Interfaces Free Bound. 24 (2022), 389–430
DOI 10.4171/IFB/475



# Curve-shortening of open elastic curves with repelling endpoints: A minimizing movements approach

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**Abstract.** We study an $L^2$-type gradient flow of an immersed elastic curve in $\mathbb{R}^2$ whose endpoints repel each other via a Coulomb potential. By De Giorgi's minimizing movements scheme we prove long-time existence of the flow. The work is complemented by several numerical experiments.

## 1. Introduction

In this paper we study an $L^2$ gradient flow of an open immersed curve $\gamma$ in $\mathbb{R}^2$ belonging to the set

$$\mathcal{AC}' := \{\gamma \in H^2([0,1];\mathbb{R}^2) \colon \gamma_s \neq 0,\ \gamma(0) \neq \gamma(1)\},$$

where $s$ denotes the curve parameter and $\gamma_s := \frac{\mathrm{d}}{\mathrm{d}s}\gamma$ the speed of the parametrization. The evolution of $\gamma$ is driven by the energy

$$E(\gamma) := L + \varepsilon W(\gamma) - \log|\gamma(0) - \gamma(1)|, \tag{1.1}$$

where $\varepsilon > 0$ is a fixed scalar, $L$ is the length of $\gamma$, and $W$ is the *Willmore energy* defined as

$$W(\gamma) := \frac{1}{2}\int_\gamma \kappa^2 \,\mathrm{d}\sigma,$$

where $\sigma$ denotes the arc length and $\kappa$ the curvature of $\gamma$. As the energy functional $E$ is invariant under reparametrizations, we restrict the class of admissible curves to the following non-linear subset of $\mathcal{AC}'$:

$$\mathcal{AC} = \{\gamma \in \mathcal{AC}' \colon |\gamma_x| = \text{const}\}. \tag{1.2}$$

The interest in the gradient flow of functionals such as the one in (1.1) is motivated by the observation that they represent one of the main energy contributions in several physical

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*2020 Mathematics Subject Classification.* Primary 35K30, 53E40; Secondary 35A15, 49J10.
*Keywords.* Geometric evolution equations, Willmore flow, minimizing movements, curve shortening flow with interacting endpoints.

systems driven by the formation of topological and geometrical defects that can be seen, roughly speaking, as codimension two and one singularities of some ad hoc chosen order parameter.

Among the models characterized by the emergence of topological singularities, of particular interest in our case are those featuring fractional vortices. They have been widely studied in the theory of spin systems as a generalization of the classical XY model studied in [2–4, 31], as well as of superconductors systems as a generalization of the Ginzburg–Landau model for which we refer the reader to [6, 9, 21, 32]. What concerns us with the geometric singularities is that they are peculiar of phase separation phenomena in which they represent phase boundaries. Regarding the time evolution of the singularities in this kind of model, we mention papers [5, 33]. In [8] the authors study an energy model describing a class of spin systems whose minimizers may develop complicated structures in the form of clusters of phase boundaries possibly connecting fractional vortices (see also [18]), thus providing a first variational analysis of a physical system exhibiting both, codimension two and one, singularities. Notice that the presence of these two types of singularities is considered to be one of the main characteristics of the ground states of physical systems like *liquid crystals* (see also [29]), in which case the singularities represent disclinations and string defects; or of *plastic crystals* (see also [19]), where they represent partial dislocations and stacking faults. Additionally, they appear also in many *micromagnetics* and *super conductors* models (see for instance [1, 34]).

Describing the gradient flow of an energy functional as in [8] turns out to be a very difficult task when considered in its full generality. Keeping the main features of the model, we perform our analysis in the simple case of a line singularity joining two equally charged vortices. In this case, the geometric part of the energy which drives the system towards equilibrium takes the form

$$\int_0^1 |\gamma_s|_1 \, \mathrm{d}s - \log|\gamma(0) - \gamma(1)|,$$

where $|\cdot|_1$ denotes the $l^1$-norm. Here, $\gamma$ parametrizes the line defects with vortices located at $\gamma(0)$ and $\gamma(1)$. Our energy defined in (1.1) can be seen as a further simplification of the one above, where we replace the crystalline length by the Euclidean one, and add the Willmore term (thus reducing to an elastic model) whose regularizing effect has been, for instance, already exploited in [15, 30].

One of the main features of the expected flow is the competition between the shortening effect due to the length energy and the endpoints repulsion due to the Coulomb potential. As an example (see the end of this introduction for more details) one might consider the simple case of a sufficiently long straight segment. Roughly speaking, if only the Coulomb part would act, the segment would evolve towards an infinite line, while if only the length would be present, it would shorten to a point. Instead, with both terms present, the curve evolves towards a segment having an optimal length which balances the two effects.

In this paper we will model an $L^2$-type gradient flow of the energy $E$ in (1.1) employing De Giorgi's *minimizing movements* technique described, for instance, in [7]. In this case one shows that the flow emerges from a sequence of time-discrete evolutions $\{\gamma^\lambda\}_{\lambda\geq 1}$ where each $\gamma^\lambda\colon [0,1]\times\mathbb{R}_+\to\mathbb{R}^2$ is a piecewise constant (in time intervals of length $\frac{1}{\lambda}$) interpolation of a sequence $\{\gamma_n^\lambda\}_n\subset\mathcal{AC}$. This sequence is constructed via a recurrent scheme: Starting from a fixed $\gamma_0^\lambda=\gamma_0\in\mathcal{AC}$, the following curves in the sequence $\{\gamma_n^\lambda\}$ try to decrease $E$ as much as possible while not straying too far away from their respective foregoing curve along the sequence. This is achieved by introducing a penalization term $D$ which can be thought as a *Dissipation* functional. More precisely, the sequence $\{\gamma_n^\lambda\}_n$ should solve the following *step-by-step minimization problem*:

$$\begin{cases} \gamma_{n+1}^\lambda \in \operatorname*{argmin}_{\gamma\in\mathcal{AC}}\{E(\gamma)+\lambda D(\gamma,\gamma_n^\lambda)\}, \\ \gamma_0^\lambda=\gamma_0, \end{cases} \tag{1.3}$$

where $D\colon\mathcal{AC}^2\to\mathbb{R}$ is defined as

$$\begin{aligned} D(\gamma,\widetilde{\gamma}) := {}& \frac{1}{4}\int_0^1\langle\gamma-\widetilde{\gamma},\widetilde{\nu}\rangle^2\widetilde{L}\,\mathrm{d}s+\frac{1}{4}\int_0^1\langle\gamma-\widetilde{\gamma},\nu\rangle^2 L\,\mathrm{d}s \\ &+\frac{1}{2}|\gamma(0)-\widetilde{\gamma}(0)|^2+\frac{1}{2}|\gamma(1)-\widetilde{\gamma}(1)|^2. \end{aligned} \tag{1.4}$$

Here, $\widetilde{\nu}$ and $\nu$ stand for the unit normal vector field of $\widetilde{\gamma}$ and $\gamma$, respectively.

Our model has several points in common with [15, 30], in which the authors study the morphological evolution of epitaxially strained two-dimensional thin films in terms of the $H^{-1}$ and the $L^2$ gradient flow structure, respectively. Also, as we do here, the authors exploit De Giorgi's minimizing movements with curvature regularization. Nevertheless, the present work has some important differences to the ones above. Firstly, instead of describing the interfaces as the graph of a function over a fixed interval, we consider curves parametrized with constant speed. Secondly, our approach must account for freely moving boundary points (*free boundary problem*). In contrast to this, the authors of [15,30], for example, assume periodic boundary conditions which are more or less equivalent to those in the case of closed curves. Simply following their approach in the present case would complicate our analysis, since we would be forced to consider the motion of graphs on evolving domains of definition. Still, the authors believe that a graph approach could also be fruitful, albeit slightly more technical, for free boundaries.

As we consider an $L^2$-type gradient flow as also done by the author of [30], it is worth comparing our choice of dissipation to that of the latter. Expressed in intrinsic coordinates, the dissipation in [30] is given by

$$\widetilde{D}(\gamma,\widetilde{\gamma})=\frac{1}{2}\int_0^1\langle\gamma-\widetilde{\gamma},\widetilde{\nu}\rangle^2\widetilde{L}\,\mathrm{d}s. \tag{1.5}$$

Regarding our dissipation in (1.4), besides the presence of additional boundary terms which are necessary to control the flow of the free boundary points, we make a different choice of the interior $L^2$ dissipation by considering a symmetrized version of (1.5), namely

$$\frac{1}{2}\widetilde{D}(\gamma,\widetilde{\gamma})+\frac{1}{2}\widetilde{D}(\widetilde{\gamma},\gamma).$$

Such a choice does not change the limit equation (as the two symmetric terms will have the same limit), but it turns out to be convenient from a technical point of view to derive the a priori bounds of the velocity of the time-discrete evolution (see also Lemmas 2.3 and 2.4).

We continue by describing the strategy of our existence argument, which can be found in Section 2. Employing the direct method in the calculus of variations, we first prove in Theorem 2.1 the well-posedness of the scheme in (1.3). The Euler–Lagrange equations satisfied at each minimization step in (1.3) lead to a weak description of the time-discrete evolution $\gamma^\lambda$; see also equation (2.56) in Theorem 2.14. The passage to the limit $\lambda\to\infty$ in the equation governing the time-discrete evolution is eventually obtained in Theorem 2.23 by combining Theorems 2.9, 2.15, 2.18, and 2.19, where several compactness results for the sequence $\{\gamma^\lambda\}_\lambda$ are proved. This part of the argument is closely related to ideas contained, for instance, in [30]. After successfully passing to the limit we arrive at the main result of this paper in Theorem 2.23, where we show the existence of a long-time solution $\gamma$ of a non-linear system of PDEs. More precisely, $\gamma$ satisfies the initial condition $\gamma(0)=\gamma_0$ and solves a system of PDEs that are better described through the arc length parameter $\sigma$. With a slight abuse of notation, let $\gamma(\sigma,t):=\gamma(\sigma^{-1}L(t),t)$, where $L(t)$ is the length of $\gamma$ at time $t$. Then $\gamma$ satisfies for almost every $t\in\mathbb{R}_+$ and for almost every $\sigma\in[0,L(t)]$ the following system:

$$\begin{cases} V^\perp(\sigma,t)=\kappa(\sigma,t)-\varepsilon\Big(\kappa_{\sigma\sigma}(\sigma,t)+\dfrac{1}{2}\kappa^3(\sigma,t)\Big),\\ \quad V^p(t)=\dfrac{\gamma^p(t)-\gamma^q(t)}{|\gamma^p(t)-\gamma^q(t)|^2}+\tau^p(t)-\varepsilon\kappa^p_\sigma(t)\nu^p(t),\\ \quad V^q(t)=\dfrac{\gamma^q(t)-\gamma^p(t)}{|\gamma^q(t)-\gamma^p(t)|^2}-\tau^q(t)+\varepsilon\kappa^q_\sigma(t)\nu^q(t),\\ \quad \kappa^p(t)=\kappa^q(t)=0. \end{cases}\tag{1.6}$$

Here, $V:=\gamma_t=\frac{\partial}{\partial t}\gamma$ denotes the velocity, $\tau$ is the unit tangent vector field of $\gamma$, and $V^\perp=\langle V,\nu\rangle$ is the orthogonal component of $V$ with respect to $\gamma$. Furthermore, given any function $f(\sigma,t)$, the notation $f^p(t)$ and $f^q(t)$ is shorthand for $f(0,t)$ and $f(L(t),t)$.

It is worth mentioning that our approach to derive the existence of the limit equation is not the only possible one. There is vast literature for results concerning the $L^2$ gradient flow of curves or more generally networks driven by elastic energies of Willmore type, as well as in the presence of free boundary points (see [16, 17, 20, 22, 25, 36]). In contrast to our case, the free boundary points of such results are usually junctions, while the outer points of the network are either fixed (Dirichlet boundary condition) and/or have fixed angles with respect to the boundary of a convex domain containing the network (Neumann boundary condition). For such boundary conditions it is possible to follow a different strategy (see, for example, [16]) based on the theory of non-linear parabolic equations. One main issue in this setting is to guess the right choice of the equation for the tangential component of the speed with respect to the curve which is not given a priori. Such a choice must be done carefully in order to guarantee a well-defined system of PDEs. In our case instead, the tangential equation arises naturally from the constant speed constraint in $\mathcal{AC}$. In fact, we prove in Theorem 2.19 that $\gamma$ satisfies for almost every $t$ and $\sigma$ the following equation:

$$V_\sigma^\top(\sigma,t) = \frac{L_t(t)}{L(t)} + \kappa(\sigma,t)V^\perp(\sigma,t), \tag{1.7}$$

where $V^\top := \langle V, \gamma_\sigma \rangle$ is the tangential component of the velocity. Lastly, we wish to mention that also the case of Willmore energies using different powers of the curvature ($p$-elastic energies) or higher-order derivatives of the curvature was investigated in works such as [11, 12, 23, 26–28].

In the simple case of a straight segment $\gamma_0$, the system of PDEs in (1.6) and (1.7) reduces to an ODE describing the motion of the endpoints. In fact, one can easily prove that the segment remains straight during the evolution and that it monotonously converges as $t \to \infty$ towards a segment of length $\frac{1}{\pi}$, which is the global minimizer of $E$ in (1.1) and for which the repulsive force of the Coulomb potential and the attractive force of the length term balance.

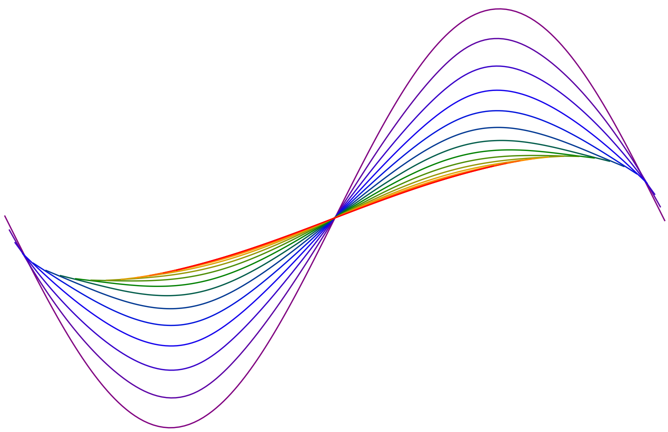


**Figure 1.** Curve-shortening motion in a special case. The color gradient shows the temporal order of the evolution from violet to red.

A more interesting example is shown in Figure 1, where we have plotted the step-by-step minimizers defined in (1.3) starting with a sinus-shaped $\gamma_0$. As time progresses, the initially sinus-shaped curve starts to straighten up while at the same time becoming shorter at its endpoints. In the limit $t \to \infty$ the curve converges towards the global minimizer of the energy functional in (1.1): a straight line with optimal length keeping the balance between the Coulomb and the length term of (1.1).

**Remark 1.1** (Notation). Throughout this paper, we will use the following shorthand notation for function spaces: A space $F([0,1];\mathbb{R}^2)$ of curves $\gamma\colon[0,1]\to\mathbb{R}^2$ of regularity described by $F$ we shortly write as $F$. So, for example, $L^2$ is shorthand for $L^2([0,1];\mathbb{R}^2)$. Furthermore, the space $F([0,\infty);G([0,1];\mathbb{R}^2))$ of time-dependent curves $\gamma\colon[0,\infty]\times[0,1]\to\mathbb{R}^2$ with time-regularity $F$ and regularity with respect to the curve-parameter given by $G$ will be shortly written as $FG$. So, for example $C_c^\infty L^2$ is shorthand for $C_c^\infty([0,\infty);L^2([0,1];\mathbb{R}^2))$. Finally, we will write $F_T G$ for $F([0,T];G([0,1];\mathbb{R}^2))$, where $T>0$.

All constants encountered in this paper are assumed to be finite and strictly positive. We will explicitly write out the dependence of constants on parameters. So, for example, a constant $C$ which possibly depends on the parameters $\alpha_1,\dots,\alpha_n$ will usually be written as $C(\alpha_1,\dots,\alpha_n)$. Lastly, the value of a constant may change during an estimate without introducing a new name for the constant.

## 2. Minimizing movements

### 2.1. Scheme

We start by introducing several objects of relevance to the minimizing movements scheme. The set of admissible curves is defined as

$$\mathcal{AC} := \{\gamma\in H^2\colon |\gamma_s|\equiv \text{const} = L, \gamma(0)\neq\gamma(1)\}.$$

Note that for any $\gamma\in\mathcal{AC}$ we can derive the following important identity at almost every (a.e.) point on $I$:

$$\gamma_{ss} = L^2\kappa\nu. \tag{2.1}$$

Here, $L$ is the length of $\gamma$, $\kappa$ is its curvature, $\nu$ is the unit normal of $\gamma$, and $I$ denotes the interval $[0,1]$. Given $\widetilde{\gamma}\in\mathcal{AC}$, the subset $\mathcal{AC}_{\widetilde{\gamma}}\subset\mathcal{AC}$ is defined as

$$\mathcal{AC}_{\widetilde{\gamma}} := \{\gamma\in\mathcal{AC}\colon \langle\gamma_s,\widetilde{\gamma}_s\rangle\geq 0\}. \tag{2.2}$$

The constraint on the sign of $\langle\gamma_s,\widetilde{\gamma}_s\rangle$ assures that the sum $\gamma+\widetilde{\gamma}$ has velocity uniformly bounded away from zero, which will be used in the proof of Lemma 2.4. For fixed $\varepsilon>0$ we define the energy $E\colon\mathcal{AC}\to\mathbb{R}$ as

$$E(\gamma) := L + \frac{\varepsilon}{2}\int_\gamma \kappa^2\,\mathrm{d}\sigma - \log|\gamma(1)-\gamma(0)|, \tag{2.3}$$

where $\sigma$ is the arc length parameter. The dissipation $D\colon\mathcal{AC}^2\to\mathbb{R}$ is defined as

$$\begin{aligned} D(\gamma,\widetilde{\gamma}) := {}& \frac{1}{4}\int_0^1 \langle\gamma-\widetilde{\gamma},\widetilde{\nu}\rangle^2\widetilde{L}\,\mathrm{d}s + \frac{1}{4}\int_0^1\langle\gamma-\widetilde{\gamma},\nu\rangle^2 L\,\mathrm{d}s + \frac{1}{2}|\gamma(0)-\widetilde{\gamma}(0)|^2 \\ & + \frac{1}{2}|\gamma(1)-\widetilde{\gamma}(1)|^2, \end{aligned} \tag{2.4}$$

where $\widetilde{\nu}$ is the unit normal vector field of $\widetilde{\gamma}$ and $\widetilde{L}$ is its length. For a given $\lambda \in [1, \infty)$ we also define $F_\lambda \colon \mathcal{AC}^2 \to \mathbb{R}$ as

$$F_\lambda(\gamma, \widetilde{\gamma}) := E(\gamma) + \lambda D(\gamma, \widetilde{\gamma}). \tag{2.5}$$

We are now able to describe our minimizing movements scheme. For a given $\gamma_0 \in \mathcal{AC}$ we define the sequence $\{\gamma_n^\lambda\}_n \subset \mathcal{AC}$ recursively as

$$\begin{cases} \gamma_{n+1}^\lambda \in \underset{\gamma \in \mathcal{AC}_{\gamma_n^\lambda}}{\operatorname{argmin}} \{E(\gamma) + \lambda D(\gamma, \gamma_n^\lambda)\}, \\ \quad \gamma_0^\lambda = \gamma_0. \end{cases} \tag{2.6}$$

Finally, for a sequence of curves $\{\gamma_n^\lambda\}_n$ we will shortly write $\kappa_n^\lambda$ for the curvature of $\gamma_n^\lambda$, $L_n^\lambda$ for its length, and $\nu_n^\lambda$ for its unit normal. We are now going to apply the direct method of the calculus of variations in order to show the well-definedness of the scheme defined above.

