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On the Rayleigh–Taylor instability for the two-phase Navier–Stokes equations in cylindrical domains

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Abstract. We are considering the two-phase Navier–Stokes equations with surface tension, modelling the dynamic behaviour of two immiscible and incompressible fluids in a cylindrical domain, which are separated by a sharp interface forming a contact angle with the fixed boundary. In the case that the heavy fluid is situated on top of the light fluid, one expects the effect which is known as *Rayleigh–Taylor instability*. Our main result implies the existence of a critical surface tension with the following property: In the case that the surface tension of the interface separating the two fluids is smaller than the critical surface tension, Rayleigh–Taylor instability occurs. On the contrary, if the surface tension of the interface is larger than the critical value, one has exponential stability of the flat interfaces. The last part of this article is concerned with the bifurcation of nontrivial equilibria in multiple eigenvalues. The invariance of the corresponding bifurcation equation with respect to rotations and reflections yields the existence of bifurcating subcritical equilibria.

1. Introduction

Let $u = u(t, x)$ and $\pi = \pi(t, x)$ denote the velocity field and the pressure field of a single incompressible fluid in a domain Ω, respectively. By saying that the fluid is incompressible, we mean that its density $\rho > 0$ is constant. Then the dynamics of the fluid are described by the Navier–Stokes equations

$$\begin{aligned} \partial_t(\rho u) - \mu\Delta u + \rho(u\cdot\nabla)u + \nabla\pi &= \rho f, \qquad t>0,\ x\in\Omega,\\ \operatorname{div} u &= 0, \qquad t>0,\ x\in\Omega, \end{aligned} \tag{1.1}$$

where $\mu > 0$ represents the viscosity of the fluid and f is some external force (e.g., gravity). The first equation reflects the balance of momentum, while the second equation states the conservation of mass.

Let us consider a more comprehensive situation, where the domain Ω is occupied by an incompressible and an immiscible fluid, *fluid* 1 and *fluid* 2, respectively, which are separated by a sharp interface $\Gamma(t)$ for each $t \geq 0$. We denote by $\Omega_j(t)$ the subset

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of Ω which is filled with *fluid* j, $j \in \{1, 2\}$ with ρ_j, μ_j being the density and viscosity, respectively, of *fluid* j. If u^j and π^j are the velocity fields and the pressure fields of *fluid* j, respectively, then, for $t \geq 0$ one sets

$$u(t,x) := \begin{cases} u^1(t,x), & x \in \Omega_1(t), \\ u^2(t,x), & x \in \Omega_2(t), \end{cases} \qquad \pi(t,x) := \begin{cases} \pi^1(t,x), & x \in \Omega_1(t), \\ \pi^2(t,x), & x \in \Omega_2(t). \end{cases}$$

Assuming that (u^j, π^j) satisfies the Navier–Stokes equations in each of the phases $\Omega_j(t)$, then we may conclude that (u, π) satisfies (1.1) for all $t > 0$ and $x \in \Omega \backslash \Gamma(t)$, where ρ and μ are defined by

$$\rho(x) := \begin{cases} \rho_1, & x \in \Omega_1(t), \\ \rho_2, & x \in \Omega_2(t), \end{cases} \qquad \mu(x) := \begin{cases} \mu_1, & x \in \Omega_1(t), \\ \mu_2, & x \in \Omega_2(t). \end{cases}$$

Clearly, one expects that the two fluids should affect each other in their dynamics. Therefore, it is natural to ask for relations that describe the coupling of the two fluids across the interface $\Gamma(t)$. If one neglects effects of phase transitions between the phases $\Omega_1(t)$ and $\Omega_2(t)$ (e.g., the exchange of mass), then the motion of the moving boundary $\Gamma(t)$ should only be caused by the velocity fields of both fluids. Therefore, it is natural to propose that $u^2|_{\Gamma(t)} = u^1|_{\Gamma(t)}$. Then, the *normal velocity* V_Γ of $\Gamma(t)$ is given by

$$V_\Gamma = u \cdot \nu_\Gamma, \tag{1.2}$$

where ν_Γ denotes the unit normal field on $\Gamma(t)$ pointing from $\Omega_1(t)$ to $\Omega_2(t)$. We call the quantity $[\![u]\!] := u^2|_{\Gamma(t)} - u^1|_{\Gamma(t)}$ the *jump of* u *across* $\Gamma(t)$. Note that $[\![u]\!] = 0$ if and only if the velocity field u is continuous across the interface $\Gamma(t)$. Another condition on $\Gamma(t)$ reads

$$-[\![\mu(\nabla u + \nabla u^{\mathsf{T}})]\!]\nu_\Gamma + [\![\pi]\!]\nu_\Gamma = \sigma H_\Gamma \nu_\Gamma, \tag{1.3}$$

where $\sigma > 0$ denotes the (constant) *surface tension* of $\Gamma(t)$ and $H_\Gamma := -\operatorname{div}_\Gamma \nu_\Gamma$ is the *mean curvature* of $\Gamma(t)$ with $\operatorname{div}_\Gamma$ being the surface divergence on $\Gamma(t)$. Condition (1.3) describes the balance of forces on the interface. To be precise, there is no contribution to the rate of change of the momentum coming from the interface $\Gamma(t)$.

If the fixed boundary $\partial\Omega$ of Ω is not empty, then system (1.1)–(1.3) with $[\![u]\!] = 0$ has to be equipped with appropriate boundary conditions on $\partial\Omega$ as well as some initial conditions on $u(0) = u_0$ and $\Gamma(0) = \Gamma_0$. There is a vast literature concerning the mathematical treatment of free boundary problems for the Navier–Stokes equations with or without surface tension. To this end, we can only give a subjective selection and refer the reader to [2, 5, 6, 8–13, 23, 24, 27–30, 32, 33, 35, 36, 38–50]. For a derivation of (1.1)–(1.3) we refer to [18] or [31].

To describe the effect of what is called *Rayleigh–Taylor instability*, let us consider the case that $\Omega = \mathbb{R}^n$ consists of two phases $\Omega_1(t)$ and $\Omega_2(t)$ which are separated by an interface $\Gamma(t)$, given by the graph of a height function h over $\mathbb{R}^{n-1}$, i.e.,

$$\Gamma(t) := \{x = (x', x_n) \in \Omega : x_n = h(t, x'),\ x' \in \mathbb{R}^{n-1}\}.$$

Assume further that $\Omega_2(t)$ is the upper phase, that is,

$$\Omega_2(t) = \{x = (x', x_n) \in \Omega : x_n > h(t, x'),\ x' \in \mathbb{R}^{n-1}\}.$$

Both phases are filled with two fluids with possibly different densities which are accelerated in the direction of $-e_n$ by the gravitational force.

Taking a close look at system (1.1)–(1.3), it turns out that the vanishing velocity fields, constant pressure fields and the flat interfaces belong to the set of *equilibria*, i.e., the set of all solutions which are constant with respect to t. Henceforth, we will speak of the trivial equilibrium whenever $u = 0$, p is constant and $h = 0$. Heuristically, one expects that the stability behaviour of the trivial equilibrium is being influenced by the densities $\rho_2 > 0$ and $\rho_1 > 0$ of the fluids. Indeed, if $[\![\rho]\!] = \rho_2 - \rho_1 > 0$, i.e., if the heavier fluid is placed above the lighter fluid, then one expects that the trivial equilibrium is unstable, while in the case that $[\![\rho]\!] \le 0$, the trivial equilibrium should be stable. In fact, if $[\![\rho]\!] > 0$ then the upper phase, which is the heavier one, should sack down into the lower phase; see Figure 1. This effect is called Rayleigh–Taylor instability and it goes back to the pioneering works of Rayleigh [33] and Taylor [50]. For more information concerning Rayleigh–Taylor instability, we refer the interested reader to Chandrasekhar [7] and Kull [24] and the references cited therein. A rigorous proof of Rayleigh–Taylor instability for the two-phase Navier–Stokes equations in the above setting has been given by Prüss and Simonett [28]. The basic strategy is to consider the full linearisation of the quasilinear problem (1.1)–(1.3) at the trivial equilibrium and to compute the spectrum of the linearisation. Due to the lack of compactness, there is a portion of approximate eigenvalues in the spectrum of the linearisation. In addition, there is no spectral gap which would allow us to apply classical tools to carry over the linear stability properties to the nonlinear case. To this end, the authors in [28] apply Henry's instability theorem [17, Theorem 5.1.5] which does not require a spectral gap.

In the periodic framework, i.e., with $\Omega = \mathbb{T}^2 \times \mathbb{R}$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the 1-torus, a rigorous proof of Rayleigh–Taylor instability has been given by Tice and Wang in [53]. Note that if $[\![\rho]\!] > 0$, then the result in [28] states that the trivial equilibrium is always unstable, no matter what the remaining parameters $\mu > 0$ and $\sigma > 0$ are. However, in the periodic setting considered in [53], the stability properties of the trivial equilibrium also depend on the surface tension. To be more precise, there exists a critical surface tension $\sigma_c > 0$ such that if $\sigma > \sigma_c$, then the trivial equilibrium is stable, while if $0 < \sigma < \sigma_c$, it is unstable. In other words, even if $[\![\rho]\!] > 0$, a sufficiently large surface tension $\sigma > 0$ of $\Gamma(t)$ prevents the heavier phase from sacking down into the lower phase.

An advantage of the approach via maximal regularity of type L_p which has been used in [28] is that one obtains a semi-flow for the free boundary problem in a natural phase space. In particular, there is no loss of regularity. With the help of functional calculus for sectorial operators and harmonic analysis, it is then shown that there exists $\lambda_\infty > 0$ such that the interval $[0, \lambda_\infty]$ is the unstable part of the spectrum of the linearisation. The functional-analytic setting used in [28] then allows us to apply Henry's instability theorem [17, Theorem 5.1.5] to conclude instability for the nonlinear problem. In contrast to the result in [28], the authors in [53] construct so-called growing mode solutions (horizontal Fourier modes growing exponentially in time) for the linearised problem and use several energy estimates to study the spectrum of the full linearisation. The passage from linear to nonlinear (in-)stability follows from a Guo–Strauss bootstrap procedure, which has been introduced by Guo and Strauss in [15]. Due to the higher-order energy estimates, the regularity of the initial values is considerably high and therefore not optimal, when one compares with the assumptions in [28]. However, the authors in [53] obtain a clear picture of the stability properties of the trivial equilibrium, which depend on $[\![\rho]\!]$ and $\sigma > 0$. Concerning further results on Rayleigh–Taylor instability for different problems, we refer the reader to the selection [3, 14, 16, 19–21, 52].

It is one purpose of this article to extend the results obtained in [28] to the framework of bounded cylindrical domains. To be precise, we assume that $\Omega = G \times (H_1, H_2)$, where $G \subset \mathbb{R}^{n-1}$, $n \in \{2, 3\}$ is a bounded domain with smooth boundary and $H_1 < 0 < H_2$. Suppose further that there is a family of hypersurfaces $\{\Gamma(t)\}_{t\geq 0}$ given by the graph of some height function h over G, i.e.,

$$\Gamma(t) = \big\{(x', x_n) \in \Omega : x_n = h(t, x'),\ x' \in G\big\}, \quad t > 0,$$

such that for each $t \geq 0$ the interface $\Gamma(t)$ divides Ω into two subdomains $\Omega_1(t)$ and $\Omega_2(t)$ which are filled with two fluids, respectively. We adopt the convention that $\Omega_2(t)$ is the upper phase. Assuming that equations (1.1)–(1.3) together with the condition $[\![u]\!] = 0$ are satisfied, we are led in a natural way to the problem of finding suitable boundary conditions on the vertical part $S_1 := \partial G \times (H_1, H_2)$ and the horizontal part $S_2 := (G \times \{H_1\}) \cup (G \times \{H_2\})$ of the boundary $\partial\Omega$ of Ω. This turns out to be a delicate question, since within the above setting we are on the one hand concerned with two parts S_1 and S_2 of the boundary such that $\partial S_1 = \partial S_2$. Therefore, the boundary conditions on S_1 and S_2 have to be chosen in such a way that they are compatible with each other. On the other hand, we have to deal with a *contact angle problem*, as $\partial\Gamma(t)$ is a moving contact line on S_1. At this point we want to emphasise that the choice of the periodic setting in [53] allows us to circumvent the formation of a contact angle. The theory of contact angle problems, in particular with a dynamic contact angle which depends on t, is not well understood yet. In fact, there exist different points of view about how to model such a problem. One party supposes that the dynamic contact angle is determined by an additional equation, while the other party assumes that the contact angle will be determined by the dynamic equations for the interface and the fluid, hence the equation for the contact angle should be redundant. We refer the reader to [4, 37] and to the references given therein for more details.

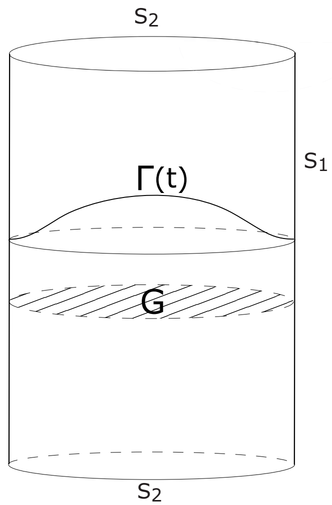

Figure 1. A cylindrical domain

Therefore, in order to avoid this lack of clarity, we assume throughout this article that the contact angle is constant and equal to 90 degrees. One can interpret this ansatz as a kind of idealisation. It is possible to translate the condition on the contact angle to a condition on the height function h from above. Indeed, if h is sufficiently smooth, then the unit normal on $\Gamma(t)$ with respect to $\Omega_1(t)$ is given by

$$\nu_\Gamma = \frac{1}{\sqrt{1+|\nabla_{x'}h|^2}}\begin{pmatrix}-\nabla_{x'}h\\ 1\end{pmatrix}.$$

Since the outer unit normal on S_1 is given by $\nu_{S_1} = (\nu_{\partial G}, 0)^\top$, the condition on the contact angle reads $\nu_\Gamma \cdot \nu_{S_1} = 0$, or equivalently, $\partial_{\nu_{\partial G}} h = 0$ at the contact line. Concerning S_1, it is not possible to propose Dirichlet boundary conditions, the so-called *no-slip* boundary conditions, since this leads to a paradox for the moving contact line (see, e.g., [32]). The next canonical choices are the so-called *Navier* boundary conditions or *partial-slip* boundary conditions

$$u \cdot \nu_{S_1} = 0, \quad P_{S_1}(\mu(\nabla u + \nabla u^\top)\nu_{S_1}) + \alpha u = 0,$$

where $P_{S_1} := I - \nu_{S_1} \otimes \nu_{S_1}$ denotes the projection to the tangent space on S_1. The parameter $\alpha \geq 0$ has the physical meaning of a friction coefficient. However, as long as $\alpha > 0$, it turns out that this kind of boundary condition does not allow the interface to move along S_1, which is not very reasonable, as numerical simulations show. For a two-dimensional analytical explanation of this pathology, see [55, Section 1].

In order to circumvent this problem, we will consider the case $\alpha = 0$, which characterises the so-called *pure-slip* boundary conditions. From a physical point of view this means that there is no friction on the boundary S_1. Having fixed the boundary conditions on S_1, we may choose suitable boundary conditions on S_2, having in mind that these conditions have to match those on S_1. It turns out that homogeneous Dirichlet boundary conditions are a good choice, since they are compatible with the pure-slip boundary conditions on S_1 and furthermore, they allow us to apply *Korn's inequality* for $Du := \nabla u + \nabla u^{\mathsf{T}}$; see Theorem A.4. Note that the no-slip boundary conditions on S_2 do not cause any problems with the moving interface, since we will always have $\Gamma(t) \cap S_2 = \emptyset$ for all $t \geq 0$. We are thus led to the problem

$$\begin{aligned}
\partial_t(\rho u) - \mu\Delta u + \rho(u\cdot\nabla)u + \nabla\pi &= -\rho\gamma_a e_n, && \text{in } \Omega\backslash\Gamma(t),\\
\operatorname{div} u &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
-[\![\mu(\nabla u + \nabla u^{\mathsf{T}})]\!]\nu_\Gamma + [\![\pi]\!]\nu_\Gamma &= \sigma H_\Gamma \nu_\Gamma, && \text{on } \Gamma(t),\\
[\![u]\!] &= 0, && \text{on } \Gamma(t),\\
V_\Gamma &= u\cdot\nu_\Gamma, && \text{on } \Gamma(t),\\
P_{S_1}\big(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_1}\big) &= 0, && \text{on } S_1\backslash\partial\Gamma(t),\\
u\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Gamma(t),\\
u &= 0, && \text{on } S_2,\\
\nu_\Gamma\cdot\nu_{S_1} &= 0, && \text{on } \partial\Gamma(t),\\
u(0) &= u_0, && \text{in } \Omega\backslash\Gamma(0),\\
\Gamma(0) &= \Gamma_0,
\end{aligned} \tag{1.4}$$

where we denote by $\gamma_a > 0$ the acceleration constant due to gravity.

With this article, we present a rather complete stability analysis of (1.4). In Section 2 we will transform the time-dependent domain $\Omega\backslash\Gamma(t)$ to a fixed domain by means of a Hanzawa transformation. For the transformed problem, we have already proven the existence and uniqueness of a strong L_p-solution in [55, Theorem 4.2]. Section 3 is devoted to the investigation of the stability properties of the trivial equilibrium, i.e., when $u = 0$, $h = 0$ and π is constant. It turns out that if $[\![\rho]\!] > 0$, then there exists a critical surface tension

$$\sigma_c := \frac{[\![\rho]\!]\gamma_a}{\lambda_1} > 0,$$

where $\lambda_1 > 0$ denotes the first nontrivial eigenvalue of the Neumann Laplacian in $L_2(G)$. If $\sigma > \sigma_c$, then the trivial equilibrium is exponentially stable in the natural phase space, while in case $\sigma \in (0, \sigma_c)$ it will be unstable. If $[\![\rho]\!] \leq 0$, then the trivial equilibrium is always exponentially stable. Specialising to the case $G = B_R(0)$, we obtain as a corollary that for fixed surface tension $\sigma > 0$ and if $[\![\rho]\!] > 0$, there exists a critical radius

$$R_c := \Big(\frac{\sigma\lambda_1^*}{[\![\rho]\!]\gamma_a}\Big)^{1/2}$$

such that if $R < R_c$, then the trivial equilibrium is exponentially stable, while for $R > R_c$ it will be unstable. Here $\lambda_1^* > 0$ denotes the first nontrivial eigenvalue of the Neumann Laplacian in $L_2(B_1(0))$, given by $\lambda_1^* = (j'_{1,1})^2$, where $j'_{1,1}$ is the first zero of the derivative J'_1 of the Bessel function J_1 (see, e.g., [1]). The proof of the stability result requires some effort, since after the transformation to a fixed domain one has to pay the price that in particular the (transformed) velocity field is no longer divergence free. Therefore, one has to split the solution into two parts in a suitable way such that one part is divergence free while the other part, whose divergence does not vanish, satisfies a nonlinear problem which can be handled by the implicit function theorem.

The results in Section 3 suggest that if σ decreases from $\sigma > \sigma_c$ to $\sigma < \sigma_c$, then an eigenvalue of the full linearisation will cross the imaginary axis. Therefore, it is natural to ask for possible bifurcations from the trivial equilibrium. In Section 4 we will see that the eigenvalue which crosses the imaginary axis through zero is, unfortunately, not simple if $n = 3$. Therefore, it is not possible to apply the bifurcation results of Crandall–Rabinowitz directly. By the choice of the boundary conditions, the equilibria of the transformed problem are such that $u = 0$, p is constant and the height function h satisfies the *capillary equation*

$$\begin{aligned} \sigma \operatorname{div}_{x'}\Big(\frac{\nabla_{x'} h}{\sqrt{1+|\nabla_{x'} h|^2}}\Big) + [\![\rho]\!]\gamma_a h &= 0, \quad x' \in B_R(0), \\ \partial_{\nu_{B_R(0)}} h &= 0, \quad x' \in \partial B_R(0). \end{aligned} \tag{1.5}$$

This equation for h exhibits certain symmetry properties; in particular, we will show that it is invariant under the group action of the orthogonal group $O(2)$. This fact enables us to reduce the bifurcation equation to a one-dimensional equation and to apply the implicit function theorem which yields the existence of *subcritical* bifurcating branches from the trivial solution.

