Interfaces Free Bound. 25 (2023), 1–36
DOI 10.4171/IFB/486

Eddy current approximation in Maxwell obstacle problems

Maurice Hensel and Irwin Yousept

Abstract. This paper analyzes the mathematical modeling of the transient eddy current approximation in the Maxwell obstacle problem. Here, the medium is assumed to be solely open, containing conducting and non-conducting materials with certain properties of anisotropy and non-smoothness. The proposed evolutionary PDE model preserves the Faraday law and excludes the displacement current from the governing Ampère–Maxwell variational inequality (VI). Our study strives to justify this model and delivers two main results: Global well-posedness of the model and its quantitative precision by uniform a priori estimates. The latter result yields an explicit bound for the smallness condition on the ratio between the electric permittivity and the electric conductivity in the region where the displacement current is disregarded. Below this threshold, the eddy current solution provides the desired reasonable approximation and justifies the proposed model.

1. Introduction

Maxwell obstacle problems describe the dynamics and propagation of electromagnetic (EM) fields under the influence of constraints. For instance, in electromagnetic shielding, certain magnetic and conducting materials may serve as a barrier to redirect or block EM fields in a specific domain of interest. From the mathematical perspective, this phenomenon falls into the class of Maxwell obstacle problems: In the free region, EM fields satisfy Maxwell's equations, while in the shielded area (unilateral or bilateral) constraints are imposed on the fields. The very first contribution to this research direction was made by Duvaut and Lions [11, Chapter 7, Section 8], who explored and analyzed the electromagnetic wave propagation in a polarizable medium through a Maxwell obstacle problem involving an electric constraint of the type

$$|\mathbf{E}(x,t)| \leq d(x) \quad \text{a.e. in } \Omega \times (0,T)$$

for some obstacle $d : \Omega \to [0,\infty]$. Based on the method of vanishing **curl curl**-viscosity and constraint penalization, they proved a well-posedness result [11, Chapter 7, Theorem 8.1] which was modified some years later by Milani [21, 22] to the case of a time-

2020 Mathematics Subject Classification. Primary 35Q60; Secondary 35L85.
Keywords. Eddy current modeling, Maxwell variational inequalities, well-posedness, a priori analysis.

dependent obstacle $d = d(x,t)$. More recently, building on [34], the second author [35] refined the developed theory by Duvaut and Lions [11] to allow a more general constraint structure.

Let us formulate the Maxwell obstacle problem we focus on in this paper: Suppose that $\Omega \subset \mathbb{R}^3$ is an open set representing an anisotropic medium in which the electric field $\mathbf{E} : \Omega \times (0,T) \to \mathbb{R}^3$ and the magnetic field $\mathbf{H} : \Omega \times (0,T) \to \mathbb{R}^3$ are acting in a finite time interval $(0,T)$. Furthermore, let $\mathbf{0} \in \mathbf{K} \subset \mathbf{L}^2(\Omega)$ denote a convex and closed subset representing the underlying feasible set for the electric field. Given initial data $(\mathbf{E}_0, \mathbf{H}_0) \in (\mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})) \times \mathbf{H}(\mathbf{curl})$ and an applied current source $\mathbf{f} \in W^{1,\infty}((0,T), \mathbf{L}^2(\Omega))$, find a unique solution $(\mathbf{E},\mathbf{H}) \in W^{1,\infty}((0,T), \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)) \cap L^\infty((0,T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega))$ such that

$$\begin{cases} \displaystyle\int_\Omega \varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) + \sigma \mathbf{E}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) - \mathbf{H}(t) \cdot \mathbf{curl}(\mathbf{v} - \mathbf{E}(t)) \,\mathrm{d}x \\ \qquad \displaystyle\geq \int_\Omega \mathbf{f}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) \,\mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl}) \text{ and for a.e. } t \in (0,T); \\ \displaystyle\mu \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}(t) + \mathbf{curl}\,\mathbf{E}(t) = \mathbf{0} \quad \text{for a.e. } t \in (0,T); \\ \mathbf{E}(t) \in \mathbf{K} \quad \text{for a.e. } t \in (0,T); \\ (\mathbf{E},\mathbf{H})(0) = (\mathbf{E}_0, \mathbf{H}_0). \end{cases} \tag{P}$$

Here, $\varepsilon, \mu, \sigma : \Omega \to \mathbb{R}^{3\times 3}$ denote the electric permittivity, the magnetic permeability, and the electric conductivity, respectively. All these coefficients are allowed to be non-smooth. Moreover, as the medium Ω may contain different conducting and non-conducting materials, the electric conductivity σ is assumed to be merely positive semi-definite. The precise mathematical assumptions for all data involved in (P) are specified in Assumption 1.1. Note that (P) preserves the Faraday law but modifies the Ampère–Maxwell equation into a variational inequality of the first kind. The global well-posedness of (P) for $\sigma \equiv 0$ is a special case of [35, Theorems 1 and 2] and can be extended to the case of a non-vanishing conductivity σ.

