{+h_x} \partial_x \widetilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \{ u_h e_i^x(x) \} \\
&\qquad \cdot e_j^y(y) \}\,\mathrm{d}x\,\mathrm{d}y \Big)^2 h_x^{-1} h_y^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\le \chi_{T_h} C {\lambda^x_{kl}}^2 \|\partial_x \mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})} \int_{\mathcal{O}} \mathcal{I}^y_h \{|\partial_x u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \chi_{T_h} C {\lambda^x_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})} \int_{\mathcal{O}} \mathcal{I}^y_h \{|\Delta^x_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \chi_{T_h} C {\lambda^x_{kl}}^2 \|\partial_x \mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})} \int_{\mathcal{O}} \mathcal{I}^{xy}_h \{|\partial^{-h_x}_x u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \chi_{T_h} C {\lambda^x_{kl}}^2 \|\partial^{+h_x}_x \partial_x \tilde{\mathfrak{g}}_{h,kl}\|^2_{L^\infty(\mathcal{O})} \int_{\mathcal{O}} \mathcal{I}^{xy}_h \{u_h^2\} \,\mathrm{d}x\,\mathrm{d}y \\
&\le \chi_{T_h} C {\lambda^x_{kl}}^2 \|\partial_x \partial_x \mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})} R(s) \\
&\quad + \chi_{T_h} C {\lambda^x_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})} \int_{\mathcal{O}} \mathcal{I}^{xy}_h \{|\Delta^x_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y. \qquad (4.28)
\end{aligned}
$$

In the final step, we used Poincaré's inequality and the pathwise conservation of $\int_{\mathcal{O}} u_h \,\mathrm{d}x\,\mathrm{d}y$ (see Remark 3.3, Assumption (I), and the norm equivalence (2.6)), as well as

$$
\begin{aligned}
\left\|\partial^{+h_x}_x \partial_x \mathcal{I}^x_h\{\mathfrak{g}_{kl}\}\right\|_{L^\infty(\mathcal{O})} &= \left\|\partial_x \mathcal{I}^x_h\{\partial^{+h_x}_x \mathfrak{g}_{kl}\}\right\|_{L^\infty(\mathcal{O})} \le \left\|\partial_x \partial^{+h_x}_x \mathfrak{g}_{kl}\right\|_{L^\infty(\mathcal{O})} \\
&\le \|\partial_x \partial_x \mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})} \le \|\mathfrak{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}, \qquad (4.29)
\end{aligned}
$$

which follows from Assumption (S) and the stability of the nodal interpolation operator. This provides

$$
\begin{aligned}
III_a &\le C\overline{p} \sum_{k\in I^x_h,\, l\in I^y_h} {\lambda^x_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{W^{2,\infty}(\mathcal{O})} \int_0^{t\wedge T_h} R(s)^{\overline{p}} \,\mathrm{d}s \\
&\quad + C\overline{p} \sum_{k\in I^x_h,\, l\in I^y_h} {\lambda^x_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})} \\
&\qquad \cdot \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}^{xy}_h \{|\Delta^x_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \qquad (4.30)
\end{aligned}
$$

Similar computations show

$$
\begin{aligned}
III_d &\le C\overline{p} \sum_{k\in I^x_h,\, l\in I^y_h} {\lambda^y_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{W^{2,\infty}(\mathcal{O})} \int_0^{t\wedge T_h} R(s)^{\overline{p}} \,\mathrm{d}s \\
&\quad + C\overline{p} \sum_{k\in I^x_h,\, l\in I^y_h} {\lambda^y_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})} \\
&\qquad \cdot \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}^{xy}_h \{|\Delta^y_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \qquad (4.31)
\end{aligned}
$$

To control III_b, we compute

$$
\int_{\mathcal{O}} \mathcal{I}^x_h \Big\{ \Big| \sum_{i=1}^{\dim U^x_h} \sum_{j=1}^{\dim U^y_h} Z^x_{ij} (\lambda^x_{kl} \mathfrak{g}_{kl}) \partial_y e^y_j(y) e^x_i(x) \Big|^2 \Big\} \,\mathrm{d}x\,\mathrm{d}y
$$

$$
\begin{aligned}
&= \int_{\mathcal{O}^x} \mathcal{I}_h^x \Big\{ \sum_{i=1}^{\dim U_h^x} \int_{\mathcal{O}^y} \Big| \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) \partial_y \mathrm{e}_j^y(y) \Big|^2 \mathrm{d}y\, \mathrm{e}_i^x(x) \Big\} \,\mathrm{d}x \\
&= \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} (Z_{i,j+1}^x - Z_{i,j}^x)^2 (\lambda_{kl}^x \mathfrak{g}_{kl}) h_x h_y^{-1} \\
&= \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \Big\{ \mathcal{I}_{h,\mathrm{loc}}^y \Big\{ \partial_x (\lambda_{kl}^x \widetilde{\mathfrak{g}}_{h,kl} u_h) \mathrm{e}_i^x(x) \\
&\qquad \cdot \frac{\mathrm{e}_{j+1}^y(y) - \mathrm{e}_j^y(y)}{h_y} \Big\} \Big\} \,\mathrm{d}x\,\mathrm{d}y \Big)^2 h_x^{-1} h_y^{-1} \\
&\leq 2\chi_{T_h} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \Big(\lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \Big\{ \partial_x \widetilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \{ u_h \mathrm{e}_i^x(x) \} \\
&\qquad \cdot \frac{\mathrm{e}_{j+1}^y(y) - \mathrm{e}_j^y(y)}{h_y} \Big\} \,\mathrm{d}x\,\mathrm{d}y \Big)^2 h_x^{-1} h_y^{-1} \\
&\quad + 2\chi_{T_h} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \Big(\lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \Big\{ \partial_x u_h \mathcal{I}_h^x \{ \widetilde{\mathfrak{g}}_{h,kl} \mathrm{e}_i^x(x) \} \\
&\qquad \cdot \frac{\mathrm{e}_{j+1}^y(y) - \mathrm{e}_j^y(y)}{h_y} \Big\} \,\mathrm{d}x\,\mathrm{d}y \Big)^2 h_x^{-1} h_y^{-1} \\
&\leq \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \Big(\lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \mathcal{I}_h^x \{ \partial_y^{-h_y} \widetilde{\mathfrak{g}}_{h,kl} \mathrm{e}_i^x(x) \} \\
&\qquad \cdot \mathrm{e}_{j+1}^y(y) \} \,\mathrm{d}x\,\mathrm{d}y \Big)^2 h_x^{-1} h_y^{-1} \\
&\quad + \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \Big(\lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_y^{+h_y} \partial_x u_h \mathcal{I}_h^x \{ \widetilde{\mathfrak{g}}_{h,kl} \mathrm{e}_i^x(x) \} \\
&\qquad \cdot \mathrm{e}_j^y(y) \} \,\mathrm{d}x\,\mathrm{d}y \Big)^2 h_x^{-1} h_y^{-1} \\
&\quad + \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \Big(\lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x \widetilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \{ \partial_y^{-h_y} u_h \mathrm{e}_i^x(x) \} \\
&\qquad \cdot \mathrm{e}_{j+1}^y(y) \} \,\mathrm{d}x\,\mathrm{d}y \Big)^2 h_x^{-1} h_y^{-1} \\
&\quad + \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \Big(\lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_y^{+h_y} \partial_x \widetilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \{ u_h \mathrm{e}_i^x(x) \} \\
&\qquad \cdot \mathrm{e}_j^y(y) \} \,\mathrm{d}x\,\mathrm{d}y \Big)^2 h_x^{-1} h_y^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \chi_{T_h} C {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|^2_{W^{1,\infty}(\mathcal{O})} \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \chi_{T_h} C {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})} \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_y^{+h_y} \partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \chi_{T_h} C {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|^2_{W^{1,\infty}(\mathcal{O})} \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|\partial_y^{-h_y} u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \chi_{T_h} C {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|^2_{W^{1,\infty}(\mathcal{O})} \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned} \tag{4.32}
$$

where we again used (2.13a) and (2.13b). Noting that

$$
\begin{aligned}
\int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_y^{+h_y} \partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y &= \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{\Delta_h^x u_h \Delta_h^y u_h\}\,\mathrm{d}y\,\mathrm{d}y \\
&\leq \frac{1}{2}\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|\Delta_h^x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|\Delta_h^y u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|\Delta_h u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned} \tag{4.33}
$$

we obtain

$$
\begin{aligned}
III_b &\leq C\overline{p} \sum_{k\in I_h^x,\, l\in I_h^y} {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|^2_{W^{1,\infty}(\mathcal{O})} \int_0^{t\wedge T_h} R(s)^{\overline{p}}\,\mathrm{d}s \\
&\quad + C\overline{p} \sum_{k\in I_h^x,\, l\in I_h^y} {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})} \\
&\qquad \cdot \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|\Delta_h u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s
\end{aligned} \tag{4.34}
$$

and analogously

$$
\begin{aligned}
III_c &\leq C\overline{p} \sum_{k\in I_h^x,\, l\in I_h^y} {\lambda_{kl}^y}^2 \|\mathfrak{g}_{kl}\|^2_{W^{1,\infty}(\mathcal{O})} \int_0^{t\wedge T_h} R(s)^{\overline{p}}\,\mathrm{d}s \\
&\quad + C\overline{p} \sum_{k\in I_h^x,\, l\in I_h^y} {\lambda_{kl}^y}^2 \|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})} \\
&\qquad \cdot \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|\Delta_h u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned} \tag{4.35}
$$

We shall apply a similar strategy to deal with III_g and III_h. We start with the decomposition

$$
\int_{\mathcal{O}} \mathcal{I}_h^{xy}\left\{\left|\Delta_h \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) e_i^x(x) e_j^y(y)\right|^2\right\}\mathrm{d}x\,\mathrm{d}y
$$

$$\leq 2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \Big\{ \sum_{j=1}^{\dim U_h^y} \Big| \sum_{i=1}^{\dim U_h^x} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) \Delta_h^x \mathrm{e}_i^x(x) \Big|^2 \mathrm{e}_j^y(y) \Big\} \,\mathrm{d}x\,\mathrm{d}y$$

$$+ 2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \Big\{ \sum_{i=1}^{\dim U_h^x} \Big| \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) \Delta_h^y \mathrm{e}_j^y(y) \Big|^2 \mathrm{e}_i^x(x) \Big\} \,\mathrm{d}x\,\mathrm{d}y$$

$$=: III_{g_1} + III_{g_2}. \tag{4.36}$$

In order to control the first term, we use $\mathrm{e}_{i-1}^x(x) = \mathrm{e}_i^x(x + h_x)$ and $\mathrm{e}_{i+1}^x(x) = \mathrm{e}_i^x(x - h_x)$, apply Lemma A.6, and compute using (2.13a) and (2.13b),

$$\sum_{i=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) \Delta_h^x \mathrm{e}_i^x(x) = \sum_{i=1}^{\dim U_h^x} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) h_x^{-2} \big(\mathrm{e}_{i-1}^x(x) - 2\mathrm{e}_i^x(x) + \mathrm{e}_{i+1}^x(x)\big)$$

$$= \sum_{i=1}^{\dim U_h^x} h_x^{-2} \mathrm{e}_i^x(x) \big(-2Z_{ij}^x + Z_{i-1,j}^x + Z_{i+1,j}^x\big)(\lambda_{kl}^x \mathfrak{g}_{kl})$$

$$= \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \mathrm{e}_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \Big\{ \lambda_{kl}^x \partial_x \widetilde{\mathfrak{g}}_{h,kl}$$

$$\cdot \mathcal{I}_h^x \Big\{ u_h \frac{\mathrm{e}_{i+1}^x(x) - 2\mathrm{e}_i^x(x) + \mathrm{e}_{i-1}^x(x)}{h_x^2} \Big\} \mathrm{e}_j^y(y) \Big\} \,\mathrm{d}x\,\mathrm{d}y$$

$$+ \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \mathrm{e}_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \Big\{ \lambda_{kl}^x \partial_x u_h$$

$$\cdot \mathcal{I}_h^x \Big\{ \widetilde{\mathfrak{g}}_{h,kl} \frac{\mathrm{e}_{i+1}^x(x) - 2\mathrm{e}_i^x(x) + \mathrm{e}_{i-1}^x(x)}{h_x^2} \Big\} \mathrm{e}_j^y(y) \Big\} \,\mathrm{d}x\,\mathrm{d}y$$

$$= \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \mathrm{e}_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \lambda_{kl}^x \partial_x \Delta_h^x \widetilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \{ u_h(x - h_x, y) \mathrm{e}_i^x(x) \}$$

$$\cdot \mathrm{e}_j^y(y) \} \,\mathrm{d}x\,\mathrm{d}y$$

$$+ \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \mathrm{e}_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \lambda_{kl}^x \partial_x \widetilde{\mathfrak{g}}_{h,kl}(x + h_x, y) \mathcal{I}_h^x \{ \Delta_h^x u_h \mathrm{e}_i^x(x) \}$$

$$\cdot \mathrm{e}_j^y(y) \} \,\mathrm{d}x\,\mathrm{d}y$$

$$+ \chi_{T_h} 2 \sum_{i=1}^{\dim U_h^x} \mathrm{e}_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \lambda_{kl}^x \partial_x \partial_x^{+h_x} \widetilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \{ \partial_x^{-h_x} u_h \mathrm{e}_i^x(x) \}$$

$$\cdot \mathrm{e}_j^y(y) \} \,\mathrm{d}x\,\mathrm{d}y$$

$$
\begin{aligned}
&+ \chi_{T_h} \sum_{i=1}^{\dim U_h^x} e_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{\lambda_{kl}^x \partial_x \Delta_h^x u_h \mathcal{I}_h^x \{\tilde{\mathfrak{g}}_{h,kl}(x - h_x, y) e_i^x(x)\} \\
&\quad \cdot e_j^y(y)\} \,\mathrm{d}x \,\mathrm{d}y \\
&+ \chi_{T_h} \sum_{i=1}^{\dim U_h^x} e_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{\lambda_{kl}^x \partial_x u_h(x + h_x, y) \mathcal{I}_h^x \{\Delta_h^x \tilde{\mathfrak{g}}_{h,kl} e_i^x(x)\} \\
&\quad \cdot e_j^y(y)\} \,\mathrm{d}x \,\mathrm{d}y \\
&+ \chi_{T_h} 2 \sum_{i=1}^{\dim U_h^x} e_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{\lambda_{kl}^x \partial_x \partial_x^{+h_x} u_h \mathcal{I}_h^x \{\partial_x^{-h_x} \tilde{\mathfrak{g}}_{h,kl} e_i^x(x)\} \\
&\quad \cdot e_j^y(y)\} \,\mathrm{d}x \,\mathrm{d}y. \qquad (4.37)
\end{aligned}
$$

Similar to (4.29), we use Assumption (S) and the stability of the nodal interpolation operator to compute

$$
\begin{aligned}
\left\| \partial_x \Delta_h^x \mathcal{I}_h^{xy} \{\mathfrak{g}_{kl}\} \right\|_{L^\infty(\mathcal{O})} &= \left\| \partial_x \mathcal{I}_h^{xy} \{\partial_x^{+h_x} \partial_x^{-h_x} \mathfrak{g}_{kl}\} \right\|_{L^\infty(\mathcal{O})} \\
&\le \left\| \partial_x \partial_x^{+h_x} \partial_x^{-h_x} \mathfrak{g}_{kl} \right\|_{L^\infty(\mathcal{O})} \\
&\le \left\| \partial_x \partial_x \partial_x \mathfrak{g}_{kl} \right\|_{L^\infty(\mathcal{O})} \le \|\mathfrak{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}. \qquad (4.38)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
III_{g_1} \le{}& \chi_{T_h} C {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{|u_h|^2\} \,\mathrm{d}x \,\mathrm{d}y \\
&+ \chi_{T_h} C {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|_{W^{1,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{|\Delta_h^x u_h|^2\} \,\mathrm{d}x \,\mathrm{d}y \\
&+ \chi_{T_h} C {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{y} \{|\partial_x u_h|^2\} \,\mathrm{d}x \,\mathrm{d}y \\
&+ \chi_{T_h} C {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|_{L^{\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{y} \{|\partial_x \Delta_h^x u_h|^2\} \,\mathrm{d}x \,\mathrm{d}y \\
&+ \chi_{T_h} C {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{y} \{|\partial_x u_h|^2\} \,\mathrm{d}x \,\mathrm{d}y \\
&+ \chi_{T_h} C {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|_{W^{1,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{|\Delta_h^x u_h|^2\} \,\mathrm{d}x \,\mathrm{d}y. \qquad (4.39)
\end{aligned}
$$

Similar computations show

$$
\begin{aligned}
III_{g_2} \le{}& \chi_{T_h} C {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{|u_h|^2\} \,\mathrm{d}x \,\mathrm{d}y \\
&+ \chi_{T_h} C {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|_{W^{1,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{|\Delta_h^y u_h|^2\} \,\mathrm{d}x \,\mathrm{d}y \\
&+ \chi_{T_h} C {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{x} \{|\partial_y u_h|^2\} \,\mathrm{d}x \,\mathrm{d}y
\end{aligned}
$$

$$+ \chi_{T_h} C {\lambda^x_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})} \int_{\mathcal{O}} \mathcal{I}^y_h\{|\partial_x \Delta^y_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y$$
$$+ \chi_{T_h} C {\lambda^x_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{W^{2,\infty}(\mathcal{O})} \int_{\mathcal{O}} \mathcal{I}^y_h\{|\partial_x u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y$$
$$+ \chi_{T_h} C {\lambda^x_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{W^{1,\infty}(\mathcal{O})} \int_{\mathcal{O}} \mathcal{I}^{xy}_h\{\Delta^x_h u_h \Delta^y_h u_h\} \,\mathrm{d}x\,\mathrm{d}y. \tag{4.40}$$

Therefore,

$$\begin{aligned} III_g \leq{} & C {\lambda^x_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{W^{3,\infty}(\mathcal{O})} h^\varepsilon \sum_{k\in I^x_h,\, l\in I^y_h} \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}} \,\mathrm{d}s \\ & + C {\lambda^x_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{W^{1,\infty}(\mathcal{O})} h^\varepsilon \\ & \cdot \sum_{k\in I^x_h,\, l\in I^y_h} \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}^{xy}_h\{|\Delta_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\ & + C {\lambda^x_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})} h^\varepsilon \\ & \cdot \sum_{k\in I^x_h,\, l\in I^y_h} \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}^y_h\{|\partial_x \Delta_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \end{aligned} \tag{4.41}$$

holds true. Analogous computations provide

$$\begin{aligned} III_h \leq{} & C {\lambda^y_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{W^{3,\infty}(\mathcal{O})} h^\varepsilon \sum_{k\in I^x_h,\, l\in I^y_h} \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}} \,\mathrm{d}s \\ & + C {\lambda^y_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{W^{1,\infty}(\mathcal{O})} h^\varepsilon \\ & \cdot \sum_{k\in I^x_h,\, l\in I^y_h} \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}^{xy}_h\{|\Delta_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\ & + C {\lambda^y_{kl}}^2 \|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})} h^\varepsilon \\ & \cdot \sum_{k\in I^x_h,\, l\in I^y_h} \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}^x_h\{|\partial_y \Delta_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{4.42}$$

To control $III_e + III_f$, we compute for all $i \in \{1, \dots, \dim U^x_h\}$, $j \in \{1, \dots, \dim U^y_h\}$, $k \in I^x_h$, and $l \in I^y_h$,

$$\begin{aligned} & \int_{\mathcal{O}} \mathcal{I}^{xy}_h\{F''(u_h)\big(|Z^x_{ij}(\lambda^x_{kl}\mathfrak{g}_{kl})|^2 + |Z^y_{ij}(\lambda^y_{kl}\mathfrak{g}_{kl})|^2\big) e^x_i(x) e^y_j(y)\} \,\mathrm{d}x\,\mathrm{d}y \\ & \quad = \chi_{T_h} F''(u_{ij}) M^{-1}_{ij} \bigg(\Big(\int_{\mathcal{O}} \mathcal{I}^y_h\{\mathcal{I}^x_{h,\mathrm{loc}}\{\lambda^x_{kl}\partial_x(u_h \widetilde{\mathfrak{g}}_{h,kl}) e^x_i(x) e^y_j(y)\}\} \,\mathrm{d}x\,\mathrm{d}y\Big)^2 \\ & \qquad + \Big(\int_{\mathcal{O}} \mathcal{I}^x_h\{\mathcal{I}^y_{h,\mathrm{loc}}\{\lambda^y_{kl}\partial_y(u_h \widetilde{\mathfrak{g}}_{h,kl}) e^x_i(x) e^y_j(y)\}\} \,\mathrm{d}x\,\mathrm{d}y\Big)^2\bigg) \\ & \quad =: \chi_{T_h} F''(u_{ij}) M^{-1}_{ij} \big({\lambda^x_{kl}}^2 A^2_{ijkl} + {\lambda^y_{kl}}^2 B^2_{ijkl}\big). \end{aligned} \tag{4.43}$$

Using the definition of $\mathcal{I}^x_{h,\mathrm{loc}}$, we obtain for the first term

$$\begin{aligned}
A^2_{ijkl} &= \Big(\int_{\mathcal{O}} \mathcal{I}^y_h\{\mathcal{I}^x_{h,\mathrm{loc}}\{\partial_x(u_h\widetilde{\mathfrak{g}}_{h,kl})\mathrm{e}^x_i(x)\mathrm{e}^y_j(y)\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2 \\
&= \Big(\sum_{Q\in\mathcal{Q}_h}\int_Q \mathcal{I}^{xy}_h\{\partial_x(u_h\widetilde{\mathfrak{g}}_{h,kl})\mathrm{e}^x_i(x)\mathrm{e}^y_j(y)\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2 \\
&\le M_{ij}\sum_{Q\in\mathcal{Q}_h}\int_Q \mathcal{I}^{xy}_h\{|\partial_x(u_h\widetilde{\mathfrak{g}}_{h,kl})|^2\mathrm{e}^x_i(x)\mathrm{e}^y_j(y)\}\,\mathrm{d}x\,\mathrm{d}y \\
&\le 2M_{ij}\sum_{Q\in\mathcal{Q}_h}\int_Q \mathcal{I}^{xy}_h\{(|\partial_x u_h|^2|\widetilde{\mathfrak{g}}_{h,kl}|^2+|u_h|^2|\partial_x\widetilde{\mathfrak{g}}_{h,kl}|^2)\mathrm{e}^x_i(x)\mathrm{e}^y_j(y)\}\,\mathrm{d}x\,\mathrm{d}y \\
&\le C\|\widetilde{\mathfrak{g}}_{h,k}\|_{W^{1,\infty}(\mathcal{O})}M_{ij}\sum_{Q\in\mathcal{Q}_h}\int_Q \mathcal{I}^{xy}_h\{|\partial_x u_h|^2\mathrm{e}^x_i(x)\mathrm{e}^y_j(y)\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + C\|\widetilde{\mathfrak{g}}_{h,k}\|_{W^{1,\infty}(\mathcal{O})}M_{ij}\sum_{Q\in\mathcal{Q}_h}\int_Q \mathcal{I}^{xy}_h\{|u_h|^2\mathrm{e}^x_i(x)\mathrm{e}^y_j(y)\}\,\mathrm{d}x\,\mathrm{d}y \\
&= C\|\widetilde{\mathfrak{g}}_{h,kl}\|_{W^{1,\infty}(\mathcal{O})}M_{ij}\int_{\mathcal{O}} \mathcal{I}^{y}_h\{|\partial_x u_h|^2\mathrm{e}^x_i(x)\mathrm{e}^y_j(y)\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + C\|\widetilde{\mathfrak{g}}_{h,kl}\|_{W^{1,\infty}(\mathcal{O})}M_{ij}\int_{\mathcal{O}} \mathcal{I}^{xy}_h\{|u_h|^2\mathrm{e}^x_i(x)\mathrm{e}^y_j(y)\}\,\mathrm{d}x\,\mathrm{d}y. \qquad (4.44)
\end{aligned}$$

Similar computations for B_{ijkl} provide

$$\begin{aligned}
III_e + III_f \le C\int_0^{t\wedge T_h} \overline{p}R(s)^{\overline{p}-1}\int_{\mathcal{O}} &\mathcal{I}^y_h\{\mathcal{I}^x_h\{F''(u_h)\}|\partial_x u_h|^2\} \\
&+ \mathcal{I}^x_h\{\mathcal{I}^y_h\{F''(u_h)\}|\partial_y u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
+ C\int_0^{t\wedge T_h} &\overline{p}R(s)^{\overline{p}-1}\int_{\mathcal{O}} \mathcal{I}^{xy}_h\{|u_h|^2F''(u_h)\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \qquad (4.45)
\end{aligned}$$

Analogously, we compute

$$\begin{aligned}
III_i + III_j \le C\kappa\int_0^{t\wedge T_h} \overline{p}R(s)^{\overline{p}-1}\int_{\mathcal{O}} &\mathcal{I}^y_h\{\mathcal{I}^x_h\{G''(u_h)\}|\partial_x u_h|^2\} \\
&+ \mathcal{I}^x_h\{\mathcal{I}^y_h\{G''(u_h)\}|\partial_y u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
+ C\kappa\int_0^{t\wedge T_h} &\overline{p}R(s)^{\overline{p}-1}\int_{\mathcal{O}} \mathcal{I}^{xy}_h\{|u_h|^2G''(u_h)\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \qquad (4.46)
\end{aligned}$$

Collecting the above estimates and applying Assumptions (B3) and (B3*), we obtain

$$\begin{aligned}
III + IV \le C\int_0^{t\wedge T_h} R(s)^{\overline{p}}\,\mathrm{d}s &+ C\int_0^{t\wedge T_h} R(s)^{\overline{p}-1}\|\Delta_h u_h\|_h^2\,\mathrm{d}s \\
+ C\int_0^{t\wedge T_h} &R(s)^{\overline{p}-1}h^{\varepsilon}\int_{\mathcal{O}} \mathcal{I}^y_h\{|\partial_x\Delta_h u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}$$

$$+ C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} h^{\varepsilon} \int_{\mathcal{O}} \mathcal{I}_h^x\{|\partial_y \Delta_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$+ C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_h^x\{F''(u_h)\}|\partial_x u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$+ C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x\{\mathcal{I}_h^y\{F''(u_h)\}|\partial_y u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$+ C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|u_h|^2 F''(u_h)\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$+ C\kappa \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_h^x\{G''(u_h)\}|\partial_x u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$+ C\kappa \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x\{\mathcal{I}_h^y\{G''(u_h)\}|\partial_y u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$+ C\kappa \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|u_h|^2 G''(u_h)\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \tag{4.47}$$

While the first term on the right-hand side is a Gronwall term, the remaining terms need to be absorbed in the negative terms provided by I. For this reason, we need the following estimates: Due to Assumption (P), we have $F''(u_h) \le C u_h^{-p-2} + C$. Recalling the oscillation lemma (Lemma 3.6), we obtain the estimates

$$\int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_h^x\{F''(u_h)\}|\partial_x u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y \le C \int_{\mathcal{O}} \mathcal{I}_h^y\{[|u_h|^{-p-2}]_x|\partial_x u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y + CR(s), \tag{4.48}$$

$$\int_{\mathcal{O}} \mathcal{I}_h^x\{\mathcal{I}_h^y\{F''(u_h)\}|\partial_y u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y \le C \int_{\mathcal{O}} \mathcal{I}_h^x\{[|u_h|^{-p-2}]_y|\partial_y u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y + CR(s). \tag{4.49}$$

Furthermore, Assumption (P), Poincaré's inequality, and our uniform control of the mass of discrete solutions (see Remark 3.3, Assumption (I), and the norm equivalence (2.6)) provide

$$\begin{aligned}\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|u_h|^2 F''(u_h)\} \,\mathrm{d}x\,\mathrm{d}y &\le C \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{F(u_h)\} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad + C \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y + C \le CR(s)\end{aligned} \tag{4.50}$$

for $\alpha > 0$. As Lemma 3.6 provides the estimate

$$\mathcal{I}_h^x\{u_h^{-2}\}^2|_{K^x} \le \big(\max_{K^x} \mathcal{I}_h^x\{u_h^{-2}\}\big)^2 \le \big(C_{\mathrm{osc}} \min_{K^x} \mathcal{I}_h^x\{u_h^{-2}\}\big)^2 \le C_{\mathrm{osc}}^2 \mathcal{I}_h^x\{u_h^{-4}\}|_{K^x},$$

we can apply Young's inequality and use $p > 2$ and $\kappa > 1$ to compute

$$\begin{aligned}
&\kappa \int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_h^x\{G''(u_h)\}|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le C_{\mathrm{osc}}^{-2} \int_{\mathcal{O}} \mathcal{I}_h^y\{|\mathcal{I}_h^x\{G''(u_h)\}|^2|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y + C\kappa^2 \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le \int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_h^x\{|u_h|^{-4}\}|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y + C\kappa^2 \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le \int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_h^x\{|u_h|^{-p-2}+1\}|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y + C\kappa^2 \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le \int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_h^x\{|u_h|^{-p-2}\}|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y + C\kappa^2 \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le C \int_{\mathcal{O}} \mathcal{I}_h^y\{[|u_h|^{-p-2}]_x|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y + C\kappa^2 R(s)
\end{aligned} \tag{4.51}$$

and

$$\begin{aligned}
\kappa \int_{\mathcal{O}} \mathcal{I}_h^x\{\mathcal{I}_h^y\{G''(u_h)\}|\partial_y u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y \le C \int_{\mathcal{O}} \mathcal{I}_h^x\{[|u_h|^{-p-2}]_y|\partial_y u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y& \\
+ C\kappa^2 R(s).&
\end{aligned} \tag{4.52}$$

