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Existence and stability of strong solutions to the Abels–Garcke–Grün model in three dimensions

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Abstract. This work is devoted to the analysis of the strong solutions to the Abels–Garcke–Grün (AGG) model in three dimensions. First, we prove the existence of local-in-time strong solutions originating from an initial datum $(\boldsymbol{u}_0, \phi_0) \in \mathbf{H}^1_\sigma \times H^2(\Omega)$ such that $\mu_0 \in H^1(\Omega)$ and $|\overline{\phi_0}| \leq 1$. For the subclass of initial data that are strictly separated from the pure phases, the corresponding strong solutions are locally unique. Finally, we show a stability estimate between the solutions to the AGG model and the model H. These results extend the analysis achieved by the author in 2021 from two-dimensional bounded domains to three-dimensional ones.

1. Introduction

Given a domain $\Omega \subset \mathbb{R}^3$, we study the Abels–Garcke–Grün (AGG) model in $\Omega \times (0, T)$

$$\begin{cases} \partial_t(\rho(\phi)\boldsymbol{u}) + \operatorname{div}(\boldsymbol{u} \otimes (\rho(\phi)\boldsymbol{u} + \widetilde{\mathbf{J}})) - \operatorname{div}(\nu(\phi)\mathbb{D}\boldsymbol{u}) + \nabla P = -\operatorname{div}(\nabla\phi \otimes \nabla\phi), \\ \operatorname{div} \boldsymbol{u} = 0, \\ \partial_t\phi + \boldsymbol{u} \cdot \nabla\phi = \Delta\mu, \\ \mu = -\Delta\phi + \Psi'(\phi), \end{cases} \tag{1.1}$$

completed with the following boundary and initial conditions:

$$\begin{cases} \boldsymbol{u} = \mathbf{0}, \quad \partial_{\boldsymbol{n}}\phi = \partial_{\boldsymbol{n}}\mu = 0 & \text{on } \partial\Omega \times (0, T), \\ \boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0, \quad \phi(\cdot, 0) = \phi_0 & \text{in } \Omega. \end{cases} \tag{1.2}$$

Here, $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega$, and $\partial_{\boldsymbol{n}}$ denotes the outer normal derivative on $\partial\Omega$. In the system, $\boldsymbol{u} = \boldsymbol{u}(x, t)$ represents the volume averaged velocity, $P = P(x, t)$ is the pressure of the mixture, and $\phi = \phi(x, t)$ is the difference of the fluids' concentrations. The operator $\mathbb{D}$ is the symmetric gradient $\frac{1}{2}(\nabla + \nabla^T)$. The flux term $\widetilde{\mathbf{J}}$, the density ρ and the viscosity ν of the mixture are defined as

$$\widetilde{\mathbf{J}} = -\frac{\rho_1 - \rho_2}{2}\nabla\mu, \quad \rho(\phi) = \rho_1\frac{1+\phi}{2} + \rho_2\frac{1-\phi}{2}, \quad \nu(\phi) = \nu_1\frac{1+\phi}{2} + \nu_2\frac{1-\phi}{2}, \tag{1.3}$$

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where ρ_1, ρ_2 and ν_1, ν_2 are the positive homogeneous density and viscosity parameters of the two fluids, respectively. The homogeneous free energy density Ψ is the Flory–Huggins potential

$$\Psi(s) = F(s) - \frac{\theta_0}{2}s^2 = \frac{\theta}{2}\big[(1+s)\log(1+s)+(1-s)\log(1-s)\big] - \frac{\theta_0}{2}s^2 \tag{1.4}$$

for all $s \in [-1,1]$, where the constant parameters θ and θ_0 fulfill the conditions $0 < \theta < \theta_0$. In what follows, we will often use the non-conservative form of $(1.1)_1$, that is,

$$\begin{aligned}\rho(\phi)\partial_t \boldsymbol{u} + \rho(\phi)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \rho'(\phi)(\nabla\mu\cdot\nabla)\boldsymbol{u} - \operatorname{div}(\nu(\phi)\mathbb{D}\boldsymbol{u}) + \nabla P \\ = -\operatorname{div}(\nabla\phi\otimes\nabla\phi).\end{aligned} \tag{1.5}$$

We also recall the total energy associated to system (1.1) given by

$$E(\boldsymbol{u},\phi) = E_{\mathrm{kin}}(\boldsymbol{u},\phi) + E_{\mathrm{free}}(\phi) = \int_\Omega \frac{1}{2}\rho(\phi)|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega \frac{1}{2}|\nabla\phi|^2 + \Psi(\phi)\,\mathrm{d}x,$$

and the corresponding energy equation that reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\boldsymbol{u},\phi) + \int_\Omega \nu(\phi)|\mathbb{D}\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega |\nabla\mu|^2\,\mathrm{d}x = 0. \tag{1.6}$$

The AGG system is a primary model in the theory of diffuse interface (phase field) modeling, which describes the motion of two viscous incompressible fluids with different densities. It was proposed in the seminal work [9] (see also [8]). The well-known model H is recovered from (1.1) in the case of matched densities $\rho_1 = \rho_2$ (see [27] for the derivation and [2, 25] for the analysis of the model H). The existence of global weak solutions (with finite energy) to the AGG model (1.1)–(1.2) has been established in the case of non-degenerate mobility in [5] and in the case of degenerate mobility in [6]. Global weak solutions were also proven for viscous non-Newtonian fluids in [4] and for the case with dynamic boundary conditions describing moving contact lines in [21]. Further generalizations to non-local versions of the AGG model have been studied in [10] for fractional free energies and in [19] and [20] for free energy with regular convolution kernels. The connection between local and non-local AGG models has recently been investigated in [11] by exploiting the arguments in [17]. Concerning the existence and uniqueness of regular solutions, far fewer results are known. In [12], the local well-posedness of strong solutions is proven in three dimensions for polynomial-like potentials Ψ provided that $\boldsymbol{u}_0 \in \mathbf{H}^1_\sigma$ and $\phi_0 \in (L^p(\Omega), W^4_{p,N}(\Omega))_{1-\frac{1}{p},p}$ for $4 < p < 6$ (in this range of p, $\phi_0 \in H^3(\Omega)$) such that $\|\phi_0\|_{L^\infty} \leq 1$. It is worth mentioning that the solution in [12] may not satisfy $|\phi(x,t)| \leq 1$ for all positive times. In [24], the local well-posedness of strong solutions in two-dimensional bounded domains has been achieved for the logarithmic potential case (see (1.4)) with initial conditions $(\boldsymbol{u}_0, \phi_0) \in \mathbf{H}^1_\sigma \times H^2(\Omega)$ such that $\mu_0 \in H^1(\Omega)$ and $|\overline{\phi_0}| \leq 1$. In this case, the solution satisfies the physical bound $|\phi(x,t)| \leq 1$ at all times. In addition, in the case of periodic boundary conditions, the strong solutions are shown to be globally defined in time in [24]. We also refer the interested reader to [14,18,23,28,30,33] and [1,3,7,15,16,26,29,35] for the modeling and the analysis of different diffuse interface models with unmatched densities.

The purpose of the present contribution is to study the well-posedness of strong solutions to the AGG model (1.1)–(1.2) in bounded domains in $\mathbb{R}^3$. In particular, we aim at generalizing the analysis obtained in [24] from the two-dimensional case to the three-dimensional one. The first result regarding the existence and uniqueness of strong solutions reads as follows:

Theorem 1.1. *Let Ω be a bounded domain of class C^3 in $\mathbb{R}^3$. Assume that $\boldsymbol{u}_0 \in \mathbf{H}^1_\sigma$ and $\phi_0 \in H^2(\Omega)$ such that $\|\phi_0\|_{L^\infty} \le 1$, $|\overline{\phi_0}| < 1$, $\mu_0 = -\Delta\phi_0 + \Psi'(\phi_0) \in H^1(\Omega)$, and $\partial_{\boldsymbol{n}}\phi_0 = 0$ on $\partial\Omega$. Then, there exist $T_0 > 0$, depending on the norms of the initial data, and (at least) a strong solution $(\boldsymbol{u}, P, \phi)$ to system (1.1)–(1.2) on $(0, T_0)$ in the following sense:*

(i) *The solution $(\boldsymbol{u}, P, \phi)$ satisfies the properties*

$$
\begin{aligned}
&\boldsymbol{u} \in C([0,T_0];\mathbf{H}^1_\sigma) \cap L^2(0,T_0;\mathbf{H}^2_\sigma) \cap W^{1,2}(0,T_0;\mathbf{L}^2_\sigma),\\
&P \in L^2(0,T_0;H^1(\Omega)),\\
&\phi \in L^\infty(0,T_0;W^{2,6}(\Omega)),\ \partial_t\phi \in L^\infty(0,T_0;(H^1(\Omega))') \cap L^2(0,T_0;H^1(\Omega)),\\
&\phi \in L^\infty(\Omega\times(0,T_0)) \text{ such that } |\phi(x,t)| < 1 \text{ a.e. in } \Omega\times(0,T_0),\\
&\mu \in L^\infty(0,T_0;H^1(\Omega)) \cap L^2(0,T_0;H^3(\Omega)),\\
&F'(\phi) \in L^\infty(0,T_0;L^6(\Omega)).
\end{aligned}
\tag{1.7}
$$

(ii) *The solution $(\boldsymbol{u}, P, \phi)$ fulfills the system (1.1) almost everywhere in $\Omega \times (0, T_0)$ and the boundary conditions $\partial_{\boldsymbol{n}}\phi = \partial_{\boldsymbol{n}}\mu = 0$ almost everywhere in $\partial\Omega \times (0, T_0)$.*

Furthermore, if additionally Ω is a bounded C^4 domain in $\mathbb{R}^3$ and $\|\phi_0\|_{L^\infty} < 1$, then the solution is locally unique, that is, there exists a time $T_1 \in (0, T_0]$, depending only on the norms of the initial data, such that the solution is unique on the time interval $[0, T_1)$.

Before proceeding with our second result, it is worth mentioning that the proof of Theorem 1.1, although still based on a semi-Galerkin approximation, differs from the one of [24, Theorem 3.1] in several aspects. First, the proof of [24, Theorem 3.1] exploited the continuity of the chemical potential and the regularity of its time derivative, which are properties available for the strong solutions of the convective Cahn–Hilliard equation in two dimensions. Since these are still an open question in three dimensions, we overcome this issue by employing an approximation procedure involving the convective viscous Cahn–Hilliard equation (see Appendix A), together with an appropriate regularization of the initial datum. Such approximations are crucial to rigorously justify the higher-order Sobolev estimates obtained for the approximate solutions. Secondly, due to the lack of global-in-time separation property in three dimensions, we show local uniqueness of solutions departing from a subclass of initial data such that $\|\phi_0\|_{L^\infty} < 1$. For such a class of solutions, the separation property holds on a (possibly short) time interval due to the regularity of the solution in Hölder spaces. We point out that the separation property (or, at least, L^p-estimates of $\Psi''(\phi)$ and $\Psi'''(\phi)$) seems to be necessary to control the additional term $\rho'(\phi)(\nabla\mu\cdot\nabla)\boldsymbol{u}$ in (1.5). Notice that the argument proposed in [25] based on estimates in dual spaces cannot be used due to the non-constant density. On the other hand, the estimate in $\mathbf{L}^2_\sigma \times H^1(\Omega)$ of the difference of the solutions in [24, Theorem 3.1] fails in three dimensions due to the above-mentioned term $\rho'(\phi)(\nabla\mu\cdot\nabla)\boldsymbol{u}$ in (1.5). To overcome these issues, the proof of the uniqueness is carried out by means of a Sobolev type estimate in $\mathbf{L}^2_\sigma \times H^2(\Omega)$ for the difference of the solutions.

Next, we prove a stability result between the strong solutions to the AGG model and to the model H departing from the same initial datum in terms of the density values.

Theorem 1.2. *Let Ω be a bounded domain of class C^3 in $\mathbb{R}^3$. Given an initial datum $(\boldsymbol{u}_0,\phi_0)$ as in Theorem 1.1, we consider the strong solution $(\boldsymbol{u},P,\phi)$ to the AGG model with density (1.3) and the strong solution $(\boldsymbol{u}_H,P_H,\phi_H)$ to the model H with constant density $\overline{\rho}>0$, both defined on $[0,T_0]$. Then, there exists a constant C, that depends on the norm of the initial data, the time T_0 and the parameters of the systems, such that*

$$\begin{aligned}\sup_{t\in[0,T_0]}\|\boldsymbol{u}(t)-\boldsymbol{u}_H(t)\|_{(\mathbf{H}^1_\sigma)'}&+\sup_{t\in[0,T_0]}\|\phi(t)-\phi_H(t)\|_{(H^1)'}\\ &\leq C\left(\left|\frac{\rho_1-\rho_2}{2}\right|+\left|\frac{\rho_1+\rho_2}{2}-\overline{\rho}\right|\right).\end{aligned}\tag{1.8}$$

Remark 1.3. Assuming that $\rho_1=\overline{\rho}$ and $\rho_2=\overline{\rho}+\varepsilon$, for (small) $\varepsilon>0$, the stability estimate (1.8) reads as

$$\sup_{t\in[0,T_0]}\|\boldsymbol{u}(t)-\boldsymbol{u}_H(t)\|_{(\mathbf{H}^1_\sigma)'}+\sup_{t\in[0,T_0]}\|\phi(t)-\phi_H(t)\|_{(H^1)'}\leq C\varepsilon.$$

Theorem 1.2 justifies the model H as the constant density approximation of the AGG model when the two viscous fluids have negligible difference between their densities. To make a comparison with [24, Theorem 3.5], we notice that the estimate holds in dual Sobolev spaces. Indeed, the main idea is to write the momentum equation for the solutions difference $(\boldsymbol{u}-\boldsymbol{u}_H,\phi-\phi_H)$ as the Navier–Stokes equations with constant density and exploit the uniqueness argument introduced in [25].

Plan of the paper. We report in Section 2 the preliminaries for the analysis. Sections 3 and 4 are devoted to the proof of Theorem 1.1, in particular, the local existence of strong solutions and their uniqueness, respectively. In Section 5 we prove the stability result contained in Theorem 1.2. Appendix A is concerned with the well-posedness results for the convective viscous Cahn–Hilliard equation.

2. Notation and functional spaces

Let X be a real Banach space. Its norm is denoted by $\|\cdot\|_X$ and the symbol $\langle\cdot,\cdot\rangle_{X',X}$ stands for the duality between X and its dual space X'. We assume that Ω is a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$ of class C^3. For $p \in [1, \infty]$, let $L^p(\Omega)$ be the Lebesgue space with norm $\|\cdot\|_{L^p}$. The inner product in $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$. For $s \in \mathbb{N}$, $p \in [1, \infty]$, $W^{s,p}(\Omega)$ is the Sobolev space with norm $\|\cdot\|_{W^{s,p}}$. If $p = 2$, we use the notation $W^{s,p}(\Omega) = H^s(\Omega)$. For every $f \in (H^1(\Omega))'$, we denote by $\overline{f}$ the generalized mean value over Ω defined by $\overline{f} = |\Omega|^{-1}\langle f, 1\rangle_{(H^1(\Omega))',H^1(\Omega)}$. If $f \in L^1(\Omega)$, then $\overline{f} = |\Omega|^{-1}\int_\Omega f \,\mathrm{d}x$. By the generalized Poincaré inequality, there exists a positive constant C such that

$$\|f\|_{H^1} \leq C\big(\|\nabla f\|_{L^2}^2 + |\overline{f}|^2\big)^{\frac{1}{2}}, \quad \forall\, f \in H^1(\Omega). \tag{2.1}$$

We recall the Ladyzhenskaya, Agmon and Gagliardo–Nirenberg inequalities in three dimensions:

$$\|f\|_{L^3} \leq C\|f\|_{L^2}^{\frac{1}{2}}\|f\|_{H^1}^{\frac{1}{2}}, \qquad \forall\, f \in H^1(\Omega), \tag{2.2}$$

$$\|f\|_{L^\infty} \leq C\|f\|_{H^1}^{\frac{1}{2}}\|f\|_{H^2}^{\frac{1}{2}}, \qquad \forall\, f \in H^2(\Omega), \tag{2.3}$$

$$\|\nabla f\|_{L^4} \leq C\|f\|_{L^\infty}^{\frac{1}{2}}\|f\|_{H^2}^{\frac{1}{2}}, \qquad \forall\, f \in H^2(\Omega), \tag{2.4}$$

$$\|f\|_{W^{1,4}} \leq C\|f\|_{H^1}^{\frac{5}{8}}\|f\|_{W^{2,6}}^{\frac{3}{8}}, \quad \forall\, f \in W^{2,6}(\Omega). \tag{2.5}$$

Next, we introduce the Hilbert spaces of solenoidal vector-valued functions:

$$\mathbf{L}^2_\sigma = \big\{\boldsymbol{u} \in \mathbf{L}^2(\Omega) : \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\big\},$$
$$\mathbf{H}^1_\sigma = \big\{\boldsymbol{u} \in \mathbf{H}^1(\Omega) : \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u} = \mathbf{0} \text{ on } \partial\Omega\big\}.$$

We also use $(\cdot, \cdot)$ and $\|\cdot\|_{L^2}$ for the inner product and the norm in $\mathbf{L}^2_\sigma$, respectively. The space $\mathbf{H}^1_\sigma$ is endowed with the inner product and norm $(\boldsymbol{u}, \boldsymbol{v})_{\mathbf{H}^1_\sigma} = (\nabla\boldsymbol{u}, \nabla\boldsymbol{v})$ and $\|\boldsymbol{u}\|_{\mathbf{H}^1_\sigma} = \|\nabla\boldsymbol{u}\|_{L^2}$, respectively. We recall the Korn inequality

$$\|\nabla\boldsymbol{u}\|_{L^2} \leq \sqrt{2}\|\mathbb{D}\boldsymbol{u}\|_{L^2}, \quad \forall\, \boldsymbol{u} \in \mathbf{H}^1_\sigma, \tag{2.6}$$

which implies that $\|\mathbb{D}\boldsymbol{u}\|_{L^2}$ is a norm on $\mathbf{H}^1_\sigma$ equivalent to $\|\boldsymbol{u}\|_{\mathbf{H}^1_\sigma}$. We introduce the space $\mathbf{H}^2_\sigma = \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_\sigma$ with inner product $(\boldsymbol{u}, \boldsymbol{v})_{\mathbf{H}^2_\sigma} = (\mathbf{A}\boldsymbol{u}, \mathbf{A}\boldsymbol{v})$ and norm $\|\boldsymbol{u}\|_{\mathbf{H}^2_\sigma} = \|\mathbf{A}\boldsymbol{u}\|_{L^2}$, where $\mathbf{A} = \mathbb{P}(-\Delta)$ is the Stokes operator and $\mathbb{P}$ is the Leray projection from $\mathbf{L}^2(\Omega)$ onto $\mathbf{L}^2_\sigma$. We recall that there exists a positive constant $C > 0$ such that

$$\|\boldsymbol{u}\|_{H^2} \leq C\|\boldsymbol{u}\|_{\mathbf{H}^2_\sigma}, \quad \forall\, \boldsymbol{u} \in \mathbf{H}^2_\sigma. \tag{2.7}$$

We denote by $\mathbf{A}^{-1} : (\mathbf{H}^1_\sigma)' \to \mathbf{H}^1_\sigma$ the inverse map of the Stokes operator; that is, given $\boldsymbol{f} \in (\mathbf{H}^1_\sigma)'$, there exists a unique $\boldsymbol{u} = \mathbf{A}^{-1}\boldsymbol{f} \in \mathbf{H}^1_\sigma$ such that $(\nabla\mathbf{A}^{-1}\boldsymbol{f}, \nabla\boldsymbol{v}) = \langle\boldsymbol{f}, \boldsymbol{v}\rangle_{(\mathbf{H}^1_\sigma)',\mathbf{H}^1_\sigma}$, for all $\boldsymbol{v} \in \mathbf{H}^1_\sigma$. As a consequence, it follows that

$$\|\boldsymbol{f}\|_\sharp := \|\nabla\mathbf{A}^{-1}\boldsymbol{f}\| = \langle\boldsymbol{f}, \mathbf{A}^{-1}\boldsymbol{f}\rangle^{\frac{1}{2}}_{(\mathbf{H}^1_\sigma)',\mathbf{H}^1_\sigma}$$

is an equivalent norm on $(\mathbf{H}^1_\sigma)'$.

Throughout this paper, we will use the following notation:

$$\rho_* = \min\{\rho_1, \rho_2\}, \quad \rho^* = \max\{\rho_1, \rho_2\}, \quad \nu_* = \min\{\nu_1, \nu_2\}, \quad \nu^* = \max\{\nu_1, \nu_2\}.$$

The symbol C will denote a generic positive constant whose value may change from line to line. The specific value depends on the domain Ω and the parameters of the system, such as ρ_*, ρ^*, ν_*, ν^*, θ and θ_0. Further dependencies will be specified when necessary.

3. Proof of Theorem 1.1. Part one: Existence of solutions

3.1. Approximation of the initial datum

First of all, we approximate the initial concentration ϕ_0 following the method introduced in [25]. For $k \in \mathbb{N}$, we consider the elliptic problem

$$\begin{cases} -\Delta\phi_{0,k} + F'(\phi_{0,k}) = \widetilde{\mu}_{0,k} & \text{in } \Omega, \\ \partial_n \phi_{0,k} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\widetilde{\mu}_{0,k} = h_k \circ \widetilde{\mu}_0$, $h_k : \mathbb{R} \to \mathbb{R}$ is the globally Lipschitz function

$$h_k(z) = \begin{cases} -k, & z < -k, \\ z, & z \in [-k, k], \\ k, & z > k, \end{cases} \tag{3.2}$$

and $\widetilde{\mu}_0 = -\Delta\phi_0 + F'(\phi_0)$. Thanks to the superposition principle [31], we have that $\widetilde{\mu}_{0,k} \in H^1(\Omega) \cap L^\infty(\Omega)$ and

$$\|\widetilde{\mu}_{0,k}\|_{H^1} \le \|\widetilde{\mu}_0\|_{H^1}. \tag{3.3}$$

As shown in [25, Lemma A.1], there exists a unique solution $\phi_{0,k}$ to (3.1) such that $\phi_{0,k} \in H^2(\Omega)$, $F'(\phi_{0,k}) \in L^2(\Omega)$, which satisfies (3.1) almost everywhere in Ω and $\partial_n \phi_{0,k} = 0$ almost everywhere on $\partial\Omega$. In addition, there exist $\widetilde{m} \in (0, 1)$, which is independent of k, and $\overline{k}$ sufficiently large such that

$$\|\phi_{0,k}\|_{H^1} \le 1 + \|\phi_0\|_{H^1}, \quad |\overline{\phi_{0,k}}| \le \widetilde{m} < 1, \quad \|\phi_{0,k}\|_{H^2} \le C(1 + \|\widetilde{\mu}_0\|_{L^2}) \tag{3.4}$$

for any $k > \overline{k}$. Furthermore, by [25, Theorem A.2] (see also [2, Lemma 2]), we have

$$\|F'(\phi_{0,k})\|_{L^\infty} \le \|\widetilde{\mu}_{0,k}\|_{L^\infty} \le k.$$

Then, there exists $\delta = \delta(k) > 0$ such that

$$\|\phi_{0,k}\|_{L^\infty} \le 1 - \delta. \tag{3.5}$$

As a consequence, since $F'(\phi_{0,k}) \in H^1(\Omega)$, it is easily seen that $\phi_{0,k} \in H^3(\Omega)$ by elliptic regularity. Finally, observing that $\widetilde{\mu}_{0,k} \to \widetilde{\mu}_0$ in $L^2(\Omega)$, it follows that $\phi_{0,k} \to \phi_0$ in $H^1(\Omega)$.