The present paper aims to explore the eddy current (magneto-quasistatic) approximation to (P) and its justification. Our analysis is mainly motivated by the profound role of eddy current modeling in electrical engineering applications and low-frequency physics. Generally speaking, the eddy current model approximates the full Maxwell system by excluding the displacement current $\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}$ while still preserving the Faraday law. Such approximations are widely used in the engineering community and are particularly reasonable if the electric permittivity is significantly smaller than the electric conductivity, and the corresponding wavelength is much larger than the diameter of Ω. From among many other contributions to the eddy current model, we refer to the monographs by Alonso and Valli [3] and Bossavit [6,8], and papers [1,4,5,10,14,15,23,26,28,29,31]. While the mathematical and numerical analysis for the eddy current equations seems to have reached an advanced stage of development, so far, we are not aware of any previous study regarding the justification of eddy current modeling for (P).

We focus on an eddy current model allowing the displacement current to be disregarded in an open conducting subregion $\Omega_\sigma \subset \Omega$. More precisely, we look for a unique solution $(\mathbf{E}, \mathbf{H}) \in W^{1,\infty}((0,T), \mathbf{L}^2(\Omega\setminus\Omega_\sigma)\times\mathbf{L}^2(\Omega)) \cap L^\infty((0,T), \mathbf{H}_0(\mathbf{curl})\times\mathbf{L}^2(\Omega))$ to

$$\begin{cases} \displaystyle\int_{\Omega\setminus\Omega_\sigma} \varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(t)\cdot(\mathbf{v}-\mathbf{E}(t))\,\mathrm{d}x + \int_\Omega \sigma\mathbf{E}(t)\cdot(\mathbf{v}-\mathbf{E}(t)) - \mathbf{H}(t)\cdot\mathbf{curl}(\mathbf{v}-\mathbf{E}(t))\,\mathrm{d}x \\ \qquad \displaystyle\geq \int_\Omega \mathbf{f}(t)\cdot(\mathbf{v}-\mathbf{E}(t))\,\mathrm{d}x, \quad \forall\, \mathbf{v} \in \mathbf{K}\cap\mathbf{H}_0(\mathbf{curl}) \text{ and for a.e. } t\in(0,T); \\ \mu\dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}(t) + \mathbf{curl}\,\mathbf{E}(t) = \mathbf{0} \quad \text{for a.e. } t\in(0,T); \\ \mathbf{E}(t)\in\mathbf{K} \quad \text{for a.e. } t\in(0,T); \\ (\mathbf{E},\mathbf{H})(0) = (\mathbf{E}_0,\mathbf{H}_0) \quad \text{a.e. in } (\Omega\setminus\Omega_\sigma)\times\Omega. \end{cases} \tag{P$_{\rm ec}$}$$

To justify the eddy current model ($\mathrm{P_{ec}}$), there are two open mathematical questions to be rigorously addressed and answered. First, the model itself (see ($\mathrm{P_{ec}}$)) has to be reasonable in the sense that there exists a unique solution to it. Second, under a suitable condition, its solution must provide a good estimation for the original problem (see (P)). In particular, inspired by the time-harmonic case [3], ($\mathrm{P_{ec}}$) should serve as a reasonable approximation if the quantity $\underline{\sigma}^{-1}\|\varepsilon\|_{L^\infty(\Omega_\sigma)^{3\times3}}$ is small enough, with $\underline{\sigma} > 0$ denoting a uniform lower bound for the lowest eigenvalues of $\sigma(x)$ for almost all $x \in \Omega_\sigma$.