In the last line of (4.51), we used that Lemma 3.6 allows us to control $\mathcal{I}_h^x\{|u_h|^{p-2}\}$ by its mean value. Noting the definition of $G''(u_h)$, we obtain

$$\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|u_h|^2 G''(u_h)\}\,\mathrm{d}x\,\mathrm{d}y \le \int_{\mathcal{O}} 1\,\mathrm{d}x\,\mathrm{d}y \le C. \tag{4.53}$$

Therefore, the last term in (4.47) can be controlled by $C\kappa \int_0^{t\wedge T_h} R(s)^{\overline{p}-1}\,\mathrm{d}s$ for $\alpha > 0$. This allows us to rewrite (4.47) for $\kappa > 1$ as

$$\begin{aligned}
III + IV \le{}& C\kappa^2 \int_0^{t\wedge T_h} R(s)^{\overline{p}}\,\mathrm{d}s + C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1}\|\Delta_h u_h\|_h^2\,\mathrm{d}s \\
&+ C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} h^{\varepsilon} \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x \Delta_h u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&+ C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} h^{\varepsilon} \int_{\mathcal{O}} \mathcal{I}_h^x\{|\partial_y \Delta_h u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&+ C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y\{[|u_h|^{-p-2}]_x|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&+ C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x\{[|u_h|^{-p-2}]_y|\partial_y u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned} \tag{4.54}$$

Using (3.6b), we compute

$$\begin{aligned}
&(D\mathcal{E}_h + \kappa D\mathcal{S}_h) \otimes (D\mathcal{E}_h + \kappa D\mathcal{S}_h)(\Phi_{h,kl}^x, \Phi_{h,kl}^x) \\
&\quad\le 2\Big(\int_{\mathcal{O}} \mathcal{I}_h^y\Big\{\partial_x u_h \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl})\partial_x e_i^x(x) e_j^y(y)\Big\}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

$$+ \int_{\mathcal{O}} \mathcal{I}_h^x \Big\{ \partial_y u_h \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) e_i^x(x) \partial_y e_j^y(y) \Big\} \,\mathrm{d}x\,\mathrm{d}y$$

$$+ \int_{\mathcal{O}} \mathcal{I}_h^{xy} \Big\{ F'(u_h) \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) e_i^x(x) e_j^y(y) \Big\} \,\mathrm{d}x\,\mathrm{d}y$$

$$+ h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \Big\{ \Delta_h u_h \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) \Delta_h (e_i^x(x) e_j^y(y)) \Big\} \,\mathrm{d}x\,\mathrm{d}y \Big)^2$$

$$+ 2\kappa^2 \Big(\int_{\mathcal{O}} \mathcal{I}_h^{xy} \Big\{ G'(u_h) \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) e_i^x(x) e_j^y(y) \Big\} \,\mathrm{d}x\,\mathrm{d}y \Big)^2$$

$$= 2 \Big(\int_{\mathcal{O}} \mathcal{I}_h^{xy} \Big\{ p_h \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) e_i^x(x) e_j^y(y) \Big\} \,\mathrm{d}x\,\mathrm{d}y \Big)^2$$

$$+ 2\kappa^2 \Big(\int_{\mathcal{O}} \mathcal{I}_h^{xy} \Big\{ G'(u_h) \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) e_i^x(x) e_j^y(y) \Big\} \,\mathrm{d}x\,\mathrm{d}y \Big)^2$$

$$= \chi_{T_h} 2 \, {\lambda_{kl}^x}^2 \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\mathrm{loc}}^x \{ \partial_x (u_h \widetilde{\mathfrak{g}}_{h,kl}) p_h \} \} \,\mathrm{d}x\,\mathrm{d}y \Big)^2$$

$$+ \chi_{T_h} 2 \kappa^2 {\lambda_{kl}^x}^2 \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\mathrm{loc}}^x \{ \partial_x (u_h \widetilde{\mathfrak{g}}_{h,kl}) G'(u_h) \} \} \,\mathrm{d}x\,\mathrm{d}y \Big)^2$$

$$=: \chi_{T_h} 2 \, {\lambda_{kl}^x}^2 A_I + \chi_{T_h} 2\kappa^2 \, {\lambda_{kl}^x}^2 A_{II}, \tag{4.55}$$

where we used (4.3) in the last step. To control the first term on the right-hand side of (4.55), we use (2.13a) and (2.13b) to compute

$$A_I \le 2 \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \mathcal{I}_h^x \{ \widetilde{\mathfrak{g}}_{h,kl} p_h \} \} \,\mathrm{d}x\,\mathrm{d}y \Big)^2 + 2 \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x \widetilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \{ u_h p_h \} \} \,\mathrm{d}x\,\mathrm{d}y \Big)^2$$
$$=: A_{I_a} + A_{I_b}. \tag{4.56}$$

Recalling (3.6b), Assumption (P), the integration by parts formula used in (4.37), Hölder's inequality, and the positivity of u_h, we obtain

$$A_{I_a} \le C \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \mathcal{I}_h^x \{ \widetilde{\mathfrak{g}}_{h,kl} \Delta_h u_h \} \} \,\mathrm{d}x\,\mathrm{d}y \Big)^2$$
$$+ C \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h| \mathcal{I}_h^x \{ |\widetilde{\mathfrak{g}}_{h,kl}| (u_h^{-p-1} + 1) \} \} \,\mathrm{d}x\,\mathrm{d}y \Big)^2$$
$$+ C \Big(h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \mathcal{I}_h^x \{ \widetilde{\mathfrak{g}}_{h,kl} \Delta_h \Delta_h u_h \} \} \,\mathrm{d}x\,\mathrm{d}y \Big)^2$$

$$
\begin{aligned}
&\leq C\Big(\int_{\mathcal{O}} \mathcal{I}_h^y\big\{|\partial_x u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\Big)\|\widetilde{\mathfrak{g}}_{h,kl}\|^2_{L^\infty(\mathcal{O})}\|\Delta_h u_h\|_h^2 \\
&\quad + C\Big(\int_{\mathcal{O}} \mathcal{I}_h^y\big\{|\partial_x u_h|\mathcal{I}_h^x\big\{|\widetilde{\mathfrak{g}}_{h,kl}|u_h^{-p/2}u_h^{-p/2-1}\big\}\big\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2 \\
&\quad + C\Big(\int_{\mathcal{O}} \mathcal{I}_h^y\big\{|\partial_x u_h|\mathcal{I}_h^x\{|\widetilde{\mathfrak{g}}_{h,kl}|\}\big\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2 \\
&\quad + C\|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})}\,h^\varepsilon\int_{\mathcal{O}} \mathcal{I}_h^y\big\{|\partial_x\Delta_h^x u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,h^\varepsilon\int_{\mathcal{O}}\mathcal{I}_h^{xy}\big\{|\Delta_h u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + C\|\partial_x\partial_x\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})}\,h^\varepsilon\int_{\mathcal{O}} \mathcal{I}_h^y\big\{|\partial_x u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,h^\varepsilon \\
&\qquad\cdot\int_{\mathcal{O}}\mathcal{I}_h^{xy}\big\{|\Delta_h u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + C\|\partial_x\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})}\,h^\varepsilon\int_{\mathcal{O}} \mathcal{I}_h^{xy}\big\{|\Delta_h^x u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,h^\varepsilon\int_{\mathcal{O}}\mathcal{I}_h^{xy}\big\{|\Delta_h u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + C\|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})}\,h^\varepsilon\int_{\mathcal{O}} \mathcal{I}_h^y\big\{|\partial_x\Delta_h^y u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,h^\varepsilon\int_{\mathcal{O}}\mathcal{I}_h^{xy}\big\{|\Delta_h u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + C\|\partial_y\partial_y\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})}\,h^\varepsilon\int_{\mathcal{O}} \mathcal{I}_h^y\big\{|\partial_x u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,h^\varepsilon\int_{\mathcal{O}}\mathcal{I}_h^{xy}\big\{|\Delta_h u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + C\|\partial_y\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})}\,h^\varepsilon\int_{\mathcal{O}} \mathcal{I}_h^{xy}\big\{\Delta_h^x u_h\Delta_h^y u_h\big\}\,\mathrm{d}x\,\mathrm{d}y\,h^\varepsilon \\
&\qquad\cdot\int_{\mathcal{O}}\mathcal{I}_h^{xy}\big\{|\Delta_h u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
\tag{4.57}
$$

where we used $\int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x\partial_y^{+h_y}u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathcal{O}}\mathcal{I}_h^{xy}\{\Delta_h^x u_h\Delta_h^y u_h\}\,\mathrm{d}x\,\mathrm{d}y$. Using a discrete version of Hölder's inequality, we obtain

$$
\int_{\mathcal{O}}\mathcal{I}_h^{xy}\{\Delta_h^x u_h\Delta_h^y u_h\}\,\mathrm{d}x\,\mathrm{d}y \leq \big\|\Delta_h^x u_h\big\|_h\big\|\Delta_h^y u_h\big\|_h \leq \|\Delta_h u_h\|_h^2. \tag{4.58}
$$

Therefore, we have

$$
\begin{aligned}
A_{I_a} &\leq CR(s)\|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})}\|\Delta_h u_h\|_h^2 \\
&\quad + C\|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})}\Big(\int_{\mathcal{O}}\mathcal{I}_h^y\big\{|\partial_x u_h|^2\mathcal{I}_h^x\big\{u_h^{-p-2}\big\}\big\}\,\mathrm{d}x\,\mathrm{d}y\Big)\Big(\int_{\mathcal{O}}\mathcal{I}_h^{xy}\big\{u_h^{-p}\big\}\,\mathrm{d}x\,\mathrm{d}y\Big) \\
&\quad + C\|\mathfrak{g}_{kl}\|^2_{L^\infty(\mathcal{O})}\int_{\mathcal{O}}\mathcal{I}_h^y\big\{|\partial_x u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + C\|\mathfrak{g}_{kl}\|^2_{W^{2,\infty}(\mathcal{O})}R(s)\Big(h^\varepsilon\int_{\mathcal{O}}\mathcal{I}_h^y\big\{|\partial_x\Delta_h u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y + R(s)\Big) \\
&\leq CR(s)\|\mathfrak{g}_{kl}\|^2_{W^{2,\infty}(\mathcal{O})}\Big(\|\Delta_h u_h\|_h^2 + \int_{\mathcal{O}}\mathcal{I}_h^y\big\{|\partial_x u_h|^2\mathcal{I}_h^x\big\{u_h^{-p-2}\big\}\big\}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + h^\varepsilon\int_{\mathcal{O}}\mathcal{I}_h^y\big\{|\partial_x\Delta_h u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y + R(s)\Big).
\end{aligned}
\tag{4.59}
$$

Using (3.6b), Hölder's inequality, Assumption (P), the computations used in (4.37), Poincaré's inequality, and $R(s) \geq \alpha$, we obtain the estimate

$$\begin{aligned}
A_{I_b} &\leq C\Big(\int_{\mathcal{O}} \mathcal{I}_h^y\{\partial_x \widetilde{\mathfrak{g}}_{h,kl}\,\mathcal{I}_h^x\{u_h \Delta_h u_h\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2 \\
&\quad + C\Big(\int_{\mathcal{O}} \mathcal{I}_h^y\{\partial_x \widetilde{\mathfrak{g}}_{h,kl}\,\mathcal{I}_h^x\{u_h F'(u_h)\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2 \\
&\quad + C\Big(h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y\{\partial_x \widetilde{\mathfrak{g}}_{h,kl}\,\mathcal{I}_h^x\{u_h \Delta_h \Delta_h u_h\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2 \\
&\leq C\|\partial_x \mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \|u_h\|_h^2 \|\Delta_h u_h\|_h^2 \\
&\quad + C\|\partial_x \mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \Big(\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{u_h^{-p}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2 + C\|\partial_x \mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \Big(\int_{\mathcal{O}} u_h\,\mathrm{d}x\,\mathrm{d}y\Big)^2 \\
&\quad + C h^\varepsilon \|\partial_x\partial_x\partial_x \mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \|u_h\|_h^2 h^\varepsilon \|\Delta_h u_h\|_h^2 \\
&\quad + C\|\partial_x \mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2\, h^\varepsilon \|\Delta_h^x u_h\|_h^2 h^\varepsilon \|\Delta_h u_h\|_h^2 \\
&\quad + C\|\partial_x\partial_x \mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y\, h^\varepsilon \|\Delta_h u_h\|_h^2 \\
&\quad + C h^\varepsilon \|\partial_x\partial_y\partial_y \mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \|u_h\|_h^2 h^\varepsilon \|\Delta_h u\|_h^2 \\
&\quad + C\|\partial_x \mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 h^\varepsilon \|\Delta_h^y u_h\|_h^2 h^\varepsilon \|\Delta_h u_h\|_h^2 \\
&\quad + C\|\partial_x\partial_y \mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2\, h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^x\{|\partial_y u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y\, h^\varepsilon \|\Delta_h u_h\|_h^2 \\
&\leq C\big(\|\mathfrak{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 + h^\varepsilon \|\mathfrak{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2\big) R(s)\big(\|\Delta_h u_h\|_h^2 + R(s)\big).
\end{aligned} \tag{4.60}$$

Here, we used that $\int_{\mathcal{O}} u_h\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathcal{O}} u_h^0\,\mathrm{d}x\,\mathrm{d}y$ is uniformly bounded $\mathbb{P}$-almost surely (cf. Assumption (I) and Remark 3.3). Using (2.13a) and (2.13b) and $G'(u) = u^{-1}$, we obtain

$$A_{II} \leq C\|\mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_h^x\{G''(u_h)\}|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y + C\|\partial_x \mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2. \tag{4.61}$$

Therefore, we have

$$\begin{aligned}
V &\leq C \sum_{k\in I_h^x, l\in I_h^y} {\lambda_{kl}^x}^2 \big(\|\mathfrak{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 + h^\varepsilon \|\mathfrak{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2\big) \int_0^{t\wedge T_h} R(s)^{\overline{p}}\,\mathrm{d}s \\
&\quad + C \sum_{k\in I_h^x, l\in I_h^y} {\lambda_{kl}^x}^2 \big(\|\mathfrak{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 + h^\varepsilon \|\mathfrak{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2\big) \\
&\qquad \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \|\Delta_h u_h\|_h^2\,\mathrm{d}s \\
&\quad + C \sum_{k\in I_h^x, l\in I_h^y} {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 \\
&\qquad \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x u_h|^2 \mathcal{I}_h^x\{u_h^{-p-2}\}\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}$$

$$+ C \sum_{k\in I_h^x, l\in I_h^y} {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2$$
$$\int_0^{t\wedge T_h} R(s)^{\overline{p}-1} h^{\varepsilon} \int_{\mathcal{O}} \mathcal{I}_h^y \{|\partial_x \Delta_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$+ C\kappa^2 \sum_{k\in I_h^x, l\in I_h^y} {\lambda_{kl}^x}^2 \|\mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2$$
$$\int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{\mathcal{I}_h^x\{G''(u_h)\}|\partial_x u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \tag{4.62}$$

Recalling Assumptions (B3) and (B3*) and Lemma 3.6 and mimicking the computations in (4.51), we obtain for $\kappa > 1$

$$V \le C\kappa^4 \int_0^{t\wedge T_h} R(s)^{\overline{p}} \,\mathrm{d}s + C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \|\Delta_h u_h\|_h^2 \,\mathrm{d}s$$
$$+ C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{[|u_h|^{-p-2}]_x |\partial_x u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$+ C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \{[|u_h|^{-p-2}]_y |\partial_y u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$+ C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} h^{\varepsilon} \int_{\mathcal{O}} \mathcal{I}_h^y \{|\partial_x \Delta_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \tag{4.63}$$

Analogously, we compute

$$VI \le C\kappa^4 \int_0^{t\wedge T_h} R(s)^{\overline{p}} \,\mathrm{d}s + C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \|\Delta_h u_h\|_h^2 \,\mathrm{d}s$$
$$+ C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{[|u_h|^{-p-2}]_x |\partial_x u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$+ C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \{[|u_h|^{-p-2}]_y |\partial_y u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$+ C \int_0^{t\wedge T_h} R(s)^{\overline{p}-1} h^{\varepsilon} \int_{\mathcal{O}} \mathcal{I}_h^x \{|\partial_y \Delta_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \tag{4.64}$$

In order to obtain bounds for the expected value of stochastic integral, we rewrite

$$II = \sum_{k\in I_h^x, l\in I_h^y} \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{\mathcal{I}_{h,\mathrm{loc}}^x \{\partial_x(\lambda_{kl}^x \tilde{\mathfrak{g}}_{h,kl} u_h) p_h\}\} \,\mathrm{d}x\,\mathrm{d}y\ \mathrm{d}\beta_{kl}^x$$
$$+ \sum_{k\in I_h^x, l\in I_h^y} \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \{\mathcal{I}_{h,\mathrm{loc}}^y \{\partial_y(\lambda_{kl}^y \tilde{\mathfrak{g}}_{h,kl} u_h) p_h\}\} \,\mathrm{d}x\,\mathrm{d}y\ \mathrm{d}\beta_{kl}^y$$
$$+ \kappa \sum_{k\in I_h^x, l\in I_h^y} \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{\mathcal{I}_{h,\mathrm{loc}}^x \{\partial_x(\lambda_{kl}^x \tilde{\mathfrak{g}}_{h,kl} u_h) G'(u_h)\}\}$$
$$\mathrm{d}x\,\mathrm{d}y\ \mathrm{d}\beta_{kl}^x$$

$$+\kappa \sum_{k\in I_h^x, l\in I_h^y} \int_0^{t\wedge T_h} \overline{p} R(s)^{\overline{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x\{\mathcal{I}_{h,\text{loc}}^y\{\partial_y(\lambda_{kl}^y \widetilde{\mathfrak{g}}_{h,kl} u_h) G'(u_h)\}\} \,\mathrm{d}x\,\mathrm{d}y\,\,\mathrm{d}\beta_{kl}^y$$
$$=: II_a + II_b + II_c + II_d \tag{4.65}$$

and apply the Burkholder–Davis–Gundy inequality. For this reason, we introduce the Hilbert–Schmidt operators

$$\mathfrak{T}_1(s)(\omega) := \chi_{T_h} R(s)^{\overline{p}-1} \sum_{k\in I_h^x, l\in I_h^y} \int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_{h,\text{loc}}^x\{\partial_x(\langle \mathfrak{g}_{kl}, \omega\rangle_{L^2} u_h \widetilde{\mathfrak{g}}_{h,kl}) p_h\}\} \,\mathrm{d}x\,\mathrm{d}y, \tag{4.66a}$$

$$\mathfrak{T}_3(s)(\omega) := \chi_{T_h} \kappa R(s)^{\overline{p}-1} \sum_{k\in I_h^x, l\in I_h^y} \int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_{h,\text{loc}}^x\{\partial_x(\langle \mathfrak{g}_{kl}, \omega\rangle_{L^2} u_h \widetilde{\mathfrak{g}}_{h,kl}) G'(u_h)\}\} \,\mathrm{d}x\,\mathrm{d}y, \tag{4.66b}$$

mapping $Q_x^{1/2} L^2(\mathcal{O})$ onto $\mathbb{R}$ and the Hilbert–Schmidt operators

$$\mathfrak{T}_2(s)(\omega) := \chi_{T_h} R(s)^{\overline{p}-1} \sum_{k\in I_h^x, l\in I_h^y} \int_{\mathcal{O}} \mathcal{I}_h^x\{\mathcal{I}_{h,\text{loc}}^y\{\partial_y(\langle \mathfrak{g}_{kl}, \omega\rangle_{L^2} u_h \widetilde{\mathfrak{g}}_{h,kl}) p_h\}\} \,\mathrm{d}x\,\mathrm{d}y, \tag{4.66c}$$

$$\mathfrak{T}_4(s)(\omega) := \chi_{T_h} \kappa R(s)^{\overline{p}-1} \sum_{k\in I_h^x, l\in I_h^y} \int_{\mathcal{O}} \mathcal{I}_h^x\{\mathcal{I}_{h,\text{loc}}^y\{\partial_y(\langle \mathfrak{g}_{kl}, \omega\rangle_{L^2} u_h \widetilde{\mathfrak{g}}_{h,kl}) G'(u_h)\}\} \,\mathrm{d}x\,\mathrm{d}y, \tag{4.66d}$$

mapping $Q_y^{1/2} L^2(\mathcal{O})$ onto $\mathbb{R}$. Recalling (4.59) and (4.60), we obtain

$$\begin{aligned}
&\Big(\sum_{k,l\in\mathbb{Z}} \big|\mathfrak{T}_1(s)(Q_x^{1/2}\mathfrak{g}_{kl})\big|^2\Big)^{1/2} \\
&\quad = \chi_{T_h} R(s)^{\overline{p}-1} \Big(\sum_{k\in I_h^x, l\in I_h^y} {\lambda_{kl}^x}^2 \Big(\int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_{h,\text{loc}}^x\{\partial_x(u_h \widetilde{\mathfrak{g}}_{h,kl}) p_h\}\} \,\mathrm{d}x\,\mathrm{d}y\Big)^2\Big)^{1/2} \\
&\quad \le \chi_{T_h} C R(s)^{\overline{p}-1} \Big(\sum_{k\in I_h^x, l\in I_h^y} {\lambda_{kl}^x}^2 \big(\|\mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 + h^\varepsilon \|\mathfrak{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2\big) \\
&\qquad \times \Big(\|\Delta_h u_h\|_h^2 + \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x u_h|^2 \mathcal{I}_h^x\{u_h^{-p-2}\}\} \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x \Delta_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y + R(s)\Big)^2\Big)^{1/2} \\
&\quad \le \chi_{T_h} C R(s)^{\overline{p}-1} \Big(\|\Delta_h u_h\|_h^2 + \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x u_h|^2 \mathcal{I}_h^x\{u_h^{-p-2}\}\} \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x \Delta_h u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y + R(s)\Big).
\end{aligned} \tag{4.67}$$

Similarly, we have

$$\Big(\sum_{k,l\in\mathbb{Z}}\big|\mathfrak{T}_2(s)(Q_y^{1/2}\mathfrak{g}_{kl})\big|^2\Big)^{1/2}$$
$$\le \chi_{T_h} C R(s)^{\overline{p}-1}\Big(\|\Delta_h u_h\|_h^2+\int_{\mathcal{O}}\mathcal{I}_h^x\{|\partial_y u_h|^2\mathcal{I}_h^y\{u_h^{-p-2}\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)$$
$$+\chi_{T_h} C R(s)^{\overline{p}-1}\Big(h^\varepsilon\int_{\mathcal{O}}\mathcal{I}_h^x\{|\partial_y\Delta_h u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y+R(s)\Big). \tag{4.68}$$

Following the computations in (4.61), we obtain

$$\Big(\sum_{k,l\in\mathbb{Z}}\big|\mathfrak{T}_3(s)(Q_x^{1/2}\mathfrak{g}_{kl})\big|^2\Big)^{1/2}\le \chi_{T_h} C\kappa R(s)^{\overline{p}-1}$$
$$\cdot\Big(\int_{\mathcal{O}}\mathcal{I}_h^y\{\mathcal{I}_h^x\{G''(u_h)\}|\partial_x u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y+C\Big), \tag{4.69}$$
$$\Big(\sum_{k,l\in\mathbb{Z}}\big|\mathfrak{T}_4(s)(Q_y^{1/2}\mathfrak{g}_{kl})\big|^2\Big)^{1/2}\le \chi_{T_h} C\kappa R(s)^{\overline{p}-1}$$
$$\cdot\Big(\int_{\mathcal{O}}\mathcal{I}_h^x\{\mathcal{I}_h^y\{G''(u_h)\}|\partial_y u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y+C\Big). \tag{4.70}$$

Therefore, the Burkholder–Davis–Gundy inequality yields together with Young's inequality and $R(s)\ge\alpha$

$$\mathbb{E}\Big[\sup_{t\in[0,T_{\max}]}|II_a|\Big]$$
$$\le C\,\mathbb{E}\Big[\Big(\int_0^{T_{\max}\wedge T_h}R(s)^{2\overline{p}-2}\Big(\|\Delta_h u_h\|_h^2+\int_{\mathcal{O}}\mathcal{I}_h^y\{|\partial_x u_h|^2\mathcal{I}_h^x\{u_h^{-p-2}\}\}\,\mathrm{d}x\,\mathrm{d}y$$
$$+h^\varepsilon\int_{\mathcal{O}}\mathcal{I}_h^y\{|\partial_x\Delta_h u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y+R(s)\Big)\,\mathrm{d}s\Big)^{1/2}\Big]$$
$$\le\frac{1}{8}\mathbb{E}\Big[\sup_{t\in[0,T_{\max}\wedge T_h]}R(t)^{\overline{p}}\Big]$$
$$+C\alpha^{-1}\mathbb{E}\Big[\int_0^{T_{\max}\wedge T_h}R(s)^{\overline{p}-1}\Big(\|\Delta_h u_h\|_h^2+\int_{\mathcal{O}}\mathcal{I}_h^y\{|\partial_x u_h|^2\mathcal{I}_h^x\{u_h^{-p-2}\}\}\,\mathrm{d}x\,\mathrm{d}y$$
$$+h^\varepsilon\int_{\mathcal{O}}\mathcal{I}_h^y\{|\partial_x\Delta_h u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y+R(s)\Big)\,\mathrm{d}s\Big], \tag{4.71}$$
$$\mathbb{E}\Big[\sup_{t\in[0,T_{\max}]}|II_b|\Big]\le\frac{1}{8}\mathbb{E}\Big[\sup_{t\in[0,T_{\max}\wedge T_h]}R(t)^{\overline{p}}\Big]$$
$$+C\alpha^{-1}\mathbb{E}\Big[\int_0^{T_{\max}\wedge T_h}R(s)^{\overline{p}-1}\Big(\|\Delta_h u_h\|_h^2+\int_{\mathcal{O}}\mathcal{I}_h^x\{|\partial_y u_h|^2\mathcal{I}_h^y\{u_h^{-p-2}\}\}\,\mathrm{d}x\,\mathrm{d}y$$
$$+h^\varepsilon\int_{\mathcal{O}}\mathcal{I}_h^x\{|\partial_y\Delta_h u_h|^2\}\,\mathrm{d}x\,\mathrm{d}y+R(s)\Big)\,\mathrm{d}s\Big], \tag{4.72}$$

and

$$\mathbb{E}\Big[\sup_{t\in[0,T_{\max}]}|II_c| + \sup_{t\in[0,T_{\max}]}|II_d|\Big] \leq \frac{1}{4}\mathbb{E}\Big[\sup_{t\in[0,T_{\max}\wedge T_h]} R(t)^{\overline{p}-1}\Big]$$
$$+ C\alpha^{-1}\kappa^2\mathbb{E}\Big[\int_0^{T_{\max}\wedge T_h} R(s)^{\overline{p}-1}\Big(\int_{\mathcal{O}} \mathcal{I}_h^y\big\{\mathcal{I}_h^x\{G''(u_h)|\partial_x u_h|^2\}\big\}\Big)\Big]$$
$$+ C\alpha^{-1}\kappa^2\mathbb{E}\Big[\int_0^{T_{\max}\wedge T_h} R(s)^{\overline{p}-1}\Big(\int_{\mathcal{O}} \mathcal{I}_h^x\big\{\mathcal{I}_h^y\{G''(u_h)\}|\partial_y u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y + C\Big)\Big]. \tag{4.73}$$

Collecting the above results, we obtain

$$\mathbb{E}\Big[\sup_{t\in[0,T_{\max}]}|II|\Big] \leq \frac{1}{2}\mathbb{E}\Big[\sup_{t\in[0,T_{\max}]} R(t)^{\overline{p}}\Big] + C\alpha^{-1}\mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}\|\Delta_h u_h\|_h^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]$$
$$+ C\alpha^{-1}\mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}\int_{\mathcal{O}} \mathcal{I}_h^y\big\{[|u_h|^{-p-2}]_x|\partial_x u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]$$
$$+ C\alpha^{-1}\mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}\int_{\mathcal{O}} \mathcal{I}_h^x\big\{[|u_h|^{-p-2}]_y|\partial_y u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]$$
$$+ C\alpha^{-1}\mathbb{E}\Big[\int_0^{t_h} R(s)^{\overline{p}-1}h^{\varepsilon}\int_{\mathcal{O}} \mathcal{I}_h^y\big\{|\partial_x\Delta_h u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]$$
$$+ C\alpha^{-1}\mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}h^{\varepsilon}\int_{\mathcal{O}} \mathcal{I}_h^x\big\{|\partial_y\Delta_h u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]$$
$$+ C\alpha^{-1}\kappa^4\mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}}\,\mathrm{d}s\Big], \tag{4.74}$$

where we again used Lemma 3.6 and mimicked (4.51). Collecting the intermediate results established above, we obtain for κ sufficiently large

$$\mathbb{E}\Big[\sup_{t\in[0,T_{\max}]} R(t)^{\overline{p}}\Big] + \mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}\int_{\mathcal{O}} \mathcal{I}_h^y\big\{[G''(u_h)]_x^{-1}|\partial_x p_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]$$
$$+ \mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}\int_{\mathcal{O}} \mathcal{I}_h^x\big\{[G''(u_h)]_y^{-1}|\partial_y u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]$$
$$+ \mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}\|\Delta_h u_h\|_h^2\,\mathrm{d}s\Big]$$
$$+ \mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}h^{\varepsilon}\int_{\mathcal{O}} \mathcal{I}_h^y\big\{|\partial_x\Delta_h u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]$$
$$+ \mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}h^{\varepsilon}\int_{\mathcal{O}} \mathcal{I}_h^x\big\{|\partial_y\Delta_h u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]$$
$$+ \mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}\int_{\mathcal{O}} \mathcal{I}_h^y\big\{[|u_h|^{-p-2}]_x|\partial_x u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]$$
$$+ \mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}-1}\int_{\mathcal{O}} \mathcal{I}_h^x\big\{[|u_h|^{-p-2}]_y|\partial_y u_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]$$

$$\leq \mathbb{E}[R(0)^{\overline{p}}] + C\kappa^4 \mathbb{E}\Big[\int_0^{T_h} R(s)^{\overline{p}}\,\mathrm{d}s\Big]. \tag{4.75}$$

Applying Gronwall's lemma concludes the proof. ∎

4.2. Hölder continuity in time

For compactness in time, the first step is to establish uniform Hölder continuity for the stochastic integral. In particular, we will prove the following lemma:

Lemma 4.3. *Let $T_{\max} > 0$, $\overline{p} > 1$, $\nu \in (0, \frac{1}{2})$ and let Assumptions (S), (I), (P), (B), (R), and (B3*) hold true. Assume $2\nu\overline{p} > 1$. Then for solutions (u_h, p_h) to (3.6), the stochastic integral*

$$I_h(t) := \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \sum_{k\in I_h^x, l\in I_h^y} I_{ijkl}(t)\mathrm{e}_i^x(\widetilde{x})\mathrm{e}_j^y(\widetilde{y}) \tag{4.76}$$

with

$$\begin{aligned} I_{ijkl}(t) := M_{ij}^{-1}\int_0^{t\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_{h,\mathrm{loc}}^x\{\partial_x(\lambda_{kl}^x u_h \widetilde{\mathfrak{g}}_{h,kl})\mathrm{e}_i^x(x)\mathrm{e}_i^y(y)\}\}\,\mathrm{d}x\,\mathrm{d}y\;\mathrm{d}\beta_{kl}^x \\ + M_{ij}^{-1}\int_0^{t\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^x\{\mathcal{I}_{h,\mathrm{loc}}^y\{\partial_y(\lambda_{kl}^x u_h \widetilde{\mathfrak{g}}_{h,kl})\mathrm{e}_i^x(x)\mathrm{e}_i^y(y)\}\}\,\mathrm{d}x\,\mathrm{d}y\;\mathrm{d}\beta_{kl}^y \end{aligned} \tag{4.77}$$

is contained in $L^{2\overline{p}}(\Omega; C^\beta([0,T_{\max}]; L^2(\mathcal{O})))$ with $\beta := \nu - \frac{1}{2\overline{p}}$.