**Theorem 2.1** (Existence of step-by-step minimizers). *For every $n \in \mathbb{N}$, the problem in (2.6) attains a minimum. Furthermore, the following a priori bounds hold true for the sequence $\{\gamma_n^\lambda\}_n$:*

$$c \le |\gamma_n^\lambda(1) - \gamma_n^\lambda(0)| \le L_n^\lambda \le C, \tag{2.7}$$

$$\int_0^1 (\kappa_n^\lambda)^2 \, \mathrm{d}s \le C(\varepsilon), \tag{2.8}$$

*for constants $C$ and $C(\varepsilon)$ independent of $\lambda$.*

*Proof.* In what follows, we will omit in our notation the explicit dependence on $\lambda$ and shortly write, for example, $\gamma_n$ for $\gamma_n^\lambda$ or $\kappa_n$ for $\kappa_n^\lambda$. Suppose that we have already proved the existence of $\gamma_0, \dots, \gamma_n$. By comparison and the definition of $F$ (see (2.5)), we have

$$\inf_{\gamma \in \mathcal{AC}_{\gamma_n}} F(\gamma, \gamma_n) \le F(\gamma_n, \gamma_n) = E(\gamma_n). \tag{2.9}$$

Furthermore, in the case $n > 1$, we iteratively derive again by comparison and the non-negativity of $D$ that

$$E(\gamma_n) \le F(\gamma_n, \gamma_{n-1}) = \inf_{\gamma \in \mathcal{AC}_{\gamma_{n-1}}} F(\gamma, \gamma_{n-1}) \le F(\gamma_{n-1}, \gamma_{n-1}) = E(\gamma_{n-1}) \le \dots \le E(\gamma_0). \tag{2.10}$$

By the basic estimate

$$-\log|\gamma(1) - \gamma(0)| \ge -|\gamma(1) - \gamma(0)| \ge -L$$

and the very definition of $E$ in (2.3), we have that $E$ is non-negative on $\mathcal{AC}$. This fact combined with (2.9) and (2.10) then leads to

$$0 \le \inf_{\gamma \in \mathcal{AC}_{\gamma_n}} F(\gamma, \gamma_n) \le E(\gamma_0).$$

Consequently, we can find a minimizing sequence $\{\mu_i\} \subset \mathcal{AC}_{\gamma_n}$ such that

$$\lim_{i\to\infty} F(\mu_i, \gamma_n) = \inf_{\gamma\in\mathcal{AC}_{\gamma_n}} F(\gamma, \gamma_n), \tag{2.11}$$

$$F(\mu_i, \gamma_n) \leq E(\gamma_0) + 1 < \infty \quad \text{for all } i \in \mathbb{N}. \tag{2.12}$$

Our main goal at this point is to show that $\sup_i \|\mu_i\|_{H^2} < \infty$. For this, note that by (2.12) and $\log t \leq \frac{t}{2}$ it follows that

$$\begin{aligned} E(\gamma_0) + 1 \geq F(\mu_i, \gamma_n) \geq E(\mu_i) &\geq -\frac{1}{2}|\mu_i(1) - \mu_i(0)| + L_{\mu_i} + \frac{\varepsilon L_{\mu_i}}{2}\int_0^1 \kappa_{\mu_i}^2 \,\mathrm{d}s \\ &\geq \frac{L_{\mu_i}}{2} + \frac{\varepsilon L_{\mu_i}}{2}\int_0^1 \kappa_{\mu_i}^2 \,\mathrm{d}s. \end{aligned} \tag{2.13}$$

Moreover, by the definition of $D$, the non-negativity of $E$, and (2.12), we also get that

$$\frac{1}{2\tau}|\mu_i(0) - \gamma_n(0)|^2 \leq D(\mu_i, \gamma_n) \leq F(\mu_i, \gamma_n) \leq E(\gamma_0) + 1.$$

Hence, by the fundamental theorem of calculus, the fact that $|(\mu_i)_s| = L_{\mu_i}$, $\lambda \geq 1$, and (2.13), we derive

$$\begin{aligned} \int_0^1 |\mu_i|^2 \,\mathrm{d}s + \int_0^1 |(\mu_i)_s|^2 \,\mathrm{d}s &\leq (|\mu_i(0)| + L_{\mu_i})^2 + L_{\mu_i}^2 \\ &\leq (|\gamma_n(0)| + |\mu_i(0) - \gamma_n(0)| + L_{\mu_i})^2 + L_{\mu_i}^2 \\ &\leq 3|\gamma_n(0)|^2 + \frac{6}{\lambda}\cdot\frac{\lambda}{2}|\mu_i(0) - \gamma_n(0)|^2 + 16\Big(\frac{L_{\mu_i}}{2}\Big)^2 \\ &\leq 3|\gamma_n(0)|^2 + 6(E(\gamma_0) + 1) + 16(E(\gamma_0) + 1)^2. \end{aligned}$$

Furthermore, with (2.1) applied to $\mu_i$ and (2.13) it follows that

$$\begin{aligned} \int_0^1 |(\mu_i)_{ss}|^2 \,\mathrm{d}s = \int_0^1 |L_{\mu_i}^2 \kappa_{\mu_i} \nu_{\mu_i}|^2 \,\mathrm{d}s &= \frac{16}{\varepsilon}\Big(\frac{1}{2}L_{\mu_i}\Big)^3\Big(\frac{\varepsilon}{2}L_{\mu_i}\int_0^1 \kappa_{\mu_i}^2 \,\mathrm{d}s\Big) \\ &\leq \frac{16}{\varepsilon}(E(\gamma_0) + 1)^4, \end{aligned}$$

where $\nu_{\mu_i}$ denotes the unit normal field of $\mu_i$. Combining the last two estimates eventually leads to $\sup_i \|\mu_i\|_{H^2} < \infty$ as desired. By the weak compactness in $H^2$ and by the Sobolev embedding theorem we can find $\mu \in H^2$ such that, up to taking a subsequence,

$$\mu_i \rightharpoonup \mu \qquad \text{weakly in } H^2, \tag{2.14}$$

$$\mu_i \to \mu \text{ in } C^{1,\alpha} \qquad \text{for any } \alpha \in (0, \tfrac{1}{2}). \tag{2.15}$$

We now wish to show that $\mu$ is admissible, which means $\mu \in \mathcal{AC}_{\gamma_n}$. By (2.15) and $\{\mu_i\} \subset \mathcal{AC}_{\gamma_n}$, we derive that $\mu$ also satisfies

$$|\mu_s| \equiv L_\mu, \quad \langle \mu_s, (\gamma_n)_s\rangle \geq 0.$$

In order to prove $\mu \in \mathcal{AC}_{\gamma_n}$, it is left to show that $\mu(0) \neq \mu(1)$. This follows from (2.15) and (2.12), which together imply

$$-\log|\mu(1) - \mu(0)| = -\lim_{i\to\infty} \log|\mu_i(1) - \mu_i(0)| \leq \limsup_{i\to\infty} F(\mu_i, \gamma_n) \leq E(\gamma_0) + 1.$$

Hence, by the monotonicity of the logarithm,

$$L_\mu \geq |\mu(1) - \mu(0)| \geq c > 0, \tag{2.16}$$

where $c$ is a constant only depending on $\gamma_0$.

It remains to show that $\mu$ is the desired minimizer. To this end, note that

$$\int_{\mu_i} \kappa_{\mu_i}^2 \, d\sigma = \int_0^1 \Big(\frac{\langle(\mu_i)_{ss}, (\mu_i)_s^\perp\rangle}{|(\mu_i)_s|^3}\Big)^2 |(\mu_i)_s| \, ds = \int_0^1 \langle(\mu_i)_{ss}, L_{\mu_i}^{-\frac{5}{2}} (\mu_i)_s^\perp\rangle^2 \, ds.$$

Furthermore, by (2.14) and (2.15) the following convergences hold true:

$$\begin{aligned} (\mu_i)_{ss} &\rightharpoonup \mu_{ss} && \text{weakly in } L^2, \\ L_{\mu_i}^{-\frac{5}{2}} (\mu_i)_s^\perp &\to L_\mu^{-\frac{5}{2}} \mu_s^\perp && \text{in } L^2. \end{aligned}$$

Hence,

$$\langle(\mu_i)_{ss}, L_{\mu_i}^{-\frac{5}{2}} (\mu_i)_s^\perp\rangle \rightharpoonup \langle\mu_{ss}, L_\mu^{-\frac{5}{2}} \mu_s^\perp\rangle \quad \text{weakly in } L^2$$

and therefore,

$$\liminf_{i\to\infty} \frac{\varepsilon}{2} \int_{\mu_i} \kappa_{\mu_i}^2 \, d\sigma \geq \frac{\varepsilon}{2} \int_\mu \kappa_\mu^2 \, d\sigma. \tag{2.17}$$

Furthermore, by (2.15) we have

$$\lim_{i\to\infty} F(\mu_i, \gamma_n) - \frac{\varepsilon}{2} \int_{\mu_i} \kappa_{\mu_i}^2 \, d\sigma = F(\mu, \gamma_n) - \frac{\varepsilon}{2} \int_\mu \kappa_\mu^2 \, d\sigma. \tag{2.18}$$

Taking (2.17) and (2.18) together and using (2.11), we derive

$$\inf_{\gamma\in\mathcal{AC}_{\gamma_n}} F(\gamma, \gamma_n) = \lim_{i\to\infty} F(\mu_i, \gamma_n) \geq F(\mu, \gamma_n).$$

Consequently, $\gamma_{n+1} := \mu$ is a desired minimizer. Passing to the limit $i \to \infty$ in (2.13), we derive the upper bounds in (2.7) and (2.8). The lower bound in (2.7) instead follows from (2.16). ∎

## 2.2. Compactness

In this subsection, we will derive important compactness results for interpolations of the sequence of step-by-step minimizers $\{\gamma_n^\lambda\}_n$ (see also (2.6) and Theorem 2.1). These interpolations are defined as follows:

**Definition 2.2** (Interpolations in time). For fixed $\lambda \geq 1$ and initial curve $\gamma_0 \in \mathcal{AC}$, let $\{\gamma_n^\lambda\}_n$ be a sequence of step-by-step minimizers (as defined in (2.6)). We first define the piecewise constant interpolations $\gamma^\lambda\colon [0,\infty) \times [0,1] \to \mathbb{R}^2$ of $\{\gamma_n^\lambda\}_n$ as

$$\gamma^\lambda(t,s) := \gamma^\lambda_{\lceil \lambda t \rceil}(s).$$

Here, $\lceil x \rceil$ is the biggest natural number $n$ satisfying $n \leq x$. We also set $L^\lambda$, $\kappa^\lambda$, $\tau^\lambda$, and $\nu^\lambda$ as the length, curvature, unit tangent vector field, and unit normal vector field of $\gamma^\lambda$, respectively. Furthermore, we set $\widetilde{\gamma}^\lambda\colon [0,\infty) \times [0,1] \to \mathbb{R}^2$ to be

$$\widetilde{\gamma}^\lambda(t,s) := \gamma^\lambda_{\lfloor \lambda t \rfloor}(s).$$

Similarly, we set $\widetilde{L}^\lambda$, $\widetilde{\kappa}^\lambda$, $\widetilde{\tau}^\lambda$, and $\widetilde{\nu}^\lambda$ to be the corresponding time-shift of the geometric quantities of $\gamma^\lambda$. We write $\widehat{\gamma}^\lambda\colon [0,\infty) \times [0,1] \to \mathbb{R}^2$ for the piecewise affine interpolation of $\{\gamma_n^\lambda\}_n$, given by

$$\widehat{\gamma}^\lambda(t,s) := (\lceil \lambda t \rceil - \lambda t)\gamma^\lambda_{\lfloor \lambda t \rfloor}(s) + (\lambda t - \lfloor \lambda t \rfloor)\gamma^\lambda_{\lceil \lambda t \rceil}(s),$$

and $\widehat{L}^\lambda\colon [0,\infty) \to \mathbb{R}$ for the piecewise affine interpolation of $\{L_n^\lambda\}_n$, given by

$$\widehat{L}^\lambda(t) := (\lceil \lambda t \rceil - \lambda t)L^\lambda_{\lfloor \lambda t \rfloor}(s) + (\lambda t - \lfloor \lambda t \rfloor)L^\lambda_{\lceil \lambda t \rceil}(s),$$

where $\lfloor x \rfloor$ is the smallest natural number $n$ satisfying $n \geq x$. Note that $\widehat{L}^\lambda \neq L_{\widehat{\gamma}^\lambda}$ in general. Finally, we set $V^\lambda\colon [0,\infty) \times [0,1] \to \mathbb{R}^2$ to be

$$V^\lambda(t,s) := \widehat{\gamma}^\lambda_t(t,s),$$

where $\widehat{\gamma}^\lambda_t$ is the weak time-derivative of $\widehat{\gamma}^\lambda$.

The next lemma is concerned with an important coupling relation between the tangential and the orthogonal projection of the velocity $V^\lambda$. It plays a crucial role in the compactness result for the sequence $\{V^\lambda\}_\lambda$ (see also Lemma 2.4).

**Lemma 2.3.** *For every $t \in [0,\infty)$ and $s \in [0,1]$, it holds that*

$$(\langle V^\lambda, \widetilde{\gamma}^\lambda_s + \gamma^\lambda_s \rangle)_s = (\widetilde{L}^\lambda + L^\lambda)\widehat{L}^\lambda_t + \langle V^\lambda, \widetilde{L}^\lambda \widetilde{\kappa}^\lambda (\widetilde{\gamma}^\lambda_s)^\perp + L^\lambda \kappa^\lambda (\gamma^\lambda_s)^\perp \rangle. \tag{2.19}$$

*Proof.* The derivation of the coupling relation (2.19) is the result of the following computation: Since $\gamma_n$ belongs to $\mathcal{AC}$ we have $\gamma^\lambda_s(t,s) \equiv L^\lambda(t)$ for all $t \in [0,\infty)$ and $s \in [0,1]$. Defining $\mu^\lambda := \gamma^\lambda_s + \widetilde{\gamma}^\lambda_s$, we derive for all $t \in [0,\infty)$ and $s \in [0,1]$

$$\begin{aligned}(\widetilde{L}^\lambda + L^\lambda)\widehat{L}^\lambda_t &= \lambda(\widetilde{L}^\lambda + L^\lambda)(L^\lambda - \widetilde{L}^\lambda) = \lambda\big((L^\lambda)^2 - (\widetilde{L}^\lambda)^2\big)\\ &= \lambda(\langle \gamma^\lambda_s, \gamma^\lambda_s\rangle - \langle \widetilde{\gamma}^\lambda_s, \widetilde{\gamma}^\lambda_s\rangle) = \langle V^\lambda_s, \mu^\lambda\rangle.\end{aligned}$$

Furthermore, by the product rule and (2.1), we have

$$\begin{aligned}\langle V^\lambda, \mu^\lambda\rangle_s &= \langle V^\lambda_s, \mu^\lambda\rangle + \langle V^\lambda, \widetilde{\gamma}^\lambda_{ss} + \gamma^\lambda_{ss}\rangle\\ &= \langle V^\lambda_s, \mu^\lambda\rangle + \langle V^\lambda, \widetilde{L}^\lambda \widetilde{\kappa}^\lambda (\widetilde{\gamma}^\lambda_s)^\perp + L^\lambda \kappa^\lambda (\gamma^\lambda_s)^\perp\rangle.\end{aligned}$$

Combining both equations readily leads to (2.19). ∎

**Lemma 2.4.** *There exists a constant $C(\varepsilon)$ only depending on $\varepsilon$ such that*

$$\int_0^\infty \int_0^1 |V^\lambda|^2 \,\mathrm{d}s\,\mathrm{d}t + \int_0^\infty |V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2 \,\mathrm{d}t \le C(\varepsilon). \tag{2.20}$$

*Proof.* In order to shorten notation, we write

$$\mu^\lambda := \widetilde{\gamma}_s^\lambda + \gamma_s^\lambda.$$

Note that as $\gamma_n^\lambda \in \mathcal{AC}_{\gamma_{n-1}^\lambda}$ for all $n$, we have by (2.2) and (2.7)

$$|\mu^\lambda|^2 = (\widetilde{L}^\lambda)^2 + 2\langle \widetilde{\gamma}_s^\lambda, \gamma_s^\lambda\rangle + (L^\lambda)^2 \ge (\widetilde{L}^\lambda)^2 + (L^\lambda)^2 \ge c. \tag{2.21}$$

Furthermore, by comparison,

$$\lambda D(\gamma_{n+1}^\lambda, \gamma_n^\lambda) \le E(\gamma_n^\lambda) - E(\gamma_{n+1}^\lambda).$$

Summing the above expression over $n$ and using the non-negativity of $E$, we get

$$\begin{aligned}\lambda \sum_{n=0}^\infty D(\gamma_{n+1}^\lambda, \gamma_n^\lambda) &\le \limsup_{N\to\infty} \sum_{n=0}^N (E(\gamma_n^\lambda) - E(\gamma_{n+1}^\lambda)) \\ &\le E(\gamma_0) - \liminf_{N\to\infty} E(\gamma_{N+1}^\lambda) \le E(\gamma_0).\end{aligned} \tag{2.22}$$

We then compute

$$\begin{aligned}\lambda \sum_{n=0}^\infty D(\gamma_{n+1}^\lambda, \gamma_n^\lambda) &= \frac{1}{4}\sum_{n=0}^\infty \lambda^{-1} \int_0^1 \langle \lambda(\gamma_{n+1}^\lambda - \gamma_n^\lambda), \nu_n^\lambda\rangle^2 L_n^\lambda \,\mathrm{d}s \\ &\quad + \frac{1}{4}\sum_{n=0}^\infty \lambda^{-1} \int_0^1 \langle \lambda(\gamma_{n+1}^\lambda - \gamma_n^\lambda), \nu_{n+1}^\lambda\rangle^2 L_{n+1}^\lambda \,\mathrm{d}s \\ &\quad + \frac{1}{2}\sum_{n=0}^\infty \lambda^{-1}(\lambda|\gamma_{n+1}^\lambda(0) - \gamma_n^\lambda(0)|)^2 \\ &\quad + \frac{1}{2}\sum_{n=0}^\infty \lambda^{-1}(\lambda|\gamma_{n+1}^\lambda(1) - \gamma_n^\lambda(1)|)^2 \\ &= \int_0^\infty \Big(\frac{1}{4\widetilde{L}^\lambda}\int_0^1 \langle V^\lambda, (\widetilde{\gamma}_s^\lambda)^\perp\rangle^2 \,\mathrm{d}s + \frac{1}{4L^\lambda}\int_0^1 \langle V^\lambda, (\gamma_s^\lambda)^\perp\rangle^2 \,\mathrm{d}s\Big)\,\mathrm{d}t \\ &\quad + \frac{1}{2}\int_0^\infty |V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2 \,\mathrm{d}t.\end{aligned} \tag{2.23}$$

Combining (2.22) (2.23) and (2.7) leads to

$$\begin{aligned}&\int_0^\infty \Big(\int_0^1 \langle V^\lambda, (\widetilde{\gamma}_s^\lambda)^\perp\rangle^2 \,\mathrm{d}s + \int_0^1 \langle V^\lambda, (\gamma_s^\lambda)^\perp\rangle^2 \,\mathrm{d}s\Big)\,\mathrm{d}t \\ &\quad + \int_0^\infty |V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2 \,\mathrm{d}t \le C.\end{aligned} \tag{2.24}$$

Since (2.24) and (2.21) imply

$$\int_0^\infty \int_0^1 \Big\langle V^\lambda \frac{(\mu^\lambda)^\perp}{|\mu^\lambda|}\Big\rangle^2 \,\mathrm{d}s\,\mathrm{d}t \le \frac{1}{c^2}\int_0^\infty \int_0^1 \langle V^\lambda, (\mu^\lambda)^\perp\rangle^2 \,\mathrm{d}s\,\mathrm{d}t \le C,$$

it follows that

$$\int_0^\infty \int_0^1 \Big\langle V^\lambda \frac{(\mu^\lambda)^\perp}{|\mu^\lambda|}\Big\rangle^2 \,\mathrm{d}s\,\mathrm{d}t + \int_0^\infty |V^\lambda(t,0)|^2 + \int_0^\infty |V^\lambda(t,1)|^2 \,\mathrm{d}t \le C.$$

In order to obtain (2.20), we are left to control

$$\int_0^\infty \int_0^1 \Big\langle V^\lambda, \frac{\mu^\lambda}{|\mu^\lambda|}\Big\rangle^2 \,\mathrm{d}s\,\mathrm{d}t$$

from above. This will be achieved by employing the relation in (2.19). To this end, we integrate (2.19) in the curve parameter over $[0,1]$ and solve for $\widehat{L}_t^\lambda$, thus obtaining

$$\widehat{L}_t^\lambda = \frac{1}{\widetilde{L}^\lambda + L^\lambda}\Big(\langle V^\lambda, \mu^\lambda\rangle|_{s=0}^1 - \int_0^1 \langle V^\lambda, \widetilde{L}^\lambda \widetilde{\kappa}^\lambda (\widetilde{\gamma}_s^\lambda)^\perp + L^\lambda \kappa^\lambda (\gamma_s^\lambda)^\perp\rangle \,\mathrm{d}s\Big).$$