This paper develops three novelties delivering positive answers to the issues mentioned above. The first novelty concerns the a priori analysis for the time-discrete approximation (P_N) of ($\mathrm{P_{ec}}$) based on the Rothe method. Here, our analysis hinges on the mild compatibility assumption (1.8) for the initial data $(\mathbf{E}_0, \mathbf{H}_0)$ in the subset Ω_σ. With the compatibility condition, we prove the stability of (P_N) (see Theorem 2.1) through the use of special correction terms developed using the variational inequality structure of ($\mathrm{P_{ec}}$). Then, the analysis for the time-discrete scheme (P_N) allows us to establish a well-posedness result for ($\mathrm{P_{ec}}$) as the second novelty of this paper (Theorem 3.1). To be more precise, applying the stability result to a specific interpolation of (P_N) and passing to the limit in the time discretization, the weak-star limit of the interpolation turns to satisfy ($\mathrm{P_{ec}}$), leading to an existence result for ($\mathrm{P_{ec}}$). We note that the standard technique of passing to the weak-star limit of the piecewise linear interpolations fails to work, as (P_N) does not admit sufficient stability of its solutions in Ω_σ. This difficulty is overcome by considering the weak-star limit of the piecewise constant interpolations. Let us point out that ($\mathrm{P_{ec}}$) does not exclude the displacement current $\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}$ in $\Omega\setminus\Omega_\sigma$. In Section 3.1, we address the case described by ($\mathrm{P^0_{ec}}$) where the displacement current is entirely neglected both in the conducting and non-conducting regions (Ω_σ and $\Omega\setminus\Omega_\sigma$). It turns out that the proposed techniques for ($\mathrm{P_{ec}}$) can be extended to ($\mathrm{P^0_{ec}}$), leading to a well-posedness result for ($\mathrm{P^0_{ec}}$) (see Theorem 3.2) under additional assumptions (see Assumption 3.2). Our final result is the justification for (P_{ec}) (see Theorem 4.1): If $|\Omega \setminus \Omega_\sigma| \neq 0$, then the solution $(\mathbf{E}_{ec}, \mathbf{H}_{ec})$ of (P_{ec}) approximates the solution $(\mathbf{E}, \mathbf{H})$ to (P) through the following uniform a priori estimate:

$$\begin{aligned} &\|(\mathbf{E}, \mathbf{H}) - (\mathbf{E}_{ec}, \mathbf{H}_{ec})\|_{\mathcal{C}([0,T],\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)\times\mathbf{L}^2_\mu(\Omega))} + \|\mathbf{E} - \mathbf{E}_{ec}\|_{L^2((0,T),\mathbf{L}^2_\sigma(\Omega_\sigma))} \qquad (1.1)\\ &\leq 2\left(\frac{L(\Omega_\sigma)^2 T}{\underline{\sigma}} + \frac{2L(\Omega\setminus\Omega_\sigma)T}{\sqrt{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)}}\sqrt{\frac{4L(\Omega\setminus\Omega_\sigma)^2T^2}{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)} + \frac{2L(\Omega_\sigma)^2T}{\underline{\sigma}}}\right)^{1/2} \left\|\frac{\varepsilon}{\underline{\sigma}}\right\|_{L^\infty(\Omega_\sigma)^{3\times3}}, \end{aligned}$$

where $L(\Omega_\sigma) > 0$ (resp. $L(\Omega \setminus \Omega_\sigma) > 0$) stands for the Lipschitz constant of $\mathbf{f}_{|\Omega_\sigma}$ (resp. $\mathbf{f}_{|\Omega\setminus\Omega_\sigma}$), and $\underline{\varepsilon}(\Omega \setminus \Omega_\sigma) > 0$ denotes a uniform lower bound for the lowest eigenvalues of $\varepsilon(x)$ for almost all $x \in \Omega \setminus \Omega_\sigma$. If $\Omega_\sigma = \Omega$, that is, if the displacement current is completely removed in the conducting medium Ω, then the following precision is obtained for the eddy current approximation:

$$\|\mathbf{H} - \mathbf{H}_{ec}\|_{\mathcal{C}([0,T],\mathbf{L}^2_\mu(\Omega))} + \|\mathbf{E} - \mathbf{E}_{ec}\|_{L^2((0,T),\mathbf{L}^2_\sigma(\Omega))} \leq \frac{2L(\Omega)\sqrt{T}}{\sqrt{\underline{\sigma}}}\left\|\frac{\varepsilon}{\underline{\sigma}}\right\|_{L^\infty(\Omega)^{3\times3}}. \qquad (1.2)$$