Proof. According to [30, Lemma 2.1], it suffices to show that

$$\begin{aligned} \widehat{Z}^x(s)(\omega) := \chi_{T_h}\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y} M_{ij}^{-1}\mathrm{e}_i^x(\widetilde{x})\mathrm{e}_j^y(\widetilde{y}) \\ \times \int_{\mathcal{O}} \mathcal{I}_h^y\Big\{\mathcal{I}_{h,\mathrm{loc}}^x\Big\{\partial_x\Big(u_h \sum_{k\in I_h^x, l\in I_h^y}\langle \mathfrak{g}_{kl},\omega\rangle_{L^2}\widetilde{\mathfrak{g}}_{h,kl}\Big)\mathrm{e}_i^x(x)\mathrm{e}_j^y(y)\Big\}\Big\}\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{4.78}$$

and

$$\begin{aligned} \widehat{Z}^y(s)(\omega) := \chi_{T_h}\sum_{i=1}^{\dim U_h^x}\sum_{j=1}^{\dim U_h^y} M_{ij}^{-1}\mathrm{e}_i^x(\widetilde{x})\mathrm{e}_j^y(\widetilde{y}) \\ \times \int_{\mathcal{O}} \mathcal{I}_h^x\Big\{\mathcal{I}_{h,\mathrm{loc}}^y\Big\{\partial_y\Big(u_h \sum_{k\in I_h^x, l\in I_h^y}\langle \mathfrak{g}_{kl},\omega\rangle_{L^2}\widetilde{\mathfrak{g}}_{h,kl}\Big)\mathrm{e}_i^x(x)\mathrm{e}_j^y(y)\Big\}\Big\}\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{4.79}$$

are progressively measurable and contained in $L^{2\overline{p}}(\Omega \times (0, T_{\max}); L_2(L^2(\mathcal{O}); L^2(\mathcal{O})))$ with a uniform bound in h to establish $I_h \in L^{2\overline{p}}(\Omega; W^{\nu,2\overline{p}}(0, T_{\max}; L^2(\mathcal{O})))$. Then, the continuous embedding

$$W^{\nu,2\overline{p}}(0, T_{\max}; L^2(\Omega)) \hookrightarrow C^{\nu-\frac{1}{2\overline{p}}}([0, T_{\max}]; L^2(\mathcal{O})) \tag{4.80}$$

completes the proof. Recalling the computations from (4.44) and using (B3), we immediately obtain the bounds

$$\|\hat{Z}^x\|^2_{L_2(L^2(\mathcal{O});L^2(\mathcal{O}))} \leq \chi_{T_h} C \int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y + C \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y, \tag{4.81}$$

$$\|\hat{Z}^y\|^2_{L_2(L^2(\mathcal{O});L^2(\mathcal{O}))} \leq \chi_{T_h} C \int_{\mathcal{O}} \mathcal{I}_h^x\{|\partial_y u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y + C \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|u_h|^2\} \,\mathrm{d}x\,\mathrm{d}y. \tag{4.82}$$

Progressive measurability is satisfied due to the pathwise continuity of the u_h $\mathbb{P}$-almost surely. Hence, the result follows by Proposition 4.2. ■

In order to show compactness in time, we shall use Lemma 4.3 to establish the Hölder continuity of u_h as a mapping from $[0, T_{\max}]$ into appropriate Sobolev spaces.

Lemma 4.4. *Let $T_{\max} > 0$ and let Assumptions (S), (I), (P), (B), (R), and (B3*) hold true. Then, for $\overline{p}$ sufficiently large, a solution u_h to system (3.6) is uniformly bounded in $L^\sigma(\Omega; C^{1/4}([0, T_{\max}]; (H^1_{\mathrm{per}}(\mathcal{O}))'))$ for $\sigma < 8/5$, i.e.,*

$$\mathbb{E}\Big[\Big(\sup_{t_1,t_2\in[0,T_{\max}]} \frac{\|u_h(t_2) - u_h(t_1)\|_{(H^1_{\mathrm{per}}(\mathcal{O}))'}}{|t_2 - t_1|^{1/4}}\Big)^\sigma\Big] \leq C. \tag{4.83}$$

Proof. Denoting the standard L^2-projection onto U_h by $\mathcal{P}_{U_h}$, we obtain

$$\begin{aligned}
\|u_h(t_2) - u_h(t_1)\|_{(H^1_{\mathrm{per}}(\mathcal{O}))'} &= \sup_{0\neq\psi\in H^1_{\mathrm{per}}(\mathcal{O})} \Big(\|\psi\|^{-1}_{H^1(\mathcal{O})}\Big|\int_{\mathcal{O}} (u_h(t_2) - u_h(t_1))\psi \,\mathrm{d}x\,\mathrm{d}y\Big|\Big) \\
&\leq \sup_{0\neq\psi\in H^1_{\mathrm{per}}(\mathcal{O})} \Big(\|\psi\|^{-1}_{H^1(\mathcal{O})}\Big|\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{(u_h(t_2) - u_h(t_1))\mathcal{P}_{U_h}\{\psi\}\} \,\mathrm{d}x\,\mathrm{d}y\Big|\Big) \\
&\quad + \sup_{0\neq\psi\in H^1_{\mathrm{per}}(\mathcal{O})} \Big(\|\psi\|^{-1}_{H^1(\mathcal{O})}\Big|\int_{\mathcal{O}} (I - \mathcal{I}_h^{xy})\{(u_h(t_2) - u_h(t_1))\mathcal{P}_{U_h}\{\psi\}\} \,\mathrm{d}x\,\mathrm{d}y\Big|\Big) \\
&=: \sup_{0\neq\psi\in H^1_{\mathrm{per}}(\mathcal{O})} \big(\|\psi\|^{-1}_{H^1(\mathcal{O})}|I|\big) + \sup_{0\neq\psi\in H^1_{\mathrm{per}}(\mathcal{O})} \big(\|\psi\|^{-1}_{H^1(\mathcal{O})}|II|\big).
\end{aligned} \tag{4.84}$$

To derive an estimate for $|I|$, we start with the following identity:

$$\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{(u_h(t_2)-u_h(t_1))\phi_h\}\,\mathrm{d}x\,\mathrm{d}y + \int_{t_1\wedge T_h}^{t_2\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^{y}\big\{[G''(u_h)]_x^{-1}\partial_x p_h \partial_x \phi_h\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$+ \int_{t_1\wedge T_h}^{t_2\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^{x}\big\{[G''(u_h)]_y^{-1}\partial_y p_h \partial_y \phi_h\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$= \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{(I_h(t_2)-I_h(t_1))\phi_h\}\,\mathrm{d}x\,\mathrm{d}y \tag{4.85}$$

resulting from (3.6a) for $0 \le t_1 < t_2$. Using (4.85) with $\phi_h = \mathcal{P}_{U_h}\{\psi\}$ and Hölder's inequality, we deduce the estimate

$$|I| \le \Big(\int_{t_1\wedge T_h}^{t_2\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^{y}\big\{[G''(u_h)]_x^{-1}|\partial_x p_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big)^{1/2}$$
$$\times \Big(\int_{t_1\wedge T_h}^{t_2\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^{y}\big\{[G''(u_h)]_x^{-1}|\partial_x \mathcal{P}_{U_h}\{\psi\}|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big)^{1/2}$$
$$+ \Big(\int_{t_1\wedge T_h}^{t_2\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^{x}\big\{[G''(u_h)]_y^{-1}|\partial_y p_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big)^{1/2}$$
$$\times \Big(\int_{t_1\wedge T_h}^{t_2\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^{x}\big\{[G''(u_h)]_y^{-1}|\partial_y \mathcal{P}_{U_h}\{\psi\}|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big)^{1/2}$$
$$+ C\|I_h(t_2)-I_h(t_1)\|_{L^2(\mathcal{O})}\|\mathcal{P}_{U_h}\{\psi\}\|_{L^2(\mathcal{O})}. \tag{4.86}$$

In order to derive bounds for the first term on the right-hand side, we use the estimate

$$\Big(\int_{t_1\wedge T_h}^{t_2\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^{y}\big\{[G''(u_h)]_x^{-1}|\partial_x \mathcal{P}_{U_h}\{\psi\}|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big)^{1/2}$$
$$\le C\Big(\int_{t_1\wedge T_h}^{t_2\wedge T_h} \|u_h\|_{L^\infty(\mathcal{O})}^2 \|\partial_x \mathcal{P}_{U_h}\{\psi\}\|_{L^2(\mathcal{O})}^2\,\mathrm{d}s\Big)^{1/2}$$
$$\le C\big((t_2-t_1)^{3/4}\|u_h\|_{L^8(0,T_{\max};L^\infty(\mathcal{O}))}^2\big)^{1/2}\|\psi\|_{H^1(\mathcal{O})}$$
$$\le C(t_2-t_1)^{3/8}\|\psi\|_{H^1(\mathcal{O})}\|\Delta_h u_h\|_{L^2(0,T_{\max};L^2(\mathcal{O}))}^{1/4}\|u_h\|_{L^\infty(0,T_{\max};H^1(\mathcal{O}))}^{3/4}$$
$$+ C(t_2-t_1)^{3/8}\|\psi\|_{H^1(\mathcal{O})}\|u_h\|_{L^\infty(0,T_{\max};H^1(\mathcal{O}))}. \tag{4.87}$$

In the last step, we used the inequality

$$\|u_h\|_{L^8(0,T_{\max};L^\infty(\mathcal{O}))} \le C\|\Delta_h u_h\|_{L^2(0,T_{\max};L^2(\mathcal{O}))}^{1/4}\|u_h\|_{L^\infty(0,T_{\max};H^1(\mathcal{O}))}^{3/4}$$
$$+ C\|u_h\|_{L^\infty(0,T_{\max};H^1(\mathcal{O}))},$$

which follows from the discrete Gagliardo–Nirenberg inequality proven in Corollary A.4. Using similar computations for the second term on the right-hand side of (4.86) and estimating the remaining term with the help of Lemma 4.3, we obtain the following:

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\neq\psi\in H^1_{per}(\mathcal{O})}\big(\|\psi\|^{-1}_{H^1(\mathcal{O})}|I|\big)^{\sigma}\Big]\\
&\quad\leq C(t_2-t_1)^{3\sigma/8}\mathbb{E}\Big[\Big(\int_{t_1\wedge T_h}^{t_2\wedge T_h}\int_{\mathcal{O}}\mathcal{I}_h^y\big\{[G''(u_h)]_x^{-1}|\partial_x p_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big)^{\sigma/2}\\
&\qquad\quad\times\big(\|\Delta_h u_h\|^{\sigma/4}_{L^2(0,T_{\max};L^2(\mathcal{O}))}\|u_h\|^{3\sigma/4}_{L^\infty(0,T_{\max};H^1(\mathcal{O}))}+\|u_h\|^{\sigma}_{L^\infty(0,T_{\max};H^1(\mathcal{O}))}\big)\Big]\\
&\qquad+C(t_2-t_1)^{3\sigma/8}\mathbb{E}\Big[\Big(\int_{t_1\wedge T_h}^{t_2\wedge T_h}\int_{\mathcal{O}}\mathcal{I}_h^x\big\{[G''(u_h)]_y^{-1}|\partial_y p_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big)^{\sigma/2}\\
&\qquad\quad\times\big(\|\Delta_h u_h\|^{\sigma/4}_{L^2(0,T_{\max};L^2(\mathcal{O}))}\|u_h\|^{3\sigma/4}_{L^\infty(0,T_{\max};H^1(\mathcal{O}))}+\|u_h\|^{\sigma}_{L^\infty(0,T_{\max};H^1(\mathcal{O}))}\big)\Big]\\
&\qquad+C(t_2-t_1)^{\sigma\beta}.
\end{aligned}\tag{4.88}$$

Applying Young's inequality, we end up with

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\neq\psi\in H^1_{per}(\mathcal{O})}\big(\|\psi\|^{-1}_{H^1(\mathcal{O})}|I|\big)^{\sigma}\Big]\\
&\quad\leq C(t_2-t_1)^{3\sigma/8}\mathbb{E}\Big[\int_{t_1\wedge T_h}^{t_2\wedge T_h}\int_{\mathcal{O}}\mathcal{I}_h^y\big\{[G''(u_h)]_x^{-1}|\partial_x p_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]\\
&\qquad+C(t_2-t_1)^{3\sigma/8}\mathbb{E}\Big[\int_{t_1\wedge T_h}^{t_2\wedge T_h}\int_{\mathcal{O}}\mathcal{I}_h^x\big\{[G''(u_h)]_y^{-1}|\partial_y p_h|^2\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big]\\
&\qquad+C(t_2-t_1)^{3\sigma/8}\mathbb{E}\Big[\|\Delta_h u_h\|^{\sigma/(4-2\sigma)}_{L^2(0,T_{\max};L^2(\mathcal{O}))}\|u_h\|^{3\sigma/(4-2\sigma)}_{L^\infty(0,T_{\max};H^1(\mathcal{O}))}\\
&\qquad+\|u_h\|^{2/(2-\sigma)}_{L^\infty(0,T_{\max};H^1(\mathcal{O}))}\Big]+C(t_2-t_1)^{\sigma\beta}.
\end{aligned}\tag{4.89}$$

As we assumed $\sigma<8/5$, we have $\sigma/(4-2\sigma)<2$. Therefore, we may apply Young's inequality once again to obtain

$$\begin{aligned}
\|\Delta_h u_h\|^{\sigma/(4-2\sigma)}_{L^2(0,T_{\max};L^2(\mathcal{O}))}\|u_h\|^{3\sigma/(4-2\sigma)}_{L^\infty(0,T_{\max};H^1(\mathcal{O}))}&\leq C\|\Delta_h u_h\|^2_{L^2(0,T_{\max};L^2(\mathcal{O}))}\\
&\quad+C\|u_h\|^{6\sigma/(8-5\sigma)}_{L^\infty(0,T_{\max};H^1(\mathcal{O}))}.
\end{aligned}\tag{4.90}$$

In view of Proposition 4.2, we conclude

$$\mathbb{E}\Big[\sup_{0\neq\psi\in H^1_{per}(\mathcal{O})}\big(\|\psi\|^{-1}_{H^1(\mathcal{O})}|I|^{\sigma}\big)\Big]\leq C(t_2-t_1)^{3\sigma/8}+C(t_2-t_1)^{\sigma\beta}\tag{4.91}$$

for $\overline{p}$ large enough.

Using (A.1e), an inverse estimate, and the stability properties of $\mathcal{P}_{U_h}$, we obtain

$$|II|\leq Ch\|u_h(t_2)-u_h(t_1)\|_{L^2(\mathcal{O})}\|\psi\|_{H^1(\mathcal{O})}.\tag{4.92}$$

To obtain an estimate for $\|u_h(t_2)-u_h(t_1)\|_{L^2(\mathcal{O})}$, we again start with identity (4.85) and choose $\phi_h=(u_h(t_2)-u_h(t_1))$. This provides the estimate

$$\|u_h(t_2)-u_h(t_1)\|_h^2\leq\Big|\int_{t_1\wedge T_h}^{t_2\wedge T_h}\int_{\mathcal{O}}\mathcal{I}_h^y\big\{[G''(u_h)]_x^{-1}\partial_x p_h\partial_x(u_h(t_2)-u_h(t_1))\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big|$$

$$+ \Big| \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \big\{ [G''(u_h)]_y^{-1} \partial_y p_h \partial_y (u_h(t_2) - u_h(t_1)) \big\} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s \Big|$$

$$+ \| I_h(t_2) - I_h(t_1) \|_{L^2(\mathcal{O})} \| u_h(t_2) - u_h(t_1) \|_{L^2(\mathcal{O})}$$

$$=: N_1 + N_2 + N_3. \tag{4.93}$$

Using Hölder's inequality and Young's inequality, we compute

$$\begin{aligned} N_1 &\le \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \big\{ |[G''(u_h)]_x^{-1} \partial_x p_h|^{4/3} \big\} \, \mathrm{d}x \, \mathrm{d}y \Big)^{3/4} \\ &\quad \times \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \big\{ |\partial_x (u(t_2) - u_h(t_1))|^4 \big\} \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/4} \mathrm{d}s \\ &\le C (t_2 - t_1)^{1/2} \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \big\{ \big| [G''(u_h)]_x^{-1} \partial_x p_h \big|^{4/3} \big\} \, \mathrm{d}x \, \mathrm{d}y \Big)^{3/2} \mathrm{d}s \\ &\quad + C (t_2 - t_1)^{-1/2} \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \big\{ |\partial_x (u_h(t_2) - u_h(t_1))|^4 \big\} \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/2} \mathrm{d}s \\ &=: (t_2 - t_1)^{1/2} N_{1_a} + (t_2 - t_1)^{-1/2} N_{1_b}. \end{aligned} \tag{4.94}$$

For the first term, we obtain from Hölder's inequality

$$\begin{aligned} N_{1_a} &\le C \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \big\{ [G''(u_h)]_x^{-1} |\partial_x p_h|^2 \big\} \, \mathrm{d}x \, \mathrm{d}y \Big) \\ &\quad \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \big\{ \big| [G''(u_h)]_x^{-1} \big|^2 \big\} \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/2} \mathrm{d}s \\ &\le C \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} R(s) \int_{\mathcal{O}} \mathcal{I}_h^y \big\{ [G''(u_h)]_x^{-1} |\partial_x p_h|^2 \big\} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s, \end{aligned} \tag{4.95}$$

where we used

$$\begin{aligned} \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \big\{ \big| [G''(u_h)]_x^{-1} \big|^2 \big\} \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/2} &\le C \Big(\int_{\mathcal{O}} |u_h|^4 \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/2} \\ &\le C \| u_h \|_{H^1(\mathcal{O})}^2 \le C R(s). \end{aligned} \tag{4.96}$$

Concerning N_{1_b}, we obtain from a discrete version of the Gagliardo–Nirenberg inequality (cf. Corollary A.4)

$$\begin{aligned} N_{1_b} &\le C (t_2 - t_1) \| u_h(t_2) - u_h(t_1) \|_{W^{1,4}(\mathcal{O})}^2 \\ &\le C (t_2 - t_1) \| \Delta_h u_h(t_2) - \Delta_h u_h(t_1) \|_h \| u_h(t_2) - u_h(t_1) \|_{H^1(\mathcal{O})} \\ &\quad + C (t_2 - t_1) \| u_h(t_2) - u_h(t_1) \|_{H^1(\mathcal{O})}^2. \end{aligned} \tag{4.97}$$

Estimates for N_2 can be derived in a similar manner. To derive bounds for N_3, we use the results from Lemma 4.3 and obtain

$$\begin{aligned}\mathbb{E}[N_3] &\le \delta\mathbb{E}\big[\|u_h(t_2)-u_h(t_1)\|_h^2\big] + C_\delta\mathbb{E}\big[\|I_h(t_2)-I_h(t_1)\|_{L^2(\mathcal{O})}^2\big]\\ &\le \delta\mathbb{E}\big[\|u_h(t_2)-u_h(t_1)\|_h^2\big] + C_\delta(t_2-t_1)^{2\beta}.\end{aligned} \tag{4.98}$$

Since the first term on the right-hand side of (4.98) can be absorbed for δ small enough, we obtain

$$\begin{aligned}&\mathbb{E}\big[h^\sigma\|u_h(t_2)-u_h(t_1)\|_{L^2(\mathcal{O})}^\sigma\big]\\ &\quad\le C(t_2-t_1)^{\sigma/4}h^\sigma\mathbb{E}\Big[\int_{t_1\wedge T_h}^{t_2\wedge T_h} R(s)\int_{\mathcal{O}}\mathcal{I}_h^y\big\{[G''(u_h)]_x^{-1}|\partial_x p_h|^2\big\}\,dx\,dy\,ds\Big]\\ &\qquad + C(t_2-t_1)^{\sigma/4}h^\sigma\mathbb{E}\Big[\int_{t_1\wedge T_h}^{t_2\wedge T_h} R(s)\int_{\mathcal{O}}\mathcal{I}_h^x\big\{[G''(u_h)]_y^{-1}|\partial_y p_h|^2\big\}\,dx\,dy\,ds\Big]\\ &\qquad + C(t_2-t_1)^{\sigma/4}h^\sigma\mathbb{E}\big[\|\Delta_h u_h(t_2)-\Delta_h u_h(t_1)\|_h^{\sigma/2}\|u_h(t_2)-u_h(t_1)\|_{H^1(\mathcal{O})}^{\sigma/2}\big]\\ &\qquad + C(t_2-t_1)^{\sigma/4}h^\sigma\mathbb{E}\big[\|u_h(t_2)-u_h(t_1)\|_{H^1(\mathcal{O})}^\sigma\big]\\ &\qquad + Ch^\sigma(t_2-t_1)^{\sigma/4} + Ch^\sigma(t_2-t_1)^{\sigma\beta}\\ &\quad\le Ch^\sigma(t_2-t_1)^{\sigma/4} + C(t_2-t_1)^{\sigma/4}\mathbb{E}\big[\sup_{s\in[0,T_{\max}]}\|u_h(s)\|_{H^1(\mathcal{O})}^\sigma\big] + Ch^\sigma(t_2-t_1)^{\sigma\beta},\end{aligned} \tag{4.99}$$

where we used (2.9). Choosing $\beta \ge 1/4$ (cf. Lemma 4.3) completes the proof. ∎

5. Passage to the limit

5.1. Compactness

As u_h is only strictly positive for $h > 0$, we lack h-independent bounds on the pressure p_h. Therefore, we consider the fluxes

$$J_h^x := \mathcal{I}_h^y\Big\{\sqrt{[G''(u_h)]_x^{-1}}\partial_x p_h\Big\}, \quad J_h^y := \mathcal{I}_h^x\Big\{\sqrt{[G''(u_h)]_y^{-1}}\partial_y p_h\Big\}, \tag{5.1}$$

which are uniformly bounded in $L^2(\Omega; L^2(0,T_{\max}; L^2(\mathcal{O})))$. Note that solutions (u_h, p_h) to (3.6) may be equivalently characterized by $(u_h, \Delta_h u_h, J_h^x, J_h^y)$. In the following, we consider these objects in the spaces

$$\mathcal{X}_u := C([0,T_{\max}]; L^q(\mathcal{O})), \quad q < \infty, \tag{5.2a}$$

$$\mathcal{X}_{\Delta u} := (L^2(0,T_{\max}; L^2(\mathcal{O})))_{\text{weak}}, \tag{5.2b}$$

$$\mathcal{X}_{J^x} := (L^2(0,T_{\max}; L^2(\mathcal{O})))_{\text{weak}}, \tag{5.2c}$$

$$\mathcal{X}_{J^y} := (L^2(0,T_{\max}; L^2(\mathcal{O})))_{\text{weak}}. \tag{5.2d}$$

Lemma 5.1. *Let $T_{\max} > 0$ be arbitrary but fixed. Let $(u_h, \Delta_h u_h, J_h^x, J_h^y)_h$ be a sequence of discrete solutions to (3.6). Then the families of laws $(\mu_{u_h})_h$, $(\mu_{\Delta_h u_h})_h$, $(\mu_{J_h^x})_h$, and $(\mu_{J_h^y})_h$ are tight.*

Proof. From Proposition 4.2 and Lemma 4.4, we obtain that $(u_h)_h$ is uniformly bounded in $L^2(\Omega; L^\infty(0, T_{\max}; H^1(\mathcal{O}))) \cap L^\sigma(\Omega; C^{1/4}([0, T_{\max}]; (H^1_{\mathrm{per}}(\mathcal{O}))'))$ for $\sigma < 8/5$. Due to the well-known compactness theorem by Simon (cf. [66]), the ball $\overline{B}_R$ in the space $L^\infty(0, T_{\max}; H^1_{\mathrm{per}}(\mathcal{O})) \cap C^{1/4}([0, T_{\max}]; (H^1_{\mathrm{per}}(\mathcal{O}))')$ is a compact subset of the space $C([0, T_{\max}]; L^q(\mathcal{O}))$. Furthermore, we have for any $R > 0$

$$\begin{aligned}\mu_{u_h}(\mathcal{X}_u \setminus \overline{B}_R) &= \mathbb{P}\big[\|u_h\|^\sigma_{L^\infty(0,T_{\max};H^1(\mathcal{O}))} + \|u_h\|^\sigma_{C^{1/4}([0,T_{\max}];(H^1_{\mathrm{per}}(\mathcal{O}))')} > R^\sigma\big] \\ &\leq R^{-\sigma}\mathbb{E}\big[\|u_h\|^2_{L^\infty(0,T_{\max};H^1(\mathcal{O}))} + C + \|u_h\|^\sigma_{C^{1/4}([0,T_{\max}];(H^1_{\mathrm{per}}(\mathcal{O}))')}\big],\end{aligned} \tag{5.3}$$

which shows the tightness of $(\mu_{u_h})_h$. As closed balls in $L^2(0, T_{\max}; L^2(\mathcal{O}))$ are compact in the weak topology, the tightness of $(\mu_{\Delta_h u_h})_h$, $(\mu_{J_h^x})_h$ and $(\mu_{J_h^y})_h$ is a direct consequence of Markov's inequality and the bound obtained in Proposition 4.2. ∎

Following the lines of [29], we introduce the Polish space

$$\mathcal{X}_{\boldsymbol{W}} := \left(C([0, T_{\max}]; L^2(\mathcal{O}))\right)^2 \tag{5.4}$$

as an additional path space. Let $\mu_{\boldsymbol{W}} := (\mu_{W^x}, \mu_{W^y})^T$ be the law of

$$\boldsymbol{W} = \Big(\sum_{k,l\in\mathbb{Z}} \lambda^x_{kl}\mathfrak{g}_{kl}\beta^x_{kl}, \sum_{k,l\in\mathbb{Z}} \lambda^y_{kl}\mathfrak{g}_{kl}\beta^y_{kl}\Big)^T. \tag{5.5}$$

As $\mathcal{X}_{\boldsymbol{W}}$ is a Polish space, $\mu_{\boldsymbol{W}}$ is a Radon measure and therefore regular from the interior, i.e.,

$$\mu_{\boldsymbol{W}}\Big(\left(C([0, T_{\max}]; L^2(\mathcal{O}))\right)^2\Big) = \sup\Big\{\mu_{\boldsymbol{W}}(K) : K \subset \left(C([0, T_{\max}]; L^2(\mathcal{O}))\right)^2 \text{ compact}\Big\}. \tag{5.6}$$

To deal with the initial data, we introduce the space $\mathcal{X}_{u_0} := H^1_{\mathrm{per}}(\mathcal{O})$. Together with the tightness results of Lemma 5.1, we obtain the following result:

Lemma 5.2. *On the path space $\mathcal{X} := \mathcal{X}_u \times \mathcal{X}_{\Delta u} \times \mathcal{X}_{J^x} \times \mathcal{X}_{J^y} \times \mathcal{X}_{\boldsymbol{W}} \times \mathcal{X}_{u_0}$ the joint laws μ_h defined by*

$$\begin{aligned}\mu_h(A \times B \times C \times D \times E \times F) := \mathbb{P}\big[&\{u_h \in A\} \cap \{\Delta_h u_h \in B\} \cap \{J_h^x \in C\} \\ &\cap \{J_h^y \in D\} \cap \{\boldsymbol{W} \in E\} \cap \{u^0 \in F\}\big]\end{aligned} \tag{5.7}$$

for $h \in (0, 1]$ are tight.