Finally, we collect all technical results which are needed for the execution of the above program in Appendix A.

Notation. The symbols H_p^s, W_p^s, $s \geq 0$ refer to the Bessel potential spaces and Sobolev–Slobodeckij spaces, respectively (Sobolev spaces for $s \in \mathbb{N}$ with $H = W$). If $J = [0, T]$ is some interval and X a suitable Banach space, then ${}_0W_p^s(J; X)$ denotes the subspace of $W_p^s(J; X)$ consisting of all functions having a vanishing trace at $t = 0$, whenever it exists. Finally, we denote by $\dot{W}_p^k(\Omega) = \dot{H}_p^k(\Omega)$ the homogeneous Sobolev space of order $k \in \mathbb{N}$, where $\Omega \subset \mathbb{R}^n$ is a sufficiently smooth domain. The symbol $(\cdot|\cdot)$ denotes the standard inner product in $\mathbb{R}^n$ and we will sometimes also make use of the notation $u \cdot v = (u|v)$ for $u, v \in \mathbb{R}^n$.

Remark 1.1. The results in this paper are partially taken from the author's habilitation thesis [54].

2. Preliminaries

For the sake of readability we will assume throughout this article that the space dimension n is equal to 3. This is the most important case from a viewpoint of applications. Furthermore, we will assume from now on that $p > 5$. In [55], an article about the well-posedness of the nonlinear model, this condition on p is needed for an application of some Sobolev embeddings. For arbitrary n one may work with the restriction $p > n + 2$.

It is convenient to introduce the *modified pressure*

$$\tilde{\pi} := \pi + \rho\gamma_a x_3$$

in (1.4). Then, we obtain the following problem:

$$\begin{aligned}
\partial_t(\rho u) - \mu\Delta u + \rho(u\cdot\nabla)u + \nabla\tilde{\pi} &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
\operatorname{div} u &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
-[\![\mu(\nabla u + \nabla u^{\mathsf{T}})]\!]\nu_\Gamma + [\![\tilde{\pi}]\!]\nu_\Gamma &= \sigma H_\Gamma \nu_\Gamma + [\![\rho]\!]\gamma_a x_3 \nu_\Gamma, && \text{on } \Gamma(t),\\
[\![u]\!] &= 0, && \text{on } \Gamma(t),\\
V_\Gamma &= u\cdot\nu_\Gamma, && \text{on } \Gamma(t),\\
P_{S_1}\left(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_1}\right) &= 0, && \text{on } S_1\backslash\partial\Gamma(t),\\
u\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Gamma(t),\\
u &= 0, && \text{on } S_2,\\
\nu_\Gamma\cdot\nu_{S_1} &= 0, && \text{on } \partial\Gamma(t),\\
u(0) &= u_0, && \text{in } \Omega\backslash\Gamma(0),\\
\Gamma(0) &= \Gamma_0.
\end{aligned} \tag{2.1}$$

Here $\Omega = G\times(H_1, H_2)$, $H_1 < 0 < H_2$, is a cylindrical domain where $G \subset \mathbb{R}^2$ is an open bounded domain with a smooth boundary ∂G. The compact free boundary $\Gamma(t)$ divides Ω into two unbounded disjoint phases $\Omega_j(t)$, $j = 1, 2$, so that $\Omega = \Omega_1(t)\cup\Gamma(t)\cup\Omega_2(t)$. The convention is that $\Omega_2(t)$ is the upper phase while $\Omega_1(t)$ is the lower one, with the unit normal ν_Γ at $x \in \Gamma(t)$ pointing from $\Omega_1(t)$ to $\Omega_2(t)$. We denote by ν_{S_1} the outer unit normal at the fixed boundary $S_1 := \partial G\times(H_1, H_2)$. The operator $P_{S_1} := I - \nu_{S_1}\otimes\nu_{S_1}$ stands for the projection to the tangential space on S_1. Finally, $S_2 := \bigcup_{j=1}^2 G\times\{H_j\}$.

2.1. Reduction to a flat interface

In this section we transform the time-dependent domain $\Omega\backslash\Gamma(t)$ to a fixed domain by means of a Hanzawa tranformation. To this end, we assume that

$$\Gamma(t) = \left\{x \in G\times(H_1, H_2) : x_3 = h(t, x'),\ x' = (x_1, x_2) \in G,\ t \geq 0\right\}.$$

Let $\varphi \in C^\infty(\mathbb{R}; [0, 1]))$ be such that $\varphi(s) = 1$ if $|s| \leq \delta/2$ and $\varphi(s) = 0$ if $|s| \geq \delta$, where we have $\delta < \min\{-H_1, H_2\}/2$. Define a mapping

$$\Theta_h(t,\bar{x}) := \bar{x} + \varphi(\bar{x}_3)h(t,\bar{x}')e_3 =: \bar{x} + \theta_h(t,\bar{x})$$

where $\bar{x} := (\bar{x}', \bar{x}_3)$, and for fixed $t > 0$ we set $x = \Theta_h(t,\bar{x})$. An easy computation shows

$$\theta_h'^{\mathsf{T}} = \begin{pmatrix} 0 & 0 & \partial_1 h\varphi \\ 0 & 0 & \partial_2 h\varphi \\ 0 & 0 & h\varphi' \end{pmatrix}.$$

It follows that Θ_h' is invertible if $\|h\|_{\infty,\infty} < 1/(2|\varphi'|_\infty)$, and

$$(\Theta_h')^{-\mathsf{T}} = (I + \theta_h'^{\mathsf{T}})^{-1} = \frac{1}{1+h\varphi'} \begin{pmatrix} 1+h\varphi' & 0 & -\partial_1 h\varphi \\ 0 & 1+h\varphi' & -\partial_2 h\varphi \\ 0 & 0 & 1 \end{pmatrix}.$$

In what follows, let $\|h\|_{\infty,\infty} < \eta$ with $0 < \eta \le 1/(2|\varphi'|_\infty)$ being sufficiently small. Then, the inverse $\Theta_h^{-1} : \Omega \to \Omega$ is well-defined and it transforms the free interface $\Gamma(t)$ to the flat and fixed interface $\Sigma := G \times \{0\}$. Now we define the transformed quantities

$$\begin{aligned} \bar{u}(t,\bar{x}) &:= u(t,\Theta_h(t,\bar{x})), \\ \bar{\pi}(t,\bar{x}) &:= \tilde{\pi}(t,\Theta_h(t,\bar{x})) \end{aligned}$$

and compute $\nu_\Gamma = (-\nabla_{x'}h, 1)^{\mathsf{T}}/\sqrt{1+|\nabla_{x'}h|^2}$, where

$$\begin{aligned} \nabla\tilde{\pi} &= \nabla\bar{\pi} - M_0(h)\nabla\bar{\pi}, \\ \operatorname{div} u &= \operatorname{div}\bar{u} - (M_0(h)\nabla|\bar{u}), \\ \Delta u &= \Delta\bar{u} - M_1(h) : \nabla^2\bar{u} - M_2(h)\nabla\bar{u}, \\ \partial_t u &= \partial_t\bar{u} - \varphi\partial_t h(1+\varphi' h)^{-1}\partial_3\bar{u}, \end{aligned}$$

with

$$\begin{aligned} M_0(h) &:= \theta_h'^{\mathsf{T}}(I + \theta_h'^{\mathsf{T}})^{-1}, \\ M_1(h) : \nabla^2\bar{u} &:= \big[2\operatorname{sym}(\theta_h'^{\mathsf{T}}[I+\theta_h']^{-\mathsf{T}}) - [I+\theta_h']^{-1}\theta_h'\theta_h'^{\mathsf{T}}[I+\theta_h']^{-\mathsf{T}}\big] : \nabla^2\bar{u}, \end{aligned}$$

and

$$M_2(h)\nabla\bar{u} := \big([\Delta\Theta_h^{-1}]\circ\Theta_h|\nabla\big)\bar{u}.$$

Furthermore, it holds that

$$V_\Gamma = (\partial_t\Theta_h|\nu_\Gamma) = \partial_t h(e_3|\nu_\Gamma) = \frac{\partial_t h}{\sqrt{1+|\nabla_{x'}h|^2}}.$$

This yields the following transformed problem for $\overline{u}$ and $\overline{\pi}$ (for convenience, we drop the bars in what follows):

$$
\begin{aligned}
\partial_t(\rho u) - \mu\Delta u + \nabla\pi &= F(u,\pi,h), && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} u &= F_d(u,h), && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu\partial_3 v]\!] - [\![\mu\nabla_{x'}w]\!] &= G_v(u,h), && \text{on } \Sigma,\\
-2[\![\mu\partial_3 w]\!] + [\![\pi]\!] - \sigma\Delta_{x'}h - [\![\rho]\!]\gamma_a h &= G_w(u,h), && \text{on } \Sigma,\\
[\![u]\!] &= 0, && \text{on } \Sigma,\\
\partial_t h - w &= H_1(u,h), && \text{on } \Sigma,\\
P_{S_1}\big(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_1}\big) &= H_2(u,h), && \text{on } S_1\backslash\partial\Sigma,\\
u\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
u &= 0, && \text{on } S_2,\\
\partial_{\nu_{\partial G}} h &= 0, && \text{on } \partial\Sigma,\\
u(0) &= u_0, && \text{in } \Omega\backslash\Sigma,\\
h(0) &= h_0, && \text{on } \Sigma.
\end{aligned}
\tag{2.2}
$$

Here,

$$
\begin{aligned}
F(u,\pi,h) &:= \rho\varphi\partial_t h(1+\varphi' h)^{-1}\partial_3 u - \mu(M_1(h):\nabla^2 u + M_2(h)\nabla u) + M_0(h)\nabla\pi,\\
F_d(u,h) &:= (M_0(h)\nabla|u),\\
G_v(u,h) &:= -[\![\mu(\nabla v + \nabla v^{\mathsf{T}})]\!]\nabla h + |\nabla h|^2[\![\mu\partial_3 v]\!]\\
&\qquad + \big((1+|\nabla h|^2)[\![\mu\partial_3 w]\!] - (\nabla h|[\![\mu\nabla w]\!])\big)\nabla h,\\
G_w(u,h) &:= -(\nabla h|[\![\mu\nabla w]\!]) - (\nabla h|[\![\mu\partial_3 v]\!]) + |\nabla h|^2[\![\mu\partial_3 w]\!] + \sigma G_\kappa(h),\\
G_\kappa(h) &:= \operatorname{div}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big) - \Delta h,\\
H_1(u,h) &:= -(v|\nabla h),\\
H_2(u,h) &:= P_{S_1}(\mu(M_0(h)\nabla u + \nabla u^{\mathsf{T}} M_0(h)^{\mathsf{T}})\nu_{S_1}),
\end{aligned}
$$

where we have set

$$
\begin{gathered}
v := (u_1,u_2), \quad w := u_3,\\
\nabla w = \nabla_{x'}w, \quad \nabla v = \nabla_{x'}v, \quad \nabla h = \nabla_{x'}h
\end{gathered}
$$

for the sake of readability. Note that $H_2(u,h) = 0$ on $S_1 \setminus \partial\Sigma$ since $u\cdot\nu_{S_1} = 0$ on $S_1 \setminus \partial\Sigma$ and $\partial_{\nu_{\partial G}} h = 0$ on ∂G; see [55, Section 4.1].

The following result on the existence and uniqueness of strong L_p-solutions having optimal regularity has been published in [55, Theorem 4.2]:

Theorem 2.1. *Let $n = 3$, $p > 5$. For each given $T > 0$ there exists a number $\eta = \eta(T) > 0$ such that for all initial values $(u_0, h_0) \in W_p^{2-2/p}(\Omega\backslash\Sigma)^3 \times W_p^{3-2/p}(\Sigma)$ satisfying the* *compatibility conditions*

$$\begin{aligned}
\operatorname{div} u_0 &= F_d(u_0, h_0),\\
-[\![\mu\partial_3 v_0]\!] - [\![\mu\nabla_{x'} w_0]\!] &= G_v(v_0, h_0),\\
[\![u_0]\!] &= 0,\\
u_0 \cdot \nu_{S_1} &= 0,\\
P_{S_1}(\mu(\nabla u_0 + \nabla u_0^{\mathsf T})\nu_{S_1}) &= 0,\\
u_0|_{S_2} &= 0,\\
\partial_{\nu_{\partial G}} h_0 &= 0,
\end{aligned}$$

as well as the smallness condition

$$\|u_0\|_{W_p^{2-2/p}(\Omega\backslash\Sigma)} + \|h_0\|_{W_p^{3-2/p}(\Sigma)} \le \eta,$$

there exists a unique solution $(u, \pi, [\![\pi]\!], h)$ *of* (2.2) *with regularity*

$$u \in H_p^1((0,T); L_p(\Omega)^3) \cap L_p((0,T); H_p^2(\Omega\backslash\Sigma)^3), \quad \pi \in L_p((0,T); \dot H_p^1(\Omega)),$$
$$[\![\pi]\!] \in W_p^{1/2-1/2p}((0,T); L_p(\Sigma)) \cap L_p((0,T); W_p^{1-1/p}(\Sigma)).$$

and

$$h \in W_p^{2-1/2p}((0,T); L_p(\Sigma)) \cap H_p^1((0,T); W_p^{2-1/p}(\Sigma)) \cap L_p((0,T); W_p^{3-1/p}(\Sigma)).$$

3. Rayleigh–Taylor instability

3.1. Equilibria and spectrum of the linearisation

In this section we compute the equilibria of (2.1) as well as the spectrum of the full linearisation of (2.1) in the trivial equilibrium.

Assume that we have a time-independent solution of (2.1). Then multiplying $(2.1)_1$ by u and integrating by parts yields the identity

$$\|\mu^{1/2} Du\|^2_{L_2(\Omega)} = 0,$$

hence $u = 0$ on $\partial\Omega$ and therefore $u = 0$ in all of Ω, by Korn's inequality (Theorem A.4). If $u = 0$, then π must be constant, with possibly different values in different phases. Hence, condition $(2.1)_3$ yields that

$$\sigma H_\Gamma + [\![\rho]\!]\gamma_a x_3 = \text{const.}$$

on Γ. In particular, if $H_\Gamma = 0$ then x_3 must be constant, hence flat interfaces belong to the set of equilibria. Assume that Γ is given by the graph of a height function h, that is,

$$\Gamma = \big\{x \in \Omega : x_3 = h(x_1, x_2),\ (x_1, x_2) \in G\big\}.$$

Then the normal ν_Γ on Γ pointing from Ω_1 ($x_3 < h(x_1, x_2)$) into Ω_2 ($x_3 > h(x_1, x_2)$) is given by

$$\nu_\Gamma(x', h(x')) = \frac{1}{\sqrt{1 + |\nabla_{x'} h(x')|^2}}[-\nabla_{x'} h(x'), 1]^\mathsf{T}, \quad x' = (x_1, x_2) \in B_R(0).$$

Since $H_\Gamma = -\operatorname{div}_\Gamma \nu_\Gamma$, we obtain the quasilinear elliptic problem

$$\begin{aligned} \sigma \operatorname{div}_{x'}\Big(\frac{\nabla_{x'} h}{\sqrt{1 + |\nabla_{x'} h|^2}}\Big) + [\![\rho]\!]\gamma_a h = c, \quad & x' \in G, \\ \partial_{\nu_{\partial G}} h = 0, \quad & x' \in \partial G, \end{aligned} \tag{3.1}$$

where $c := \frac{[\![\rho]\!]\gamma_a}{|G|} \int_G h\,dx'$. All admissible height functions which solve (3.1) belong to the set of equilibria.

We are interested in the stability properties of the flat interface $\Sigma = G \times \{0\}$ in $\Omega = G \times (H_1, H_2)$. After transformation of (2.1) to the fixed domain $\Omega \backslash \Sigma$, we consider the full linearisation around the equilibrium $(0, \Sigma)$:

$$\begin{aligned} \partial_t(\rho u) - \mu \Delta u + \nabla \pi &= 0, && \text{in } \Omega\backslash\Sigma, \\ \operatorname{div} u &= 0, && \text{in } \Omega\backslash\Sigma, \\ -[\![\mu(\nabla u + \nabla u^\mathsf{T})]\!] e_3 + [\![\pi]\!] e_3 &= \sigma(\Delta_{x'} h) e_3 + [\![\rho]\!]\gamma_a h e_3, && \text{on } \Sigma, \\ [\![u]\!] &= 0, && \text{on } \Sigma, \\ \partial_t h - u_3 &= 0, && \text{on } \Sigma, \\ P_{S_1}\big(\mu(\nabla u + \nabla u^\mathsf{T})\nu_{S_1}\big) &= 0, && \text{on } S_1\backslash\partial\Sigma, \\ (u|\nu_{S_1}) &= 0, && \text{on } S_1\backslash\partial\Sigma, \\ u &= 0, && \text{on } S_2, \\ \partial_{\nu_{\partial G}} h &= 0, && \text{on } \partial\Sigma, \\ u(0) &= u_0, && \text{in } \Omega\backslash\Sigma, \\ h(0) &= h_0, && \text{on } \Sigma. \end{aligned} \tag{3.2}$$

Observe that by *conservation of mass*, it holds that

$$\int_G h(t)\, dx' = \int_G h_0\, dx'$$

for $t > 0$. Indeed, this follows from an integration of $(3.2)_5$ over $\Sigma = G \times \{0\}$ and the fact that

$$\int_G u_3\, dx' = \int_{\Omega_1} \operatorname{div} u\, dx = 0$$

by $(3.2)_{2,4,7,8}$ and the divergence theorem for Lipschitz domains. Therefore, if h_0 is mean value free, the solution $h(t)$ inherits this property for $t > 0$.

Define a linear operator $L : X_1 \to X_0$ by

$$L(u, h) := [(\mu/\rho)\Delta u - (1/\rho)\nabla\pi, u \cdot e_3], \tag{3.3}$$

where $X_0 := L_{p,\sigma}(\Omega) \times \{h \in W_p^{2-1/p}(\Sigma) : \int_G h\, dx' = 0,\ \partial_{\nu_{\partial G}} h = 0\}$,

$$L_{p,\sigma}(\Omega) := \overline{\{u \in C_c^\infty(\Omega)^3 : \operatorname{div} u = 0\}}^{\|\cdot\|_{L_p}}, \quad \bar{X}_1 = H_p^2(\Omega\backslash\Sigma)^3 \times W_p^{3-1/p}(\Sigma),$$

and

$$\begin{aligned} X_1 := D(L) = \big\{(u,h) \in X_0 \cap \bar{X}_1 : P_\Sigma([\![\mu(\nabla u + \nabla u^{\mathsf{T}})]\!]e_3) = 0,\ [\![u]\!] = 0,\ u|_{S_2} = 0, \\ P_{S_1}\big(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_1}\big) = 0, (u|\nu_{S_1}) = 0, \partial_{\nu_{\partial G}} h = 0\big\}. \end{aligned} \tag{3.4}$$

The function $\pi \in \dot{H}_p^1(\Omega\backslash\Sigma)$ in the definition of L is determined as the solution of the weak transmission problem

$$\begin{aligned} \Big(\frac{1}{\rho}\nabla\pi|\nabla\phi\Big)_{L_2(\Omega)} &= \Big(\frac{\mu}{\rho}\Delta u|\nabla\phi\Big)_{L_2(\Omega)}, \\ [\![\pi]\!] &= \sigma\Delta_{x'}h + [\![\rho]\!]\gamma_a h + ([\![\mu(\nabla u + \nabla u^{\mathsf{T}})]\!]e_3|e_3) \quad \text{on } \Sigma, \end{aligned}$$

where $\phi \in H_{p'}^1(\Omega)$ and $p' = p/(p-1)$, which we know is well-defined thanks to [55, Lemma 5.7]. We will sometimes make use of the notation via solution operators, i.e.,

$$\frac{1}{\rho}\nabla\pi = T_1[(\mu/\rho)\Delta u] + T_2[\sigma\Delta_{x'}h + [\![\rho]\!]\gamma_a h + ([\![\mu(\nabla u + \nabla u^{\mathsf{T}})]\!]e_3|e_3)], \tag{3.5}$$

where $T_1 : L_p(\Omega)^3 \to L_p(\Omega)^3$ and $T_2 : W_p^{1-1/p}(\Sigma) \to L_p(\Omega)^3$ are bounded linear operators.