3.2. Definition of the approximated problem

Let us consider the family of eigenfunctions $\{\boldsymbol{w}_j\}_{j=1}^{\infty}$ and eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ of the Stokes operator $\mathbf{A}$. For any integer $m \geq 1$, let $\mathbf{V}_m$ denote the finite-dimensional subspaces of $\mathbf{L}^2_\sigma$ defined as $\mathbf{V}_m = \text{span}\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\}$. The finite-dimensional spaces $\mathbf{V}_m$ are endowed with the norm of $\mathbf{L}^2_\sigma$. The orthogonal projection on $\mathbf{V}_m$ with respect to the inner product in $\mathbf{L}^2_\sigma$ is denoted by $\mathbb{P}_m$. Recalling that Ω is of class C^3, the regularity theory of the Stokes operator yields that $\boldsymbol{w}_j \in \mathbf{H}^3(\Omega) \cap \mathbf{H}^1_\sigma$ for all $j \in \mathbb{N}$. As a consequence, the following inverse Sobolev embedding inequalities hold for all $\boldsymbol{v} \in \mathbf{V}_m$:

$$\|\boldsymbol{v}\|_{H^1} \leq C_m \|\boldsymbol{v}\|_{L^2}, \quad \|\boldsymbol{v}\|_{H^2} \leq C_m \|\boldsymbol{v}\|_{L^2}, \quad \|\boldsymbol{v}\|_{H^3} \leq C_m \|\boldsymbol{v}\|_{L^2}. \tag{3.6}$$

Let us set $T > 0$. For any $k > 0, \alpha \in (0,1)$ and $m \in \mathbb{N}$, we claim that there exists an approximate solution $(\boldsymbol{u}_m, \phi_m)$ to system (1.1)–(1.2) in the following sense:

$$\begin{aligned} &\boldsymbol{u}_m \in C^1([0,T]; \mathbf{V}_m), \\ &\phi_m \in L^\infty(0,T; H^3(\Omega)),\ \partial_t \phi_m \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)), \\ &\phi_m \in L^\infty(\Omega \times (0,T)) \text{ such that } |\phi_m(x,t)| \leq 1-\delta \text{ a.e. in } \Omega \times (0,T), \\ &\mu_m \in L^\infty(0,T; H^2(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega)), \end{aligned} \tag{3.7}$$

for some $\delta > 0$, such that

$$\begin{aligned} &(\rho(\phi_m)\partial_t \boldsymbol{u}_m, \boldsymbol{w}) + (\rho(\phi_m)(\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, \boldsymbol{w}) + (\nu(\phi_m)\mathbb{D}\boldsymbol{u}_m, \nabla \boldsymbol{w}) \\ &\quad - \frac{\rho_1 - \rho_2}{2}((\nabla \mu_m \cdot \nabla)\boldsymbol{u}_m, \boldsymbol{w}) = (\mu_m \nabla \phi_m, \boldsymbol{w}), \end{aligned} \tag{3.8}$$

for all $\boldsymbol{w} \in \mathbf{V}_m$ and $t \in [0,T]$, and

$$\partial_t \phi_m + \boldsymbol{u}_m \cdot \nabla \phi_m = \Delta \mu_m, \quad \mu_m = \alpha \partial_t \phi_m - \Delta \phi_m + \Psi'(\phi_m) \quad \text{a.e. in } \Omega \times (0,T), \tag{3.9}$$

together with

$$\begin{cases} \boldsymbol{u}_m = \mathbf{0}, \quad \partial_{\boldsymbol{n}} \phi_m = \partial_{\boldsymbol{n}} \mu_m = 0 & \text{on } \partial\Omega \times (0,T), \\ \boldsymbol{u}_m(\cdot,0) = \mathbb{P}_m \boldsymbol{u}_0,\ \phi(\cdot,0) = \phi_{0,k} & \text{in } \Omega. \end{cases} \tag{3.10}$$

3.3. Existence of approximate solutions

We now exploit a fixed point argument to show the existence of $(\boldsymbol{u}_m, \phi_m)$ satisfying (3.7)–(3.10). For this purpose, we fix $\boldsymbol{v} \in W^{1,2}(0,T; \mathbf{V}_m)$. We consider the convective viscous Cahn–Hilliard system

$$\begin{cases} \partial_t \phi_m + \boldsymbol{v} \cdot \nabla \phi_m = \Delta \mu_m \\ \mu_m = \alpha \partial_t \phi_m - \Delta \phi_m + F'(\phi_m) - \theta_0 \phi_m \end{cases} \quad \text{in } \Omega \times (0,T), \tag{3.11}$$

which is equipped with the boundary and initial conditions

$$\partial_{\boldsymbol{n}} \phi_m = \partial_{\boldsymbol{n}} \mu_m = 0 \quad \text{on } \partial\Omega \times (0,T), \qquad \phi_m(\cdot,0) = \phi_{0,k} \quad \text{in } \Omega. \tag{3.12}$$

Thanks to Theorem A.1, there exists a unique solution ϕ_m to (3.11)–(3.12) such that

$$\begin{aligned}&\phi_m \in L^\infty(0,T;H^3(\Omega)), \quad \partial_t\phi_m \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),\\ &\phi_m \in L^\infty(\Omega\times(0,T)) \text{ such that } |\phi_m(x,t)| \le 1-\widetilde{\delta} \text{ a.e. in } \Omega\times(0,T),\\ &\mu_m \in L^\infty(0,T;H^2(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)),\end{aligned} \tag{3.13}$$

for some $\widetilde{\delta}$ which depends on α and k. We report the following estimates for system (3.11)–(3.12):

(1) L^2 estimate: for any $T>0$,

$$\sup_{t\in[0,T]}\left(\|\phi_m(t)\|_{L^2}^2 + \alpha\|\nabla\phi_m(t)\|_{L^2}^2\right) + \int_0^T \|\Delta\phi_m(\tau)\|_{L^2}^2\,\mathrm{d}\tau \le \|\phi_{0,k}\|_{L^2}^2 + \alpha\|\nabla\phi_{0,k}\|_{L^2}^2 + \theta_0^2|\Omega|T;$$

(2) Energy estimate: for any $T>0$,

$$\sup_{t\in[0,T]} E_{\text{free}}(\phi(t)) + \frac{1}{2}\int_0^T \|\nabla\mu_m(\tau)\|_{L^2}^2\,\mathrm{d}\tau + \alpha\int_0^T \|\partial_t\phi_m(\tau)\|_{L^2}^2\,\mathrm{d}\tau \le E_{\text{free}}(\phi_{0,k}) + \frac{1}{2}\int_0^T \|\boldsymbol{v}(\tau)\|_{L^2}^2\,\mathrm{d}\tau. \tag{3.14}$$

We now make the ansatz that

$$\boldsymbol{u}_m(x,t) = \sum_{j=1}^m a_j^m(t)\boldsymbol{w}_j(x)$$

is the solution to the Galerkin approximation of $(1.1)_1$ which reads as

$$(\rho(\phi_m)\partial_t\boldsymbol{u}_m, \boldsymbol{w}_l) + (\rho(\phi_m)(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}_m, \boldsymbol{w}_l) + (\nu(\phi_m)\mathbb{D}\boldsymbol{u}_m, \nabla\boldsymbol{w}_l) - \frac{\rho_1-\rho_2}{2}((\nabla\mu_m\cdot\nabla)\boldsymbol{u}_m, \boldsymbol{w}_l) = (\mu_m\nabla\phi_m, \boldsymbol{w}_l), \quad \forall\, l = 1,\dots,m, \tag{3.15}$$

such that $\boldsymbol{u}_m(\cdot,0) = \mathbb{P}_m\boldsymbol{u}_0$. Setting $\mathbf{A}^m(t) = (a_1^m(t),\dots,a_m^m(t))^T$, (3.15) is equivalent to the system of differential equations

$$\mathbf{M}^m(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^m + \mathbf{L}^m(t)\mathbf{A}^m = \mathbf{G}^m(t), \tag{3.16}$$

where the matrices $\mathbf{M}^m(t)$, $\mathbf{L}^m(t)$ and the vector $\mathbf{G}^m(t)$ are defined as

$$\begin{aligned}(\mathbf{M}^m(t))_{l,j} &= \int_\Omega \rho(\phi_m)\boldsymbol{w}_l\cdot\boldsymbol{w}_j\,\mathrm{d}x,\\ (\mathbf{L}^m(t))_{l,j} &= \int_\Omega \Big(\rho(\phi_m)(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}_j\cdot\boldsymbol{w}_l + \nu(\phi_m)\mathbb{D}\boldsymbol{w}_j : \nabla\boldsymbol{w}_l\\ &\qquad - \Big(\frac{\rho_1-\rho_2}{2}\Big)(\nabla\mu_m\cdot\nabla)\boldsymbol{w}_j\cdot\boldsymbol{w}_l\Big)\,\mathrm{d}x,\end{aligned}$$

$$(\mathbf{G}^m(t))_l = \int_\Omega \mu_m \nabla\phi_m \cdot \boldsymbol{w}_l \,\mathrm{d}x,$$

and $\mathbf{A}^m(0) = ((\mathbb{P}_m \boldsymbol{u}_0, \boldsymbol{w}_1), \dots, (\mathbb{P}_m \boldsymbol{u}_0, \boldsymbol{w}_m))^T$. The regularity properties (3.13) imply the continuity of $\phi_m \in C([0,T]; W^{1,4}(\Omega))$ and $\mu_m \in C([0,T]; H^1(\Omega))$. In turn, we have that $\rho(\phi_m), \nu(\phi_m) \in C(\overline{\Omega \times [0,T]})$. Moreover, we observe that $\boldsymbol{v} \in C([0,T]; \mathbf{L}^2_\sigma)$. Thus, we infer that $\mathbf{M}^m$ and $\mathbf{L}^m$ belong to $C([0,T]; \mathbb{R}^{m\times m})$, and $\mathbf{G}^m \in C([0,T]; \mathbb{R}^m)$. Since the matrix $\mathbf{M}^m(\cdot)$ is positive definite on $[0,T]$ (see [26, Appendix A]), the inverse $(\mathbf{M}^m)^{-1} \in C([0,T]; \mathbb{R}^{m\times m})$. Thus, the existence and uniqueness theorem for systems of linear ODEs guarantees that there exists a unique solution $\mathbf{A}^m \in C^1([0,T]; \mathbb{R}^m)$ to (3.16) on $[0,T]$. As a result, the problem (3.15) has a unique solution $\boldsymbol{u}_m \in C^1([0,T]; \mathbf{V}_m)$.

Next, multiplying (3.15) by a_l^m and summing over l, we find

$$\begin{aligned}\int_\Omega \rho(\phi_m)\partial_t\Big(\frac{|\boldsymbol{u}_m|^2}{2}\Big)\,\mathrm{d}x + \int_\Omega \rho(\phi_m)\boldsymbol{v}\cdot\nabla\Big(\frac{|\boldsymbol{u}_m|^2}{2}\Big)\,\mathrm{d}x + \int_\Omega \nu(\phi_m)|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x \\ - \frac{\rho_1-\rho_2}{2}\int_\Omega \nabla\mu_m\cdot\nabla\Big(\frac{|\boldsymbol{u}_m|^2}{2}\Big)\,\mathrm{d}x = \int_\Omega \mu_m\nabla\phi_m\cdot\boldsymbol{u}_m\,\mathrm{d}x.\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\phi_m)\frac{|\boldsymbol{u}_m|^2}{2}\,\mathrm{d}x - \int_\Omega \big(\partial_t\rho(\phi_m) + \mathrm{div}\,(\rho(\phi_m)\boldsymbol{v})\big)\frac{|\boldsymbol{u}_m|^2}{2}\,\mathrm{d}x \\ &\quad + \int_\Omega \nu(\phi_m)|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x + \frac{\rho_1-\rho_2}{2}\int_\Omega \Delta\mu_m\frac{|\boldsymbol{u}_m|^2}{2}\,\mathrm{d}x = \int_\Omega \phi_m\nabla\mu_m\cdot\boldsymbol{u}_m\,\mathrm{d}x.\end{aligned}$$

Recalling that $\rho'(\phi_m) = \frac{\rho_1-\rho_2}{2}$ and $\mathrm{div}\,\boldsymbol{v} = 0$, by using $(3.11)_1$, we have

$$-\int_\Omega \big(\partial_t\rho(\phi_m) + \mathrm{div}\,(\rho(\phi_m)\boldsymbol{v})\big)\frac{|\boldsymbol{u}_m|^2}{2}\,\mathrm{d}x + \frac{\rho_1-\rho_2}{2}\int_\Omega \Delta\mu_m\frac{|\boldsymbol{u}_m|^2}{2}\,\mathrm{d}x = 0.$$

Thus, we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\phi_m)\frac{|\boldsymbol{u}_m|^2}{2}\,\mathrm{d}x + \int_\Omega \nu(\phi_m)|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x = \int_\Omega \phi_m\nabla\mu_m\cdot\boldsymbol{u}_m\,\mathrm{d}x. \tag{3.17}$$

By using $(3.13)_2$ and the Poincaré inequality, we get

$$\int_\Omega \phi_m\nabla\mu_m\cdot\boldsymbol{u}_m\,\mathrm{d}x \le \|\phi_m\|_{L^\infty}\|\nabla\mu_m\|_{L^2}\|\boldsymbol{u}_m\|_{L^2} \le \frac{\nu_*}{2}\|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^2 + \frac{1}{\lambda_1\nu_*}\|\nabla\mu_m\|_{L^2}^2.$$

So, we find the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\phi_m)\frac{|\boldsymbol{u}_m|^2}{2}\,\mathrm{d}x + \frac{\nu_*}{2}\int_\Omega |\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x \le \frac{1}{\lambda_1\nu_*}\|\nabla\mu_m\|_{L^2}^2. \tag{3.18}$$

Integrating the above inequality on $[0,s]$ with $s \in [0,T]$, and using (3.14), it follows that

$$\begin{aligned}\int_\Omega \frac{\rho_*}{2}|\boldsymbol{u}_m(s)|^2\,\mathrm{d}x \le \int_\Omega \rho(\phi_{0,k})\frac{|\mathbb{P}_m\boldsymbol{u}_0|^2}{2}\,\mathrm{d}x + \frac{2}{\lambda_1\nu_*}E_{\mathrm{free}}(\phi_{0,k}) \\ + \frac{1}{\lambda_1\nu_*}\int_0^s \|\boldsymbol{v}(\tau)\|_{L^2}^2\,\mathrm{d}\tau,\end{aligned} \tag{3.19}$$

which, in turn, entails that

$$\|\boldsymbol{u}_m(s)\|_{L^2}^2 \leq \frac{\rho^*}{\rho_*}\|\boldsymbol{u}_0\|_{L^2}^2 + \frac{4}{\lambda_1\rho_*\nu_*}E_{\text{free}}(\phi_{0,k}) + \frac{2}{\lambda_1\rho_*\nu_*}\int_0^s \|\boldsymbol{v}(\tau)\|_{L^2}^2\,\mathrm{d}\tau. \tag{3.20}$$

At this point, setting

$$C_1 = \frac{\rho^*}{\rho_*}\|\boldsymbol{u}_0\|_{L^2}^2 + \frac{4}{\lambda_1\rho_*\nu_*}E_{\text{free}}(\phi_{0,k}), \quad C_2 = \frac{2}{\lambda_1\rho_*\nu_*},$$

and assuming

$$\int_0^t \|\boldsymbol{v}(\tau)\|_{L^2}^2\,\mathrm{d}\tau \leq C_3\mathrm{e}^{C_2 t}, \quad t \in [0,T], \tag{3.21}$$

where $C_3 = C_1 T$, we deduce that

$$\int_0^t \|\boldsymbol{u}_m(s)\|_{L^2}^2\,\mathrm{d}s \leq C_3 + C_2\int_0^t\int_0^s \|\boldsymbol{v}(\tau)\|_{L^2}^2\,\mathrm{d}\tau\,\mathrm{d}s \leq C_3\mathrm{e}^{C_2 t}, \quad \forall\, t\in[0,T]. \tag{3.22}$$

Furthermore, thanks to (3.20) and (3.21), we also infer that

$$\sup_{t\in[0,T]} \|\boldsymbol{u}_m(t)\|_{L^2} \leq \big(C_1 + C_3C_2\mathrm{e}^{C_2T}\big)^{\frac{1}{2}} =: K_0. \tag{3.23}$$

Now, we control the time derivative of $\boldsymbol{u}_m$. Multiplying (3.15) by $\frac{\mathrm{d}}{\mathrm{d}t}a_l^m$ and summing over l, we find

$$\begin{aligned}\rho_*\|\partial_t\boldsymbol{u}_m\|_{L^2}^2 \leq& -(\rho(\phi_m)(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}_m, \partial_t\boldsymbol{u}_m) - (\nu(\phi_m)\mathbb{D}\boldsymbol{u}_m, \nabla\partial_t\boldsymbol{u}_m)\\ &+ \frac{\rho_1-\rho_2}{2}((\nabla\mu_m\cdot\nabla)\boldsymbol{u}_m, \partial_t\boldsymbol{u}_m) + (\phi_m\nabla\mu_m, \partial_t\boldsymbol{u}_m).\end{aligned}$$

By exploiting (3.6), we obtain

$$\begin{aligned}\rho_*\|\partial_t\boldsymbol{u}_m\|_{L^2}^2 \leq& \rho^*\|\boldsymbol{v}\|_{L^2}\|\nabla\boldsymbol{u}_m\|_{L^\infty}\|\partial_t\boldsymbol{u}_m\|_{L^2} + \nu^*\|\mathbb{D}\boldsymbol{u}_m\|_{L^2}\|\nabla\partial_t\boldsymbol{u}_m\|_{L^2}\\ &+ \left|\frac{\rho_1-\rho_2}{2}\right|\|\nabla\boldsymbol{u}_m\|_{L^\infty}\|\nabla\mu_m\|_{L^2}\|\partial_t\boldsymbol{u}_m\|_{L^2}\\ &+ \|\phi_m\|_{L^\infty}\|\nabla\mu_m\|_{L^2}\|\nabla\partial_t\boldsymbol{u}_m\|_{L^2}\\ \leq& \rho^*C\|\boldsymbol{v}\|_{L^2}\|\boldsymbol{u}_m\|_{H^3}\|\partial_t\boldsymbol{u}_m\|_{L^2} + \nu^*C_m^2\|\boldsymbol{u}_m\|_{L^2}\|\partial_t\boldsymbol{u}_m\|_{L^2}\\ &+ C\left|\frac{\rho_1-\rho_2}{2}\right|\|\boldsymbol{u}_m\|_{H^3}\|\nabla\mu_m\|_{L^2}\|\partial_t\boldsymbol{u}_m\|_{L^2} + C_m\|\nabla\mu_m\|_{L^2}\|\partial_t\boldsymbol{u}_m\|_{L^2}\\ \leq& \rho^*C_m\|\boldsymbol{v}\|_{L^2}\|\boldsymbol{u}_m\|_{L^2}\|\partial_t\boldsymbol{u}_m\|_{L^2} + \nu^*C_m^2\|\boldsymbol{u}_m\|_{L^2}\|\partial_t\boldsymbol{u}_m\|_{L^2}\\ &+ C_m\left|\frac{\rho_1-\rho_2}{2}\right|\|\boldsymbol{u}_m\|_{L^2}\|\nabla\mu_m\|_{L^2}\|\partial_t\boldsymbol{u}_m\|_{L^2}\\ &+ C_m\|\nabla\mu_m\|_{L^2}\|\partial_t\boldsymbol{u}_m\|_{L^2}.\end{aligned}$$

Then, by using (3.14), (3.21), (3.22) and (3.23), we infer that

$$\begin{aligned}\int_0^T \|\partial_t\boldsymbol{u}_m(\tau)\|_{L^2}^2\,\mathrm{d}\tau \leq& 4\Big(\frac{\rho^*}{\rho_*}C_mK_0\Big)^2\int_0^T \|\boldsymbol{v}(\tau)\|_{L^2}^2\,\mathrm{d}\tau + 4\Big(\frac{\nu^*}{\rho_*}C_m^2\Big)^2C_3\mathrm{e}^{C_2T}\\ &+ 4\Big(\Big(\frac{C_m}{\rho_*}\left|\frac{\rho_1-\rho_2}{2}\right|K_0\Big)^2 + \frac{C_m^2}{\rho_*^2}\Big)\int_0^T \|\nabla\mu_m(\tau)\|_{L^2}^2\,\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&\le 4\Big(\Big(\frac{\rho^*}{\rho_*}C_m K_0\Big)^2 + \Big(\frac{\nu_*}{\rho_*}C_m^2\Big)^2\Big)C_3 e^{C_2 T} \\
&\qquad + 4\Big(\Big(\frac{C_m}{\rho_*}\Big|\frac{\rho_1-\rho_2}{2}\Big|K_0\Big)^2 + \frac{C_m^2}{\rho_*^2}\Big)\big(2E_{\text{free}}(\phi_{0,k}) + C_3 e^{C_2 T}\big) \\
&=: K_1^2, \qquad (3.24)
\end{aligned}
$$

where K_1 depends only on ρ_*, ρ^*, ν_*, θ_0, $\|\boldsymbol{u}_0\|_{L^2}$, $E_{\text{free}}(\phi_0)$, T, Ω, m.