Using Jakubowski's theorem (cf. [52]), which is a generalization of Skorokhod's theorem (cf. [67]), we obtain the following result:

Proposition 5.3. *Let $(u_h, \Delta_h u_h, J_h^x, J_h^y)$ be solutions to (3.6) in the sense of Lemma 3.7 defined on the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with respect to the Wiener process $\boldsymbol{W}$. Then there exists a subsequence which we again denote by $(u_h, \Delta_h u_h, J_h^x, J_h^y)$ such that there exist a stochastic basis $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, a sequence of random variables*

$$\widetilde{u}_h : \widetilde{\Omega} \to C([0, T_{\max}]; L^q(\mathcal{O})) \quad (q < \infty), \tag{5.8a}$$

$$\widetilde{\Delta_h u_h} : \widetilde{\Omega} \to L^2(0, T_{\max}; L^2(\mathcal{O})), \tag{5.8b}$$

$$\widetilde{J}_h^x : \widetilde{\Omega} \to L^2(0, T_{\max}; L^2(\mathcal{O})), \tag{5.8c}$$

$$\widetilde{J}_h^y : \widetilde{\Omega} \to L^2(0, T_{\max}; L^2(\mathcal{O})), \tag{5.8d}$$

$$\widetilde{u}_h^0 : \widetilde{\Omega} \to H^1_{\text{per}}(\mathcal{O}), \tag{5.8e}$$

a sequence of $(L^2(\mathcal{O}))^2$-valued processes $\widetilde{\boldsymbol{W}}_h$ on $\widetilde{\Omega}$, random variables

$$\widetilde{u} \in L^2(\widetilde{\Omega}; C([0, T_{\max}]; L^q(\mathcal{O}))) \quad (q < \infty), \tag{5.9a}$$

$$\widetilde{\Delta u} \in L^2(\widetilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O}))), \tag{5.9b}$$

$$\widetilde{J}^x \in L^2(\widetilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O}))), \tag{5.9c}$$

$$\widetilde{J}^y \in L^2(\widetilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O}))), \tag{5.9d}$$

$$\widetilde{u}^0 \in L^2(\widetilde{\Omega}; H^2_{\text{per}}(\mathcal{O})), \tag{5.9e}$$

and an $L^2(\mathcal{O})$-valued process $\widetilde{W}$ on $\widetilde{\Omega}$ which satisfy the following properties:

(i) *The law of $(\widetilde{u}_h, \widetilde{\Delta_h u_h}, \widetilde{J}_h^x, \widetilde{J}_h^y, \widetilde{W}_h, \widetilde{u}_h^0)$ on $\mathcal{X}_u \times \mathcal{X}_{\Delta u} \times \mathcal{X}_{J^x} \times \mathcal{X}_{J^y} \times \mathcal{X}_{\boldsymbol{W}} \times \mathcal{X}_{u_0}$ under $\widetilde{\mathbb{P}}$ coincides for any h with the law of $(u_h, \Delta_h u_h, J_h^x, J_h^y, W, u_h^0)$ under $\mathbb{P}$.*

(ii) *The sequence $(\widetilde{u}_h, \widetilde{\Delta_h u_h}, \widetilde{J}_h^x, \widetilde{J}_h^y, \widetilde{W}_h, \widetilde{u}_h^0)$ converges $\widetilde{\mathbb{P}}$-almost surely towards $(\widetilde{u}, \widetilde{\Delta u}, \widetilde{J}^x, \widetilde{J}^y, \widetilde{W}, \widetilde{u}^0)$ in the topology of $\mathcal{X}$.*

Remark 5.4. In particular, one may use the interval $[0, 1]$ for $\widetilde{\Omega}$, its standard Borel σ-algebra for $\widetilde{\mathcal{F}}$, and the Lebesgue measure for $\widetilde{\mathbb{P}}$ (cf. [52]).

Similar to the definition of T_h, we introduce the random stopping times

$$\widetilde{T}_h := T_{\max} \wedge \inf\left\{t \geq 0 : \mathcal{E}_h(\widetilde{u}_h(t)) \geq \mathcal{E}_{\max,h}\right\}. \tag{5.10}$$

Lemma 5.5. *Along a subsequence, the convergence $\lim_{h\searrow 0} \widetilde{T}_h = T_{\max}$ holds $\widetilde{\mathbb{P}}$-almost surely.*

Proof. Following the lines of [29], we compute for each $\tau \in (0, T_{\max}]$

$$\widetilde{\mathbb{P}}\left[\left\{\widetilde{T}_h < \tau\right\}\right] = \mathbb{P}\left[\left\{T_h < \tau\right\}\right] \leq C h^{\rho/(2+p)}. \tag{5.11}$$

Hence, $\widetilde{T}_h \to T_{\max}$ in probability for $h \searrow 0$, which implies the $\widetilde{\mathbb{P}}$-almost sure convergence for a subsequence. ∎

Lemma 5.6. *Under Assumptions (S), (I), (P), (B), (R), and (B3*), $\widetilde{\Delta_h u_h}$ can be identified $\widetilde{\mathbb{P}}$-almost surely as the discrete Laplacian of $\tilde{u}_h$, i.e.,*

$$\widetilde{\Delta_h u_h} = \Delta_h \tilde{u}_h. \tag{5.12a}$$

Furthermore, the flux components $\tilde{J}_h^x$ and $\tilde{J}_h^y$ can be identified $\widetilde{\mathbb{P}}$-almost surely as

$$\tilde{J}_h^x = \chi_{\tilde{T}_h} \mathcal{I}_h^y \Big\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \partial_x \big(-\Delta_h \tilde{u}_h + \mathcal{I}_h^{xy}\{F'(\tilde{u}_h)\} + h^\varepsilon \Delta_h \Delta_h \tilde{u}_h \big) \Big\}, \tag{5.12b}$$

$$\tilde{J}_h^y = \chi_{\tilde{T}_h} \mathcal{I}_h^x \Big\{ \sqrt{[G''(\tilde{u}_h)]_y^{-1}} \partial_y \big(-\Delta_h \tilde{u}_h + \mathcal{I}_h^{xy}\{F'(\tilde{u}_h)\} + h^\varepsilon \Delta_h \Delta_h \tilde{u}_h \big) \Big\}. \tag{5.12c}$$

Proof. As $\Delta_h u_h$ depends continuously on u_h (cf. (2.9)), (5.12a) follows by equality of laws.

As for every fixed $h > 0$, the functions $u_h, \tilde{u}_h$ are almost surely in $C([0, T_{\max}]; C(\overline{\mathcal{O}}))$, the stopping times $T_h(\omega)$ and $\tilde{T}_h(\tilde{\omega})$ are also continuous functions of u_h and $\tilde{u}_h$, respectively. By inverse estimates (cf. [10, Theorem 4.5.11]) and the oscillation lemma (Lemma 3.6), the same holds true for the terms on the right-hand side of equations (5.12b) and (5.12c). In particular, the expectation

$$\widetilde{\mathbb{E}}\Big[\Big|\int_0^{T_{\max}} \int_{\mathcal{O}} \tilde{J}_h^x \phi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \int_0^{T_{\max}} \chi_{\tilde{T}_h} \mathcal{I}_h^y \Big\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \partial_x \big(-\Delta_h \tilde{u}_h + \mathcal{I}_h^{xy}\{F'(\tilde{u}_h)\} + h^\varepsilon \Delta_h \Delta_h \tilde{u}_h \big) \Big\} \Big|\Big] \tag{5.13}$$

coincides with

$$\widetilde{\mathbb{E}}\Big[\Big|\int_0^{T_{\max}} \int_{\mathcal{O}} J_h^x \phi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \int_0^{T_{\max}} \chi_{T_h} \mathcal{I}_h^y \Big\{ \sqrt{[G''(u_h)]_x^{-1}} \partial_x \big(-\Delta_h u_h + \mathcal{I}_h^{xy}\{F'(u_h)\} + h^\varepsilon \Delta_h \Delta_h u_h \big) \Big\} \Big|\Big] \tag{5.14}$$

for arbitrary $\phi \in C^\infty([0, T_{\max}]; (C^\infty_{\mathrm{per}}(\overline{\mathcal{O}})))$. As the latter is equal to zero, this gives the claim with respect to $\tilde{J}_h^x$. The argumentation for $\tilde{J}_h^y$ is the same. ∎

Corollary 5.7. *Let the assumptions of Proposition 5.3 and Lemma 5.6 hold true. Then the limit function $\widetilde{\Delta u}$ introduced in (5.8b) can be identified with the Laplacian of $\tilde{u}$ $\widetilde{\mathbb{P}}$-almost surely.*

Proof. Choosing $\phi \in C^\infty_{\mathrm{per}}(\overline{\mathcal{O}})$, a Taylor expansion provides

$$\partial_x^{-h_x} \partial_x^{+h_x} \phi + \partial_y^{-h_y} \partial_y^{+h_y} \phi \to \Delta \phi$$

in $L^\infty(\mathcal{O})$. Therefore, shifting the discrete Laplacian onto the test function before passing to the limit provides the desired result. ∎

We proceed by showing that $\widetilde{\boldsymbol{W}}$ and $\widetilde{\boldsymbol{W}}_h$ are Q-Wiener processes adapted to suitably defined filtrations $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ and $(\widetilde{\mathcal{F}}_{h,t})_{t\geq 0}$, respectively. We define $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ to be the $\widetilde{\mathbb{P}}$-augmented canonical filtration associated with $(\widetilde{u}, \widetilde{\boldsymbol{W}}, \widetilde{u}^0)$, i.e.,

$$\widetilde{\mathcal{F}}_t := \sigma\big(\sigma(r_t\widetilde{u}, r_t\widetilde{\boldsymbol{W}}) \cup \big\{N \in \widetilde{\mathcal{F}} : \widetilde{\mathbb{P}}(N) = 0\big\} \cup \sigma(\widetilde{u}^0)\big). \tag{5.15}$$

Here, r_t is the restriction of a function defined on $[0, T_{\max}]$ to the interval $[0, t]$ with $t \in [0, T_{\max}]$. Analogously, we introduce the filtrations $(\widetilde{\mathcal{F}}_{h,t})_{t\geq 0}$ as the $\widetilde{\mathbb{P}}$-augmented canonical filtration associated with $(\widetilde{u}_h, \widetilde{\boldsymbol{W}}_h, \widetilde{u}_h^0)$,

$$\widetilde{\mathcal{F}}_{h,t} := \sigma\big(\sigma(r_t\widetilde{u}_h, r_t\widetilde{\boldsymbol{W}}_h) \cup \big\{N \in \widetilde{\mathcal{F}} : \widetilde{\mathbb{P}}(N) = 0\big\} \cup \sigma(\widetilde{u}_h^0)\big). \tag{5.16}$$

Lemma 5.8. *The processes $\widetilde{\boldsymbol{W}}_h$ and $\widetilde{\boldsymbol{W}}$ are Q-Wiener processes adapted to the filtrations $(\widetilde{\mathcal{F}}_{h,t})_{t\geq 0}$ and $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$. They can be written as*

$$\widetilde{\boldsymbol{W}}_h(t) = \sum_{\alpha\in\{x,y\}}\sum_{k,l\in\mathbb{Z}} \lambda_{kl}^{\alpha} \mathfrak{g}_{kl} \widetilde{\beta}_{h,kl}^{\alpha} \boldsymbol{b}_\alpha \quad \text{and} \quad \widetilde{\boldsymbol{W}}(t) = \sum_{\alpha\in\{x,y\}}\sum_{k,l\in\mathbb{Z}} \lambda_{kl}^{\alpha} \mathfrak{g}_{kl} \widetilde{\beta}_{kl}^{\alpha} \boldsymbol{b}_\alpha. \tag{5.17}$$

Here, $(\widetilde{\beta}_{h,kl}^{\alpha})_{\alpha\in\{x,y\},k,l\in\mathbb{Z}}$ and $(\widetilde{\beta}_{kl}^{\alpha})_{\alpha\in\{x,y\},k,l\in\mathbb{Z}}$ are families of i.i.d. Brownian motions with respect to $(\widetilde{\mathcal{F}}_{h,t})_{t\geq 0}$ and $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$.

For a proof we refer to [29]. Combining the results of Proposition 5.3 and Lemma 5.6 with the discrete Gagliardo–Nirenberg inequality allows us to establish improved convergence results.

Lemma 5.9. *Let $\widetilde{u}_h$ and $\Delta_h\widetilde{u}_h$ be the random variables identified in Proposition 5.3. Furthermore, let Assumptions (S), (I), (P), (B), (R), and (B3*) hold true and let $q \in (1, \infty)$. Then $\widetilde{u}_h$ converges strongly along a subsequence towards $\widetilde{u}$ in $L^2(0, T_{\max}; W^{1,q}(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-almost surely.*

Proof. We follow the lines of [60, Lemma 5.1]. Using Hölder's inequality, we compute

$$\|\widetilde{u}_h - \widetilde{u}\|_{L^2(0,T_{\max};W^{1,q}(\mathcal{O}))} \leq C\,\|\widetilde{u}_h - \widetilde{u}\|_{L^2(0,T_{\max};H^1(\mathcal{O}))}^{1/q} \|\widetilde{u}_h - \widetilde{u}\|_{L^2(0,T_{\max};W^{1,2q-2}(\mathcal{O}))}^{(q-1)/q}. \tag{5.18}$$

Due to the discrete Gagliardo–Nirenberg inequality (cf. Lemma A.2), we have that $\widetilde{u} \in L^2(0, T_{\max}; W_{\text{per}}^{1,q}(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-almost surely. As $\widetilde{u}_h$ is $\widetilde{\mathbb{P}}$-almost surely in the space $U_h \cap L^\infty(0, T_{\max}; H_{\text{per}}^1(\mathcal{O}))$ with $\Delta_h\widetilde{u}_h \in L^2(0, T_{\max}; L^2(\mathcal{O}))$, we may use the discrete Gagliardo–Nirenberg inequality (cf. Corollary A.4) to show that $\widetilde{u}$ is also $\widetilde{\mathbb{P}}$-almost surely in $L^2(0, T_{\max}; W_{\text{per}}^{1,q}(\mathcal{O}))$. Therefore, it suffices to show that $\widetilde{u}_h$ converges strongly towards $\widetilde{u}$ in $L^2(0, T_{\max}; H^1(\mathcal{O}))$. Using the triangle inequality, we derive

$$\begin{aligned}\|\widetilde{u}_h - \widetilde{u}\|_{L^2(0,T_{\max};H^1(\mathcal{O}))} \leq{}& \|\widetilde{u}_h - \mathcal{R}\{\widetilde{u}\}\|_{L^2(0,T_{\max};H^1(\mathcal{O}))} \\ &+ \|\mathcal{R}\{\widetilde{u}\} - \widetilde{u}\|_{L^2(0,T_{\max};H^1(\mathcal{O}))},\end{aligned} \tag{5.19}$$

where $\mathcal{R}$ is the Ritz projection operator defined in (2.17). As $\mathcal{R}\{\widetilde{u}\}$ converges strongly towards $\widetilde{u}$ in $H^1(\mathcal{O})$ and since $\widetilde{u}_h$ is $L^\infty(H^1)$-regular, it only remains to show that the first term on the right-hand side vanishes. We define $A_h : U_h \to U_h \cap H^1_*(\mathcal{O})$ via $\int_{\mathcal{O}} (A_h\phi_h)\psi_h \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathcal{O}} \nabla\phi_h \cdot \nabla\psi_h \,\mathrm{d}x\,\mathrm{d}y$ for all $\phi_h, \psi_h \in U_h$ and compute

$$
\begin{aligned}
\|\nabla\tilde{u}_h - \nabla\mathcal{R}\{\tilde{u}\}\|^2_{L^2(0,T;L^2(\mathcal{O}))} &\le \Big|\int_0^{T_{\max}}\int_{\mathcal{O}} \big(A_h(\tilde{u}_h - \mathcal{R}\{\tilde{u}\})\big)\cdot\big(\tilde{u}_h - \mathcal{R}\{\tilde{u}\}\big)\,\mathrm{d}x\,\mathrm{d}y\Big| \\
&\le \|A_h(\tilde{u}_h - \mathcal{R}\{\tilde{u}\})\|_{L^2(0,T_{\max};L^2(\mathcal{O}))}\|\tilde{u}_h - \mathcal{R}\{\tilde{u}\}\|_{L^2(0,T_{\max};L^2(\mathcal{O}))} \\
&\le C\|\tilde{u}_h - \mathcal{R}\{\tilde{u}\}\|_{L^2(0,T_{\max};L^2(\mathcal{O}))}. \qquad (5.20)
\end{aligned}
$$

Together with the strong convergence of $\tilde{u}_h$ in $L^2(0,T_{\max};L^2(\mathcal{O}))$ $\tilde{\mathbb{P}}$-almost surely, which we have from Proposition 5.3, and the strong convergence of $\mathcal{R}\{\tilde{u}\}$ towards $\tilde{u}$, we complete the proof. ∎

5.2. Convergence of the deterministic terms

In this section, we identify the limit functions $\tilde{J}^x_h$ and $\tilde{J}^y_h$ introduced in Proposition 5.3 and use the a priori estimates from Proposition 4.2 to establish additional (weak) convergence properties. In order to identify the limit of the fluxes, we consider the discrete pressure

$$U_h \ni \tilde{p}_h := -\chi_{\tilde{T}_h}\Delta_h\tilde{u}_h + \chi_{\tilde{T}_h}\mathcal{I}^{xy}_h\{F'(\tilde{u}_h)\} + \chi_{\tilde{T}_h}h^{\varepsilon}\Delta_h\Delta_h\tilde{u}_h$$

$\tilde{\mathbb{P}}$-almost everywhere. In addition, we introduce the sets

$$S_\delta := \big\{(\tilde{\omega}, t, (x,y)) \in \tilde{\Omega}\times(0,T_{\max})\times\mathcal{O} : \tilde{u}(\omega,t,x,y) > \delta\big\}, \qquad (5.21a)$$

$$S_\delta(\tilde{\omega},t) := \big\{(x,y)\in\mathcal{O} : \tilde{u}(\tilde{\omega},t,(x,y)) > \delta\big\}, \qquad (5.21b)$$

$$
\begin{aligned}
S^{\mathcal{Q}_h}_\delta := \big\{&(\tilde{\omega}, t, (x,y)) \in \tilde{\Omega}\times(0,T_{\max})\times\mathcal{O} : \exists Q \in \mathcal{Q}_h \text{ s.t. } (x,y)\in Q \\
&\text{and}\, \tilde{u}_h(\tilde{\omega},t,\cdot)|_Q > \delta\big\},
\end{aligned} \qquad (5.21c)
$$

$$S^{\mathcal{Q}_h}_\delta(\tilde{\omega},t) := \big\{(x,y)\in\mathcal{O} : \exists Q\in\mathcal{Q}_h \text{ s.t. } (x,y)\in Q \text{ and } \tilde{u}_h(\tilde{\omega},t,\cdot)|_Q > \delta\big\}. \quad (5.21d)$$

On these superlevel sets, we will be able to identify the limit functions of $\tilde{p}_h$, $\tilde{J}^x_h$, and $\tilde{J}^y_h$. In particular, the following lemma holds true:

Lemma 5.10. *Let $\tilde{u}_h$ and $\tilde{u}$ be the random variables identified in Proposition 5.3. Furthermore, let Assumptions (S), (I), (P), (B), (R), and (B3*) hold true. Then, there exists a subsequence, again denoted by $\tilde{u}_h$, such that for all $q < \infty$ and $\overline{q} < \frac{2q}{q-2}$ the following convergence properties hold true:*

$$\tilde{u}_h \to \tilde{u} \qquad \text{in } L^q(\tilde{\Omega}; C([0,T_{\max}]; L^q(\mathcal{O}))), \qquad (5.22a)$$

$$\tilde{u}_h \to \tilde{u} \qquad \text{in } L^{\overline{q}}(\tilde{\Omega}; L^2(0,T_{\max}; W^{1,q}(\mathcal{O}))), \qquad (5.22b)$$

$$\tilde{u}_h \overset{*}{\rightharpoonup} \tilde{u} \qquad \text{in } L^{2\overline{p}}_{\text{weak-}(*)}(\tilde{\Omega}; L^\infty(0,T_{\max}; H^1_{\text{per}}(\mathcal{O}))), \qquad (5.22c)$$

$$\Delta_h\tilde{u}_h \rightharpoonup \Delta\tilde{u} \qquad \text{in } L^2(\tilde{\Omega}; L^2(0,T_{\max}; L^2(\mathcal{O}))), \qquad (5.22d)$$

$$\mathcal{I}^y_h\Big\{\sqrt{[G''(\tilde{u}_h)]^{-1}_x}\Big\} \to \tilde{u} \qquad \text{in } L^q(\tilde{\Omega}; L^\infty(0,T_{\max}; L^q(\mathcal{O}))), \qquad (5.22e)$$

$$\mathcal{I}^x_h\Big\{\sqrt{[G''(\tilde{u}_h)]^{-1}_y}\Big\} \to \tilde{u} \qquad \text{in } L^q(\tilde{\Omega}; L^\infty(0,T_{\max}; L^q(\mathcal{O}))), \qquad (5.22f)$$

where we have used the notation used in [25, Chapter 0.3] *to denote the dual space of* $L^{2\overline{p}/(2\overline{p}-1)}(\widetilde{\Omega}; L^1(0,T_{\max}; (H^1_{per}(\mathcal{O}))'))$.

In addition, we have

$$\chi_{[S^{\mathcal{Q}_h}_{\delta/4}]}\chi[S_\delta]\partial_x\widetilde{p}_h \rightharpoonup \chi[S_\delta]\partial_x\widetilde{p} \quad \text{in } L^2(\widetilde{\Omega}; L^2(0,T_{\max}; L^2(\mathcal{O}))), \tag{5.23a}$$

$$\chi_{[S^{\mathcal{Q}_h}_{\delta/4}]}\chi[S_\delta]\partial_y\widetilde{p}_h \rightharpoonup \chi[S_\delta]\partial_y\widetilde{p} \quad \text{in } L^2(\widetilde{\Omega}; L^2(0,T_{\max}; L^2(\mathcal{O}))), \tag{5.23b}$$

with $\widetilde{p} = -\Delta\widetilde{u} + F'(\widetilde{u})$ *on* S_δ *for all* $\delta > 0$ *and*

$$\widetilde{J}^x_h \rightharpoonup \widetilde{J}^x \quad \text{in } L^2(\widetilde{\Omega}; L^2(0,T_{\max}; L^2(\mathcal{O}))), \tag{5.24a}$$

$$\widetilde{J}^y_h \rightharpoonup \widetilde{J}^y \quad \text{in } L^2(\widetilde{\Omega}; L^2(0,T_{\max}; L^2(\mathcal{O}))), \tag{5.24b}$$

where $\widetilde{J}^x$ *and* $\widetilde{J}^y$ *are the limit functions introduced in Proposition* 5.3. *For every* $\delta > 0$, *we are able to identify these limit functions on the superlevel sets* S_δ *as*

$$\widetilde{J}^x = \widetilde{u}\partial_x(-\Delta\widetilde{u} + F'(\widetilde{u})) \quad \text{and} \quad \widetilde{J}^y = \widetilde{u}\partial_y(-\Delta\widetilde{u} + F'(\widetilde{u})).$$

Proof. By Proposition 5.3, we have in particular $\widetilde{u}_h \to \widetilde{u}$. Choosing $\overline{p}$ sufficiently large in Proposition 4.2 and combining Proposition 5.3 with Vitali's convergence theorem and with the bounds on $\widetilde{u}_h$ and on $\Delta_h\widetilde{u}_h$ (see Proposition 4.2), we obtain the strong convergence of $\widetilde{u}_h$ towards $\widetilde{u}$ in $L^q(\widetilde{\Omega}; C([0,T_{\max}]; L^q(\Omega)))$, which gives (5.22a). To establish (5.22b), we use the continuous Gagliardo–Nirenberg inequality and Lemma A.2 to show that

$$\begin{aligned}\|\widetilde{u}_h - \widehat{u}\|^{\widehat{q}}_{L^2(0,T_{\max};W^{1,q}(\mathcal{O}))} \leq{}& C\|\Delta_h\widetilde{u}_h\|^{\widehat{q}(q-2)/q}_{L^2(0,T_{\max};L^2(\mathcal{O}))}\|\widetilde{u}_h\|^{\widehat{q}-\widehat{q}(q-2)/q}_{L^\infty(0,T_{\max};H^1(\mathcal{O}))}\\ &+ C\|\widetilde{u}\|^{\widehat{q}(q-2)/q}_{L^2(0,T_{\max};H^2(\mathcal{O}))}\|\widetilde{u}\|^{\widehat{q}-\widehat{q}(q-2)/q}_{L^\infty(0,T_{\max};H^1(\mathcal{O}))}\\ &+ C\|\widetilde{u}_h\|^{\widehat{q}}_{L^2(0,T_{\max};H^1(\mathcal{O}))}\\ &+ C\|\widetilde{u}\|^{\widehat{q}}_{L^2(0,T_{\max};H^1(\mathcal{O}))}\end{aligned} \tag{5.25}$$

for all $\overline{q} < \widehat{q} < \frac{2q}{q-2}$. As this choice of $\widehat{q}$ in particular implies that $\widehat{q}(q-2)/q < 2$, we may use Hölder's inequality to show that $\widetilde{\mathbb{E}}[\|\widetilde{u}_h - \widetilde{u}\|^{\widehat{q}}_{L^2(0,T_{\max};W^{1,q}(\mathcal{O}))}]$ is uniformly bounded. As we already established the $\widetilde{\mathbb{P}}$-almost sure convergence in Proposition 5.3, an application of Vitali's convergence theorem provides the result.

From the bounds on $\widetilde{u}_h$ stated in Proposition 4.2, we obtain the weak convergence $\widetilde{u}_h \overset{*}{\rightharpoonup} \mathfrak{u}$ in $L^{2\overline{p}}_{\text{weak-}(*)}(\widetilde{\Omega}; L^\infty(0,T_{\max}; H^1_{per}(\mathcal{O})))$ along a subsequence. As the strong convergence of $\widetilde{u}_h$ towards $\widetilde{u}$ is already established in (5.22a), we are able to identify $\mathfrak{u}$ and $\widetilde{u}$.

To establish the weak convergence expressed in (5.22d), we again start with the uniform $L^2(\widetilde{\Omega}; L^2(0,T_{\max}; L^2(\mathcal{O})))$-bounds on $\Delta_h\widetilde{u}_h$ which are stated in Proposition 5.3. These bounds provide the existence of a subsequence converging weakly in the space $L^2(\widetilde{\Omega}; L^2(0,T_{\max}; L^2(\mathcal{O})))$ towards some limit function $\mathfrak{v}$. As we also have $\Delta_h\widetilde{u}_h \rightharpoonup \Delta\widetilde{u}$ $\widetilde{\mathbb{P}}$-almost surely (cf. Proposition 5.3 and Corollary 5.7), combining the aforementioned uniform bounds and Vitali's convergence theorem provides

$$\int_0^{T_{\max}} \int_{\mathcal{O}} \Delta_h \widetilde{u}_h \phi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \to \int_0^{T_{\max}} \int_{\mathcal{O}} \Delta \widetilde{u} \phi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \tag{5.26}$$

strongly in $L^r(\widetilde{\Omega})$ for $r < 2$. Therefore, we have $\mathfrak{v} = \Delta_h \widetilde{u}$.