Squaring both sides of the equality above, integrating them over $t \in [0,\infty)$, and using (2.7), (2.24), and Hölder's inequality, we get

$$\begin{aligned}
\int_0^\infty (\widehat{L}_t^\lambda)^2 \,\mathrm{d}t &\le C\int_0^\infty |V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2 \,\mathrm{d}t \\
&\quad + C\int_0^\infty \Big(\int_0^1 \langle V^\lambda, \widetilde{L}^\lambda \widetilde{\kappa}^\lambda (\widetilde{\gamma}_s^\lambda)^\perp + L^\lambda \kappa^\lambda (\gamma_s^\lambda)^\perp\rangle \,\mathrm{d}s\Big)^2 \,\mathrm{d}t \\
&\le C + C\int_0^\infty \Big(\int_0^1 (\widetilde{\kappa}^\lambda)^2 \,\mathrm{d}s\Big)\Big(\int_0^1 \langle V^\lambda, (\widetilde{\gamma}_s^\lambda)^\perp\rangle^2 \,\mathrm{d}s\Big)\,\mathrm{d}t \\
&\quad + C\int_0^\infty \Big(\int_0^1 (\kappa^\lambda)^2 \,\mathrm{d}s\Big)\Big(\int_0^1 \langle V^\lambda, (\gamma_s^\lambda)^\perp\rangle^2 \,\mathrm{d}s\Big)\,\mathrm{d}t \\
&\le C(\varepsilon)\Big(1 + \int_0^\infty \int_0^1 \langle V^\lambda, (\widetilde{\gamma}_s^\lambda)^\perp\rangle^2 + \langle V^\lambda, (\gamma_s^\lambda)^\perp\rangle^2 \,\mathrm{d}s\,\mathrm{d}t\Big) \le C(\varepsilon),
\end{aligned} \tag{2.25}$$

where in the last inequality we used (2.8). Next, we integrate (2.19), again, in the curve parameter but now over $[0,s]$, thus obtaining

$$\begin{aligned}
\langle V^\lambda, \mu^\lambda\rangle(t,s) &= \langle V^\lambda, \mu^\lambda\rangle(t,0) + (\widetilde{L}^\lambda + L^\lambda)\widehat{L}_t^\lambda s \\
&\quad + \int_0^s \langle V^\lambda, \widetilde{L}^\lambda \widetilde{\kappa}^\lambda (\widetilde{\gamma}_s^\lambda)^\perp + L^\lambda \kappa^\lambda (\gamma_s^\lambda)^\perp\rangle \,\mathrm{d}\widetilde{s}.
\end{aligned}$$

We then square both sides of the equality above, integrate over $(t,s)$ in $[0,1]\times[0,\infty)$, as well as employ (2.7), (2.8), (2.24), (2.25), and Hölder's inequality to derive

$$\begin{aligned}
&\int_0^\infty\int_0^1 \langle V^\lambda,\mu^\lambda\rangle^2\,\mathrm{d}s\,\mathrm{d}t\\
&\quad\le C\int_0^\infty |V^\lambda(t,0)|^2+(\widehat{L}_t^\lambda)^2\,\mathrm{d}t\\
&\qquad + C\int_0^\infty\int_0^1\Big(\int_0^s(\widetilde{\kappa}^\lambda)^2(t,\widetilde{s})\,\mathrm{d}\widetilde{s}\Big)\Big(\int_0^s\langle V^\lambda,(\widetilde{\gamma}_s^\lambda)^\perp\rangle^2(t,\widetilde{s})\,\mathrm{d}\widetilde{s}\Big)\,\mathrm{d}s\,\mathrm{d}t\\
&\qquad + C\int_0^\infty\int_0^1\Big(\int_0^s(\kappa^\lambda)^2(t,\widetilde{s})\,\mathrm{d}\widetilde{s}\Big)\Big(\int_0^s\langle V^\lambda,(\gamma_s^\lambda)^\perp\rangle^2(t,\widetilde{s})\,\mathrm{d}\widetilde{s}\Big)\,\mathrm{d}s\,\mathrm{d}t\\
&\quad\le C(\varepsilon)\Big(1+\int_0^\infty\int_0^1\langle V^\lambda,(\widetilde{\gamma}_s^\lambda)^\perp\rangle^2+\langle V^\lambda,(\gamma_s^\lambda)^\perp\rangle^2\,\mathrm{d}s\,\mathrm{d}t\Big)\le C(\varepsilon).
\end{aligned}$$

Hence, by (2.21),

$$\int_0^\infty\int_0^1\langle V^\lambda,\frac{\mu^\lambda}{|\mu^\lambda|}\rangle^2\,\mathrm{d}s\,\mathrm{d}t\le C\int_0^\infty\int_0^1\langle V^\lambda,\mu^\lambda\rangle^2\,\mathrm{d}s\,\mathrm{d}t\le C(\varepsilon).$$

With (2.24), this finally leads to (2.20). ∎

We continue by showing uniform Hölder continuity for the sequence of piecewise affine interpolations.

**Lemma 2.5.** *For $0\le t_1<t_2<\infty$ it holds that*

$$\|\widehat{\gamma}^\lambda(t_2,\cdot)-\widehat{\gamma}^\lambda(t_1,\cdot)\|_{L^2}\le C(\varepsilon)(t_2-t_1)^{\frac{1}{2}} \tag{2.26}$$

*and*

$$\begin{aligned}
|\widehat{\gamma}^\lambda(t_2,0)-\widehat{\gamma}^\lambda(t_1,0)|&\le C(t_2-t_1)^{\frac{1}{2}},\\
|\widehat{\gamma}^\lambda(t_2,1)-\widehat{\gamma}^\lambda(t_1,1)|&\le C(t_2-t_1)^{\frac{1}{2}}.
\end{aligned} \tag{2.27}$$

*Furthermore, for any $T>0$ we have*

$$\|\gamma^\lambda\|_{L_T^\infty H^2}\le C(\varepsilon,T). \tag{2.28}$$

*Proof.* By the absolute continuity of $\widehat{\gamma}^\lambda(\cdot,s)$ for every $s\in[0,1]$, (2.20), Hölder's inequality, and Fubini's theorem, we derive for all $0\le t_1<t_2<\infty$ that

$$\begin{aligned}
\|\widehat{\gamma}^\lambda(t_2,\cdot)-\widehat{\gamma}^\lambda(t_1,\cdot)\|_{L^2}&=\Big(\int_0^1\Big|\int_{t_1}^{t_2}V^\lambda\,\mathrm{d}t\Big|^2\mathrm{d}s\Big)^{\frac{1}{2}}\\
&\le\Big(\int_0^1(t_2-t_1)\int_{t_1}^{t_2}|V^\lambda|^2\,\mathrm{d}t\,\mathrm{d}s\Big)^{\frac{1}{2}}\\
&\le\Big(\int_0^\infty\|V^\lambda\|_{L^2}^2\,\mathrm{d}t\Big)^{\frac{1}{2}}(t_2-t_1)^{\frac{1}{2}}\le C(\varepsilon)(t_2-t_1)^{\frac{1}{2}}.
\end{aligned}$$

Hence, (2.26) follows. The proof of (2.27) follows similarly. Let us now fix $T > 0$, then with the definition of $\widehat{\gamma}^\lambda$ and (2.26) we can derive for any $0 \leq t \leq T$

$$\|\widehat{\gamma}^\lambda(t,\cdot)\|_{L^2} \leq \|\widehat{\gamma}^\lambda(t,\cdot) - \widehat{\gamma}^\lambda(0,\cdot)\|_{L^2} + \|\gamma_0\|_{L^2} \leq C(\varepsilon)T^{\frac{1}{2}} + C \leq C(\varepsilon, T).$$

Applying this for $t = \lambda^{-1}n$ ($n \in \mathbb{N}$) and using the definition of $\widehat{\gamma}^\lambda$, we see that

$$\|\gamma^\lambda\|_{L^2_T L^2} \leq C(\varepsilon, T).$$

Furthermore, by (2.7) and (2.8) we have

$$\|\gamma^\lambda_s\|_{L^\infty H^1} \leq C(\varepsilon),$$

which leads to (2.28). ∎

Throughout the paper we will employ the following formulation of the Gagliardo–Nirenberg interpolation inequality (see also [24, Theorem 1]); For a proof we refer to [15, Theorem 6.4]:

**Theorem 2.6** (Interpolation inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the cone condition. Let $i, j$, and $m$ be integers such that $0 \leq i \leq j \leq m$. Let $1 \leq p \leq q < \infty$ if $(m-j)p \geq n$, or let $1 \leq p \leq q \leq \infty$ if $(m-j)p > n$. Then, there exists a constant $C$ such that for all $u \in W^{m,p}(\Omega)$, it holds that*

$$\|D^j u\|_{L^q(\Omega)} \leq C\big(\|D^m u\|^\theta_{L^p(\Omega)} \|D^i u\|^{1-\theta}_{L^p(\Omega)} + \|D^i u\|_{L^p(\Omega)}\big), \tag{2.29}$$

*where*

$$\theta := \frac{1}{m-i}\Big(\frac{n}{p} - \frac{n}{q} + j - i\Big).$$

Thanks to the uniform $L^2$ bound on the curvature in (2.8), we will improve the Hölder continuity results from the previous lemma by interpolation.

**Lemma 2.7.** *For any $\alpha \in (0, \frac{1}{2})$, $T > 0$, and $0 \leq t_1 < t_2 \leq T$ it holds that*

$$\|\widehat{\gamma}^\lambda(t_2,\cdot) - \widehat{\gamma}^\lambda(t_1,\cdot)\|_{C^{1,\alpha}} \leq C(\varepsilon, T)(t_2 - t_1)^{\frac{1-2\alpha}{8}}. \tag{2.30}$$

**Remark 2.8** (to Lemma 2.7). Take any $\alpha \in (0, \frac{1}{2})$ and $T > 0$. Using Definition 2.2 and (2.30), we derive for any $t \in [0, T]$

$$\begin{aligned}\|\gamma^\lambda_s(t,\cdot) - \widetilde{\gamma}^\lambda_s(t,\cdot)\|_{L^\infty} &= \|\widehat{\gamma}^\lambda_s(\lceil\lambda t\rceil\lambda^{-1},\cdot) - \widehat{\gamma}^\lambda_s(\lfloor\lambda t\rfloor\lambda^{-1},\cdot)\|_{L^\infty} \\ &\leq C(\varepsilon, T)|(\lceil\lambda t\rceil - \lfloor\lambda t\rfloor)\lambda^{-1}|^{\frac{1-2\alpha}{8}} \leq C(\varepsilon, T)\lambda^{\frac{2\alpha-1}{8}}.\end{aligned}$$

Consequently,

$$\begin{aligned}|\langle\gamma^\lambda_s, \widetilde{\gamma}^\lambda_s\rangle - L^\lambda\widetilde{L}^\lambda| &= |\langle\gamma^\lambda_s - \widetilde{\gamma}^\lambda_s, \widetilde{\gamma}^\lambda_s\rangle + (\widetilde{L}^\lambda)^2 - L^\lambda\widetilde{L}^\lambda| \\ &\leq \widetilde{L}^\lambda\Big(\|\gamma^\lambda_s - \widetilde{\gamma}^\lambda_s\|_{L^\infty} + \int_0^1 |\gamma^\lambda_s - \widetilde{\gamma}^\lambda_s|\,\mathrm{d}s\Big)\end{aligned}$$

$$\leq C\|\gamma_s^\lambda - \widetilde{\gamma}_s^\lambda\|_{L^\infty} \leq C(\varepsilon, T)\lambda^{\frac{2\alpha-1}{8}} \stackrel{\lambda\to\infty}{\to} 0.$$

Hence, by (2.7) there exists $\lambda_0 := \lambda(\varepsilon, T) \geq 1$ big enough such that for all $\lambda > \lambda_0$ we have

$$\langle \gamma_s^\lambda, \widetilde{\gamma}_s^\lambda \rangle > 0.$$

In particular, we derive the following crucial result: For $\lambda > \lambda_0$ and $n \leq \lfloor \lambda T \rfloor$, the step-by-step minimizer $\gamma_{n+1}^\lambda$ satisfies

$$\gamma_{n+1}^\lambda \in \operatorname*{argmin}_{\gamma \in \mathcal{AC}} F(\gamma, \gamma_n^\lambda). \tag{2.31}$$

This will become relevant once we compute the Euler–Lagrange equation corresponding to the step-by-step minimization (2.6), as (2.31) tells us that the additional angle constraint in (2.6) is not influencing the minimization, at least for $\lambda > \lambda_0$ and $n \leq \lfloor \lambda T \rfloor$.

*Proof of Lemma 2.7.* Fix $\alpha \in (0, \frac{1}{2})$, $T > 0$ and $0 \leq t_1 < t_2 \leq T$. In order to shorten notation, we define

$$\Delta\gamma^\lambda := \widehat{\gamma}^\lambda(t_2, \cdot) - \widehat{\gamma}^\lambda(t_1, \cdot).$$

Using the interpolation inequality (2.29) for $\Delta\gamma^\lambda$ with $n = 1, i = 0, j = 1, m = 2, p = 2$, and $q = \infty$, we derive

$$\|\Delta\gamma_s^\lambda\|_{L^\infty} \leq C\big(\|\Delta\gamma_{ss}^\lambda\|_{L^2}^{\frac{3}{4}} \|\Delta\gamma^\lambda\|_{L^2}^{\frac{1}{4}} + \|\Delta\gamma^\lambda\|_{L^2}\big).$$

By (2.26), (2.28), and the very definition of $\Delta\gamma^\lambda$, we can control the right-hand side of the previous equation as follows:

$$\begin{aligned}
\|\Delta\gamma_s^\lambda\|_{L^\infty} &\leq C(\varepsilon, T)\big((t_2 - t_1)^{\frac{1}{8}} + (t_2 - t_1)^{\frac{1}{2}}\big) \\
&= C(\varepsilon, T)\big(1 + (t_2 - t_1)^{\frac{1}{2} - \frac{1}{8}}\big)(t_2 - t_1)^{\frac{1}{8}} \\
&\leq C(\varepsilon, T)(t_2 - t_1)^{\frac{1}{8}}.
\end{aligned} \tag{2.32}$$

Note that in the last inequality we have used the fact that $t_1, t_2$ are contained in the bounded interval $[0, T]$. By the fundamental theorem of calculus, (2.27), and (2.32), we also derive

$$\begin{aligned}
\|\Delta\gamma^\lambda\|_{L^\infty} &\leq |\Delta\gamma^\lambda(0)| + \int_0^1 |\Delta\gamma_s^\lambda| \, \mathrm{d}s \\
&\leq C(t_2 - t_1)^{\frac{1}{2}} + \|\Delta\gamma_s^\lambda\|_{L^\infty} \\
&\leq C(t_2 - t_1)^{\frac{1}{2}} + C(\varepsilon, T)(t_2 - t_1)^{\frac{1}{8}} \\
&\leq C(\varepsilon, T)(t_2 - t_1)^{\frac{1}{8}}.
\end{aligned} \tag{2.33}$$

In order to conclude the proof, it remains to control the Hölder seminorm $|\Delta\gamma_s^\lambda|_\alpha$. By Morrey's inequality, (2.32), and (2.28), we have that

$$\begin{aligned}
|\Delta\gamma_s^\lambda|_\alpha &= \sup_{s_1, s_2 \in I} \frac{|\Delta\gamma_s^\lambda(s_2) - \Delta\gamma_s^\lambda(s_1)|}{|s_2 - s_1|^\alpha} \\
&= \Big(\sup_{s_1, s_2 \in I} \frac{|\Delta\gamma_s^\lambda(s_2) - \Delta\gamma_s^\lambda(s_1)|}{|s_2 - s_1|^{\frac{1}{2}}}\Big)^{2\alpha} \sup_{s_1, s_2 \in I} |\Delta\gamma_s^\lambda(s_2) - \Delta\gamma_s^\lambda(s_1)|^{1-2\alpha}
\end{aligned}$$

$$\le C|\Delta\gamma_s^\lambda|_{\frac{1}{2}}^{2\alpha}\|\Delta\gamma_s^\lambda\|_{L^\infty}^{1-2\alpha} \le C\|\Delta\gamma^\lambda\|_{H^2}^{2\alpha}\|\Delta\gamma_s^\lambda\|_{L^\infty}^{1-2\alpha}$$
$$\le C(\varepsilon, T)(t_2 - t_1)^{\frac{1-2\alpha}{8}}. \tag{2.34}$$

Combining (2.33), (2.32), and (2.34) results in (2.30). ∎

We combine the previous lemmas of this subsection to derive the initial compactness result.

**Theorem 2.9** (Initial Compactness). *Given $\hat{\gamma}^\lambda$ and $\gamma^\lambda$ as in Definition 2.2, there exists $\gamma: [0, \infty) \times [0, 1] \to \mathbb{R}^2$ such that for any $\alpha \in (0, \frac{1}{2})$ and $\beta \in (0, \frac{1-2\alpha}{8})$, up to subsequences, it holds that*

$$\hat{\gamma}^\lambda \to \gamma \quad \text{in } C^{0,\beta}_{\mathrm{loc}} C^{1,\alpha}, \tag{2.35}$$
$$\gamma^\lambda \to \gamma \quad \text{in } L^\infty_{\mathrm{loc}} C^{1,\alpha}, \tag{2.36}$$

*and*

$$\hat{\gamma}^\lambda \rightharpoonup \gamma \qquad \text{weakly in } H^1_{\mathrm{loc}} L^2, \tag{2.37}$$
$$\begin{aligned} \hat{\gamma}^\lambda(0,\cdot) &\rightharpoonup \gamma(0,\cdot) \quad \text{weakly in } H^1_{\mathrm{loc}}([0,\infty); \mathbb{R}^2), \\ \hat{\gamma}^\lambda(1,\cdot) &\rightharpoonup \gamma(1,\cdot) \quad \text{weakly in } H^1_{\mathrm{loc}}([0,\infty); \mathbb{R}^2). \end{aligned} \tag{2.38}$$

*Proof.* The proof of (2.35) follows from (2.30) and a standard diagonal sequence argument. For this, let $(T_k) \subset [0, \infty)$ be an auxiliary sequence with $T_k \uparrow \infty$. By (2.30) and the Arzelà–Ascoli Theorem, there exist $(\lambda_n^{(0)})$ converging to $\infty$ and $\gamma^{(0)}: [0, T_0] \times [0, 1] \to \mathbb{R}^2$ such that for any $\alpha \in (0, \frac{1}{2})$ and $\beta \in (0, \frac{1-2\alpha}{8})$, we have as $n \to \infty$ that

$$\hat{\gamma}^{\lambda_n^{(0)}} \to \gamma^{(0)} \quad \text{in } C^{0,\beta}_{T_0} C^{1,\beta}.$$

Now, for every $k \in \mathbb{N}$ we apply the Arzelà–Ascoli theorem to the sequence $\hat{\gamma}^{\lambda_n^{(k)}}$ to construct $(\lambda_n^{(k+1)})_n$, as a subsequence of $(\lambda_n^{(k)})_n$, and $\gamma^{(k+1)}: [0, T_{k+1}] \times [0, 1] \to \mathbb{R}^2$ such that for any $\alpha \in (0, \frac{1}{2})$ and $\beta \in (0, \frac{1-2\alpha}{8})$, we have as $n \to \infty$ that

$$\hat{\gamma}^{\tau_n^{(k+1)}} \to \gamma^{(k+1)} \quad \text{in } C^{0,\beta}_{T_{k+1}} C^{1,\alpha}.$$

Note that, as convergence in $C^{0,\alpha}_{T_{k+1}} C^{1,\alpha}$ implies convergence in $C^{0,\alpha}_{T_k} C^{1,\alpha}$, we have

$$\gamma^{(k+1)}|_{[0,T_k]} = \gamma^{(k)}$$

for all $k \in \mathbb{N}$. Hence, we can define $\gamma: [0, \infty) \times [0, 1] \to \mathbb{R}^2$ through

$$\gamma|_{[0,T_k]} := \gamma^{(k)} \qquad \text{for all } k \in \mathbb{N}.$$

Along the diagonal sequence $\lambda_n := \lambda_n^{(n)}$ we then have for any $\alpha \in (0, \frac{1}{2})$ and $\beta \in (0, \frac{1-2\alpha}{8})$ that

$$\hat{\gamma}^{\lambda_n} \to \gamma \quad \text{in } C^{0,\beta}_{\mathrm{loc}} C^{1,\alpha}.$$

From this point on we assume that we have already extracted the subsequence $(\widehat{\gamma}^{\lambda_n})$ and we will denote it, for the sake of shorter notation, just by $(\widehat{\gamma}^{\lambda})$. By the definition of $\widehat{\gamma}^{\lambda}$ and $\gamma^{\lambda}$, and thanks to (2.30), for any $\alpha \in (0, \frac{1}{2})$, $T > 0$, and $0 \leq t \leq T$ we have that

$$\|\gamma^{\lambda}(t,\cdot) - \widehat{\gamma}^{\lambda}(t,\cdot)\|_{C^{1,\alpha}} \leq C(\varepsilon, T)\lambda^{\frac{2\alpha-1}{8}} \overset{\lambda\to\infty}{\to} 0.$$

Consequently, by (2.30) we can deduce (2.36). Thanks to (2.20) and the already proven convergence (2.30) we also have, up to a further subsequence, that (2.37) and (2.38) hold true. ■

We wish to compute the first variation of the minimization problem

$$\min_{\gamma \in \mathcal{AC}} F(\gamma, \widetilde{\gamma}),$$

for some fixed $\widetilde{\gamma} \in \mathcal{AC}$. Due to the non-linearity of the speed constraint of $\mathcal{AC}$, the additive variation $\gamma + \delta\eta$, with $\gamma \in \mathcal{AC}$, $\delta > 0$, and $\eta \in H^2$, is in general not admissible. In the next lemma we will show that there exists a reparametrization $P\colon [0,1] \to [0,1]$ (depending on $\delta$) such that $(\gamma + \delta\eta) \circ P \in \mathcal{AC}$.