We define the setting of the fixed point argument. We introduce the set

$$
\begin{aligned}
S = \Big\{&\boldsymbol{u} \in W^{1,2}(0,T;\mathbf{V}_m) : \int_0^t \|\boldsymbol{u}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \le C_3 e^{C_2 t},\ t \in [0,T], \\
&\|\partial_t \boldsymbol{u}\|_{L^2(0,T;\mathbf{V}_m)} \le K_1\Big\},
\end{aligned}
$$

which is a subset of $L^2(0,T;\mathbf{V}_m)$. We define the map

$$
\Lambda : S \to L^2(0,T;\mathbf{V}_m), \quad \Lambda(\boldsymbol{v}) = \boldsymbol{u}_m,
$$

where $\boldsymbol{u}_m$ is the solution to the system (3.15). In light of (3.22) and (3.24), we deduce that $\Lambda : S \to S$. It is easily seen that S is convex and closed. Furthermore, S is a compact set in $L^2(0,T;\mathbf{V}_m)$. We are left to prove that the map Λ is continuous. This is done by adapting the argument in [24, proof of Theorem 3.1] to the viscous case. Let us consider a sequence $\{\boldsymbol{v}_n\} \subset S$ such that $\boldsymbol{v}_n \to \widetilde{\boldsymbol{v}}$ in $L^2(0,T;\mathbf{V}_m)$. By arguing as above, there exist a sequence $\{(\psi_n,\mu_n)\}$ and a pair $(\widetilde{\psi},\widetilde{\mu})$ that solve the convective viscous Cahn–Hilliard equation (3.11)–(3.12), where $\boldsymbol{v}$ is replaced by $\boldsymbol{v}_n$ and $\widetilde{\boldsymbol{v}}$, respectively. Repeating the uniqueness argument in the proof of Theorem A.1, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla A^{-1}(\psi_n - \widetilde{\psi})\|_{L^2}^2 + \alpha\|\psi_n - \widetilde{\psi}\|_{L^2}^2\big) + \|\nabla(\psi_n - \widetilde{\psi})\|_{L^2}^2 \\
&\quad \le \int_\Omega \psi_n(\boldsymbol{v}_n - \widetilde{\boldsymbol{v}}) \cdot \nabla A^{-1}(\psi_n - \widetilde{\psi})\,\mathrm{d}x + \int_\Omega (\psi_n - \widetilde{\psi})\widetilde{\boldsymbol{v}} \cdot \nabla A^{-1}(\psi_n - \widetilde{\psi})\,\mathrm{d}x \\
&\qquad + \theta_0\|\psi_n - \widetilde{\psi}\|_{L^2}^2,
\end{aligned}
$$

where the operator A is the Laplace operator $-\Delta$ with homogeneous Neumann boundary conditions. Since $\widetilde{\boldsymbol{v}}$ belong to S, we infer that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} f(t) + \frac{1}{2}\|\nabla(\psi_n - \widetilde{\psi})\|_{L^2}^2 \le C f(t) + \|\boldsymbol{v}_n - \widetilde{\boldsymbol{v}}\|_{L^2}^2,
$$

where $f(t) = \|\nabla A^{-1}(\psi_n(t) - \widetilde{\psi}(t))\|_{L^2}^2 + \alpha\|\psi_n(t) - \widetilde{\psi}(t)\|_{L^2}^2$, for some constant C depending on C_1, C_2, K_1 and θ_0. Observing that $\psi_n(0) - \widetilde{\psi}(0) = 0$, by the Gronwall lemma we obtain

$$
\|\psi_n - \widetilde{\psi}\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le e^{CT}\int_0^T \|\boldsymbol{v}_n(\tau) - \widetilde{\boldsymbol{v}}(\tau)\|_{L^2}^2\,\mathrm{d}\tau \to 0 \qquad (3.25)
$$

as $n \to \infty$. On the other hand, using that $\{\boldsymbol{v}_n\}$ and $\widetilde{\boldsymbol{v}}$ belong to S, the continuous embedding $W^{1,2}(0,T;\mathbf{V}_m) \hookrightarrow Y_T$ (see Appendix A for the definition of Y_T) and the properties of the initial condition $\phi_{0,k}$ (cf. $\phi_{0,k} \in H^3(\Omega)$ and (3.5)), it follows from Theorem A.1 that

$$\|\partial_t \psi_n\|_{L^\infty(0,T;H^1(\Omega))} + \|\partial_t \psi_n\|_{L^2(0,T;H^2(\Omega))} \le C, \tag{3.26}$$

$$\|\partial_t \widetilde{\psi}\|_{L^\infty(0,T;H^1(\Omega))} + \|\partial_t \widetilde{\psi}\|_{L^2(0,T;H^2(\Omega))} \le C, \tag{3.27}$$

for some C independent of n. Moreover, we also have

$$\|\mu_n\|_{L^\infty(0,T;H^2(\Omega))} + \|\psi_n\|_{L^\infty(0,T;H^3(\Omega))} \le C, \tag{3.28}$$

$$\|\widetilde{\mu}\|_{L^\infty(0,T;H^2(\Omega))} + \|\widetilde{\psi}\|_{L^\infty(0,T;H^3(\Omega))} \le C, \tag{3.29}$$

$$\|\partial_t \mu_n\|_{L^2(0,T;L^2(\Omega))} + \|\partial_t \widetilde{\mu}\|_{L^2(0,T;L^2(\Omega))} \le C \tag{3.30}$$

and

$$\max_{(x,t)\in\Omega\times(0,T)} |\psi_n(x,t)| \le 1-\delta^*, \qquad \max_{(x,t)\in\Omega\times(0,T)} |\widetilde{\psi}(x,t)| \le 1-\delta^*, \tag{3.31}$$

for some positive C and $\delta^* \in (0,1)$ independent of n. Our claim is that $\mu_n \to \widetilde{\mu}$ in $L^\infty(0,T;H^1(\Omega))$. To this end, in light of the above estimates, we first deduce from the Aubin–Lions compactness result that there exists a subsequence μ_{n_j} with the property that $\mu_{n_j} - \widetilde{\mu} \to \mu^*$ in $L^\infty(0,T;H^1(\Omega))$. Let us show that $\mu^* = 0$ by using the equation

$$\mu_n - \widetilde{\mu} = \varepsilon \partial_t(\psi_n - \widetilde{\psi}) - \Delta(\psi_n - \widetilde{\psi}) + \Psi'(\psi_n) - \Psi'(\widetilde{\psi}).$$

By interpolation, we infer from (3.25), (3.28) and (3.29) that

$$\|\psi_n - \widetilde{\psi}\|_{L^\infty(0,T;H^2(\Omega))} \to 0, \quad \text{as } n \to \infty. \tag{3.32}$$

As a consequence, thanks to (3.31), $\|\Psi'(\psi_n) - \Psi'(\widetilde{\psi})\|_{L^\infty(0,T;L^2(\Omega))} \to 0$, as $n \to \infty$. On the other hand, it follows from (3.25), (3.26) and (3.27) that (up to a subsequence) $\partial_t(\psi_{n_j} - \widetilde{\psi}) \rightharpoonup 0$ weakly in $L^2(0,T;H^2(\Omega))$. Thus, we obtain that $\mu^* = 0$. Besides, by uniqueness of the (weak) limit point, we conclude that

$$\|\mu_n - \widetilde{\mu}\|_{L^\infty(0,T;H^1(\Omega))} \to 0, \quad \text{as } n \to \infty. \tag{3.33}$$

We now define $\boldsymbol{u}_n = \Lambda(\boldsymbol{v}_n) \in S$, for any $n \in \mathbb{N}$, and $\widetilde{\boldsymbol{u}} = \Lambda(\widetilde{\boldsymbol{v}}) \in S$. We consider $\boldsymbol{u} = \boldsymbol{u}_n - \widetilde{\boldsymbol{u}}$, $\psi = \psi_n - \widetilde{\psi}$, $\boldsymbol{v} = \boldsymbol{v}_n - \widetilde{\boldsymbol{v}}$, and $\mu = \mu_n - \widetilde{\mu}$ that solve

$$\begin{aligned}
&(\rho(\psi_n)\partial_t \boldsymbol{u}, \boldsymbol{w}) + ((\rho(\psi_n) - \rho(\widetilde{\psi}))\partial_t \widetilde{\boldsymbol{u}}, \boldsymbol{w}) + (\rho(\psi_n)(\boldsymbol{v}_n \cdot \nabla)\boldsymbol{u}_n - \rho(\widetilde{\psi})(\widetilde{\boldsymbol{v}} \cdot \nabla)\widetilde{\boldsymbol{u}}, \boldsymbol{w}) \\
&\quad + (\nu(\psi_n)\mathbb{D}\boldsymbol{u}, \nabla \boldsymbol{w}) + ((\nu(\psi_n) - \nu(\widetilde{\psi}))\mathbb{D}\widetilde{\boldsymbol{u}}, \nabla \boldsymbol{w}) \\
&\quad - \frac{\rho_1 - \rho_2}{2}((\nabla \mu_n \cdot \nabla)\boldsymbol{u}_n - (\nabla \widetilde{\mu} \cdot \nabla)\widetilde{\boldsymbol{u}}, \boldsymbol{w}) = (\mu_n \nabla \psi_n - \widetilde{\mu} \nabla \widetilde{\psi}, \boldsymbol{w})
\end{aligned} \tag{3.34}$$

for all $\boldsymbol{w} \in \mathbf{V}_m$ and for all $t \in [0, T]$. Taking $\boldsymbol{w} = \boldsymbol{u}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\psi_n)|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega \nu(\psi_n)|\mathbb{D}\boldsymbol{u}|^2\,\mathrm{d}x\\
&\quad= \frac{\rho_1-\rho_2}{4}\int_\Omega \partial_t\psi_n|\boldsymbol{u}|^2\,\mathrm{d}x - \frac{\rho_1-\rho_2}{2}\int_\Omega \psi(\partial_t\widetilde{\boldsymbol{u}}\cdot\boldsymbol{u})\,\mathrm{d}x\\
&\qquad- \int_\Omega \big(\rho(\psi_n)(\boldsymbol{v}_n\cdot\nabla)\boldsymbol{u}_n - \rho(\widetilde{\psi})(\widetilde{\boldsymbol{v}}\cdot\nabla)\widetilde{\boldsymbol{u}}\big)\cdot\boldsymbol{u}\,\mathrm{d}x - \frac{\nu_1-\nu_2}{2}\int_\Omega \psi(\mathbb{D}\widetilde{\boldsymbol{u}}:\mathbb{D}\boldsymbol{u})\,\mathrm{d}x\\
&\qquad+ \frac{\rho_1-\rho_2}{2}\int_\Omega\big((\nabla\mu_n\cdot\nabla)\boldsymbol{u}_n - (\nabla\widetilde{\mu}\cdot\nabla)\widetilde{\boldsymbol{u}}\big)\cdot\boldsymbol{u}\,\mathrm{d}x + \int_\Omega\big(\mu_n\nabla\psi_n - \widetilde{\mu}\nabla\widetilde{\psi}\big)\cdot\boldsymbol{u}\,\mathrm{d}x.
\end{aligned}$$

Thanks to (2.6) and (3.26), we have

$$\frac{\rho_1-\rho_2}{4}\int_\Omega \partial_t\psi_n|\boldsymbol{u}|^2\,\mathrm{d}x \le C\|\partial_t\psi_n\|_{L^6}\|\boldsymbol{u}\|_{L^2}\|\boldsymbol{u}\|_{L^3} \le \frac{\nu_*}{10}\|\mathbb{D}\boldsymbol{u}\|_{L^2}^2 + C\|\boldsymbol{u}\|_{L^2}^2$$

and

$$-\frac{\rho_1-\rho_2}{2}\int_\Omega \psi(\partial_t\widetilde{\boldsymbol{u}}\cdot\boldsymbol{u})\,\mathrm{d}x \le C\|\psi\|_{L^\infty}\|\partial_t\widetilde{\boldsymbol{u}}\|_{L^2}\|\boldsymbol{u}\|_{L^2} \le C\|\boldsymbol{u}\|_{L^2}^2 + C\|\partial_t\widetilde{\boldsymbol{u}}\|_{L^2}^2\|\psi\|_{H^2}^2.$$

Noticing that $\boldsymbol{v}_n, \widetilde{\boldsymbol{v}}, \boldsymbol{u}_n \in S$, by exploiting (2.6) and (3.6), we find

$$\begin{aligned}
&-\int_\Omega \big(\rho(\psi_n)(\boldsymbol{v}_n\cdot\nabla)\boldsymbol{u}_n - \rho(\widetilde{\psi})(\widetilde{\boldsymbol{v}}\cdot\nabla)\widetilde{\boldsymbol{u}}\big)\cdot\boldsymbol{u}\,\mathrm{d}x\\
&\quad= -\frac{\rho_1-\rho_2}{2}\int_\Omega \psi((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{u}_n)\cdot\boldsymbol{u}\ dx - \int_\Omega \rho(\widetilde{\psi})((\boldsymbol{v}\cdot\nabla)\boldsymbol{u}_n)\cdot\boldsymbol{u}\,\mathrm{d}x\\
&\qquad- \int_\Omega \rho(\widetilde{\psi})((\widetilde{\boldsymbol{v}}\cdot\nabla)\boldsymbol{u})\cdot\boldsymbol{u}\,\mathrm{d}x\\
&\quad\le C\|\psi\|_{L^\infty}\|\boldsymbol{v}_n\|_{L^\infty}\|\nabla\boldsymbol{u}_n\|_{L^2}\|\boldsymbol{u}\|_{L^2} + C\|\boldsymbol{v}\|_{L^2}\|\nabla\boldsymbol{u}_n\|_{L^\infty}\|\boldsymbol{u}\|_{L^2}\\
&\qquad+ C\|\widetilde{\boldsymbol{v}}\|_{L^\infty}\|\nabla\boldsymbol{u}\|_{L^2}\|\boldsymbol{u}\|_{L^2}\\
&\quad\le C_m\|\psi\|_{H^2}\|\boldsymbol{u}\|_{L^2} + C_m\|\boldsymbol{v}\|_{L^2}\|\boldsymbol{u}\|_{L^2} + C\|\nabla\boldsymbol{u}\|_{L^2}\|\boldsymbol{u}\|_{L^2}\\
&\quad\le \frac{\nu_*}{10}\|\mathbb{D}\boldsymbol{u}\|_{L^2}^2 + C_m\|\boldsymbol{u}\|_{L^2}^2 + C_m\|\psi\|_{H^2}^2 + C_m\|\boldsymbol{v}\|_{L^2}^2.
\end{aligned}$$

In addition, we deduce that

$$-\frac{\nu_1-\nu_2}{2}\int_\Omega \psi(\mathbb{D}\widetilde{\boldsymbol{u}}:\mathbb{D}\boldsymbol{u})\,\mathrm{d}x \le C\|\psi\|_{L^\infty}\|\mathbb{D}\widetilde{\boldsymbol{u}}\|_{L^2}\|\mathbb{D}\boldsymbol{u}\|_{L^2} \le \frac{\nu_*}{10}\|\mathbb{D}\boldsymbol{u}\|_{L^2}^2 + C_m\|\psi\|_{H^2}^2$$

and

$$\begin{aligned}
&\frac{\rho_1-\rho_2}{2}\int_\Omega ((\nabla\mu_n\cdot\nabla)\boldsymbol{u}_n - (\nabla\widetilde{\mu}\cdot\nabla)\widetilde{\boldsymbol{u}})\cdot\boldsymbol{u}\,\mathrm{d}x\\
&\quad= -\frac{\rho_1-\rho_2}{2}\int_\Omega (\mu_n\Delta\boldsymbol{u}_n - \widetilde{\mu}\Delta\widetilde{\boldsymbol{u}})\cdot\boldsymbol{u}\,\mathrm{d}x - \frac{\rho_1-\rho_2}{2}\int_\Omega (\mu_n\nabla\boldsymbol{u}_n - \widetilde{\mu}\nabla\widetilde{\boldsymbol{u}}):\nabla\boldsymbol{u}\,\mathrm{d}x\\
&\quad= -\frac{\rho_1-\rho_2}{2}\int_\Omega (\mu\Delta\boldsymbol{u}_n + \widetilde{\mu}\Delta\boldsymbol{u})\cdot\boldsymbol{u}\,\mathrm{d}x - \frac{\rho_1-\rho_2}{2}\int_\Omega (\mu\nabla\boldsymbol{u}_n + \widetilde{\mu}\nabla\boldsymbol{u}):\nabla\boldsymbol{u}\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\mu\|_{L^2}\|\Delta \boldsymbol{u}_n\|_{L^2}\|\boldsymbol{u}\|_{L^\infty} + C\|\widetilde{\mu}\|_{L^6}\|\Delta \boldsymbol{u}\|_{L^2}\|\boldsymbol{u}\|_{L^3}\\
&\qquad + C\|\mu\|_{L^2}\|\nabla \boldsymbol{u}_n\|_{L^6}\|\nabla \boldsymbol{u}\|_{L^3} + C\|\widetilde{\mu}\|_{L^6}\|\nabla \boldsymbol{u}\|_{L^6}\|\nabla \boldsymbol{u}\|_{L^3}\\
&\le C_m\|\mu\|_{L^2}\|\nabla \boldsymbol{u}\|_{L^2} + C_m\|\nabla \boldsymbol{u}\|_{L^2}\|\boldsymbol{u}\|_{L^2}\\
&\le \frac{\nu_*}{10}\|\mathbb{D}\boldsymbol{u}\|_{L^2}^2 + C_m\|\mu\|_{L^2}^2 + C_m\|\boldsymbol{u}\|_{L^2}^2.
\end{aligned}
$$

Finally, by (3.28)–(3.29), we have

$$
\begin{aligned}
\int_\Omega \left(\mu_n \nabla \psi_n - \widetilde{\mu}\nabla\widetilde{\psi}\right)\cdot \boldsymbol{u}\,\mathrm{d}x &\le \left(\|\mu\|_{L^2}\|\nabla\psi_n\|_{L^6} + \|\widetilde{\mu}\|_{L^2}\|\nabla\psi\|_{L^6}\right)\|\boldsymbol{u}\|_{L^3}\\
&\le C(\|\mu\|_{L^2} + \|\psi\|_{H^2})\|\nabla \boldsymbol{u}\|_{L^2}\\
&\le \frac{\nu_*}{10}\|\mathbb{D}\boldsymbol{u}\|_{L^2}^2 + C\|\mu\|_{L^2}^2 + C\|\psi\|_{H^2}^2.
\end{aligned}
$$

Combining the above inequalities, we are led to the differential inequality

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\psi_n)|\boldsymbol{u}|^2\,\mathrm{d}x \le h_1(t)\int_\Omega \rho(\psi_n)|\boldsymbol{u}|^2\,\mathrm{d}x + h_2(t),
$$

where

$$
h_1(t) = C_m\left(1 + \|\partial_t\psi_n(t)\|_{H^1}^2\right)
$$

and

$$
h_2(t) = C_m\left(\|\partial_t\widetilde{\boldsymbol{u}}(t)\|_{L^2}^2\|\psi(t)\|_{H^2}^2 + \|\psi(t)\|_{H^2}^2 + \|\boldsymbol{v}(t)\|_{L^2}^2 + \|\mu(t)\|_{L^2}^2\right).
$$

Thus, the Gronwall lemma entails

$$
\sup_{t\in[0,T]}\|\boldsymbol{u}(t)\|_{L^2}^2 \le \frac{1}{\rho_*}\mathrm{e}^{\int_0^T h_1(\tau)\mathrm{d}\tau}\int_0^T h_2(\tau)\,\mathrm{d}\tau.
$$

On account of (3.26), (3.32), (3.33), and the convergence $\boldsymbol{v}_n \to \widetilde{\boldsymbol{v}}$ in $L^2(0,T;\mathbf{V}_m)$, we deduce that $\boldsymbol{u}_n \to \widetilde{\boldsymbol{u}}$ in $L^\infty(0,T;\mathbf{V}_m)$, implying that the map Λ is continuous. Finally, we are in the position to apply the Schauder fixed point theorem and conclude that the map Λ has a fixed point in S, which gives the existence of the approximate solution $(\boldsymbol{u}_m,\phi_m)$ on $[0,T]$ satisfying (3.7)–(3.10) for any $m\in\mathbb{N}$.

3.4. Uniform estimates independent of the approximation parameters

Integrating $(3.9)_1$ over Ω, we get

$$
\int_\Omega \phi_m(t)\,\mathrm{d}x = \int_\Omega \phi_{0,k}\,\mathrm{d}x, \quad \forall\, t\in[0,T]. \tag{3.35}
$$

Owing to (3.4), for $k>\overline{k}$, $|\overline{\phi_m}(t)| \le \widetilde{m} < 1$ for all $t\in[0,T]$. Taking $\boldsymbol{w} = \boldsymbol{u}_m$ in (3.8) and integrating by parts, we have (cf. (3.17))

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\rho(\phi_m)|\boldsymbol{u}_m|^2\,\mathrm{d}x + \int_\Omega \nu(\phi_m)|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x = \int_\Omega \mu_m\nabla\phi_m\cdot\boldsymbol{u}_m\,\mathrm{d}x. \tag{3.36}
$$

Multiplying (3.11) by μ_m, integrating over Ω and exploiting the definition of μ_m, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \frac{1}{2}|\nabla\phi_m|^2 + \Psi(\phi_m)\,\mathrm{d}x\Big) + \int_\Omega |\nabla\mu_m|^2 + \alpha|\partial_t\phi_m|^2\,\mathrm{d}x \\ + \int_\Omega \boldsymbol{u}_m\cdot\nabla\phi_m\mu_m\,\mathrm{d}x = 0. \tag{3.37}$$

By summing (3.36) and (3.37), we reach

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\boldsymbol{u}_m,\phi_m) + \int_\Omega \nu(\phi_m)|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x + \int_\Omega |\nabla\mu_m|^2\,\mathrm{d}x + \int_\Omega \alpha|\partial_t\phi_m|^2\,\mathrm{d}x = 0. \tag{3.38}$$

An integration in time on $[0,t]$, with $0<t\le T$, yields

$$E(\boldsymbol{u}_m(t),\phi_m(t)) + \int_0^t \|\sqrt{\nu(\phi_m(s))}\mathbb{D}\boldsymbol{u}_m(s)\|_{L^2}^2 + \|\nabla\mu_m(s)\|_{L^2}^2 + \alpha\|\partial_t\phi_m(s)\|_{L^2}^2\,\mathrm{d}s \\ = E(\mathbb{P}_m\boldsymbol{u}_0,\phi_{0,k}).$$

Thanks to (3.4) and (3.5), we observe that

$$E(\mathbb{P}_m\boldsymbol{u}_0,\phi_{0,k}) \le \frac{\rho^*}{2}\|\boldsymbol{u}_0\|_{L^2}^2 + \frac{1}{2}\|\phi_0\|_{H^1}^2 + \theta_0\big(1+|\Omega|\max_{s\in[-1,1]}|\Psi(s)|\big).$$

Since $\phi_m\in L^\infty(\Omega\times(0,T))$ such that $|\phi_m(x,t)|<1$ almost everywhere in $\Omega\times(0,T)$, we obtain

$$\|\boldsymbol{u}_m\|_{L^\infty(0,T;\mathbf{L}^2_\sigma)} + \|\boldsymbol{u}_m\|_{L^2(0,T;\mathbf{H}^1_\sigma)} \le C, \tag{3.39}$$
$$\|\phi_m\|_{L^\infty(0,T;H^1(\Omega))} \le C, \tag{3.40}$$
$$\|\nabla\mu_m\|_{L^2(0,T;L^2(\Omega))} \le C, \tag{3.41}$$
$$\sqrt{\alpha}\|\partial_t\phi_m\|_{L^2(0,T;L^2(\Omega))} \le C, \tag{3.42}$$

where the constant C depends on $\|\boldsymbol{u}_0\|_{L^2}$ and $\|\phi_0\|_{H^1}$, but is independent of m, α and k. Multiplying (3.11) by $-\Delta\phi_m$, integrating over Ω and using (3.13), we get

$$\|\Delta\phi_m\|_{L^2}^2 + \int_\Omega F''(\phi_m)|\nabla\phi_m|^2\,\mathrm{d}x = \alpha\int_\Omega \partial_t\phi_m\Delta\phi_m\,\mathrm{d}x + \int_\Omega \nabla\mu_m\cdot\nabla\phi_m\,\mathrm{d}x \\ + \theta_0\|\nabla\phi_m\|_{L^2}^2.$$

Since $F''(s)>0$ for $s\in(-1,1)$, by using (3.40), we have

$$\|\Delta\phi_m\|_{L^2}^2 \le C\big(1+\|\nabla\mu_m\|_{L^2}^2 + \alpha^2\|\partial_t\phi_m\|_{L^2}^2\big), \tag{3.43}$$

for some C independent of m. Then, it follows from (3.41) and (3.42) that

$$\|\phi_m\|_{L^2(0,T;H^2(\Omega))} \le C. \tag{3.44}$$

We now recall the well-known inequality (see [32])

$$\int_\Omega |F'(\phi_m)|\,\mathrm{d}x \le C_1 \int_\Omega F'(\phi_m)\big(\phi_m - \overline{\phi_{0,k}}\big)\,\mathrm{d}x + C_2, \tag{3.45}$$

where the positive constants C_1, C_2 depends only on $\overline{\phi_{0,k}}$, thereby they are independent of k (for large k). Then, multiplying $(3.9)_2$ by $\phi_m - \overline{\phi_{0,k}}$ (cf. (3.35)), we find

$$\begin{aligned}\int_\Omega |\nabla\phi_m|^2\,\mathrm{d}x &+ \int_\Omega F'(\phi_m)\big(\phi_m - \overline{\phi_{0,k}}\big)\,\mathrm{d}x\\ &= -\alpha\int_\Omega \partial_t\phi_m\big(\phi_m - \overline{\phi_{0,k}}\big)\,\mathrm{d}x + \int_\Omega \big(\mu_m - \overline{\mu_m}\big)\phi_m\,\mathrm{d}x\\ &\quad + \theta_0\int_\Omega \phi_m\big(\phi_m - \overline{\phi_{0,k}}\big)\,\mathrm{d}x.\end{aligned}$$

By the Poincaré inequality and (3.40), we obtain

$$\left|\int_\Omega F'(\phi_m)(\phi_m - \overline{\phi_{0,k}})\,\mathrm{d}x\right| \le C\big(1 + \|\nabla\mu_m\|_{L^2} + \alpha\|\partial_t\phi_m\|_{L^2}\big). \tag{3.46}$$

Since $\overline{\mu_m} = \overline{F'(\phi_m)} - \theta_0\overline{\phi_{0,k}}$, we infer from (3.45) and (3.46) that

$$|\overline{\mu_m}| \le C\big(1 + \|\nabla\mu_m\|_{L^2} + \alpha\|\partial_t\phi_m\|_{L^2}\big).$$

Thanks to (2.1), we have

$$\|\mu_m\|_{H^1} \le C\big(1 + \|\nabla\mu_m\|_{L^2} + \alpha\|\partial_t\phi_m\|_{L^2}\big). \tag{3.47}$$