In what follows, we will analyse the spectrum of the operator L. Note that L has a compact resolvent. This implies that the spectrum of L is discrete and it consists solely of eigenvalues with finite multiplicity. Consider the eigenvalue problem $\lambda(u,h) = L(u,h)$, that is,

$$\begin{aligned} \lambda\rho u - \mu\Delta u + \nabla\pi &= 0, && \text{in } \Omega\backslash\Sigma, \\ \operatorname{div} u &= 0, && \text{in } \Omega\backslash\Sigma, \\ -[\![\mu(\nabla u + \nabla u^{\mathsf{T}})]\!]e_3 + [\![\pi]\!]e_3 &= \sigma(\Delta_{x'}h)e_3 + [\![\rho]\!]\gamma_a h e_3, && \text{on } \Sigma, \\ [\![u]\!] &= 0, && \text{on } \Sigma, \\ \lambda h - u_3 &= 0, && \text{on } \Sigma, \\ P_{S_1}\big(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{\partial\Omega}\big) &= 0, && \text{on } S_1\backslash\partial\Sigma, \\ (u|\nu_{S_1}) &= 0, && \text{on } S_1\backslash\partial\Sigma, \\ u &= 0, && \text{on } S_2, \\ \partial_{\nu_{\partial G}} h &= 0, && \text{on } \partial\Sigma. \end{aligned} \tag{3.6}$$

We test the first equation with u and integrate by parts to obtain

$$\lambda|\rho^{1/2}u|_{L_2(\Omega)}^2 + \frac{1}{2}|\mu^{1/2}Du|_{L_2(\Omega)}^2 + \bar{\lambda}\big[\sigma|\nabla_{x'}h|_{L_2(G)}^2 - [\![\rho]\!]\gamma_a|h|_{L_2(G)}^2\big] = 0. \tag{3.7}$$

The above identity for $\lambda = 0$ implies $u = 0$, by Korn's inequality (Theorem A.4), hence p as well as $[\![p]\!]$ are constant. Therefore, h is a solution of the linear elliptic problem

$$\begin{aligned} \Delta_{x'} h + \frac{[\![\rho]\!]\gamma_a}{\sigma} h &= 0, \quad x' \in G, \\ \partial_{\nu_{\partial G}} h &= 0, \quad x' \in \partial G, \end{aligned} \tag{3.8}$$

since h is mean value free. Let $\sigma(-\Delta_N) \subset (0,\infty)$ denote the spectrum of the negative Neumann Laplacian in the space

$$X := \left\{ h \in W_p^{1-1/p}(G) : \int_G h \, dx' = 0 \right\}$$

and let $E(\eta)$ denote the eigenspace corresponding to the eigenvalue $\eta \in \sigma(-\Delta_N)$. It follows that $h = 0$ is the unique solution of (3.8) if and only if

$$\frac{[\![\rho]\!]\gamma_a}{\sigma} \notin \sigma(-\Delta_N),$$

and there exists $0 \neq h \in E(\eta)$ if and only if

$$\eta := \frac{[\![\rho]\!]\gamma_a}{\sigma} \in \sigma(-\Delta_N).$$

This shows that

$$0 \in \sigma(L) \quad \text{if and only if} \quad \frac{[\![\rho]\!]\gamma_a}{\sigma} \in \sigma(-\Delta_N).$$

Suppose that $0 \neq \lambda \in \sigma(L)$ with $\mathrm{Re}\,\lambda = 0$. Taking real parts in (3.7), it follows that $u = 0$ by Korn's inequality (Theorem A.4), hence h must be nontrivial. By equation $(3.6)_5$, it follows that $\lambda = 0$. This shows that $\lambda = 0$ is the only eigenvalue of L on the imaginary axis.

In particular, if

$$\frac{[\![\rho]\!]\gamma_a}{\sigma} < \lambda_1,$$

with $\lambda_1 > 0$ being the first nontrivial eigenvalue of $-\Delta_N$ in X, then

$$\sigma(L) \subset \left\{ \lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \leq -\omega < 0 \right\}$$

for some $\omega > 0$, since

$$|\nabla_{x'} h|_{L_2(G)}^2 - \frac{[\![\rho]\!]\gamma_a}{\sigma} |h|_{L_2(G)}^2 \geq 0$$

by the Poincaré inequality for functions h with mean value zero. To see this, observe that (by a bootstrap argument) e_j is an eigenfunction of $-\Delta_N$ with eigenvalue λ_j in X if and only if e_j is an eigenfunction of $-\Delta_N$ with eigenvalue λ_j in

$$L_2^{(0)}(G) := \left\{ h \in L_2(G) : \int_G h \, dx' = 0 \right\}.$$

As $-\Delta_N$ is self-adjoint in $L_2^{(0)}(G)$ with compact resolvent, the spectral mapping theorem yields

$$|\nabla_{x'} h|^2_{L_2(G)} \geq \lambda_1 |h|^2_{L_2(G)}$$

for all $h \in H_2^1(G) \cap L_2^{(0)}(G)$.

Note that there exists $\kappa > 0$ such that $\kappa - L$ is a sectorial operator, since L has maximal L_p-regularity. In particular, it holds that $\sigma(L-\kappa) \subset \Sigma_{\pi/2+\delta}$, or equivalently, $\sigma(L) \subset \Sigma_{\pi/2+\delta} + \kappa$ for some $\delta \in (0, \pi/2)$. This concludes the proof of existence of the number $\omega > 0$ above.

Now, we aim to show that $\sigma(L) \cap \mathbb{C}_+ \neq \emptyset$ whenever $\frac{[\![\rho]\!]\gamma_a}{\sigma} > \lambda_1$. To this end, for $\lambda \geq 0$ and given $g \in W_p^{1-1/p}(G)$, $p > 2$, we solve the elliptic two-phase Stokes problem

$$\begin{aligned} \lambda\rho u - \mu\Delta u + \nabla\pi &= 0, && \text{in } \Omega\backslash\Sigma,\\ \operatorname{div} u &= 0, && \text{in } \Omega\backslash\Sigma,\\ -[\![\mu(\nabla u + \nabla u^{\mathsf T})]\!]e_3 + [\![\pi]\!]e_3 &= ge_3, && \text{on } \Sigma,\\ [\![u]\!] &= 0, && \text{on } \Sigma,\\ P_{S_1}\left(\mu(\nabla u + \nabla u^{\mathsf T})\nu_{S_1}\right) &= 0, && \text{on } S_1\backslash\partial\Sigma,\\ (u|\nu_{S_1}) &= 0, && \text{on } S_1\backslash\partial\Sigma,\\ u &= 0, && \text{on } S_2 \end{aligned} \tag{3.9}$$

by Theorem A.3 to obtain a unique solution $u \in H_p^2(\Omega\backslash\Sigma) \cap H_p^1(\Omega)$. Define the (reduced) *Neumann-to-Dirichlet operator* $N_\lambda : W_p^{1-1/p}(G) \to W_p^{2-1/p}(G)$ by $N_\lambda g := (u|e_3)$. With the compact operator N_λ at hand, we may rewrite the eigenvalue problem (3.6) as

$$\lambda h + N_\lambda(A_* h) = 0, \tag{3.10}$$

where $A_* h := -\sigma\Delta_N h - [\![\rho]\!]\gamma_a h$ is the shifted Neumann Laplacian with domain

$$D(A_*) = \left\{h \in W_p^{3-1/p}(G) : \int_G h\, dx' = 0,\ \partial_{\nu_{\partial G}} h = 0 \text{ on } \partial G\right\}.$$

We remark that for $\lambda \geq 0$ problems (3.6) and (3.10) are equivalent. Therefore, it suffices to show that for $\frac{[\![\rho]\!]\gamma_a}{\sigma} > \lambda_1$ there exists $\lambda > 0$ such that equation (3.10) has a nontrivial solution $h \in D(A_*)$.

Concerning N_λ, we have the following result (see also [31, Section 10.5] for the case of a bounded smooth domain with $\Gamma(t) \cap \partial\Omega = \emptyset$):

Proposition 3.1. *The Neumann-to-Dirichlet operator N_λ of the Stokes problem (3.9) admits a compact self-adjoint extension to $L_2(G)$ which has the following properties:*

(1) *If u denotes the solution of (3.9), then*

$$(N_\lambda g|g)_2 = \lambda|\rho^{1/2}u|^2_{L_2(\Omega)} + \frac{1}{2}|\mu^{1/2}Du|^2_{L_2(\Omega)}$$

for all $g \in W_p^{1-1/p}(G)$ and $\lambda \geq 0$.

(2) *For each* $\alpha \in (0, 1/2)$ *there is a constant* $C > 0$ *such that*

$$(N_\lambda g|g)_2 \geq \frac{(1+\lambda)^\alpha}{C} |N_\lambda g|^2_{L_2(G)}$$

for all $g \in L_2(G)$ *and* $\lambda \geq 0$. *In particular,*

$$|N_\lambda|_{\mathcal{B}(L_2(G))} \leq \frac{C}{(1+\lambda)^\alpha}$$

for all $\lambda \geq 0$.

(3) $N_\lambda g$ *has mean value zero for all* $\lambda \geq 0$ *and each* $g \in L_2(G)$.

Proof. The first assertion follows from integration by parts, while for the proof of the second assertion one uses trace theory, interpolation theory and Korn's inequality (Theorem A.4). To show the third assertion, observe that for each $\lambda \geq 0$ we have

$$\int_G N_\lambda g \, dx' = \int_G (u|e_3) \, dx' = \int_{\Omega_1} \operatorname{div} u_1 \, dx = 0$$

by the divergence theorem, where $u_1 := u|_{\Omega_1}$. ■

Proposition 3.1 combined with Korn's inequality (Theorem A.4) implies that whenever $N_\lambda g = 0$, then $u = 0$, hence g must be constant. Therefore, the restriction of N_λ to functions with mean value zero is injective. Therefore, we may rewrite equation (3.10) as

$$\lambda N_\lambda^{-1} h + A_* h = 0 \tag{3.11}$$

for each $h \in D(A_*)$. Let us consider (3.11) in $L_2^{(0)}(G)$, the subspace of $L_2(G)$ consisting of functions with vanishing mean value. Define $B_\lambda := \lambda N_\lambda^{-1} + A_*$ with

$$D(B_\lambda) = D(A_*) = \left\{h \in W_2^2(G) \cap L_2^{(0)}(G) : \partial_{\nu_{\partial G}} h = 0 \text{ on } \partial G\right\},$$

since N_λ^{-1} is a relatively compact perturbation of A_*. We will show that the operator B_λ is positive definite provided $\lambda > 0$ is large enough. Let $\mu_j > 0$ be an eigenvalue of N_λ^{-1} in $L_2^{(0)}(G)$ with corresponding eigenfunction e_j. Then

$$\frac{1}{\mu_j}|e_j|_2 = |N_\lambda e_j|_2 \leq \frac{C}{(1+\lambda)^\alpha}|e_j|_2,$$

hence $\mu_j \geq \frac{1}{C} > 0$ for each $\lambda \geq 0$. It follows that

$$(B_\lambda h|h)_2 = \lambda(N_\lambda^{-1} h|h)_2 + (A_* h|h)_2 \geq (\lambda/C - [\![\rho]\!]\gamma_a)\,|h|_2^2 > 0$$

for each $h \in D(A_*)$, if $\lambda > 0$ is sufficiently large.

On the other hand, let $0 \neq h_* \in D(A_*)$ be an eigenfunction of $-\Delta_N$ to the first nontrivial eigenvalue $\lambda_1 > 0$ of $-\Delta_N$, i.e., $-\Delta_N h_* = \lambda_1 h_*$. This yields

$$(B_\lambda h_*|h_*)_2 = \lambda(N_\lambda^{-1} h_*|h_*)_2 - \sigma\Big(\frac{[\![\rho]\!]\gamma_a}{\sigma} - \lambda_1\Big)|h_*|_2^2.$$

Since $\lim_{\lambda\to 0_+}\lambda(N_\lambda^{-1}h_*|h_*)=0$, it follows that $(B_\lambda h_*|h_*)_2<0$ provided $\lambda>0$ is sufficiently small and $\frac{[\![\rho]\!]\gamma_a}{\sigma}>\lambda_1$. Let $\frac{[\![\rho]\!]\gamma_a}{\sigma}>\lambda_1$ and define

$$\lambda_*:=\sup\big\{\lambda>0: B_\mu \text{ is not positive semi-definite for each } \mu\in(0,\lambda]\big\}.$$

Then, $\lambda_*>0$ by what we have shown above and B_λ has a negative eigenvalue for each $\lambda<\lambda_*$, since the resolvent of B_λ is compact. It follows that $0\in\sigma(B_{\lambda_*})$, hence there exists a solution $0\neq h\in D(A_*)$ in $L_2^{(0)}(G)$ of (3.11). A bootstrap argument finally shows that $h\in D(A_*)\cap W_p^{3-1/p}(G)$. This in turn yields that $\sigma(L)\cap\mathbb{C}_+\neq\emptyset$ whenever $\frac{[\![\rho]\!]\gamma_a}{\sigma}>\lambda_1$. We have proven the following result:

Proposition 3.2. *The operator L defined above has the following spectral properties:*

(1) $\sigma(L)\cap i\mathbb{R}\subset\{0\}$ *and* $0\in\sigma(L)$ *if and only if* $[\![\rho]\!]\gamma_a/\sigma\in\sigma(-\Delta_N)$.

(2) *If* $[\![\rho]\!]\le 0$ *then* $\sigma(L)\subset\mathbb{C}_-$.

(3) *If* $[\![\rho]\!]>0$ *and* $\frac{[\![\rho]\!]\gamma_a}{\sigma}<\lambda_1$, *then* $\sigma(L)\subset\mathbb{C}_-$.

(4) *If* $[\![\rho]\!]>0$ *and* $\frac{[\![\rho]\!]\gamma_a}{\sigma}>\lambda_1$, *then* $\sigma(L)\cap\mathbb{C}_+\neq\emptyset$.

3.2. Parametrisation of the nonlinear phase manifold

We have already seen that after a Hanzawa transformation, the transformed velocity field is no longer divergence free. Moreover, the jump condition of the stress tensor as well as the divergence condition are transformed into some nonlinear terms. It is the aim of this section to parametrise the nonlinear phase manifold

$$\begin{aligned}\mathcal{PM}:=\big\{&(u,h)\in W_p^{2-2/p}(\Omega\backslash\Sigma)^3\times[W_p^{3-2/p}(\Sigma)\cap X]:\\ &u|_{S_2}=0,\ u|_{S_1}\cdot\nu_{S_1}=0,\ P_{S_1}(\mu(\nabla u+\nabla u^{\mathsf{T}})\nu_{S_1})=0,\ [\![u]\!]=0,\\ &P_\Sigma(\mu(\nabla u+\nabla u^{\mathsf{T}})e_3)=(G_v(u,h),0),\ \partial_{\nu_{\partial G}}h=0,\ \operatorname{div}u=F_d(u,h)\big\}\end{aligned}$$

as a subset of the set $X_\gamma:=W_p^{2-2/p}(\Omega\backslash\Sigma)^3\times W_p^{3-2/p}(\Sigma)$, near the trivial equilibrium $(u_*,h_*)=(0,0)$ over the linear phase manifold

$$\begin{aligned}X_\gamma^0:=\big\{&(u,h)\in[W_p^{2-2/p}(\Omega\backslash\Sigma)^3\times W_p^{3-2/p}(\Sigma)]\cap X_0: u|_{S_2}=0,\ u|_{S_1}\cdot\nu_{S_1}=0,\\ &P_{S_1}(\mu(\nabla u+\nabla u^{\mathsf{T}})\nu_{S_1})=0,\ [\![u]\!]=0,\ P_\Sigma(\mu(\nabla u+\nabla u^{\mathsf{T}})e_3)=0,\ \partial_{\nu_{\partial G}}h=0\big\}.\end{aligned}$$

Let $\mathbb{E}_\pi:=\dot W_p^{1-2/p}(\Omega\backslash\Sigma)$, $\mathbb{E}_q:=W_p^{1-3/p}(\Sigma)$,

$$\begin{aligned}\mathbb{E}_u:=\big\{&u\in W_p^{2-2/p}(\Omega\backslash\Sigma)^3:[\![u]\!]=0,\ u|_{S_1}\cdot\nu_{S_1}=0,\ u|_{S_2}=0,\\ &P_{S_1}(\mu(\nabla u+\nabla u^{\mathsf{T}})\nu_{S_1})=0\big\},\end{aligned}$$

$\mathbb{E}:=\{(u,\pi,q)\in\mathbb{E}_u\times\mathbb{E}_\pi\times\mathbb{E}_q: q=[\![\pi]\!]\}$, and

$$\mathbb{F}:=\big\{(f_1,f_2)\in[W_p^{1-2/p}(\Omega\backslash\Sigma)\cap\hat H_p^{-1}(\Omega)]\times W_p^{1-3/p}(\Sigma)^3:(P_\Sigma f_2)\cdot\nu_{S_1}=0 \text{ at } \partial\Sigma\big\}.$$

We will need the following auxiliary result for the Stokes problem:

$$
\begin{aligned}
\rho\omega u - \mu\Delta u + \nabla\pi &= 0, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} u &= f_d, && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu\partial_3 v]\!] - [\![\mu\nabla_{x'} w]\!] &= g_v, && \text{on } \Sigma,\\
-2[\![\mu\partial_3 w]\!] + [\![\pi]\!] &= g_w, && \text{on } \Sigma,\\
[\![u]\!] &= 0, && \text{on } \Sigma,\\
P_{S_1}(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_1}) &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
u\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
u &= 0, && \text{on } S_2.
\end{aligned}
\tag{3.12}
$$

Proposition 3.3. *Let $n = 3$, $p > 5$ and $\rho_j, \mu_j > 0$. If $\omega > 0$ is sufficiently large, then there exists a unique solution $(u, \pi, [\![\pi]\!]) \in \mathbb{E}$ of (3.12) if and only if $(f_d, (g_v, g_w)) \in \mathbb{F}$. Moreover, there exists a constant $M_\omega > 0$ such that*

$$
\|(u, \pi, [\![\pi]\!])\|_{\mathbb{E}} \le M_\omega \|(f_d, (g_v, g_w))\|_{\mathbb{F}}.
$$

Proof. For the proof of this result one may apply the same strategy which was used in the proof of Theorem A.3. We omit the details. ∎

Let us consider the elliptic problem

$$
\begin{aligned}
\rho\omega\bar{u} - \mu\Delta\bar{u} + \nabla\bar{\pi} &= 0, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div}\bar{u} &= P_0 F_d(\bar{u}+\tilde{u}, \tilde{h}), && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu\partial_3\bar{v}]\!] - [\![\mu\nabla_{x'}\bar{w}]\!] &= G_v(\bar{u}+\tilde{u}, \tilde{h}), && \text{on } \Sigma,\\
-2[\![\mu\partial_3\bar{w}]\!] + [\![\bar{\pi}]\!] &= G_w(\bar{u}+\tilde{u}, \tilde{h}), && \text{on } \Sigma,\\
[\![\bar{u}]\!] &= 0, && \text{on } \Sigma,\\
P_{S_1}(\mu(\nabla\bar{u} + \nabla\bar{u}^{\mathsf{T}})\nu_{S_1}) &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
\bar{u}\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
\bar{u} &= 0, && \text{on } S_2
\end{aligned}
\tag{3.13}
$$

for $(\bar{u}, \bar{\pi}, [\![\bar{\pi}]\!])$, where $\omega > 0$ and $(\tilde{u}, \tilde{h}) \in rB_{X_\gamma^0}(0)$ are given. Here we have set

$$
P_0 f := f - \frac{1}{|\Omega|}\int_\Omega f\,dx
$$

for $f \in L_1(\Omega)$.