**Lemma 2.10** (Admissible variations in $\mathcal{AC}$). *For $\gamma \in \mathcal{AC}$, $\eta \in H^2$, and $0 < \delta < \frac{L}{\|\eta_s\|_{L^\infty}}$, where $L$ is the length of $\gamma$, there exists a unique map $P(\delta,\cdot)\colon [0,1] \to [0,1]$ such that $\mu(\delta,\cdot)\colon [0,1] \to \mathbb{R}^2$ defined as*

$$\mu(\delta, s) := (\gamma + \delta\eta)(P(\delta, s)), \tag{2.39}$$

*satisfies*

$$\mu(\delta,\cdot) \in \mathcal{AC}, \quad \mu(\delta, 0) = \gamma(0) + \delta\eta(0). \tag{2.40}$$

*Furthermore, we have*

$$P_\delta(\delta, s) = \frac{1}{L^2}\Big(s\int_0^1 \langle\gamma_s, \eta_s\rangle \,\mathrm{d}\widetilde{s} - \int_0^s \langle\gamma_s, \eta_s\rangle \,\mathrm{d}\widetilde{s}\Big), \tag{2.41}$$

$$P_s(0, s) = 1, \tag{2.42}$$

$$P_{s\delta}(\delta, s) = \frac{1}{L^2}\Big(\int_0^1 \langle\gamma_s, \eta_s\rangle \,\mathrm{d}\widetilde{s} - \langle\gamma_s(s), \eta_s(s)\rangle\Big). \tag{2.43}$$

*Proof.* Let us consider the auxiliary differentiable function $F(\delta,\cdot)\colon [0,1] \to \mathbb{R}$ given by

$$F(\delta, s') := \frac{\int_0^{s'} |\gamma_s + \delta\eta_s| \,\mathrm{d}\widetilde{s}}{L_\delta}, \tag{2.44}$$

where $L_\delta$ is the length of $\gamma + \delta\eta$. Then, for $0 < \delta < \frac{L}{\|\eta_s\|_{L^\infty}}$ we have

$$|\gamma_s + \delta\eta_s| \geq |\gamma_s| - \delta\|\eta_s\|_{L^\infty} = L - \delta\|\eta_s\|_{L^\infty} > 0.$$

Therefore, it follows that $F_{s'} > 0$. Together with $F(\delta, 0) = 0$ and $F(\delta, 1) = 1$, this implies that $F(\delta,\cdot)$ is a diffeomorphism.

We now consider $P(\delta, \cdot)\colon [0,1] \to [0,1]$ defined as

$$P(\delta, s) := F(\delta, \cdot)^{-1}(s).$$

Let us check that for such a choice of $P$ all statements of the lemma hold true. As $P(\delta, 0) = 0$, we see that

$$\mu(\delta, 0) = \gamma(0) + \delta\eta(0).$$

From $F(\delta, P(\delta, s)) = s$ and the chain rule, we also derive that

$$F_{s'}(\delta, P(\delta, s))P_s(\delta, s) = 1, \tag{2.45}$$

$$F_\delta(\delta, P(\delta, s)) + F_{s'}(\delta, P(\delta, s))P_\delta(\delta, s) = 0. \tag{2.46}$$

Moreover,

$$F_{s'}(\delta, s') = \frac{1}{L_\delta}|\gamma_s(s') + \delta\eta_s(s')|, \tag{2.47}$$

$$\begin{aligned} F_\delta(\delta, s') = {}& \frac{1}{L_\delta}\int_0^{s'} \Big\langle \frac{\gamma_s + \delta\eta_s}{|\gamma_s + \delta\eta_s|}, \eta_s \Big\rangle \, d\widetilde{s} \\ & - \frac{1}{L_\delta^2}\int_0^{s'} |\gamma_s + \delta\eta_s| \, d\widetilde{s} \int_0^1 \Big\langle \frac{\gamma_s + \delta\eta_s}{|\gamma_s + \delta\eta_s|}, \eta_s \Big\rangle \, d\widetilde{s}. \end{aligned} \tag{2.48}$$

Hence, (2.45) and (2.47) imply

$$P_s(\delta, s) = \frac{L_\delta}{|(\gamma_s + \delta\eta_s)(P(\delta, s))|}, \tag{2.49}$$

by which (2.42) follows. From (2.42), we have

$$|(\mu(\delta, \cdot))_s(s)| = |(\gamma + \delta\eta)(P(\delta, s))||P_s(\delta, s)| = L_\delta$$

and therefore, $\mu \in \mathcal{AC}$. It remains to check (2.41) and (2.43). We use (2.46), (2.47), (2.48), and $P(0, s) = s$ in order to compute

$$P_\delta(\delta, s) = -\frac{F_\delta(0, P(0, s))}{F_{s'}(0, P(0, s))} = -F_\delta(0, s) = \frac{1}{L^2}\Big(s\int_0^1 \langle \gamma_s, \eta_s\rangle \, d\widetilde{s} - \int_0^s \langle \gamma_s, \eta_s\rangle \, d\widetilde{s}\Big),$$

that is, (2.41). In order to conclude the proof, we differentiate (2.49) with respect to $\delta$, thus obtaining

$$\begin{aligned} P_{s\delta}(\delta, s) = {}& \frac{1}{|(\gamma_s + \delta\eta_s)(P(\delta, s))|}\int_0^1 \Big\langle \frac{\gamma_s + \delta\eta_s}{|\gamma_s + \delta\eta_s|}, \eta_s \Big\rangle \, d\widetilde{s} \\ & - \frac{L_\delta}{|(\gamma_s + \delta\eta_s)(P(\delta, s))|^3}\langle (\gamma_s + \delta\eta_s)(P(\delta, s)), \gamma_{ss}(P(\delta, s))P_\delta(\delta, s) \\ & + \eta_s(P(\delta, s)) + \delta\eta_{ss}(P(\delta, s))P_\delta(\delta, s)\rangle. \end{aligned}$$

Plugging in $\delta = 0$ above and using $\langle \gamma_s, \gamma_{ss}\rangle = 0$ eventually leads to (2.43). ■

**Remark 2.11.** We wish to provide the intuition behind formula (2.44). Suppose that there exists $P(\delta,\cdot)\colon[0,1]\to[0,1]$ such that $\mu(\delta,\cdot)$, as defined in (2.39), satisfies (2.40). It then follows that

$$\int_0^{P(\delta,s)}|\gamma_s+\delta\eta_s|\,\mathrm{d}\tilde{s}=\int_0^s|\mu_s(\delta,\cdot)|\,\mathrm{d}\tilde{s}=L_\delta s$$

for all $s\in[0,1]$. After dividing by $L_\delta$ above, we see that $P(\delta,\cdot)$ is the inverse of

$$F(\delta,s'):=\frac{\int_0^{s'}|\gamma_s+\delta\eta_s|\,\mathrm{d}\tilde{s}}{L_\delta},$$

as long as one such inverse exists.

**Definition 2.12.** Given $\gamma,\ \eta\in H^1([0,1];\mathbb{R}^2)$ we define $P_1(\gamma,\eta)\colon[0,1]\to\mathbb{R}$ and $P_2(\gamma,\eta)\colon[0,1]\to\mathbb{R}$ as

$$P_1(\gamma,\eta)(s):=\frac{1}{L^2}\Big(s\int_0^1\langle\gamma_s,\eta_s\rangle\,\mathrm{d}\tilde{s}-\int_0^s\langle\gamma_s,\eta_s\rangle\,\mathrm{d}\tilde{s}\Big),\tag{2.50}$$

$$P_2(\gamma,\eta)(s)=\frac{1}{L^2}\Big(\int_0^1\langle\gamma_s,\eta_s\rangle\,\mathrm{d}\tilde{s}-\langle\gamma_s,\eta_s\rangle\Big).\tag{2.51}$$

We are finally ready to compute the first variation of the minimization problem (2.31), which eventually leads to the weak formulation of the time-discrete evolution in Theorem 2.14.

**Lemma 2.13** (First variation). *Fix $\tilde{\gamma}\in\mathcal{AC}$ and let*

$$\gamma\in\operatorname*{argmin}_{\mu\in\mathcal{AC}}F(\mu,\tilde{\gamma}).$$

*Then for all $\eta\in C^\infty$ it holds that*

$$E(\gamma,\eta)+\mathrm{Diss}(\gamma,\eta)+\mathrm{Err}(\gamma,\eta)=0,\tag{2.52}$$

*where*

$$\begin{aligned}E(\gamma,\eta)&:=\int_0^1\frac{\varepsilon}{L^3}\langle\gamma_{ss},\eta_{ss}\rangle+\frac{1}{L}\Big(1-\frac{3\varepsilon}{2}\kappa_\gamma^2\Big)\langle\gamma_s,\eta_s\rangle\,\mathrm{d}s\\&\quad-\Big\langle\frac{\gamma(1)-\gamma(0)}{|\gamma(1)-\gamma(0)|^2},\eta(1)-\eta(0)\Big\rangle,\end{aligned}\tag{2.53}$$

$$\begin{aligned}\mathrm{Diss}(\gamma,\eta)&:=\frac{1}{2\tilde{L}}\int_0^1\langle\lambda(\gamma-\tilde{\gamma}),\tilde{\gamma}_s^\perp\rangle\langle\tilde{\gamma}_s^\perp,\eta\rangle\,\mathrm{d}s+\frac{1}{2L}\int_0^1\langle\lambda(\gamma-\tilde{\gamma}),\gamma_s^\perp\rangle\langle\gamma_s^\perp,\eta\rangle\,\mathrm{d}s\\&\quad+\langle\lambda(\gamma(0)-\tilde{\gamma}(0)),\eta(0)\rangle+\langle\lambda(\gamma(1)-\tilde{\gamma}(1)),\eta(1)\rangle,\end{aligned}\tag{2.54}$$

*where $\kappa$ is the curvature of $\gamma$, $L$ is its length, and $\tilde{L}$ the length of $\tilde{\gamma}$. Furthermore, the last* *term* $\mathrm{Err}(\gamma,\eta)$ *is given by*

$$
\begin{aligned}
\mathrm{Err}(\gamma,\eta) := {}& \frac{1}{2\widetilde{L}}\int_0^1 \langle\lambda(\gamma-\widetilde{\gamma}),\widetilde{\gamma}_s^{\perp}\rangle\langle\widetilde{\gamma}_s^{\perp},P_1(\gamma,\eta)\gamma_s\rangle\,\mathrm{d}s \\
&+\frac{1}{2\widetilde{L}}\int_0^1 \langle\lambda(\gamma-\widetilde{\gamma}),\widetilde{\gamma}_s^{\perp}\rangle\langle\gamma-\widetilde{\gamma},P_1(\gamma,\eta)\gamma_{ss}^{\perp}+P_2(\gamma,\eta)\gamma_s^{\perp}+\eta_s^{\perp}\rangle\,\mathrm{d}s \\
&-\frac{1}{4L^3}\int_0^1\langle\gamma_s,\eta_s\rangle\,\mathrm{d}s\int_0^1\langle\lambda(\gamma-\widetilde{\gamma}),\gamma_s^{\perp}\rangle\langle\gamma-\widetilde{\gamma},\gamma_s^{\perp}\rangle\,\mathrm{d}s. \qquad (2.55)
\end{aligned}
$$

*Proof.* By the minimality of $\gamma$ we must have $\frac{\mathrm{d}}{\mathrm{d}\delta}\big|_{\delta=0}F(\mu(\delta,\cdot))=0$, where $\mu(\delta,\cdot)$ is as in (2.39). It remains to show that

$$
\frac{\mathrm{d}}{\mathrm{d}\delta}\bigg|_{\delta=0}F(\mu(\delta,\cdot)) = E(\gamma,\eta)+\mathrm{Diss}(\gamma,\eta)+\mathrm{Err}(\gamma,\eta).
$$

Given any $\mu\in\mathcal{AC}$, for the reader's convenience, we split Diss defined in (2.4) into the following three terms:

$$
\begin{aligned}
D_1(\mu) &:= \frac{\lambda}{2}|\mu(0)-\widetilde{\gamma}(0)|^2+\frac{\lambda}{2}|\mu(1)-\widetilde{\gamma}(1)|^2, \\
D_2(\mu) &:= \frac{\lambda}{4\widetilde{L}}\int_0^1\langle\mu-\widetilde{\gamma},\widetilde{\gamma}_s^{\perp}\rangle^2\,\mathrm{d}s, \\
D_3(\mu) &:= \frac{\lambda}{4L_\mu}\int_0^1\langle\mu-\widetilde{\gamma},\mu_s^{\perp}\rangle^2\,\mathrm{d}s,
\end{aligned}
$$

where $L_\mu$ denotes the length of $\mu$. From the very definition in (2.5) of $F$, we can then write

$$
F(\mu,\widetilde{\gamma}) = E(\mu)+D_1(\mu)+D_2(\mu)+D_3(\mu).
$$

We wish to compute the first variation of each term on the right-hand side in the equation above separately.

*First variation of $E$*:

$$
\begin{aligned}
E(\mu(\delta,\cdot)) = {}& -\log|\gamma(1)-\gamma(0)+\delta(\eta(1)-\eta(0))| \\
&+\int_0^1\frac{\varepsilon}{2}\frac{\langle\gamma_{ss}+\delta\eta_{ss},\gamma_s^{\perp}+\delta\eta_s^{\perp}\rangle^2}{|\gamma_s+\delta\eta_s|^5}+|\gamma_s+\delta\eta_s|\,\mathrm{d}s.
\end{aligned}
$$

By the dominated convergence theorem and thanks to $\gamma_{ss}=L\kappa_\gamma\gamma_s^{\perp}$, we derive

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\delta}\bigg|_{\delta=0}E(\mu(\delta,\cdot)) = {}& \int_0^1 \varepsilon\frac{\langle\gamma_{ss},\gamma_s^{\perp}\rangle}{|\gamma_s|^5}(\langle\gamma_s^{\perp},\eta_{ss}\rangle+\langle\gamma_{ss},\eta_s^{\perp}\rangle)-\frac{5\varepsilon}{2}\frac{\langle\gamma_{ss},\gamma_s^{\perp}\rangle^2}{|\gamma_s|^7}\langle\gamma_s,\eta_s\rangle \\
&+\frac{\langle\gamma_s,\eta_s\rangle}{|\gamma_s|}\,\mathrm{d}s-\Big\langle\frac{\gamma(1)-\gamma(0)}{|\gamma(1)-\gamma(0)|^2},\eta(1)-\eta(0)\Big\rangle
\end{aligned}
$$

$$= \int_0^1 \frac{\varepsilon}{L^3}\langle \gamma_{ss}, \eta_{ss}\rangle - \frac{3\varepsilon}{2}\frac{\kappa_\gamma^2}{L}\langle \gamma_s, \eta_s\rangle + \frac{1}{L}\langle \gamma_s, \eta_s\rangle \, \mathrm{d}s$$
$$- \left\langle \frac{\gamma(1)-\gamma(0)}{|\gamma(1)-\gamma(0)|^2}, \eta(1)-\eta(0)\right\rangle = E(\gamma, \eta),$$

where we have used

$$\varepsilon \frac{\langle \gamma_{ss}, \gamma_s^\perp\rangle}{|\gamma_s|^5}\langle \gamma_s^\perp, \eta_{ss}\rangle = \varepsilon \frac{L\kappa_\gamma \langle \gamma_s^\perp, \gamma_s^\perp\rangle}{L^5}\langle \gamma_s^\perp, \eta_{ss}\rangle = \frac{\varepsilon}{L}\langle \gamma_{ss}, \eta_{ss}\rangle,$$

$$\varepsilon \frac{\langle \gamma_{ss}, \gamma_s^\perp\rangle}{|\gamma_s|^5}\langle \gamma_{ss}, \eta_s^\perp\rangle - \frac{5\varepsilon}{2}\frac{\langle \gamma_{ss}, \gamma_s^\perp\rangle^2}{|\gamma_s|^7}\langle \gamma_s, \eta_s\rangle = \varepsilon \frac{L^3\kappa_\gamma}{L^5}\langle L\kappa_\gamma \gamma_s, \eta_s\rangle - \frac{5\varepsilon}{2}\frac{(L^3\kappa_\gamma)^2}{L^7}\langle \gamma_s, \eta_s\rangle$$
$$= -\frac{3\varepsilon}{2}\frac{\kappa_\gamma^2}{L}\langle \gamma_s, \eta_s\rangle.$$

*First variation of $D_1$*:

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\bigg|_{\eta=0} D_1(\mu(\delta,\cdot)) = \frac{\mathrm{d}}{\mathrm{d}\delta}\bigg|_{\delta=0} D_1(\gamma+\delta\eta)$$
$$= \langle \lambda(\gamma(0)-\widetilde{\gamma}(0)), \eta(0)\rangle + \langle \lambda(\gamma(1)-\widetilde{\gamma}(1)), \eta(1)\rangle.$$

*First variation of $D_2$*: Note that by comparing (2.40), (2.41), and Definition 2.12, we see that

$$P_\delta(0,\cdot) = P_1(\gamma,\eta), \quad P_{s\delta}(0,\cdot) = P_2(\gamma,\eta).$$

Furthermore, we preliminarily compute

$$\mu_\delta(\delta,s) = \gamma_s(P(\delta,s))P_\delta(\delta,s) + \eta(P(\delta,s)) + \delta\eta_s(P(\delta,s))P_\delta(\delta,s).$$

Hence, by the dominated convergence theorem we have

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\bigg|_{\delta=0} D_2(\mu(\delta,\cdot)) = \frac{1}{2\widetilde{L}}\int_0^1 \langle \lambda(\gamma-\widetilde{\gamma}), \widetilde{\gamma}_s^\perp\rangle\langle \widetilde{\gamma}_s^\perp, \eta\rangle \, \mathrm{d}s$$
$$+ \frac{1}{2\widetilde{L}}\int_0^1 \langle \lambda(\gamma-\widetilde{\gamma}), \widetilde{\gamma}_s^\perp\rangle\langle \widetilde{\gamma}_s^\perp, P_1(\gamma,\eta)\gamma_s\rangle \, \mathrm{d}s.$$