To establish (5.22e), it suffices to show

$$\widetilde{\mathbb{E}}\Big[\big\| \mathcal{I}_h^y\big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\big\} - \widetilde{u}_h \big\|^q_{L^\infty(0,T_{\max};L^q(\mathcal{O}))}\Big]$$

vanishes for $h \searrow 0$. Using Hölder's inequality, we obtain

$$\begin{aligned}
&\widetilde{\mathbb{E}}\Big[\sup_{t\in[0,T_{\max}]} \int_{\mathcal{O}} \big|\mathcal{I}_h^y\big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\big\} - \widetilde{u}_h\big|^q \, \mathrm{d}x \, \mathrm{d}y\Big] \\
&\quad \le \Big(\widetilde{\mathbb{E}}\Big[\sup_{t\in[0,T_{\max}]} \int_{\mathcal{O}} \big|\mathcal{I}_h^y\big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\big\} - \widetilde{u}_h\big|^2 \, \mathrm{d}x \, \mathrm{d}y\Big]\Big)^{1/2} \\
&\qquad \times \Big(\widetilde{\mathbb{E}}\Big[\sup_{t\in[0,T_{\max}]} \int_{\mathcal{O}} \big|\mathcal{I}_h^y\big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\big\} - \widetilde{u}_h\big|^{2q-2} \, \mathrm{d}x \, \mathrm{d}y\Big]\Big)^{1/2}.
\end{aligned} \tag{5.27}$$

In the following, we will show that the first factor converges towards zero while the second factor remains bounded. As $[G''(\widetilde{u}_h)]_x^{-1}|_{K^x}(y) \in [\min_{x\in K^x} \widetilde{u}_h^2(x,y), \max_{x\in K^x} \widetilde{u}_h^2(x,y)]$, we compute on each $Q \in \mathcal{Q}_h$

$$\begin{aligned}
\int_Q \big|\mathcal{I}_h^y\big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\big\} - \widetilde{u}_h\big|^2 \, \mathrm{d}x \, \mathrm{d}y &\le \int_Q \mathcal{I}_h^y\big\{\big|\max_{K^x} \widetilde{u}_h - \min_{K^x} \widetilde{u}_h\big|^2\big\} \, \mathrm{d}x \, \mathrm{d}y \\
&\le h^2 \int_Q \mathcal{I}_h^y\big\{|\partial_x \widetilde{u}_h|^2\big\} \, \mathrm{d}x \, \mathrm{d}y,
\end{aligned} \tag{5.28}$$

which provides the first result. To show that the second integral remains bounded, we use that on each element $K^x = (ih_x, (i+1)h_x)$, we have $[G''(\widetilde{u}_h)]_x^{-1}|_{K^x}(y) = \widetilde{u}_h(ih_x, y) \cdot \widetilde{u}_h((i+1)h_x, y)$. Applying the inequality of arithmetic and geometric means and Jensen's inequality, we obtain on each $Q := (ih_x, (i+1)h_x) \times K^y$

$$\begin{aligned}
&\int_Q \big|\mathcal{I}_h^y\big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\big\}\big|^{2q-2} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad \le \int_Q \big|\mathcal{I}_h^y\big\{\sqrt{\widetilde{u}_h(ih_x, y)\widetilde{u}_h((i+1)h_x, y)}\big\}\big|^{2q-2} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad \le \int_Q \Big|\frac{1}{2}\mathcal{I}_h^y\{\widetilde{u}_h(ih_x, y) + \widetilde{u}_h((i+1)h_x, y)\}\Big|^{2q-q} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad \le \int_Q \frac{1}{2}\mathcal{I}_h^y\big\{\widetilde{u}_h^{2q-2}(ih_x, y) + \widetilde{u}_h^{2q-2}((i+1)h_x, y)\big\} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad = \int_Q \mathcal{I}_h^{xy}\big\{\widetilde{u}_h^{2q-2}\big\} \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned} \tag{5.29}$$

Summing over all $Q \in \mathcal{Q}_h$ and applying (2.6), we obtain the result for $\overline{p}$ large enough. The convergence expressed in (5.22f) can be shown with analogous computations.

To address (5.23a), we combine the bounds in Proposition 5.3 with the estimate

$$\int_{\mathcal{O}} \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \chi_{[S_\delta]} |\partial_x \widetilde{p}_h|^2 \,\mathrm{d}x\,\mathrm{d}y \le \delta^{-2} C \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [G''(\widetilde{u}_h)]_x^{-1} |\partial_x \widetilde{p}_h|^2 \right\} \mathrm{d}x\,\mathrm{d}y, \tag{5.30}$$

which indicates that

$$\chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \chi_{[S_\delta]} \partial_x \widetilde{p}_h$$

is uniformly bounded in $L^2(\widetilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))$. Therefore, there exists a subsequence converging towards a limit function η_δ. In the following, we have to show that $\eta_\delta = \partial_x \widetilde{p}$ on S_δ:

We shall approximate the characteristic function $\chi_{[S_\delta]}$ by a family of functions $\widetilde{\chi}_N(\omega, t)_{N \in \mathbb{N}} : \widetilde{\Omega} \times [0, T_{\max}] \to C^\infty_{\text{per}}(\mathcal{O})$ satisfying

$$\begin{aligned} \widetilde{\chi}_N(\omega, t, (x, y)) &= 0 \quad \text{for } (x, y) \in S_\delta(\omega, t)^c, \\ \widetilde{\chi}_N(\omega, t, (x, y)) &= \begin{cases} 1 & \text{if } \operatorname{dist}((x, y), \partial S_\delta(\omega, t)) \ge \dfrac{1}{N}, \\ 0 & \text{if } \operatorname{dist}((x, y), \partial S_\delta(\omega, t)) \le \dfrac{1}{N+1}. \end{cases} \end{aligned} \tag{5.31}$$

To identify η_δ on S_δ with $\partial_x \widetilde{p}$, we will show that

$$\widetilde{\mathbb{E}}\Big[\int_0^{T_{\max}} \int_{\mathcal{O}} \eta_\delta \widetilde{\phi} \widetilde{\chi}_N \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \Big] = -\widetilde{\mathbb{E}}\Big[\int_0^{T_{\max}} \int_{\mathcal{O}} \widetilde{p} \partial_x (\widetilde{\phi} \widetilde{\chi}_N) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \Big] \tag{5.32}$$

for sufficiently regular test functions $\widetilde{\phi}$. In the following, we will show that this equation is valid even in the slightly more general case when $\widetilde{\phi}\widetilde{\chi}$ is replaced by the test function $\widetilde{\zeta} \in L^\infty(\widetilde{\Omega}; C^\infty([0, T_{\max}]; C^\infty_{\text{per}}(\overline{\mathcal{O}})))$ with $\operatorname{supp} \widetilde{\zeta}(\omega, t) \subset S_\delta(\omega, t)$:

As $\widetilde{u}_h$ converges strongly in $W^{1,q}_{\text{per}}(\mathcal{O})$ ($q < \infty$) $\widetilde{\mathbb{P}}$-almost surely for almost all $t \in (0, T_{\max})$ and hence almost everywhere in $\widetilde{\Omega} \times (0, T_{\max})$, we have $\widetilde{u}_h \to \widetilde{u}$ in $C^\gamma_{\text{per}}(\overline{\mathcal{O}})$ with $\gamma < 1$ almost everywhere in $\widetilde{\Omega} \times (0, T_{\max})$. This allows us to apply an appropriate version of Egorov's theorem (cf. [19, Theorem 42]) to deduce for all $\iota > 0$ the existence of a subset $\mathfrak{E}_\iota \subset \widetilde{\Omega} \times (0, T_{\max})$ with measure smaller than ι such that $\widetilde{u}_h$ converges uniformly in $(\widetilde{\Omega} \times (0, T_{\max})) \setminus \mathfrak{E}_\iota =: \mathfrak{E}_\iota^c$. Therefore, we compute

$$\begin{aligned} &\widetilde{\mathbb{E}}\Big[\int_0^{T_{\max}} \int_{\mathcal{O}} \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \chi_{[S_\delta]} \partial_x \widetilde{p}_h \widetilde{\zeta} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \Big] \\ &\quad = \int_{\widetilde{\Omega}} \int_0^{T_{\max}} \int_{\mathcal{O}} \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \partial_x \widetilde{p}_h \widetilde{\zeta} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}] \\ &\quad = \int_{\mathfrak{E}_\iota^c} \int_{\mathcal{O}} \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \partial_x \widetilde{p}_h \widetilde{\zeta} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}] + \int_{\mathfrak{E}_\iota} \int_{\mathcal{O}} \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \partial_x \widetilde{p}_h \widetilde{\zeta} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}] \\ &\quad =: A + B. \end{aligned} \tag{5.33}$$

Applying Hölder's inequality, we immediately obtain $|B| \leq C\iota^{1/2}$. As $\widetilde{u}_h$ converges uniformly towards $\widetilde{u}$ in $\mathfrak{E}_\iota^c$, we have supp $\widetilde{\zeta}(\omega,t) \subset S_\delta(\omega,t) \subset S_{\delta/4}^{\mathcal{Q}_h}(\omega,t)$ for h sufficiently small. Using the definition of $\widetilde{p}_h$, we obtain

$$\begin{aligned} A &= -\int_{\mathfrak{E}_\iota^c}\int_{\mathcal{O}} \widetilde{p}_h \partial_x \widetilde{\zeta} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}] = \int_{\mathfrak{E}_\iota^c}\int_{\mathcal{O}} \chi_{\widetilde{T}_h}\Delta_h \widetilde{u}_h \partial_x \widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}] \\ &\quad - \int_{\mathfrak{E}_\iota^c}\int_{\mathcal{O}} \chi_{\widetilde{T}_h}\mathcal{I}_h^{xy}\{F'(\widetilde{u}_h)\}\partial_x\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}] \\ &\quad - h^\varepsilon \int_{\mathfrak{E}_\iota^c}\int_{\mathcal{O}} \chi_{\widetilde{T}_h}\Delta_h\Delta_h\widetilde{u}_h\partial_x\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}] \\ &=: A_1 + A_2 + A_3. \end{aligned} \tag{5.34}$$

The convergence of A_1 towards $\int_{\mathfrak{E}_\iota^c}\int_{\mathcal{O}} \Delta\widetilde{u}\partial_x\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}]$ is a direct consequence of the weak convergence (5.22d). To obtain the convergence of A_2, we use

$$\begin{aligned} A_2 &= -\int_{\mathfrak{E}_\iota^c}\int_{\mathcal{O}} \chi_{\widetilde{T}_h} F'(\widetilde{u}_h)\partial_x\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}] \\ &\quad + \int_{\mathfrak{E}_\iota^c}\int_{\mathcal{O}} \chi_{\widetilde{T}_h}(I-\mathcal{I}_h^{xy})\{F'(\widetilde{u}_h)\}\partial_x\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}]. \end{aligned}$$

As supp $\widetilde{\zeta}(\omega,t) \subset S_{\delta/4}^{\mathcal{Q}_h}(\omega,t)$, the second term vanishes for $h \searrow 0$ due to Lemma A.5 and the first term converges towards $-\int_{\mathfrak{E}_\iota^c}\int_{\mathcal{O}} F'(\widetilde{u})\partial_x\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}]$ due to Vitali's convergence theorem. The treatment of A_3 is more delicate, as the bounds on $h^\varepsilon\Delta_h\Delta_h\widetilde{u}_h$ are not obvious. We begin by splitting A_3 into

$$\begin{aligned} A_3 &= -h^\varepsilon \int_{\mathfrak{E}_\iota^c}\int_{\mathcal{O}} \chi_{\widetilde{T}_h}\mathcal{I}_h^{xy}\left\{\Delta_h\Delta_h\widetilde{u}_h\mathcal{I}_h^{xy}\{\partial_x\widetilde{\zeta}\}\right\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}] \\ &\quad - h^\varepsilon \int_{\mathfrak{E}_\iota^c}\int_{\mathcal{O}} \chi_{\widetilde{T}_h}(I-\mathcal{I}_h^{xy})\left\{\Delta_h\Delta_h\widetilde{u}_h\mathcal{I}_h^{xy}\{\partial_x\widetilde{\zeta}\}\right\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}] \\ &\quad - h^\varepsilon \int_{\mathfrak{E}_\iota^c}\int_{\mathcal{O}} \chi_{\widetilde{T}_h}\Delta_h\Delta_h\widetilde{u}_h(I-\mathcal{I}_h^{xy})\{\partial_x\widetilde{\zeta}\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}] \\ &=: A_{3_a} + A_{3_b} + A_{3_c}. \end{aligned} \tag{5.35}$$

Recalling (2.7) and applying Hölder's inequality shows that A_{3_a} vanishes, since

$$\begin{aligned} |A_{3_a}| &\leq h^{\varepsilon/2}\|h^{\varepsilon/2}\partial_x\Delta_h\widetilde{u}_h\|_{L^2(\widetilde{\Omega};L^2(0,T_{\max};L^2(\mathcal{O})))}\left\|\partial_x\mathcal{I}_h^{xy}\{\partial_x\widetilde{\zeta}\}\right\|_{L^2(\widetilde{\Omega};L^2(0,T_{\max};L^2(\mathcal{O})))} \\ &\quad + h^{\varepsilon/2}\|h^{\varepsilon/2}\partial_y\Delta_h\widetilde{u}_h\|_{L^2(\widetilde{\Omega};L^2(0,T_{\max};L^2(\mathcal{O})))}\left\|\partial_y\mathcal{I}_h^{xy}\{\partial_x\widetilde{\zeta}\}\right\|_{L^2(\widetilde{\Omega};L^2(0,T_{\max};L^2(\mathcal{O})))} \\ &\leq h^{\varepsilon/2}C. \end{aligned} \tag{5.36}$$

Due to (2.9), we have $h\|h^{\varepsilon/2}\Delta_h\Delta_h\widetilde{u}_h\|_{L^2(\widetilde{\Omega};L^2(0,T_{\max};L^2(\mathcal{O})))} \leq C$. Therefore, standard estimates for $\mathcal{I}_h^{xy}$ which can be found, e.g., in [10, Theorem 4.4.20] provide

$$\begin{aligned} |A_{3_b}| &\leq Ch^{\varepsilon/2}h\|h^{\varepsilon/2}\Delta_h\Delta_h\widetilde{u}_h\|_{L^2(\widetilde{\Omega};L^2(0,T_{\max};L^2(\mathcal{O})))}\left\|\nabla\mathcal{I}_h^{xy}\{\partial_x\widetilde{\zeta}\}\right\|_{L^2(\widetilde{\Omega};L^2(0,T_{\max};L^2(\mathcal{O})))} \\ &\leq Ch^{\varepsilon/2}. \end{aligned} \tag{5.37}$$

Similar considerations based on Lemma A.1 provide

$$\begin{aligned}|A_{3_c}| &\le h^{\varepsilon/2}\|h^{\varepsilon/2}\Delta_h\Delta_h\widetilde{u}_h\|_{L^2(\widetilde{\Omega};L^2(0,T_{\max};L^2(\mathcal{O})))}\big\|(I-\mathcal{I}_h^{xy})\{\partial_x\widetilde{\zeta}\}\big\|_{L^2(\widetilde{\Omega};L^2(0,T_{\max};L^2(\mathcal{O})))}\\ &\le Ch^{1+\varepsilon/2}\|\partial_x\widetilde{\zeta}\|_{L^2(\widetilde{\Omega};L^2(0,T_{\max};H^2(\mathcal{O})))}.\end{aligned} \tag{5.38}$$

Collecting the results above, we have

$$\begin{aligned}A &\to \int_{\mathfrak{E}_\iota^c}\int_{\mathcal{O}}\Delta\widetilde{u}\partial_x\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}] - \int_{\mathfrak{E}_\iota^c}\int_{\mathcal{O}}F'(\widetilde{u})\partial_x\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}]\\ &= -\int_{\widetilde{\Omega}}\int_0^{T_{\max}}\int_{\mathcal{O}}\widetilde{p}\partial_x\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}] - \int_{\mathfrak{E}_\iota}\int_{\mathcal{O}}\Delta\widetilde{u}\partial_x\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}]\\ &\quad + \int_{\mathfrak{E}_\iota}\int_{\mathcal{O}}F'(\widetilde{u})\partial_x\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\widetilde{\mathbb{P}}[\widetilde{\omega}].\end{aligned} \tag{5.39}$$

As $\widetilde{u} > \delta$ in supp $\partial_x\widetilde{\zeta}$, the last two integrals can be bounded by $C\iota^{1/2}$. Therefore, we may identify η_δ with $\chi_{[S_\delta]}\partial_x\widetilde{p}$, which provides (5.23a). The convergence expressed in (5.23b) can be proven by similar computations.

The weak convergence expressed in (5.24a) and (5.24b) can be established analogously to (5.22d). Therefore, it remains to show that the fluxes $\widetilde{J}^x$ and $\widetilde{J}^y$ coincide with $\widetilde{u}\partial_x\widetilde{p}$ and $\widetilde{u}\partial_y\widetilde{p}$, respectively. Reusing the ideas of the proof of (5.23a), we choose $\widetilde{\zeta} \in L^\infty(\widetilde{\Omega}; C^\infty([0,T_{\max}]; C^\infty_{\text{per}}(\overline{\mathcal{O}})))$ with supp $\widetilde{\zeta}(\omega,t) \subset S_\delta(\omega,t)$ and compute

$$\begin{aligned}\widetilde{\mathbb{E}}\Big[\int_0^{T_{\max}}\int_{\mathcal{O}}\widetilde{J}_h^y\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big] &= \widetilde{\mathbb{E}}\Big[\int_0^{T_{\max}}\int_{\mathcal{O}}\mathcal{I}_h^y\Big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\partial_x\widetilde{p}_h\Big\}\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big]\\ &= \widetilde{\mathbb{E}}\Big[\int_0^{T_{\max}}\int_{\mathcal{O}}\chi_{[S_{\delta/4}^{\mathcal{Q}_h}]}\chi_{[S_\delta]}\mathcal{I}_h^y\Big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\partial_x\widetilde{p}_h\Big\}\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big]\\ &\quad + \widetilde{\mathbb{E}}\Big[\int_0^{T_{\max}}\int_{\mathcal{O}}(1-\chi_{[S_{\delta/4}^{\mathcal{Q}_h}]})\chi_{[S_\delta]}\mathcal{I}_h^y\Big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\partial_x\widetilde{p}_h\Big\}\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big]\\ &= \widetilde{\mathbb{E}}\Big[\int_0^{T_{\max}}\int_{\mathcal{O}}\chi_{[S_{\delta/4}^{\mathcal{Q}_h}]}\chi_{[S_\delta]}\mathcal{I}_h^y\Big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\Big\}\partial_x\widetilde{p}_h\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big]\\ &\quad - \widetilde{\mathbb{E}}\Big[\int_0^{T_{\max}}\int_{\mathcal{O}}\chi_{[S_{\delta/4}^{\mathcal{Q}_h}]}\chi_{[S_\delta]}(I-\mathcal{I}_h^y)\Big\{\mathcal{I}_h^y\Big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\Big\}\partial_x\widetilde{p}_h\Big\}\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big]\\ &\quad + \widetilde{\mathbb{E}}\Big[\int_0^{T_{\max}}\int_{\mathcal{O}}(1-\chi_{[S_{\delta/4}^{\mathcal{Q}_h}]})\chi_{[S_\delta]}\mathcal{I}_h^y\Big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\partial_x\widetilde{p}_h\Big\}\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big]\\ &=: B_1 + B_2 + B_3.\end{aligned} \tag{5.40}$$

The convergence

$$B_1 \to \widetilde{\mathbb{E}}\Big[\int_0^{T_{\max}}\int_{\mathcal{O}}\chi_{[S_\delta]}\widetilde{u}\partial_x\widetilde{p}\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big] = \widetilde{\mathbb{E}}\Big[\int_0^{T_{\max}}\int_{\mathcal{O}}\widetilde{u}\partial_x\widetilde{p}\widetilde{\zeta}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big] \tag{5.41}$$

follows directly from (5.22e) and (5.23a). Combining (A.1c) (cf. Lemma A.1) with standard inverse estimates (cf. [10, Theorem 4.5.11]) and (5.30) provides the estimate

$$B_2 \le Ch\Big\|\partial_y\mathcal{I}_h^y\Big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\Big\}\Big\|_{L^2(\widetilde{\Omega};L^2(0,T_{\max};L^2(\mathcal{O})))}\|\partial_x\widetilde{p}_h\|_{L^2(S_\delta)}$$

$$\leq C\delta^{-1}\big(\big\| \mathcal{I}_h^y\big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\big\} - \widetilde{u_h}\big\|_{L^2(\widetilde{\Omega};L^2(0,T_{\max};L^2(\mathcal{O})))} + h\|\partial_y \widetilde{u}_h\|_{L^2(\widetilde{\Omega};L^2(0,T_{\max};L^2(\mathcal{O})))}\big). \tag{5.42}$$

Therefore, B_2 vanishes for $h \searrow 0$ due to (5.22e) and (5.22a). Applying Hölder's inequality and the bounds established in Proposition 5.3, we obtain

$$B_3 \leq C\big\|\big(\chi_{[S_\delta]} - \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]}\big)\widetilde{\zeta}\big\|_{L^2(\widetilde{\Omega};L^2(0,T_{\max};L^2(\mathcal{O})))}. \tag{5.43}$$

Similar to the arguments used in the proof of (5.23a), we may use Egorov's theorem to obtain the existence of a subset $\mathfrak{E}_\iota \subset \widetilde{\Omega} \times (0, T_{\max})$ with measure smaller than ι such that $\widetilde{u}_h$ converges uniformly in $C^{0,\gamma}_{\mathrm{per}}(\overline{\mathcal{O}})$ in $(\widetilde{\Omega} \times (0, T_{\max})) \setminus \mathfrak{E}_\iota =: \mathfrak{E}_\iota^c$. As we have

$$\chi_{[S_\delta]} \subset \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]}$$

in $\mathfrak{E}_\iota^c$ for h small enough and $\operatorname{supp}\widetilde{\zeta} \subset S_\delta$, we obtain $B_3 \leq C\iota^{1/2}$ for all $\iota > 0$. As we also have $\widetilde{J}_h^x \rightharpoonup \widetilde{J}^x$ in $L^2(\widetilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))$, we obtain $\widetilde{J}^x = \widetilde{u}\partial_x(-\Delta\widetilde{u} + F'(\widetilde{u}))$ on S_δ. The identification of $\widetilde{J}^y$ on S_δ follows by similar arguments. ∎

5.3. Convergence of the stochastic integral

We consider for arbitrary but fixed $v \in C^\infty_{\mathrm{per}}(\overline{\mathcal{O}})$ the operator $\mathcal{M}_{h,v} : \Omega \times [0, T_{\max}] \to \mathbb{R}$ defined by

$$\begin{aligned}
\mathcal{M}_{h,v}(t) :=& \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{(u_h(t) - u_h(0))v\}\,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_0^{t\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^y\Big\{\sqrt{[G''(u_h)]_x^{-1}}\,J_h^x\,\partial_x\mathcal{I}_h^{xy}\{v\}\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau \\
&+ \int_0^{t\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^x\Big\{\sqrt{[G''(u_h)]_y^{-1}}\,J_h^y\,\partial_y\mathcal{I}_h^{xy}\{v\}\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau \\
=& \sum_{k\in I_h^x}\sum_{l\in I_h^y}\int_0^{t\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^y\big\{\mathcal{I}_{h,\mathrm{loc}}^x\{\partial_x(u_h\tilde{\mathfrak{g}}_{h,kl})\mathcal{I}_h^{xy}\{v\}\}\big\}\,\mathrm{d}x\,\mathrm{d}y\;\mathrm{d}\beta_{kl}^x \\
&+ \sum_{k\in I_h^x}\sum_{l\in I_h^y}\int_0^{t\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^x\big\{\mathcal{I}_{h,\mathrm{loc}}^y\{\partial_y(u_h\tilde{\mathfrak{g}}_{h,kl})\mathcal{I}_h^{xy}\{v\}\}\big\}\,\mathrm{d}x\,\mathrm{d}y\;\mathrm{d}\beta_{kl}^y.
\end{aligned} \tag{5.44}$$

Remark 5.11. In contrast to [29], we define $\mathcal{M}_{h,v}$ using functions v in $C^\infty_{\mathrm{per}}(\overline{\mathcal{O}})$ instead of $H^2_{\mathrm{per}}(\mathcal{O})$. This seems to be necessary, as the limit process in Lemma 5.14 requires convergence and stability properties of the projection of v that are not provided by the L^2-projection. Hence, the requirements on the regularity of v are also higher.

By the optional stopping theorem, $\mathcal{M}_{h,v}$ is a real-valued martingale, i.e., we have

$$\mathbb{E}\big[(\mathcal{M}_{h,v}(t) - \mathcal{M}_{h,v}(s))\Psi(r_s u_h,\, r_s W)\big] = 0 \tag{5.45}$$

for all $0 \le s \le t \le T_{\max}$ and for all $[0,1]$-valued functions Ψ defined on $\mathcal{X}_u$. Here, r_s denotes the restriction of a function defined on $[0, T_{\max}]$ to $[0, s]$.

Lemma 5.12. *For the quadratic variation of $\mathcal{M}_{h,v}$, we have*

$$\begin{aligned}\langle\!\langle \mathcal{M}_{h,v} \rangle\!\rangle_t &= \sum_{k\in I_h^x}\sum_{l\in I_h^y}\int_0^{t\wedge T_h} {\lambda_{kl}^x}^2 \Big(\int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_{h,\text{loc}}^x\{\partial_x(u_h\tilde{\mathfrak{g}}_{h,kl})\mathcal{I}_h^{xy}\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\,\mathrm{d}s\\ &\quad+\sum_{k\in I_h^x}\sum_{l\in I_h^y}\int_0^{t\wedge T_h} {\lambda_{kl}^y}^2 \Big(\int_{\mathcal{O}} \mathcal{I}_h^x\{\mathcal{I}_{h,\text{loc}}^y\{\partial_y(u_h\tilde{\mathfrak{g}}_{h,kl})\mathcal{I}_h^{xy}\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\,\mathrm{d}s\\ &\le C\|v\|_{H^2(\mathcal{O})}^2\int_0^{t\wedge T_h}\|u_h(s)\|_{H^1(\mathcal{O})}^2\,\mathrm{d}s. \end{aligned}\tag{5.46}$$

Proof. We consider the mapping $R(u_h, v) : \Omega \times [0, T_{\max}] \times (L^2(\mathcal{O}))^2 \to \mathbb{R}$ defined by

$$\begin{aligned}&(\omega, t, (z_x, z_y)) \mapsto\\ &\quad \chi_{T_h}(t,\omega)\int_{\mathcal{O}}\mathcal{I}_h^y\Big\{\mathcal{I}_{h,\text{loc}}^x\Big\{\partial_x\Big(u_h\sum_{k\in I_h^x}\sum_{l\in I_h^y}\langle\mathfrak{g}_{kl}, z_x\rangle_{L^2}\tilde{\mathfrak{g}}_{h,kl}\Big)\mathcal{I}_h^{xy}\{v\}\Big\}\Big\}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+\chi_{T_h}(t,\omega)\int_{\mathcal{O}}\mathcal{I}_h^x\Big\{\mathcal{I}_{h,\text{loc}}^y\Big\{\partial_y\Big(u_h\sum_{k\in I_h^x}\sum_{l\in I_h^y}\langle\mathfrak{g}_{kl}, z_y\rangle_{L^2}\tilde{\mathfrak{g}}_{h,kl}\Big)\mathcal{I}_h^{xy}\{v\}\Big\}\Big\}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}\tag{5.47}$$

We obtain for the Hilbert–Schmidt norm

$$\begin{aligned}&\|R(u_h,v)(t,\omega)\|_{L_2(Q^{1/2}(L^2(\mathcal{O}))^2;\mathbb{R})}^2\\ &\quad=\chi_{T_h}(t,\omega)\sum_{k\in I_h^x}\sum_{l\in I_h^y}{\lambda_{kl}^x}^2\Big(\int_{\mathcal{O}}\mathcal{I}_h^y\{\mathcal{I}_{h,\text{loc}}^x\{\partial_x(u_h\tilde{\mathfrak{g}}_{h,kl})\mathcal{I}_h^{xy}\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\\ &\qquad+\chi_{T_h}(t,\omega)\sum_{k\in I_h^x}\sum_{l\in I_h^y}{\lambda_{kl}^x}^2\Big(\int_{\mathcal{O}}\mathcal{I}_h^x\{\mathcal{I}_{h,\text{loc}}^y\{\partial_y(u_h\tilde{\mathfrak{g}}_{h,kl})\mathcal{I}_h^{xy}\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\\ &\quad=:\chi_{T_h}(t,\omega)\sum_{k\in I_h^x}\sum_{l\in I_h^y}{\lambda_{kl}^x}^2A^2+\chi_{T_h}(t,\omega)\sum_{k\in I_h^x}\sum_{l\in I_h^y}{\lambda_{kl}^y}^2B^2. \end{aligned}\tag{5.48}$$

To estimate the first term, we use (2.13a) and (2.13b) and compute

$$\begin{aligned}A^2 &\le 2\Big(\int_{\mathcal{O}}\mathcal{I}_h^y\{\partial_x u_h\mathcal{I}_h^x\{\tilde{\mathfrak{g}}_{h,kl}\mathcal{I}_h^{xy}\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\\ &\qquad+2\Big(\int_{\mathcal{O}}\mathcal{I}_h^y\{\partial_x\tilde{\mathfrak{g}}_{h,kl}\mathcal{I}_h^x\{u_h\mathcal{I}_h^{xy}\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\\ &\le C\|\tilde{\mathfrak{g}}_{h,kl}\|_{L^\infty(\mathcal{O})}^2\|\partial_x u_h\|_{L^2(\mathcal{O})}^2\|\mathcal{I}_h^{xy}\{v\}\|_{L^2(\mathcal{O})}^2\\ &\qquad+\|\partial_x\tilde{\mathfrak{g}}_{h,kl}\|_{L^\infty(\mathcal{O})}^2\|u_h\|_{L^2(\mathcal{O})}^2\|\mathcal{I}_h^{xy}\{v\}\|_{L^2(\mathcal{O})}^2. \end{aligned}\tag{5.49}$$

Using Assumption (B3), the standard error estimates for $\mathcal{I}_h^{xy}$, and a similar estimate for B^2, we conclude

$$\|R(u_h, v)(t,\omega)\|^2_{L_2(Q^{1/2}(L^2(\mathcal{O}))^2;\mathbb{R})} \le C\chi_{T_h}(t,\omega)\|u_h\|^2_{H^1(\mathcal{O})}\|v\|^2_{H^2(\mathcal{O})}. \tag{5.50}$$

Applying [62, Lemma 2.4.3] concludes the proof. ∎

In the next lemma, we will study cross variation of $\mathcal{M}_{h,v}$ with the processes

$$\beta_{kl}^{\alpha}(t) = \int_{\mathcal{O}}\int_0^t (\lambda_{kl}^{\alpha})^{-1}\mathfrak{g}_{kl}\boldsymbol{b}_\alpha \cdot \mathrm{d}\boldsymbol{W}\,\mathrm{d}x\,\mathrm{d}y \tag{5.51}$$

for $k,l \in \mathbb{Z}$ and $\alpha \in \{x,y\}$.