As a direct consequence, we deduce that

$$\|\mu_m\|_{L^2(0,T;H^1(\Omega))} \le C, \tag{3.48}$$

for some constant C independent of m, α and k. In addition, using the boundary conditions (3.10) and (3.39), we find

$$\|\partial_t\phi_m\|_{(H^1)'} \le C\big(1 + \|\nabla\mu_m\|_{L^2}\big), \tag{3.49}$$

which, in turn, implies that

$$\|\partial_t\phi_m\|_{L^2(0,T;(H^1(\Omega))')} \le C.$$

Next, taking $\boldsymbol{w} = \partial_t\boldsymbol{u}_m$ in (3.8), we find

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nu(\phi_m)|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x &+ \int_\Omega \rho(\phi_m)|\partial_t\boldsymbol{u}_m|^2\,\mathrm{d}x\\ = -\int_\Omega \rho(\phi_m)((\boldsymbol{u}_m\cdot\nabla)\boldsymbol{u}_m)\cdot\partial_t\boldsymbol{u}_m\,\mathrm{d}x &+ \frac{\nu_1-\nu_2}{2}\int_\Omega \partial_t\phi_m|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x\\ + \frac{\rho_1-\rho_2}{2}\int_\Omega ((\nabla\mu_m\cdot\nabla)\boldsymbol{u}_m)\cdot\partial_t\boldsymbol{u}_m\,\mathrm{d}x &+ \int_\Omega \mu_m\nabla\phi_m\cdot\partial_t\boldsymbol{u}_m\,\mathrm{d}x.\end{aligned} \tag{3.50}$$

Thanks to the regularity of μ_m (cf. (3.13)), we multiply $(3.9)_1$ by $\partial_t \mu_m$ and integrate over Ω to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla \mu_m|^2 \,\mathrm{d}x + (\partial_t \mu_m, \partial_t \phi_m) + (\partial_t \mu_m, \boldsymbol{u}_m \cdot \nabla \phi_m) = 0.$$

Direct computations give that

$$\begin{aligned}(\partial_t \mu_m, \partial_t \phi_m) = \alpha(\partial_{tt}\phi_m, \partial_t \phi_m) + \|\nabla \partial_t \phi_m\|_{L^2}^2 + \int_\Omega F''(\phi_m)|\partial_t \phi_m|^2 \,\mathrm{d}x \\ - \theta_0 \|\partial_t \phi_m\|_{L^2}^2\end{aligned}$$

and

$$\begin{aligned}(\partial_t \mu_m, \boldsymbol{u}_m \cdot \nabla \phi_m) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \mu_m \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x\Big) - \int_\Omega \mu_m \partial_t \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x \\ - \int_\Omega \mu_m \boldsymbol{u}_m \cdot \nabla \partial_t \phi_m \,\mathrm{d}x.\end{aligned}$$

As a result, we find

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \frac{1}{2}|\nabla \mu_m|^2 \,\mathrm{d}x + \int_\Omega \frac{\alpha}{2}|\partial_t \phi_m|^2 \,\mathrm{d}x + \int_\Omega \mu_m \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x\Big) + \|\nabla \partial_t \phi_m\|_{L^2}^2 \\ &\quad \le \theta_0 \|\partial_t \phi_m\|_{L^2}^2 + \int_\Omega \mu_m \partial_t \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x + \int_\Omega \mu_m \boldsymbol{u}_m \cdot \nabla \partial_t \phi_m \,\mathrm{d}x. \end{aligned} \tag{3.51}$$

By summing (3.50) and (3.51), we arrive at

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} H_m + \rho_* \|\partial_t \boldsymbol{u}_m\|_{L^2}^2 + \|\nabla \partial_t \phi_m\|_{L^2}^2 \\ &\quad \le - \int_\Omega \rho(\phi_m)((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m) \cdot \partial_t \boldsymbol{u}_m \,\mathrm{d}x + \frac{\nu_1 - \nu_2}{2}\int_\Omega \partial_t \phi_m |\mathbb{D}\boldsymbol{u}_m|^2 \,\mathrm{d}x \\ &\qquad + \frac{\rho_1 - \rho_2}{2}\int_\Omega ((\nabla \mu_m \cdot \nabla)\boldsymbol{u}_m) \cdot \partial_t \boldsymbol{u}_m \,\mathrm{d}x + 2\int_\Omega \mu_m \nabla \phi_m \cdot \partial_t \boldsymbol{u}_m \,\mathrm{d}x \\ &\qquad + \theta_0 \|\partial_t \phi_m\|_{L^2}^2 + \int_\Omega \mu_m \boldsymbol{u}_m \cdot \nabla \partial_t \phi_m \,\mathrm{d}x \\ &\quad = \sum_{k=1}^{6} R_i, \end{aligned} \tag{3.52}$$

where

$$\begin{aligned}H_m(t) = \frac{1}{2}\int_\Omega \nu(\phi_m)|\mathbb{D}\boldsymbol{u}_m|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla \mu_m|^2 \, dx + \frac{\alpha}{2}\int_\Omega |\partial_t \phi_m|^2 \,\mathrm{d}x \\ + \int_\Omega \mu_m \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x.\end{aligned}$$

By exploiting (2.2), (2.6), (3.39), (3.40), and (3.47), we observe that

$$
\begin{aligned}
\left|\int_\Omega \mu_m \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x\right| &\leq \|\mu_m\|_{L^6}\|\boldsymbol{u}_m\|_{L^3}\|\nabla \phi_m\|_{L^2} \\
&\leq C\left(1+\|\nabla \mu_m\|_{L^2}+\alpha\|\partial_t \phi_m\|_{L^2}\right)\|\nabla \boldsymbol{u}_m\|_{L^2}^{\frac12} \\
&\leq \frac14 \int_\Omega \nu(\phi_m)|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x + \frac14\|\nabla\mu_m\|_{L^2}^2+\frac{\alpha}{4}\|\partial_t\phi_m\|_{L^2}^2 + C_0,
\end{aligned}
$$

for some C_0 independent of m, α and k. Thus, it follows that

$$
H_m \geq \frac14 \int_\Omega \nu(\phi_m)|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x + \frac14\|\nabla\mu_m\|_{L^2}^2+\frac{\alpha}{4}\|\partial_t\phi_m\|_{L^2}^2 - C_0. \tag{3.53}
$$

Similarly, it is easily seen that

$$
H_m \leq \int_\Omega \nu(\phi_m)|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x + \|\nabla\mu_m\|_{L^2}^2+\alpha\|\partial_t\phi_m\|_{L^2}^2 + \widetilde{C}_0, \tag{3.54}
$$

for some $\widetilde{C}_0$ independent of m, α and k. Before proceeding with the estimate of the terms R_i, $i=1,\dots,7$, we need to control the norms $\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}$ and $\|\mu_m\|_{H^3}$. To this end, taking $\boldsymbol{w}=\mathbf{A}\boldsymbol{u}_m$ in (3.15), we have

$$
\begin{aligned}
-\frac12(\nu(\phi_m)\Delta\boldsymbol{u}_m,\mathbf{A}\boldsymbol{u}_m) &= -(\rho(\phi_m)\partial_t\boldsymbol{u}_m,\mathbf{A}\boldsymbol{u}_m) - (\rho(\phi_m)(\boldsymbol{u}_m\cdot\nabla)\boldsymbol{u}_m,\mathbf{A}\boldsymbol{u}_m) \\
&\quad + \frac{\rho_1-\rho_2}{2}((\nabla\mu_m\cdot\nabla)\boldsymbol{u}_m,\mathbf{A}\boldsymbol{u}_m) + (\mu_m\nabla\phi_m,\mathbf{A}\boldsymbol{u}_m) \\
&\quad + \frac{\nu_1-\nu_2}{2}(\mathbb{D}\boldsymbol{u}_m\nabla\phi_m,\mathbf{A}\boldsymbol{u}_m).
\end{aligned} \tag{3.55}
$$

By arguing as in [25] (see also [24]), there exists $\pi_m\in C([0,T];H^1(\Omega))$ such that $-\Delta\boldsymbol{u}_m+\nabla\pi_m=\mathbf{A}\boldsymbol{u}_m$ almost everywhere in $\Omega\times(0,T)$. Moreover, π_m satisfies

$$
\|\pi_m\|_{L^2}\leq C\|\nabla\boldsymbol{u}_m\|_{L^2}^{\frac12}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^{\frac12}, \quad \|\pi_m\|_{H^1}\leq C\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}, \tag{3.56}
$$

where C is independent of m, α and k. Therefore, we obtain

$$
\begin{aligned}
\frac12(\nu(\phi_m)\mathbf{A}\boldsymbol{u}_m,\mathbf{A}\boldsymbol{u}_m) &= -(\rho(\phi_m)\partial_t\boldsymbol{u}_m,\mathbf{A}\boldsymbol{u}_m) - (\rho(\phi_m)(\boldsymbol{u}_m\cdot\nabla)\boldsymbol{u}_m,\mathbf{A}\boldsymbol{u}_m) \\
&\quad + \frac{\rho_1-\rho_2}{2}((\nabla\mu_m\cdot\nabla)\boldsymbol{u}_m,\mathbf{A}\boldsymbol{u}_m) + (\mu_m\nabla\phi_m,\mathbf{A}\boldsymbol{u}_m) \\
&\quad + \frac{\nu_1-\nu_2}{2}(\mathbb{D}\boldsymbol{u}_m\nabla\phi_m,\mathbf{A}\boldsymbol{u}_m) - \frac{\nu_1-\nu_2}{4}(\pi_m\nabla\phi_m,\mathbf{A}\boldsymbol{u}_m) \\
&= \sum_{i=7}^{12} R_i.
\end{aligned} \tag{3.57}
$$

On the other hand, taking the gradient of $(3.9)_1$, multiplying it by $\nabla\Delta\mu_m$ and integrating over Ω, we find

$$
\|\nabla\Delta\mu_m\|_{L^2}^2 = (\nabla\partial_t\phi_m,\nabla\Delta\mu_m) + (\nabla(\boldsymbol{u}_m\cdot\nabla\phi_m),\nabla\Delta\mu_m). \tag{3.58}
$$

Then, in light of $(3.9)_1$ and $(3.10)_1$, it follows that

$$\|\mu_m\|_{H^3}^2 \le C\big(\|\mu_m\|_{H^1}^2 + \|\nabla\Delta\mu_m\|_{L^2}^2\big),$$

which, in turn, by (3.53) gives that

$$\begin{aligned}\|\mu_m\|_{H^3}^2 &\le C\big(1 + \|\nabla\mu_m\|_{L^2}^2 + \alpha^2\|\partial_t\phi_m\|_{L^2}^2 + (\nabla\partial_t\phi_m, \nabla\Delta\mu_m) \\ &\qquad + (\nabla(\boldsymbol{u}_m\cdot\nabla\phi_m), \nabla\Delta\mu_m)\big) \\ &= C(1 + C_0 + H_m) + \sum_{i=13}^{14} R_i,\end{aligned} \tag{3.59}$$

where C is independent of m, α and k. Now, multiplying (3.57) and (3.59) by two positive constants ϖ_1 and ϖ_2 (which will be chosen later on), respectively, and summing them with (3.52), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}H_m + \rho_*\|\partial_t\boldsymbol{u}_m\|_{L^2}^2 + \|\nabla\partial_t\phi_m\|_{L^2}^2 + \frac{\nu_*\varpi_1}{2}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 + \varpi_2\|\mu_m\|_{H^3}^2 \\ &\quad\le C(1+\varpi_2)(1 + C_0 + H_m) + \sum_{i=1}^{6} R_i + \varpi_1\sum_{i=7}^{12} R_i + \varpi_2\sum_{i=13}^{14} R_i.\end{aligned} \tag{3.60}$$

Let us proceed with the estimate of the terms R_i, $i = 1, \dots, 14$. In what follows, the generic constant C may depend on ϖ_1 and ϖ_2. Exploiting (2.2), (2.6), (3.39) and (3.53), we have

$$\begin{aligned}\left|-\int_\Omega \rho(\phi_m)((\boldsymbol{u}_m\cdot\nabla)\boldsymbol{u}_m)\cdot\partial_t\boldsymbol{u}_m\,\mathrm{d}x\right| &\le \rho^*\|\boldsymbol{u}_m\|_{L^6}\|\nabla\boldsymbol{u}_m\|_{L^3}\|\partial_t\boldsymbol{u}_m\|_{L^2} \\ &\le \frac{\rho_*}{8}\|\partial_t\boldsymbol{u}_m\|_{L^2}^2 + C\|\nabla\boldsymbol{u}_m\|_{L^2}^3\|\mathbf{A}\boldsymbol{u}_m\|_{L^2} \\ &\le \frac{\rho_*}{8}\|\partial_t\boldsymbol{u}_m\|_{L^2}^2 + \frac{\nu_*\varpi_1}{32}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 + C\|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^6 \\ &\le \frac{\rho_*}{8}\|\partial_t\boldsymbol{u}_m\|_{L^2}^2 + \frac{\nu_*\varpi_1}{32}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 \\ &\qquad + C(C_0 + H_m)^3.\end{aligned}$$

By the Sobolev embedding, (2.2) and (3.53), we obtain

$$\begin{aligned}\left|\frac{\nu_1-\nu_2}{2}\int_\Omega \partial_t\phi_m|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x\right| &\le C\|\partial_t\phi_m\|_{L^6}\|\mathbb{D}\boldsymbol{u}_m\|_{L^3}\|\mathbb{D}\boldsymbol{u}_m\|_{L^2} \\ &\le \frac{1}{8}\|\nabla\partial_t\phi_m\|_{L^2}^2 + C\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}\|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^3 \\ &\le \frac{1}{8}\|\nabla\partial_t\phi_m\|_{L^2}^2 + \frac{\nu_*\varpi_1}{32}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 + C\|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^3 \\ &\le \frac{1}{8}\|\nabla\partial_t\phi_m\|_{L^2}^2 + \frac{\nu_*\varpi_1}{32}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 + C(C_0 + H_m)^3.\end{aligned}$$

By the Sobolev interpolation, (2.3) and (3.47), we get

$$
\begin{aligned}
\left|\frac{\rho_1-\rho_2}{2}\int_\Omega ((\nabla\mu_m\cdot\nabla)\boldsymbol{u}_m)\cdot\partial_t\boldsymbol{u}_m\,\mathrm{d}x\right| &\le C\|\nabla\mu_m\|_{L^\infty}\|\nabla\boldsymbol{u}_m\|_{L^2}\|\partial_t\boldsymbol{u}_m\|_{L^2}\\
&\le C\|\nabla\mu_m\|_{H^1}^{\frac12}\|\mu_m\|_{H^3}^{\frac12}\|\nabla\boldsymbol{u}_m\|_{L^2}\|\partial_t\boldsymbol{u}_m\|_{L^2}\\
&\le \frac{\rho_*}{8}\|\partial_t\boldsymbol{u}_m\|_{L^2}^2 + C\|\nabla\mu_m\|_{L^2}^{\frac12}\|\mu_m\|_{H^3}^{\frac32}\|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^2\\
&\le \frac{\rho_*}{8}\|\partial_t\boldsymbol{u}_m\|_{L^2}^2 + \frac{\varpi_2}{6}\|\mu_m\|_{H^3}^2\\
&\quad + C\|\nabla\mu_m\|_{L^2}^2\|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^8\\
&\le \frac{\rho_*}{8}\|\partial_t\boldsymbol{u}_m\|_{L^2}^2 + \frac{\varpi_2}{6}\|\mu_m\|_{H^3}^2 + C(C_0+H_m)^5.
\end{aligned}
$$

Exploiting (3.43), (3.47), (3.49) and (3.53), we find

$$
\begin{aligned}
\left|2\int_\Omega \mu_m\nabla\phi_m\cdot\partial_t\boldsymbol{u}_m\,\mathrm{d}x\right| &\le 2\|\mu_m\|_{L^6}\|\nabla\phi_m\|_{L^3}\|\partial_t\boldsymbol{u}_m\|_{L^2}\\
&\le \frac{\rho_*}{8}\|\partial_t\boldsymbol{u}_m\|_{L^2}^2 + C\|\phi_m\|_{H^2}^2\|\mu_m\|_{H^1}^2\\
&\le \frac{\rho_*}{8}\|\partial_t\boldsymbol{u}_m\|_{L^2}^2 + C\left(1+\|\nabla\mu_m\|_{L^2}^2+\alpha^2\|\partial_t\phi_m\|_{L^2}^2\right)^2\\
&\le \frac{\rho_*}{8}\|\partial_t\boldsymbol{u}_m\|_{L^2}^2 + C(1+C_0+H_m)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\theta_0\|\partial_t\phi_m\|_{L^2}^2 &\le C\|\partial_t\phi_m\|_{(H^1)'}\|\nabla\partial_t\phi_m\|_{L^2}\\
&\le \frac18\|\nabla\partial_t\phi_m\|_{L^2}^2 + C(1+C_0+H_m),
\end{aligned}
$$

as well as

$$
\begin{aligned}
\left|\int_\Omega \mu_m\boldsymbol{u}_m\cdot\nabla\partial_t\phi_m\,\mathrm{d}x\right| &\le \|\mu_m\|_{L^6}\|\boldsymbol{u}_m\|_{L^3}\|\nabla\partial_t\phi_m\|_{L^2}\\
&\le \frac18\|\nabla\partial_t\phi_m\|_{L^2}^2 + C\|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^2\left(1+\|\nabla\mu_m\|_{L^2}^2+\alpha^2\|\partial_t\phi_m\|_{L^2}^2\right)\\
&\le \frac18\|\nabla\partial_t\phi_m\|_{L^2}^2 + C(1+C_0+H_m)^2.
\end{aligned}
$$

By Young's inequality, we have

$$
\begin{aligned}
\left|-\int_\Omega \rho(\phi_m)\partial_t\boldsymbol{u}_m\cdot\mathbf{A}\boldsymbol{u}_m\,\mathrm{d}x\right| &\le \varpi_1\rho^*\|\partial_t\boldsymbol{u}_m\|_{L^2}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}\\
&\le \frac{\rho_*}{8\varpi_1}\|\partial_t\boldsymbol{u}_m\|_{L^2}^2 + \frac{2(\rho^*)^2\varpi_1}{\rho_*}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2.
\end{aligned}
$$

By using (2.2), (2.3), (2.6) and (3.53), we find

$$
\begin{aligned}
\left| - \int_\Omega \rho(\phi_m)(\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m \cdot \mathbf{A}\boldsymbol{u}_m \,\mathrm{d}x \right| &\le \rho^* \|\boldsymbol{u}_m\|_{L^6} \|\nabla \boldsymbol{u}_m\|_{L^3} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2} \\
&\le C \|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^{\frac{3}{2}} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^{\frac{3}{2}} \\
&\le \frac{\nu_*}{32} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 + C \|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^6 \\
&\le \frac{\nu_*}{32} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 + C(C_0 + H_m)^3,
\end{aligned}
$$

and

$$
\begin{aligned}
\left| \frac{\rho_1 - \rho_2}{2} \int_\Omega (\nabla \mu_m \cdot \nabla)\boldsymbol{u}_m \cdot \mathbf{A}\boldsymbol{u}_m \,\mathrm{d}x \right| &\le C \|\nabla \mu_m\|_{L^\infty} \|\nabla \boldsymbol{u}_m\|_{L^2} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2} \\
&\le C \|\nabla \mu_m\|_{H^1}^{\frac{1}{2}} \|\mu_m\|_{H^3}^{\frac{1}{2}} \|\nabla \boldsymbol{u}_m\|_{L^2} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2} \\
&\le \frac{\nu_*}{32} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 + C \|\nabla \mu_m\|_{L^2}^{\frac{1}{2}} \|\mu_m\|_{H^3}^{\frac{3}{2}} \|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^2 \\
&\le \frac{\nu_*}{32} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 + \frac{\varpi_2}{6\varpi_1} \|\mu_m\|_{H^3}^2 \\
&\quad + C \|\nabla \mu_m\|_{L^2}^2 \|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^8 \\
&\le \frac{\nu_*}{32} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 + \frac{\varpi_2}{6\varpi_1} \|\mu_m\|_{H^3}^2 + C(C_0 + H_m)^5.
\end{aligned}
$$

In light of (3.43) and (3.47), we have

$$
\begin{aligned}
\left| \int_\Omega \mu_m \nabla \phi_m \cdot \mathbf{A}\boldsymbol{u}_m \,\mathrm{d}x \right| &\le \|\mu_m\|_{L^6} \|\nabla \phi_m\|_{L^3} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2} \\
&\le \frac{\nu_*}{32} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 + C \|\mu_m\|_{H^1}^2 \|\phi_m\|_{H^2}^2 \\
&\le \frac{\nu_*}{32} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 + C \left(1 + \|\nabla \mu_m\|_{L^2}^2 + \alpha^2 \|\partial_t \phi_m\|_{L^2}^2\right)^2 \\
&\le \frac{\nu_*}{32} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 + C(1 + C_0 + H_m)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\left| \frac{\nu_1 - \nu_2}{2} \int_\Omega \mathbb{D}\boldsymbol{u}_m \nabla \phi_m \cdot \mathbf{A}\boldsymbol{u}_m \,\mathrm{d}x \right| &\le C \|\mathbb{D}\boldsymbol{u}_m\|_{L^3} \|\nabla \phi_m\|_{L^6} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2} \\
&\le C \|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^{\frac{1}{2}} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^{\frac{3}{2}} \|\phi_m\|_{H^2} \\
&\le \frac{\nu_*}{32} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 + C \|D\boldsymbol{u}_m\|_{L^2}^2 \|\phi_m\|_{H^2}^4 \\
&\le \frac{\nu_*}{32} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2 + C \left(1 + C_0 + H_m\right)^3.
\end{aligned}
$$

Owing to (3.43) and (3.56), we obtain

$$
\begin{aligned}
\left|\frac{\nu_1-\nu_2}{4}\int_\Omega \pi_m \nabla\phi_m\cdot \mathbf{A}\boldsymbol{u}_m \,\mathrm{d}x\right| &\le C\|\pi_m\|_{L^3}\|\nabla\phi_m\|_{L^6}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}\\
&\le C\|\pi_m\|_{L^2}^{\frac12}\|\pi_m\|_{H^1}^{\frac12}\|\phi_m\|_{H^2}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}\\
&\le C\|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^{\frac14}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^{\frac74}\big(1+\|\nabla\mu_m\|_{L^2}^2\\
&\qquad +\alpha^2\|\partial_t\phi_m\|_{L^2}^2\big)^{\frac12}\\
&\le \frac{\nu_*}{32}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2+C\|D\boldsymbol{u}_m\|_{L^2}^2\big(1+\|\nabla\mu_m\|_{L^2}^2\\
&\qquad +\alpha^2\|\partial_t\phi_m\|_{L^2}^2\big)^4\\
&\le \frac{\nu_*}{32}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2+C(1+C_0+H_m)^5.
\end{aligned}
$$

By using the Young inequality, it easily follows that

$$
\left|\int_\Omega \nabla\partial_t\phi_m\cdot\nabla\Delta\mu_m\,\mathrm{d}x\right|\le \frac{1}{8\varpi_2}\|\nabla\partial_t\phi_m\|_{L^2}^2+2\varpi_2\|\mu_m\|_{H^3}^2.
$$

Finally, by exploiting (2.2), (2.3), (2.6), (3.43) and (3.53), we infer that

$$
\begin{aligned}
\left|\int_\Omega \nabla(\boldsymbol{u}_m\cdot\nabla\phi_m)\cdot\nabla\Delta\mu_m\,\mathrm{d}x\right| &\le C(\|\mathbb{D}\boldsymbol{u}_m\|_{L^3}\|\nabla\phi_m\|_{L^6}\\
&\qquad +\|\nabla^2\phi_m\|_{L^2}\|\boldsymbol{u}_m\|_{L^\infty})\|\nabla\Delta\mu_m\|_{L^2}\\
&\le C\|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^{\frac12}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^{\frac12}\|\phi_m\|_{H^2}\|\mu_m\|_{H^3}\\
&\le \frac{\nu_*\varpi_1}{32\varpi_2}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2+\frac16\|\mu_m\|_{H^3}^2+C\|\mathbb{D}\boldsymbol{u}_m\|_{L^2}^2\|\phi_m\|_{H^2}^4\\
&\le \frac{\nu_*\varpi_1}{32\varpi_2}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2+\frac16\|\mu_m\|_{H^3}^2+C(1+C_0+H_m)^3.
\end{aligned}
$$