Define a nonlinear mapping $N : \mathbb{E}_u \times X_\gamma^0 \to \mathbb{F}$ via

$$
N(\bar{u}, \tilde{u}, \tilde{h}) := \begin{pmatrix} P_0 F_d(\bar{u}+\tilde{u}, \tilde{h}) \\ \big(G_v(\bar{u}+\tilde{u}, \tilde{h}), G_w(\bar{u}+\tilde{u}, \tilde{h})\big)^{\mathsf{T}} \end{pmatrix}.
$$

Let S_ω denote the solution operator which is induced by Proposition 3.3 and define a mapping $H := \mathbb{E} \times X_\gamma^0 \to \mathbb{E}$ by

$$H((\bar{u},\bar{\pi},\bar{q}),(\tilde{u},\tilde{h})) := (\bar{u},\bar{\pi},\bar{q}) - S_\omega N(\bar{u},\tilde{u},\tilde{h}),$$

where $\bar{q}$ is a dummy variable representing $[\![\bar{\pi}]\!]$. Since $N(0) = 0$, it follows that the equation $H(0,0) = 0$ holds. Since $N \in C^2$, it holds that $H \in C^2$, too. Differentiating H with respect to $(\bar{u},\bar{\pi},\bar{q})$, we obtain

$$D_{(\bar{u},\bar{\pi},\bar{q})} H(0,0) = I_{\mathbb{E}},$$

where we used the fact that $D_{\bar{u}} N(0) = 0$. The implicit function theorem implies the existence of a C^2-function $\phi_0 : rB_{X_\gamma^0} \to \mathbb{E}$ with $\phi_0(0) = 0$ and $\phi_0'(0) = 0$, such that $H(\phi_0(\tilde{u},\tilde{h}),(\tilde{u},\tilde{h})) = 0$ whenever $(\tilde{u},\tilde{h}) \in rB_{X_\gamma^0}(0)$. In other words, this means that $(\bar{u},\bar{\pi},\bar{q}) = \phi_0(\tilde{u},\tilde{h})$ is the unique solution of (3.13) for a given $(\tilde{u},\tilde{h}) \in rB_{X_\gamma^0}(0)$. Furthermore, it can be shown that $P_0 F_d(\bar{u}+\tilde{u},\tilde{h}) = F_d(\bar{u}+\tilde{u},\tilde{h})$ (see proof of [55, Theorem 4.2]).

Let $P(\bar{u},\bar{\pi},\bar{q}) := \bar{u}$ and define $\phi(\tilde{u},\tilde{h}) := P\phi_0(\tilde{u},\tilde{h})$ as well as

$$\Phi(\tilde{u},\tilde{h}) := (\tilde{u},\tilde{h}) + (\phi(\tilde{u},\tilde{h}),0).$$

It follows that $\Phi(rB_{X_\gamma^0}(0)) \subset \mathcal{PM}$ and that Φ is injective. We will now show that Φ is locally surjective near 0. To this end, we assume that $(u,h) \in \mathcal{PM}$ is given and close to 0 in X_γ. Then we solve the linear problem

$$\begin{aligned}
\rho\omega\bar{u} - \mu\Delta\bar{u} + \nabla\bar{\pi} &= 0, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div}\bar{u} &= P_0 F_d(u,h), && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu\partial_3\bar{v}]\!] - [\![\mu\nabla_{x'}\bar{w}]\!] &= G_v(u,h), && \text{on } \Sigma,\\
-2[\![\mu\partial_3\bar{w}]\!] + [\![\bar{\pi}]\!] &= G_w(u,h), && \text{on } \Sigma,\\
[\![\bar{u}]\!] &= 0, && \text{on } \Sigma,\\
P_{S_1}(\mu(\nabla\bar{u} + \nabla\bar{u}^{\mathsf{T}})\nu_{S_1}) &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
\bar{u}\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
\bar{u} &= 0, && \text{on } S_2
\end{aligned} \tag{3.14}$$

by Proposition 3.3 to obtain $\bar{u} \in \mathbb{E}_u$. Define $(\tilde{u},\tilde{h}) := (u-\bar{u},h)$ and observe that

$$\operatorname{div}\tilde{u} = F_d(u,h) - P_0 F_d(u,h) = \frac{1}{|\Omega|}\int_\Omega F_d(u,h)\,dx.$$

Since $\tilde{u} \in H_p^1(\Omega)^3$ with $\tilde{u}|_{S_1}\cdot\nu_{S_1} = 0$, $\tilde{u}|_{S_2} = 0$ and $[\![\tilde{u}]\!] = 0$, it follows that the equation $P_0 F_d(u,h) = F_d(u,h)$ holds, hence $\operatorname{div}\tilde{u} = 0$.

This in turn yields $(\tilde{u},\tilde{h}) \in X_\gamma^0$ and $\phi(\tilde{u},\tilde{h}) = \bar{u}$, showing that Φ is locally surjective near 0.

3.3. Main result on Rayleigh–Taylor instability

In this section we are going to prove the following main result:

Theorem 3.4. *Let $n = 3$, $p > 5$ and $\rho_j, \mu_j, \gamma_j, \sigma > 0$. Denote by $(u_*, h_*) = (0,0)$ the trivial equilibrium and let $s(L)$ denote the spectral bound of the linearisation L (see equation (3.3)). Furthermore, let $\lambda_1 > 0$ be the first eigenvalue of $-\Delta_N$ in*

$$\Big\{h \in W_p^{1-1/p}(G) : \int_G h\, dx' = 0\Big\}.$$

Then, the following assertions hold:

(1) *If $[\![\rho]\!]\gamma_a/\sigma < \lambda_1$, then (u_*, h_*) is exponentially stable in the sense that there exist constants $\eta \in [0, -s(L))$ and $\delta > 0$ such that whenever $(u_0, h_0) \in \mathcal{PM}$ with*

$$\|(u_0, h_0)\|_{X_\gamma} \le \delta,$$

the solution (u, h) of (2.2) exists globally and satisfies the estimate

$$\|(u(t), h(t))\|_{X_\gamma} \le e^{-\eta t}\|(u_0, h_0)\|_{X_\gamma}$$

for all $t \ge 0$.

(2) *If $[\![\rho]\!] > 0$ and $[\![\rho]\!]\gamma_a/\sigma > \lambda_1$, then (u_*, h_*) is unstable in the sense that there is a constant $\varepsilon_0 > 0$ such that for each $\delta > 0$ there are initial values $(u_0, h_0) \in \mathcal{PM}$ with*

$$\|(u_0, h_0)\|_{X_\gamma} \le \delta$$

such that the solution (u, h) of (2.2) satisfies

$$\|(u(t_0), h(t_0))\|_{X_\gamma} \ge \varepsilon_0$$

for some $t_0 > 0$.

Proof. For $\eta \ge 0$, let

$$\begin{aligned}
e^{-\eta}\mathbb{E}_u(\mathbb{R}_+) &:= \big\{u \in e^{-\eta}[H_p^1(\mathbb{R}_+; L_p(\Omega)^3) \cap L_p(\mathbb{R}_+; H_p^2(\Omega\backslash\Sigma)^3)] : [\![u]\!] = 0,\\
&\qquad u \cdot \nu_{S_1} = 0,\ P_{S_1}(\mu(\nabla u + \nabla u^\mathsf{T})\nu_{S_1}) = 0,\ u|_{S_2} = 0\big\},\\
e^{-\eta}\mathbb{E}_\pi(\mathbb{R}_+) &:= e^{-\eta}L_p(\mathbb{R}_+; \dot{H}_p^1(\Omega\backslash\Sigma)),\\
e^{-\eta}\mathbb{E}_q(\mathbb{R}_+) &:= e^{-\eta}[W_p^{1/2-1/2p}(\mathbb{R}_+; L_p(\Sigma)) \cap L_p(\mathbb{R}_+; W_p^{1-1/p}(\Sigma))],\\
e^{-\eta}\mathbb{E}_h(\mathbb{R}_+) &:= \big\{h \in e^{-\eta}[W_p^{2-1/2p}(\mathbb{R}_+; L_p(\Sigma))\\
&\qquad \cap H_p^1(\mathbb{R}_+; W_p^{2-1/p}(\Sigma)) \cap L_p(\mathbb{R}_+; W_p^{3-1/p}(\Sigma))] : \partial_{\nu_{\partial G}} h = 0\big\},
\end{aligned}$$

and

$$e^{-\eta}\mathbb{E}(\mathbb{R}_+) := \big\{(u,\pi,q,h) \in e^{-\eta}[\mathbb{E}_u(\mathbb{R}_+) \times \mathbb{E}_\pi(\mathbb{R}_+) \times \mathbb{E}_q(\mathbb{R}_+) \times \mathbb{E}_h(\mathbb{R}_+)] : q = [\![\pi]\!]\big\}.$$

Moreover, we define the data spaces as follows:

$$
\begin{aligned}
e^{-\eta}\mathbb{F}_1(\mathbb{R}_+) &:= e^{-\eta}L_p(\mathbb{R}_+;L_p(\Omega)^3),\\
e^{-\eta}\mathbb{F}_2(\mathbb{R}_+) &:= e^{-\eta}[H_p^1(\mathbb{R}_+;\widehat{H}_p^{-1}(\Omega))\cap L_p(\mathbb{R}_+;H_p^1(\Omega\backslash\Sigma))],\\
e^{-\eta}\mathbb{F}_3(\mathbb{R}_+) &:= \big\{f_3\in e^{-\eta}[W_p^{1/2-1/2p}(\mathbb{R}_+;L_p(\Sigma)^3)\cap L_p(\mathbb{R}_+;W_p^{1-1/p}(\Sigma)^3)]:\\
&\qquad P_\Sigma(f_3)\cdot\nu_{S_1}=0\big\},\\
e^{-\eta}\mathbb{F}_4(\mathbb{R}_+) &:= \big\{f_4\in e^{-\eta}[W_p^{1-1/2p}(\mathbb{R}_+;L_p(\Sigma))\\
&\qquad \cap L_p(\mathbb{R}_+;W_p^{2-1/p}(\Sigma))]:\partial_{\nu_{\partial G}}f_4=0\big\},
\end{aligned}
$$

and $e^{-\eta}\mathbb{F}(\mathbb{R}_+):=\bigtimes_{j=1}^4 e^{-\eta}\mathbb{F}_j(\mathbb{R}_+)$. We can now prove each of the assertions.

(1) Let $(u_0,h_0)\in\mathcal{PM}$ be fixed such that $\|u_0\|_{W_p^{2-2/p}}+\|h_0\|_{W_p^{3-2/p}}\le\delta$ for some sufficiently small $\delta>0$ to be determined later. It follows from the results of Section 3.2 that $(u_0,h_0)=(\tilde u_0,\tilde h_0)+(\phi(\tilde u_0,\tilde h_0),0)$, i.e., we have $\tilde h_0=h_0$, where $(\tilde u_0,\tilde h_0)\in rB_{X_\gamma^0}(0)$. For $h\in L_1(\Sigma)$, we define

$$P_0^\Sigma h:=h-\frac{1}{|\Sigma|}\int_\Sigma h\,dx',$$

and consider the linear evolution equation

$$\partial_t(\tilde u,\tilde h)-L(\tilde u,\tilde h)=\omega\big((I-T_1)\bar u,\,P_0^\Sigma\bar h\big),\quad(\tilde u,\tilde h)|_{t=0}=(\tilde u_0,\tilde h_0)\tag{3.15}$$

in the space

$$X_0:=L_{p,\sigma}(\Omega)\times\Big\{h\in W_p^{2-1/p}(\Sigma):\int_G h\,dx'=0,\ \partial_{\nu_{\partial G}}h=0\Big\},$$

where L has been defined in Section 3.1 and $(\bar u,\bar h)\in e^{-\eta}[\mathbb{E}_u(\mathbb{R}_+)\times\mathbb{E}_h(\mathbb{R}_+)]$ are given functions. Here $\eta\in[0,-s(L))$, where $s(L)$ denotes the spectral bound of L.

By [55, Corollary 3.3] and Proposition 3.2, it follows that the operator L has the property of L_p-maximal regularity on $\mathbb{R}_+$ provided that $[\![\rho]\!]\gamma_a/\sigma<\lambda_1$. Since $(f,g):=\omega((I-T_1)\bar u,P_0^\Sigma\bar h)\in e^{-\eta}L_p(\mathbb{R}_+;X_0)$ and $(\tilde u_0,\tilde h_0)\in X_\gamma^0$, we obtain a unique solution

$$(\tilde u,\tilde h)\in e^{-\eta}[H_p^1(\mathbb{R}_+;X_0)\cap L_p(\mathbb{R}_+;X_1)]=:e^{-\eta}\widetilde{\mathbb{E}}(\mathbb{R}_+)$$

for each $\eta\in[0,-s(L))$, where $X_1=D(L)$ is given by (3.4). We denote by

$$\Xi:=(\partial_t-L,\operatorname{tr}|_{t=0})^{-1}:e^{-\eta}L_p(\mathbb{R}_+;X_0)\times X_\gamma^0\to e^{-\eta}\widetilde{\mathbb{E}}(\mathbb{R}_+)$$

the corresponding solution operator which satisfies the estimate

$$\|\Xi((f,g),(\tilde u_0,\tilde h_0))\|_{e^{-\eta}\widetilde{\mathbb{E}}(\mathbb{R}_+)}\le M\|((f,g),(\tilde u_0,\tilde h_0))\|_{e^{-\eta}L_p(\mathbb{R}_+;X_0)\times X_\gamma^0}.$$

In particular, by (3.5), we obtain on the one hand that $\nabla\tilde\pi$ is given in terms of $(\bar u,\bar h)$ and

$$\|\nabla\tilde\pi\|_{e^{-\eta}L_p(\mathbb{R}_+;L_p(\Omega))}\le CM\|((f,g),(\tilde u_0,\tilde h_0))\|_{e^{-\eta}L_p(\mathbb{R}_+;X_0)\times X_\gamma^0}.$$

At this point, we remark that the function $\widetilde{h}$ possesses some more regularity. Indeed, it holds that

$$\partial_t \widetilde{h} = \widetilde{u}_3|_\Sigma + \omega P_0^\Sigma \overline{h} \in e^{-\eta} W_p^{1-1/2p}(\mathbb{R}_+; L_p(\Sigma)),$$

hence $\widetilde{h} \in e^{-\eta} W_p^{2-1/2p}(\mathbb{R}_+; L_p(\Sigma))$ holds in addition.

Next, we consider the problem

$$\begin{aligned}
\omega\rho\overline{u} + \partial_t \rho\overline{u} - \mu\Delta\overline{u} + \nabla\overline{\pi} &= F(\widetilde{u}+\overline{u}, \widetilde{\pi}+\overline{\pi}, \widetilde{h}+\overline{h}), && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div}\overline{u} &= P_0 F_d(\widetilde{u}+\overline{u}, \widetilde{h}+\overline{h}), && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu\partial_3\overline{v}]\!] - [\![\mu\nabla_{x'}\overline{w}]\!] &= G_v(\widetilde{u}+\overline{u}, \widetilde{h}+\overline{h}), && \text{on } \Sigma,\\
-2[\![\mu\partial_3\overline{w}]\!] + [\![\overline{\pi}]\!] - \sigma\Delta_{x'}\overline{h} - [\![\rho]\!]\gamma_a\overline{h} &= G_w(\widetilde{u}+\overline{u}, \widetilde{h}+\overline{h}), && \text{on } \Sigma,\\
[\![\overline{u}]\!] &= 0, && \text{on } \Sigma,\\
\omega\overline{h} + \partial_t\overline{h} - u\cdot e_3 &= H_1(\widetilde{u}+\overline{u}, \widetilde{h}+\overline{h}), && \text{on } \Sigma,\\
P_{S_1}\left(\mu(\nabla\overline{u} + \nabla\overline{u}^{\mathsf{T}})\nu_{S_1}\right) &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
\overline{u}\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
\overline{u} &= 0, && \text{on } S_2,\\
\partial_{\nu_{\partial G}}\overline{h} &= 0, && \text{on } \partial\Sigma,\\
\overline{u}(0) &= \phi(\widetilde{u}_0, \widetilde{h}_0), && \text{in } \Omega\backslash\Sigma,\\
\overline{h}(0) &= 0, && \text{on } \Sigma,
\end{aligned} \tag{3.16}$$

where $(\widetilde{u}, \widetilde{h}) = \omega\Xi((I - T_1)\overline{u}, P_0^\Sigma\overline{h})$ and $\nabla\widetilde{\pi}$ is given by (3.5), with (u, h) being replaced by $(\widetilde{u}, \widetilde{h})$.

Define an operator $\mathbb{L}_\omega : e^{-\eta}\mathbb{E}(\mathbb{R}_+) \to e^{-\eta}\mathbb{F}(\mathbb{R}_+)$ by

$$\mathbb{L}_\omega(\overline{u}, \overline{\pi}, \overline{q}, \overline{h}) := \begin{pmatrix} \omega\rho\overline{u} + \partial_t\rho\overline{u} - \mu\Delta\overline{u} + \nabla\overline{\pi} \\ \operatorname{div}\overline{u} \\ -[\![\mu(\nabla\overline{u} + \nabla\overline{u}^{\mathsf{T}})]\!]e_3 + \overline{q}e_3 - \sigma\Delta_{x'}\overline{h}e_3 - [\![\rho]\!]\gamma_a\overline{h}e_3 \\ \omega\overline{h} + \partial_t\overline{h} - \overline{u}\cdot e_3 \end{pmatrix},$$

where $\overline{q} = [\![\overline{\pi}]\!]$. Set

$$\begin{aligned}
\bar{X}_\gamma := \big\{&(u, h) \in W_p^{2-2/p}(\Omega\backslash\Sigma)^3 \times W_p^{3-2/p}(\Sigma) : u|_{S_2} = 0,\\
&u|_{S_1}\cdot\nu_{S_1} = 0,\ P_{S_1}(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_1}) = 0,\ [\![u]\!] = 0,\ \partial_{\nu_{\partial G}} h = 0\big\}
\end{aligned}$$

and denote by

$$\operatorname{ext}_\eta : \bar{X}_\gamma \to e^{-\eta}[\mathbb{E}_u(\mathbb{R}_+) \times \mathbb{E}_h(\mathbb{R}_+)]$$

a linear extension operator such that $\operatorname{ext}_\eta(\hat{u}, \hat{h})|_{t=0} = (\hat{u}, \hat{h})$. The existence of such an extension operator can be seen in [55, Section 4.2], by solving the corresponding auxiliary problems in exponentially weighted spaces.

Furthermore, we define a nonlinear mapping

$$N : e^{-\eta}[\mathbb{E}_u(\mathbb{R}_+) \times \mathbb{E}_\pi(\mathbb{R}_+) \times \mathbb{E}_h(\mathbb{R}_+)] \times X_\gamma^0 \to e^{-\eta}\mathbb{F}(\mathbb{R}_+)$$

by

$$N((\bar{u}, \bar{\pi}, \bar{h}), (\tilde{u}_0, \tilde{h}_0)) := \begin{pmatrix} \bar{F}(\bar{u}, \bar{\pi}, \bar{h}) \\ P_0 \bar{F}_d\big((\bar{u}, \bar{h}) + \mathrm{ext}_\eta[(\phi(\tilde{u}_0, \tilde{h}_0), 0) \\ \qquad -(\bar{u}(0), \bar{h}(0))]\big) \\ \big(\bar{G}_v\big((\bar{u}, \bar{h}) + \mathrm{ext}_\eta[(\phi(\tilde{u}_0, \tilde{h}_0), 0) \\ \qquad -(\bar{u}(0), \bar{h}(0))]\big), \bar{G}_w(\bar{u}, \bar{h})\big)^{\mathsf{T}} \\ \bar{H}_1(\bar{u}, \bar{h}) \end{pmatrix}.$$

Here the functions $(\bar{F}, \bar{F}_d, \bar{G}_j, \bar{H}_1)$ result from (F, F_d, G_j, H_1) by replacing $(\tilde{u}, \tilde{h})$ and $\nabla\tilde{\pi}$ by $\omega\Xi((I - T_1)\bar{u}, P_0^\Sigma \bar{h})$ and (3.5), respectively.

Consider the equation

$$\mathbb{L}_\omega(\bar{u}, \bar{\pi}, \bar{q}, \bar{h}) = N((\bar{u}, \bar{\pi}, \bar{h}), (\tilde{u}_0, \tilde{h}_0)),$$

subject to the initial condition $(\bar{u}, \bar{h})|_{t=0} = (\phi(\tilde{u}_0, \tilde{h}_0), 0)$. If we can show that this problem has a unique solution $(\bar{u}, \bar{\pi}, \bar{q}, \bar{h}) \in e^{-\eta}\mathbb{E}(\mathbb{R}_+)$, then, by construction, $(\bar{u}, \bar{\pi}, \bar{q}, \bar{h})$ is a solution of (3.16). Here, we have set $\bar{q} = [\![\bar{\pi}]\!]$.