*First variation of $D_3$*: We preliminarily compute

$$\mu_s(\delta,s) = \gamma_s(P(\delta,s))P_s(\delta,s) + \delta\eta_s(P(\delta,s))P_s(\delta,s),$$
$$\mu_{s\delta}(\delta,s) = \gamma_{ss}(P(\delta,s))P_\delta(\delta,s)P_s(\delta,s) + \gamma_s(P(\delta,s))P_{s\delta}(\delta,s)$$
$$+ \eta_s(P(\delta,s))P_s(\delta,s) + \delta\eta_{ss}(P(\delta,s))P_\delta(\delta,s)P_s(\delta,s)$$
$$+ \delta\eta_s(P(\delta,s))P_{s\delta}(\delta,s),$$
$$\frac{\mathrm{d}}{\mathrm{d}\delta}\bigg|_{\delta=0}\frac{1}{L_\delta} = -\frac{1}{L^2}\int_0^1 \left\langle \frac{\gamma_s}{L}, \eta_s\right\rangle \mathrm{d}s,$$

where $L_\delta$ is the length of $\mu(\delta,\cdot)$. With the computation above and the dominated convergence theorem we derive

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta}\bigg|_{\delta=0} D_3(\mu(\delta,\cdot)) &= \frac{1}{2L}\int_0^1 \langle \lambda(\gamma-\widetilde{\gamma}),\gamma_s^\perp\rangle\langle\gamma_s^\perp, P_1(\gamma,\eta)\gamma_s\eta\rangle\,\mathrm{d}s\\ &+\frac{1}{2L}\int_0^1\langle\lambda(\gamma-\widetilde{\gamma}),\gamma_s^\perp\rangle\langle\gamma-\widetilde{\gamma}, P_1(\gamma,\eta)\gamma_{ss}^\perp + P_2(\gamma,\eta)\gamma_s^\perp+\eta_s^\perp\rangle\,\mathrm{d}s\\ &-\frac{1}{L^2}\int_0^1\Big\langle\frac{\gamma_s}{L},\eta_s\Big\rangle\,\mathrm{d}s\int_0^1\frac{\lambda}{4}\langle\gamma-\widetilde{\gamma},\gamma_s^\perp\rangle^2\,\mathrm{d}s.\end{aligned}$$

By collecting all the aforementioned results, (2.54) and (2.55) follow. ■

**Theorem 2.14** (Time-discrete geometric evolution). *For any $T>0$ there exists $\lambda_0=\lambda_0(\varepsilon,T)>0$ such that for every $\eta\in C^\infty([0,\infty),C^\infty)$ and for every $\lambda>\lambda_0$, it holds that*

$$\int_0^T E(\gamma^\lambda(t,\cdot),\eta(t,\cdot))+\mathrm{Diss}(\gamma^\lambda(t,\cdot),\eta(t,\cdot))+\mathrm{Err}(\gamma^\lambda(t,\cdot),\eta(t,\cdot))\,\mathrm{d}t=0, \quad (2.56)$$

*where $E$,* Diss *and* Err *are as in* (2.53)*,* (2.54)*, and* (2.55)*, respectively.*

*Proof.* The proof follows using Remark 2.8, (2.52), and a simple induction argument. ■

The weak formulation in (2.56) of the time-discrete evolution will now be used to derive further compactness results. We start with

**Theorem 2.15.** *Let $(\gamma^\lambda)$ and $\gamma$ be as in Theorem 2.9. Then, up to a subsequence,*

$$\gamma^\lambda\to\gamma \quad \text{in } L^2_{\mathrm{loc}}H^2. \quad (2.57)$$

*Proof.* Fix $T>0$ and let $\lambda_0$ be as in Theorem 2.14. We wish to show that $\{\gamma^\lambda\}_\lambda$ is a Cauchy sequence in $L^2_T H^2$. Due to (2.36) there exists $\lambda_1=\lambda_1(\delta)>0$ such that for all $\lambda,\Lambda\in[0,\infty)$ satisfying $\lambda_1<\lambda<\Lambda<\infty$, we have for $\Delta\gamma:=\gamma^\Lambda-\gamma^\lambda$ that

$$\|\Delta\gamma\|_{L^\infty_T C^1}<\delta. \quad (2.58)$$

Let us instead consider $\lambda,\Lambda\in[0,\infty)$ satisfying $0<\lambda<\Lambda<\min\{\lambda_0,\lambda_1\}$, with $\lambda_0$, $\lambda_1$ as above. We first write

$$\begin{aligned}\frac{\varepsilon}{(L^\Lambda)^3}\int_0^T\int_0^1|\Delta\gamma_{ss}|^2\,\mathrm{d}s\,\mathrm{d}t &= \int_0^T\int_0^1\frac{\varepsilon}{(L^\Lambda)^3}\langle\gamma_{ss}^\Lambda,\Delta\gamma_{ss}\rangle-\frac{\varepsilon}{(L^\lambda)^3}\langle\gamma_{ss}^\lambda,\Delta\gamma_{ss}\rangle\,\mathrm{d}s\,\mathrm{d}t\\ &+\int_0^T\int_0^1\varepsilon((L^\lambda)^{-3}-(L^\Lambda)^{-3})\langle\gamma_{ss}^\lambda,\Delta\gamma_{ss}\rangle\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Subtracting (2.56) with penalization $\lambda$ and $\eta=\Delta\gamma$ from (2.56) with penalization $\Lambda$ and again $\eta=\Delta\gamma$, we rewrite the above equation as

$$\frac{\varepsilon}{(L^\Lambda)^3}\int_0^T\int_0^1|\Delta\gamma_{ss}|^2\,\mathrm{d}s\,\mathrm{d}t=A+B_1^\lambda-B_1^\Lambda+B_2^\lambda-B_2^\Lambda+B_3^\lambda-B_3^\Lambda, \quad (2.59)$$

where we have set

$$A = \int_0^T \int_0^1 \varepsilon\big((L^\lambda)^{-3} - (L^\Lambda)^{-3}\big)\langle \gamma_{ss}^\lambda, \Delta\gamma_{ss}\rangle \,\mathrm{d}s\,\mathrm{d}t,$$

and

$$\begin{aligned}
B_1^\lambda &= \int_0^T \int_0^1 \frac{1}{L^\lambda}\Big(1 - \frac{3\varepsilon}{2}(\kappa^\lambda)^2\Big)\langle \gamma_s^\lambda, \Delta\gamma_s\rangle \,\mathrm{d}s\,\mathrm{d}t \\
&\quad - \int_0^T \Big\langle \frac{\gamma^\lambda(t,1) - \gamma^\lambda(t,0)}{|\gamma^\lambda(t,1) - \gamma^\lambda(t,0)|^2}, \Delta\gamma(t,1) - \Delta\gamma(t,0)\Big\rangle \,\mathrm{d}t, \\
B_2^\lambda &= \int_0^T \frac{1}{2\widetilde{L}^\lambda}\int_0^1 \langle V^\lambda, (\widetilde{\gamma}_s^\lambda)^\perp\rangle\langle (\widetilde{\gamma}_s^\lambda)^\perp, \Delta\gamma\rangle \,\mathrm{d}s \\
&\quad + \frac{1}{2L^\lambda}\int_0^1 \langle V^\lambda, (\gamma_s^\lambda)^\perp\rangle\langle (\gamma_s^\lambda)^\perp, \Delta\gamma\rangle \,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \int_0^T \langle V^\lambda(t,0), \Delta\gamma(t,0)\rangle + \langle V^\lambda(t,1), \Delta\gamma(t,1)\rangle \,\mathrm{d}t, \\
B_3^\lambda &= \int_0^T \frac{1}{2L^\lambda}\int_0^1 \langle V^\lambda, (\widetilde{\gamma}_s^\lambda)^\perp\rangle\langle (\widetilde{\gamma}_s^\lambda)^\perp, \gamma_s^\lambda\rangle P_1(\gamma^\lambda, \Delta\gamma) \,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \int_0^T \frac{1}{2L^\lambda}\int_0^1 \langle V^\lambda, (\widetilde{\gamma}_s^\lambda)^\perp\rangle\langle \gamma^\lambda - \widetilde{\gamma}^\lambda, P_1(\gamma^\lambda, \Delta\gamma)(\gamma_{ss}^\lambda)^\perp + P_2(\gamma^\lambda, \Delta\gamma)(\gamma_s^\lambda)^\perp \\
&\qquad + (\Delta\gamma)_s^\perp\rangle \,\mathrm{d}s\,\mathrm{d}t \\
&\quad - \int_0^T \frac{1}{4(L^\lambda)^3}\int_0^1 \langle \gamma_s^\lambda, \Delta\gamma_s\rangle \,\mathrm{d}s \int_0^1 \langle V^\lambda, (\gamma_s^\lambda)^\perp\rangle\langle \gamma^\lambda - \widetilde{\gamma}^\lambda, (\gamma_s^\lambda)^\perp\rangle \,\mathrm{d}s\,\mathrm{d}t,
\end{aligned}$$

and $B_i^\Lambda$, $i \in \{1,2,3\}$, are defined by the same formula as $B_i^\lambda$, but with each $\lambda$ exchanged with $\Lambda$. We wish to bound the right-hand side of (2.59). This will be achieved by taking advantage of (2.58), thanks to which we can bound every $\Delta\gamma$- and $\Delta\gamma_s$-term appearing on the right-hand side of (2.59) by $\delta$ from above. For all the remaining terms, it will be enough to find an upper bound independent of $\Lambda$ and $\lambda$. We first begin with the $A$-term.

*$A$-term:* Since $L^\lambda, L^\Lambda \geq c > 0$ for a constant $c$ independent of $\lambda$ or $\Lambda$ (see also (2.7)), and due to the Lipschitz continuity of $x \mapsto 1/x^3$ away from 0, we have

$$|(L^\lambda)^{-3} - (L^\Lambda)^{-3}| \leq C|L^\lambda - L^\Lambda| = C\Big|\int_0^1 |\gamma_s^\lambda| - |\gamma_s^\Lambda| \,\mathrm{d}s\Big| \leq C\int_0^1 |\gamma_s^\lambda - \gamma_s^\Lambda| \,\mathrm{d}s \leq C\delta.$$

Using (2.7) we also see that

$$\begin{aligned}
|\langle \gamma_{ss}^\lambda, \Delta\gamma_{ss}\rangle| &\leq C(|\gamma_{ss}^\lambda|^2 + |\gamma_{ss}^\Lambda|^2) = C((L^\lambda)^2(\kappa^\lambda)^2 + (L^\Lambda)^2(\kappa^\Lambda)^2) \\
&\leq C((\kappa^\lambda)^2 + (\kappa^\Lambda)^2).
\end{aligned}$$

Consequently, by (2.8) it follows

$$|A| \leq C\delta \int_0^T \int_0^1 \varepsilon((\kappa^\lambda)^2 + (\kappa^\Lambda)^2) \,\mathrm{d}s \leq C(\varepsilon)\delta.$$

$B_1^\lambda$*-term:* By (2.7) we have

$$|\langle \gamma_s^\lambda, \Delta\gamma_s\rangle| \leq L^\lambda |\Delta\gamma_s| \leq C|\Delta\gamma_s|.$$

Consequently, by (2.8)

$$\begin{aligned}|B_1^\lambda| &\leq C\int_0^T\int_0^1 (1+\varepsilon(\kappa^\lambda)^2)|\Delta\gamma_s|\,\mathrm{d}s\,\mathrm{d}t + C\int_0^T |\Delta\gamma(t,1)| + |\Delta\gamma(t,0)|\,\mathrm{d}t\\ &\leq C(\varepsilon)\delta + C\delta = C(\varepsilon)\delta.\end{aligned}$$

$B_2^\lambda$*-term:* Due to (2.7), (2.8), and (2.20), we derive

$$\begin{aligned}|B_2^\lambda| &\leq C\int_0^T\int_0^1 |V^\lambda||\Delta\gamma|\,\mathrm{d}s\,\mathrm{d}t + \int_0^T |V^\lambda(t,0)||\Delta\gamma(t,0)| + |V^\lambda(t,1)||\Delta\gamma(t,1)|\,\mathrm{d}t\\ &\leq \sqrt{T}\delta\Big(\int_0^T\int_0^1 |V^\lambda|^2\,\mathrm{d}t\Big)^{\frac12} + \sqrt{T}\delta\Big(\int_0^T |V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2\,\mathrm{d}t\Big)^{\frac12}\\ &\leq C(\varepsilon,T)\delta.\end{aligned}$$

$B_3^\lambda$*-term:* The bound on the $B_3^\lambda$-term can be obtained, similarly to one of the $B_2^\lambda$-terms, by using (2.7), (2.8), (2.20), and noticing that $P_i(\gamma^\lambda,\Delta\gamma)$, $i=1,2$, can be bounded as follows:

$$\max_{i=1,2}|P_i(\gamma^\lambda,\Delta\gamma)| \leq C\int_0^1 |\langle\gamma_s^\lambda,\Delta\gamma_s\rangle|\,\mathrm{d}s \leq C\delta.$$

As all constants above do not depend on $\lambda$ or $\Lambda$, the same bounds also hold true for $B_i^\Lambda$, $i\in\{1,2,3\}$.

Exploiting again (2.7) and taking into account all the previous estimates, we eventually get

$$c(\varepsilon)\int_0^T\int_0^1 |\Delta\gamma_{ss}|^2\,\mathrm{d}s\,\mathrm{d}t \leq \frac{\varepsilon}{(L^\Lambda)^3}\int_0^T\int_0^1 |\Delta\gamma_{ss}|^2\,\mathrm{d}s\,\mathrm{d}t \leq C(\varepsilon,T)\delta. \tag{2.60}$$

By (2.58) and (2.60), we have that $\{\gamma^\lambda\}_\lambda$ is a Cauchy sequence in $L_T^2H^2$, whose limit being $\gamma$ is due to (2.36). ■

**Corollary 2.16.** *Let $\{\gamma^\lambda\}_\lambda$ and $\gamma$ be as in Theorem 2.9. As $\gamma^\lambda(t,\cdot)\in\mathcal{AC}$ for all $t$, and by (2.36) and (2.57), we see that $\gamma(t,\cdot)\in\mathcal{AC}$ for almost all $t$.*

We continue by employing a boot-strapping argument in order to show boundedness of higher order $s$-derivatives of $\{\gamma^\lambda\}_\lambda$.

**Lemma 2.17** (Boot-strapping). *Let $T>0$ be fixed and $\lambda>\lambda(\varepsilon,T)$ with $\lambda(\varepsilon,T)$ as in Remark 2.8. Then $\gamma_{ssss}^\lambda(t,\cdot)$ exists for all $t\in[0,T]$ and*

$$\|\gamma_{ssss}^\lambda\|_{L_T^{\frac32}L^{\frac32}} \leq C(\varepsilon,T) < \infty. \tag{2.61}$$

*Proof.* Let us, for the moment, fix $t \in [0, T]$. In order not to overburden the reader, we write $\gamma^\lambda(\cdot) := \gamma^\lambda(t, \cdot)$, $\widetilde{\gamma}^\lambda(\cdot) := \widetilde{\gamma}^\lambda(t, \cdot)$, $\kappa^\lambda(\cdot) := \kappa^\lambda(t, \cdot)$, $L^\lambda := L^\lambda(t)$, $\widetilde{L}^\lambda := \widetilde{L}^\lambda(t)$, and $V^\lambda(\cdot) := V^\lambda(t, \cdot)$. Furthermore, we define for any $f \colon [0, 1] \to \mathbb{R}^2$

$$D_s^{-1} f(s) := \int_0^s f(\widetilde{s}) \,\mathrm{d}\widetilde{s},$$

and $D_s^{-(n+1)}$ recursively as $D_s^{-1} D_s^{-n}$. Integrating by parts in (2.53) and (2.54) for a fixed $\eta \in C_c^\infty([0, 1]; \mathbb{R}^2)$ leads to

$$E(\gamma^\lambda, \eta) = \int_0^1 \Big\langle \frac{\varepsilon}{(L^\lambda)^3} \gamma^\lambda_{ss} + D_s^{-1} A_1, \eta_{ss} \Big\rangle \,\mathrm{d}s, \tag{2.62}$$

$$\mathrm{Diss}(\gamma^\lambda, \eta) = \int_0^1 \langle D_s^{-2} A_2, \eta_{ss} \rangle \,\mathrm{d}s, \tag{2.63}$$

where

$$\begin{aligned}
A_1 &:= -\frac{1}{L^\lambda}\Big(1 - \frac{3\varepsilon}{2}\kappa_\gamma^2\Big)\gamma^\lambda_s, \\
A_2 &:= \frac{1}{2\widetilde{L}^\lambda}\langle V^\lambda, (\widetilde{\gamma}^\lambda_s)^\perp\rangle (\widetilde{\gamma}^\lambda_s)^\perp + \frac{1}{2L^\lambda}\langle V^\lambda, (\gamma^\lambda_s)^\perp\rangle (\gamma^\lambda_s)^\perp.
\end{aligned}$$

Setting

$$\begin{aligned}
A_3 &:= \frac{1}{2\widetilde{L}^\lambda (L^\lambda)^2}\langle V^\lambda, (\widetilde{\gamma}^\lambda_s)^\perp\rangle \big(\langle (\widetilde{\gamma}^\lambda_s)^\perp, \gamma^\lambda_s\rangle + \langle \gamma^\lambda - \widetilde{\gamma}^\lambda, (\gamma^\lambda_{ss})^\perp\rangle\big), \\
A_4 &:= \frac{1}{2\widetilde{L}^\lambda (L^\lambda)^2}\langle V^\lambda, (\widetilde{\gamma}^\lambda_s)^\perp\rangle \langle \gamma^\lambda - \widetilde{\gamma}^\lambda, (\gamma^\lambda_s)^\perp\rangle, \\
A_5 &:= \frac{1}{4(L^\lambda)^3}\int_0^1 \langle V^\lambda, (\gamma^\lambda_s)^\perp\rangle \langle \gamma^\lambda - \widetilde{\gamma}^\lambda, (\gamma^\lambda_s)^\perp\rangle \,\mathrm{d}s, \\
A_6 &:= -\frac{1}{2\widetilde{L}^\lambda}\langle V^\lambda, (\widetilde{\gamma}^\lambda_s)^\perp\rangle (\gamma^\lambda - \widetilde{\gamma}^\lambda)^\perp,
\end{aligned}$$

we can write

$$\mathrm{Err}(\gamma^\lambda, \eta) = \int_0^1 (L^\lambda)^2 A_3 P_1(\gamma^\lambda, \eta) + (L^\lambda)^2 A_4 P_2(\gamma^\lambda, \eta) - A_5 \langle \gamma^\lambda_s, \eta_s\rangle - \langle A_6, \eta_s\rangle \,\mathrm{d}s.$$

Using the definition of $P_1$ (see (2.50)), Fubini's theorem, and integrating by parts, it follows that

$$\begin{aligned}
\int_0^1 (L^\lambda)^2 A_3 P_1(\gamma^\lambda, \eta) \,\mathrm{d}s &= \int_0^1 A_3(s)\Big(s\int_0^1 \langle \gamma^\lambda_s(\widetilde{s}), \eta_s(\widetilde{s})\rangle \,\mathrm{d}\widetilde{s} - \int_0^s \langle \gamma^\lambda_s(\widetilde{s}), \eta_s(\widetilde{s})\rangle \,\mathrm{d}\widetilde{s}\Big) \,\mathrm{d}s \\
&= \int_0^1 sA_3(s) \,\mathrm{d}s \int_0^1 \langle \gamma^\lambda_s, \eta_s\rangle \,\mathrm{d}\widetilde{s} \\
&\quad - \int_0^1 \Big(\int_{\widetilde{s}}^1 A_3(s) \,\mathrm{d}s\Big) \langle \gamma^\lambda_s(\widetilde{s}), \eta_s(\widetilde{s})\rangle \,\mathrm{d}\widetilde{s} \\
&= \int_0^1 \Big\langle D_s^{-1}\Big\{\Big(\int_{\widetilde{s}}^1 A_3(s) \,\mathrm{d}s - \int_0^1 sA_3(s) \,\mathrm{d}s\Big)\gamma^\lambda_s\Big\}, \eta_{ss}\Big\rangle \,\mathrm{d}\widetilde{s}.
\end{aligned}$$