Combining (3.60) with the above estimates, we arrive at

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}H_m+\frac{\rho_*}{2}\|\partial_t\boldsymbol{u}_m\|_{L^2}^2+\frac12\|\nabla\partial_t\phi_m\|_{L^2}^2+\Big(\frac{\nu_*\varpi_1}{4}-\frac{2(\rho^*)^2\varpi_1^2}{\rho_*}\Big)\|\mathbf{A}\boldsymbol{u}_m\|_{L^2}^2\\
&\quad+\Big(\frac{\varpi_2}{2}-2\varpi_2^2\Big)\|\mu_m\|_{H^3}^2\le C(1+C_0+H_m)^5, \qquad (3.61)
\end{aligned}
$$

where the positive constant C depends on ϖ_1 and ϖ_2, but is independent of m, α and k. Therefore, by setting

$$
\varpi_1=\frac{\rho_*\nu_*}{16(\rho^*)^2},\qquad \varpi_2=\frac18,
$$

we deduce the differential inequality

$$
\frac{\mathrm{d}}{\mathrm{d}t}H_m+F_m\le C(1+C_0+H_m)^5, \qquad (3.62)
$$

where

$$F_m(t) = \frac{\rho_*}{2}\|\partial_t \boldsymbol{u}_m(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla \partial_t \phi_m(t)\|_{L^2}^2 + \frac{\varpi_1 \nu_*}{8}\|\mathbf{A}\boldsymbol{u}_m(t)\|_{L^2}^2 + \frac{1}{32}\|\mu_m(t)\|_{H^3}^2,$$

and the constant C is independent of the approximation parameters α, m and k. Hence, whenever $\widetilde{T} > 0$ satisfies

$$1 - 4C\widetilde{T}(1 + C_0 + H_m(0))^4 > 0,$$

we infer that

$$C_0 + H_m(t) \leq \frac{1 + C_0 + H_m(0)}{\left(1 - 4Ct(C_1 + H_m(0))^4\right)^{\frac{1}{4}}}, \quad \forall\, t \in [0, \widetilde{T}]. \tag{3.63}$$

To deduce an estimate of H_m which is independent of m, α and k, we are left to control $\alpha\|\partial_t \phi_m(0)\|_{L^2}^2$ (cf. the definition of H_m and (3.54)). For this purpose, we first observe that $\partial_t \phi_m \in C([0, T]; H^1(\Omega))$, $\mu_m \in C([0, T]; H^1(\Omega))$ due to the regularity in Theorem A.1. By comparison with terms in $(3.9)_2$, it follows that $-\Delta\phi_m + \Psi'(\phi_m) \in C([0, T]; H^1(\Omega))$. Now, multiplying $(3.9)_2$ by $\partial_t \phi_m$ and integrating over Ω, we have

$$\alpha\|\partial_t \phi_m\|_{L^2}^2 + (-\Delta\phi_m + \Psi'(\phi_m), \partial_t \phi_m) = (\mu_m, \partial_t \phi_m).$$

By using $(3.9)_1$, we find

$$\alpha\|\partial_t \phi_m\|_{L^2}^2 + (-\Delta\phi_m + \Psi'(\phi_m), \Delta\mu_m - \boldsymbol{u}_m \cdot \nabla\phi_m) = (\mu_m, \Delta\mu_m - \boldsymbol{u}_m \cdot \nabla\phi_m).$$

Integrating by parts, we arrive at

$$\alpha\|\partial_t \phi_m\|_{L^2}^2 + \|\nabla\mu_m\|_{L^2}^2 = (\nabla(-\Delta\phi_m + \Psi'(\phi_m)), \nabla\mu_m - \phi_m \boldsymbol{u}_m) + (\nabla\mu_m, \phi_m\, \boldsymbol{u}_m).$$

By continuity, we obtain

$$\begin{aligned}&\alpha\|\partial_t \phi_m(0)\|_{L^2}^2 + \|\nabla\mu_m(0)\|_{L^2}^2 \\ &\quad = (\nabla(-\Delta\phi_{0,k} + \Psi'(\phi_{0,k})), \nabla\mu_m(0) - \phi_{0,k}\, \boldsymbol{u}_m(0)) + (\nabla\mu_m(0), \phi_{0,k}\, \boldsymbol{u}_m(0)),\end{aligned}$$

which, in turn, implies that

$$\alpha\|\partial_t \phi_m(0)\|_{L^2}^2 + \|\nabla\mu_m(0)\|_{L^2}^2 \leq C\|\nabla(-\Delta\phi_{0,k} + \Psi'(\phi_{0,k}))\|_{L^2}^2 + C\|\boldsymbol{u}_m(0)\|_{L^2}^2. \tag{3.64}$$

Thus, we conclude from (3.1), (3.3), (3.4) and (3.54) that

$$H_m(0) \leq C\left(1 + \|\boldsymbol{u}_0\|_{\mathbf{H}^1_\sigma}^2 + \|-\Delta\phi_0 + F'(\phi_0)\|_{H^1}^2 + \|\phi_0\|_{H^1}^2\right) + \widetilde{C}_0 := \widetilde{K}_0,$$

where the constant C is independent of m, α and k. Therefore, setting $\widetilde{T}_0 = \frac{1}{4C(C_1 + \widetilde{K}_0))^4}$ yields that

$$0 \leq C_0 + H_m(t) \leq \frac{1 + C_0 + \widetilde{K}_0}{\left(1 - 4Ct(C_1 + \widetilde{K}_0)^4\right)^{\frac{1}{4}}}, \quad \forall\, t \in [0, \widetilde{T}_0).$$

Notice that $\widetilde{T}_0$ is independent of m, α and k. Let us now fix $T_0 \in (0, \widetilde{T}_0)$. Thanks to (3.53), we infer that

$$\sup_{t\in[0,T_0]} \|\nabla \boldsymbol{u}_m(t)\|_{L^2} + \sup_{t\in[0,T_0]} \|\nabla \mu_m(t)\|_{L^2} + \sup_{t\in[0,T_0]} \sqrt{\alpha}\|\partial_t \phi_m(t)\|_{L^2} \leq K_1, \quad (3.65)$$

where K_1 is a positive constant that depends on $E(\boldsymbol{u}_0, \phi_0)$, $\|\boldsymbol{u}_0\|_{\mathbf{H}^1_\sigma}$, $\|\mu_0\|_{H^1}$, and the parameters of the system, but is independent of m, α and k. Recalling (3.43) and (3.47), we immediately obtain

$$\sup_{t\in[0,T_0]} \|\phi_m(t)\|_{H^2} + \sup_{t\in[0,T_0]} \|\mu_m(t)\|_{H^1} + \sup_{t\in[0,T_0]} \|F'(\phi_m(t))\|_{L^2} \leq K_2. \quad (3.66)$$

Integrating (3.60) on $[0, T_0]$, we deduce that

$$\int_0^{T_0} \|\partial_t \boldsymbol{u}_m(\tau)\|_{L^2}^2 + \|\nabla \partial_t \phi_m(\tau)\|_{L^2}^2 + \|\mathbf{A}\boldsymbol{u}_m(\tau)\|_{L^2}^2 + \|\mu_m(\tau)\|_{H^3}^2 \, \mathrm{d}\tau \leq K_3. \quad (3.67)$$

Finally, in light of (3.65) and (3.67), we observe that separation property $(3.13)_2$ (cf. Theorem A.1) depends on α and k, but is independent of m, that is,

$$\phi_m \in L^\infty(\Omega \times (0,T)) \text{ is such that } |\phi_m(x,t)| \leq 1 - \widetilde{\delta} \text{ a.e. in } \Omega \times (0, T_0) \quad (3.68)$$

for some $\widetilde{\delta} = \widetilde{\delta}(\alpha, k)$.

3.5. Passage to the limit and existence of strong solutions

Thanks to estimates (3.65)–(3.67) given above, we deduce the following convergences (up to a subsequence) as $m \to \infty$:

$$\begin{aligned}
\boldsymbol{u}_m &\rightharpoonup \boldsymbol{u}_\alpha && \text{weak-star in } L^\infty(0, T_0; \mathbf{H}^1_\sigma),\\
\boldsymbol{u}_m &\rightharpoonup \boldsymbol{u}_\alpha && \text{weakly in } L^2(0, T_0; H^2) \cap W^{1,2}(0, T_0; \mathbf{L}^2_\sigma),\\
\phi_m &\rightharpoonup \phi_\alpha && \text{weak-star in } L^\infty(0, T_0; H^2(\Omega)),\\
\phi_m &\rightharpoonup \phi_\alpha && \text{weakly in } W^{1,2}(0, T_0; H^1(\Omega)),\\
\mu_m &\rightharpoonup \mu_\alpha && \text{weak-star in } L^\infty(0, T_0; H^1(\Omega)),\\
\mu_m &\rightharpoonup \mu_\alpha && \text{weakly in } L^2(0, T_0; H^3(\Omega)).
\end{aligned} \quad (3.69)$$

The strong convergences of $\boldsymbol{u}_m$ and ϕ_m are recovered through the Aubin–Lions lemma, which implies that

$$\begin{aligned}
\boldsymbol{u}_m &\to \boldsymbol{u}_\alpha && \text{strongly in } L^2(0, T_0; \mathbf{H}^1_\sigma),\\
\phi_m &\to \phi_\alpha && \text{strongly in } C([0, T_0]; W^{1,p}(\Omega)), \ \forall\, p \in [2, 6).
\end{aligned} \quad (3.70)$$

As a consequence, we infer that

$$\rho(\phi_m) \to \rho(\phi_\alpha), \quad \nu(\phi_m) \to \nu(\phi_\alpha) \qquad \text{strongly in } C([0, T_0]; W^{1,p}(\Omega)), \quad (3.71)$$

for all $p \in [2,6)$. Additionally, we have

$$\phi_\alpha \in L^\infty(\Omega \times (0,T)) \text{ is such that } |\phi_\alpha(x,t)| \le 1-\delta \text{ a.e. in } \Omega \times (0,T_0) \tag{3.72}$$

for some $\delta = \delta(\alpha,k)$. The above properties entail the convergence of the non-linear terms in (3.8) and of the logarithmic potential $\Psi'(\phi)$ in (3.9), and thereby we pass to the limit in the Galerkin formulation as $m \to \infty$ in (3.8)–(3.9). The limit solution $(\boldsymbol{u}_\alpha, \phi_\alpha)$ satisfies

$$\begin{aligned}(\rho(\phi_\alpha)\partial_t \boldsymbol{u}_\alpha, \boldsymbol{w}) + (\rho(\phi_\alpha)(\boldsymbol{u}_\alpha \cdot \nabla)\boldsymbol{u}_\alpha, \boldsymbol{w}) - (\operatorname{div}(\nu(\phi_\alpha)\mathbb{D}\boldsymbol{u}_\alpha), \boldsymbol{w})\\ - (\rho'(\phi_\alpha)(\nabla\mu_\alpha \cdot \nabla)\boldsymbol{u}_\alpha, \boldsymbol{w}) - (\mu_\alpha \nabla\phi_\alpha, \boldsymbol{w}) = 0,\end{aligned} \tag{3.73}$$

for all $\boldsymbol{w} \in \mathbf{L}^2_\sigma$ and almost every $t \in (0,T_0)$, and

$$\partial_t\phi_\alpha + \boldsymbol{u}_\alpha \cdot \nabla\phi_\alpha = \Delta\mu_\alpha, \quad \mu_\alpha = \alpha\partial_t\phi_\alpha - \Delta\phi_\alpha + \Psi'(\phi_\alpha) \qquad \text{a.e. in } \Omega \times (0,T_0). \tag{3.74}$$

Moreover, we have

$$\begin{cases} \boldsymbol{u}_\alpha = \mathbf{0},\ \partial_{\boldsymbol{n}}\phi_\alpha = \partial_{\boldsymbol{n}}\mu_\alpha = 0 & \text{a.e. on } \partial\Omega \times (0,T_0),\\ \boldsymbol{u}_\alpha(\cdot,0) = \boldsymbol{u}_0,\ \phi(\cdot,0) = \phi_{0,k} & \text{in } \Omega. \end{cases} \tag{3.75}$$

Next, we proceed with the vanishing viscosity limit in the Cahn–Hilliard equation. Thanks to the lower semicontinuity of the norm, we obtain from (3.65)–(3.67) that

$$\operatorname*{ess\,sup}_{t\in[0,T_0]} \|\nabla\boldsymbol{u}_\alpha(t)\|_{L^2} + \operatorname*{ess\,sup}_{t\in[0,T_0]} \|\mu_\alpha(t)\|_{H^1} + \operatorname*{ess\,sup}_{t\in[0,T_0]} \sqrt{\alpha}\|\partial_t\phi_\alpha(t)\|_{L^2} \le K_1, \tag{3.76}$$

$$\operatorname*{ess\,sup}_{t\in[0,T_0]} \|\phi_\alpha(t)\|_{H^2} + \operatorname*{ess\,sup}_{t\in[0,T_0]} \|F'(\phi_\alpha(t))\|_{L^2} \le K_2, \tag{3.77}$$

and

$$\int_0^{T_0} \|\partial_t\boldsymbol{u}_\alpha(\tau)\|_{L^2}^2 + \|\nabla\partial_t\phi_\alpha(\tau)\|_{L^2}^2 + \|\mathbf{A}\boldsymbol{u}_\alpha(\tau)\|_{L^2}^2 + \|\mu_\alpha(\tau)\|_{H^3}^2 \,\mathrm{d}\tau \le K_3. \tag{3.78}$$

Therefore, we can infer that

$$\begin{aligned} &\boldsymbol{u}_\alpha \rightharpoonup \boldsymbol{u}_k && \text{weak-star in } L^\infty(0,T_0;\mathbf{H}^1_\sigma),\\ &\boldsymbol{u}_\alpha \rightharpoonup \boldsymbol{u}_k && \text{weakly in } L^2(0,T_0;H^2) \cap W^{1,2}(0,T_0;\mathbf{L}^2_\sigma),\\ &\phi_\alpha \rightharpoonup \phi_k && \text{weak-star in } L^\infty(0,T_0;H^2(\Omega)),\\ &\phi_\alpha \rightharpoonup \phi_k && \text{weakly in } W^{1,2}(0,T_0;H^1(\Omega)),\\ &\mu_\alpha \rightharpoonup \mu_k && \text{weak-star in } L^\infty(0,T_0;H^1(\Omega)),\\ &\mu_\alpha \rightharpoonup \mu_k && \text{weakly in } L^2(0,T_0;H^3(\Omega)). \end{aligned} \tag{3.79}$$

In a similar manner as above, we have

$$\begin{aligned} \boldsymbol{u}_\alpha &\to \boldsymbol{u}_k && \text{strongly in } L^2(0,T_0;\mathbf{H}^1_\sigma),\\ \phi_\alpha &\to \phi_k && \text{strongly in } C([0,T_0];W^{1,p}(\Omega)),\\ \rho(\phi_\alpha) &\to \rho(\phi_k) && \text{strongly in } C([0,T_0];W^{1,p}(\Omega)),\\ \nu(\phi_\alpha) &\to \nu(\phi_k) && \text{strongly in } C([0,T_0];W^{1,p}(\Omega)), \end{aligned} \tag{3.80}$$

for all $p \in [2,6)$. In order to pass to the limit in F', we observe that

$$\phi_\alpha \in L^\infty(\Omega \times (0,T_0)) \text{ is such that } |\phi_\alpha(x,t)| < 1 \text{ a.e. in } \ \Omega \times (0,T_0).$$

Thanks to $(3.80)_2$, it follows that $\phi_\alpha \to \phi_k$ almost everywhere in $\Omega \times (0,T_0)$, and thereby,

$$\phi_k \in L^\infty(\Omega \times (0,T_0)) \text{ is such that } |\phi_k(x,t)| \leq 1 \text{ a.e. in } \ \Omega \times (0,T_0).$$

Then, since $F'(\phi_\alpha) \to F'(\phi_k)$ almost everywhere in $\Omega \times (0,T_0)$, by the Fatou lemma we have $F'(\phi_k) \in L^2(\Omega \times (0,T_0))$, which also implies that

$$\phi_k \in L^\infty(\Omega \times (0,T_0)) \text{ is such that } |\phi_k(x,t)| < 1 \text{ a.e. in } \ \Omega \times (0,T_0).$$

Owing to this and (3.77), we conclude that

$$F'(\phi_\alpha) \rightharpoonup F'(\phi_k) \quad \text{weak-star in } L^\infty(0,T_0;L^2(\Omega)).$$

Thus, letting $\alpha \to 0$ in (3.74)–(3.73), we obtain

$$\begin{aligned}(\rho(\phi_k)\partial_t \boldsymbol{u}_k, \boldsymbol{w}) + (\rho(\phi_k)(\boldsymbol{u}_k \cdot \nabla)\boldsymbol{u}_k, \boldsymbol{w}) - (\operatorname{div}(\nu(\phi_k)\mathbb{D}\boldsymbol{u}_k), \boldsymbol{w}) \\ - (\rho'(\phi_k)(\nabla\mu_k \cdot \nabla)\boldsymbol{u}_k, \boldsymbol{w}) - (\mu_k \nabla\phi_k, \boldsymbol{w}) = 0,\end{aligned} \tag{3.81}$$

for all $\boldsymbol{w} \in \mathbf{L}^2_\sigma$ and almost every $t \in (0,T_0)$, and

$$\partial_t \phi_k + \boldsymbol{u}_k \cdot \nabla\phi_k = \Delta\mu_k, \quad \mu_k = -\Delta\phi_k + \Psi'(\phi_k) \qquad \text{a.e. in } \ \Omega \times (0,T_0), \tag{3.82}$$

together with

$$\begin{cases} \boldsymbol{u}_k = \mathbf{0}, \ \partial_{\boldsymbol{n}}\phi_k = \partial_{\boldsymbol{n}}\mu_k = 0 & \text{a.e. on } \partial\Omega \times (0,T_0), \\ \boldsymbol{u}_k(\cdot,0) = \boldsymbol{u}_0, \ \phi(\cdot,0) = \phi_{0,k} & \text{in } \Omega. \end{cases} \tag{3.83}$$

Finally, since the estimates (3.76)–(3.78) are independent of k, we can further pass to the limit as $k \to \infty$. The argument readily follows the one above, and so it is left to the reader. As a result, we obtain

$$\big(\rho(\phi)\partial_t \boldsymbol{u} + \rho(\phi)(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \operatorname{div}(\nu(\phi)\mathbb{D}\boldsymbol{u}) - \rho'(\phi)(\nabla\mu \cdot \nabla)\boldsymbol{u} - \mu\nabla\phi, \boldsymbol{w}\big) = 0, \tag{3.84}$$

for all $\boldsymbol{w} \in \mathbf{L}^2_\sigma$ and almost every $t \in (0,T_0)$, and

$$\partial_t \phi + \boldsymbol{u} \cdot \nabla\phi = \Delta\mu, \quad \mu = -\Delta\phi + \Psi'(\phi) \qquad \text{a.e. in } \ \Omega \times (0,T_0), \tag{3.85}$$

together with

$$\begin{cases} \boldsymbol{u} = \mathbf{0}, \ \partial_{\boldsymbol{n}}\phi = \partial_{\boldsymbol{n}}\mu = 0 & \text{a.e. on } \partial\Omega \times (0,T_0), \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \ \phi(\cdot,0) = \phi_0 & \text{in } \Omega. \end{cases} \tag{3.86}$$

Recalling the well-known relation

$$\mu \nabla \phi = -\mathrm{div}\,(\nabla \phi \otimes \nabla \phi) + \nabla \Big(\frac{1}{2}|\nabla \phi|^2 + \Psi(\phi)\Big),$$

in a canonical way, there exists $P \in L^2(0, T_0; H^1(\Omega))$ (see, e.g., [22]) such that $\overline{P}(t) = 0$ and

$$\nabla P = -\rho(\phi)\partial_t \boldsymbol{u} - \rho(\phi)(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \mathrm{div}\,(\nu(\phi)\mathbb{D}\boldsymbol{u}) + \rho'(\phi)\nabla \boldsymbol{u} \nabla \mu - \mathrm{div}\,(\nabla \phi \otimes \nabla \phi).$$

Moreover, exploiting the regularity theory of the Cahn–Hilliard equation with logarithmic potential (see [2, Lemma 2] or [25, Theorem A.2]), we have that $\phi \in L^\infty(0, T; W^{2,6}(\Omega))$ and $F'(\phi) \in L^\infty(0, T; L^6(\Omega))$.

4. Proof of Theorem 1.1. Part two: Uniqueness

Let $(\boldsymbol{u}_1, P_1, \phi_1)$ and $(\boldsymbol{u}_2, P_2, \phi_2)$ be two strong solutions to system (1.1)–(1.2) defined on the interval $[0, T_0]$ as stated in Theorem 1.1. We define $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $P = P_1 - P_2$, $\phi = \phi_1 - \phi_2$, which solve

$$\begin{aligned}
&\rho(\phi_1)\partial_t \boldsymbol{u} + (\rho(\phi_1) - \rho(\phi_2))\partial_t \boldsymbol{u}_2 + \big(\rho(\phi_1)(\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{u}_1 - \rho(\phi_2)(\boldsymbol{u}_2 \cdot \nabla)\boldsymbol{u}_2\big)\\
&\quad - \frac{\rho_1 - \rho_2}{2}\big((\nabla \mu_1 \cdot \nabla)\boldsymbol{u}_1 - (\nabla \mu_2 \cdot \nabla)\boldsymbol{u}_2\big) - \mathrm{div}\,(\nu(\phi_1)\mathbb{D}\boldsymbol{u})\\
&\quad - \mathrm{div}\,((\nu(\phi_1) - \nu(\phi_2))\mathbb{D}\boldsymbol{u}_2) + \nabla P = -\mathrm{div}\,(\nabla \phi_1 \otimes \nabla \phi_1 - \nabla \phi_2 \otimes \nabla \phi_2), \quad (4.1)
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t \phi + \boldsymbol{u}_1 \cdot \nabla \phi + \boldsymbol{u} \cdot \nabla \phi_2 = \Delta \mu,\\
&-\Delta \phi + \Psi'(\phi_1) - \Psi'(\phi_2) = \mu,
\end{aligned} \quad (4.2)$$

almost everywhere in $\Omega \times (0, T_0)$, where $\mu_i = -\Delta \phi_i + \Psi'(\phi_i)$, for $i = 1, 2$, and subject to the boundary conditions

$$\boldsymbol{u} = \boldsymbol{0}, \quad \partial_{\boldsymbol{n}} \phi = \partial_{\boldsymbol{n}} \mu = 0 \qquad \text{a.e. on } \partial\Omega \times (0, T). \quad (4.3)$$

We recall that

$$\|\phi_i\|_{L^\infty(0,T_0;W^{2,6}(\Omega))} + \|\partial_t \phi_i\|_{L^2(0,T_0;H^1(\Omega))} \le K, \quad i = 1, 2, \quad (4.4)$$

where K is a positive constant depending only on $E(\boldsymbol{u}_0, \phi_0)$, $\|\boldsymbol{u}_0\|_{\mathbf{H}^1_\sigma}$, $\|\mu_0\|_{H^1}$, T_0, and and the parameters of the system. As a consequence, we claim that

$$\|\phi_i\|_{C^{\frac{5}{16}}([0,T_0];C(\overline{\Omega}))} \le CK, \quad i = 1, 2,$$

for some constant C depending only on Ω. Indeed, by (2.5), we have

$$\|\phi_i(t_1) - \phi_i(t_2)\|_{C(\overline{\Omega})} \le C\|\phi_i(t_1) - \phi_i(t_2)\|_{W^{1,4}}$$

$$
\begin{aligned}
&\le C\|\phi_i(t_1)-\phi_i(t_2)\|_{H^1}^{\frac{5}{8}}\|\phi_i(t_1)-\phi_i(t_2)\|_{W^{2,6}}^{\frac{3}{8}}\\
&\le CK^{\frac{3}{8}}\Big(\int_{t_1}^{t_2}\|\partial_t\phi_i(\tau)\|_{H^1}\,\mathrm{d}\tau\Big)^{\frac{5}{8}}\\
&\le CK^{\frac{3}{8}}\|\partial_t\phi_i\|_{L^2(0,T_0;H^1(\Omega))}^{\frac{5}{8}}|t_1-t_2|^{\frac{5}{16}},\quad \forall\, t_1,t_2\in[0,T_0],\ i=1,2.
\end{aligned}
$$

In light of the assumption $\|\phi_0\|_{L^\infty}<1$, we infer that

$$
\max_{t\in[0,T_1]}\|\phi_i(t)\|_{L^\infty}<1,\quad \text{where}\quad T_1<\min\Big\{\Big(\frac{1-\|\phi_0\|_{L^\infty}}{CK}\Big)^{\frac{16}{5}},T_0\Big\},\ i=1,2. \tag{4.5}
$$

Owing to (4.5), it immediately follows that $\Psi''(\phi_i),\Psi'''(\phi_i)\in L^\infty(\Omega\times(0,T_1))$. Since Ω is a C^4 domain, by the elliptic regularity theory of the Laplacian and (4.4), we deduce that $\phi_i\in L^\infty(0,T_1;H^3(\Omega))\cap L^2(0,T_1;H^4(\Omega))$, for $i=1,2$. In addition, it follows from (4.3) that $\partial_{\boldsymbol{n}}\Delta\phi=0$ almost everywhere on $\partial\Omega\times(0,T_1)$.