Let $(f, f_d, g_v, g_w, g_h) \in e^{-\eta}\mathbb{F}(\mathbb{R}_+)$ and $(u_0, h_0) \in \bar{X}_\gamma$ be given in such a way that $\operatorname{div} u_0 = f_d|_{t=0}$ and $-[\![\mu\nabla_{x'}w_0]\!] - [\![\mu\partial_3 v_0]\!] = g_v|_{t=0}$, where $u_0 = (v_0, w_0)$. Consider the linear problem to find a unique $w = (u, \pi, q, h) \in e^{-\eta}\mathbb{E}(\mathbb{R}_+)$, $q = [\![\pi]\!]$, such that

$$\mathbb{L}_\omega w = F, \quad z(0) = z_0 = (u_0, h_0),$$

for a sufficiently large $\omega > 0$, where $F := (f, f_d, g_v, g_w, g_h)$ and $z := (u, h)$. Indeed, by Corollary A.2 we may assume without loss of generality that $f = u_0 = 0$, $f_d = g_w = 0$ and $g_v = 0$. The remaining problem with $\tilde{F} = (0, 0, 0, 0, g_h)$ (g_h has been modified but not relabelled) and $\tilde{z}_0 = (0, h_0)$ can be written in the abstract form

$$\omega z + \dot{z} + Lz = (0, g_h), \quad t > 0, \qquad z(0) = \tilde{z}_0,$$

where the operator L has been defined in Section 3.1. If $\omega > 0$ is chosen sufficiently large, then there exists a unique solution $z \in e^{-\omega}[\mathbb{E}_u(\mathbb{R}_+) \times \mathbb{E}_h(\mathbb{R}_+)]$, since L has the property of maximal regularity of type L_p on $\mathbb{R}_+$ in

$$L_{p,\sigma}(\Omega) \times \big\{h \in W_p^{2-1/p}(\Sigma) : \partial_{\nu_{\partial G}} h = 0\big\},$$

by [55, Corollary 3.3].

Therefore, it makes sense to define a function $H : e^{-\eta}\mathbb{E}(\mathbb{R}_+) \times X_\gamma^0 \to e^{-\eta}\mathbb{E}(\mathbb{R}_+)$ by

$$\begin{aligned} H((\bar{u}, \bar{\pi}, \bar{q}, \bar{h}), (\tilde{u}_0, \tilde{h}_0)) &:= (\bar{u}, \bar{\pi}, \bar{q}, \bar{h}) \\ &\quad - (\mathbb{L}_\omega, \mathrm{tr}|_{t=0})^{-1}[N((\bar{u}, \bar{\pi}, \bar{h}), (\tilde{u}_0, \tilde{h}_0)), (\phi(\tilde{u}_0, \tilde{h}_0), 0)]. \end{aligned}$$

Note that H is well-defined, since all compatibility conditions at $t=0$ as well as at $\partial\Sigma$ and ∂S_2 are satisfied by construction. It follows from [55, Proposition 4.1] and the results in Section 3.2 that H is a C^2-mapping with $H(0,0)=0$ and

$$D_{(\bar{u},\bar{\pi},\bar{q},\bar{h})}H(0,0)=I_{e^{-\eta}\mathbb{E}(\mathbb{R}_+)}.$$

Therefore, applying the implicit function theorem yields the existence of a C^2-function $\psi: X_\gamma^0\to e^{-\eta}\mathbb{E}(\mathbb{R}_+)$ with $\psi(0)=0$ and $\psi'(0)=0$ such that $H(\psi(\tilde{u}_0,\tilde{h}_0),(\tilde{u}_0,\tilde{h}_0))=0$, whenever $(\tilde{u}_0,\tilde{h}_0)\in rB_{X_\gamma^0}(0)$ for some sufficiently small $r>0$.

Let

$$(u,\pi,q,h):=(\tilde{u},\tilde{\pi},\tilde{q},\tilde{h})+(\bar{u},\bar{\pi},\bar{q},\bar{h}).$$

As in the proof of [55, Theorem 4.2], one can show that $P_0F_d(u,h)=F_d(u,h)$, since $\operatorname{div} u=\operatorname{div}(\tilde{u}+\bar{u})=\operatorname{div}\bar{u}$. Integrating $\tilde{w}=\tilde{u}\cdot e_3$ over Σ yields

$$\int_\Sigma \tilde{w}\,dx'=\int_{\Omega_1}\operatorname{div}\tilde{u}_1\,dx=0.$$

This in turn implies that

$$\begin{aligned}\Big(\omega+\frac{d}{dt}\Big)\int_\Sigma \bar{h}\,dx' &= \int_\Sigma[\bar{w}-(v|\nabla h)]\,dx'\\ &= \int_\Sigma[w-(v|\nabla h)]\,dx'\\ &= \int_\Sigma (u|\nu_{\Gamma(t)})\sqrt{1+|\nabla h|^2}\,dx'\\ &= \int_{\Gamma(t)}((u\circ\Theta_h^{-1})|\nu_{\Gamma(t)})\,d\Gamma(t)\\ &= \int_{\Omega_1(t)}\operatorname{div}(u\circ\Theta_h^{-1})\,d\Omega_1(t)\\ &= 0,\end{aligned}$$

since

$$\operatorname{div}(u\circ\Theta_h^{-1})=(\operatorname{div}u-F_d(u,h))\circ\Theta_h^{-1}=(\operatorname{div}\bar{u}-F_d(u,h))\circ\Theta_h^{-1}=0.$$

Since $\bar{h}|_{t=0}=0$, this readily yields that $\bar{h}$ is mean value free, hence $P_0^\Sigma\bar{h}=\bar{h}$ and therefore (u,π,q,h) is a solution of (2.2) which is unique, by Theorem 2.1. The component (u,h) of the solution has the representation

$$(u,h)=\bar{\psi}(\tilde{u}_0,\tilde{h}_0)+\bar{\Xi}(\tilde{u}_0,\tilde{h}_0),$$

where $\bar{\psi}(\tilde{u}_0,\tilde{h}_0):=(\bar{u},\bar{h})$ and $\bar{\Xi}$ results by replacing $(\bar{u},\bar{h})$ by $\bar{\psi}(\tilde{u}_0,\tilde{h}_0)$ in the definition of Ξ. This yields the estimate

$$\|(u,h)\|_{e^{-\eta}[\mathbb{E}_u\times\mathbb{E}_h]}\le M\|(\tilde{u}_0,\tilde{h}_0)\|_{X_\gamma^0},$$

where $M > 0$ does not depend on $(\widetilde{u}_0, \widetilde{h}_0) \in rB_{X_\gamma^0}(0)$ as long as $r > 0$ is sufficiently small. This follows from smoothness of the function ψ. Since $(\widetilde{u}_0, \widetilde{h}_0) = (u_0, h_0) - \phi(\widetilde{u}_0, \widetilde{h}_0)$, $\phi(0) = 0$ and $\phi'(0) = 0$, we find for each $\varepsilon > 0$ a number $r(\varepsilon) > 0$ such that the estimate

$$\begin{aligned}\|(\widetilde{u}_0, \widetilde{h}_0)\|_{X_\gamma} &\le \|(u_0, h_0)\|_{X_\gamma} + \|\phi(\widetilde{u}_0, \widetilde{h}_0)\|_{X_\gamma} \\ &\le \|(u_0, h_0)\|_{X_\gamma} + \varepsilon\|(\widetilde{u}_0, \widetilde{h}_0)\|_{X_\gamma}\end{aligned}$$

is valid. This implies the final estimate

$$\|(u, h)\|_{e^{-\eta}[\mathbb{E}_u \times \mathbb{E}_h]} \le M_\varepsilon \|(u_0, h_0)\|_{X_\gamma},$$

proving the first assertion.

(2) Denote by σ^+ the collection of the eigenvalues of L with positive real parts and let P^+ be the spectral projection related to σ^+. Define $P^- := I - P^+$ and $X_0^\pm := P^\pm X_0$. Since σ^+ is finite, it follows that X_0^+ is finite-dimensional and the decompositions

$$X_0 = X_0^+ \oplus X_0^-, \quad L = L^+ \oplus L^-$$

hold, where L^+ is a bounded linear operator from X_0^+ to X_0^+. Note further that the spaces $D(L^+)$ and X_0^+ coincide and that

$$\|z\| := \|P^+ z\|_{X_0} + \|P^- z\|_{X_0}$$

defines an equivalent norm in X_0, since $P^\pm$ are bounded linear operators. By spectral theory, it holds that $\sigma^\pm = \sigma(L^\pm)$ and $\sigma^- \subset \overline{\mathbb{C}_-}$. Let $\lambda_* \in \sigma^+$ denote the eigenvalue with the smallest real part and choose numbers $\kappa, \eta > 0$ such that $[\kappa - \eta, \kappa + \eta] \subset (0, \operatorname{Re} \lambda_*)$. It follows that the strip

$$\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \in [\kappa - \eta, \kappa + \eta]\}$$

does not contain any eigenvalues of L. Therefore, the restricted semigroups $e^{\mp L^\pm t}$ satisfy the estimates

$$\|e^{L^- t}\| \le M e^{(\kappa - \eta)t}, \quad \|e^{-L^+ t}\| \le M e^{-(\kappa + \eta)t}, \quad t \ge 0 \tag{3.17}$$

for some constant $M > 0$.

Our aim is to prove the second assertion by a contradiction argument. To this end, we assume that $(u_*, h_*) = (0, 0)$ is stable. Then there exists a global solution $(u(t), \pi(t), q(t), h(t))$ of (2.2) such that $(u, \pi, q, h) \in \mathbb{E}(T)$ for each finite interval $J = [0, T] \subset [0, \infty)$, $q = [\![\pi]\!]$. Also, for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that whenever $\|(u_0, h_0)\|_{X_\gamma} \le \delta$ then $\|(u(t), h(t))\|_{X_\gamma} \le \varepsilon$ for all $t \ge 0$. Note that the solution admits the decomposition

$$(u, \pi, q, h) = (\widetilde{u}, \widetilde{\pi}, \widetilde{q}, \widetilde{h}) + (\overline{u}, \overline{\pi}, \overline{q}, \overline{h}),$$

where $(\widetilde{u}, \widetilde{h})$ solves (3.15) with $\widetilde{\pi}$, $\widetilde{q} = [\![\widetilde{\pi}]\!]$ given in terms of $(\widetilde{u}, \widetilde{h})$ (see (3.5)) and $(\overline{u}, \overline{\pi}, \overline{q}, \overline{h})$ solves (3.16) with a given right hand side (u, π, q, h). Observe that in this case, $P_0^\Sigma \bar{h} = \bar{h}$ by integration of (3.16)$_6$ over Σ, since

$$\int_\Sigma (\bar{u}|e_n)\, d\Sigma = \int_{\Omega_1} \operatorname{div} \bar{u}^1\, dx = \int_{\Omega_1} F_d(u^1, h)\, dx = \int_{\Omega_1} \operatorname{div} u^1\, dx$$

and

$$\int_\Sigma H_1(u, h)\, d\Sigma = \int_\Sigma (\partial_t h - (u|e_3))\, d\Sigma = -\int_{\Omega_1} \operatorname{div} u^1\, dx,$$

where $u^1 := u|_{\Omega_1}$ and where we made use of the fact that $P_0^\Sigma h = h$.

To shorten the notation we introduce the new functions $\tilde{z} := (\tilde{u}, \tilde{h})$, $\bar{z} = (\bar{u}, \bar{h})$, $\tilde{w} = (\tilde{u}, \tilde{\pi}, \tilde{q}, \tilde{h})$ and $\bar{w} = (\bar{u}, \bar{\pi}, \bar{q}, \bar{h})$. The functions $P^\pm \tilde{z}$ solve the evolutionary problem

$$\frac{d}{dt} P^\pm \tilde{z} - L^\pm P^\pm \tilde{z} = \omega P^\pm Q \bar{z}, \quad P^\pm \tilde{z}|_{t=0} = P^\pm \tilde{z}_0, \tag{3.18}$$

where $Q\bar{z} := ((I - T_1)\bar{u}, \bar{h})$ and $\tilde{z}_0 := (\tilde{u}_0, \tilde{h}_0)$. In the first step, we show that $P^+ \tilde{z}$ is given by the formula

$$P^+ \tilde{z}(t) = -\int_t^\infty e^{L^+(t-s)} \omega P^+ Q \bar{z}(s)\, ds. \tag{3.19}$$

Since P^+ is bounded and $X_\gamma^0 \hookrightarrow X_0$, it follows from the assumption that

$$\|P^+ \tilde{z}(t)\|_{X_0^+} \le \|P^+ z(t)\|_{X_0^+} + \|P^+ \bar{z}(t)\|_{X_0^+} \le C(\varepsilon + \|\bar{z}(t)\|_{X_0})$$

for all $t \ge 0$. This implies the estimate

$$\begin{aligned} \|e^{-\kappa t} P^+ \tilde{z}\|_{L_p(0,T;X_0^+)} &\le C\Big(\varepsilon \Big(\int_0^T e^{-\kappa p t}\, dt\Big)^{1/p} + \|e^{-\kappa t} \bar{z}\|_{L_p(0,T;X_0)}\Big) \\ &\le C(\kappa, p)\big(\varepsilon + \|e^{-\kappa t} \bar{z}\|_{\tilde{\mathbb{E}}(T)}\big), \end{aligned} \tag{3.20}$$

where

$$\tilde{\mathbb{E}}(T) := \mathbb{E}_u(T) \times \mathbb{E}_h(T),$$

and $\tilde{\mathbb{E}}(T) \hookrightarrow L_p(0, T; X_0)$, with an embedding constant being independent of $T > 0$. Employing the relation

$$\frac{d}{dt}(e^{-\kappa t} P^+ \tilde{z}(t)) = (-\kappa I + L^+) e^{-\kappa t} P^+ \tilde{z}(t) + e^{-\kappa t} P^+ Q \bar{z}(t), \tag{3.21}$$

we obtain that

$$\|e^{-\kappa t} P^+ \tilde{z}\|_{\mathbb{Z}(T)} \le C_1(\varepsilon + \|e^{-\kappa t} \bar{z}\|_{\tilde{\mathbb{E}}(T)}), \tag{3.22}$$

where the constant $C_1 > 0$ does not depend on $T > 0$. Here we have set

$$\mathbb{Z}(T) := H_p^1(0, T; X_0) \cap L_p(0, T; D(L)).$$

For the function $e^{-\kappa t}P^-\tilde{z}(t)$, the identity

$$\frac{d}{dt}(e^{-\kappa t}P^-\tilde{z}(t)) = (-\kappa I + L^-)e^{-\kappa t}P^-\tilde{z}(t) + e^{-\kappa t}P^-Q\overline{z}(t) \tag{3.23}$$

holds. Since by (3.17) the semigroup generated by $(-\kappa I + L^-)$ is exponentially stable in X_0^-, we obtain from L_p-maximal regularity theory that the estimate

$$\begin{aligned}\|e^{-\kappa t}P^-\tilde{z}\|_{\mathbb{Z}(T)} &\le M\big(\|P^-\tilde{z}_0\|_{X_\gamma^0} + \|e^{-\kappa t}P^-Q\overline{z}\|_{L_p(0,T;X_0)}\big)\\ &\le M\big(\|P^-\tilde{z}_0\|_{X_\gamma^0} + \|e^{-\kappa t}\overline{z}\|_{\widetilde{\mathbb{E}}(T)}\big)\end{aligned} \tag{3.24}$$

is valid for some constant $M > 0$ that does not depend on $T > 0$. A combination of (3.22) and (3.24) implies

$$\|e^{-\kappa t}\tilde{z}\|_{\mathbb{Z}(T)} \le C_2\big(\varepsilon + \|P^-\tilde{z}_0\|_{X_\gamma^0} + \|e^{-\kappa t}\overline{z}\|_{\widetilde{\mathbb{E}}(T)}\big), \tag{3.25}$$

with $C_2 > 0$ being independent of $T > 0$. In what follows, we want to reproduce the norm of $e^{-\kappa t}\tilde{z}$ in $\widetilde{\mathbb{E}}(T)$ on the left hand side of (3.25). To this end, we have to estimate $e^{-\kappa t}\tilde{h}$ and $e^{-\kappa t}\partial_t\tilde{h}$ in $W_p^{1-1/2p}(0,T;L_p(\Sigma))$.

To estimate $e^{-\kappa t}\tilde{h}$ in $W_p^{1-1/2p}(0,T;L_p(\Sigma))$, we cannot simply use interpolation of $H_p^1(0,T;L_p(\Sigma))$ with $L_p(0,T;L_p(\Sigma))$, since the interpolation constant would depend on $T > 0$. The following proposition takes care of this problem:

Proposition 3.5. *Let $T \in (0,\infty)$, $\kappa > 0$ and let $\tilde{z} \in \mathbb{Z}(T)$ be the unique solution to (3.15). Then there exists $\hat{z} \in \mathbb{Z}(\mathbb{R}_+)$ with $\hat{z}|_{[0,T]} = \tilde{z}$ such that the estimate*

$$\|e^{-\kappa t}\hat{z}\|_{\mathbb{Z}(\mathbb{R}_+)} \le M\big(\|\tilde{z}_0\|_{X_\gamma^0} + \|e^{-\kappa t}\overline{z}\|_{L_p(0,T;X_0)} + \|e^{-\kappa t}\tilde{z}\|_{L_p(0,T;X_0)}\big)$$

is valid, with a constant $M > 0$ being independent of $T > 0$.

Proof. We fix $a > 0$ large enough such that the operator $L - aI$ has the property of L_p-maximal regularity on $\mathbb{R}_+$. Define a function $f : \mathbb{R}_+ \to X_0$ by

$$f(t) := \begin{cases} \omega Q\overline{z}(t) + a\tilde{z}(t), & \text{if } t \in [0,T],\\ 0, & \text{if } t > T.\end{cases}$$

Then $f \in L_p(\mathbb{R}_+;X_0)$ and we may solve the problem

$$\partial_t\hat{z} - (L - aI)\hat{z} = f, \quad \hat{z}|_{t=0} = \tilde{z}_0 \tag{3.26}$$

to obtain a unique solution $\hat{z} \in \mathbb{Z}(\mathbb{R}_+)$. Observe that by the uniqueness of the solution of (3.15), it holds that $\hat{z}|_{[0,T]} = \tilde{z}$.

Multiplying (3.26) by $e^{-\kappa t}$, it follows that the function $e^{-\kappa t}\hat{z}(t)$ solves the initial value problem

$$\partial_t(e^{-\kappa t}\hat{z}) - (L - (a+\kappa)I)e^{-\kappa t}\hat{z} = e^{-\kappa t}f, \quad \hat{z}|_{t=0} = \tilde{z}_0.$$

Since the operator $L - (a+\kappa)I$ has L_p-maximal regularity on $\mathbb{R}_+$ as well, we obtain the desired estimate. The independence of the constant $M > 0$ from t follows from the exponential stability of the analytic semigroup which is generated by $L - (a+\kappa)I$. ■

Since $\|e^{-\kappa t}\tilde{z}\|_{W_p^{1-1/2p}(0,T;X_0)} \le \|e^{-\kappa t}\hat{z}\|_{W_p^{1-1/2p}(\mathbb{R}_+;X_0)}$ (here we use the intrinsic norm in $W_p^{1-1/2p}$), it follows by the real interpolation method and Proposition 3.5 that the estimate

$$\begin{aligned}\|e^{-\kappa t}\tilde{z}\|_{W_p^{1-1/2p}(0,T;X_0)} &\le M\big(\|\tilde{z}_0\|_{X_\gamma^0} + \|e^{-\kappa t}\bar{z}\|_{L_p(0,T;X_0)} + \|e^{-\kappa t}\tilde{z}\|_{L_p(0,T;X_0)}\big)\\ &\le M\big(\|\tilde{z}_0\|_{X_\gamma^0} + \|e^{-\kappa t}\bar{z}\|_{\widetilde{\mathbb{E}}(T)} + \|e^{-\kappa t}\tilde{z}\|_{\mathbb{Z}(T)}\big) \qquad (3.27)\end{aligned}$$

is valid. The second equation in (3.15) and Proposition 3.5 together with trace theory imply

$$\begin{aligned}&\|e^{-\kappa t}\partial_t\tilde{h}\|_{W_p^{1-1/2p}(0,T;L_p(\Sigma))}\\ &\quad\le C_3\big(\|e^{-\kappa t}\tilde{u}\|_{W_p^{1-1/2p}(0,T;L_p(\Sigma))} + \|e^{-\kappa t}\bar{h}\|_{W_p^{1-1/2p}(0,T;L_p(\Sigma))}\big)\\ &\quad\le C_4\big(\|\tilde{z}_0\|_{X_\gamma^0} + \|e^{-\kappa t}\bar{z}\|_{\widetilde{\mathbb{E}}(T)} + \|e^{-\kappa t}\tilde{z}\|_{\mathbb{Z}(T)}\big). \qquad (3.28)\end{aligned}$$

Observe that for the estimate of $e^{-\kappa t}\bar{h}$, we have used the fact that

$$\mathbb{E}_h(T) \hookrightarrow W_p^{1-1/2p}(0,T;L_p(\Sigma))$$

with an embedding constant being independent of $T > 0$, since the norm in the last space is a part of the norm in $\mathbb{E}_h(T)$. Combining (3.25) with (3.27) and (3.28), we obtain

$$\|e^{-\kappa t}\tilde{z}\|_{\widetilde{\mathbb{E}}(T)} \le C_5\big(\varepsilon + \|\tilde{z}_0\|_{X_\gamma^0} + \|P^-\tilde{z}_0\|_{X_\gamma^0} + \|e^{-\kappa t}\bar{z}\|_{\widetilde{\mathbb{E}}(T)}\big), \qquad (3.29)$$

with a constant $C_5 > 0$ being independent of $T > 0$.