We emphasize that in many electromagnetic applications (see, e.g., [3, 20]), the ratio $\|\varepsilon/\underline{\sigma}\|_{L^\infty(\Omega_\sigma)^{3\times3}}$ is often negligibly small. For instance, stainless steel and copper admit the value $6.14 \cdot 10^{-18}$ and $1.56 \cdot 10^{-19}$ for the corresponding ratio, respectively. This property is in particular satisfied by every good conductor Ω_σ (see [20]) as the electric permittivity ε is in this case very close to the one in a vacuum ($\approx 8.85 \cdot 10^{-12}$), and the electric conductivity σ is in the order of 10^6–10^7. Therefore, the achieved estimation reveals the desired approximation by the eddy current solution with a specific bound for the smallness condition on the quantity $\|\varepsilon/\underline{\sigma}\|_{L^\infty(\Omega_\sigma)^{3\times3}}$. At the same time, it guarantees the strong convergence of (P_{ec}) towards (P) with a linear convergence rate in terms of $\|\varepsilon\|_{L^\infty(\Omega_\sigma)^{3\times3}}$. Last but not least, all theoretical results of this paper, in particular uniform estimates (1.1) and (1.2), also apply to the classical Maxwell equations by simply considering $\mathbf{K} = \mathbf{L}^2(\Omega)$.

Another important application of the eddy current model arises in the context of type-II superconductivity. The corresponding model leads to parabolic obstacle problems with first-order gradient or **curl** constraints. We refer to [7,12,24,27,30,32,33] for contributions in this research direction. More recently, a unified analysis for non-linear parabolic obstacle problems, including those with **curl**-type constraints, has been recently developed by Miranda et al. [25].

The remainder of this paper is organized as follows: In the upcoming section, we introduce our notation as well as basic properties and present the required assumptions for our analysis. The subsequent section includes the formulation of the time-discrete scheme together with its associated a priori stability analysis (Theorem 2.1). Section 3 is devoted to the existence and uniqueness analysis (Theorems 3.1 and 3.2) for (P_{ec}) and its full eddy current version (P^0_{ec}). In Section 4, we prove Theorem 4.1 for the justification of the eddy current model, and the final section features a numerical test verifying the a priori estimate and the predicted convergence rate (see Theorem 4.1).

1.1. Preliminaries

Given a real Hilbert space H, we denote by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ its scalar product and induced norm, respectively. In the case of $H = \mathbb{R}^d$, we simply write a dot and $|\cdot|$ for the Euclidean scalar product and norm, respectively. Note that given $a_0, \dots, a_{n_0} \in H$ for $n_0 \in \mathbb{N}$, the binomial-type formula

$$\sum_{n=1}^{n_0} (a_n - a_{n-1}, a_n)_H = \frac{1}{2}\Big(\|a_{n_0}\|_H^2 - \|a_0\|_H^2 + \sum_{n=1}^{n_0} \|a_n - a_{n-1}\|_H^2\Big) \qquad (1.3)$$

holds. Discussing problems of Maxwell-type, there naturally arise function spaces of $\mathbb{R}^3$-valued functions. We will therefore use a bold typeface to indicate them. Given an open set $\Omega \subset \mathbb{R}^3$, let $\mathbf{L}^2(\Omega)$ denote the space of all (equivalence classes of) $\mathbb{R}^3$-valued Lebesgue square-integrable functions. We introduce the Hilbert space

$$\mathbf{H}(\mathbf{curl}) := \big\{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl}\,\mathbf{u} \in \mathbf{L}^2(\Omega)\big\}$$

endowed with its natural graph norm. Here the **curl** operator is to be understood in the sense of distributions. Furthermore, let $\boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ denote the space of infinitely differentiable $\mathbb{R}^3$-valued functions with compact support in Ω. The subspace $\mathbf{H}_0(\mathbf{curl})$ stands for the closure of $\boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ with respect to the $\mathbf{H}(\mathbf{curl})$ topology. We recall that $\mathbf{H}_0(\mathbf{curl})$ admits the useful characterization

$$\mathbf{H}_0(\mathbf{curl}) = \big\{\mathbf{z} \in \mathbf{H}(\mathbf{curl}) \mid (\mathbf{z}, \mathbf{curl}\,\mathbf{v})_{\mathbf{L}^2(\Omega)} = (\mathbf{curl}\,\mathbf{z}, \mathbf{v})_{\mathbf{L}^2(\Omega)}\ \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl})\big\}. \qquad (1.4)$$