Next, multiplying (4.1) by $\boldsymbol{u}$ and integrating over Ω, we find

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho(\phi_1)|\boldsymbol{u}|^2\,\mathrm{d}x+\int_\Omega\nu(\phi_1)|\mathbb{D}\boldsymbol{u}|^2\,\mathrm{d}x\\
&\quad=-\int_\Omega(\rho(\phi_1)-\rho(\phi_2))\partial_t\boldsymbol{u}_2\cdot\boldsymbol{u}\,\mathrm{d}x-\int_\Omega\rho(\phi_1)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_2\cdot\boldsymbol{u}\,\mathrm{d}x\\
&\qquad-\int_\Omega(\rho(\phi_1)-\rho(\phi_2))(\boldsymbol{u}_2\cdot\nabla)\boldsymbol{u}_2\cdot\boldsymbol{u}\,\mathrm{d}x+\frac{\rho_1-\rho_2}{2}\int_\Omega((\nabla\mu\cdot\nabla)\boldsymbol{u}_2)\cdot\boldsymbol{u}\,\mathrm{d}x\\
&\qquad-\int_\Omega(\nu(\phi_1)-\nu(\phi_2))\mathbb{D}\boldsymbol{u}_2:\nabla\boldsymbol{u}\,\mathrm{d}x+\int_\Omega(\nabla\phi_1\otimes\nabla\phi+\nabla\phi\otimes\nabla\phi_2):\nabla\boldsymbol{u}\,\mathrm{d}x\\
&\quad=\sum_{i=1}^{6}Z_i.
\end{aligned}\tag{4.6}
$$

Here, we have used that

$$
-\int_\Omega\partial_t\rho(\phi_1)\frac{|\boldsymbol{u}|^2}{2}\,\mathrm{d}x+\int_\Omega\rho(\phi_1)\boldsymbol{u}_1\cdot\nabla\frac{|\boldsymbol{u}|^2}{2}\,\mathrm{d}x-\frac{\rho_1-\rho_2}{2}\int_\Omega\nabla\mu_1\cdot\nabla\frac{|\boldsymbol{u}|^2}{2}\,\mathrm{d}x=0.
$$

Taking the gradient of $(4.2)_1$, multiplying by $\nabla\Delta\phi$ and integrating over Ω, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\phi\|_{L^2}^2+\|\Delta^2\phi\|_{L^2}^2&=\int_\Omega\boldsymbol{u}_1\cdot\nabla\phi\Delta^2\phi\,\mathrm{d}x+\int_\Omega\boldsymbol{u}\cdot\nabla\phi_2\Delta^2\phi\,\mathrm{d}x\\
&\quad+\int_\Omega\Delta(\Psi'(\phi_1)-\Psi'(\phi_2))\Delta^2\phi\,\mathrm{d}x\\
&=\sum_{i=7}^{9}Z_i.
\end{aligned}
$$

Therefore, we arrive at

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_\Omega\rho(\phi_1)|\boldsymbol{u}|^2\,\mathrm{d}x+\frac{1}{2}\|\Delta\phi\|_{L^2}^2\Big)+\int_\Omega\nu(\phi_1)|\mathbb{D}\boldsymbol{u}|^2\,\mathrm{d}x+\|\Delta^2\phi\|_{L^2}^2=\sum_{i=1}^{9}Z_i.
$$

Arguing in a similar way as in [24, Section 6], it is easily seen that

$$
\begin{aligned}
|Z_1+Z_2+Z_3+Z_5+Z_6| &\le \frac{\nu_*}{2}\|\mathbb{D}\boldsymbol{u}\|_{L^2}^2\\
&\quad + C\big(1+\|\boldsymbol{u}_2\|_{H^2}^2+\|\partial_t\boldsymbol{u}_2\|_{L^2}^2\big)\big(\|\boldsymbol{u}\|_{L^2}^2+\|\Delta\phi\|_{L^2}^2\big).
\end{aligned}
$$

By (4.4) and (4.5), together with Sobolev embeddings, we find

$$
\begin{aligned}
|Z_4| &\le \int_\Omega |(\nabla\Delta\phi\cdot\nabla)\boldsymbol{u}_2\cdot\boldsymbol{u}|\,\mathrm{d}x + \int_\Omega \big|\big(\nabla(\Psi'(\phi_1)-\Psi'(\phi_2))\cdot\nabla\big)\boldsymbol{u}_2\cdot\boldsymbol{u}\big|\,\mathrm{d}x\\
&\le \|\nabla\Delta\phi\|_{L^6}\|\nabla\boldsymbol{u}_2\|_{L^3}\|\boldsymbol{u}\|_{L^2} + \|\Psi''(\phi_1)\|_{L^\infty}\|\nabla\phi\|_{L^6}\|\nabla\boldsymbol{u}_2\|_{L^3}\|\boldsymbol{u}\|_{L^2}\\
&\quad + (\|\Psi'''(\phi_1)\|_{L^\infty}+\|\Psi'''(\phi_2)\|_{L^\infty})\|\phi\|_{L^\infty}\|\nabla\phi_2\|_{L^\infty}\|\nabla\boldsymbol{u}_2\|_{L^2}\|\boldsymbol{u}\|_{L^2}\\
&\le \frac{1}{6}\|\Delta^2\phi\|_{L^2}^2 + C\|\nabla\boldsymbol{u}_2\|_{L^3}^2\|\boldsymbol{u}\|_{L^2}^2 + C(1+\|\nabla\boldsymbol{u}_2\|_{L^3})\big(\|\boldsymbol{u}\|_{L^2}^2+\|\Delta\phi\|_{L^2}^2\big).
\end{aligned}
$$

As for the remaining terms, by using (4.4) and (4.5) once more, we have

$$
\begin{aligned}
|Z_7+Z_8| &\le \|\boldsymbol{u}_1\|_{L^3}\|\nabla\phi\|_{L^6}\|\Delta^2\phi\|_{L^2} + \|\boldsymbol{u}\|_{L^2}\|\nabla\phi_2\|_{L^\infty}\|\Delta^2\phi\|_{L^2}\\
&\le \frac{1}{6}\|\Delta^2\phi\|_{L^2}^2 + C\big(\|\boldsymbol{u}\|_{L^2}^2+\|\Delta\phi\|_{L^2}^2\big)
\end{aligned}
$$

and

$$
\begin{aligned}
|Z_9| &\le \int_\Omega \big|\big(\Psi''(\phi_1)\Delta\phi + (\Psi''(\phi_1)-\Psi''(\phi_2))\Delta\phi_2\big)\Delta^2\phi\big|\,\mathrm{d}x\\
&\quad + \int_\Omega \big|\big(\Psi'''(\phi_1)(|\nabla\phi_1|^2-|\nabla\phi_2|^2) + (\Psi'''(\phi_1)-\Psi'''(\phi_2))|\nabla\phi_2|^2\big)\Delta^2\phi\big|\,\mathrm{d}x\\
&\le C\|\Delta\phi\|_{L^2}\|\Delta^2\phi\|_{L^2} + C\big(\|\Psi'''(\phi_1)\|_{L^\infty}\\
&\quad + \|\Psi'''(\phi_2)\|_{L^\infty}\big)\|\phi\|_{L^\infty}\|\Delta\phi_2\|_{L^2}\|\Delta^2\phi\|_{L^2}\\
&\quad + C\big(\|\nabla\phi_1\|_{L^\infty)} + \|\nabla\phi_2\|_{L^\infty}\big)\|\nabla\phi\|_{L^2}\|\Delta^2\phi\|_{L^2}\\
&\quad + \big(\|\Psi''''(\phi_1)\|_{L^\infty} + \|\Psi''''(\phi_2)\|_{L^\infty}\big)\|\phi\|_{L^\infty}\|\nabla\phi_2\|_{L^\infty}^2\|\Delta^2\phi\|_{L^2}\\
&\le \frac{1}{6}\|\Delta^2\phi\|_{L^2}^2 + C\|\Delta\phi\|_{L^2}^2.
\end{aligned}
$$

In conclusion, we find the differential inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_\Omega \rho(\phi_1)|\boldsymbol{u}|^2\,\mathrm{d}x + \frac{1}{2}\|\Delta\phi\|_{L^2}^2\Big) + \frac{\nu_*}{2}\|\mathbb{D}\boldsymbol{u}\|_{L^2}^2 + \frac{1}{2}\|\Delta^2\phi\|_{L^2}^2\\
&\qquad \le C(K)\big(1+\|\boldsymbol{u}_2\|_{H^2}^2+\|\partial_t\boldsymbol{u}_2\|_{L^2}^2\big)\big(\|\boldsymbol{u}\|_{L^2}^2+\|\Delta\phi\|_{L^2}^2\big).
\end{aligned}
$$

An application of the Gronwall lemma implies the desired uniqueness of strong solutions on the time interval $[0, T_1]$.

5. Proof of Theorem 1.2: Stability

Let $(\boldsymbol{u}, P, \phi)$ and $(\boldsymbol{u}_H, P_H, \phi_H)$ be the strong solutions to the AGG model with density $\rho(\phi)$ and to the model H with constant density $\overline{\rho}$, respectively, defined on a common interval $[0, T_0]$. We recall that the existence of $(\boldsymbol{u}_H, P_H, \phi_H)$ fulfilling the same regularity properties of $(\boldsymbol{u}, P, \phi)$, as stated in Theorem 1.1, has been proven in [25, Theorem 5.1]. For simplicity, we assume that the viscosity function is given by $\nu(s) = \nu_1 \frac{1+s}{2} + \nu_2 \frac{1-s}{2}$ (cf. (1.3)) for both systems. We define $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_H$, $Q = P - P_H$, $\varphi = \phi - \phi_H$, and the difference of the chemical potentials $w = \mu - \mu_H$. They clearly solve the problem

$$\begin{aligned}
&\Big(\frac{\rho_1 + \rho_2}{2}\Big)\partial_t \boldsymbol{v} + \Big(\frac{\rho_1 - \rho_2}{2}\phi\Big)\partial_t \boldsymbol{u} + \Big(\frac{\rho_1 + \rho_2}{2} - \overline{\rho}\Big)\partial_t \boldsymbol{u}_H \\
&\quad + \big(\rho(\phi)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \overline{\rho}(\boldsymbol{u}_H\cdot\nabla)\boldsymbol{u}_H\big) - \Big(\frac{\rho_1 - \rho_2}{2}\Big)\big((\nabla\mu\cdot\nabla)\boldsymbol{u}\big) - \operatorname{div}(\nu(\phi)\mathbb{D}\boldsymbol{v}) \\
&\quad - \operatorname{div}((\nu(\phi) - \nu(\phi_H))\mathbb{D}\boldsymbol{u}_H) + \nabla Q = -\operatorname{div}(\nabla\phi\otimes\nabla\phi - \nabla\phi_H\otimes\nabla\phi_H), \qquad (5.1)
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t\varphi + \boldsymbol{u}\cdot\nabla\varphi + \boldsymbol{v}\cdot\nabla\phi_H = \Delta w, \\
&-\Delta\varphi + \Psi'(\phi) - \Psi'(\phi_H) = w,
\end{aligned} \qquad (5.2)$$

almost everywhere in $\Omega \times (0, T_0)$. In addition, we have the boundary and initial conditions

$$\begin{aligned}
&\boldsymbol{v} = \boldsymbol{0}, \quad \partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}} w = 0 \quad \text{a.e. on } \partial\Omega \times (0, T), \\
&\boldsymbol{v}(\cdot, 0) = \boldsymbol{0}, \quad \varphi(\cdot, 0) = 0 \quad \text{in } \Omega.
\end{aligned} \qquad (5.3)$$

Multiplying (5.1) by $\mathbf{A}^{-1}\boldsymbol{v}$ and integrating over Ω, we obtain

$$\begin{aligned}
&\Big(\frac{\rho_1 + \rho_2}{4}\Big)\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}\|_\sharp^2 + \int_\Omega \nu(\phi)\mathbb{D}\boldsymbol{v} : \nabla\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x = -\int_\Omega \Big(\frac{\rho_1 - \rho_2}{2}\phi\Big)\partial_t\boldsymbol{u}\cdot\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x \\
&\quad - \int_\Omega \Big(\frac{\rho_1 + \rho_2}{2} - \overline{\rho}\Big)\partial_t\boldsymbol{u}_H\cdot\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x - \int_\Omega \big(\rho(\phi)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \overline{\rho}(\boldsymbol{u}_H\cdot\nabla)\boldsymbol{u}_H\big)\cdot\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x \\
&\quad + \int_\Omega \Big(\frac{\rho_1 - \rho_2}{2}\Big)\big((\nabla\mu\cdot\nabla)\boldsymbol{u}\big)\cdot\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x - \int_\Omega (\nu(\phi) - \nu(\phi_H))\mathbb{D}\boldsymbol{u}_H : \nabla\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x \\
&\quad + \int_\Omega \nabla\phi\otimes\nabla\phi - \nabla\phi_H\otimes\nabla\phi_H : \nabla\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x.
\end{aligned}$$

Following [25, proof of Theorem 3.1], we infer that

$$\begin{aligned}
\int_\Omega \nu(\phi)\mathbb{D}\boldsymbol{v} : \nabla\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x \geq \frac{\nu_*}{2}\|\boldsymbol{u}\|_{L^2}^2 - \int_\Omega \nu'(\phi)\mathbb{D}\mathbf{A}^{-1}\boldsymbol{v}\nabla\phi\cdot\boldsymbol{v}\,\mathrm{d}x \\
+ \frac{1}{2}\int_\Omega \nu'(\phi)\nabla\phi\cdot\boldsymbol{v}\,\Pi\,\mathrm{d}x, \qquad (5.4)
\end{aligned}$$

where $\Pi \in L^\infty(0, T_0; H^1(\Omega))$ is such that $-\Delta\mathbf{A}^{-1}\boldsymbol{v} + \nabla\Pi = \boldsymbol{v}$ a.e. in $\Omega \times (0, T_0)$. In addition, Π fulfills the estimates

$$\|\Pi\|_{L^2} \leq C\|\nabla\mathbf{A}^{-1}\boldsymbol{v}\|_{L^2}^{\frac{1}{2}}\|\boldsymbol{v}\|_{L^2}^{\frac{1}{2}}, \quad \|\Pi\|_{H^1} \leq C\|\boldsymbol{v}\|_{L^2}. \qquad (5.5)$$

Therefore, we are led to

$$
\begin{aligned}
&\Big(\frac{\rho_1+\rho_2}{4}\Big)\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}\|_\sharp^2+\frac{\nu_*}{2}\|\boldsymbol{v}\|_{L^2}^2\\
&\quad=-\int_\Omega\Big(\frac{\rho_1-\rho_2}{2}\phi\Big)\partial_t\boldsymbol{u}\cdot\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x-\int_\Omega\Big(\frac{\rho_1+\rho_2}{2}-\overline{\rho}\Big)\partial_t\boldsymbol{u}_H\cdot\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x\\
&\qquad-\int_\Omega\big(\rho(\phi)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\overline{\rho}(\boldsymbol{u}_H\cdot\nabla)\boldsymbol{u}_H\big)\cdot\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x\\
&\qquad+\int_\Omega\Big(\frac{\rho_1-\rho_2}{2}\Big)\big((\nabla\mu\cdot\nabla)\boldsymbol{u}\big)\cdot\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x-\int_\Omega(\nu(\phi)-\nu(\phi_H))\mathbb{D}\boldsymbol{u}_H:\nabla\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x\\
&\qquad+\int_\Omega\nabla\phi\otimes\nabla\phi-\nabla\phi_H\otimes\nabla\phi_H:\nabla\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x+\int_\Omega\nu'(\phi)\mathbb{D}\mathbf{A}^{-1}\boldsymbol{v}\nabla\phi\cdot\boldsymbol{v}\,\mathrm{d}x\\
&\qquad-\frac{1}{2}\int_\Omega\nu'(\phi)\nabla\phi\cdot\boldsymbol{v}\,\Pi\,\mathrm{d}x.
\end{aligned}
\tag{5.6}
$$

On the other hand, multiplying (5.2)$_2$ by $A^{-1}\varphi$, where A is the Laplace operator with homogeneous Neumann boundary conditions, and integrating over Ω, we get

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_*^2+\frac{1}{2}\|\nabla\varphi\|_{L^2}^2\le C\|\varphi\|_*^2+\int_\Omega\varphi\,\boldsymbol{u}\cdot\nabla A^{-1}\varphi\,\mathrm{d}x+\int_\Omega\phi_H\,\boldsymbol{v}\cdot\nabla A^{-1}\varphi\,\mathrm{d}x \tag{5.7}
$$

(see [25, proof of Theorem 3.1] for more details). We proceed with the estimate of the terms on the right-hand side of (5.6) and (5.7). To this end, we will exploit the following bounds on the solution:

$$
\begin{aligned}
&\|(\boldsymbol{u},\boldsymbol{u}_H)\|_{L^\infty(0,T_0;\mathbf{H}^1_\sigma)\cap L^2(0,T_0;\mathbf{H}^2_\sigma(\Omega))\cap W^{1,2}(0,T_0;\mathbf{L}^2_\sigma)}\le K_0,\\
&\|(\phi,\phi_H)\|_{L^\infty(0,T_0;W^{2,6}(\Omega))}+\|\nabla\mu\|_{L^\infty(0,T_0;L^2(\Omega))}\le K_0,
\end{aligned}
\tag{5.8}
$$

where K_0 is a constant depending on the norms of the initial conditions. Exploiting these estimates, we have

$$
\begin{aligned}
\Big|\int_\Omega\Big(\frac{\rho_1-\rho_2}{2}\phi\Big)\partial_t\boldsymbol{u}\cdot\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x\Big|&\le\Big|\frac{\rho_1-\rho_2}{2}\Big|\|\phi\|_{L^\infty}\|\partial_t\boldsymbol{u}\|_{L^2}\|\mathbf{A}^{-1}\boldsymbol{v}\|_{L^2}\\
&\le C\|\boldsymbol{v}\|_\sharp^2+C\Big|\frac{\rho_1-\rho_2}{2}\Big|^2\|\partial_t\boldsymbol{u}\|_{L^2}^2
\end{aligned}
$$

and

$$
\Big|\int_\Omega\Big(\frac{\rho_1+\rho_2}{2}-\overline{\rho}\Big)\partial_t\boldsymbol{u}_H\cdot\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x\Big|\le C\|\boldsymbol{v}\|_\sharp^2+C\Big|\frac{\rho_1+\rho_2}{2}-\overline{\rho}\Big|^2\|\partial_t\boldsymbol{u}_H\|_{L^2}^2.
$$

By Sobolev embeddings, we find

$$
\begin{aligned}
&\Big|\int_\Omega\big(\rho(\phi)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\overline{\rho}(\boldsymbol{u}_H\cdot\nabla)\boldsymbol{u}_H\big)\cdot\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x\Big|\\
&\quad\le\Big|\int_\Omega\rho(\phi)(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\cdot\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x\Big|+\Big|\int_\Omega\rho(\phi)(\boldsymbol{u}_H\cdot\nabla)\boldsymbol{v}\cdot\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x\Big|\\
&\qquad+\Big|\int_\Omega\big(\rho(\phi)-\overline{\rho}\big)(\boldsymbol{u}_H\cdot\nabla)\boldsymbol{u}_H\cdot\mathbf{A}^{-1}\boldsymbol{v}\,\mathrm{d}x\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\le \rho^* \|\boldsymbol{v}\|_{L^2} \|\nabla \boldsymbol{u}\|_{L^6} \|\mathbf{A}^{-1}\boldsymbol{v}\|_{L^3} \\
&\quad + \left| \int_\Omega \rho(\phi)(\boldsymbol{u}_H \cdot \nabla)\mathbf{A}^{-1}\boldsymbol{v} \cdot \boldsymbol{v} \, \mathrm{d}x + \int_\Omega \rho'(\phi)(\nabla\phi \cdot \boldsymbol{u}_H)\big(\boldsymbol{v} \cdot \mathbf{A}^{-1}\boldsymbol{v}\big) \, \mathrm{d}x \right| \\
&\quad + \|\rho(\phi) - \overline{\rho}\|_{L^\infty} \|\boldsymbol{u}_H\|_{L^6} \|\nabla \boldsymbol{u}_H\|_{L^2} \|\mathbf{A}^{-1}\boldsymbol{v}\|_{L^3} \\
&\le \frac{\nu_*}{16} \|\boldsymbol{v}\|_{L^2}^2 + C(1 + \|\boldsymbol{u}\|_{H^2}^2)\|\boldsymbol{v}\|_\sharp^2 + \rho^* \|\nabla \mathbf{A}^{-1}\boldsymbol{v}\|_{L^2} \|\boldsymbol{u}_H\|_{L^\infty} \|\boldsymbol{v}\|_{L^2} \\
&\quad + \left| \frac{\rho_1 - \rho_2}{2} \right| \|\nabla\phi\|_{L^\infty} \|\boldsymbol{u}_H\|_{L^6} \|\boldsymbol{v}\|_{L^2} \|\mathbf{A}^{-1}\boldsymbol{v}\|_{L^3} \\
&\quad + C(K_0)\left(\left| \frac{\rho_1 - \rho_2}{2} \right|^2 + \left| \frac{\rho_1 + \rho_2}{2} - \overline{\rho} \right|^2 \right) \\
&\le \frac{\nu_*}{8} \|\boldsymbol{v}\|_{L^2}^2 + C(K_0)\big(1 + \|\boldsymbol{u}\|_{H^2}^2 + \|\boldsymbol{u}_H\|_{H^2}^2\big)\|\boldsymbol{v}\|_\sharp^2 \\
&\quad + C(K_0)\left(\left| \frac{\rho_1 - \rho_2}{2} \right|^2 + \left| \frac{\rho_1 + \rho_2}{2} - \overline{\rho} \right|^2 \right),
\end{aligned}
$$

and

$$
\begin{aligned}
\left| \int_\Omega \left(\frac{\rho_1 - \rho_2}{2} \right) \big((\nabla\mu \cdot \nabla)\boldsymbol{u} \big) \cdot \mathbf{A}^{-1}\boldsymbol{v} \, \mathrm{d}x \right| &\le \left| \frac{\rho_1 - \rho_2}{2} \right| \|\nabla\mu\|_{L^2} \|\nabla\boldsymbol{u}\|_{L^3} \|\mathbf{A}^{-1}\boldsymbol{v}\|_{L^6} \\
&\le C\|\boldsymbol{v}\|_\sharp^2 + C(K_0)\left| \frac{\rho_1 - \rho_2}{2} \right|^2 \|\nabla\boldsymbol{u}\|_{L^3}^2.
\end{aligned}
$$