We are now turning our attention to system (3.16) for $\bar{w} = (\bar{u}, \bar{\pi}, \bar{q}, \bar{h})$, which we write shortly as $\mathbb{L}_\omega\bar{w} = N(\tilde{w} + \bar{w})$ with initial condition $\bar{z}|_{t=0} = (\phi(\tilde{z}_0), 0)$. It will be convenient to write $N(w) = N_1(z) + N_2(z,\pi)$, where all components of $N_2(z,\pi)$ are zero except for the first one, which is given by $M_0(h)\nabla\pi$.

Proposition 3.6. *Let $\kappa \ge 0$. There exists a nondecreasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that*

(i) *if $z \in \mathbb{Z}(\mathbb{R}_+)$, then*

$$\|e^{-\kappa t}N_1(z)\|_{\mathbb{F}(\mathbb{R}_+)} \le \alpha(\varepsilon)\|e^{-\kappa t}z\|_{\mathbb{Z}(\mathbb{R}_+)},$$

whenever $\|z(t)\|_{X_\gamma} \le \varepsilon$ for all $t \ge 0$;

(ii) *if $\hat{z} \in {}_0\mathbb{Z}(T)$ and $z_* \in \mathbb{Z}(\mathbb{R}_+)$, then*

$$\|e^{-\kappa t}N_1(\hat{z} + z_*)\|_{\mathbb{F}(T)} \le \alpha(\varepsilon)C\big(\|e^{-\kappa t}\hat{z}\|_{\mathbb{Z}(T)} + \|e^{-\kappa t}z_*\|_{\mathbb{Z}(\mathbb{R}_+)}\big),$$

whenever

$$\|\hat{z}(t)\|_{X_\gamma} \le C\varepsilon$$

for all $t \in [0,T]$ and

$$\|z_*(t)\|_{X_\gamma} \le C\varepsilon$$

for all $t \ge 0$. The constant $C > 0$ does not depend on $T > 0$.

Proof. The proof of the first assertion follows by similar arguments as in [25, Proposition 9]. Therefore, we concentrate on the proof of the second assertion. For $\hat{z} \in {}_0\widetilde{\mathbb{E}}(T)$ we define a bounded linear extension operator $E : {}_0\mathbb{Z}(T) \to {}_0\mathbb{Z}(\mathbb{R}_+)$ by

$$(E\hat{z})(t) := \begin{cases} \hat{z}(t), & t \in [0,T], \\ \hat{z}(2T-t), & t \in [T,2T], \\ 0, & t \geq 2T. \end{cases}$$

For the norm of $e^{-\kappa t}(E\hat{z})$ in $\mathbb{Z}(\mathbb{R}_+)$, we then obtain

$$\begin{aligned} \|e^{-\kappa t} E\hat{z}\|^p_{\mathbb{Z}(\mathbb{R}_+)} &= \int_0^T e^{-\kappa t p}\|\hat{z}(t)\|^p_{X_1}\,dt + \int_T^{2T} e^{-\kappa t p}\|\hat{z}(2T-t)\|^p_{X_1}\,dt \\ &\quad + \int_0^T e^{-\kappa t p}\|\dot{\hat{z}}(t)\|^p_{X_0}\,dt + \int_T^{2T} e^{-\kappa t p}\|\dot{\hat{z}}(2T-t)\|^p_{X_0}\,dt \\ &= \int_0^T e^{-\kappa t p}\|\hat{z}(t)\|^p_{X_1}\,dt + \int_0^T e^{-\kappa(2T-\tau)p}\|\hat{z}(\tau)\|^p_{X_1}\,d\tau \\ &\quad + \int_0^T e^{-\kappa t p}\|\dot{\hat{z}}(t)\|^p_{X_0}\,dt + \int_0^T e^{-\kappa(2T-\tau)p}\|\dot{\hat{z}}(\tau)\|^p_{X_0}\,d\tau \\ &\leq \|e^{-\kappa t}\hat{z}\|_{\mathbb{Z}(T)}, \end{aligned}$$

since $2T-\tau \geq \tau$ for $\tau \in [0,T]$.

In addition, there holds $\|(E\hat{z})(t)\|_{W_p^{2-2/p}\times W_p^{3-2/p}} \leq C\varepsilon$ for all $t \geq 0$. Then, the first assertion yields

$$\begin{aligned} \|e^{-\kappa t} N_1(\hat{z}+z_*)\|_{\mathbb{F}(T)} &\leq \|e^{-\kappa t} N_1(E\hat{z}+z_*)\|_{\mathbb{F}(\mathbb{R}_+)} \\ &\leq \alpha(\varepsilon)C\|e^{-\kappa t}(E\hat{z}+z_*)\|_{\mathbb{Z}(\mathbb{R}_+)} \\ &\leq \alpha(\varepsilon)C\big(\|e^{-\kappa t}\hat{z}\|_{\mathbb{Z}(T)} + \|e^{-\kappa t} z_*\|_{\mathbb{Z}(\mathbb{R}_+)}\big). \end{aligned}$$ ■

In order to apply this proposition to the system $\mathbb{L}_\omega \overline{w} = N(\overline{w}+\widetilde{w})$, let z_* be an extension of z_0 such that $e^{-\kappa t} z_* \in \widetilde{\mathbb{E}}(\mathbb{R}_+)$ and $\|z_*\|_{\mathbb{Z}(\mathbb{R}_+)} \leq C\|z_0\|_{X_\gamma}$. The existence of such an extension can be seen as in the proof of the first assertion. Then we use the representation $N(w) = N_1(z) + N_2(z,\pi)$ as well as the identity $N_1(z) = N_1(z - z_* + z_*) = N_1(\hat{z}+z_*)$, where $\hat{z} := (z - z_*) \in {}_0\mathbb{Z}(T)$. Finally, note that

$$\|e^{-\kappa t} N_2(z,\pi)\|_{L_p(0,T;L_p(\Omega))} \leq C\varepsilon\|e^{-\kappa t}\pi\|_{\mathbb{E}_\pi(T)}.$$

Therefore, the second assertion of Proposition 3.6 implies the estimate

$$\begin{aligned} \|e^{-\kappa t} N(\overline{w}+\widetilde{w})\|_{\mathbb{F}(T)} &\leq \alpha(\varepsilon)C\big(\|e^{-\kappa t}\widetilde{z}\|_{\mathbb{Z}(T)} + \|e^{-\kappa t}\overline{z}\|_{\mathbb{Z}(T)} + \|e^{-\kappa t} z_*\|_{\mathbb{Z}(\mathbb{R}_+)}\big) \\ &\quad + \varepsilon C\big(\|e^{-\kappa t}\widetilde{\pi}\|_{\mathbb{E}_\pi(T)} + \|e^{-\kappa t}\overline{\pi}\|_{\mathbb{E}_\pi(T)}\big) \\ &\leq \alpha_1(\varepsilon)\big(\|e^{-\kappa t}\widetilde{z}\|_{\widetilde{\mathbb{E}}(T)} + \|e^{-\kappa t}\overline{z}\|_{\widetilde{\mathbb{E}}(T)} + \|e^{-\kappa t}\overline{\pi}\|_{\mathbb{E}_\pi(T)} + \|z_0\|_{X_\gamma}\big), \end{aligned}$$

where $\alpha_1(\varepsilon) := \alpha(\varepsilon) + \varepsilon \to 0$ as $\varepsilon \to 0$. Here we have used the estimates

$$\|e^{-\kappa t} z_*\|_{\mathbb{Z}(\mathbb{R}_+)} \le C\|z_0\|_{X_\gamma}$$

and

$$\|e^{-\kappa t}\widetilde{\pi}\|_{\mathbb{E}_\pi(T)} \le C\big(\|e^{-\kappa t}\widetilde{z}\|_{\widetilde{\mathbb{E}}(T)} + \|e^{-\kappa t}\overline{z}\|_{\widetilde{\mathbb{E}}(T)}\big),$$

which hold for some constant $C > 0$ that does not depend on $T > 0$. Note also that $\widetilde{\mathbb{E}}(T) \hookrightarrow \mathbb{Z}(T)$ with a universal embedding constant being independent of $T > 0$ and $\|\hat{z}(t)\|_{X_\gamma} \le (1+C)\varepsilon$ for all $t \in [0,T]$, $\|z_*(t)\|_{X_\gamma} \le C\varepsilon$ for all $t \ge 0$.

By the invertibility of $\mathbb{L}_\omega$, we obtain

$$\begin{aligned}\|e^{-\kappa t}\overline{w}\|_{\mathbb{E}(T)} &\le C_6\big(\|\phi(\widetilde{z}_0)\|_{X_\gamma} + \|e^{-\kappa t}N(\overline{w}+\widetilde{w})\|_{\mathbb{F}(T)}\big)\\ &\le C_6\big(\|\phi(\widetilde{z}_0)\|_{X_\gamma} + \alpha_1(\varepsilon)(\|e^{-\kappa t}\overline{z}\|_{\widetilde{\mathbb{E}}(T)} + \|e^{-\kappa t}\widetilde{z}\|_{\widetilde{\mathbb{E}}(T)}\\ &\qquad + \|e^{-\kappa t}\overline{\pi}\|_{\mathbb{E}_\pi(T)} + \|z_0\|_{X_\gamma})\big).\end{aligned} \tag{3.30}$$

Choose $\varepsilon > 0$ sufficiently small such that $C_6\alpha_1(\varepsilon) \le 1/2$ and note that

$$\|e^{-\kappa t}\overline{w}\|_{\mathbb{E}(T)} = \|e^{-\kappa t}\overline{z}\|_{\widetilde{\mathbb{E}}(T)} + \|e^{-\kappa t}\overline{\pi}\|_{\mathbb{E}_\pi(T)} + \|e^{-\kappa t}[\![\overline{\pi}]\!]\|_{\mathbb{E}_q(T)}.$$

This implies the estimate

$$\|e^{-\kappa t}\overline{z}\|_{\widetilde{\mathbb{E}}(T)} \le 2C_6\big(\|\phi(\widetilde{z}_0)\|_{X_\gamma} + \alpha_1(\varepsilon)(\|e^{-\kappa t}\widetilde{z}\|_{\widetilde{\mathbb{E}}(T)} + \|z_0\|_{X_\gamma})\big). \tag{3.31}$$

If $\varepsilon > 0$ is sufficiently small, we obtain from (3.29) and (3.31) that

$$\|e^{-\kappa t}\widetilde{z}\|_{\widetilde{\mathbb{E}}(T)} + \|e^{-\kappa t}\overline{z}\|_{\widetilde{\mathbb{E}}(T)} \le C_7\big(\varepsilon + \|\widetilde{z}_0\|_{X_\gamma^0} + \|P^-\widetilde{z}_0\|_{X_\gamma^0} + \|\phi(\widetilde{z}_0)\|_{X_\gamma}\big) \tag{3.32}$$

with $C_7 > 0$ being independent of $T > 0$ and where we made use of the fact that $z_0 = \widetilde{z}_0 + \phi(\widetilde{z}_0)$. In particular, this shows that

$$e^{-\kappa t}\widetilde{z}, e^{-\kappa t}\overline{z} \in \widetilde{\mathbb{E}}(\mathbb{R}_+).$$

This in turn yields that

$$\begin{aligned}&e^{-\kappa t}\int_t^\infty \|e^{L^+(t-s)}P^+\omega Q\overline{z}(s)\|_{X_0}\,ds\\ &\qquad\le M\Big(\int_t^\infty e^{\eta p'(t-s)}\,ds\Big)^{1/p'}\|e^{-\kappa t}\omega\overline{z}\|_{L_p(\mathbb{R}_+;X_0)}\\ &\qquad\le C(\eta,p')\|e^{-\kappa t}\omega\overline{z}\|_{\widetilde{\mathbb{E}}(\mathbb{R}_+)} < \infty.\end{aligned}$$

For the projection of the solution $\widetilde{z}$ of (3.15) to X_0^+, we have the variation of parameters formula

$$\begin{aligned}P^+\widetilde{z}(t) &= P^+e^{L^+t}\widetilde{z}_0 + \int_0^t e^{L^+(t-s)}P^+\omega Q\overline{z}(s)ds\\ &= P^+e^{L^+t}\widetilde{z}_0 + \int_0^\infty e^{L^+(t-s)}P^+\omega Q\overline{z}(s)ds - \int_t^\infty e^{L^+(t-s)}P^+\omega Q\overline{z}(s)ds\end{aligned}$$

at our disposal. Since e^{L^+t} extends to a C_0-group, we obtain the identity

$$e^{-L^+t}\Big(P^+\tilde{z}(t)+\int_t^\infty e^{L^+(t-s)}P^+\omega Q\overline{z}(s)ds\Big)=P^+\tilde{z}_0+\int_0^\infty e^{-L^+s}P^+\omega Q\overline{z}(s)ds,$$

which holds for all $t\geq 0$. The left hand side of this equation may be estimated in X_0 as follows:

$$\begin{aligned}&\Big\|e^{-L^+t}\Big(P^+\tilde{z}(t)+\int_t^\infty e^{L^+(t-s)}P^+\omega Q\overline{z}(s)ds\Big)\Big\|_{X_0}\\ &\quad\leq Me^{-(\kappa+\eta)t}\Big(\|\tilde{z}(t)\|_{X_0}+\int_t^\infty \|e^{L^+(t-s)}P^+\omega Q\overline{z}(s)\|_{X_0}\,ds\Big)\\ &\quad\leq Me^{-\eta t}\big(\|e^{-\kappa t}\tilde{z}(t)\|_{X_0}+C\big).\end{aligned}$$

Here we made use of the fact that the integral does not grow faster than $e^{\kappa t}$ by the computations above. Since the function $[t\mapsto\|e^{-\kappa t}\tilde{z}(t)\|_{X_0}]$ is bounded (see above), it follows that

$$e^{-\eta t}\big(\|e^{-\kappa t}\tilde{z}(t)\|_{X_0}+C\big)\to 0$$

as $t\to\infty$. This shows in particular that $P^+\tilde{z}_0+\int_0^\infty e^{-L^+s}P^+\omega Q\overline{z}(s)ds=0$, hence the relation (3.19) holds.

From (3.19) and Young's inequality, we obtain the estimate

$$\|e^{-\kappa t}P^+\tilde{z}\|_{L_p(\mathbb{R}_+;X_0)}\leq M(\eta)\|e^{-\kappa t}P^+\overline{z}\|_{L_p(\mathbb{R}_+;X_0)}.$$

By (3.21), this yields

$$\|e^{-\kappa t}P^+\tilde{z}\|_{\mathbb{Z}(\mathbb{R}_+)}\leq M(\eta)\|e^{-\kappa t}P^+\overline{z}\|_{\tilde{\mathbb{E}}(\mathbb{R}_+)}.\tag{3.33}$$

One may now mimic the above estimates with the interval $[0,T]$ being replaced by $\mathbb{R}_+$ to obtain the relation

$$\|e^{-\kappa t}\tilde{z}\|_{\tilde{\mathbb{E}}(\mathbb{R}_+)}+\|e^{-\kappa t}\overline{z}\|_{\tilde{\mathbb{E}}(\mathbb{R}_+)}\leq C\big(\|P^-\tilde{z}_0\|_{X_\gamma}+\|\phi(\tilde{z}_0)\|_{X_\gamma}\big).\tag{3.34}$$

At this point, we want to emphasise that the term $\|\tilde{z}_0\|_{X_\gamma^0}$ does not appear on the right hand side of (3.34), since on $\mathbb{R}_+$ there is no need to apply Proposition 3.5. Furthermore, since we estimate norms on the half-line $\mathbb{R}_+$, we may use the first assertion of Proposition 3.6 instead of the second one.

Then, formula (3.19) for $t=0$ and (3.34) imply

$$\begin{aligned}\|P^+\tilde{z}_0\|_{X_\gamma^0}&\leq M(\omega,\eta)\|e^{-\kappa t}\overline{z}\|_{L_\infty(\mathbb{R}_+;X_\gamma^0)}\leq M_1(\omega,\eta)\|e^{-\kappa t}\overline{z}\|_{\tilde{\mathbb{E}}(\mathbb{R}_+)}\\ &\leq C\big(\|P^-\tilde{z}_0\|_{X_\gamma^0}+\|\phi(\tilde{z}_0)\|_{X_\gamma}\big),\end{aligned}$$

since $\tilde{\mathbb{E}}(\mathbb{R}_+)\hookrightarrow BUC(\mathbb{R}_+;X_\gamma^0)$. Due to the fact that $\phi(0)=0$ and $\phi'(0)=0$, we may decrease $\delta>0$ (if necessary) to obtain

$$\|\phi(\tilde{z}_0)\|_{X_\gamma}\leq\frac{1}{2}\big(\|P^-\tilde{z}_0\|_{X_\gamma^0}+\|P^+\tilde{z}_0\|_{X_\gamma^0}\big),$$

whenever $\widetilde{z}_0 \in \delta B_{X_\gamma^0}(0)$. Finally, this yields the relation

$$\|P^+\widetilde{z}_0\|_{X_\gamma^0} \le C\|P^-\widetilde{z}_0\|_{X_\gamma^0}.$$

Choosing $\widetilde{z}_0 \in \delta B_{X_\gamma^0}(0)$ in such a way that $P^-\widetilde{z}_0 = 0$ and $P^+\widetilde{z}_0 \neq 0$, we have a contradiction. The proof is complete. ■

We complete this section by considering the special case $G = B_R(0)$ and give a result on stability which is dependent on the radius $R > 0$.

Corollary 3.7. *Let the conditions of Theorem 3.4 be satisfied and let the surface tension $\sigma > 0$ be fixed. Denote by $\lambda_1^* > 0$ the first nontrivial eigenvalue of the negative Neumann Laplacian in $L_2(B_1(0))$. Then the following assertions hold:*

(1) *If $R^2[\![\rho]\!]\gamma_a/\sigma < \lambda_1^*$, then $(u_*, h_*) = (0,0)$ is exponentially stable in the sense of Theorem 3.4.*

(2) *If $[\![\rho]\!] > 0$ and $R^2[\![\rho]\!]\gamma_a/\sigma > \lambda_1^*$, then $(u_*, h_*) = (0,0)$ is unstable in the sense of Theorem 3.4.*

Proof. The assertions follow from Theorem 3.4. Indeed, denoting by $\lambda_1(R) > 0$ the first nontrivial eigenvalue of the Neumann Laplacian on $B_R(0)$, Theorem 3.4 yields that $(0,0)$ is exponentially stable if $[\![\rho]\!]\gamma_a/\sigma < \lambda_1(R)$ and unstable if $[\![\rho]\!]\gamma_a/\sigma > \lambda_1(R)$ and $[\![\rho]\!] > 0$. An easy computation yields that $\lambda_1(R) = \lambda_1^*/R^2$. This concludes the proof of the corollary. ■

4. Bifurcation at a multiple eigenvalue

In this section we consider the special case $G = B_R := B_R(0) \subset \mathbb{R}^2$ for some radius $R > 0$. Proposition 3.2 implies that an eigenvalue of the linearisation L crosses the imaginary axis through zero if $[\![\rho]\!]\gamma_a/\sigma = \lambda_1$, where $\lambda_1 > 0$ is the first nontrivial eigenvalue of the negative Neumann Laplacian in $L_2(G)$. This suggests that $(\lambda_1, 0)$ is a bifurcation point for the nonlinear Navier–Stokes system (2.2). Unfortunately, the eigenvalue $\lambda_1 > 0$ is not simple. Indeed, it is a double eigenvalue, being semi-simple. Therefore, we cannot directly apply the results of Crandall and Rabinowitz. Instead, we will use certain symmetry properties of the bifurcation equation to reduce it to a purely one-dimensional bifurcation equation which then can be solved by the implicit function theorem. For a general theory concerning bifurcation at multiple eigenvalues, we refer the reader to [22, 34, 51].