Lemma 5.13. *For $k,l \in \mathbb{Z}$ and $\alpha \in \{x,y\}$, the cross variation $\langle\langle \mathcal{M}_{h,v}, \beta_{kl}^{\alpha}\rangle\rangle_t$ is given by*

$$\langle\langle \mathcal{M}_{h,v}, \beta_{kl}^{x}\rangle\rangle_t \tag{5.52a}$$
$$= \begin{cases} \lambda_{kl}^{x}\int_0^{t\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^{y}\{\mathcal{I}_{h,\mathrm{loc}}^{x}\{\partial_x(u_h\tilde{\mathfrak{g}}_{h,kl})\mathcal{I}_h^{xy}\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s & \text{if } k \in I_h^x, l \in I_h^y,\\ 0 & \text{else},\end{cases}$$

$$\langle\langle \mathcal{M}_{h,v}, \beta_{kl}^{y}\rangle\rangle_t \tag{5.52b}$$
$$= \begin{cases} \lambda_{kl}^{y}\int_0^{t\wedge T_h}\int_{\mathcal{O}} \mathcal{I}_h^{x}\{\mathcal{I}_{h,\mathrm{loc}}^{y}\{\partial_y(u_h\tilde{\mathfrak{g}}_{h,kl})\mathcal{I}_h^{xy}\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s & \text{if } k \in I_h^x,\ l \in I_h^y,\\ 0 & \text{else}.\end{cases}$$

Proof. The proof follows the lines of [29, Lemma 5.12]. We will only prove (5.52b), as (5.53) follows by similar computations.

To compute the cross variation with β_{kl}^{x}, we consider for given k, l the mappings $S_{\pm}^{x} : \Omega \times [0, T_{\max}] \times Q_x^{1/2}L^2(\Omega) \to \mathbb{R}$ which are defined as

$$z \mapsto \chi_{T_h}\sum_{\overline{k}\in I_h^x}\sum_{\overline{l}\in I_h^y}\Big(\int_{\mathcal{O}} \mathcal{I}_h^{y}\{\mathcal{I}_{h,\mathrm{loc}}^{x}\{\partial_x(u_h\langle \mathfrak{g}_{\overline{kl}}, z\rangle_{L^2})\mathcal{I}_h^{xy}\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y$$
$$\pm (\lambda_{kl}^{x})^{-1}\int_{\mathcal{O}} \mathfrak{g}_{kl} z\,\mathrm{d}x\,\mathrm{d}y\Big). \tag{5.53}$$

Computing the Hilbert–Schmidt norm of $S_{\pm}^{x}$, we obtain

$$\big\|S_{\pm}^{x}(u_h, v)\big\|_{L_2(Q_x^{1/2}L^2(\mathcal{O});\mathbb{R})}$$
$$= \chi_{T_h}\sum_{\overline{k}\in I_h^x}\sum_{\overline{l}\in I_h^y}\lambda_{\overline{kl}}^{x}\Big(\int_{\mathcal{O}} \mathcal{I}_h^{y}\{\mathcal{I}_{h,\mathrm{loc}}^{x}\{\partial_x(u_h\tilde{\mathfrak{g}}_{h,\overline{kl}})\mathcal{I}_h^{xy}\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2$$
$$\pm \chi_{T_h}2\sum_{\overline{k}\in I_h^x}\sum_{\overline{l}\in I_h^y}\lambda_{\overline{kl}}^{x}\int_{\mathcal{O}} \mathcal{I}_h^{y}\{\mathcal{I}_{h,\mathrm{loc}}^{x}\{\partial_x(u_h\tilde{\mathfrak{g}}_{h,\overline{kl}})\mathcal{I}_h^{xy}\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y$$
$$\cdot \frac{\lambda_{\overline{kl}}^{x}}{\lambda_{kl}^{x}}\int_{\mathcal{O}} \mathfrak{g}_{kl}\mathfrak{g}_{\overline{kl}}\,\mathrm{d}x\,\mathrm{d}y$$

$$+\,\chi_{T_h}\sum_{\overline{k},\overline{l}\in\mathbb{Z}}\Big(\frac{\lambda^x_{\overline{kl}}}{\lambda^x_{kl}}\int_{\mathcal{O}}\mathfrak{g}_{kl}\mathfrak{g}_{\overline{kl}}\,\mathrm{d}x\,\mathrm{d}y\Big)^2. \tag{5.54}$$

By using $\langle\!\langle \mathcal{M}_{h,v},\beta^x_{kl}\rangle\!\rangle_t=\frac{1}{4}(\langle\!\langle S^x_+(u_h,v)\rangle\!\rangle_t-\langle\!\langle S^x_-(u_h,v)\rangle\!\rangle_t)$ and recalling the identity $\int_{\mathcal{O}}\mathfrak{g}_{kl}\mathfrak{g}_{\overline{kl}}\,\mathrm{d}x\,\mathrm{d}y=\delta_{k\overline{k}}\delta_{l\overline{l}}$, we deduce (5.52b). ■

In addition to $\mathcal{M}_{h,v}$, the processes

$$\begin{aligned}\mathcal{M}^2_{h,v}&-\sum_{k\in I^x_h}\sum_{l\in I^y_h}\int_0^{(\cdot)\wedge T_h}{\lambda^x_{kl}}^2\Big(\int_{\mathcal{O}}\mathcal{I}^y_h\big\{\mathcal{I}^x_{h,\mathrm{loc}}\{\partial_x(u_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}^{xy}_h\{v\}\}\big\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\mathrm{d}s\\&-\sum_{k\in I^x_h}\sum_{l\in I^y_h}\int_0^{(\cdot)\wedge T_h}{\lambda^y_{kl}}^2\Big(\int_{\mathcal{O}}\mathcal{I}^x_h\big\{\mathcal{I}^y_{h,\mathrm{loc}}\{\partial_y(u_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}^{xy}_h\{v\}\}\big\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\mathrm{d}s\end{aligned} \tag{5.55}$$

and

$$\begin{cases}\mathcal{M}_{h,v}\beta^x_{kl}\\ \quad-\lambda^x_{kl}\int_0^{(\cdot)\wedge T_h}\int_{\mathcal{O}}\mathcal{I}^y_h\big\{\mathcal{I}^x_{h,\mathrm{loc}}\{\partial_x(u_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}^{xy}_h\{v\}\}\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s & \text{if } k\in I^x_h,\ l\in I^y_h,\\ \mathcal{M}_{h,v}\beta^x_{kl} & \text{else,}\end{cases} \tag{5.56}$$

$$\begin{cases}\mathcal{M}_{h,v}\beta^y_{kl}\\ \quad-\lambda^y_{kl}\int_0^{(\cdot)\wedge T_h}\int_{\mathcal{O}}\mathcal{I}^y_h\big\{\mathcal{I}^x_{h,\mathrm{loc}}\{\partial_y(u_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}^{xy}_h\{v\}\}\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s & \text{if } k\in I^x_h,\ l\in I^y_h,\\ \mathcal{M}_{h,v}\beta^y_{kl} & \text{else}\end{cases} \tag{5.57}$$

are also martingales.

By equality of laws, we deduce that the following processes are also $(\widetilde{\mathcal{F}}_{h,t})$-martingales:

$$\begin{aligned}\widetilde{\mathcal{M}}_{h,v}(t):=&\int_{\mathcal{O}}\mathcal{I}^{xy}_h\{(\widetilde{u}_h(t)-\widetilde{u}_h(0))\mathcal{I}^{xy}_h\{v\}\}\,\mathrm{d}x\,\mathrm{d}y\\&+\int_0^{t\wedge\widetilde{T}_h}\int_{\mathcal{O}}\mathcal{I}^y_h\Big\{\sqrt{[G''(\widetilde{u}_h)]^{-1}_x}\,\widetilde{J}^x_h\partial_x\mathcal{I}^{xy}_h\{v\}\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\\&+\int_0^{t\wedge\widetilde{T}_h}\int_{\mathcal{O}}\mathcal{I}^x_h\Big\{\sqrt{[G''(\widetilde{u}_h)]^{-1}_y}\,\widetilde{J}^y_h\partial_y\mathcal{I}^{xy}_h\{v\}\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\end{aligned} \tag{5.58a}$$

and

$$\begin{aligned}\widetilde{\mathcal{M}}^2_{h,v}(t)&-\sum_{k\in I^x_h}\sum_{l\in I^y_h}{\lambda^x_{kl}}^2\int_0^{t\wedge\widetilde{T}_h}\Big(\int_{\mathcal{O}}\mathcal{I}^y_h\big\{\mathcal{I}^x_{h,\mathrm{loc}}\{\partial_x(\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}^{xy}_h\{v\}\}\big\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\mathrm{d}s\\&-\sum_{k\in I^x_h}\sum_{l\in I^y_h}{\lambda^y_{kl}}^2\int_0^{t\wedge\widetilde{T}_h}\Big(\int_{\mathcal{O}}\mathcal{I}^x_h\big\{\mathcal{I}^y_{h,\mathrm{loc}}\{\partial_y(\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}^{xy}_h\{v\}\}\big\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\mathrm{d}s,\end{aligned} \tag{5.58b}$$

as well as

$$\begin{cases} \widetilde{\mathcal{M}}_{h,v}\widetilde{\beta}^x_{h,kl} \\ \quad -\lambda^x_{kl}\int_0^{(\cdot)\wedge\widetilde{T}_h}\int_{\mathcal{O}} \mathcal{I}^y_h\{\mathcal{I}^x_{h,\mathrm{loc}}\{\partial_x(\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}^{xy}_h\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s & \text{if } k\in I^x_h,\ l\in I^y_h, \\ \widetilde{\mathcal{M}}_{h,v}\widetilde{\beta}^x_{h,kl} & \text{else,} \end{cases} \tag{5.58c}$$

$$\begin{cases} \widetilde{\mathcal{M}}_{h,v}\widetilde{\beta}^y_{h,kl} - \\ \quad \lambda^y_{kl}\int_0^{(\cdot)\wedge\widetilde{T}_h}\int_{\mathcal{O}} \mathcal{I}^x_h\{\mathcal{I}^y_{h,\mathrm{loc}}\{\partial_y(\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}^{xy}_h\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s & \text{if } k\in I^x_h,\ l\in I^y_h, \\ \widetilde{\mathcal{M}}_{h,v}\widetilde{\beta}^y_{h,kl} & \text{else.} \end{cases} \tag{5.58d}$$

Here, we used

$$\widetilde{\beta}^\alpha_{h,kl}(t) = \int_{\mathcal{O}}\int_0^t (\lambda^\alpha_{kl})^{-1}\mathfrak{g}_{kl}\boldsymbol{b}_\alpha\cdot\mathrm{d}\widetilde{\boldsymbol{W}}_h\,\mathrm{d}x\,\mathrm{d}y \tag{5.59}$$

for $k,l\in\mathbb{Z}$ and $\alpha\in\{x,y\}$. Furthermore, the quadratic variation of $\widetilde{\mathcal{M}}_{h,v}$ and the cross variations with $\widetilde{\beta}^\alpha_{h,kl}$ are given by

$$\begin{aligned} \langle\!\langle\widetilde{\mathcal{M}}_{h,v}\rangle\!\rangle_t &= \sum_{k\in I^x_h}\sum_{l\in I^y_h}\int_0^{t\wedge\widetilde{T}_h}\lambda^{x\,2}_{kl}\Big(\int_{\mathcal{O}}\mathcal{I}^y_h\{\mathcal{I}^x_{h,\mathrm{loc}}\{\partial_x(\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}^{xy}_h\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\mathrm{d}s \\ &\quad+\sum_{k\in I^x_h}\sum_{l\in I^y_h}\int_0^{t\wedge\widetilde{T}_h}\lambda^{y\,2}_{kl}\Big(\int_{\mathcal{O}}\mathcal{I}^x_h\{\mathcal{I}^y_{h,\mathrm{loc}}\{\partial_y(\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}^{xy}_h\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\mathrm{d}s \end{aligned} \tag{5.60}$$

and

$$\begin{aligned} &\langle\!\langle\widetilde{\mathcal{M}}_{h,v},\widetilde{\beta}^x_{h,kl}\rangle\!\rangle_t \\ &\quad=\begin{cases} \lambda^x_{kl}\int_0^{t\wedge\widetilde{T}_h}\int_{\mathcal{O}}\mathcal{I}^y_h\{\mathcal{I}^x_{h,\mathrm{loc}}\{\partial_x(\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}^{xy}_h\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s & \text{if } k\in I^x_h,\ l\in I^y_h, \\ 0 & \text{else,} \end{cases} \end{aligned} \tag{5.61}$$

$$\begin{aligned} &\langle\!\langle\widetilde{\mathcal{M}}_{h,v},\widetilde{\beta}^y_{h,kl}\rangle\!\rangle_t \\ &\quad=\begin{cases} \lambda^y_{kl}\int_0^{t\wedge\widetilde{T}_h}\int_{\mathcal{O}}\mathcal{I}^y_h\{\mathcal{I}^x_{h,\mathrm{loc}}\{\partial_y(\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}^{xy}_h\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s & \text{if } k\in I^x_h,\ l\in I^y_h, \\ 0 & \text{else.} \end{cases} \end{aligned} \tag{5.62}$$

Lemma 5.14. *Let Assumptions (S), (I), (P), (B), (R), and (B3*) hold true. Then, for all* $[0,1]$*-valued continuous functions* Ψ *defined on* $\mathcal{X}_u\times\mathcal{X}_{\boldsymbol{W}}$*, we have*

$$\widetilde{\mathbb{E}}\Big[\Big(\int_{\mathcal{O}} (\widetilde{u}(t)-\widetilde{u}(s))v \,\mathrm{d}x\,\mathrm{d}y + \int_s^t \int_{\mathcal{O}} \widetilde{u} J^x \partial_x v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau + \int_s^t \int_{\mathcal{O}} \widetilde{u} J^y \partial_y v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\Big)\Psi(r_s\widetilde{u}, r_s\widetilde{W})\Big] = 0 \tag{5.63}$$

for all $0 \le s \le t \le T_{\max}$.

Proof. To establish the claim, we pass to the limit in the identity

$$\widetilde{\mathbb{E}}\big[\big(\widetilde{\mathcal{M}}_{h,v}(t) - \widetilde{\mathcal{M}}_{h,v}(s)\big)\Psi(r_s\widetilde{u}_h, r_s\widetilde{W}_h)\big] = 0. \tag{5.64}$$

Due to Lemma 5.5, we have $\chi_{\widetilde{T}_h} = 1$ on $[s,t]$ for h small enough depending on $\widetilde{\omega} \in \widetilde{\Omega}$. We start with the decomposition

$$\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{(\widetilde{u}_h(t)-\widetilde{u}_h(s))v\}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathcal{O}} (\widetilde{u}_h(t)-\widetilde{u}_h(s))\mathcal{I}_h^{xy}\{v\}\,\mathrm{d}x\,\mathrm{d}y - \int_{\mathcal{O}} (I-\mathcal{I}_h^{xy})\{(\widetilde{u}_h(t)-\widetilde{u}_h(s))\mathcal{I}_h^{xy}\{v\}\}\,\mathrm{d}x\,\mathrm{d}y. \tag{5.65}$$

Due to the strong convergence of $\widetilde{u}_h$ in $C([0,T_{\max}];L^q(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-almost surely and the convergence properties of $\mathcal{I}_h^{xy}$ (cf. [10, Theorem 4.4.20]), the first term on the right-hand side converges $\widetilde{\mathbb{P}}$-almost surely towards $\int_{\mathcal{O}}(\widetilde{u}(t)-\widetilde{u}(s))v\,\mathrm{d}x\,\mathrm{d}y$, while the second term vanishes due to Lemma A.1, as the following computation shows:

$$\begin{aligned}
&\Big|\int_{\mathcal{O}} (I-\mathcal{I}_h^{xy})\{(\widetilde{u}_h(t)-\widetilde{u}_h(s))\mathcal{I}_h^{xy}\{v\}\}\,\mathrm{d}x\,\mathrm{d}y\Big| \\
&\quad \le Ch\|\widetilde{u}_h(t)-\widetilde{u}_h(s)\|_{L^1(\mathcal{O})}\|\nabla \mathcal{I}_h^{xy}\{v\}\|_{L^\infty(\mathcal{O})} \\
&\quad \le Ch\int_{\mathcal{O}} \widetilde{u}_h^0\,\mathrm{d}x\,\mathrm{d}y\,\|\nabla v\|_{L^\infty(\mathcal{O})} \le Ch.
\end{aligned} \tag{5.66}$$

Here, we used a standard inverse estimate (cf. [10, Theorem 4.5.11]) and the nonnegativity of $\widetilde{u}_h$. In order to deal with the remaining terms, we use the decomposition

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathcal{O}} \mathcal{I}_h^y\Big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\,\widetilde{J}_h^x\partial_x\mathcal{I}_h^{xy}\{v\}\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&= \int_{t_1}^{t_2}\int_{\mathcal{O}} \mathcal{I}_h^y\Big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\Big\}\widetilde{J}_h^x\partial_x\mathcal{I}_h^{xy}\{v\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad - \int_{t_1}^{t_2}\int_{\mathcal{O}} (I-\mathcal{I}_h^y)\Big\{\widetilde{J}_h^x\mathcal{I}_h^y\Big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\,\partial_x\mathcal{I}_h^{xy}\{v\}\Big\}\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad - \int_{t_1}^{t_2}\int_{\mathcal{O}} \widetilde{J}_h^x(I-\mathcal{I}_h^y)\Big\{\mathcal{I}_h^y\Big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\Big\}\partial_x\mathcal{I}_h^{xy}\{v\}\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&= A+B+C.
\end{aligned} \tag{5.67}$$

Applying (A.1c) from Lemma A.1 pointwise in x together with an inverse estimate (cf. [10, Theorem 4.5.11]) and applying Hölder's inequality, we obtain

$$|B| \le Ch \int_{t_1}^{t_2} \|\widetilde{\mathcal{J}}_h^x\|_{L^2(\mathcal{O})} \|\partial_y \mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\partial_x \mathcal{I}_h^{xy}\{v\}\}\|_{L^2(\mathcal{O})}\, \mathrm{d}t. \tag{5.68}$$

Applying (A.1d) from Lemma A.1 and standard inverse estimates (cf. [10, Theorem 4.5.11]), we deduce the following:

$$\begin{aligned}
\|\partial_y \mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\partial_x \mathcal{I}_h^{xy}\{v\}\}\|_{L^2(\mathcal{O})} &\le \|\partial_y(\mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\}\partial_x \mathcal{I}_h^{xy}\{v\})\|_{L^2(\mathcal{O})} \\
&\quad + \|\partial_y(I-\mathcal{I}_h^y)\{\mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\}\partial_x \mathcal{I}_h^{xy}\{v\}\}\|_{L^2(\mathcal{O})} \\
&\le \|\partial_y(\mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\})\partial_x \mathcal{I}_h^{xy}\{v\}\|_{L^2(\mathcal{O})} \\
&\quad + \|\mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\}\partial_y\partial_x \mathcal{I}_h^{xy}\{v\}\|_{L^2(\mathcal{O})} \\
&\quad + C\|\mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\}\|_{L^2(\mathcal{O})}\|\partial_y\partial_x \mathcal{I}_h^{xy}\{v\}\|_{L^\infty(\mathcal{O})}.
\end{aligned} \tag{5.69}$$

As v was chosen to be sufficiently regular to control $\|\partial_y\partial_x \mathcal{I}_h^{xy}\{v\}\|_{L^\infty(\mathcal{O})}$, the only problematic term is the first one on the right-hand side. In view of (5.22e), we use the regularity of v and apply an inverse estimate (cf. [10, Theorem 4.5.11]) to obtain

$$\begin{aligned}
&\|\partial_y(\mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\})\partial_x \mathcal{I}_h^{xy}\{v\}\|_{L^2(\mathcal{O})} \\
&\quad \le C\|\partial_y(\mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\} - \widetilde{u}_h)\|_{L^2(\mathcal{O})} + C\|\partial_y \widetilde{u}_h\|_{L^2(\mathcal{O})} \\
&\quad \le Ch^{-1}\|\mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\} - \widetilde{u}_h\|_{L^2(\mathcal{O})} + C\|\partial_y \widetilde{u}_h\|_{L^2(\mathcal{O})}.
\end{aligned} \tag{5.70}$$

In conclusion, by Hölder's inequality, we have

$$\begin{aligned}
|B| \le Ch\|\widetilde{\mathcal{J}}_h^x\|_{L^2(0,T_{\max};L^2(\mathcal{O}))}\big(&\|\mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\}\|_{L^2(0,T_{\max};L^2(\mathcal{O}))} \\
&+ \|\partial_y \widetilde{u}_h\|_{L^2(0,T_{\max};L^2(\mathcal{O}))}\big) \\
+ C\|\widetilde{\mathcal{J}}_h^x\|_{L^2(0,T_{\max};L^2(\mathcal{O}))}&\|\mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\} - \widetilde{u}_h\|_{L^2(0,T_{\max};L^2(\mathcal{O}))}.
\end{aligned} \tag{5.71}$$

Therefore, in view of Lemma 5.10, B vanishes in $L^q(\widetilde{\Omega})$ for any $q < \infty$. In the same spirit, we use Hölder's inequality and (A.1c) in Lemma A.1 to prove that C vanishes in $L^q(\widetilde{\Omega})$:

$$\begin{aligned}
|C| &\le \int_{t_1}^{t_2} \|\widetilde{\mathcal{J}}_h^x\|_{L^2(\mathcal{O})}\|(I-\mathcal{I}_h^y)\{\mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\}\partial_x \mathcal{I}_h^{xy}\{v\}\}\|_{L^2(\mathcal{O})}\, \mathrm{d}t \\
&\le Ch \int_{t_1}^{t_2} \|\widetilde{\mathcal{J}}_h^x\|_{L^2(\mathcal{O})}\|\mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\}\|_{L^2(\mathcal{O})}\|\partial_y\partial_x \mathcal{I}_h^{xy}\{v\}\|_{L^\infty(\mathcal{O})}\, \mathrm{d}t \\
&\le Ch.
\end{aligned} \tag{5.72}$$

Therefore, it remains to analyze the convergence properties of A. As $\widetilde{J}_h^x$ converges weakly towards $\widetilde{J}^x$ $\widetilde{\mathbb{P}}$-almost surely in $L^2(0, T_{\max}; L^2(\mathcal{O}))$, we may use the strong convergence of $\mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\}$ in $L^\infty(0, T_{\max}; L^q(\mathcal{O}))$ with $q < \infty$ (cf. (5.22e) in Lemma 5.10) and $\partial_x \mathcal{I}_h^{xy}\{v\}$ in $L^\infty(\mathcal{O})$ to conclude that

$$A \to \int_{t_1}^{t_2} \int_{\mathcal{O}} \widetilde{u} \widetilde{J}^x \partial_x v \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t \quad \widetilde{\mathbb{P}}\text{-almost surely.} \tag{5.73}$$

Analogous arguments provide the $\widetilde{\mathbb{P}}$-almost sure convergence

$$\int_{t_1}^{t_2} \int_{\mathcal{O}} \mathcal{I}_h^x\{\sqrt{[G''(\widetilde{u}_h)]_y^{-1}} \widetilde{J}_h^y \partial_y \mathcal{I}_h^{xy}\{v\}\} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t \to \int_{t_1}^{t_2} \int_{\mathcal{O}} \widetilde{u} \widetilde{J}^y \partial_y v \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t. \tag{5.74}$$

As the Ψ-term is continuous, uniformly bounded, and converges $\widetilde{\mathbb{P}}$-almost surely, it remains to control higher moments to conclude the proof by applying Vitali's convergence theorem.

As $\widetilde{u}_h$ is uniformly bounded in $L^q(\widetilde{\Omega}; C([0, T_{\max}]; L^q(\mathcal{O})))$ for arbitrary $q < \infty$ and the error terms B and C, which were introduced by the nodal interpolation operators, vanish also in $L^q(\widetilde{\Omega})$, it remains to establish the integrability of appropriate higher moments of

$$\int_{t_1}^{t_2} \int_{\mathcal{O}} \mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\} \widetilde{J}_h^x \partial_x \mathcal{I}_h^{xy}\{v\} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t \tag{5.75a}$$

and

$$\int_{t_1}^{t_2} \int_{\mathcal{O}} \mathcal{I}_h^x\{\sqrt{[G''(\widetilde{u}_h)]_y^{-1}}\} \widetilde{J}_h^y \partial_y \mathcal{I}_h^{xy}\{v\} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t. \tag{5.75b}$$

The desired integrability of higher moments of the first integral follows from the estimate

$$\begin{aligned} &\Big| \int_{t_1}^{t_2} \int_{\mathcal{O}} \mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\} \widetilde{J}_h^x \partial_x \mathcal{I}_h^{xy}\{v\} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t \Big| \\ &\quad \leq C \|\widetilde{J}_h^x\|_{L^2(0,T_{\max};L^2(\mathcal{O}))} \|\mathcal{I}_h^y\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\}\|_{L^\infty(0,T_{\max};L^q(\mathcal{O}))} \|\partial_x \mathcal{I}_h^{xy}\{v\}\|_{L^\infty(\mathcal{O})}. \end{aligned} \tag{5.76}$$

Together with the bounds from Proposition 5.3 and Lemma 5.10, we obtain the uniform integrability of a q-moment for $q > 1$. A similar argument provides the uniform integrability of the second term in (5.75b), which concludes the proof. ■

Lemma 5.15. *Let Assumptions (S), (I), (P), (B), (R), (B3*), and (B4) hold true. Then for all* $[0, 1]$*-valued continuous functions* Ψ *defined on* $\mathcal{X}_u \times \mathcal{X}_{\mathbf{W}}$*, we have*

$$\begin{aligned} \widetilde{\mathbb{E}}\Big[\Big(\widetilde{\mathcal{M}}_v^2(t) - \widetilde{\mathcal{M}}_v^2(s) - \int_s^t \sum_{k,l\in\mathbb{Z}} {\lambda_{kl}^x}^2 \Big(\int_{\mathcal{O}} \widetilde{u} \mathfrak{g}_{kl} \partial_x v \,\mathrm{d}x \,\mathrm{d}y\Big)^2 \mathrm{d}\tau \\ - \int_s^t \sum_{k,l\in\mathbb{Z}} {\lambda_{kl}^y}^2 \Big(\int_{\mathcal{O}} \widetilde{u} \mathfrak{g}_{kl} \partial_y v \,\mathrm{d}x \,\mathrm{d}y\Big)^2 \mathrm{d}\tau\Big) \Psi(r_s \widetilde{u}, r_s \widetilde{W})\Big] = 0 \end{aligned} \tag{5.77}$$

for all $0 \le s \le t \le T_{\max}$, *where*

$$\widetilde{\mathcal{M}}_v(t) := \int_{\mathcal{O}} (\widetilde{u}(t) - \widetilde{u}(0)) v \,\mathrm{d}x\,\mathrm{d}y + \int_0^t \int_{\mathcal{O}} \widetilde{u}\widetilde{J}^x \partial_x v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau + \int_0^t \int_{\mathcal{O}} \widetilde{u}\widetilde{J}^y \partial_y v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau. \tag{5.78}$$

Proof. We will prove this by passing to the limit in the martingale (5.58b). Recalling the arguments from the proof of Lemma 5.14, we obtain that

$$\begin{aligned}\widetilde{\mathcal{M}}_{h,v}^2(t) := \Big(\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{(\widetilde{u}_h(t) - \widetilde{u}_h(0))v\} \,\mathrm{d}x\,\mathrm{d}y &+ \int_0^{t\wedge\widetilde{T}_h} \int_{\mathcal{O}} \mathcal{I}_h^y\Big\{\sqrt{[G''(\widetilde{u}_h)]_x^{-1}}\,\widetilde{J}_h^x \partial_x \mathcal{I}_h^{xy}\{v\}\Big\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau \\ &+ \int_0^{t\wedge\widetilde{T}_h} \int_{\mathcal{O}} \mathcal{I}_h^x\Big\{\sqrt{[G''(\widetilde{u}_h)]_y^{-1}}\,\widetilde{J}_h^y \partial_y \mathcal{I}_h^{xy}\{v\}\Big\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\Big)^2\end{aligned} \tag{5.79}$$

converges along a subsequence $\widetilde{\mathbb{P}}$-almost surely towards

$$\widetilde{\mathcal{M}}_v^2(t) = \Big(\int_{\mathcal{O}} (\widetilde{u}(t) - \widetilde{u}(0)) v \,\mathrm{d}x\,\mathrm{d}y + \int_0^t \int_{\mathcal{O}} \widetilde{u}\widetilde{J}^x \partial_x v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau + \int_0^t \int_{\mathcal{O}} \widetilde{u}\widetilde{J}^y \partial_y v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\Big)^2. \tag{5.80}$$