Hence, employing a similar argument for the remaining terms, we arrive at

$$\begin{aligned}\operatorname{Err}(\gamma^\lambda,\eta) = \int_0^1 \Big\langle D_s^{-1}\Big\{\Big(\int_s^1 A_3\,\mathrm{d}\widetilde{s} - \int_0^1 \widetilde{s}A_3\,\mathrm{d}\widetilde{s} + A_4 \\ - \int_0^1 A_4\,\mathrm{d}\widetilde{s} + A_5\Big)\gamma_s^\lambda + A_6\Big\}, \eta_{ss}\Big\rangle\,\mathrm{d}s.\end{aligned} \tag{2.64}$$

By (2.62), (2.63), (2.64), and the Euler–Lagrange equation (see (2.52)), there exist $v$, $w \in \mathbb{R}^2$ such that

$$-\frac{\varepsilon}{(L^\lambda)^3}\gamma_{ss}^\lambda = v + ws + D_s^{-1}A_7 + D_s^{-2}A_2, \tag{2.65}$$

where

$$A_7 := A_1 + \Big(\int_s^1 A_3\,\mathrm{d}\widetilde{s} - \int_0^1 \widetilde{s}A_3\,\mathrm{d}\widetilde{s} + A_4 - \int_0^1 A_4\,\mathrm{d}\widetilde{s} + A_5\Big)\gamma_s^\lambda + A_6.$$

As the right-hand side of (2.65) is weakly differentiable (in $s$) we can further differentiate $\gamma_{ss}^\lambda$ to obtain

$$-\frac{\varepsilon}{(L^\lambda)^3}\gamma_{sss}^\lambda = w + A_7 + D_s^{-1}A_2. \tag{2.66}$$

By the very definition of $A_7$ and thanks to the regularity of $\gamma^\lambda$, (2.66) shows that $\gamma^\lambda$ is four times weakly differentiable (in $s$). Consequently, we can compute

$$-\frac{\varepsilon}{(L^\lambda)^3}\gamma_{ssss}^\lambda = (A_7)_s + A_2.$$

For convenience, we will now split up the right-hand side of the equation above as follows:

$$-\frac{\varepsilon}{(L^\lambda)^3}\gamma_{ssss}^\lambda = \sum_{i=1}^5 B_i, \tag{2.67}$$

where

$$\begin{aligned}
B_1 &:= (A_1)_s = \frac{1}{L^\lambda}\Big(3\varepsilon\kappa^\lambda\kappa_s^\lambda\gamma_s^\lambda - \Big(1 - \frac{3\varepsilon}{2}(\kappa^\lambda)^2\Big)\gamma_{ss}^\lambda\Big),\\
B_2 &:= A_2,\\
B_3 &:= A_3\gamma_s^\lambda + \Big(\int_s^1 A_3\,\mathrm{d}\widetilde{s} - \int_0^1 \widetilde{s}A_3\,\mathrm{d}\widetilde{s}\Big)\gamma_{ss}^\lambda,\\
B_4 &:= (A_4)_s\gamma_s^\lambda + \Big(A_4 - \int_0^1 A_4\,\mathrm{d}\widetilde{s}\Big)\gamma_{ss}^\lambda,\\
B_5 &:= (A_5)_s\gamma_s^\lambda + A_5\gamma_{ss}^\lambda + (A_6)_s.
\end{aligned}$$

We will now estimate each term on the right-hand side of (2.67) separately, where we repeatedly make use of (2.7), (2.8), (2.30), and of the boundedness implied by the convergence in (2.36).

*$B_1$-term*:

$$\int_0^1 |B_1|^{\frac{3}{2}}\,\mathrm{d}s \le C(\varepsilon)\int_0^1 |\kappa^\lambda|^{\frac{3}{2}}|\kappa_s^\lambda|^{\frac{3}{2}} + |\gamma_{ss}^\lambda|^{\frac{3}{2}} + |\kappa^\lambda|^3|\gamma_{ss}^\lambda|^{\frac{3}{2}}\,\mathrm{d}s$$
$$\le C(\varepsilon)\int_0^1 |\gamma_{ss}^\lambda|^{\frac{3}{2}}(|\gamma_{sss}^\lambda|^{\frac{3}{2}}+1) + |\gamma_{ss}^\lambda|^{\frac{9}{2}}\,\mathrm{d}s$$
$$\le C(\varepsilon)\int_0^1 1 + |\gamma_{sss}^\lambda|^{\frac{9}{4}} + |\gamma_{ss}^\lambda|^{\frac{9}{2}}\,\mathrm{d}s, \tag{2.68}$$

where in the third line we employed Young's inequality with powers 3 and $\frac{3}{2}$. Using the interpolation inequality (2.29) with parameters $i=2$, $j=3$, $m=4$, $p=\frac{3}{2}$, $q=\frac{9}{4}$, $\theta=\frac{11}{18}$, and eventually Young's inequality with arbitrary constant $\delta>0$ and powers $\frac{12}{11}$ and 12 leads to

$$\|\gamma_{sss}^\lambda\|_{L^{\frac{9}{4}}}^{\frac{9}{4}} \le C\Big(\|\gamma_{ssss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{11}{18}}\|\gamma_{ss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{7}{18}} + \|\gamma_{ss}^\lambda\|_{L^{\frac{3}{2}}}\Big)^{\frac{9}{4}}$$
$$\le C\Big(\|\gamma_{ssss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{11}{8}}\|\gamma_{ss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{7}{8}} + \|\gamma_{ss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{9}{4}}\Big)$$
$$\le C\Big(\delta\|\gamma_{ssss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + C(\delta)\|\gamma_{ss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{21}{2}} + \|\gamma_{ss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{9}{4}}\Big)$$
$$\le C\delta\|\gamma_{ssss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + C(\delta,\varepsilon). \tag{2.69}$$

Furthermore, by (2.29) with parameters $i=j=2$, $m=4$, $p=\frac{3}{2}$, $q=\frac{9}{2}$, $\theta=\frac{2}{9}$, and eventually Young's inequality with arbitrary constant $\delta>0$ and powers $\frac{3}{2}$ and 3, we derive that

$$\|\gamma_{ss}^\lambda\|_{L^{\frac{9}{2}}}^{\frac{9}{2}} \le C\Big(\|\gamma_{ssss}^\lambda\|_{L^{\frac{3}{2}}}\|\gamma_{ss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{7}{2}} + \|\gamma_{ss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{9}{2}}\Big)$$
$$\le C\Big(\delta\|\gamma_{ssss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + C(\delta)\|\gamma_{ss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{21}{2}} + \|\gamma_{ss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{9}{2}}\Big)$$
$$\le C\delta\|\gamma_{ssss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + C(\delta,\varepsilon). \tag{2.70}$$

By (2.68), (2.69), and (2.70) we eventually get

$$\int_0^1 |B_1|^{\frac{3}{2}}\,\mathrm{d}s \le C(\varepsilon,\delta) + C(\varepsilon)\delta\|\gamma_{ssss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}.$$

*$B_2$-term*:

$$\int_0^1 |B_2|^{\frac{3}{2}}\,\mathrm{d}s \le C\|B_2\|_{L^2}^{\frac{3}{2}} \le C\|V^\lambda\|_{L^2}^{\frac{3}{2}} \le C(1+\|V^\lambda\|_{L^2}^2).$$

*$B_3$-term*: By the definition of $A_3$, (2.7), the boundedness implied by the convergence in (2.36), and Young's inequality with arbitrary constant $\delta>0$ and powers $\frac{4}{3}$ and 4 we estimate

$$|A_3|^{\frac{3}{2}} \le C|V^\lambda|^{\frac{3}{2}}(1+C(T)|\gamma_{ss}^\lambda|)^{\frac{3}{2}} \le C(T)|V^\lambda|^{\frac{3}{2}}(1+|\gamma_{ss}^\lambda|^{\frac{3}{2}})$$
$$\le C(T)|V^\lambda|^{\frac{3}{2}} + C(T,\delta)|V^\lambda|^2 + \delta|\gamma_{ss}^\lambda|^6 \le C(T,\delta)(1+|V^\lambda|^2) + \delta|\gamma_{ss}^\lambda|^6.$$

Further, by (2.8), the Cauchy–Schwarz inequality, and Young's inequality with powers $\frac{4}{3}$ and 4, we have

$$\begin{aligned}\Big(\int_0^1 |A_3|\,\mathrm{d}\widetilde{s}\Big)^{\frac{3}{2}} &\leq C(T)\Big(\int_0^1 |V^\lambda|(1+|\gamma^\lambda_{ss}|)\,\mathrm{d}\widetilde{s}\Big)^{\frac{3}{2}} \\ &\leq C(T)\Big(\|V^\lambda\|_{L^1}^{\frac{3}{2}} + \|V^\lambda\|_{L^2}^{\frac{3}{2}}\|\gamma^\lambda_{ss}\|_{L^2}^{\frac{3}{2}}\Big) \\ &\leq C(T)\Big(\|V^\lambda\|_{L^2}^2 + 1 + \|\gamma^\lambda_{ss}\|_{L^2}^6\Big) \leq C(\varepsilon,T)(1+\|V^\lambda\|_{L^2}^2).\end{aligned}$$

With the last two estimates, (2.8), and the definition of $B_3$, we derive that

$$\begin{aligned}\int_0^1 |B_3|^{\frac{3}{2}}\,\mathrm{d}s &\leq C\int_0^1 |A_3|^{\frac{3}{2}} + \Big(\int_0^1 |A_3|\,\mathrm{d}\widetilde{s}\Big)^{\frac{3}{2}}|\gamma^\lambda_{ss}|^{\frac{3}{2}}\,\mathrm{d}s \\ &\leq C\int_0^1 C(T,\delta)(1+|V^\lambda|^2) + \delta|\gamma^\lambda_{ss}|^6 + C(\varepsilon,T)(1+\|V^\lambda\|_{L^2}^2)|\gamma^\lambda_{ss}|^{\frac{3}{2}}\,\mathrm{d}s \\ &\leq C(\varepsilon,T,\delta)\big(1+\|V^\lambda\|_{L^2}^2\big) + C\delta\|\gamma^\lambda_{ss}\|_{L^6}^6. \end{aligned} \tag{2.71}$$

Making use of the interpolation inequality (2.29) with parameters $i=2$, $j=2$, $m=4$, $p=\frac{3}{2}$, $q=6$, $\theta=\frac{1}{4}$, it follows

$$\begin{aligned}\|\gamma^\lambda_{ss}\|_{L^6}^6 &\leq C\Big(\|\gamma^\lambda_{ssss}\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}\|\gamma^\lambda_{ss}\|_{L^{\frac{3}{2}}}^{\frac{9}{2}} + \|\gamma^\lambda_{ss}\|_{L^{\frac{3}{2}}}^6\Big) \\ &\leq C\Big(\|\gamma^\lambda_{ssss}\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}\|\gamma^\lambda_{ss}\|_{L^2}^{\frac{9}{2}} + \|\gamma^\lambda_{ss}\|_{L^2}^6\Big).\end{aligned} \tag{2.72}$$

Combining (2.71) and (2.72) eventually leads to

$$\int_0^1 |B_3|^{\frac{3}{2}}\,\mathrm{d}s \leq C(\varepsilon,T,\delta)(1+\|V^\lambda\|_{L^2}^2) + C(\varepsilon)\delta\|\gamma^\lambda_{ssss}\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}.$$

*$B_4$-term*:

$$\int_0^1 |B_4|^{\frac{3}{2}}\,\mathrm{d}s \leq C\int_0^1 \Big(|(A_4)_s|^{\frac{3}{2}} + \Big(\int_0^1 |A_4|\,\mathrm{d}\widetilde{s}\Big)^{\frac{3}{2}}|\gamma^\lambda_{ss}|^{\frac{3}{2}} + |A_4|^{\frac{3}{2}}|\gamma^\lambda_{ss}|^{\frac{3}{2}}\Big)\,\mathrm{d}s. \tag{2.73}$$

Using

$$\begin{aligned}2\widetilde{L}^\lambda(L^\lambda)^2(A_4)_s = {}&\langle\gamma^\lambda_s - \widetilde{\gamma}^\lambda_s, (\widetilde{\gamma}^\lambda_s)^\perp\rangle\langle V^\lambda, (\gamma^\lambda_s)^\perp\rangle + \langle V^\lambda, (\widetilde{\gamma}^\lambda_{ss})^\perp\rangle\langle\gamma^\lambda - \widetilde{\gamma}^\lambda, (\gamma^\lambda_s)^\perp\rangle \\ &+ \langle V^\lambda, (\widetilde{\gamma}^\lambda_s)^\perp\rangle\langle\gamma^\lambda_s - \widetilde{\gamma}^\lambda_s, (\gamma^\lambda_s)^\perp\rangle + \langle V^\lambda, (\widetilde{\gamma}^\lambda_s)^\perp\rangle\langle\gamma^\lambda - \widetilde{\gamma}^\lambda, (\gamma^\lambda_{ss})^\perp\rangle\end{aligned}$$

and (2.72), it follows

$$\begin{aligned}\int_0^1 |(A_4)_s|^{\frac{3}{2}}\,\mathrm{d}s &\leq C(T)\int_0^1 |V^\lambda|^{\frac{3}{2}}(1+|\gamma^\lambda_{ss}|^{\frac{3}{2}})\,\mathrm{d}s \\ &\leq C(T)\int_0^1 1 + C(\delta)|V^\lambda|^2 + \delta|\gamma^\lambda_{ss}|^6\,\mathrm{d}s \\ &= C(T,\delta)(1+\|V^\lambda\|_{L^2}^2) + C(\varepsilon,T)\delta\|\gamma^\lambda_{ssss}\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}.\end{aligned} \tag{2.74}$$

Furthermore, by (2.72) and Young's inequality, we have

$$\begin{aligned}
&\int_0^1 \Big(\int_0^1 |A_4|\,\mathrm{d}\tilde{s}\Big)^{\frac{3}{2}} |\gamma_{ss}^\lambda|^{\frac{3}{2}} + |A_4|^{\frac{3}{2}} |\gamma_{ss}^\lambda|^{\frac{3}{2}}\,\mathrm{d}s \\
&\quad \le C(T)\Big(\|V^\lambda\|_{L^1}^{\frac{3}{2}} \|\gamma_{ss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \int_0^1 |V^\lambda|^{\frac{3}{2}} |\gamma_{ss}^\lambda|^{\frac{3}{2}}\,\mathrm{d}s\Big) \\
&\quad \le C(\varepsilon, T)(1 + \|V^\lambda\|_{L^2}^2) + C(T,\delta)\|V^\lambda\|_{L^2}^2 + \delta\|\gamma_{ss}^\lambda\|_{L^6}^6 \\
&\quad \le C(\varepsilon, T, \delta)(1 + \|V^\lambda\|_{L^2}^2) + C(\varepsilon, T)\delta\|\gamma_{ssss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}.
\end{aligned} \tag{2.75}$$

Combining (2.73), (2.74), and (2.75), we have

$$\int_0^1 |B_4|^{\frac{3}{2}}\,\mathrm{d}s \le C(\varepsilon, T, \delta)(1 + \|V^\lambda\|_{L^2}^2) + C(\varepsilon, T)\delta\|\gamma_{ssss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}.$$

*$B_5$-term*: Repeating the same argument as in the previous steps, we derive that

$$\int_0^1 |B_5|^{\frac{3}{2}}\,\mathrm{d}s \le C(\varepsilon, T, \delta)(1 + \|V^\lambda\|_{L^2}^2) + C(\varepsilon, T)\delta\|\gamma_{ssss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}.$$

Hence, with (2.67) and the bounds we found for the $B_i$-terms, we have

$$c(\varepsilon)\|\gamma_{ssss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} \le C(\varepsilon, T, \delta)\big(1 + \|V^\lambda\|_{L^2}^2\big) + C(\varepsilon, T)\delta\|\gamma_{ssss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}$$

and hence, for $\delta > 0$ small enough,

$$\|\gamma_{ssss}^\lambda\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} \le C(\varepsilon, T)\big(1 + \|V^\lambda\|_{L^2}^2\big).$$

By the arbitrariness of $t \in [0, T]$, we have

$$\|\gamma_{ssss}^\lambda(t,\cdot)\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} \le C(\varepsilon, T)\big(1 + \|V^\lambda(t,\cdot)\|_{L^2}^2\big)$$

for all $t \in [0, T]$. Integrating the above equation over $t \in [0, T]$ and employing (2.20) finally leads to (2.61). ∎

The previously derived bound results in the following compactness result:

**Theorem 2.18.** *Let $\{\gamma^\lambda\}_\lambda$ and $\gamma$ be as in Theorem 2.9, then*

$$\gamma^\lambda \rightharpoonup \gamma \quad \text{weakly in } L^{\frac{3}{2}}_{\mathrm{loc}} W^{4,\frac{3}{2}}, \tag{2.76}$$

$$\gamma^\lambda \to \gamma \quad \text{in } L^{\frac{39}{23}}_{\mathrm{loc}} W^{3,\frac{39}{23}}. \tag{2.77}$$

*In particular, for almost all $t \in [0, \infty)$,*

$$\gamma^\lambda(t,\cdot) \to \gamma(t,\cdot) \quad \text{in } C^3. \tag{2.78}$$

*Proof.* Let us fix $T > 0$. Then, (2.61) directly leads to

$$\gamma^\lambda \rightharpoonup \gamma \quad \text{weakly in } L^{\frac{3}{2}}(0,T;W^{4,\frac{3}{2}}).$$

We will now show that $\{\gamma^\lambda\}_\lambda$ is a Cauchy sequence in $L_T^{\frac{39}{23}} W^{2,\frac{39}{23}}$. Fix $\delta \geq \widetilde{\delta} > 0$. As $\frac{39}{23} \leq 2$, we know by Theorem 2.15 that there exists $\lambda_0 = \lambda_0(\widetilde{\delta}) > 0$ big enough such that for any $\lambda_0 < \lambda < \Lambda < \infty$, we have for $\Delta\gamma := \gamma^\Lambda - \gamma^\lambda$ that

$$\|\Delta\gamma\|_{L_T^{\frac{39}{23}} W^{2,\frac{39}{23}}} \leq C\|\Delta\gamma\|_{L^2(0,T;H^2)} < \widetilde{\delta}. \tag{2.79}$$

Furthermore, using the interpolation inequality (2.29) with parameters $i = 2$, $j = 3$, $m = 4$, $p = \frac{3}{2}$, $q = \frac{39}{23}$, $\theta = \frac{7}{13}$, we have

$$\|\Delta\gamma_{sss}\|_{L^{\frac{39}{23}}} \leq C\Big(\|\Delta\gamma_{ssss}\|_{L^{\frac{3}{2}}}^{\frac{7}{13}} \|\Delta\gamma_{ss}\|_{L^{\frac{3}{2}}}^{\frac{6}{13}} + \|\Delta\gamma_{ss}\|_{L^{\frac{3}{2}}}\Big).$$

Hence, by Hölder's inequality, Lemma 2.17, and (2.79), we derive for all $\lambda_0 < \lambda < \Lambda < \infty$

$$\begin{aligned}
\int_0^T \|\Delta\gamma_{sss}\|_{L^{\frac{39}{23}}}^{\frac{39}{23}} \,\mathrm{d}t &\leq C\Big(\int_0^T \|\Delta\gamma_{ssss}\|_{L^{\frac{3}{2}}}^{\frac{21}{23}} \|\Delta\gamma_{ss}\|_{L^{\frac{3}{2}}}^{\frac{18}{23}} + \|\Delta\gamma_{ss}\|_{L^{\frac{3}{2}}}^{\frac{39}{23}} \,\mathrm{d}t\Big) \\
&\leq C\|\Delta\gamma_{ssss}\|_{L^{\frac{3}{2}}(0,T;L^{\frac{3}{2}})} \|\Delta\gamma_{ss}\|_{L^2(0,T;L^2)} + C(T)\|\Delta\gamma_{ss}\|_{L^2(0,T;L^2)}^{\frac{39}{46}} \\
&\leq C(\varepsilon,T)(\widetilde{\delta} + \widetilde{\delta}^{\frac{39}{46}}).
\end{aligned}$$

Therefore, for $\widetilde{\delta}$ small enough, we have for $\lambda_0 < \lambda < \Lambda \leq \tau_0$ that

$$\|\Delta\gamma_{sss}\|_{L_T^{\frac{39}{23}} L^{\frac{39}{23}}} < \eta. \tag{2.80}$$

Due to (2.79) and (2.80), we conclude with (2.77) through a diagonal sequence argument, similar to the one in the proof of Theorem 2.9. Finally, (2.78) directly follows from the Sobolev embedding theorem. ∎

Our last compactness result is derived by employing the coupling relation (2.19). At the same time, we will also derive the equation satisfied by the tangential component of the velocity of $\gamma$.