In a similar way as in [25, proof of Theorem 5.1], we obtain

$$
\begin{aligned}
\left| \int_\Omega (\nu(\phi) - \nu(\phi_H))\mathbb{D}\boldsymbol{u}_H : \nabla\mathbf{A}^{-1}\boldsymbol{v} \, \mathrm{d}x \right| &\le C\|\varphi\|_{L^6} \|\mathbb{D}\boldsymbol{u}_H\|_{L^3} \|\nabla\mathbf{A}^{-1}\boldsymbol{v}\|_{L^2} \\
&\le \frac{1}{6}\|\nabla\varphi\|_{L^2}^2 + C\|\boldsymbol{u}_H\|_{H^2}^2 \|\boldsymbol{v}\|_\sharp^2, \\
\left| \int_\Omega (\nabla\phi \otimes \nabla\phi - \nabla\phi_H \otimes \nabla\phi_H) : \nabla\mathbf{A}^{-1}\boldsymbol{v} \, \mathrm{d}x \right| &\le \big(\|\nabla\phi\|_{L^\infty} + \|\nabla\phi_H\|_{L^\infty} \big) \|\nabla\varphi\|_{L^2} \|\nabla\mathbf{A}^{-1}\boldsymbol{v}\|_{L^2} \\
&\le \frac{1}{6}\|\nabla\varphi\|_{L^2}^2 + C(K_0)\|\boldsymbol{v}\|_\sharp^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\left| \int_\Omega \nu'(\phi)\mathbb{D}\mathbf{A}^{-1}\boldsymbol{v}\nabla\phi \cdot \boldsymbol{v} \, \mathrm{d}x \right| &\le C\|\mathbb{D}\mathbf{A}^{-1}\boldsymbol{v}\|_{L^2} \|\nabla\phi\|_{L^\infty} \|\boldsymbol{v}\|_{L^2} \le \frac{\nu_*}{8}\|\boldsymbol{v}\|_{L^2}^2 + C(K_0)\|\boldsymbol{v}\|_\sharp^2, \\
\left| \frac{1}{2} \int_\Omega \nu'(\phi)\,(\nabla\phi \cdot \boldsymbol{v})\,\Pi \, \mathrm{d}x \right| &\le C\|\nabla\phi\|_{L^\infty} \|\boldsymbol{v}\|_{L^2} \|\Pi\|_{L^2} \le \frac{\nu_*}{8}\|\boldsymbol{v}\|_{L^2}^2 + C(K_0)\|\boldsymbol{v}\|_\sharp^2, \\
\left| \int_\Omega \varphi\, \boldsymbol{u} \cdot \nabla A^{-1}\varphi \, \mathrm{d}x \right| &\le \frac{1}{6}\|\nabla\varphi\|_{L^2}^2 + C\|\boldsymbol{u}\|_{H^2(\Omega)}^2 \|\varphi\|_*^2, \\
\left| \int_\Omega \phi_H\, \boldsymbol{v} \cdot \nabla A^{-1}\varphi \, \mathrm{d}x \right| &\le \frac{\nu_*}{8}\|\boldsymbol{v}\|_{L^2}^2 + C\|\varphi\|_*^2.
\end{aligned}
$$

Collecting the above estimates together, we find the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big(\frac{\rho_1+\rho_2}{4}\Big)\|\boldsymbol{v}\|_\sharp^2+\frac{1}{2}\|\varphi\|_*^2\Big)\le f_1(t)\big(\|\boldsymbol{v}\|_\sharp^2+\|\varphi\|_*^2\big)$$
$$+f_2(t)\Big(\Big|\frac{\rho_1-\rho_2}{2}\Big|^2+\Big|\frac{\rho_1+\rho_2}{2}-\overline{\rho}\Big|^2\Big),$$

where

$$f_1(t)=C(K_0)\big(1+\|\boldsymbol{u}_H\|_{H^2}^2+\|\boldsymbol{u}\|_{H^2}^2\big),$$
$$f_2(t)=C(K_0)\big(1+\|\partial_t\boldsymbol{u}_H\|_{L^2}^2+\|\boldsymbol{u}_H\|_{H^2}^2+\|\partial_t\boldsymbol{u}\|_{L^2}^2+\|\boldsymbol{u}\|_{H^2}^2\big).$$

Here, the positive constant C depends on the norm of the initial data and the time T_0. By using the Gronwall lemma, together with initial conditions $(5.3)_2$, we infer that

$$\|\boldsymbol{v}(t)\|_\sharp^2+\|\varphi(t)\|_*^2\le\frac{\big(\big|\frac{\rho_1-\rho_2}{2}\big|^2+\big|\frac{\rho_1+\rho_2}{2}-\overline{\rho}\big|^2\big)}{\min\{\frac{\rho_1+\rho_2}{4},\frac{1}{2}\}}\int_0^t e^{\int_s^t f_1(r)\,\mathrm{d}r}f_2(s)\,\mathrm{d}s,\quad\forall\, t\in[0,T_0].$$

Thus, the above inequality implies that

$$\|\boldsymbol{u}(t)-\boldsymbol{u}_H(t)\|_{(\mathbf{H}^1_\sigma)'}+\|\phi(t)-\phi_H(t)\|_{(H^1)'}$$
$$\le\frac{C(K_0)}{\min\{\sqrt{\rho_*},1\}}\Big(\Big|\frac{\rho_1-\rho_2}{2}\Big|+\Big|\frac{\rho_1+\rho_2}{2}-\overline{\rho}\Big|\Big),\qquad\forall\, t\in[0,T_0],$$

where the positive constant $C(K_0)$ depends on the norm of the initial data, the time T_0 and the parameters of the systems.

A. On the convective viscous Cahn–Hilliard system

Given $\alpha>0$ and an incompressible velocity field $\boldsymbol{u}$, we consider the convective viscous Cahn–Hilliard (cvCH) system

$$\begin{cases}\partial_t\phi+\boldsymbol{u}\cdot\nabla\phi=\Delta\mu\\ \mu=\alpha\partial_t\phi-\Delta\phi+\Psi'(\phi)\end{cases}\qquad\text{in }\Omega\times(0,T),\tag{A.1}$$

with boundary and initial conditions

$$\begin{aligned}\partial_{\boldsymbol{n}}\phi=\partial_{\boldsymbol{n}}\mu=0&\quad\text{on }\partial\Omega\times(0,T),\\ \phi(\cdot,0)=\phi_0&\quad\text{in }\Omega.\end{aligned}\tag{A.2}$$

We observe that (A.1) can be rewritten as

$$\partial_t(\phi-\alpha\Delta\phi)+\boldsymbol{u}\cdot\nabla\phi=\Delta(-\Delta\phi+F'(\phi)-\theta_0\phi)\quad\text{in }\Omega\times(0,T).$$

We state well-posedness and regularity results for system (A.1). The aim of this appendix is to extend the analysis performed in [32] to the convective case under minimal assumptions on the velocity field. In particular, we focus on the regularity of the chemical potential μ.

Theorem A.1. *Let Ω be a bounded domain of class C^3 in $\mathbb{R}^3$. We assume that $\boldsymbol{u} \in L^\infty(0,T;\mathbf{L}^2_\sigma(\Omega)\cap L^3(\Omega))$, $\phi_0 \in H^1(\Omega)\cap L^\infty(\Omega)$ such that $\|\phi_0\|_{L^\infty} \leq 1$ and $|\overline{\phi_0}| < 1$. Then, there exists a unique weak solution to (A.1)–(A.2) such that*

$$\begin{aligned} &\phi \in L^\infty(0,T;H^1(\Omega)\cap L^\infty(\Omega)) \text{ with } |\phi(x,t)|<1 \text{ a.e. in } \Omega\times(0,T),\\ &\phi \in L^2(0,T;H^2(\Omega))\cap W^{1,2}(0,T;L^2(\Omega)),\\ &\mu \in L^2(0,T;H^2(\Omega)), \quad F'(\phi)\in L^2(0,T;L^2(\Omega)), \end{aligned} \tag{A.3}$$

which satisfies (A.1) almost everywhere in $\Omega\times(0,T)$, (A.2) almost everywhere on $\partial\Omega\times(0,T)$ and $\phi(\cdot,0)=\phi_0(\cdot)$ in Ω. In addition, the following regularity results hold:

(R1) *If $-\Delta\phi_0 + F'(\phi_0)\in L^2(\Omega)$ and $\partial_t \boldsymbol{u} \in L^{\frac{4}{3}}(0,T;L^1(\Omega))$, we have*

$$\begin{aligned} \partial_t\phi \in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)),\\ \phi\in L^\infty(0,T;H^2(\Omega)), \quad \mu\in L^\infty(0,T;H^2(\Omega)). \end{aligned}$$

(R2) *Let the assumptions of (R1) hold. Suppose that $\|\phi_0\|_{L^\infty}\leq 1-\delta_0$, for some $\delta_0\in(0,1)$. Then, there exists $\delta>0$ such that*

$$\max_{(x,t)\in\Omega\times[0,T]} |\phi(x,t)| \leq 1-\delta, \tag{A.4}$$

and

$$\phi\in L^2(0,T;H^3(\Omega)).$$

(R3) *Let the assumption of (R2) hold. Suppose that $\phi_0\in H^3(\Omega)$ such that $\partial_{\boldsymbol{n}}\phi=0$ on $\partial\Omega$, and $\partial_t\boldsymbol{u}\in L^2(0,T;L^{\frac{6}{5}}(\Omega))$. Then, we have*

$$\begin{aligned} \phi\in L^\infty(0,T;H^3(\Omega)), \quad \partial_t\phi\in L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega)),\\ \partial_t^2\phi\in L^2(0,T;L^2(\Omega)), \quad \partial_t\mu\in L^2(0,T;L^2(\Omega)). \end{aligned}$$

Proof. The proof is divided into several parts. We notify the reader that the estimates proved herein are not independent of the viscous parameter α.

Existence. The existence of a weak solution satisfying (A.3) is proved in a classical way[1]. We proceed here by proving the basic *energy* estimates. First, we observe that, by integrating $(A.1)_1$ over Ω and using the boundary conditions, we have

$$\overline{\phi}(t)=\overline{\phi_0} \quad \text{and} \quad \overline{\partial_t\phi}(t)=0 \qquad \forall\, t\in[0,T]. \tag{A.5}$$

Multiplying $(A.1)_1$ by μ, integrating over Ω, and using the boundary conditions (A.2) and [34, Ch. IV, Lemma 4.3], we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \frac{1}{2}|\nabla\phi|^2+\Psi(\phi)\,\mathrm{d}x\Big)+\|\nabla\mu\|_{L^2}^2+\alpha\|\partial_t\phi\|_{L^2}^2=\int_\Omega \phi\,\boldsymbol{u}\cdot\nabla\mu\,\mathrm{d}x.$$

[1]The interested reader might exploit the combination of the Galerkin method with the approximation of the logarithmic potential by smooth potentials (see, e.g., [32], or [13] for a different approach).

By the Hölder inequality and the boundedness of ϕ, we simply obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \frac{1}{2}|\nabla\phi|^2 + \Psi(\phi)\,\mathrm{d}x\Big)+\frac{1}{2}\|\nabla\mu\|_{L^2}^2 + \alpha\|\partial_t\phi\|_{L^2}^2 \le \frac{1}{2}\|\boldsymbol{u}\|_{L^2}^2.$$

Thus, integrating over $[0,T]$ and using the continuity of Ψ, we have

$$\begin{aligned}&\|\nabla\phi\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla\mu\|_{L^2(0,T;L^2(\Omega))} + \|\partial_t\phi\|_{L^2(0,T;L^2(\Omega))}\\ &\quad\le C_\alpha\big(\sqrt{E_{\mathrm{free}}(\phi_0)} + \|\boldsymbol{u}\|_{L^2(0,T;L^2(\Omega))}\big).\end{aligned} \tag{A.6}$$

In light of (2.1) and (A.5), we infer that

$$\|\phi\|_{L^\infty(0,T;H^1(\Omega))} \le C_\alpha\big(\sqrt{E_{\mathrm{free}}(\phi_0)} + \|\boldsymbol{u}\|_{L^2(0,T;L^2(\Omega))} + |\overline{\phi_0}|\big). \tag{A.7}$$

Now, multiplying $(\text{A.1})_2$ by $-\Delta\phi$ and integrating over Ω, we get

$$\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi\|_{L^2}^2 + \|\Delta\phi\|_{L^2}^2 + \int_\Omega -F'(\phi)\Delta\phi\,\mathrm{d}x = \int_\Omega \nabla\mu\cdot\nabla\phi\,\mathrm{d}x + \theta_0\|\nabla\phi\|_{L^2}^2.$$

The third term on the left-hand side is clearly positive by monotonicity. Then, using (A.7) we obtain

$$\int_0^T \|\Delta\phi(\tau)\|_{L^2}^2\,\mathrm{d}\tau \le \frac{\alpha}{2}\|\nabla\phi_0\|_{L^2}^2 + C_\alpha(1+T)\big(\sqrt{E_{\mathrm{free}}(\phi_0)} + \|\boldsymbol{u}\|_{L^2(0,T;L^2(\Omega))}\big)^2, \tag{A.8}$$

which entails that

$$\|\phi\|_{L^2(0,T;H^2(\Omega))} \le C_\alpha\big(1+\|\nabla\phi_0\|_{L^2} + \sqrt{1+T}\big(\sqrt{E_{\mathrm{free}}(\phi_0)}+\|\boldsymbol{u}\|_{L^2(0,T;L^2(\Omega))}\big)\big). \tag{A.9}$$

Next, we control the total mass of the chemical potential. Arguing as for the Cahn–Hilliard equation, we multiply $(\text{A.1})_2$ by $\phi-\overline{\phi}$ and integrate over Ω. We find

$$\begin{aligned}&\int_\Omega |\nabla\phi|^2\,\mathrm{d}x + \int_\Omega F'(\phi)(\phi-\overline{\phi})\,\mathrm{d}x\\ &\quad= \int_\Omega \mu(\phi-\overline{\phi})\,\mathrm{d}x + \theta_0\|\phi-\overline{\phi}\|_{L^2}^2 - \alpha\int_\Omega \partial_t\phi(\phi-\overline{\phi})\,\mathrm{d}x.\end{aligned}$$

By using the Poincaré inequality and $(\text{A.3})_1$, we find

$$\int_\Omega F'(\phi)(\phi-\overline{\phi})\,\mathrm{d}x \le C_\alpha(1+\|\nabla\mu\|_{L^2} + \|\partial_t\phi\|_{L^2}),$$

for some C_α depending on Ω, θ_0 and α. We are now in position to control a full Sobolev norm of μ. Thanks to [32, Proposition A.1], there exist two positive constants C_1, C_2 (depending only on $\overline{\phi_0}$) such that

$$\int_\Omega |F'(\phi)|\,\mathrm{d}x \le C_1\int_\Omega F'(\phi)(\phi-\overline{\phi_0})\,\mathrm{d}x + C_2,$$

thus we infer that

$$\|F'(\phi)\|_{L^1} \le C_\alpha(1 + \|\nabla\mu\|_{L^2} + \|\partial_t\phi\|_{L^2}).$$

Since $\overline{\mu} = \overline{F'(\phi)} - \theta_0\overline{\phi_0}$, the above control yields

$$|\overline{\mu}| \le C_\alpha(1 + \|\nabla\mu\|_{L^2} + \|\partial_t\phi\|_{L^2}). \tag{A.10}$$

As a result, it immediately follows that

$$\|\mu\|_{L^2(0,T;H^1(\Omega))} \le C_\alpha\big(\sqrt{T} + \sqrt{E_{\text{free}}(\phi_0)} + \|\boldsymbol{u}\|_{L^2(0,T;L^2(\Omega))}\big). \tag{A.11}$$

In addition, by using (A.1)$_1$, we observe that

$$\|\Delta\mu\|_{L^2} \le \|\partial_t\phi\|_{L^2} + \|\boldsymbol{u}\|_{L^3}\|\nabla\phi\|_{L^6}.$$

Then, combining the elliptic regularity with (A.6) and (A.9), we find

$$\begin{aligned}\|\mu\|_{L^2(0,T;H^2(\Omega))} \le C(\alpha, E_{\text{free}}(\phi_0), T)\big((1 + \|\boldsymbol{u}\|_{L^\infty(0,T;L^3(\Omega))})\\ \times (1 + \|\boldsymbol{u}\|_{L^2(0,T;L^2(\Omega))})\big).\end{aligned} \tag{A.12}$$

By comparison with terms in (A.1)$_2$, a similar estimate can be obtained for $F'(\phi)$ in $L^2(0,T;L^2(\Omega))$.

Uniqueness. Let ϕ_1, ϕ_2 be two weak solutions. We define the solutions difference by $\psi = \phi_1 - \phi_2$ which solves

$$\partial_t\psi + \boldsymbol{u}\cdot\nabla\psi = \Delta\big(\alpha\partial_t\psi - \Delta\psi + \Psi'(\phi_1) - \Psi'(\phi_2)\big) \quad \text{in } \Omega\times(0,T).$$

Since $\overline{\psi}(t) = 0$ for all $t \in [0,T]$, multiplying by $A^{-1}\psi$, where the operator A is the Laplace operator $-\Delta$ with homogeneous Neumann boundary conditions, and integrating over Ω, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla A^{-1}\psi\|_{L^2}^2 + \alpha\|\psi\|_{L^2}^2\big) + \|\nabla\psi\|_{L^2}^2 \le \int_\Omega \psi\boldsymbol{u}\cdot\nabla A^{-1}\psi\,\mathrm{d}x + \theta_0\|\psi\|_{L^2}^2.$$

Here, we have used that F' is a monotone function. Observing that

$$\left|\int_\Omega \psi\boldsymbol{u}\cdot\nabla A^{-1}\psi\,\mathrm{d}x\right| \le \|\psi\|_{L^2}\|\boldsymbol{u}\|_{L^3}\|\nabla A^{-1}\psi\|_{L^6} \le C\|\boldsymbol{u}\|_{L^3}\|\psi\|_{L^2}^2,$$

it is easily seen that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla A^{-1}\psi\|_{L^2}^2 + \alpha\|\psi\|_{L^2}^2\big) \le C(1 + \|\boldsymbol{u}\|_{L^3})\|\psi\|_{L^2}^2.$$

An application of the Gronwall lemma yields

$$\|\nabla A^{-1}\psi(t)\|_{L^2}^2 + \alpha\|\psi(t)\|_{L^2}^2 \le \big(\|\nabla A^{-1}\psi(0)\|_{L^2}^2 + \alpha\|\psi(0)\|_{L^2}^2\big)\mathrm{e}^{C_\alpha\int_0^t(1+\|\boldsymbol{u}(\tau)\|_{L^3})\,\mathrm{d}\tau}$$

for all $t \in [0,T]$, which implies the uniqueness of the solution.

Regularity 1. For $h \in (0,1)$, we define the notation $\partial_t^h f(\cdot,t) = \frac{1}{h}(f(\cdot,t+h) - f(\cdot,t))$. We observe that $\phi \in C([0,T];H^1(\Omega))$ and $\boldsymbol{u} \in C([0,T];L^1(\Omega))$; thereby, we can extend both ϕ and $\boldsymbol{u}$ on $[0,T+1]$ by $\phi(t) = \phi(T)$ and $\boldsymbol{u}(t) = \boldsymbol{u}(T)$ for $t \in (T,T+1]$. It follows from (A.1) that

$$\partial_t \partial_t^h \phi + \partial_t^h \boldsymbol{u} \cdot \nabla \phi(\cdot + h) + \boldsymbol{u} \cdot \nabla \partial_t^h \phi = \Delta(\varepsilon \partial_t \partial_t^h \phi - \Delta \partial_t^h \phi + \partial_t^h \Psi'(\phi)) \tag{A.13}$$

in $\Omega \times (0,T)$. We multiply the above equation by $A^{-1}\partial_t^h \phi$ and integrate over Ω. Exploiting the monotonicity of F', the boundary condition of $\boldsymbol{u}$ and the Agmon inequality (2.3), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla A^{-1}\partial_t^h\phi\|_{L^2}^2 + \alpha\|\partial_t^h\phi\|_{L^2}^2\right) + \|\nabla\partial_t^h\phi\|_{L^2}^2 \\
&\quad \le \int_\Omega \phi(\cdot+h)\partial_t^h\boldsymbol{u}\cdot\nabla A^{-1}\partial_t^h\phi\,\mathrm{d}x + \int_\Omega \partial_t^h\phi\,\boldsymbol{u}\cdot\nabla A^{-1}\partial_t^h\phi\,\mathrm{d}x + \theta_0\|\partial_t^h\phi\|_{L^2}^2 \\
&\quad \le \|\partial_t^h\boldsymbol{u}\|_{L^1}\|\nabla A^{-1}\partial_t^h\phi\|_{L^\infty} + \|\partial_t^h\phi\|_{L^2}\|\boldsymbol{u}\|_{L^3}\|\nabla A^{-1}\partial_t^h\phi\|_{L^6} + \theta_0\|\partial_t^h\phi\|_{L^2}^2 \\
&\quad \le C\|\partial_t^h\boldsymbol{u}\|_{L^1}\|\partial_t^h\phi\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_t^h\phi\|_{L^2} + C(1+\|\boldsymbol{u}\|_{L^3})\|\partial_t^h\phi\|_{L^2}^2 \\
&\quad \le \frac{1}{2}\|\nabla\partial_t^h\phi\|_{L^2}^2 + C\|\partial_t^h\boldsymbol{u}\|_{L^1}^{\frac{4}{3}}\left(1+\|\partial_t^h\phi\|_{L^2}^2\right) + C(1+\|\boldsymbol{u}\|_{L^3})\|\partial_t^h\phi\|_{L^2}^2.
\end{aligned}$$

The Gronwall lemma entails

$$\begin{aligned}
&\alpha\|\partial_t^h\phi(t)\|_{L^2}^2 + \int_0^t \|\nabla\partial_t^h\phi(\tau)\|_{L^2}^2\,\mathrm{d}\tau \\
&\quad \le \left(\|\nabla A^{-1}\partial_t^h\phi(0)\|_{L^2}^2 + \alpha\|\partial_t^h\phi(0)\|_{L^2}^2 + C\int_0^t \|\partial_t^h\boldsymbol{u}(\tau)\|_{L^1}^{\frac{4}{3}}\,\mathrm{d}\tau\right)\mathrm{e}^{\int_0^t g(\tau)\,\mathrm{d}\tau}
\end{aligned} \tag{A.14}$$

for all $t \in [0,T]$, where $g(\tau) = C_\alpha(1 + \|\boldsymbol{u}\|_{L^3} + \|\partial_t^h\boldsymbol{u}\|_{L^1}^{\frac{4}{3}})$. In order to control the right-hand side, we compute

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla A^{-1}(\phi-\phi_0)\|_{L^2}^2 + \alpha\|\phi-\phi_0\|_{L^2}^2\right) \\
&\quad = (\alpha\partial_t\phi - \mu, \phi-\phi_0) + (\phi\,\boldsymbol{u}, \nabla A^{-1}(\phi-\phi_0)) \\
&\quad = (\Delta\phi - \Psi'(\phi), \phi-\phi_0) + (\phi\,\boldsymbol{u}, \nabla A^{-1}(\phi-\phi_0)) \\
&\quad = \underbrace{(\Delta(\phi-\phi_0) - (F'(\phi - F'(\phi_0)), \phi-\phi_0)}_{\le 0} + (\Delta\phi_0 - F'(\phi_0), \phi-\phi_0) \\
&\qquad\quad + \theta_0(\phi, \phi-\phi_0) + (\phi\,\boldsymbol{u}, \nabla A^{-1}(\phi-\phi_0)).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla A^{-1}(\phi-\phi_0)\|_{L^2}^2 + \alpha\|\phi-\phi_0\|_{L^2}^2\right) \\
&\quad \le C\left(1 + \|\Delta\phi_0 - F'(\phi_0)\|_{L^2} + \|\boldsymbol{u}\|_{L^2}\right)\|\phi-\phi_0\|_{L^2}.
\end{aligned}$$

Thanks to [34, Chap. IV, Lemma 4.1], we arrive at

$$\begin{aligned}&\|\nabla A^{-1}(\phi(t)-\phi_0)\|_{L^2}^2+\alpha\|\phi(t)-\phi_0\|_{L^2}^2\\&\quad\le\Big(C_\alpha(1+\|\Delta\phi_0-F'(\phi_0)\|_{L^2})t+C_\alpha\int_0^t\|\boldsymbol{u}(\tau)\|_{L^2}\,\mathrm{d}\tau\Big)^2\end{aligned}$$

for all $t\in[0,T]$. By choosing $t=h$, we deduce that

$$\|\nabla A^{-1}\partial_t^h\phi(0)\|_{L^2}^2+\alpha\|\partial_t^h\phi(0)\|_{L^2}^2\le C_\alpha\big(1+\|\Delta\phi_0-F'(\phi_0)\|_{L^2}^2+\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))}^2\big). \tag{A.15}$$