In order to deduce the convergence of the corresponding expected values, we need to establish higher regularity of $\widetilde{\mathcal{M}}_{h,v}$. Starting from the representation

$$\begin{aligned}\widetilde{\mathcal{M}}_{h,v}(t) = &-\sum_{k\in I_h^x}\sum_{l\in I_h^y} \lambda_{kl}^x \int_0^{t\wedge\widetilde{T}_h} \int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_{h,\mathrm{loc}}^x\{\partial_x(\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}_h^{xy}\{v\}\}\} \,\mathrm{d}x\,\mathrm{d}y\ \mathrm{d}\widetilde{\beta}_{h,kl}^x \\ &-\sum_{k\in I_h^x}\sum_{l\in I_h^y} \lambda_{kl}^y \int_0^{t\wedge\widetilde{T}_h} \int_{\mathcal{O}} \mathcal{I}_h^x\{\mathcal{I}_{h,\mathrm{loc}}^y\{\partial_y(\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}_h^{xy}\{v\}\}\} \,\mathrm{d}x\,\mathrm{d}y\ \mathrm{d}\widetilde{\beta}_{h,kl}^y\end{aligned} \tag{5.81}$$

and combining the martingale moment inequality

$$\widetilde{\mathbb{E}}\big[|\widetilde{\mathcal{M}}_{h,v}(t)|^{2q}\big] \le C_q \widetilde{\mathbb{E}}\big[\langle\!\langle \widetilde{\mathcal{M}}_{h,v}\rangle\!\rangle_t^q\big] \quad \text{for any } q > 0 \tag{5.82}$$

(see, e.g., [54, Chapter 3, Proposition 3.26]) with Lemma 5.12 formulated for $\widetilde{\mathcal{M}}_{h,v}$, we obtain

$$\begin{aligned}\widetilde{\mathbb{E}}\big[|\widetilde{\mathcal{M}}_{h,v}(t)|^{2q}\big] \le C\widetilde{\mathbb{E}}\big[\langle\!\langle \widetilde{\mathcal{M}}_{h,v}\rangle\!\rangle_t^q\big] &\le C\|v\|_{H^2(\mathcal{O})}^{2q}\widetilde{\mathbb{E}}\Big[\Big(\int_0^{t\wedge\widetilde{T}_h} \|\widetilde{u}_h(s)\|_{H^1(\mathcal{O})}^2 \,\mathrm{d}s\Big)^q\Big] \\ &\le C\|v\|_{H^2(\mathcal{O})}^{2q} T_{\max}^q \widetilde{\mathbb{E}}\big[\sup_{s\in[0,T_{\max}]} \mathcal{E}_h(\widetilde{u}_h)^q + C\big].\end{aligned} \tag{5.83}$$

It remains to pass to the limit in the remaining integrals in (5.58b), i.e., we have to analyze the convergence behavior of $\langle\langle \widetilde{\mathcal{M}}_{h,v}\rangle\rangle_t$ for $h \searrow 0$ (cf. (5.60)). The first step is to show that we may neglect the interpolation operators when passing to the limit. Applying (2.13a) and (2.13b), we may rewrite the first component of $\widetilde{\mathcal{M}}_{h,v}$ using

$$\begin{aligned}
&\int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_{h,\mathrm{loc}}^x\{\partial_x(\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}_h^{xy}\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \int_{\mathcal{O}} \mathcal{I}_h^y\{\partial_x\widetilde{u}_h\mathcal{I}_h^x\{\widetilde{\mathfrak{g}}_{h,kl}v\}\}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathcal{O}} \mathcal{I}_h^y\{\partial_x\widetilde{\mathfrak{g}}_{h,kl}\mathcal{I}_h^x\{\widetilde{u}_h v\}\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \int_{\mathcal{O}} \partial_x\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl}\mathcal{I}_h^{xy}\{v\}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathcal{O}} \partial_x\widetilde{\mathfrak{g}}_{h,kl}\widetilde{u}_h\mathcal{I}_h^{xy}\{v\}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad - \int_{\mathcal{O}} (I-\mathcal{I}_h^y)\{\partial_x\widetilde{u}_h\mathcal{I}_h^{xy}\{\widetilde{\mathfrak{g}}_{h,kl}v\}\}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad - \int_{\mathcal{O}} \partial_x\widetilde{u}_h(I-\mathcal{I}_h^{xy})\{\widetilde{\mathfrak{g}}_{h,kl}\mathcal{I}_h^{xy}\{v\}\}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad - \int_{\mathcal{O}} (I-\mathcal{I}_h^y)\{\partial_x\widetilde{\mathfrak{g}}_{h,kl}\mathcal{I}_h^{xy}\{\widetilde{u}_h v\}\}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad - \int_{\mathcal{O}} \partial_x\widetilde{\mathfrak{g}}_{h,kl}(I-\mathcal{I}_h^{xy})\{\widetilde{u}_h\mathcal{I}_h^{xy}\{v\}\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad=: \int_{\mathcal{O}} \partial_x(\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}_h^{xy}\{v\}\,\mathrm{d}x\,\mathrm{d}y + A_{kl} + B_{kl} + C_{kl} + D_{kl},
\end{aligned} \tag{5.84}$$

where we used that $\widetilde{\mathfrak{g}}_{h,kl} \in U_h$. Combining (5.84) with the binomial theorem and (5.60), we get

$$\begin{aligned}
&\Big|\sum_{k\in I_h^x}\sum_{l\in I_h^y} {\lambda_{kl}^x}^2 \int_0^{t\wedge\widetilde{T}_h}\Big(\int_{\mathcal{O}} \mathcal{I}_h^y\{\mathcal{I}_{h,\mathrm{loc}}^x\{\partial_x(\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}_h^{xy}\{v\}\}\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\,\mathrm{d}\tau \\
&\qquad - \sum_{k\in I_h^x}\sum_{l\in I_h^y} {\lambda_{kl}^x}^2 \int_0^{t\wedge\widetilde{T}_h}\Big(\int_{\mathcal{O}} \partial_x(\widetilde{u}_h\widetilde{\mathfrak{g}}_{h,kl})\mathcal{I}_h^{xy}\{v\}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\,\mathrm{d}\tau\Big| \\
&\quad\le C\sqrt{\langle\langle \widetilde{\mathcal{M}}_{h,v}\rangle\rangle_t}\Big(\sum_{k\in I_h^x}\sum_{l\in I_h^y} {\lambda_{kl}^x}^2 \int_0^{t\wedge\widetilde{T}_h} A_{kl}^2 + B_{kl}^2 + C_{kl}^2 + D_{kl}^2\,\mathrm{d}\tau\Big)^{1/2} \\
&\qquad + C\sum_{k\in I_h^x}\sum_{l\in I_h^y} {\lambda_{kl}^x}^2 \int_0^{t\wedge\widetilde{T}_h} A_{kl}^2 + B_{kl}^2 + C_{kl}^2 + D_{kl}^2\,\mathrm{d}\tau \;=: (\#).
\end{aligned} \tag{5.85}$$

Using Lemma A.1 and standard inverse estimates (cf. [10, Theorem 4.5.11]), we obtain the estimates

$$\begin{aligned}
A_{kl} &\le Ch\|\partial_x\widetilde{u}_h\|_{L^2(\mathcal{O})}\|\partial_y\mathcal{I}_h^{xy}\{\widetilde{\mathfrak{g}}_{h,kl}v\}\|_{L^2(\mathcal{O})} \\
&\le Ch\|\partial_x\widetilde{u}_h\|_{L^2(\mathcal{O})}\|\partial_y(\widetilde{\mathfrak{g}}_{h,kl}v)\|_{L^\infty(\mathcal{O})},
\end{aligned} \tag{5.86}$$

$$B_{kl} \le Ch \|\partial_x \widetilde{u}_h\|_{L^2(\mathcal{O})} \|\widetilde{\mathfrak{g}}_{h,kl}\|_{L^4(\mathcal{O})} \|\nabla \mathcal{I}_h^{xy}\{v\}\|_{L^4(\mathcal{O})}, \tag{5.87}$$

$$\begin{aligned} C_{kl} &\le Ch \|\partial_x \partial_y \widetilde{\mathfrak{g}}_{h,kl}\|_{L^\infty(\mathcal{O})} \|\mathcal{I}_h^{xy}\{\widetilde{u}_h v\}\|_{L^1(\mathcal{O})} \\ &\le Ch \|\partial_x \partial_y \mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})} \|\widetilde{u}_h\|_{L^2(\mathcal{O})} \|\mathcal{I}_h^{xy}\{v\}\|_{L^2(\mathcal{O})}, \end{aligned} \tag{5.88}$$

$$D_{kl} \le Ch \|\partial_x \widetilde{\mathfrak{g}}_{h,kl}\|_{L^\infty(\mathcal{O})} \|\widetilde{u}_h\|_{L^2(\mathcal{O})} \|\nabla \mathcal{I}_h^{xy}\{v\}\|_{L^2(\mathcal{O})}. \tag{5.89}$$

Recalling Assumption (B3), (2.6), and the regularity of v, we obtain

$$(\#) \le Ch^{1/2} \sqrt{\langle\!\langle \widetilde{\mathcal{M}}_{h,v} \rangle\!\rangle_t} \Big(\int_0^{t\wedge \widetilde{T}_h} \|\widetilde{u}_h\|_{H^1(\mathcal{O})}^2 \,\mathrm{d}\tau \Big)^{1/2} + Ch \int_0^{t\wedge \widetilde{T}_h} \|\widetilde{u}_h\|_{H^1(\mathcal{O})}^2 \,\mathrm{d}\tau. \tag{5.90}$$

Therefore, the $\overline{p}$-th moment of the left-hand side of (5.85) vanishes, which in particular provides convergence $\widetilde{\mathbb{P}}$-almost surely after restricting ourselves to appropriate subsequences. It remains to discuss the convergence properties of

$$\sum_{k\in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^x \int_0^{t\wedge \widetilde{T}_h} \Big(\int_{\mathcal{O}} \partial_x (\widetilde{u}_h \widetilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy}\{v\} \,\mathrm{d}x \,\mathrm{d}y \Big)^2 \,\mathrm{d}\tau. \tag{5.91}$$

After integrating by parts, we may use the standard error estimates for the nodal interpolation operator (cf. [10, Theorem 4.4.20]) and Assumption (B3) to obtain the strong convergence of $\lambda_{kl}^x \widetilde{\mathfrak{g}}_{h,kl}$ towards $\lambda_{kl}^x \mathfrak{g}_{kl}$ in $L^\infty(\mathcal{O})$ and the strong convergence of $\partial_x \mathcal{I}_h^{xy}\{v\}$ towards $\partial_x v$ in $L^\infty(\mathcal{O})$. Together with (5.22a), this provides the $\widetilde{\mathbb{P}}$-almost sure convergence. Similar considerations provide the convergence of

$$\sum_{k\in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^x \int_0^{t\wedge \widetilde{T}_h} \Big(\int_{\mathcal{O}} \mathcal{I}_h^x\{\mathcal{I}_{h,\mathrm{loc}}^y\{\partial_y (\widetilde{u}_h \widetilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy}\{v\}\}\} \,\mathrm{d}x \,\mathrm{d}y \Big)^2 \,\mathrm{d}\tau$$

$\widetilde{\mathbb{P}}$-almost surely. As we already established the higher integrability of $\langle\!\langle \widetilde{\mathcal{M}}_{h,v} \rangle\!\rangle_t$ in (5.83), we may conclude by applying Vitali's theorem. ∎

In the same spirit, we get the following result:

Lemma 5.16. *Let Assumptions (S), (I), (P), (B), (R), (B3*), and (B4) hold true. Then for all* $[0,1]$*-valued continuous functions* Ψ *defined on* $\mathcal{X}_u \times \mathcal{X}_{\boldsymbol{W}}$*, we have*

$$\widetilde{\mathbb{E}}\Big[\Big(\widetilde{\mathcal{M}}_v(t) \widetilde{\beta}_{kl}^\alpha(t) - \widetilde{\mathcal{M}}_v(s) \widetilde{\beta}_{kl}^\alpha(s) - \lambda_{kl}^\alpha \int_s^t \int_{\mathcal{O}} \partial_\alpha (\widetilde{u} \mathfrak{g}_{kl}) v \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}\tau \Big) \Psi(r_s \widetilde{u}, r_s \widetilde{\boldsymbol{W}}) \Big] = 0 \tag{5.92}$$

for all $k, l \in \mathbb{Z}$*,* $\alpha \in \{x, y\}$*, and all* $s \le t \in [0, T_{\max}]$.

Lemma 5.17. *Let Assumptions (S), (I), (P), (B), (R), (B3*), and (B4) hold true. Then, we have*

$$\widetilde{\mathcal{M}}_v(t) = \sum_{k,l\in\mathbb{Z}} \lambda_{kl}^x \int_0^t \int_{\mathcal{O}} \partial_x(\widetilde{u}\mathfrak{g}_{kl})v \,\mathrm{d}x\,\mathrm{d}y\;\mathrm{d}\widetilde{\beta}_{kl}^x + \sum_{k,l\in\mathbb{Z}} \lambda_{kl}^y \int_0^t \int_{\mathcal{O}} \partial_y(\widetilde{u}\mathfrak{g}_{kl})v \,\mathrm{d}x\,\mathrm{d}y\;\mathrm{d}\widetilde{\beta}_{kl}^y. \tag{5.93}$$

Proof. As a martingale with vanishing quadratic variation is almost surely constant, it is sufficient to show that

$$0 = \langle\langle \widetilde{\mathcal{M}}_v(\cdot)\rangle\rangle_T + \Big\langle\Big\langle \sum_{\alpha\in\{x,y\}}\sum_{k,l\in\mathbb{Z}} \int_0^{(\cdot)} \lambda_{kl}^\alpha \int_{\mathcal{O}} \partial_\alpha(\widetilde{u}\mathfrak{g}_{kl})v\,\mathrm{d}x\,\mathrm{d}y\;\mathrm{d}\widetilde{\beta}_{kl}^\alpha \Big\rangle\Big\rangle_T - 2\Big\langle\Big\langle \widetilde{\mathcal{M}}_v(\cdot), \sum_{\alpha\in\{x,y\}}\sum_{k,l\in\mathbb{Z}} \int_0^{(\cdot)} \lambda_{kl}^\alpha \int_{\mathcal{O}} \partial_\alpha(\widetilde{u}\mathfrak{g}_{kl})v\,\mathrm{d}x\,\mathrm{d}y\;\mathrm{d}\widetilde{\beta}_{kl}^\alpha \Big\rangle\Big\rangle_T. \tag{5.94}$$

To compute the last term on the right-hand side, we use the cross variation formula, which can be found, e.g., in [54, Section 3.2, Lemma 2.16], to obtain

$$\Big\langle\Big\langle \widetilde{\mathcal{M}}_v(\cdot), \sum_{\alpha\in\{x,y\}}\sum_{k,l\in\mathbb{Z}} \int_0^{(\cdot)} \lambda_{kl}^\alpha \int_{\mathcal{O}} \partial_\alpha(\widetilde{u}\mathfrak{g}_{kl})v\,\mathrm{d}x\,\mathrm{d}y\;\mathrm{d}\widetilde{\beta}_{kl}^\alpha \Big\rangle\Big\rangle_T = \sum_{\alpha\in\{x,y\}}\sum_{k,l\in\mathbb{Z}} \int_0^{T} \lambda_{kl}^\alpha \int_{\mathcal{O}} \partial_\alpha(\widetilde{u}\mathfrak{g}_{kl})v\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\langle\langle \widetilde{\mathcal{M}}_v(\cdot), \widetilde{\beta}_{kl}^\alpha(\cdot)\rangle\rangle_s. \tag{5.95}$$

Following the arguments in [29], it is possible to show that the process $s \mapsto \langle\langle \widetilde{\mathcal{M}}_v(\cdot), \widetilde{\beta}_{kl}^\alpha\rangle\rangle_s$ is absolutely continuous $\widetilde{\mathbb{P}}$-almost surely, and consequently,

$$\mathrm{d}\langle\langle \widetilde{\mathcal{M}}_v(\cdot), \widetilde{\beta}_{kl}^\alpha\rangle\rangle_s = \lambda_{kl}^\alpha \int_{\mathcal{O}} \partial_\alpha(\widetilde{u}(s)\mathfrak{g}_{kl})v\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \tag{5.96}$$

Using the identities

$$\langle\langle \widetilde{\mathcal{M}}_v(\cdot)\rangle\rangle_T = \sum_{\alpha\in\{x,y\}}\sum_{k,l\in\mathbb{Z}} \lambda_{kl}^\alpha \int_0^T \Big(\int_{\mathcal{O}} \partial_\alpha(\widetilde{u}\mathfrak{g}_{kl})v\,\mathrm{d}x\,\mathrm{d}y\Big)^2 \mathrm{d}s = \Big\langle\Big\langle \sum_{\alpha\in\{x,y\}}\sum_{k,l\in\mathbb{Z}} \lambda_{kl}^\alpha \int_0^{(\cdot)} \int_{\mathcal{O}} \partial_\alpha(\widetilde{u}\mathfrak{g}_{kl})v\,\mathrm{d}x\,\mathrm{d}y\;\mathrm{d}\widetilde{\beta}_{kl}^\alpha \Big\rangle\Big\rangle_T, \tag{5.97}$$

we have

$$\Big\langle\Big\langle \widetilde{\mathcal{M}}_v(\cdot) - \sum_{\alpha\in\{x,y\}}\sum_{k,l\in\mathbb{Z}} \lambda_{kl}^\alpha \int_0^{(\cdot)} \int_{\mathcal{O}} \partial_\alpha(\widetilde{u}\mathfrak{g}_{kl})v\,\mathrm{d}x\,\mathrm{d}y\;\mathrm{d}\widetilde{\beta}_{kl}^\alpha \Big\rangle\Big\rangle_T \equiv 0. \tag{5.98}$$

Thus, we obtain the desired result. ■

Having established the previous results, we are now in the position to prove Theorem 3.5.

Proof of Theorem 3.5. From Proposition 5.3, Corollary 5.7, and Lemma 5.8, we infer the existence of a stochastic basis $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$ of a Wiener process

$$\widetilde{W}(t) = \sum_{\alpha\in\{x,y\}} \sum_{k,l\in\mathbb{Z}} \lambda_{kl}^{\alpha} \mathfrak{g}_{kl} \widetilde{\beta}_{kl}^{\alpha} \boldsymbol{b}_{\alpha}, \tag{5.99}$$

and of random variables

$$\begin{aligned} \widetilde{u} &\in L^q(\widetilde{\mathcal{O}}; L^\infty(0,T_{\max}; H^1_{\mathrm{per}}(\mathcal{O}))) \cap L^2(\widetilde{\Omega}; L^2(0,T_{\max}; H^2_{\mathrm{per}}(\mathcal{O}))) \\ &\quad \cap L^\sigma(\widetilde{\Omega}; C^{1/4}([0,T_{\max}]; (H^1_{\mathrm{per}}(\mathcal{O}))')), \qquad (5.100a) \\ \widetilde{J}^x &\in L^2(\widetilde{\Omega}; L^2(0,T_{\max}; L^2(\mathcal{O}))), \qquad (5.100b) \\ \widetilde{J}^y &\in L^2(\widetilde{\Omega}; L^2(0,T_{\max}; L^2(\mathcal{O}))) \qquad (5.100c) \end{aligned}$$

with $q < \infty$ and $\sigma < 8/5$. As shown in Lemma 5.10, these random variables satisfy

$$\begin{aligned} \widetilde{J}^x &= \widetilde{u}\partial_x(-\Delta\widetilde{u} + F'(\widetilde{u})) \quad \widetilde{\mathbb{P}}\text{-almost surely in } [\widetilde{u} > 0], \qquad (5.101a) \\ \widetilde{J}^y &= \widetilde{u}\partial_y(-\Delta\widetilde{u} + F'(\widetilde{u})) \quad \widetilde{\mathbb{P}}\text{-almost surely in } [\widetilde{u} > 0]. \qquad (5.101b) \end{aligned}$$

Furthermore, we have $\Lambda = \widetilde{\mathbb{P}} \circ (\widetilde{u}^0)^{-1}$ by construction. Lemma 5.14 implies that

$$\widetilde{\mathcal{M}}_v(t) = \int_{\mathcal{O}} (\widetilde{u}(t) - \widetilde{u}^0) v \,\mathrm{d}x\,\mathrm{d}y + \int_0^t \int_{\mathcal{O}} \widetilde{u}\widetilde{J}^x \partial_x v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s + \int_0^t \int_{\mathcal{O}} \widetilde{u}\widetilde{J}^y \partial_y v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \tag{5.102}$$

is an $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$-martingale, and by Lemma 5.17, we obtain

$$\begin{aligned} &\int_{\mathcal{O}} (\widetilde{u}(t) - \widetilde{u}^0) v \,\mathrm{d}x\,\mathrm{d}y + \int_0^t \int_{\mathcal{O}} \widetilde{u}\widetilde{J}^x \partial_x v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s + \int_0^t \int_{\mathcal{O}} \widetilde{u}\widetilde{J}^y \partial_y v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\ &\quad = \sum_{k,l\in\mathbb{Z}} \lambda_{kl}^x \int_0^t \int_{\mathcal{O}} \partial_x(\widetilde{u}\mathfrak{g}_{kl}) v \,\mathrm{d}x\,\mathrm{d}y \;\mathrm{d}\widetilde{\beta}_{kl}^x \\ &\qquad + \sum_{k,l\in\mathbb{Z}} \lambda_{kl}^y \int_0^t \int_{\mathcal{O}} \partial_y(\widetilde{u}\mathfrak{g}_{kl}) v \,\mathrm{d}x\,\mathrm{d}y \;\mathrm{d}\widetilde{\beta}_{kl}^y. \end{aligned} \tag{5.103}$$

It remains to establish the energy estimate (3.9). Starting from Proposition 4.2, using Fatou's lemma and the definition of the fluxes in (5.1), we find

$$\begin{aligned} &\widetilde{\mathbb{E}}\Big[\liminf_{h\searrow 0}\Big(\sup_{t\in[0,T_{\max}]}\Big(\frac{1}{2}\int_{\mathcal{O}} \mathcal{I}_h^y\{|\partial_x\widetilde{u}_h|^2\} + \mathcal{I}_h^x\{|\partial_y\widetilde{u}_h|^2\}\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad + \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{F(\widetilde{u}_h)\}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\mathfrak{p}}\Big)\Big] + \widetilde{\mathbb{E}}\Big[\liminf_{h\searrow 0}\int_0^{T_{\max}}\int_{\mathcal{O}} |\widetilde{J}_h^x|^2 + |\widetilde{J}_h^y|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big] \end{aligned}$$

$$\leq \liminf_{h\searrow 0} \widetilde{\mathbb{E}}\Big[\sup_{t\in[0,T_{\max}]} \mathcal{E}_h(\widetilde{u}_h)^{\overline{p}} + \int_0^{T_{\max}} \int_{\mathcal{O}} \mathcal{I}_h^y\{|\widetilde{J}_h^x|^2\} + \mathcal{I}_h^x\{|\widetilde{J}_h^y|^2\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big]$$
$$\leq C(\overline{p}, u^0, T_{\max}), \tag{5.104}$$

where we used $\frac{1}{2}h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{|\Delta_h\widetilde{u}_h|^2\}\,\mathrm{d}x\,\mathrm{d}y \geq 0$. By the norm equivalence (2.6), we obtain

$$\widetilde{\mathbb{E}}\Big[\liminf_{h\searrow 0}\Big(\sup_{t\in[0,T_{\max}]}\Big(\frac{1}{2}\int_{\mathcal{O}} |\nabla\widetilde{u}_h|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{F(\widetilde{u}_h)\}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\overline{p}}\Big)\Big]$$
$$+ \widetilde{\mathbb{E}}\Big[\liminf_{h\searrow 0} \int_0^{T_{\max}} \int_{\mathcal{O}} |\widetilde{J}_h^x|^2 + |\widetilde{J}_h^y|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big] \leq C(\overline{p}, u^0, T_{\max}). \tag{5.105}$$

Similar to the proof of [29, Theorem 3.2], we may use the lower semi-continuity in appropriate topologies to find that

$$\mathbb{E}\Big[\sup_{t\in[0,T_{\max}]}\Big(\frac{1}{2}\int_{\mathcal{O}} |\nabla\widetilde{u}_h|^2\,\mathrm{d}x\,\mathrm{d}y\Big)^{\overline{p}} + \int_0^{T_{\max}} \int_{\mathcal{O}} |\widetilde{J}_h^x|^2 + |\widetilde{J}_h^y|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big]$$
$$\leq C(\overline{p}, u^0, T_{\max}). \tag{5.106}$$

So, let us focus on the remaining term $\int_{\mathcal{O}} \mathcal{I}_h^{xy}\{F(\widetilde{u}_h)\}\,\mathrm{d}x\,\mathrm{d}y$.

Step 1: There is a positive constant C such that

$$\int_{\mathcal{O}} F(\widetilde{u}_h)\,\mathrm{d}x\,\mathrm{d}y \leq \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{F(\widetilde{u}_h)\}\,\mathrm{d}x\,\mathrm{d}y + C. \tag{5.107}$$

Indeed, with the notation of Assumption (P), we find F to be convex for $s \in [0, \widehat{u}]$, where $\widehat{u}$ is given by

$$\widehat{u} = \sqrt[p+2]{\frac{\widetilde{c}_1}{\widetilde{c}_2}} \tag{5.108}$$

with the constants $\widetilde{c}_1$ and $\widetilde{c}_2$ introduced in Assumption (P). Introducing $\widehat{\delta} := \widehat{u}/C_{\mathrm{osc}}$, with C_{osc} being the constant in (3.10), and using the definition for $S_{\widehat{\delta}}^{\mathcal{Q}_h}$ from (5.21d), we find that

$$\max_{(x,y)\in\mathcal{O}\setminus S_{\widehat{\delta}}^{\mathcal{Q}_h}} \widetilde{u}_h \leq \widehat{u}. \tag{5.109}$$

Therefore, we have

$$F(\widetilde{u}_h) \leq \mathcal{I}_h^{xy}\{F(\widetilde{u}_h)\} \quad \text{on } \mathcal{O} \setminus S_{\widehat{\delta}}^{\mathcal{Q}_h} \tag{5.110}$$

due to the convexity of F on $(0, \widehat{u})$.

Using (P) once more, we find

$$0 \leq F(\widetilde{u}_h) \leq C(\widehat{\delta}^{-p} + 1) \quad \text{on } S_{\widehat{\delta}}^{\mathcal{Q}_h}. \tag{5.111}$$

Combining (5.110) and (5.111), (5.107) is established.

Step 2: Estimate for $\widetilde{\mathbb{E}}[\operatorname{ess\,sup}_{t\in[0,T_{\max}]}(\int_{\mathcal{O}} F(\tilde{u})\,\mathrm{d}x\,\mathrm{d}y)^{\overline{p}}]$.

From (5.22a) and (5.22b) together with Sobolev's embedding result, we infer $\widetilde{\mathbb{P}}$-a.s. that for almost all $t\in[0,T_{\max}]$,

$$\tilde{u}_h(\tilde{\omega},t,\cdot)\to\tilde{u}(\tilde{\omega},t,\cdot)\quad\text{in } C^{\gamma}(\overline{\mathcal{O}}). \tag{5.112}$$

Hence, $F(\tilde{u}_h(\tilde{\omega},t,\cdot))\to F(\tilde{u}(\tilde{\omega},t,\cdot))$ pointwise, which implies together with Fatou's lemma that

$$\operatorname*{ess\,sup}_{t\in[0,T_{\max}]}\int_{\mathcal{O}} F(\tilde{u}(\tilde{\omega},t,\cdot))\,\mathrm{d}x\,\mathrm{d}y\le\liminf_{h\searrow 0}\sup_{t\in[0,T_{\max}]}\int_{\mathcal{O}} F(\tilde{u}_h(\tilde{\omega},t,\cdot))\,\mathrm{d}x\,\mathrm{d}y. \tag{5.113}$$

As both sides of inequality (5.113) are nonnegative, we can take the $\overline{p}$-th power on both sides. Taking the expectation concludes this step.

Step 3: Estimate for $\widetilde{\mathbb{E}}[\sup_{t\in[0,T_{\max}]}(\int_{\mathcal{O}} F(\tilde{u})\,\mathrm{d}x\,\mathrm{d}y)^{\overline{p}}]$.

We fix $t_0\in[0,T_{\max}]$ arbitrarily and choose $\tilde{\omega}\in\widetilde{\Omega}$ such that $\tilde{u}(\tilde{\omega},\cdot,\cdot)\in C([0,T_{\max}];L^q(\mathcal{O}))$ and $M(\tilde{\omega}):=\operatorname{ess\,sup}_{t\in[0,T_{\max}]}\int_{\mathcal{O}} F(\tilde{u})\,\mathrm{d}x\,\mathrm{d}y<\infty$. The latter limitations are satisfied by almost all $\tilde{\omega}\in\widetilde{\Omega}$ due to Proposition 5.3 and Step 2.