We recall that the set of equilibria $\mathcal{E}$ for height functions h with vanishing mean value is given by

$$\mathcal{E} = \big\{(u_*, \pi_*, q_*, h_*) : u_* = 0,\ \pi_* = \text{const.},\ q_* = [\![\pi_*]\!] = 0,\ h_* \text{ solves } (4.1)\big\}.$$

Note that if there exist nontrivial equilibria, i.e., $h_* \neq 0$, then these equilibria are determined by the nontrivial solutions of the quasilinear elliptic boundary value problem

$$\begin{aligned} \sigma \operatorname{div}_{x'}\Big(\frac{\nabla_{x'} h}{\sqrt{1+|\nabla_{x'} h|^2}}\Big) + [\![\rho]\!]\gamma_a h = 0, \quad & x' \in B_R(0), \\ \partial_{\nu_{B_R(0)}} h = 0, \quad & x' \in \partial B_R(0). \end{aligned} \tag{4.1}$$

Here the differential operators $\nabla_{x'}$ and $\operatorname{div}_{x'}$ act only in the variables $x' \in G$. We intend to show that if $[\![\rho]\!]\gamma_a/\sigma = \lambda_1$, then there exist bifurcating nontrivial solutions h_* of (4.1) from the trivial solution $h = 0$. To this end, let

$$\begin{aligned} X &:= \Big\{h \in W_p^{1-1/p}(B_R) : \int_{B_R} h\, dx' = 0\Big\}, \\ Y &:= \Big\{h \in W_p^{3-1/p}(B_R) \cap X : \partial_{\nu_{\partial B_R}} h = 0\Big\} \end{aligned} \tag{4.2}$$

and define $F : \mathbb{R}_+ \times Y \to X$ by

$$F(\alpha, h) := \operatorname{div}_{x'}\Big(\frac{\nabla_{x'} h}{\sqrt{1+|\nabla_{x'} h|^2}}\Big) + \alpha h. \tag{4.3}$$

For $h \in W_p^s(B_R)$, $s > 0$, define $(\Gamma_{\mathcal{O}_\phi} h)(\bar{x}') := h(\mathcal{O}_\phi \bar{x}')$, where

$$\mathcal{O}_\phi := \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

describes a two-dimensional rotation of $\bar{x}' \in B_R$ through the angle ϕ. Note that $\mathcal{O}_\phi$ is an orthogonal matrix, i.e., $\mathcal{O}_\phi^\mathsf{T} = \mathcal{O}_\phi^{-1}$. Furthermore, we define $(\Gamma_\mathcal{R} h)(\bar{x}') := h(\mathcal{R}\bar{x}')$, where $\mathcal{R}\bar{x}' := (\bar{x}_1, -\bar{x}_2)^\mathsf{T}$. It is easily seen that Γ_j leaves both spaces X and Y invariant and one readily computes $\nabla_{\bar{x}'}(\Gamma_{\mathcal{O}_\phi} h) = \mathcal{O}_\phi^\mathsf{T}(\Gamma_{\mathcal{O}_\phi} \nabla_{x'} h)$, $\Delta_{\bar{x}'}(\Gamma_{\mathcal{O}_\phi} h) = \Gamma_{\mathcal{O}_\phi} \Delta_{x'} h$ and $\nabla^2_{\bar{x}'}(\Gamma_{\mathcal{O}_\phi} h) = \mathcal{O}_\phi^\mathsf{T}(\Gamma_{\mathcal{O}_\phi} \nabla^2_{x'} h)\mathcal{O}_\phi$, where $\bar{x}' = \mathcal{O}_\phi^\mathsf{T} x'$. Therefore, the identity

$$\operatorname{div}_{x'}\Big(\frac{\nabla_{x'} h}{\sqrt{1+|\nabla_{x'} h|^2}}\Big) = \frac{\Delta_{x'} h}{\sqrt{1+|\nabla_{x'} h|^2}} - \frac{(\nabla^2_{x'} h \nabla_{x'} h | \nabla_{x'} h)}{\sqrt{1+|\nabla_{x'} h|^2}^3}$$

implies that $\Gamma_{\mathcal{O}_\phi} F(\alpha, h) = F(\alpha, \Gamma_{\mathcal{O}_\phi} h)$. Similarly, it holds that $\Gamma_\mathcal{R} F(\alpha, h) = F(\alpha, \Gamma_\mathcal{R} h)$. This shows that F is invariant with respect to the group operations of the orthogonal group $O(2)$.

4.1. Lyapunov–Schmidt reduction

By smoothness of the mapping $[\mathbb{R} \ni s \mapsto (1+s^2)^{-1/2}]$, it holds that $F \in C^\infty(\mathbb{R}_+ \times Y; X)$ and the first Fréchet derivative of F is given by

$$[D_h F(\alpha, h)]\hat{h} = \operatorname{div}_{x'}\Big(\frac{\nabla_{x'} \hat{h}}{\sqrt{1+|\nabla_{x'} h|^2}}\Big) - \operatorname{div}_{x'}\Big(\frac{\nabla_{x'} h (\nabla_{x'} \hat{h} | \nabla_{x'} h)}{\sqrt{1+|\nabla_{x'} h|^2}^3}\Big) + \alpha \hat{h}.$$

Therefore, it holds that $D_h F(\lambda_1, 0) = \Delta_N + \lambda_1 I$, where Δ_N denotes the Neumann Laplacian and $\lambda_1 > 0$ is the first eigenvalue of $-\Delta_N$ in X (note that $0 \notin \sigma(-\Delta_N)$, since all functions in X have a vanishing mean value). For convenience, we set $A := D_h F(\lambda_1, 0)$. We claim that $0 \in \sigma(A)$ is a semi-simple eigenvalue. Since the operator A has a compact resolvent, it follows that the spectrum consists only of discrete eigenvalues having finite multiplicity. Therefore, it suffices to show that $N(A) = N(A^2)$. To this end, let $0 \neq v \in N(A^2)$ and $u := Av$. Then $u \in N(A)$ and we compute

$$\|u\|^2_{L_2(B_R)} = (Av|u)_{L_2(B_R)} = (v|Au)_{L_2(B_R)} = 0,$$

since A is self-adjoint in $L_2(B_R)$. This shows that $u = 0$, hence $v \in N(A)$ and $0 \in \sigma(A)$ is semi-simple. We note here that this implies $X = N(A) \oplus R(A)$. Rewriting the eigenvalue problem $-\Delta_N h = \lambda h$ in polar coordinates (r, φ), it follows that the kernel $N(A)$ of A is spanned by the two linearly independent functions

$$u_1^*(x') := J_1(j'_{1,1} r/R) \cos\varphi, \quad u_2^*(x') := J_1(j'_{1,1} r/R) \sin\varphi, \tag{4.4}$$

for $r \in [0, R]$, $\varphi \in [0, 2\pi)$, where J_1 is a Bessel function of first order and $j'_{1,1}$ denotes the first zero of the derivative J_1' of J_1. Hence, $\dim N(A) = 2$ (notably, A is a Fredholm operator of index zero). In particular, each $h \in X$ can be written in a unique way as $h = u + v$, where $u \in N(A)$ and $v \in R(A)$. Defining $Ph := u$, it follows that the mapping $P : X \to N(A)$ is a projection onto $N(A)$. With $Q := I - P$ we also have that the mapping $Q : X \to R(A)$ is onto and $Qh = v$. Moreover, it holds that $Y = U \oplus V$, where $U := N(A)$ and $V := R(A) \cap Y$.

Let us now split the equation $F(\alpha, h) = 0$ into two parts: $PF(\alpha, u + v) = 0$ and $QF(\alpha, u + v) = 0$. Since the operator $D_v QF(\lambda_1, 0) = QD_h F(\lambda_1, 0) : V \to R(A)$ is an isomorphism, we may solve the equation $QF(\alpha, u + v) = 0$ in a neighbourhood of $(\lambda_1, 0)$, by making use of the implicit function theorem, to obtain a unique smooth function $v_* : \mathbb{R}_+ \times U \to V$ such that $QF(\alpha, u + v_*(\alpha, u)) = 0$ for all (α, u) close to $(\lambda_1, 0)$. The function $v_* = v_*(\alpha, u)$ has the following properties:

(1) $v_*(\alpha, 0) = 0$ if $\alpha > 0$ is close to λ_1;

(2) $D_\alpha v_*(\lambda_1, 0) = 0$, $D_u v_*(\lambda_1, 0) = 0$;

(3) $\Gamma_j v_*(\alpha, u) = v_*(\alpha, \Gamma_j u)$ for $j \in \{\mathcal{R}, \mathcal{O}_\phi\}$ if (α, u) is close to $(\lambda_1, 0)$.

The first two properties follow directly from the equation $QF(\alpha, u + v_*, \alpha, u)) = 0$ after differentiation, and the fact that $F(\alpha, 0) = 0$ for each $\alpha \in \mathbb{R}_+$. The last property follows from the uniqueness of v_* and the fact that $\Gamma_j QF(\alpha, u + v) = QF(\alpha, \Gamma_j u + \Gamma_j v)$ for $j \in \{\mathcal{R}, \mathcal{O}_\phi\}$. To see this, we differentiate the identity $\Gamma_j F(\alpha, u) = F(\alpha, \Gamma_j u)$ with respect to u and evaluate the result at $(\alpha, u) = (\lambda_1, 0)$ to obtain the relation

$$\Gamma_j A = A\Gamma_j.$$

In other words, Γ_j commutes with the operator A. It follows readily that Γ_j leaves $N(A)$ as well as $R(A)$ invariant, hence we have $\Gamma_j P = P\Gamma_j$ as well as $\Gamma_j Q = Q\Gamma_j$.

4.2. Reduction to a one-dimensional bifurcation equation

It remains to study the equation $0 = G(\alpha, u)$ for $(\alpha, u) \in \mathbb{R}_+ \times U$ in some neighbourhood of $(\lambda_1, 0)$, where $G(\alpha, u) := PF(\alpha, u + v_*(\alpha, u))$. Let us remark that this equation is purely two-dimensional. Similar to the above, it holds that $\Gamma_j G(\alpha, u) = G(\alpha, \Gamma_j u)$ for $j \in \{\mathcal{R}, \mathcal{O}_\phi\}$. Let $\Psi : U \to \mathbb{R}^2$ be defined by $\Psi(u) := (b_1, b_2)^\mathsf{T}$ for $u = b_1 u_1 + b_2 u_2 \in U$, $b_k := (u|u_k)_{L_2(B_R)} \in \mathbb{R}$, where $u_j := u_j^*/\|u_j^*\|_{L_2}$. It follows that Ψ is an isomorphism with inverse Ψ^{-1} given by $\Psi^{-1}(b_1, b_2) = b_1 u_1 + b_2 u_2$. Consider now the equation

$$g(\alpha, b) := \Psi G(\alpha, \Psi^{-1} b) = 0, \quad b \in \mathbb{R}^2,$$

and define $\Gamma_j^0 := \Psi \Gamma_j \Psi^{-1}$ on $\mathbb{R}^2$ for $j \in \{\mathcal{R}, \mathcal{O}_\phi\}$. With these definitions, it holds that $\Gamma_j^0 g(\alpha, b) = g(\alpha, \Gamma_j^0 b)$ for $j \in \{\mathcal{R}, \mathcal{O}_\phi\}$. A short computation also shows that the identities

- $\Gamma_{\mathcal{O}_\phi}^0 b = \mathcal{O}_\phi b$;
- $\Gamma_{\mathcal{R}}^0 b = \mathcal{R} b$

hold for each $b \in \mathbb{R}^2$. We will use these two properties to reduce $g(\alpha, b) = 0$ to a purely one-dimensional equation. Choose ϕ in such a way that $\mathcal{O}_\phi b = s e_1 = (s, 0)^\mathsf{T}$ for some $s \in \mathbb{R}$ close to 0. Then $g(\alpha, b) = 0$ if and only if $g(\alpha, s e_1) = 0$, by the first property. Furthermore, $\mathcal{R} e_1 = e_1$, hence

$$g(\alpha, s e_1) = g(\alpha, s \mathcal{R} e_1) = \mathcal{R} g(\alpha, s e_1).$$

This in turn yields that $g_2(\alpha, s e_1) = 0$ is always satisfied and therefore, we have reduced the equation $g(\alpha, b) = 0$ to $g_1(\alpha, s e_1) = 0$ for $(\alpha, s) \in \mathbb{R}_+ \times \mathbb{R}$ close to $(\lambda_1, 0)$.

Since $D_\alpha g_1(\lambda_1, 0) = 0$, we cannot simply solve the equation $g_1(\alpha, s e_1) = 0$ for α in a neighbourhood of $(\lambda_1, 0)$ by the implicit function theorem. Instead, we define a new function

$$\widetilde{g}(\alpha, s) := \begin{cases} g_1(\alpha, s e_1)/s, & s \neq 0, \\ D_b g_1(\alpha, 0) e_1, & s = 0. \end{cases}$$

Since $D_b g_1(\lambda_1, 0) = 0$, we have $\widetilde{g}(\lambda_1, 0) = 0$. Moreover, we compute

$$D_\alpha \widetilde{g}(\lambda_1, 0) = D_\alpha D_b g_1(\lambda_1, 0) e_1.$$

Since $D_\alpha D_h F(\lambda_1, 0) = I$ and

$$D_\alpha D_b g(\lambda_1, 0) e_1 = \Psi P D_\alpha D_h F(\lambda_1, 0) \Psi^{-1} e_1 = e_1,$$

it follows that $D_\alpha D_b g_1(\lambda_1, 0) e_1 = 1 \neq 0$. Hence, the implicit function theorem yields the existence of a smooth function $\alpha : (-\eta, \eta) \to \mathbb{R}$ with $\alpha(0) = \lambda_1$ such that $\widetilde{g}(\alpha(s), s) = 0$ for all $s \in (-\eta, \eta)$ and some (small) $\eta > 0$. This in turn yields the following result:

Theorem 4.1. *Modulo the action in $O(2)$, all solutions of $F(\alpha,h)=0$ in a neighbourhood $\mathcal{U}$ of $(\lambda_1,0)$ in $\mathbb{R}_+\times Y$ are given by*

$$F^{-1}(0)\cap\mathcal{U}=\big\{(\alpha(s),su_1+y(s)):|s|<\eta\big\}\cup\big\{(\alpha,0):(\alpha,0)\in\mathcal{U}\big\},$$

where $\alpha\in C^\infty((-\eta,\eta);\mathbb{R})$ with $\alpha(0)=\lambda_1>0$ and $y\in C^\infty((-\eta,\eta);R(A)\cap Y)$ with $y(0)=y'(0)=0$ are uniquely determined.

Proof. Define $y(s):=v_*(\alpha(s),su_1)$. Then the assertions for y follow from the properties of the function v_*. ∎

Let us now show that the bifurcation in $(\lambda_1,0)$ is of *subcritical type*, i.e., $s\alpha'(s)<0$ for $0<|s|<\delta$ and some $\delta>0$. We first prove that $\alpha'(0)=0$. To this end, we differentiate the expression $F(\alpha(s),su_1+y(s))=0$ with respect to s twice and evaluate at $s=0$ to obtain

$$0=\Delta_N y''(0)+\lambda_1 y''(0)+2\alpha'(0)u_1.$$

By multiplying this identity by u_1 in $L_2(B_R)$ and integrating by parts, we obtain $\alpha'(0)\|u_1\|^2_{L_2(B_R)}=0$, since $u_1\in N(A)$. This implies that $\alpha'(0)=0$, since $u_1\neq 0$. Differentiating $F(\alpha(s),su_1+y(s))=0$ a third time yields at $s=0$

$$0=\Delta_N y'''(0)+\lambda_1 y'''(0)-3\operatorname{div}(\nabla u_1|\nabla u_1|^2)+3\alpha''(0)u_1,$$

where we have used the fact that $\alpha'(0)=0$. We test the latter equation by u_1 in $L_2(B_R)$ and integrate by parts to obtain

$$0=\alpha''(0)\|u_1\|^2_{L_2(B_R)}+\|u_1\|^4_{L_4(B_R)},$$

hence $\alpha''(0)<0$, since $u_1\neq 0$.

Corollary 4.2. *The bifurcation in Theorem 4.1 at $(\lambda_1,0)$ is of subcritical type, i.e., $s\alpha'(s)<0$ for $0<|s|<\delta$ and some $\delta>0$.*

Remark 4.3. One can prove that the bifurcating equilibria induced by Theorem 4.1 are unstable with respect to the flow that is generated by problem (2.2). To this end, one defines an operator $\mathcal{L}(s)$, $|s|<\delta$, as an analogue of the operator L from Section 3.1, representing the full linearisation of (2.2) in one of the bifurcating equilibria. For sufficiently small $\delta>0$, the operator $\mathcal{L}(s)$ possesses a positive eigenvalue, implying the instability of the bifurcating equilibria. We refrain from giving the details and refer the interested reader to [54, Section 5.3] for the proof.