**Theorem 2.19.** *Let $\{\gamma^\lambda\}_\lambda$ and $\gamma$ be as in Theorem 2.9 and let $V = \gamma_t$. Then, up to a subsequence, it holds that*

$$\widehat{L}^\lambda \rightharpoonup L \qquad \text{weakly in } H^1_{\mathrm{loc}}([0,\infty)), \tag{2.81}$$

$$\langle V^\lambda, \widetilde{\gamma}_s^\lambda + \gamma_s^\lambda\rangle \rightharpoonup 2LV^\top \quad \text{weakly in } L^{\frac{3}{2}}_{\mathrm{loc}}([0,\infty); W^{1,\frac{3}{2}}), \tag{2.82}$$

*where $\widehat{L}^\lambda$ is as in Definition 2.2, $L$ is the length of $\gamma$, and $V^\top := \langle \gamma_t, \tau\rangle$ is the tangential component of the velocity of $\gamma$ with $\tau$ being the unit tangent vector field of $\gamma$. Furthermore, for almost all $t \in [0,\infty)$ and $s \in [0,1]$, it holds that*

$$V_s^\top(t,s) = L_t(t) + L(t)\kappa(t,s)V^\perp(t,s), \tag{2.83}$$

*where $L_t$ denotes the weak derivative of $L$, $\kappa$ is the curvature of $\gamma$, and $V^\perp$ is the orthogonal component of the velocity of $\gamma$ with $\nu$ being the unit normal vector field of $\gamma$.*

*Proof.* Let us fix $T > 0$. We start by integrating (2.19), for fixed $t$, over $s \in [0,1]$. Consequently, solving for $\widehat{L}_t^\lambda$ leads to

$$\widehat{L}_t^\lambda = \frac{1}{\widetilde{L}^\lambda + L^\lambda}\Big(\langle V^\lambda, \widetilde{\gamma}_s^\lambda + \gamma_s^\lambda\rangle|_{s=0}^1 - \int_0^1 \langle V^\lambda, \widetilde{L}^\lambda \widetilde{\kappa}^\lambda (\widetilde{\gamma}_s^\lambda)^\perp + L^\lambda \kappa^\lambda (\gamma_s^\lambda)^\perp\rangle \,\mathrm{d}s\Big).$$

Integrating the square of the equation above over $t \in [0,T]$; using (2.7), (2.8), (2.20); and the Cauchy–Schwarz inequality, we see that

$$\begin{aligned}
\int_0^T (\widehat{L}_t^\lambda)^2 \,\mathrm{d}t &\le C \int_0^T |V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2 \,\mathrm{d}t \\
&\quad + C \int_0^T \Big(\int_0^1 |\widetilde{\kappa}^\lambda V^\lambda| \,\mathrm{d}s\Big)^2 + \Big(\int_0^1 |\kappa^\lambda V^\lambda| \,\mathrm{d}s\Big)^2 \,\mathrm{d}t \\
&\le C \int_0^T |V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2 \,\mathrm{d}t \\
&\quad + C \int_0^T \int_0^1 (\widetilde{\kappa}^\lambda)^2 + (\kappa^\lambda)^2 \,\mathrm{d}s \int_0^1 |V^\lambda|^2 \,\mathrm{d}s \,\mathrm{d}t \\
&\le C \int_0^T |V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2 \,\mathrm{d}t + C(\varepsilon) \int_0^T \int_0^1 |V^\lambda|^2 \,\mathrm{d}s \,\mathrm{d}t \le C(\varepsilon, T).
\end{aligned}$$

Therefore, $\{\widehat{L}^\lambda\}_\lambda$ is uniformly bounded in $H^1(0,T)$ and

$$\widehat{L}^\lambda \rightharpoonup L \quad \text{weakly in } H^1(0,T).$$

We now take the absolute value of both sides of (2.19) to the power $\frac{3}{2}$ and integrate over $t \in [0,T]$ and $s \in [0,1]$. By the $L^2$ bound on $\{\widehat{L}_t^\lambda\}$, (2.20), (2.72), and (2.61), we have

$$\begin{aligned}
\int_0^T \int_0^1 |\langle V^\lambda, \widetilde{\gamma}_s^\lambda + \gamma_s^\lambda\rangle_s|^{\frac{3}{2}} \,\mathrm{d}s \,\mathrm{d}t &\le C \int_0^T (\widehat{L}_t^\lambda)^{\frac{3}{2}} \,\mathrm{d}t + C \int_0^T \int_0^1 |V^\lambda|^{\frac{3}{2}} |\gamma_{ss}^\lambda|^{\frac{3}{2}} \,\mathrm{d}s \,\mathrm{d}t \\
&\le C(T) \|\widehat{L}_t^\lambda\|_{L^2}^{\frac{3}{2}} + \int_0^T \|V^\lambda\|_{L^2}^2 + \|\gamma_{ss}^\lambda\|_{L^6}^6 \,\mathrm{d}t \\
&\le C(\varepsilon, T),
\end{aligned}$$

where in the second line we have employed Young's inequality. Hence, $\langle V^\lambda, \widetilde{\gamma}_s^\lambda + \gamma_s^\lambda\rangle_s$ is bounded in $L^{\frac{3}{2}}([0,T]; L^{\frac{3}{2}})$ and therefore, up to a subsequence,

$$\langle V^\lambda, \widetilde{\gamma}_s^\lambda + \gamma_s^\lambda\rangle \rightharpoonup \langle V, 2\gamma_s\rangle = 2LV^\top \quad \text{weakly in } L^{\frac{3}{2}}(0,T; W^{1,\frac{3}{2}}).$$

By a diagonal argument and the reasoning above, we see that (2.81) and (2.82) hold true. Finally, the equation in (2.83) follows by combining the convergences in (2.81) and (2.82) with the coupling relation (2.19). ∎

### 2.3. Convergence

In this subsection we derive the equations stated in (1.6). We start by employing the compactness results of the previous subsection in order to pass to the limit $\lambda \to \infty$ in the weak formulation (2.56) of the time-discrete evolution.

**Theorem 2.20** (Weak form of the geometric evolution). *Let $\gamma$ be as in Theorem 2.9. For all $\eta \in C_c^\infty([0,\infty); C^\infty)$, it holds that*

$$
\begin{aligned}
0 = &\int_0^\infty \int_0^1 \frac{\varepsilon}{L^3}\langle \gamma_{ss}, \eta_{ss}\rangle + \frac{1}{L}(1 - \frac{3\varepsilon}{2}\kappa^2)\langle \gamma_s, \eta_s\rangle \, \mathrm{d}s \, \mathrm{d}t \\
&- \int_0^\infty \Big\langle \frac{\gamma(t,1)-\gamma(t,0)}{|\gamma(t,1)-\gamma(t,0)|^2}, \eta(t,1)-\eta(t,0)\Big\rangle \mathrm{d}t + \int_0^\infty \int_0^1 \frac{1}{L}\langle V, \gamma_s^\perp\rangle\langle \gamma_s^\perp, \eta\rangle \, \mathrm{d}s \, \mathrm{d}t \\
&+ \int_0^\infty \langle V(t,0), \eta(t,0)\rangle + \langle V(t,1), \eta(t,1)\rangle \, \mathrm{d}t. \qquad (2.84)
\end{aligned}
$$

*Proof.* By (2.56), in order to show (2.84) it is enough to prove the following convergences:

$$
\int_0^T E(\gamma^\lambda, \eta)\, \mathrm{d}t \to \int_0^T E(\gamma, \eta), \qquad (2.85)
$$

$$
\begin{aligned}
\int_0^T \mathrm{Diss}(\gamma^\lambda, \eta)\, \mathrm{d}t \to &\int_0^T \int_0^1 \frac{1}{L}\langle V, \gamma_s^\perp\rangle\langle \gamma_s^\perp, \eta\rangle \, \mathrm{d}s \, \mathrm{d}t \\
&+ \int_0^T \langle V(t,0), \eta(t,0)\rangle + \langle V(t,1), \eta(t,1)\rangle \, \mathrm{d}t \qquad (2.86)
\end{aligned}
$$

$$
\int_1^T \mathrm{Err}(\gamma^\lambda, \eta)\, \mathrm{d}t \to 0, \qquad (2.87)
$$

where $\{\gamma^\lambda\}_\lambda$ is as in Theorem 2.9 and $E$, $D$, and Err are defined in (2.53), (2.54), and (2.55), respectively. In the following, let $T > 0$ such that $\mathrm{supp}(\eta) \subset [0,T] \times [0,1]$. Here, $\mathrm{supp}(\eta)$ denotes the support of $\eta$.

*Proof of* (2.85): By (2.7) and the convergence in (2.36), we see that

$$
\begin{aligned}
&\Big|\Big\langle \frac{\gamma^\lambda(t,1)-\gamma^\lambda(t,0)}{|\gamma^\lambda(t,1)-\gamma^\lambda(t,0)|}, \eta(t,1)-\eta(t,0)\Big\rangle - \Big\langle \frac{\gamma(t,1)-\gamma(t,0)}{|\gamma(t,1)-\gamma(t,0)|}, \eta(t,1)-\eta(t,0)\Big\rangle\Big| \\
&\quad \le C|\gamma^\lambda(t,1) - \gamma^\lambda(t,0) - \gamma(t,1) + \gamma(t,0)| \|\eta\|_{L_T^\infty L^\infty} \\
&\quad \le C(\eta)\|\gamma^\lambda - \gamma\|_{L_T^\infty L^\infty} \to 0 \text{ as } \lambda \to \infty. \qquad (2.88)
\end{aligned}
$$

Employing (2.7) and (2.8), it follows

$$
\int_0^1 \frac{\varepsilon}{(L^\lambda)^3}\langle \gamma_{ss}^\lambda, \eta_{ss}\rangle + \frac{1}{L^\lambda}\Big(1 - \frac{3\varepsilon}{2}(\kappa^\lambda)^2\Big)\langle \gamma_s^\lambda, \eta_s\rangle \, \mathrm{d}s \le C(\varepsilon, \eta).
$$

Hence by (2.78), the dominated convergence theorem, and (2.88), we see that the convergence in (2.85) holds true.

*Proof of* (2.86): By (2.36) we have that

$$\frac{\langle(\widetilde{\gamma}_s^\lambda)^\perp, \eta\rangle}{\widetilde{L}^\lambda}(\widetilde{\gamma}_s^\lambda)^\perp \to \frac{\langle\gamma_s^\perp, \eta\rangle}{L}\gamma_s^\perp,$$
$$\frac{\langle(\gamma_s^\lambda)^\perp, \eta\rangle}{L^\lambda}(\gamma_s^\lambda)^\perp \to \frac{\langle\gamma_s^\perp, \eta\rangle}{L}\gamma_s^\perp$$

strongly in $L_T^\infty L^\infty$ and therefore also strongly in $L_T^2 L^2$. Hence, by weak-strong convergence, we derive

$$\int_0^T \int_0^1 \Big\langle V^\lambda, \frac{\langle(\widetilde{\gamma}_s^\lambda)^\perp, \eta\rangle}{2\widetilde{L}^\lambda}(\widetilde{\gamma}_s^\lambda)^\perp\Big\rangle + \Big\langle V^\lambda, \frac{\langle(\gamma_s^\lambda)^\perp, \eta\rangle}{2L^\lambda}(\gamma_s^\lambda)^\perp\Big\rangle \,\mathrm{d}s\,\mathrm{d}t$$
$$\to \int_0^T \int_0^1 \frac{1}{L}\langle V, \gamma_s^\perp\rangle\langle\gamma_s^\perp, \eta\rangle \,\mathrm{d}s\,\mathrm{d}t.$$

Therefore, by additionally using (2.38), we conclude the proof of (2.86).

*Proof of* (2.87): From (2.36) and Definition 2.12 of $P_1$ and $P_2$, we derive that

$$P_1(\gamma^\lambda, \eta) \to P_1(\gamma, \eta),$$
$$P_2(\gamma^\lambda, \eta) \to P_2(\gamma, \eta)$$

as $\lambda \to \infty$ strongly in $L_T^\infty L^\infty$. Hence, (2.36) and (2.15) imply

$$\frac{1}{\widetilde{L}^\lambda}\langle(\widetilde{\gamma}_s^\lambda)^\perp, P_1(\gamma^\lambda, \eta)(\gamma_s^\lambda)^\perp\rangle(\widetilde{\gamma}_s^\lambda)^\perp \to 0,$$
$$\frac{1}{L}\langle\gamma^\lambda - \widetilde{\gamma}^\lambda, P_2(\gamma^\lambda, \eta)(\gamma_s^\lambda)^\perp + P_1(\gamma^\lambda, \eta)(\gamma_{ss}^\lambda)^\perp + (\eta_s)^\perp\rangle(\gamma_s^\lambda)^\perp \to 0,$$
$$\Big(\frac{1}{L^3}\int_0^1 \langle\gamma_s^\lambda, \eta_s\rangle \,\mathrm{d}\widetilde{s}\Big)\langle\gamma^\lambda - \widetilde{\gamma}^\lambda, (\gamma_s^\lambda)\rangle(\gamma_s^\lambda)^\perp \to 0$$

strongly in $L_T^2 L^2$. Therefore, as in the previous step, the result follows by the weak-strong convergence. ∎

**Corollary 2.21.** *The time continuous evolution of $\gamma$ from Theorem 2.9 satisfies*

$$\gamma \in L_{\mathrm{loc}}^2 H^4. \tag{2.89}$$

**Remark 2.22.** A priori, from the bound in (2.61) we can only derive that

$$\gamma \in L_{\mathrm{loc}}^{\frac{3}{2}} W^{4,\frac{3}{2}}.$$

In order to improve the integrability from $\frac{3}{2}$ to 2 we need to repeat the strategy of the proof of Lemma 2.17 in the time-continuous setting. Instead of (2.56), we will employ (2.84). The main difference between these two is the absence of all Lagrange multiplier terms contained in the error term of (2.56), which vanish in the limit $\lambda \to \infty$. Their absence will allow us to improve the regularity of $\gamma$.

*Proof.* The argument is similar to the one in the proof of Lemma 2.17: Testing (2.84) with $\eta(t,s) := \phi(s)\psi(t)$, where $\phi \in C_c^\infty((0,1);\mathbb{R}^2)$ and $\psi \in C_c^\infty([0,\infty))$, and using the arbitrariness of $\psi$, we derive that for a.e. $t \in [0,\infty)$ it holds that

$$\int_0^1 \frac{\varepsilon}{L^3}\langle \gamma_{ss}, \phi_{ss}\rangle + \frac{1}{L}\Big(1 - \frac{3\varepsilon}{2}\kappa^2\Big)\langle \gamma_s, \phi_s\rangle + \frac{1}{L}\langle V, \gamma_s^\perp\rangle\langle \gamma_s^\perp, \phi\rangle \,\mathrm{d}s = 0. \tag{2.90}$$

Integrating by parts in (2.90) and employing the notation from the proof of Lemma 2.17 leads to

$$\int_0^1 \Big\langle \frac{\varepsilon}{L^3}\gamma_{ss} - D_s^{-1}\Big\{\frac{1}{L}\Big(1 - \frac{3\varepsilon}{2}\kappa^2\Big)\gamma_s\Big\} + D_s^{-2}\Big\{\frac{1}{L}\langle V, \gamma_s^\perp\rangle\gamma_s^\perp\Big\}, \phi_{ss}\Big\rangle \mathrm{d}s = 0$$

for a.e. $t \in [0,\infty)$. Hence, for a.e. $t \in [0,\infty)$, there exist $v(t), w(t) \in \mathbb{R}^2$ such that for all such $t$ and a.e. $s \in [0,1]$, it holds that

$$-\frac{\varepsilon}{L^3}\gamma_{ss} = v + ws - D_s^{-1}\Big\{\frac{1}{L}\Big(1 - \frac{3\varepsilon}{2}\kappa^2\Big)\gamma_s\Big\} + D_s^{-2}\Big\{\frac{1}{L}\langle V, \gamma_s^\perp\rangle\gamma_s^\perp\Big\}.$$

We differentiate the equation above twice in $s$, which results in

$$-\frac{\varepsilon}{L^3}\gamma_{ssss} = 3\varepsilon\kappa\kappa_s\gamma_s - \frac{1}{L}\Big(1 - \frac{3\varepsilon}{2}\kappa^2\Big)\gamma_{ss} + \frac{1}{L}\langle V, \gamma_s^\perp\rangle\gamma_s^\perp, \tag{2.91}$$

again for a.e. $t$ and $s$. By Young's inequality, (2.91), (2.7), and (2.8), we have for a.e. $t \in [0,\infty)$

$$\begin{aligned}
\|\gamma_{ssss}\|_{L^2}^2 &\le C(\varepsilon)\int_0^1 |\kappa|^2|\kappa_s|^2 + |\gamma_{ss}|^2 + |\kappa|^4|\gamma_{ss}|^2 + |V|^2 \,\mathrm{d}s \\
&\le C(\varepsilon)\int_0^1 |\gamma_{ss}|^2(|\gamma_{sss}|^2 + 1) + |\gamma_{ss}|^6 + |V|^2 \,\mathrm{d}s \\
&\le C(\varepsilon)\int_0^1 1 + |\gamma_{ss}|^6 + |\gamma_{sss}|^3 + |V|^2 \,\mathrm{d}s.
\end{aligned}$$

Furthermore, by interpolation with parameters $i = 2$, $j = 3$, $m = 4$, $p = 2$, $q = 3$, $\theta = \frac{7}{12}$, Young's inequality with arbitrary $\delta > 0$ as well as powers $\frac{8}{7}$ and 8, and (2.8), we see that

$$\begin{aligned}
\|\gamma_{sss}\|_{L^3}^3 &\le C\Big(\|\gamma_{ssss}\|_{L^2}^{\frac{7}{4}}\|\gamma_{ss}\|_{L^2}^{\frac{5}{4}} + \|\gamma_{ss}\|_{L^2}^3\Big) \\
&\le C\left(\delta\|\gamma_{ssss}\|_{L^2}^2 + C(\delta)\|\gamma_{ss}\|_{L^2}^{10} + \|\gamma_{ss}\|_{L^2}^3\right) \\
&\le C\delta\|\gamma_{ssss}\|_{L^2}^2 + C(\delta,\varepsilon).
\end{aligned}$$

Furthermore, by the interpolation inequality with parameters $i = 2$, $j = 2$, $m = 4$, $p = 2$, $q = 6$, $\theta = \frac{1}{6}$,

$$\begin{aligned}
\|\gamma_{ss}\|_{L^6}^6 &\le C\left(\|\gamma_{ssss}\|_{L^2}\|\gamma_{ss}\|_{L^2}^5 + \|\gamma_{ss}\|_{L^2}^6\right) \\
&\le C\left(\delta\|\gamma_{ssss}\|_{L^2}^2 + C(\delta)\|\gamma_{ss}\|_{L^2}^{10} + \|\gamma_{ss}\|_{L^2}^6\right) \\
&\le C\delta\|\gamma_{ssss}\|_{L^2}^2 + C(\delta,\varepsilon).
\end{aligned}$$

Combining the above estimates leads to

$$\|\gamma_{ssss}\|_{L^2}^2 \leq C(\varepsilon)\delta\|\gamma_{ssss}\|_{L^2}^2 + C(\delta,\varepsilon)\int_0^1 1 + |V|^2 \,\mathrm{d}s.$$

Hence, choosing $\delta$ small enough, we derive for a.e. $t \in [0,\infty)$

$$\|\gamma_{ssss}\|_{L^2}^2 \leq C(\varepsilon)\int_0^1 1 + |V|^2 \,\mathrm{d}s. \tag{2.92}$$

By integrating (2.92) over $t \in [0,T]$ with arbitrary $T > 0$ and using (2.20), we conclude the proof. ∎

Due to the higher regularity of $\gamma$ derived in Theorem 2.17 and Corollary 2.21, we will be able to integrate by parts in equation (2.84), which leads to the main result of this paper: Theorem 2.23.