By $L^\infty_{\mathrm{sym}}(\Omega)^{3\times3}$ we denote the space of all (equivalence classes of) symmetric $\mathbb{R}^{3\times3}$-valued Lebesgue measurable and essentially bounded functions with respect to the spectral norm, that is,

$$\|\alpha\|_{L^\infty(\Omega)^{3\times3}} := \operatorname*{ess\,sup}_{x\in\Omega} \max_{|\xi|\le1} |\alpha(x)\xi| < \infty, \qquad \forall \alpha \in L^\infty_{\mathrm{sym}}(\Omega)^{3\times3}. \qquad (1.5)$$

For a given uniformly positive definite matrix-valued function $\alpha \in L^\infty_{\mathrm{sym}}(\Omega)^{3\times3}$, that is, there exists a constant $\underline{\alpha} > 0$ such that

$$\alpha(x)\xi\cdot\xi \ge \underline{\alpha}|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^3,$$

we denote by $\mathbf{L}^2_\alpha(\Omega)$ the vector space $\mathbf{L}^2(\Omega)$ equipped with the weighted scalar product $(\alpha\cdot,\cdot)_{\mathbf{L}^2(\Omega)}$. We close this section by presenting the mathematical assumptions for our analysis.

Assumption 1.1. We make the following assumptions in our analysis:

(1) The set $\Omega \subset \mathbb{R}^3$ is open and contains a given (possibly empty) open subset $\Omega_\sigma \subset \Omega$.

(2) Both the electric permittivity $\varepsilon : \Omega \to \mathbb{R}^{3\times3}$ and the magnetic permeability $\mu : \Omega \to \mathbb{R}^{3\times3}$ are of class $L^\infty_{\mathrm{sym}}(\Omega)^{3\times3}$ and uniformly positive definite, that is, there exist constants $\underline{\varepsilon}, \underline{\mu} > 0$ such that

$$\varepsilon(x)\xi\cdot\xi \ge \underline{\varepsilon}|\xi|^2 \ \text{ and } \ \mu(x)\xi\cdot\xi \ge \underline{\mu}|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^3. \qquad (1.6)$$

The electric conductivity $\sigma : \Omega \to \mathbb{R}^{3\times 3}$ is of class $L^\infty_{\mathrm{sym}}(\Omega)^{3\times 3}$ and positive semi-definite. Furthermore, it is uniformly positive definite on Ω_σ, that is, there exists a constant $\underline{\sigma} > 0$ such that

$$\sigma(x)\xi \cdot \xi \geq \underline{\sigma}|\xi|^2 \quad \text{for a.e. } x \in \Omega_\sigma \text{ and all } \xi \in \mathbb{R}^3. \tag{1.7}$$

(3) The obstacle set $\mathbf{K} \subset \mathbf{L}^2(\Omega)$ is assumed to be closed and convex containing $\mathbf{0}$.

(4) The applied current source fulfills $\mathbf{f} \in W^{1,\infty}((0,T), \mathbf{L}^2(\Omega))$ with Lipschitz constant $L \geq 0$.

(5) The initial value satisfies $(\mathbf{E}_0, \mathbf{H}_0) \in (\mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})) \times \mathbf{H}(\mathbf{curl})$ and

$$\int_{\Omega_\sigma} (\sigma \mathbf{E}_0 - \mathbf{curl}\,\mathbf{H}_0) \cdot (\mathbf{v} - \mathbf{E}_0)\,\mathrm{d}x \geq \int_{\Omega_\sigma} \mathbf{f}(0) \cdot (\mathbf{v} - \mathbf{E}_0)\,\mathrm{d}x, \quad \forall\, \mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl}). \tag{1.8}$$

Remark 1.2. Condition (1.8) is obviously satisfied if

$$\sigma \mathbf{E}_0 - \mathbf{curl}\,\mathbf{H}_0 = \mathbf{f}(0) \quad \text{a.e. in } \Omega_\sigma. \tag{1.9}$$

If (1.9) fails to hold, then (1.8) may still be valid. For instance, if the feasible set $\mathbf{K}$ fulfills

$$\mathbf{e} \in \mathbf{K} \quad \Longrightarrow \quad \mathbf{e}(x) = 0 \quad \text{for a.e. } x \in \Omega_\sigma, \tag{1.10}$$

then condition (1.8) is satisfied for all $\mathbf{f}(0) \in \mathbf{L}^2(\Omega)$ and $(\mathbf{E}_0, \mathbf{H}_0) \in (\mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})) \times \mathbf{H}(\mathbf{curl})$. Note that (1.10) is highly relevant to the physical phenomenon of electric shielding, such as the Faraday cage to block the effects of external electric fields in the material Ω_σ. Another example is the case where $\mathbf{K}$ only permits a certain feasible direction of the electric field in Ω_σ, such as

$$\mathbf{e} \in \mathbf{K} \quad \Longrightarrow \quad e_1(x) \geq 0 \quad \text{and} \quad e_2(x) = e_3(x) = 0 \quad \text{for a.e. } x \in \Omega_\sigma. \tag{1.11}$$

If $\mathbf{E}_0(x) = \mathbf{0}$ holds for a.e. $x \in \Omega_\sigma$ and the first components of $\mathbf{curl}\,\mathbf{H}_0$ and $\mathbf{f}(0)$ are non-positive a.e. in Ω_σ, then condition (1.8) is satisfied in the case of (1.11).