A. Appendix

A.1. The two-phase Stokes problem on the half-line

Let $G\subset\mathbb{R}^2$ be open and bounded with $\partial G\in C^4$. Define $\Omega:=G\times(H_1,H_2)$ and let $\Sigma:=G\times\{0\}$. Let $S_1:=\partial G\times(H_1,H_2)$ and $S_2:=(G\times\{H_1\})\cup(G\times\{H_2\})$. In this section we consider the two-phase Stokes problem

$$\begin{aligned}
\omega\rho u + \partial_t(\rho u) - \mu\Delta u + \nabla\pi &= f, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} u &= f_d, && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu\partial_3 v]\!] - [\![\mu\nabla_{x'} w]\!] &= g_v, && \text{on } \Sigma,\\
-2[\![\mu\partial_3 w]\!] + [\![\pi]\!] &= g_w, && \text{on } \Sigma,\\
[\![u]\!] &= u_\Sigma, && \text{on } \Sigma,\\
P_{S_1}\big(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_1}\big) &= P_{S_1} g_1, && \text{on } S_1\backslash\partial\Sigma,\\
u\cdot\nu_{S_1} &= g_2, && \text{on } S_1\backslash\partial\Sigma,\\
u &= g_3, && \text{on } S_2,\\
u(0) &= u_0, && \text{in } \Omega\backslash\Sigma
\end{aligned} \tag{A.1}$$

on the half-line $\mathbb{R}_+$ for $\omega > 0$. Define the function spaces

$$\begin{aligned}
\mathbb{F}_1 &:= L_p(\mathbb{R}_+; L_p(\Omega)^3),\\
\mathbb{F}_2 &:= L_p(\mathbb{R}_+; H^1_p(\Omega\backslash\Sigma)),\\
\mathbb{F}_3 &:= W_p^{1/2-1/2p}(\mathbb{R}_+; L_p(\Sigma)^2) \cap L_p(\mathbb{R}_+; W_p^{1-1/p}(\Sigma)^2),\\
\mathbb{F}_4 &:= W_p^{1/2-1/2p}(\mathbb{R}_+; L_p(\Sigma)) \cap L_p(\mathbb{R}_+; W_p^{1-1/p}(\Sigma)),\\
\mathbb{F}_5 &:= W_p^{1-1/2p}(\mathbb{R}_+; L_p(\Sigma)^3) \cap L_p(\mathbb{R}_+; W_p^{2-1/p}(\Sigma)^3),\\
\mathbb{F}_6 &:= W_p^{1/2-1/2p}(\mathbb{R}_+; L_p(S_1)^3) \cap L_p(\mathbb{R}_+; W_p^{1-1/p}(S_1\backslash\partial\Sigma)^3),\\
\mathbb{F}_7 &:= W_p^{1-1/2p}(\mathbb{R}_+; L_p(S_1)) \cap L_p(\mathbb{R}_+; W_p^{2-1/p}(S_1\backslash\partial\Sigma)),\\
\mathbb{F}_8 &:= W_p^{1-1/2p}(\mathbb{R}_+; L_p(S_2)) \cap L_p(\mathbb{R}_+; W_p^{2-1/p}(S_2)),
\end{aligned}$$

and $\widetilde{\mathbb{F}} := \bigtimes_{j=1}^{8}\mathbb{F}_j$, as well as

$$\mathbb{F} := \big\{(f_1,\ldots,f_8) \in \widetilde{\mathbb{F}} : (f_2, f_5, f_7, f_8) \in H^1_p(\mathbb{R}_+; \hat{H}_p^{-1}(\Omega))\big\}.$$

Furthermore, we set $X_\gamma := W_p^{2-2/p}(\Omega\backslash\Sigma)^3$. Then we have the following result:

Theorem A.1. *Let $\mu_j, \rho_j, H_j, \sigma > 0$, $p > 2$, $p \neq 3$. Then there exists $\omega_0 > 0$ such that for each $\omega \geq \omega_0$, problem (A.1) has a unique solution*

$$u \in H^1_p(\mathbb{R}_+; L_p(\Omega)^3) \cap L_p(\mathbb{R}_+; H^2_p(\Omega\backslash\Sigma)^3), \quad \pi \in L_p(\mathbb{R}_+; \dot{H}^1_p(\Omega\backslash\Sigma)),$$

and

$$[\![\pi]\!] \in W_p^{1/2-1/2p}(\mathbb{R}_+; L_p(\Sigma)) \cap L_p(\mathbb{R}_+; W_p^{1-1/p}(\Sigma))$$

if and only if the data are subject to the following regularity and compatibility conditions:

(1) $(f, f_d, g_v, g_w, u_\Sigma, g_1, g_2, g_3) \in \mathbb{F}$,

(2) $u_0 \in X_\gamma$,

(3) $\operatorname{div} u_0 = f_d|_{t=0}$, $-[\![\mu\nabla_{x'}w_0]\!] - [\![\mu\partial_3 v_0]\!] = g_v|_{t=0}$, $[\![u_0]\!] = u_\Sigma|_{t=0}$,

(4) $P_{S_1}(\mu(\nabla u_0 + \nabla u_0^T)\nu_{S_1}) = P_{S_1}g_1|_{t=0}$ $(p > 3)$, $u_0 \cdot \nu_{S_1} = g_2|_{t=0}$, $u_0 = g_3|_{t=0}$,

(5) $[\![g_2]\!] = u_\Sigma \cdot \nu_{S_1}$,

(6) $[\![(g_1 \cdot e_3)/\mu - \partial_3 g_2]\!] = \partial_{\nu_{S_1}}(u_\Sigma \cdot e_3)$,

(7) $P_{\partial\Sigma}[(D'v_\Sigma)\nu'] = [\![P_{\partial\Sigma}g_1'/\mu]\!]$,

(8) $(g_v|\nu_{S_1}) = -[\![g_1 \cdot e_3]\!]$, $(g_3|\nu_{S_1}) = g_2$,

(9) $P_{\partial G}[\mu(D'g_3')\nu'] = (P_{\partial G}g_1')$,

(10) $\mu\partial_{\nu_{S_1}}(g_3 \cdot e_3) + \mu\partial_3 g_2 = g_1 \cdot e_3$.

Here we have set $g_j = (g_j^1, g_j^2, g_j^3) =: (g_j', g_j^3)$ *for* $j \in \{1, 3\}$, $D'k = \nabla_{x'}k + \nabla_{x'}k^T$ *for* $k \in \{v_\Sigma, g_3'\}$ *and* $\nu' := \nu_{\partial G}$.

Proof. The proof may be based on a localisation procedure. Making use of reflection arguments as in [55], shifted quarter-space problems and two-phase half-space problems are traced to shifted half-space problems and two-phase full-space problems, which may then be solved by [31, Theorem 7.2.1] and [31, Theorem 8.2.2], respectively.

Note that in contrast to the proof of [55, Theorem 3.2], we are able to control all commutator terms which appear during the localisation procedure by C/ω^a for some uniform $a > 0$ and some $C > 0$ being independent of ω, by means of interpolation and trace theory. Choosing $\omega > 0$ large enough, the norms of the lower order terms will become small. This yields the linear well-posedness of (A.1) on the half-line $\mathbb{R}_+$ for sufficiently large $\omega > 0$. Since the strategy of the proof parallels the one used in the proof of [55, Theorem 3.2] to a large extent, we refrain from giving the details. ∎

As an immediate consequence of the last theorem, one obtains maximal regularity of type L_p of (A.1) in exponentially weighted spaces. To see this, we define

$$e^{-\delta}\mathbb{F}_j := \left\{f \in \mathbb{F}_j : [t \mapsto e^{\delta t}f(t)] \in \mathbb{F}_j\right\},$$

where $\delta \in \mathbb{R}$. We define $e^{-\delta}\widetilde{\mathbb{F}}$ and $e^{-\delta}\mathbb{F}$ similarly.

We write $\omega = \omega - \delta + \delta$ in (A.1), multiply each equation by $e^{\delta t}$ and use the formula $\partial_t(e^{\delta t}u(t)) = e^{\delta t}(\delta u(t) + \partial_t u(t))$ to obtain the following result:

Corollary A.2. *Let the conditions of Theorem A.1 be satisfied. Suppose that* $\delta \in \mathbb{R}$ *and let* $\omega \geq \max\{\omega_0, \omega_0 + \delta\}$. *Then there exists a unique solution*

$$u \in e^{-\delta}[H_p^1(\mathbb{R}_+; L_p(\Omega)^3) \cap L_p(\mathbb{R}_+; H_p^2(\Omega\backslash\Sigma)^3)], \quad \pi \in e^{-\delta}[L_p(\mathbb{R}_+; \dot{H}_p^1(\Omega\backslash\Sigma))],$$

and

$$[\![\pi]\!] \in e^{-\delta}[W_p^{1/2-1/2p}(\mathbb{R}_+; L_p(\Sigma)) \cap L_p(\mathbb{R}_+; W_p^{1-1/p}(\Sigma))]$$

of (A.1) *if and only if the data are subject to the conditions in Theorem A.1 with* $\mathbb{F}$ *being replaced by* $e^{-\delta}\mathbb{F}$.

A.2. Elliptic two-phase Stokes problems

Let $\hat{f} \in L_p(\Omega)^3$, $\hat{f}_d \in H_p^1(\Omega\backslash\Sigma)$, $(\hat{g}_v, \hat{g}_w) \in W_p^{1-1/p}(\Sigma)^3$, $\hat{u}_\Sigma \in W_p^{2-1/p}(\Sigma)^3$, $\hat{g}_1 \in W_p^{1-1/p}(S_1\backslash\partial\Sigma)$, and $\hat{g}_2 \in W_p^{2-1/p}(S_1\backslash\partial\Sigma)$ as well as $\hat{g}_3 \in W_p^{2-1/p}(S_2)$ be given such that $(\hat{f}_d, \hat{u}_\Sigma, \hat{g}_2, \hat{g}_3) \in \hat{H}_p^{-1}(\Omega)$ and such that the compatibility conditions (5)–(10) in Theorem A.1 are satisfied at $\partial S_1 \cap \partial S_2$ and $S_1 \cap \partial\Sigma$.

Define $f(t) := te^{-t}\hat{f}$ and in the same way define $f_d(t), u_\Sigma(t)$ and $g_j(t)$ for $j \in \{v, w, 1, 2, 3\}$. Then it holds that

$$(f, f_d, g_v, g_w, u_\Sigma, g_1, g_2, g_3) \in e^{-\delta}\mathbb{F}$$

for each $\delta \in (0, 1)$ and the compatibility conditions (3)–(10) in Theorem A.1 are satisfied with $u_0 = 0$. By Corollary A.2, there exists a unique solution $(u, \pi, [\![\pi]\!])$ of (A.1) with $\omega \geq \omega_0 + \delta$ such that

$$u \in e^{-\delta}[{}_0H_p^1(\mathbb{R}_+; L_p(\Omega)^3) \cap L_p(\mathbb{R}_+; H_p^2(\Omega\backslash\Sigma)^3)], \quad \pi \in e^{-\delta}[L_p(\mathbb{R}_+; \dot{H}_p^1(\Omega\backslash\Sigma))],$$

and

$$[\![\pi]\!] \in e^{-\delta}[{}_0W_p^{1/2-1/2p}(\mathbb{R}_+; L_p(\Sigma)) \cap L_p(\mathbb{R}_+; W_p^{1-1/p}(\Sigma))].$$

Therefore, the Laplace transform $\mathcal{L}$ of each term in (A.1) is well-defined. Observe that

$$(\mathcal{L}f)(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt = \hat{f} \int_0^\infty te^{-(\lambda+1)t}\, dt = \frac{1}{(\lambda+1)^2}\hat{f}$$

for $\operatorname{Re}\lambda > -1$, hence $(\mathcal{L}f)(0) = \hat{f}$. Doing the same for all the other data and defining $(\hat{u}, \hat{\pi}, [\![\hat{\pi}]\!]) := \mathcal{L}(u, \pi, [\![\pi]\!])$, we obtain that $(\hat{u}, \hat{\pi}, [\![\hat{\pi}]\!])$ solves the elliptic problem

$$\begin{aligned}
\omega\rho\hat{u} - \mu\Delta\hat{u} + \nabla\hat{\pi} &= \hat{f}, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div}\hat{u} &= \hat{f}_d, && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu\partial_3\hat{v}]\!] - [\![\mu\nabla_{x'}\hat{w}]\!] &= \hat{g}_v, && \text{on } \Sigma,\\
-2[\![\mu\partial_3\hat{w}]\!] + [\![\hat{\pi}]\!] &= \hat{g}_w, && \text{on } \Sigma,\\
[\![\hat{u}]\!] &= \hat{u}_\Sigma, && \text{on } \Sigma,\\
P_{S_1}\left(\mu(\nabla\hat{u} + \nabla\hat{u}^\mathsf{T})\nu_{S_1}\right) &= P_{S_1}\hat{g}_1, && \text{on } S_1\backslash\partial\Sigma,\\
\hat{u}\cdot\nu_{S_1} &= \hat{g}_2, && \text{on } S_1\backslash\partial\Sigma,\\
\hat{u} &= \hat{g}_3, && \text{on } S_2
\end{aligned} \tag{A.2}$$

whenever $\omega \geq \omega_0 + \delta$. Let $Au := (\mu/\rho)\Delta u - (1/\rho)\nabla\pi$ with domain

$$\begin{aligned}
D(A) = \big\{u \in H_p^2(\Omega\backslash\Sigma)^3 \cap L_{p,\sigma}(\Omega) : [\![\mu\partial_3 v]\!] + [\![\mu\nabla_{x'}w]\!] = 0,\ [\![u]\!] = 0,\\
P_{S_1}(\mu(Du)\nu_{S_1}) = 0,\ u\cdot\nu_{S_1} = 0,\ u|_{S_2} = 0\big\},
\end{aligned}$$

and $\pi \in \dot{W}_p^1(\Omega\backslash\Sigma)$ be the unique solution of the weak transmission problem

$$\begin{aligned}
\Big(\frac{1}{\rho}\nabla\pi|\nabla\phi\Big)_{L_2(\Omega)} &= \Big(\frac{\mu}{\rho}\Delta u|\nabla\phi\Big)_{L_2(\Omega)}, \quad \phi \in W_{p'}^1(\Omega),\\
[\![\pi]\!] &= 2[\![\mu\partial_3 w]\!], \quad \text{on } \Sigma,
\end{aligned}$$

which we know exists, thanks to [55, Lemma 5.7]. Since A has a compact resolvent, the spectrum $\sigma(A)$ of A consists solely of isolated eigenvalues having a finite multiplicity. Furthermore, it holds that $\operatorname{Re}\sigma(A) = \sigma(A) \subset (-\infty, 0)$ by Korn's inequality (Theorem A.4). Indeed, multiplying the eigenvalue problem $Au = \lambda u$ by u and integrating by parts, we obtain the identity

$$\lambda \|u\|^2_{L_2(\Omega)} = -\|\mu^{1/2} Du\|^2_{L_2(\Omega)}.$$

This yields the following result:

Theorem A.3. *Let $\omega \geq 0$, $\mu_j, \rho_j, \sigma > 0$, $p > 2$, $p \neq 3$ and let Ω and Σ be as in Theorem A.1. Then there exists a unique solution $(\hat{u}, \hat{\pi}, [\![\hat{\pi}]\!])$ with*

$$\hat{u} \in H^2_p(\Omega\backslash\Sigma)^3, \quad \hat{\pi} \in \dot{H}^1_p(\Omega\backslash\Sigma), \quad [\![\hat{\pi}]\!] \in W^{1-1/p}_p(\Sigma)$$

of (A.2) if and only if the data are subject to the following regularity and compatibility conditions:

(1) $\hat{f} \in L_p(\Omega)^3$, $\hat{f}_d \in H^1_p(\Omega\backslash\Sigma)$,

(2) $(\hat{g}_v, \hat{g}_w) \in W^{1-1/p}_p(\Sigma)^3$, $\hat{u}_\Sigma \in W^{2-1/p}_p(\Sigma)^3$,

(3) $\hat{g}_1 \in W^{1-1/p}_p(S_1\backslash\partial\Sigma)$, $\hat{g}_2 \in W^{2-1/p}_p(S_1\backslash\partial\Sigma)$,

(4) $\hat{g}_3 \in W^{2-1/p}_p(S_2)$, $(\hat{f}_d, \hat{u}_\Sigma, \hat{g}_2, \hat{g}_3) \in \hat{H}^{-1}_p(\Omega)$,

(5) $[\![\hat{g}_2]\!] = \hat{u}_\Sigma \cdot \nu_{S_1}$,

(6) $[\![(\hat{g}_1 \cdot e_3)/\mu - \partial_3 \hat{g}_2]\!] = \partial_{\nu_{S_1}}(\hat{u}_\Sigma \cdot e_3)$,

(7) $P_{\partial\Sigma}[(D'\hat{v}_\Sigma)\nu'] = [\![P_{\partial\Sigma}\hat{g}_1'/\mu]\!]$,

(8) $(\hat{g}_v|\nu_{S_1}) = -[\![\hat{g}_1 \cdot e_3]\!]$, $(\hat{g}_3|\nu_{S_1}) = \hat{g}_2$,

(9) $P_{\partial G}[\mu(D'\hat{g}_3')\nu'] = (P_{\partial G}\hat{g}_1')$,

(10) $\mu\partial_{\nu_{S_1}}(\hat{g}_3 \cdot e_3) + \mu\partial_3\hat{g}_2 = \hat{g}_1 \cdot e_3$,

where $\nu' = \nu_{\partial G}$.

A.3. A Korn inequality

For $u \in H^1_2(\Omega)^n$, let $Du := \nabla u + \nabla u^\mathsf{T}$. The following result is well known: There exists a constant $C > 0$ such that

$$\|u\|_{H^1_2(\Omega)} \leq C\|Du\|_{L_2(\Omega)}$$

for all $u \in H^1_2(\Omega)^n$ such that $u = 0$ on $\partial\Omega$ (in the sense of traces). The proof of this inequality relies on integration by parts. We will show that the estimate remains true if $u = 0$ on some subset of $\partial\Omega$ having a positive $(n-1)$-dimensional Hausdorff measure.

Theorem A.4 (Korn's inequality). *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a connected, bounded Lipschitz domain. Then there exists $C > 0$ which depends only on Ω such that the estimate*

$$\|\nabla u\|_{L_2(\Omega)} \leq C\|Du\|_{L_2(\Omega)} \tag{A.3}$$

holds for each $u \in H_2^1(\Omega)^n$ *with* $u = 0$ *on some subset* $\partial_D\Omega$ *of the boundary* $\partial\Omega$ *of* Ω *such that* $\mathcal{H}^{n-1}(\partial_D\Omega) > 0$*, where* $\mathcal{H}^d$ *denotes the* d*-dimensional Hausdorff measure.*

Proof. Let us first show that we have some kind of Poincaré type estimate, that is, there exists a constant $C > 0$ such that the estimate

$$\|u\|_{L_2(\Omega)} \le C\|Du\|_{L_2(\Omega)}$$

holds for all $u \in H_2^1(\Omega)^n$ with $u = 0$ on some subset $\partial_D\Omega$ of the boundary $\partial\Omega$ of Ω such that $\mathcal{H}^{n-1}(\partial_D\Omega) > 0$.

Assume on the contrary that for each $m \in \mathbb{N}$ there exists $u_m \in H_2^1(\Omega)^n$ with $u_m = 0$ on $\partial_D\Omega$ and $\|u_m\|_{L_2(\Omega)} = 1$ such that

$$\|u_m\|_{L_2(\Omega)} \ge m\|Du_m\|_{L_2(\Omega)}.$$

It follows that $Du_m \to 0$ in $L_2(\Omega)$ as $m \to \infty$. By Korn's inequality for functions in $H_2^1(\Omega)^n$ (see [26]), we obtain

$$\|u_m\|_{H_2^1(\Omega)} \le C_0(\|Du_m\|_{L_2(\Omega)} + \|u_m\|_{L_2(\Omega)}) \tag{A.4}$$

for some constant $C_0 > 0$. It follows that $(u_m) \subset H_2^1(\Omega)^n$ is bounded. By Rellich's theorem, there exists a subsequence (u_{m_k}) such that $u_{m_k} \to u_*$ in $L_2(\Omega)$. Then we have $\|u_*\|_{L_2(\Omega)} = 1$ and by trace theory it holds that $u_*(x) = 0$ for a.e. $x \in \partial_D\Omega$. We make use of (A.4) one more time to conclude that (u_{m_k}) is a Cauchy sequence in $H_2^1(\Omega)^n$, since $Du_{m_k} \to 0$ in $L_2(\Omega)$. Therefore, we obtain $u_{m_k} \to u_*$ even in $H_2^1(\Omega)$. Since

$$\|Du_{m_k} - Du_*\|_{L_2(\Omega)} \le C\|\nabla u_{m_k} - \nabla u_*\|_{L_2(\Omega)} \to 0$$

as $k \to \infty$, it follows readily that $Du_* = 0$.

Therefore, there exists a skew-symmetric matrix $A \in \mathbb{R}^{n\times n}$ and some $b \in \mathbb{R}^n$ such that $u_*(x) = Ax + b$ for a.e. $x \in \Omega$ (see [26]). Define $U := \{x \in \mathbb{R}^n : Ax + b = 0\}$. Then $U \neq \emptyset$ is an affine subspace of $\mathbb{R}^n$, since $\partial_D\Omega \subset U$. Fix any $x_0 \in U$ and define

$$U_0 := U - x_0 := \big\{x - x_0 : x \in U\big\}.$$

Observe that $\dim U_0 = n - 1$ (by the assumption on the surface measure of $\partial_D\Omega$) and $Ax = 0$ for each $x \in U_0$. Let $U_0^\perp$ be the orthogonal complement of U_0 and let $y \in U_0^\perp$. Then $(x|Ay) = -(Ax|y) = 0$ for each $x \in U_0$, since A is skew-symmetric, wherefore $Ay \in U_0^\perp$. Furthermore, we have $(Ay|y) = 0$, since A is skew-symmetric. It follows from $\dim U_0^\perp = 1$ that $Ay \in (U_0^\perp)^\perp = U_0$ and therefore $Ay = 0$ for each $y \in U_0^\perp$. But, this means that $Ax = 0$ for each $x \in \mathbb{R}^n$, since $\mathbb{R}^n = U_0 \oplus U_0^\perp$. Thus, we have shown that $A = 0$, hence $u_*(x) = b$ for some $b \in \mathbb{R}^n$. Since $\|u_*\|_{L_2(\Omega)} = 1$ and $u_*(x) = 0$ for a.e. $x \in \partial_D\Omega$, we have a contradiction.

Finally, the assertion of the proposition follows from the Poincaré type estimate combined with Korn's inequality for functions in $H_2^1(\Omega)^n$. ∎

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