**Theorem 2.23** (Long-time existence). *Let $\gamma$ be as in Theorem 2.9. Then*

$$\begin{aligned} \gamma &\in C^{0,\beta}_{\mathrm{loc}}C^{1,\alpha} \cap H^1_{\mathrm{loc}}L^2 \cap L^2_{\mathrm{loc}}H^4, \\ V^\top &\in L^{\frac{3}{2}}_{\mathrm{loc}}([0,\infty); W^{1,\frac{3}{2}}([0,1])), \\ \gamma(\cdot,0),\ \gamma(\cdot,1) &\in H^1_{\mathrm{loc}}([0,\infty);\mathbb{R}^2), \end{aligned}$$

*where $\alpha \in (0,\frac{1}{2})$ and $\beta \in (0,\frac{1-2\alpha}{8})$. Furthermore, for a.e. $s \in [0,1]$ and a.e. $t \in [0,\infty)$ we have*

$$V^\perp(t,s) = \kappa(t,s) - \varepsilon\Big(\frac{1}{L^2(t)}\kappa_{ss}(t,s) + \frac{1}{2}\kappa^3(t,s)\Big), \tag{2.93}$$

$$V_s^\top(t,s) = L_t(t) + L(t)\kappa(t,s)V^\perp(t,s), \tag{2.94}$$

$$V(t,0) = -\frac{\gamma(t,1)-\gamma(t,0)}{|\gamma(t,1)-\gamma(t,0)|^2} + \frac{1}{L(t)}\gamma_s(t,0) - \frac{\varepsilon}{L^2(t)}\kappa_s(t,0)\gamma_s^\perp(t,s), \tag{2.95}$$

$$V(t,1) = \frac{\gamma(t,1)-\gamma(t,0)}{|\gamma(t,1)-\gamma(t,0)|^2} - \frac{1}{L(t)}\gamma_s(t,1) + \frac{\varepsilon}{L^2(t)}\kappa_s(t,1)\gamma_s^\perp(t,s). \tag{2.96}$$

*Moreover, for a.e. $t \in [0,\infty)$ the following natural boundary condition holds true:*

$$\kappa(t,0) = \kappa(t,1) = 0. \tag{2.97}$$

*Proof.* The regularity statements directly follow from (2.36), (2.37), (2.38), as well as (2.89). Furthermore, we note that (2.94) was already proved in Theorem 2.19 (see (2.83)).

We now test (2.84) with $\eta = \phi\gamma_s^\perp$ for some $\phi \in C_c^\infty([0,\infty); C^\infty([0,1]))$. As $\gamma$ is in general not smooth, we need to construct an argument by approximation: Due to (2.61) there exists a sequence $\{\mu^n\}_n \subset C_c^\infty C^\infty \cap L_c^{\frac{3}{2}} W^{4,\frac{3}{2}}$ such that

$$\mu^n \overset{n\to\infty}{\to} \gamma \quad \text{strongly in } L_c^{\frac{3}{2}} W^{4,\frac{3}{2}},$$

so in particular

$$\mu^n \overset{n\to\infty}{\to} \gamma \quad \text{strongly in } L_c^{\frac{3}{2}} C^3. \tag{2.98}$$

Furthermore, by (2.36) we can also assume that

$$\mu^n \overset{n\to\infty}{\to} \gamma \quad \text{strongly in } L_c^{\infty} C^1. \tag{2.99}$$

Let $T > 0$ be such that $\operatorname{supp}(\phi) \subset [0,T] \times [0,1]$. By (2.98), the definition of $E$ in (2.53), Hölder's inequality, (2.7), and (2.72), we derive

$$\begin{aligned}
&\Big|\int_0^\infty E(\gamma, \phi(\mu_s^n)^\perp)\,\mathrm{d}t - \int_0^\infty E(\gamma, \phi\gamma_s^\perp)\,\mathrm{d}t\Big| \\
&\quad\le \int_0^T \int_0^1 C(\phi)(1 + |\gamma_{ss}| + |\gamma_{ss}|^2)\|\mu^n(\cdot,t) - \gamma(\cdot,t)\|_{C^3}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad\le C(\phi)\Big(\int_0^T 1 + \|\gamma_{ss}(\cdot,t)\|_{L^6}^6\,\mathrm{d}t\Big)^{\frac{1}{3}}\Big(\int_0^T \|\mu^n(\cdot,t) - \gamma(\cdot,t)\|_{C^3}^{\frac{3}{2}}\,\mathrm{d}t\Big)^{\frac{2}{3}} \\
&\quad\le C(\phi,\varepsilon,T)\|\mu^n - \gamma\|_{L_T^{\frac{3}{2}} C^3} \overset{n\to\infty}{\to} 0.
\end{aligned}$$

Moreover, by (2.7) and (2.99) we have

$$\begin{aligned}
&\Big|\int_0^\infty \int_0^1 V^\perp\langle\gamma_s^\perp, \phi(\mu_s^n)^\perp\rangle\,\mathrm{d}s\,\mathrm{d}t - \int_0^\infty \int_0^1 V^\perp\langle\gamma_s^\perp, \phi(\gamma_s)^\perp\rangle\,\mathrm{d}s\,\mathrm{d}t\Big| \\
&\quad\le C(\phi)\|V^\perp\|_{L^1(0,T;L^1)}\|\mu^n - \gamma\|_{L^\infty(0,T;C^1)} \overset{n\to\infty}{\to} 0.
\end{aligned}$$

In a similar fashion we can derive

$$\begin{aligned}
&\Big|\int_0^\infty \langle V(t,0), \phi(t,0)(\mu_s^n)^\perp\rangle + \langle V(t,0), \phi(t,0)(\mu_s^n)^\perp\rangle\,\mathrm{d}t \\
&\quad - \int_0^\infty \langle V(t,0), \phi(t,0)\gamma_s^\perp\rangle - \langle V(t,0), \phi(t,0)\gamma_s^\perp\rangle\,\mathrm{d}t\Big| \overset{n\to\infty}{\to} 0.
\end{aligned}$$

Hence, by testing (2.84) with $\eta = \phi(\mu_s^n)^\perp$ and by passing to the limit $n \to \infty$, we see that

$$\begin{aligned}
&\int_0^\infty \int_0^1 \varepsilon\kappa\phi_{ss} - \varepsilon L^2\kappa^3\phi - L^2\Big(\kappa - \frac{3\varepsilon}{2}\kappa^3\Big)\phi + L^2 V^\perp\phi\,\mathrm{d}s\,\mathrm{d}t \\
&\quad - \int_0^T \Big\langle\frac{\gamma(t,1) - \gamma(t,0)}{|\gamma(t,1) - \gamma(t,0)|^2}, \phi(t,1)\gamma_s^\perp(t,1) - \phi(t,0)\gamma_s^\perp(t,0)\Big\rangle\,\mathrm{d}t \\
&\quad + \int_0^T L\phi(t,0)V^\perp(t,0) + L\phi(t,1)V^\perp(t,1) = 0.
\end{aligned} \tag{2.100}$$

Let us first consider only test functions $\phi \in C_c^\infty([0,\infty); C_c^\infty(0,1))$. Integrating by parts for such $\phi$ in (2.100) results in

$$\int_0^\infty \int_0^1 \Big(\frac{\varepsilon}{L^2}\kappa_{ss} + \frac{\varepsilon}{2}\kappa^3 - \kappa + V^\perp\Big) L^2 \phi \,\mathrm{d}s\,\mathrm{d}t = 0.$$

Consequently, (2.93) is satisfied by the arbitrariness of $\phi$.

We consider now more general $\phi \in C_c^\infty([0,\infty); C^\infty([0,1]))$. Integrating by parts in (2.100) and using (2.93), we derive that

$$\int_0^T \varepsilon\kappa\phi_s - \varepsilon\kappa_s\phi|_{s=0}^1 \,\mathrm{d}t - \int_0^T \Big\langle \frac{\gamma(t,1)-\gamma(t,0)}{|\gamma(t,1)-\gamma(t,0)|^2}, \phi(t,1)\gamma_s^\perp(t,1) - \phi(t,0)\gamma_s^\perp(t,0)\Big\rangle \mathrm{d}t \\ + \int_0^T \langle V(t,0), \phi(t,0)\gamma_s^\perp(t,0)\rangle + \langle V(t,1), \phi(t,1)\gamma_s^\perp(t,1)\rangle \,\mathrm{d}t = 0. \tag{2.101}$$

Choosing $\phi$ such that $\phi(\cdot,0) = \phi(\cdot,1) = \phi_s(\cdot,1) = 0$ in (2.101) leads to

$$\int_0^T \varepsilon\kappa(t,0)\phi_s(t,0)\,\mathrm{d}t = 0,$$

and then due to the arbitrariness of $\phi_s(\cdot,0)$ to

$$\kappa(t,0) = 0$$

for almost all $t$. In a similar fashion, one can derive the same natural boundary condition at $s = 1$, and (2.97) follows.

Plugging (2.97) into (2.101) and choosing $\phi$ satisfying $\phi(\cdot,1) = 0$ leads to

$$\int_0^T \varepsilon\kappa_s(t,0)\phi(t,0) + \Big\langle \frac{\gamma(t,1)-\gamma(t,0)}{|\gamma(t,1)-\gamma(t,0)|^2}, \gamma_s^\perp(t,0)\Big\rangle \phi(t,0)\,\mathrm{d}t \\ + \int_0^\infty \langle V(t,0), \gamma_s^\perp(t,0)\rangle\phi(t,0)\,\mathrm{d}t = 0.$$

Hence, by the arbitrariness of $\phi(\cdot,0)$, we have for almost all $t \in [0,\infty)$

$$V^\perp(t,0) = -\Big\langle \frac{\gamma(t,1)-\gamma(t,0)}{|\gamma(t,1)-\gamma(t,0)|^2}, \frac{\gamma_s^\perp}{L}\Big\rangle - \varepsilon\frac{1}{L}\kappa_s(t,0). \tag{2.102}$$

We next test (2.84) with $\eta = \phi\mu_s^n$, where $\phi \in C_c^\infty([0,\infty); C^\infty([0,1]))$ with $\phi(\cdot,1) \equiv 0$. Passing to the limit $n \to \infty$ as was done previously results in

$$0 = \int_0^\infty \int_0^1 \frac{\varepsilon}{L^3}\langle L\kappa\gamma_s^\perp, (2L\kappa\phi_s + L\kappa_s\phi)\gamma_s^\perp\rangle + \frac{1}{L}\Big(1 - \frac{3\varepsilon}{2}\kappa^2\Big)\langle\gamma_s, \phi_s\gamma_s\rangle \,\mathrm{d}s\,\mathrm{d}t \\ - \int_0^\infty \Big\langle \frac{\gamma(t,1)-\gamma(t,0)}{|\gamma(t,1)-\gamma(t,0)|^2}, -\phi(t,0)\gamma_s(t,0)\Big\rangle \mathrm{d}t + \int_0^\infty \langle V(t,0), \phi(t,0)\gamma_s(t,0)\rangle \,\mathrm{d}t$$

$$= \int_0^\infty \int_0^1 \Big(\Big(\frac{\varepsilon}{2}\kappa^2 + 1\Big)\phi_s + \varepsilon\kappa\kappa_s\phi\Big) L \,\mathrm{d}s\,\mathrm{d}t$$
$$+ \int_0^\infty \Big(\Big\langle \frac{\gamma(t,1)-\gamma(t,0)}{|\gamma(t,1)-\gamma(t,0)|^2}, \frac{\gamma_s}{L}\Big\rangle + V^\top(t,0)\Big)\phi(t,0) L \,\mathrm{d}t$$
$$= \int_0^\infty \Big(-1 + \Big\langle \frac{\gamma(t,1)-\gamma(t,0)}{|\gamma(t,1)-\gamma(t,0)|^2}, \frac{\gamma_s}{L}\Big\rangle + V^\top(t,0)\Big)\phi(t,0) L \,\mathrm{d}t.$$

Due to the arbitrariness of $\phi(t,0)$, we have

$$V^\top(t,0) = 1 - \Big\langle \frac{\gamma(t,1)-\gamma(t,0)}{|\gamma(t,1)-\gamma(t,0)|^2}, \frac{\gamma_s}{L}\Big\rangle \tag{2.103}$$

for almost every $t \in [0,\infty)$. By (2.102) and (2.103), equation (2.95) follows. The proof of (2.96) works similarly. ■

## 3. Numerical experiments

In this section we present some numerical experiments with the aim of showing different examples of the curve-shortening evolution derived in the previous sections. In order to make numerical computations, we will discretize our curves as it is customary in the framework of discrete differential geometry; see also [13]. Hereby, a discrete curve in $\mathbb{R}^2$ is defined as a finite sequence of $N$ points $\mathbf{x} = (x_i)_{i=1}^N \subset \mathbb{R}^2$. They define a zig-zag curve build up from the edges $E_i = [x_i, x_{i+1}]$, where $1 \le i < N$. In this framework the constant speed constraint, as employed in the previous sections, has the following discrete counterpart: There exists an $l > 0$ such that

$$|x_{i+1} - x_i| = \text{const} = l \qquad \text{for all } 1 \le i < N.$$

Hence, given $N \in \mathbb{N}$, we will consider the following set of admissible discrete curves:

$$\mathcal{AC}_N^{\text{discr}} = \big\{\mathbf{x} = \{x_1, \dots, x_N\} \subset \mathbb{R}^2 : \exists\, l \ge 0 \text{ s.t. } |x_{i+1} - x_i| = l \text{ for all } 1 \le i < N\big\}.$$

For any $i \in 1, \dots, N-1$, we define the discrete unit tangent vector $\tau_i$ and normal vector $\nu_i$ as

$$\tau_i = \frac{x_{i+1} - x_i}{|x_{i+1} - x_i|} = l^{-1}(x_{i+1} - x_i),$$
$$\nu_i = \tau_i^\perp.$$

For any $i = 1, \dots, N-1$, let $\alpha_i \in [0,\pi]$ be the unique angle between $\tau_{i-1}$ and $\tau_i$; this means it satisfies $\cos(\alpha_i) = \langle \tau_{i-1}, \tau_i\rangle$. With this, we can define the discrete curvature as

$$\kappa_i = 2l^{-1}\tan\Big(\frac{\alpha_i}{2}\Big) = 2l^{-1}\frac{\sin(\alpha_i)}{1+\cos(\alpha_i)} = 2l^{-1}\frac{\tau_{i-1} \times \tau_i}{1 + \langle \tau_{i-1}, \tau_i\rangle}.$$

Here, $\times\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ denotes the cross product in $\mathbb{R}^2$ defined as

$$v \times w = v_1 w_2 - v_2 w_1.$$

Let us fix $\varepsilon > 0$, $\lambda > 0$, and $N \in \mathbb{N}$. The discrete version of the energy functional in (2.5) is $F^{\mathrm{discr}}\colon \mathcal{AC}_N^{\mathrm{discr}} \times \mathcal{AC}_N^{\mathrm{discr}} \to \mathbb{R}$ defined as

$$\begin{aligned} F^{\mathrm{discr}}(\mathbf{x}, \tilde{\mathbf{x}}) = -\log|x_1 - x_N| + Nl + \frac{\varepsilon l}{2} \sum_{i=2}^{N-1} \kappa_i^2 \\ + \frac{\lambda l}{4} \sum_{i=1}^{N-1} \langle x_i - \tilde{x}_i, \tilde{\nu}_i \rangle^2 + \frac{\lambda l}{4} \sum_{i=1}^{N-1} \langle x_i - \tilde{x}_i, \nu_i \rangle^2 \\ + \frac{\lambda}{2} |x_1 - \tilde{x}_1|^2 + \frac{\lambda}{2} |x_N - \tilde{x}_N|^2. \end{aligned}$$

We can now describe the discrete-in-space minimizing movements scheme: Consider an arbitrary choice of initial discrete curve

$$\mathbf{x}^{(0)} = \left\{ x_1^{(0)}, \dots, x_N^{(0)} \right\} \in \mathcal{AC}_N^{\mathrm{discr}}.$$

Then, $\mathbf{x}^{(1)}$ is defined as

$$\mathbf{x}^{(1)} \in \operatorname*{argmin}_{\mathbf{x} \in \mathcal{AC}_N^{\mathrm{discr}}} F^{\mathrm{discr}}(\mathbf{x}, \mathbf{x}^{(0)}),$$

and we continue by defining $\mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \dots$ in a similar way to the space-continuous setting.

In the following, we will show plots of discrete-in-space minimizing movements for two different initial curves. We remark that all numerical computations were done in the

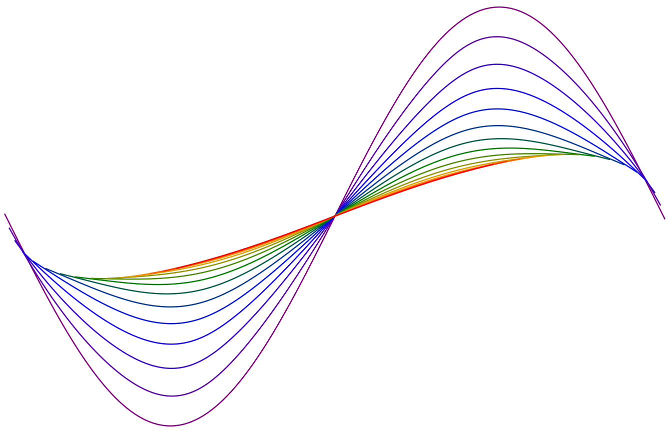
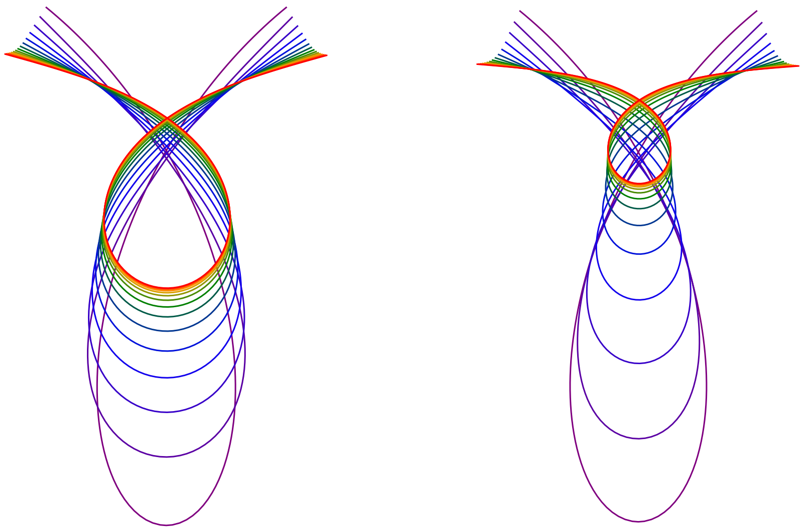


**Figure 2.** The first 15 steps of the minimizing movement scheme starting from a sinus-shaped curve for $\lambda = 5$ and $\varepsilon = 0.01$.



**(a)** $\varepsilon = 0.1$. **(b)** $\varepsilon = 0.025$.

**Figure 3.** First 15 steps of the minimizing movements scheme starting from a $\gamma$-shaped curve for $\lambda = 30$ and either $\varepsilon = 0.1$ or $\varepsilon = 0.025$.

programming language *Julia* (see also [10]). The step-by-step minimization was solved via the *JuMP* (see also [14]) interface and the software package *Ipopt* (see also [35]).

We start with a curve $\mathbf{x}^{(0)}$ discretizing the graph of the sinus-function restricted on the interval $[-\pi, \pi]$. Figure 2 shows several steps of the minimizing movements scheme. The coloring of the curves is used to clarify the temporal order. The curves close to the start are violet, while the curves close to the end are red. One can see the straightening motion of the sinus-curve. In the limit $k \to \infty$, the curve converges towards a straight line with unit length.

We next consider a curve in the shape of the letter $\gamma$. In Figure 3a, one can see that the center loop of the curve shrinks until a point where the curvature term becomes dominant. As $k \to \infty$ the curve doesn't unfold and converges towards an "optimal" $\gamma$-shaped curve. This is a good opportunity to show the dependence of the flow on the size of $\varepsilon$. In Figure 3b, we computed the minimizing movements of the $\gamma$-shaped curve from before with a smaller $\varepsilon$. One can see that for smaller values of $\varepsilon$ (smaller curvature regularization) the size of the center loop of the limit curve is smaller.

**Acknowledgments.** This work was supported by the DFG Collaborative Research Center TRR 109, "Discretization in Geometry and Dynamics". I wish to thank Marco Cicalese, Nicola Fusco, and Carlo Mantegazza for many fruitful discussions.

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Received 8 July 2021; revised 12 November 2021.

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