Lastly, we note that in the case given by (P), that is to say $\Omega_\sigma = \emptyset$, (1.8) is always satisfied since all integrals over Ω_σ vanish. In particular, for (P) there is no restriction on the initial value other than $(\mathbf{E}_0, \mathbf{H}_0) \in (\mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})) \times \mathbf{H}(\mathbf{curl})$.

2. Analysis of the time-discrete approximation to ($\mathrm{P_{ec}}$)

This section is devoted to the analysis of the time-discrete approximation to ($\mathrm{P_{ec}}$) based on the Rothe method. Let us begin by introducing an equidistant partition of the time interval $[0,T]$ as follows: Given $N \in \mathbb{N}$, we set

$$\tau := \frac{T}{N}, \quad 0 = t_0 < t_1 < \cdots < t_N = T \quad \text{with} \quad t_n := n\tau, \quad n \in \{0, \ldots, N\}.$$

Furthermore, we introduce the backward Euler difference quotients

$$\delta \mathbf{E}_n := \frac{\mathbf{E}_n - \mathbf{E}_{n-1}}{\tau}, \quad \delta \mathbf{H}_n := \frac{\mathbf{H}_n - \mathbf{H}_{n-1}}{\tau}, \quad \forall n \in \{1, \ldots, N\} \tag{2.1}$$

and set $\mathbf{f}_n := \mathbf{f}(t_n) \in \mathbf{L}^2(\Omega)$ for all $n \in \{0, \ldots, N\}$. Invoking these quantities, the time-discrete (Euler) approximation to ($\mathrm{P_{ec}}$) reads as follows: Find elements $\{(\mathbf{E}_n, \mathbf{H}_n)\}_{n=1}^N \subset (\mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})) \times \mathbf{H}(\mathbf{curl})$ such that

$$\begin{cases} \displaystyle\int_{\Omega\setminus\Omega_\sigma} \varepsilon \delta \mathbf{E}_n \cdot (\mathbf{v} - \mathbf{E}_n)\, \mathrm{d}x + \int_\Omega \sigma \mathbf{E}_n \cdot (\mathbf{v} - \mathbf{E}_n) - \mathbf{H}_n \cdot \mathbf{curl}(\mathbf{v} - \mathbf{E}_n)\, \mathrm{d}x \\ \qquad \displaystyle\geq \int_\Omega \mathbf{f}_n \cdot (\mathbf{v} - \mathbf{E}_n)\, \mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl}) \quad \forall n \in \{1, \ldots, N\}; \\ \mu \delta \mathbf{H}_n + \mathbf{curl}\, \mathbf{E}_n = \mathbf{0}, \quad \forall n \in \{1, \ldots, N\}. \end{cases} \tag{P_N}$$

To derive an existence and uniqueness result for (P_N), let us consider a bounded and coercive bilinear form $a : \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}_0(\mathbf{curl}) \to \mathbb{R}$ given by

$$(\mathbf{E}, \mathbf{v}) \mapsto \int_{\Omega\setminus\Omega_\sigma} \varepsilon \mathbf{E} \cdot \mathbf{v}\, \mathrm{d}x + \int_\Omega \tau \sigma \mathbf{E} \cdot \mathbf{v} + \tau^2 \mu^{-1}\, \mathbf{curl}\, \mathbf{E} \cdot \mathbf{curl}\, \mathbf{v}\, \mathrm{d}x$$

and bounded linear forms $F_n : \mathbf{H}_0(\mathbf{curl}) \to \mathbb{R}$ given by

$$\mathbf{v} \mapsto \int_\Omega \tau \mathbf{f}_n \cdot \mathbf{v} + \tau \mathbf{H}_{n-1} \cdot \mathbf{curl}\, \mathbf{v}\, \mathrm{d}x + \int_{\Omega\setminus\Omega_\sigma} \varepsilon \mathbf{E}_{n-1} \cdot \mathbf{v}\, \mathrm{d}x, \quad \forall n \in \{1, \ldots, N\}.$$