We will now show that $\tilde{u}(\tilde{\omega},t_0,\cdot)$ vanishes only on a set of measure zero. Therefore, we take a sequence $(s_n)_{n\in\mathbb{N}}$ in $[0,T_{\max}]$ which satisfies $s_n\to t_0$ for $n\nearrow\infty$,

$$\int_{\mathcal{O}} F(\tilde{u}(\tilde{\omega},s_n,x,y))\,\mathrm{d}x\,\mathrm{d}y\le M(\tilde{\omega}), \tag{5.114}$$

and $\tilde{u}(\tilde{\omega},s_n,\cdot)\to\tilde{u}(\tilde{\omega},t_0,\cdot)$ pointwise almost everywhere. Assuming that the set $\mathcal{N}$, where $\tilde{u}(\tilde{\omega},t_0,\cdot)$ vanishes, has positive measure, we obtain by Egorov's theorem the existence of a set $\mathcal{N}^{\delta}\subset\mathcal{N}\subset\mathcal{O}$ such that $\mu(\mathcal{N}^{\delta})>(1-\delta)\mu(\mathcal{N})$ for $\delta\in(0,1)$ such that $\tilde{u}(\tilde{\omega},s_n,\cdot)\to\tilde{u}(\tilde{\omega},t_0,\cdot)$ uniformly in $\mathcal{N}^{\delta}$. This provides

$$\int_{\mathcal{N}^{\delta}} F(\tilde{u}(\tilde{\omega},s_n,x,y))\,\mathrm{d}x\,\mathrm{d}y\overset{n\to\infty}{\to}+\infty. \tag{5.115}$$

However, at the same time we have

$$\int_{\mathcal{N}^{\delta}} F(\tilde{u}(\tilde{\omega},s_n,x,y))\,\mathrm{d}x\,\mathrm{d}y\le\int_{\mathcal{O}} F(\tilde{u}(\tilde{\omega},s_n,x,y))\,\mathrm{d}x\,\mathrm{d}y\le M(\tilde{\omega}). \tag{5.116}$$

This contradiction provides $\mu(\mathcal{N})=0$ and $F(\tilde{u}(\tilde{\omega},s_n,\cdot))\to F(\tilde{u}(\tilde{\omega},t_0,\cdot))$ pointwise almost everywhere. Hence, by applying Fatou's lemma, we obtain

$$\begin{aligned}\int_{\mathcal{O}} F(\tilde{u}(\tilde{\omega},t_0,x,y))\,\mathrm{d}x\,\mathrm{d}y&\le\liminf_{n\to\infty}\int_{\mathcal{O}} F(\tilde{u}(\tilde{\omega},s_n,x,y))\,\mathrm{d}x\,\mathrm{d}y\\&\le M(\tilde{\omega})=\operatorname*{ess\,sup}_{t\in[0,T_{\max}]}\int_{\mathcal{O}} F(\tilde{u}(\tilde{\omega},t,x,y))\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{5.117}$$

Again, taking the $\overline{p}$-th power and the expectation concludes this step.

Together with (5.104) and (5.107), the energy inequality (3.9) is established. Finally, we may combine (5.100a) and (3.9) to deduce the positivity properties of $\tilde{u}$ claimed in the theorem. ∎

Remark 5.18. As estimate (3.9) is only of a qualitative character, we did not strive for an optimal result. In fact, it is—being based on (5.111)—a rather coarse estimate. If more information is available on F, for instance number and height of local maxima, much better estimates are available, based on appropriate convexity arguments.

6. Conclusion

We have proven the existence of martingale solutions to stochastic thin-film equations with conservative linear multiplicative noise in two space dimensions. As our result covers driving noise both in the Itô- and in the Stratonovich-sense, we expect it to be a starting point to construct solutions for solely surface tension driven thin-film evolution (i.e., $F \equiv 0$) subject to compactly supported initial data.[1] This raises questions about the impact of noise on the evolution of the solution's support ("finite speed of propagation" and "waiting time phenomena"). It is well-known in the (deterministic) theory of thin-film equations that analytical concepts in two space dimensions carry over to higher dimensions, while the argumentation in dimension $d = 1$ takes advantage of the Sobolev embedding $H^1 \hookrightarrow C^{1/2}$ and is therefore much less involved. Hence, a generalization to $d = 3$ (with the perspective of applications to models for phase separation) is feasible as well. Finally, the implementation of numerical schemes related to the finite element approach presented here will provide further insight into the impact of noise on thin-film evolution.

A. Auxiliary results

Lemma A.1. *Let $\mathcal{Q}_h$ satisfy Assumption (S) and let I be the identity operator. Furthermore, let $p \in [1,\infty)$, $q, r \in [1,\infty]$, $q^* := \frac{q}{q-1}$, and $r^* := \frac{r}{r-1}$. Then the estimates*

$$\|(I - \mathcal{I}_h^x)\{f_h^x g_h^x\}\|_{L^p(\mathcal{O}^x)} \le C h_x^2 \|\partial_x f_h^x\|_{L^{pq}(\mathcal{O}^x)} \|\partial_x g_h^x\|_{L^{pq^*}(\mathcal{O}^x)}, \tag{A.1a}$$

$$\|\partial_x (I - \mathcal{I}_h^x)\{f_h^x g_h^x\}\|_{L^p(\mathcal{O}^x)} \le C h_x \|\partial_x f_h^x\|_{L^{pq}(\mathcal{O}^x)} \|\partial_x g_h^x\|_{L^{pq^*}(\mathcal{O}^x)}, \tag{A.1b}$$

$$\|(I - \mathcal{I}_h^y)\{f_h^y g_h^y\}\|_{L^p(\mathcal{O}^y)} \le C h_y^2 \|\partial_y f_h^y\|_{L^{pq}(\mathcal{O}^y)} \|\partial_y g_h^y\|_{L^{pq^*}(\mathcal{O}^y)}, \tag{A.1c}$$

$$\|\partial_y (I - \mathcal{I}_h^y)\{f_h^y g_h^y\}\|_{L^p(\mathcal{O}^y)} \le C h_y \|\partial_y f_h^y\|_{L^{pq}(\mathcal{O}^y)} \|\partial_y g_h^y\|_{L^{pq^*}(\mathcal{O}^y)} \tag{A.1d}$$

hold true for all $f_h^x, g_h^x \in U_h^x$ and $f_h^y, g_h^y \in U_h^y$. In addition, the estimates

$$\begin{aligned}\|(I - \mathcal{I}_h^{xy})\{f_h g_h\}\|_{L^p(\mathcal{O})} \le{}& C h_x^2 \|\partial_x f_h\|_{L^{pq}(\mathcal{O})} \|\partial_x g_h\|_{L^{pq^*}(\mathcal{O})} \\ &+ C h_y^2 \|\partial_y f_h\|_{L^{pr}(\mathcal{O})} \|\partial_y g_h\|_{L^{pr^*}(\mathcal{O})},\end{aligned} \tag{A.1e}$$

$$\|(I - \mathcal{I}_h^x)\{f_h g_h\}\|_{L^p(\mathcal{O})} \le C h_x^2 \|\partial_x f_h\|_{L^{pq}(\mathcal{O})} \|\partial_x g_h\|_{L^{pq^*}(\mathcal{O})}, \tag{A.1f}$$

[1] In the case of Itô-noise, no formal a priori estimates are known for the case of compactly supported initial data.

$$\begin{aligned}
\|\mathcal{I}_h^y\{(I-\mathcal{I}_h^x)\{f_h g_h\}\}\|_{L^p(\mathcal{O})} &\le C h_x^2 \|\partial_x f_h\|_{L^{pq}(\mathcal{O})} \|\partial_x g_h\|_{L^{pq^*}(\mathcal{O})},\\
\|\partial_x (I-\mathcal{I}_h^x)\{f_h g_h\}\|_{L^p(\mathcal{O})} &\le C h_x \|\partial_x f_h\|_{L^{pq}(\mathcal{O})} \|\partial_x g_h\|_{L^{pq^*}(\mathcal{O})},\\
\|(I-\mathcal{I}_h^y)\{f_h g_h\}\|_{L^p(\mathcal{O})} &\le C h_y^2 \|\partial_y f_h\|_{L^{pr}(\mathcal{O})} \|\partial_y g_h\|_{L^{pr^*}(\mathcal{O})},\\
\|\mathcal{I}_h^x\{(I-\mathcal{I}_h^y)\{f_h g_h\}\}\|_{L^p(\mathcal{O})} &\le C h_y^2 \|\partial_y f_h\|_{L^{pr}(\mathcal{O})} \|\partial_y g_h\|_{L^{pr^*}(\mathcal{O})},\\
\|\partial_y (I-\mathcal{I}_h^y)\{f_h g_h\}\|_{L^p(\mathcal{O})} &\le C h_y \|\partial_y f_h\|_{L^{pr}(\mathcal{O})} \|\partial_y g_h\|_{L^{pr^*}(\mathcal{O})}
\end{aligned} \tag{A.1g}$$

hold true for all $f_h, g_h \in U_h$.

Proof. Estimates (A.1a)–(A.1d) are proven in [60, Lemma 2.1]. To prove (A.1e), we reduce the problem to the one-dimensional setting using

$$\begin{aligned}
\|(I-\mathcal{I}_h^{xy})\{f_h g_h\}\|_{L^p(\mathcal{O})}^p &\le \|(I-\mathcal{I}_h^y)\{f_h g_h\}\|_{L^p(\mathcal{O})}^p + \|\mathcal{I}_h^y\{(I-\mathcal{I}_h^x)\{f_h g_h\}\}\|_{L^p(\mathcal{O})}^p\\
&=: A + B
\end{aligned} \tag{A.2}$$

and apply (A.1a)–(A.1d). This provides

$$\begin{aligned}
A &\le C h_y^{2p} \int_{\mathcal{O}^x} \|\partial_y f_h\|_{L^{pq}(\mathcal{O}^y)}^p \|\partial_y g_h\|_{L^{pq^*}(\mathcal{O}^y)}^p \,\mathrm{d}x\\
&\le C h_y^{2p} \|\partial_y f_h\|_{L^{pq}(\mathcal{O})}^p \|\partial_y g_h\|_{L^{pq^*}(\mathcal{O})}^p.
\end{aligned} \tag{A.3}$$

Using Jensen's inequality, (2.6), and Hölder's inequality, we obtain for $r \in (1,\infty)$

$$\begin{aligned}
B &\le \int_{\mathcal{O}^y} \mathcal{I}_h^y \Big\{ \int_{\mathcal{O}^x} |(I-\mathcal{I}_h^x)\{f_h g_h\}|^p \,\mathrm{d}x \Big\} \,\mathrm{d}y\\
&\le C h_x^{2p} \int_{\mathcal{O}^y} \mathcal{I}_h^y \{ \|\partial_x f_h\|_{L^{pr}(\mathcal{O}^x)}^p \|\partial_x g_h\|_{L^{pr^*}(\mathcal{O}^x)}^p \} \,\mathrm{d}y\\
&\le C h_x^{2p} \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x f_h|^{pr} \} \,\mathrm{d}x\,\mathrm{d}y \Big)^{1/r} \Big(\int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x g_h|^{pr^*} \} \,\mathrm{d}x\,\mathrm{d}y \Big)^{1/r^*}\\
&\le C h_x^{2p} \|\partial_x f_h\|_{L^{pr}(\mathcal{O})}^p \|\partial_x g_h\|_{L^{pr^*}(\mathcal{O})}^p.
\end{aligned} \tag{A.4}$$

The corresponding estimate for $r \in \{1,\infty\}$ is straightforward. Estimates (A.1f)–(A.1g) can be derived from (A.1a)–(A.1d) in a similar manner. ∎

Lemma A.2. *Let Assumption (S) hold true and let the operator* $A_h : U_h \to U_h \cap H_*^1(\mathcal{O})$ *be defined by*

$$\int_{\mathcal{O}} (A_h \phi_h) \psi_h \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathcal{O}} \nabla \phi_h \cdot \nabla \psi_h \,\mathrm{d}x\,\mathrm{d}y \quad \forall \psi_h \in U_h. \tag{A.5}$$

Then, there exists a constant $C > 0$ *such that for all* $\phi_h \in U_h$*, the estimates*

$$\|\phi_h\|_{L^\infty(\mathcal{O})} \le C \|A_h \phi_h\|_{L^2(\mathcal{O})}^{\kappa} \|\phi_h\|_{H^1(\mathcal{O})}^{1-\kappa} + C \|\phi_h\|_{H^1(\mathcal{O})}, \tag{A.6a}$$

$$\|\phi_h\|_{W^{1,p}(\mathcal{O})} \le C \|A_h \phi_h\|_{L^2(\mathcal{O})}^{\nu} \|\phi_h\|_{H^1(\mathcal{O})}^{1-\nu} + C \|\phi_h\|_{H^1(\mathcal{O})} \tag{A.6b}$$

hold true with $\kappa = \frac{1}{2}$, $\nu = \frac{3p-6}{2p}$ *and* $p \in [2,6]$ *for* $d = 3$*; and* $\kappa = \frac{1}{4}$, $\nu = \frac{p-2}{p}$ *and* $p \in [2,\infty)$ *for* $d = 2$.

Proof. A similar result for continuous, piecewise linear finite element functions defined on a simplicial triangulation has been proven in [60, Lemma A.1]. This proof relies solely on standard error estimates for second-order problems (see, e.g., [13, Chapter 3]), standard inverse estimates for finite element functions (see, e.g., [10, Theorem 4.5.11]), and the properties of the Clément interpolation operator (see, e.g., [23, Lemma 1.127]). As the first two results remain valid for quadrilateral finite element spaces satisfying Assumption (S) and the Clément interpolation operator, which was originally only defined for simplicial elements, can be replaced by the more general operator proposed in [4] that satisfies the same error estimates as the Clément interpolation operator, we refer the reader to [60]. ∎

Lemma A.3. *Let Assumption (S) hold true and let the operator $A_h : U_h \to U_h \cap H^1_*(\mathcal{O})$ be defined by*

$$\int_{\mathcal{O}} (A_h\phi_h)\psi_h \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathcal{O}} \nabla\phi_h \cdot \nabla\psi_h \,\mathrm{d}x\,\mathrm{d}y \quad \forall \psi_h \in U_h. \tag{A.7}$$

Then, there exists a constant $C > 0$ such that for all $\phi_h \in U_h$, the estimate

$$\|A_h\phi_h\|_{L^2(\mathcal{O})} \le C\|\Delta_h\phi_h\|_{L^2(\mathcal{O})} \tag{A.8}$$

holds true, where Δ_h denotes the discrete Laplacian defined in (2.7).

Proof. Noting that the definitions of Δ_h^x and Δ_h^y in (2.7) imply

$$\begin{aligned} -\int_{\mathcal{O}^x} \mathcal{I}_h^x\{\Delta_h^x\phi_h\psi_h\}\,\mathrm{d}x &= \int_{\mathcal{O}^x} \partial_x\phi_h\partial_x\psi_h\,\mathrm{d}x \quad \text{pointwise for all } y \in \mathcal{O}^y,\\ -\int_{\mathcal{O}^y} \mathcal{I}_h^y\{\Delta_h^y\phi_h\psi_h\}\,\mathrm{d}y &= \int_{\mathcal{O}^y} \partial_y\phi_h\partial_y\psi_h\,\mathrm{d}y \quad \text{pointwise for all } x \in \mathcal{O}^x, \end{aligned} \tag{A.9}$$

we compute

$$\begin{aligned} \|A_h\phi_h\|^2_{L^2(\mathcal{O})} &= \int_{\mathcal{O}} \partial_x\phi_h\partial_x A_h\phi_h\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathcal{O}} \partial_y\phi_h\partial_y A_h\phi_h\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{\mathcal{O}} \mathcal{I}_h^x\{(-\Delta_h^x\phi_h)A_h\phi_h\}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathcal{O}} \mathcal{I}_h^y\{(-\Delta_h^y\phi_h)A_h\phi_h\}\,\mathrm{d}x\,\mathrm{d}y\\ &\le C\|\Delta_h^x\phi_h\|_{L^2(\mathcal{O})}\|A_h\phi_h\|_{L^2(\mathcal{O})} + C\|\Delta_h^y\phi_h\|_{L^2(\mathcal{O})}\|A_h\phi_h\|_{L^2(\mathcal{O})}, \end{aligned} \tag{A.10}$$

which concludes the proof. ∎

The following corollary is a direct consequence of Lemma A.2 and Lemma A.3:

Corollary A.4. *Let Assumption (S) hold true and let Δ_h be the discrete Laplacian defined in (2.7). Then, there exists a constant $C > 0$ such that for all $\phi_h \in U_h$, the estimates*

$$\|\phi_h\|_{L^\infty(\mathcal{O})} \le C\|\Delta_h\phi_h\|^{\kappa}_{L^2(\mathcal{O})}\|\phi_h\|^{1-\kappa}_{H^1(\mathcal{O})} + C\|\phi_h\|_{H^1(\mathcal{O})}, \tag{A.11a}$$

$$\|\phi_h\|_{W^{1,p}(\mathcal{O})} \le C\|\Delta_h\phi_h\|^{\nu}_{L^2(\mathcal{O})}\|\phi_h\|^{1-\nu}_{H^1(\mathcal{O})} + C\|\phi_h\|_{H^1(\mathcal{O})} \tag{A.11b}$$

hold true with $\kappa = \frac{1}{2}$, $\nu = \frac{3p-6}{2p}$ and $p \in [2,6]$ for $d = 3$; and $\kappa = \frac{1}{4}$, $\nu = \frac{p-2}{p}$ and $p \in [2,\infty)$ for $d = 2$.

Lemma A.5. *Let $G : \mathbb{R} \supset I \to \mathbb{R}$ be of class $W^{1,\infty}$ and let a partition $\mathcal{Q}_h$ of a domain $\mathcal{O}$ satisfying Assumption (S) be given. Furthermore, let U_h be the corresponding space defined in (2.2c). Then there exists a constant $C > 0$ such that the estimate*

$$\|\mathcal{I}_h^{xy}\{G(u_h)\} - G(u_h)\|_{L^2(\mathcal{O})}^2 \le C\|G\|_{W^{1,\infty}}^2 h^2 \int_{\mathcal{O}} |\nabla u_h|^2 \,\mathrm{d}x\,\mathrm{d}y \tag{A.12}$$

holds true for arbitrary $u_h \in U_h$.

Proof. We will prove that the claim holds true on every $Q \in \mathcal{Q}_h$. After an appropriate translation, we may assume that Q is given by $Q = \mathrm{co}\{(0,0), (h_x, 0), (h_x, h_y), (0, h_y)\}$. For arbitrary $(x, y) \in Q$, we have

$$\begin{aligned} G(u_h(x,y)) &= \frac{1}{2}G(u_h(x,0)) + \frac{1}{2}(\partial_y G(u_h))(x,\xi_1^y)\cdot y \\ &\quad + \frac{1}{2}G(u_h(x,h_y)) + \frac{1}{2}(\partial_y G(u_h))(x,\xi_2^y)\cdot(y-h_y) \\ &= \frac{1}{2}G(u_h(0,0)) + \frac{1}{2}(\partial_x G(u_h))(\xi_1^x,0)\cdot x + \frac{1}{2}(\partial_y G(u_h))(x,\xi_1^y)\cdot y \\ &\quad + \frac{1}{2}G(u_h(h_x,h_y)) + \frac{1}{2}(\partial_x G(u_h))(\xi_2^x,h_y)\cdot(x-h_x) \\ &\quad + \frac{1}{2}(\partial_y G(u_h))(x,\xi_2^y)\cdot(y-h_y) \end{aligned} \tag{A.13}$$

with $\xi_1^x, \xi_2^x \in [0, h_x]$ and $\xi_1^y, \xi_2^y \in [0, h_y]$. Similarly, we have

$$\begin{aligned} (\mathcal{I}_h^{xy}\{G(u_h)\})(x,y) &= \frac{1}{2}(\mathcal{I}_h^{xy}\{G(u_h)\})(0,0) + \frac{1}{2}(\partial_x \mathcal{I}_h^{xy}\{G(u_h)\})(\zeta_1^x,0)\cdot x \\ &\quad + \frac{1}{2}(\partial_y \mathcal{I}_h^{xy}\{G(u_h)\})(x,\zeta_1^y)\cdot y + \frac{1}{2}(\mathcal{I}_h^{xy}\{G(u_h)\})(h_x,h_y) \\ &\quad + \frac{1}{2}(\partial_x \mathcal{I}_h^{xy}\{G(u_h)\})(\zeta_2^x,h_y)\cdot(x-h_x) \\ &\quad + \frac{1}{2}(\partial_y \mathcal{I}_h^{xy}\{G(u_h)\})(x,\zeta_2^y)\cdot(y-h_y) \end{aligned} \tag{A.14}$$

with $\zeta_1^x, \zeta_2^x \in [0, h_x]$ and $\zeta_1^y, \zeta_2^y \in [0, h_y]$. Therefore, we have

$$\begin{aligned} &|G(u_h(x,y)) - (\mathcal{I}_h^{xy}\{G(u_h)\})(x,y)|^2 \\ &\quad \le Ch_x^2|(\partial_x G(u_h))(\xi_1^x,0)|^2 + Ch_y^2|(\partial_y G(u_h))(x,\xi_1^y)|^2 \\ &\qquad + Ch_x^2|(\partial_x G(u_h))(\xi_2^x,h_y)|^2 + Ch_y^2|(\partial_y G(u_h))(x,\xi_2^y)|^2 \\ &\qquad + Ch_x^2|(\partial_x \mathcal{I}_h^{xy}\{G(u_h)\})(\zeta_1^x,0)|^2 + Ch_y^2|(\partial_y \mathcal{I}_h^{xy}\{G(u_h)\})(x,\zeta_1^y)|^2 \\ &\qquad + Ch_x^2|(\partial_x \mathcal{I}_h^{xy}\{G(u_h)\})(\zeta_2^x,h_y)|^2 + Ch_y^2|(\partial_y \mathcal{I}_h^{xy}\{G(u_h)\})(x,\zeta_2^y)|^2 \\ &\quad \le Ch^2\|G\|_{W^{1,\infty}}^2(|\partial_x u_h(\cdot,0)|^2 + |\partial_x u_h(\cdot,h_y)|^2 + |\partial_y u_h(x,\cdot)|^2) \\ &\qquad + Ch^2\|G\|_{W^{1,\infty}}^2\big(|\partial_x u_h(\cdot,0)|^2 + |\partial_x u_h(\cdot,h_y)|^2 + \mathcal{I}_h^x\{|\partial_y u_h(x,\cdot)|^2\}\big). \end{aligned} \tag{A.15}$$

Integrating over Q and using the norm equivalence stated in (2.6) provides the result. ∎

Lemma A.6. *Let Assumption (S) hold true. Then the identities*

$$-\int_{\mathcal{O}^x} a_h^x(x)\mathcal{I}_h^x\{b_h^x(x)\partial_x^{-h_x}c_h^x(x)\}\,\mathrm{d}x = \int_{\mathcal{O}^x} a_h^x(x)\mathcal{I}_h^x\{\partial_x^{-h_x}b_h^x(x)c_h^x(x-h_x)\}\,\mathrm{d}x + \int_{\mathcal{O}^x}\partial_x^{+h_x}a_h^x(x)\mathcal{I}_h^x\{b_h^x(x)c_h^x(x)\}\,\mathrm{d}x, \tag{A.16a}$$

$$\int_{\mathcal{O}^x} a_h^x(x)\mathcal{I}_h^x\{b_h^x(x)\Delta_h^x c_h^x(x)\}\,\mathrm{d}x = \int_{\mathcal{O}^x}\Delta_h^x a_h^x(x)\mathcal{I}_h^x\{b_h^x(x-h_x)c_h^x(x)\}\,\mathrm{d}x + \int_{\mathcal{O}^x} a_h^x(x+h_x)\mathcal{I}_h^x\{\Delta_h^x b_h^x(x)c_h^x(x)\}\,\mathrm{d}x + 2\int_{\mathcal{O}^x}\partial_x^{+h_x}a_h^x(x)\mathcal{I}_h^x\{\partial_x^{-h_x}b_h^x(x)c_h^x(x)\}\,\mathrm{d}x, \tag{A.16b}$$

$$-\int_{\mathcal{O}^y} a_h^y(y)\mathcal{I}_h^y\{b_h^y(y)\partial_y^{-h_y}c_h^y(y)\}\,\mathrm{d}y = \int_{\mathcal{O}^y} a_h^y(y)\mathcal{I}_h^y\{\partial_y^{-h_y}b_h^y(y)c_h^y(y-h_y)\}\,\mathrm{d}y + \int_{\mathcal{O}^y}\partial_y^{+h_y}a_h^y(y)\mathcal{I}_h^y\{b_h^y(y)c_h^y(y)\}\,\mathrm{d}y, \tag{A.16c}$$

$$\int_{\mathcal{O}^y} a_h^y(y)\mathcal{I}_h^y\{b_h^y(y)\Delta_h^y c_h^y(y)\}\,\mathrm{d}y = \int_{\mathcal{O}^y}\Delta_h^y a_h^y(y)\mathcal{I}_h^y\{b_h^y(y-h_y)c_h^y(y)\}\,\mathrm{d}y + \int_{\mathcal{O}^y} a_h^y(y+h_y)\mathcal{I}_h^y\{\Delta_h^y b_h^y(y)c_h^y(y)\}\,\mathrm{d}y + 2\int_{\mathcal{O}^y}\partial_y^{+h_y}a_h^y(y)\mathcal{I}_h^y\{\partial_y^{-h_y}b_h^y(y)c_h^y(y)\}\,\mathrm{d}y \tag{A.16d}$$

hold true for periodic functions $a_h^x \in C_{\mathrm{per},\mathcal{T}_h^x}$, $b_h^x, c_h^x \in U_h^x$, $a_h^y \in C_{\mathrm{per},\mathcal{T}_h^y}$, *and* $b_h^y, c_h^y \in U_h^y$.

Proof. To show the second identity in (A.16a), we use the periodicity of the considered functions and compute

$$\begin{aligned}
&-h^x\int_{\mathcal{O}^x} a_h^x(x)\mathcal{I}_h^x\{b_h^x(x)\partial_x^{-h_x}c_h^x(x)\}\,\mathrm{d}x\\
&\quad=\int_{\mathcal{O}^x} a_h^x(x+h_x)\mathcal{I}_h^x\{b_h^x(x+h_x)c_h^x(x)\}\,\mathrm{d}x\\
&\qquad-\int_{\mathcal{O}^x} a_h^x(x)\mathcal{I}_h^x\{b_h^x(x)c_h^x(x)\}\,\mathrm{d}x\\
&\quad=\int_{\mathcal{O}^x} a_h^x(x+h_x)\mathcal{I}_h^x\{(b_h^x(x+h_x)-b_h^x(x))c_h^x(x)\}\,\mathrm{d}x\\
&\qquad+\int_{\mathcal{O}^x}(a_h^x(x+h_x)-a_h^x(x))\mathcal{I}_h^x\{b_h^x(x)c_h^x(x)\}\,\mathrm{d}x
\end{aligned}$$

$$= \int_{\mathcal{O}^x} a_h^x(x)\mathcal{I}_h^x\{(b_h^x(x) - b_h^x(x-h_x))c_h^x(x-h_x)\}\,\mathrm{d}x$$
$$+ \int_{\mathcal{O}^x} (a_h^x(x+h_x) - a_h^x(x))\mathcal{I}_h^x\{b_h^x(x)c_h^x(x)\}\,\mathrm{d}x. \tag{A.17}$$

We will now use (A.16a) to prove (A.16b).

We have

$$\begin{aligned}
&\int_{\mathcal{O}^x} a_h^x(x)\mathcal{I}_h^x\{b_h^x(x)\partial_x^{-h_x}\partial_x^{+h_x}c_h^x(x)\}\,\mathrm{d}x \\
&= -\int_{\mathcal{O}^x} a_h^x(x)\mathcal{I}_h^x\{\partial_x^{-h_x}b_h^x(x)\partial_x^{+h_x}c_h^x(x-h_x)\}\,\mathrm{d}x \\
&\quad - \int_{\mathcal{O}^x} \partial_x^{+h_x}a_h^x(x)\mathcal{I}_h^x\{b_h^x(x)\partial_x^{+h_x}c_h^x(x)\}\,\mathrm{d}x \\
&= \int_{\mathcal{O}^x} a_h^x(x)\mathcal{I}_h^x\{\partial_x^{-h_x}b_h^x(x)\partial_x^{-h_x}c_h^x(x)\}\,\mathrm{d}x \\
&\quad - \int_{\mathcal{O}^x} \partial_x^{+h_x}a_h^x(x-h_x)\mathcal{I}_h^x\{b_h^x(x-h_x)\partial_x^{-h_x}c_h^x(x)\}\,\mathrm{d}x \\
&= \int_{\mathcal{O}^x} a_h^x(x)\mathcal{I}_h^x\{\partial_x^{-h_x}\partial_x^{-h_x}b_h^x(x)c_h^x(x-h_x)\}\,\mathrm{d}x \\
&\quad + \int_{\mathcal{O}^x} \partial_x^{+h_x}a_h^x(x)\mathcal{I}_h^x\{\partial_x^{-h_x}b_h^x c_h^x(x)\}\,\mathrm{d}x \\
&\quad + \int_{\mathcal{O}^x} \partial_x^{+h_x}a_h^x(x-h_x)\mathcal{I}_h^x\{\partial_x^{-h_x}b_h^x(x-h_x)c_h^x(x-h_x)\}\,\mathrm{d}x \\
&\quad + \int_{\mathcal{O}^x} \partial_x^{+h_x}\partial_x^{+h_x}a_h^x(x-h_x)\mathcal{I}_h^x\{b_h^x(x-h_x)c_h^x(x)\}\,\mathrm{d}x \\
&= \int_{\mathcal{O}^x} a_h^x(x+h_x)\mathcal{I}_h^x\{\Delta_h^x b_h^x(x)c_h^x(x)\}\,\mathrm{d}x \\
&\quad + 2\int_{\mathcal{O}^x} \partial_x^{+h_x}a_h^x(x)\mathcal{I}_h^x\{\partial_x^{-h_x}b_h^x(x)c_h^x(x)\}\,\mathrm{d}x \\
&\quad + \int_{\mathcal{O}^x} \Delta_h^x a_h^x(x)\mathcal{I}_h^x\{b_h^x(x-h_x)c_h^x(x)\}\,\mathrm{d}x.
\end{aligned} \tag{A.18}$$

Identities (A.16c) and (A.16d) can be established in a similar manner. ∎

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