Since

$$\|\partial_t^h\boldsymbol{u}\|_{L^{\frac43}(0,T;L^1(\Omega))}\le\|\partial_t\boldsymbol{u}\|_{L^{\frac43}(0,T;L^1(\Omega))},$$

by combining (A.14) and (A.15), we obtain

$$\alpha\|\partial_t^h\phi(t)\|_{L^2}^2+\int_0^t\|\nabla\partial_t^h\phi(\tau)\|_{L^2}^2\,\mathrm{d}\tau\le C_\alpha D(T)\mathrm{e}^{G(T)}, \tag{A.16}$$

for all $t\in[0,T]$, where

$$\begin{aligned}D(T)&=1+\|\Delta\phi_0-F'(\phi_0)\|_{L^2}^2+\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))}^2+\|\partial_t\boldsymbol{u}\|_{L^{\frac43}(0,T;L^1(\Omega))}^{\frac43},\\G(T)&=\int_0^T C_\alpha(1+\|\boldsymbol{u}(\tau)\|_{L^3})\,\mathrm{d}\tau+C_\alpha\int_0^T\|\partial_t\boldsymbol{u}(\tau)\|_{L^1}^{\frac43}\,\mathrm{d}\tau.\end{aligned}$$

In light of the convergence $\partial_t^h\phi\to\partial_t\phi$ in $L^2(0,T;L^2(\Omega))$ as $h\to0$, we infer that

$$\|\partial_t\phi\|_{L^\infty(0,T;L^2(\Omega))}+\|\partial_t\phi\|_{L^2(0,T;H^1(\Omega))}\le C\big(\alpha,T,\|\Delta\phi_0-F'(\phi_0)\|_{L^2},\|\boldsymbol{u}\|_{X_T}\big), \tag{A.17}$$

where $X_T=L^\infty(0,T;L^3(\Omega))\cap W^{1,\frac43}(0,T;L^1(\Omega))$. Next, we derive further regularity properties on ϕ and μ. By the incompressibility constraint, we recall that $\|\nabla\mu\|_{L^2}\le C(\|\partial_t\phi\|_{L^2}+\|\boldsymbol{u}\|_{L^2})$. Then, thanks to (A.10) and (A.17), we easily have

$$\|\mu\|_{L^\infty(0,T;H^1(\Omega))}\le C\big(\alpha,T,\|\Delta\phi_0-F'(\phi_0)\|_{L^2(\Omega)},\|\boldsymbol{u}\|_{X_T}\big). \tag{A.18}$$

As a consequence, by [25, Theorem A.1], we get

$$\|\phi\|_{L^\infty(0,T;H^2(\Omega))}+\|F'(\phi)\|_{L^\infty(0,T;L^2(\Omega))}\le C\big(\alpha,T,\|\Delta\phi_0-F'(\phi_0)\|_{L^2(\Omega)},\|\boldsymbol{u}\|_{X_T}\big). \tag{A.19}$$

Finally, since we have $\boldsymbol{u}\in L^\infty(0,T;L^3(\Omega))$ and $\nabla\phi\in L^\infty(0,T;L^6(\Omega))$, by comparison with terms in $(A.1)_1$, we also find

$$\|\mu\|_{L^\infty(0,T;H^2(\Omega))}\le C\big(\alpha,T,\|\Delta\phi_0-F'(\phi_0)\|_{L^2(\Omega)},\|\boldsymbol{u}\|_{X_T}\big). \tag{A.20}$$

Regularity 2. Let us now write $(A.1)_2$ as follows:

$$\alpha\partial_t\phi-\Delta\phi+F'(\phi)=h\quad\text{in }\Omega\times(0,T), \tag{A.21}$$

where $h=\mu+\theta_0\phi$. Thanks to (A.20), $h\in L^\infty(0,T;L^\infty(\Omega))$. Next, we consider the

ODE problems

$$\begin{cases} \alpha\partial_t U + F'(U) = \overline{H}, \\ U(0) = 1-\delta_0, \end{cases} \qquad \begin{cases} \alpha\partial_t V + F'(V) = \underline{H}, \\ V(0) = -1+\delta_0 \end{cases} \qquad \text{in } (0,T), \tag{A.22}$$

where $\overline{H} = \|h\|_{L^\infty}$ and $\underline{H} = -\|h\|_{L^\infty}$. It is not difficult to show that there exist two unique solutions $U, V \in C([0,T])$ with $U_t, V_t \in L^\infty(0,T)$. In particular, since we have $\lim_{s\to\pm1} F'(s) = \pm\infty$ and $\overline{H}, \underline{H} \in L^\infty(0,T)$, a simple comparison argument entails that there exists $\delta > 0$ such that

$$-1+\delta \le V(t) \le U(t) \le 1-\delta, \quad \forall\, t \in [0,T].$$

More precisely, it can be checked that $1-\delta \le \max\{1-\delta_0, (F')^{-1}(\|\overline{H}\|_{L^\infty(0,T)})\}$. We are left to show that $V(t) \le \phi(x,t) \le U(t)$ in $\Omega \times [0,T]$. To this aim, we use the Stampacchia method. We define $w = \phi - U$ and we consider the problem

$$\begin{cases} \alpha\partial_t w + \boldsymbol{u}\cdot\nabla\phi - \Delta\phi + F'(\phi) - F'(U) = h - \overline{H} & \text{in } \Omega\times(0,T), \\ w(0) = \phi_0 - 1 + \delta_0 & \text{in } \Omega. \end{cases} \tag{A.23}$$

Multiplying the equation by $w^+ = \max\{\phi - U, 0\}$ and integrating over Ω, and using that $\nabla\phi = \nabla w^+$ on the set $\{x \in \Omega : \phi \le U\}$, we find

$$\begin{aligned} &\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^+\|_{L^2}^2 + \int_\Omega (\boldsymbol{u}\cdot\nabla w^+)w^+\,\mathrm{d}x + \|\nabla w^+\|_{L^2}^2 + \int_\Omega (F'(\phi) - F'(U))w^+\,\mathrm{d}x \\ &\quad = \int_\Omega (h - \overline{H})w^+\,\mathrm{d}x. \end{aligned}$$

By the monotonicity of F', it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w^+\|_{L^2}^2 \le 0 \implies \|w^+(t)\|_{L^2}^2 \le \|w^+(0)\|_{L^2}^2 = 0, \quad \forall\, t \in [0,T],$$

which in turn gives the desired result, namely, $\phi(x,t) \le U(t)$ in $\Omega \times [0,T]$. A similar argument entails that $V(t) \le \phi(x,t)$ in $\Omega \times [0,T]$. Therefore, we obtain by continuity the so-called separation property

$$\max_{(x,t)\in\overline{\Omega}\times[0,T]} |\phi(x,t)| \le 1-\delta. \tag{A.24}$$

As a consequence, it follows from (A.19) that $\Psi'(\phi) \in L^\infty(0,T;H^1(\Omega))$. Then, we deduce by comparison with terms in $(\text{A.1})_2$ and by elliptic regularity that

$$\|\phi\|_{L^2(0,T;H^3(\Omega))} \le C\big(\alpha, T, \delta, \|\Delta\phi_0 - F'(\phi_0)\|_{L^2}, \|\boldsymbol{u}\|_{X_T}\big).$$

Regularity 3. Thanks to the above regularity, we rewrite (A.13) as follows

$$\int_\Omega \partial_t \partial_t^h \phi\, v + \alpha \nabla \partial_t \partial_t^h \phi \cdot \nabla v \,\mathrm{d}x + \int_\Omega \partial_t^h (\boldsymbol{u} \cdot \nabla \phi) v \,\mathrm{d}x \\ = \int_\Omega (\nabla \Delta \partial_t^h \phi - \nabla \partial_t^h \Psi'(\phi)) \cdot \nabla v \,\mathrm{d}x \tag{A.25}$$

for all $v \in H^1(\Omega)$. Taking $v = \partial_t^h \phi$ and exploiting the boundary conditions of ϕ and $\boldsymbol{u}$, we find

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\partial_t^h \phi\|_{L^2}^2 + \alpha \|\nabla \partial_t^h \phi\|_{L^2}^2\big) + \int_\Omega |\Delta \partial_t^h \phi|^2 \,\mathrm{d}x \\ &\quad = \int_\Omega \partial_t^h (\boldsymbol{u}\phi) \cdot \nabla \partial_t^h \phi \,\mathrm{d}x + \int_\Omega \partial_t^h F'(\phi) \Delta \partial_t^h \phi \,\mathrm{d}x + \theta_0 \|\nabla \partial_t^h \phi\|_{L^2}^2 \\ &\quad \leq \|\partial_t^h \boldsymbol{u}\|_{L^{\frac{6}{5}}} \|\nabla \partial_t^h \phi\|_{L^6} + \|\boldsymbol{u}\|_{L^3} \|\partial_t^h \phi\|_{L^6} \|\nabla \partial_t^h \phi\|_{L^2} + C \|\partial_t^h \phi\|_{L^2} \|\Delta \partial_t^h \phi\|_{L^2} \\ &\qquad + \theta_0 \|\nabla \partial_t^h \phi\|_{L^2(\Omega)}^2 \\ &\quad \leq \frac{1}{2}\|\Delta \partial_t^h \phi\|_{L^2}^2 + C \|\partial_t^h \boldsymbol{u}\|_{L^{\frac{6}{5}}}^2 + C\,(1 + \|\boldsymbol{u}\|_{L^3})\, \|\nabla \partial_t^h \phi\|_{L^2}^2 + C \|\partial_t^h \phi\|_{L^2}^2. \end{aligned}$$

Here, we used separation property (A.24) and the inequality $\|\partial_t^h \phi\|_{H^2} \leq C \|\Delta \partial_t^h \phi\|_{L^2}$. Then, we infer from the Gronwall lemma that

$$\|\partial_t^h \phi(t)\|_{L^2}^2 + \alpha \|\nabla \partial_t^h \phi(t)\|_{L^2}^2 + \int_0^t \|\Delta \partial_t^h \phi(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \\ \leq \Big(\|\partial_t^h \phi(0)\|_{L^2}^2 + \alpha \|\nabla \partial_t^h \phi(0)\|_{L^2}^2 + C \int_0^t \|\partial_t^h \boldsymbol{u}(\tau)\|_{L^{\frac{6}{5}}}^2 \,\mathrm{d}\tau\Big) \mathrm{e}^{\widetilde{G}(T)} \tag{A.26}$$

for all $t \in [0, T]$, where

$$\widetilde{G}(T) = C_\alpha \int_0^T (1 + \|\boldsymbol{u}(\tau)\|_{L^3}) \,\mathrm{d}\tau.$$

Since $\partial_n \phi_0 = 0$ on $\partial\Omega$ by assumption, we observe that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\phi - \phi_0\|_{L^2}^2 + \alpha \|\nabla(\phi - \phi_0)\|_{L^2}^2\big) \\ &\quad = \int_\Omega \phi\, \boldsymbol{u} \cdot \nabla(\phi - \phi_0) \,\mathrm{d}x + \int_\Omega \nabla(\Delta \phi - F'(\phi) + \theta_0 \phi) \cdot \nabla(\phi - \phi_0) \,\mathrm{d}x \\ &\quad = \int_\Omega \phi\, \boldsymbol{u} \cdot \nabla(\phi - \phi_0) \,\mathrm{d}x - \|\Delta(\phi - \phi_0)\|_{L^2}^2 + \int_\Omega \nabla \Delta \phi_0 \cdot \nabla(\phi - \phi_0) \,\mathrm{d}x \\ &\qquad + \int_\Omega \nabla(-F'(\phi) + \theta_0 \phi) \cdot \nabla(\phi - \phi_0) \,\mathrm{d}x. \end{aligned}$$

Thus, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\phi - \phi_0\|_{L^2}^2 + \alpha \|\nabla(\phi - \phi_0)\|_{L^2}^2\big) \leq C\big(1 + \|\boldsymbol{u}\|_{L^2} + \|\phi_0\|_{H^3}\big) \|\nabla(\phi - \phi_0)\|_{L^2}.$$

By using [34, Chap. IV, Lemma 4.1] and taking $t = h$, we arrive at

$$\|\partial_t^h \phi(0)\|_{L^2}^2 + \alpha \|\nabla \partial_t^h \phi(0)\|_{L^2}^2 \le C_\alpha \big(1 + \|\phi_0\|_{H^3}^2 + \|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))}^2\big). \tag{A.27}$$

Combining the above inequality with (A.26), we are led to

$$\begin{aligned}&\|\partial_t^h \phi(t)\|_{L^2}^2 + \alpha \|\nabla \partial_t^h \phi(t)\|_{L^2}^2 + \int_0^t \|\Delta \partial_t^h \phi(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \\ &\quad \le C_\alpha \big(1 + \|\phi_0\|_{H^3}^2 + \|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\partial_t \boldsymbol{u}\|^2_{L^2(0,T+1;L^{\frac{6}{5}}(\Omega))}\big) \mathrm{e}^{C \int_0^t (1+\|\boldsymbol{u}(\tau)\|_{L^3})\,\mathrm{d}\tau}\end{aligned}$$

for all $t \in [0,T]$, which, in turn, implies

$$\|\partial_t \phi\|_{L^\infty(0,T;H^1(\Omega))} + \|\partial_t \phi\|_{L^2(0,T;H^2(\Omega))} \le C(\alpha, T, \delta, \|\phi_0\|_{H^3}, \|\boldsymbol{u}\|_{Y_T}), \tag{A.28}$$

where $Y_T = L^\infty(0,T;L^3(\Omega)) \cap W^{1,2}(0,T;L^{\frac{6}{5}}(\Omega))$. As an immediate consequence, in light of (A.19), (A.20) and (A.24), we infer by comparison with terms in $(\text{A.1})_2$ and by elliptic regularity (cf. the fact that Ω is C^3) that

$$\|\phi\|_{L^\infty(0,T;H^3(\Omega))} \le C(\alpha, T, \delta, \|\phi_0\|_{H^3(\Omega)}, \|\boldsymbol{u}\|_{Y_T}). \tag{A.29}$$

Next, we take $v = A^{-1} \partial_t^h \partial_t \phi$ in (A.25). Exploiting (A.24) and (A.28), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \partial_t^h \phi\|_{L^2}^2 + \|\nabla A^{-1} \partial_t^h \partial_t \phi\|_{L^2}^2 + \alpha \|\partial_t^h \partial_t \phi\|_{L^2}^2 \\ &\quad \le \int_\Omega \partial_t^h (\phi \boldsymbol{u}) \cdot \nabla A^{-1} \partial_t^h \partial_t \phi \, \mathrm{d}x - \int_\Omega \partial_t^h \Psi'(\phi) \partial_t^h \partial_t \phi \, \mathrm{d}x \\ &\quad \le C \|\partial_t \boldsymbol{u}\|_{L^{\frac{6}{5}}} \|\partial_t^h \partial_t \phi\|_{L^2} + C \|\boldsymbol{u}\|_{L^3} \|\partial_t \phi\|_{L^2} \|\nabla A^{-1} \partial_t^h \partial_t \phi\|_{L^6} + C \|\partial_t^h \phi\|_{L^2} \|\partial_t^h \partial_t \phi\|_{L^2} \\ &\quad \le \frac{1}{2} \|\partial_t^h \partial_t \phi\|_{L^2}^2 + C\big(1 + \|\partial_t \boldsymbol{u}\|_{L^{\frac{6}{5}}}^2 + \|\boldsymbol{u}\|_{L^3}^2\big).\end{aligned}$$

By recalling (A.27), the Gronwall lemma entails

$$\int_0^T \|\partial_t^h \partial_t \phi\|_{L^2}^2 \, \mathrm{d}\tau \le C(\alpha, T, \delta, \|\phi_0\|_{H^3}, \|\boldsymbol{u}\|_{Y_T}), \tag{A.30}$$

which, in turn, gives that there exists $\partial_t^2 \phi \in L^2(0,T;L^2(\Omega))$ such that

$$\|\partial_t^2 \phi\|_{L^2(0,T;L^2(\Omega))} \le C(\alpha, T, \delta, \|\phi_0\|_{H^3}, \|\boldsymbol{u}\|_{Y_T}).$$

Thus, by comparison with terms in (A.1), we conclude that there exists $\partial_t \mu \in L^2(0,T; L^2(\Omega))$ such that

$$\|\partial_t \mu\|_{L^2(0,T;L^2(\Omega))} \le C(\alpha, T, \delta, \|\phi_0\|_{H^3(\Omega)}, \|\boldsymbol{u}\|_{Y_T}).$$

The proof is complete. ∎

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References

[1] H. Abels, Existence of weak solutions for a diffuse interface model for viscous, incompressible fluids with general densities. *Comm. Math. Phys.* **289** (2009), no. 1, 45–73 Zbl 1165.76050 MR 2504845

[2] H. Abels, On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities. *Arch. Ration. Mech. Anal.* **194** (2009), no. 2, 463–506 Zbl 1254.76158 MR 2563636

[3] H. Abels, Strong well-posedness of a diffuse interface model for a viscous, quasi-incompressible two-phase flow. *SIAM J. Math. Anal.* **44** (2012), no. 1, 316–340 Zbl 1333.76079 MR 2888290

[4] H. Abels and D. Breit, Weak solutions for a non-Newtonian diffuse interface model with different densities. *Nonlinearity* **29** (2016), no. 11, 3426–3453 Zbl 1354.35084 MR 3567095

[5] H. Abels, D. Depner, and H. Garcke, Existence of weak solutions for a diffuse interface model for two-phase flows of incompressible fluids with different densities. *J. Math. Fluid Mech.* **15** (2013), no. 3, 453–480 Zbl 1273.76421 MR 3084319

[6] H. Abels, D. Depner, and H. Garcke, On an incompressible Navier-Stokes/Cahn-Hilliard system with degenerate mobility. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **30** (2013), no. 6, 1175–1190 Zbl 1347.76052 MR 3132421

[7] H. Abels and E. Feireisl, On a diffuse interface model for a two-phase flow of compressible viscous fluids. *Indiana Univ. Math. J.* **57** (2008), no. 2, 659–698 Zbl 1144.35041 MR 2414331

[8] H. Abels and H. Garcke, Weak solutions and diffuse interface models for incompressible two-phase flows. In *Handbook of mathematical analysis in mechanics of viscous fluids*, pp. 1267–1327, Springer, Cham, 2018 MR 3916798

[9] H. Abels, H. Garcke, and G. Grün, Thermodynamically consistent, frame indifferent diffuse interface models for incompressible two-phase flows with different densities. *Math. Models Methods Appl. Sci.* **22** (2012), no. 3, 1150013 Zbl 1242.76342 MR 2890451

[10] H. Abels and Y. Terasawa, Weak solutions for a diffuse interface model for two-phase flows of incompressible fluids with different densities and nonlocal free energies. *Math. Methods Appl. Sci.* **43** (2020), no. 6, 3200–3219 Zbl 1454.35251 MR 4085477

[11] H. Abels and Y. Terasawa, Convergence of a nonlocal to a local diffuse interface model for two-phase flow with unmatched densities. *Discrete Contin. Dyn. Syst. Ser. S* **15** (2022), no. 8, 1871–1881. MR 4438771

[12] H. Abels and J. Weber, Local well-posedness of a quasi-incompressible two-phase flow. *J. Evol. Equ.* **21** (2021), no. 3, 3477–3502 Zbl 07451412 MR 4350280

[13] E. Bonetti, P. Colli, L. Scarpa, and G. Tomassetti, Bounded solutions and their asymptotics for a doubly nonlinear Cahn-Hilliard system. *Calc. Var. Partial Differential Equations* **59** (2020), no. 2, paper no. 88 Zbl 1445.35053 MR 4090479

[14] F. Boyer, A theoretical and numerical model for the study of incompressible mixture flows. *Comput. Fluids* **31** (2001), no. 1, 41–68 Zbl 1057.76060

[15] F. Boyer, Nonhomogeneous Cahn-Hilliard fluids. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **18** (2001), no. 2, 225–259 Zbl 1037.76062 MR 1808030

[16] L. Cherfils, E. Feireisl, M. Michálek, A. Miranville, M. Petcu, and D. Pražák, The compressible Navier-Stokes-Cahn-Hilliard equations with dynamic boundary conditions. *Math. Models Methods Appl. Sci.* **29** (2019), no. 14, 2557–2584 MR 4053238

[17] E. Davoli, L. Scarpa, and L. Trussardi, Local asymptotics for nonlocal convective Cahn-Hilliard equations with $W^{1,1}$ kernel and singular potential. *J. Differential Equations*, **289** (2021), 35—58. Zbl 1465.45014 MR 4248454

[18] H. Ding, P. D. M. Spelt, and C. Shu, Diffuse interface model for incompressible two-phase flows with large density ratios, *J. Comput. Phys.* **226** (2007), no. 2, 2078–2095 Zbl 1388.76403

[19] S. Frigeri, Global existence of weak solutions for a nonlocal model for two-phase flows of incompressible fluids with unmatched densities. *Math. Models Methods Appl. Sci.* **26** (2016), no. 10, 1955–1993 Zbl 1346.76203 MR 3540647

[20] S. Frigeri, On a nonlocal Cahn-Hilliard/Navier-Stokes system with degenerate mobility and singular potential for incompressible fluids with different densities. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **38** (2021), no. 3, 647–687 Zbl 1464.35179 MR 4227048

[21] C. G. Gal, M. Grasselli, and H. Wu, Global weak solutions to a diffuse interface model for incompressible two-phase flows with moving contact lines and different densities. *Arch. Ration. Mech. Anal.* **234** (2019), no. 1, 1–56 Zbl 1444.76001 MR 3981392

[22] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I.* Springer Tracts Nat. Philos. 39, Springer, New York, 1994 Zbl 0949.35004 MR 1284206

[23] M.-H. Giga, A. Kirshtein, and C. Liu, Variational modeling and complex fluids. In *Handbook of mathematical analysis in mechanics of viscous fluids*, pp. 73–113, Springer, Cham, 2018 MR 3916774

[24] A. Giorgini, Well-posedness of the two-dimensional Abels-Garcke-Grün model for two-phase flows with unmatched densities. *Calc. Var. Partial Differential Equations* **60** (2021), no. 3, paper no. 100 Zbl 1468.35134 MR 4249878

[25] A. Giorgini, A. Miranville, and R. Temam, Uniqueness and regularity for the Navier-Stokes-Cahn-Hilliard system. *SIAM J. Math. Anal.* **51** (2019), no. 3, 2535–2574 Zbl 1419.35160 MR 3968246

[26] A. Giorgini and R. Temam, Weak and strong solutions to the nonhomogeneous incompressible Navier-Stokes-Cahn-Hilliard system. *J. Math. Pures Appl. (9)* **144** (2020), 194–249 Zbl 1452.35151 MR 4175449

[27] M. E. Gurtin, D. Polignone, and J. Viñals, Two-phase binary fluids and immiscible fluids described by an order parameter. *Math. Models Methods Appl. Sci.* **6** (1996), no. 6, 815–831 Zbl 0857.76008 MR 1404829

[28] M. Heida, J. Málek, and K. R. Rajagopal, On the development and generalizations of Cahn-Hilliard equations within a thermodynamic framework. *Z. Angew. Math. Phys.* **63** (2012), no. 1, 145–169 Zbl 1295.76002 MR 2878737

[29] M. Kotschote and R. Zacher, Strong solutions in the dynamical theory of compressible fluid mixtures. *Math. Models Methods Appl. Sci.* **25** (2015), no. 7, 1217–1256 Zbl 1329.76060 MR 3335530

[30] J. Lowengrub and L. Truskinovsky, Quasi-incompressible Cahn-Hilliard fluids and topological transitions. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **454** (1998), no. 1978, 2617–2654 Zbl 0927.76007 MR 1650795

[31] M. Marcus and V. J. Mizel, Absolute continuity on tracks and mappings of Sobolev spaces. *Arch. Rational Mech. Anal.* **45** (1972), 294–320 Zbl 0236.46033 MR 338765

[32] A. Miranville and S. Zelik, Robust exponential attractors for Cahn-Hilliard type equations with singular potentials. *Math. Methods Appl. Sci.* **27** (2004), no. 5, 545–582 Zbl 1050.35113 MR 2041814

[33] M. Shokrpour Roudbari, G. Şimşek, E. H. van Brummelen, and K. G. van der Zee, Diffuse-interface two-phase flow models with different densities: a new quasi-incompressible form and a linear energy-stable method. *Math. Models Methods Appl. Sci.* **28** (2018), no. 4, 733–770 Zbl 1390.76047 MR 3786793

[34] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*. Math. Surv. Monogr. 49, American Mathematical Society, Providence, RI, 1997 Zbl 0870.35004 MR 1422252

[35] X. Zhao, Strong solutions to the density-dependent incompressible Cahn-Hilliard-Navier-Stokes system. *J. Hyperbolic Differ. Equ.* **16** (2019), no. 4, 701–742 Zbl 1441.35205 MR 4070940

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