In view of (2.1), (P_N) is equivalent to the problem of successively finding elements $(\mathbf{E}_1, \mathbf{H}_1), \ldots, (\mathbf{E}_N, \mathbf{H}_N) \in (\mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})) \times \mathbf{H}(\mathbf{curl})$ such that

$$a(\mathbf{E}_n, \mathbf{v} - \mathbf{E}_n) \geq F_n(\mathbf{v} - \mathbf{E}_n), \ \forall \mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl}) \quad \text{and} \quad \mathbf{H}_n = -\tau \mu^{-1}\, \mathbf{curl}\, \mathbf{E}_n + \mathbf{H}_{n-1}.$$

The well-posedness of (P_N) therefore follows from the classical theory of elliptic variational inequalities (see [18, Theorem 2.1] or [13, Theorem 3.1]), which we summarize in the following lemma:

Lemma 2.1. *Let Assumption 1.1 hold. Then, for every $N \in \mathbb{N}$, the time-discrete problem (P_N) admits a unique solution $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^N \subset (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$.*

In the upcoming theorem, we prove our first main result on the stability for (P_N). For the convenience of the reader, we recall Gronwall's lemma in its discrete version in the following auxiliary lemma (see [9, page 280]):

Lemma 2.2. *Let $\{a_k\}_{k=0}^\infty$ and $\{b_k\}_{k=0}^\infty$ be sequences of non-negative real numbers satisfying*

$$a_n \leq c + \sum_{k=0}^{n-1} a_k b_k, \quad \forall n \in \mathbb{N}$$

for some constant $c > 0$. *Then, it holds that*

$$a_n \leq c \exp\Big(\sum_{k=0}^{n-1} b_k\Big), \quad \forall n \in \mathbb{N}.$$

As pointed out in the introduction, our upcoming stability proof is based on the use of certain correction terms $\mathbf{z} \in \mathbf{L}^2(\Omega \setminus \Omega_\sigma)$ and $\mathbf{w} \in \mathbf{L}^2(\Omega)$ for the initial data, which are defined as follows:

$$\begin{aligned} \mathbf{z} &:= \varepsilon \mathbf{E}_0 + \sigma \mathbf{E}_0 - \mathbf{curl}\, \mathbf{H}_0 - \mathbf{f}_0 && \text{a.e. on } \Omega \setminus \Omega_\sigma, \\ \mathbf{w} &:= \mu \mathbf{H}_0 + \mathbf{curl}\, \mathbf{E}_0 && \text{a.e. on } \Omega. \end{aligned}$$

Theorem 2.1. *Let Assumption 1.1 be satisfied. Then, there exists a positive real constant* C_0 *depending only on* T, ε, μ $\sigma, \boldsymbol{f}, \boldsymbol{E}_0, \boldsymbol{H}_0$ *such that for every* $N \in \mathbb{N}$ *the unique solution* $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^N \subset (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ *of* (P_N) *satisfies*

$$\begin{aligned} \max_{1 \leq n \leq N} \big[&\|\boldsymbol{E}_n\|_{L^2(\Omega)} + \|\boldsymbol{H}_n\|_{L^2(\Omega)} + \|\delta \boldsymbol{E}_n\|_{L^2(\Omega \setminus \Omega_\sigma)} \\ &+ \|\delta \boldsymbol{H}_n\|_{L^2(\Omega)} + \|\,\mathbf{curl}\, \boldsymbol{E}_n\|_{L^2(\Omega)} \big] \leq C_0. \end{aligned} \tag{2.2}$$

Proof. Let $N \in \mathbb{N}$ be arbitrarily fixed and let $\{(\mathbf{E}_n, \mathbf{H}_n)\}_{n=1}^N \subset (\mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})) \times \mathbf{H}(\mathbf{curl})$ denote the unique solution to (P_N). Now let $\mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})$ be arbitrarily fixed. Multiplying the above equation for $\mathbf{z}$ by $\mathbf{v} - \mathbf{E}_0$ and integrating the resulting equality over $\Omega \setminus \Omega_\sigma$, we obtain that

$$\int_{\Omega \setminus \Omega_\sigma} \varepsilon \mathbf{E}_0 \cdot (\mathbf{v} - \mathbf{E}_0)\, \mathrm{d}x + \int_{\Omega \setminus \Omega_\sigma} (\sigma \mathbf{E}_0 - \mathbf{curl}\, \mathbf{H}_0) \cdot (\mathbf{v} - \mathbf{E}_0)\, \mathrm{d}x = \int_{\Omega \setminus \Omega_\sigma} (\mathbf{f}_0 + \mathbf{z}) \cdot (\mathbf{v} - \mathbf{E}_0)\, \mathrm{d}x.$$

Then, combining the above equality with (1.8), it follows that

$$\begin{aligned} &\int_{\Omega \setminus \Omega_\sigma} \varepsilon \mathbf{E}_0 \cdot (\mathbf{v} - \mathbf{E}_0)\, \mathrm{d}x + \int_{\Omega} (\sigma \mathbf{E}_0 - \mathbf{curl}\, \mathbf{H}_0) \cdot (\mathbf{v} - \mathbf{E}_0)\, \mathrm{d}x \\ &\qquad \geq \int_{\Omega} \mathbf{f}_0 \cdot (\mathbf{v} - \mathbf{E}_0)\, \mathrm{d}x + \int_{\Omega \setminus \Omega_\sigma} \mathbf{z} \cdot (\mathbf{v} - \mathbf{E}_0)\, \mathrm{d}x, \end{aligned}$$

and consequently, applying characterization (1.4) to the previous inequality, the initial data $(\mathbf{E}_0, \mathbf{H}_0) \in (\mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})) \times \mathbf{H}(\mathbf{curl})$ satisfy

$$\begin{cases} \displaystyle\int_{\Omega \setminus \Omega_\sigma} \varepsilon \mathbf{E}_0 \cdot (\mathbf{v} - \mathbf{E}_0)\, \mathrm{d}x + \int_{\Omega} \sigma \mathbf{E}_0 \cdot (\mathbf{v} - \mathbf{E}_0) - \mathbf{H}_0 \cdot \mathbf{curl}(\mathbf{v} - \mathbf{E}_0)\, \mathrm{d}x \\ \qquad \displaystyle\geq \int_{\Omega} \mathbf{f}_0 \cdot (\mathbf{v} - \mathbf{E}_0)\, \mathrm{d}x + \int_{\Omega \setminus \Omega_\sigma} \mathbf{z} \cdot (\mathbf{v} - \mathbf{E}_0)\, \mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl}); \\ \mu \mathbf{H}_0 + \mathbf{curl}\, \mathbf{E}_0 = \mathbf{w}. \end{cases} \tag{2.3}$$

Now, the underlying system (see (2.3)) allows us to incorporate the initial data $(\mathbf{E}_0, \mathbf{H}_0)$ to the time-discrete scheme (P_N) and preserve its pivotal structure for our stability analysis. To realize this, we employ the quantities

$$\begin{aligned} &\delta\mathbf{E}_0 := \mathbf{E}_0, && \delta\mathbf{H}_0 := \mathbf{H}_0, \\ &\mathbf{z}_n^N := \begin{cases} \mathbf{z}, & n = 0, \\ \mathbf{0}, & n \in \{1, \dots, N\}, \end{cases} && \mathbf{w}_n^N := \begin{cases} \mathbf{w}, & n = 0, \\ \mathbf{0}, & n \in \{1, \dots, N\} \end{cases} \end{aligned} \tag{2.4}$$

and deduce from (2.3) that the unique solution to (P_N) fulfills

$$\begin{cases} \displaystyle\int_{\Omega\setminus\Omega_\sigma} \varepsilon\delta\mathbf{E}_n \cdot (\mathbf{v} - \mathbf{E}_n)\,\mathrm{d}x + \int_\Omega \sigma\mathbf{E}_n \cdot (\mathbf{v} - \mathbf{E}_n) - \mathbf{H}_n \cdot \mathbf{curl}(\mathbf{v} - \mathbf{E}_n)\,\mathrm{d}x \\ \displaystyle\quad \geq \int_\Omega \mathbf{f}_n \cdot (\mathbf{v} - \mathbf{E}_n)\,\mathrm{d}x + \int_{\Omega\setminus\Omega_\sigma} \mathbf{z}_n^N \cdot (\mathbf{v} - \mathbf{E}_n)\,\mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl}),\ \forall n \in \{0, \dots, N\}; \\ \mu\delta\mathbf{H}_n + \mathbf{curl}\,\mathbf{E}_n = \mathbf{w}_n^N, \quad \forall n \in \{0, \dots, N\}. \end{cases} \tag{2.5}$$

For every $n \in \{1, \dots, N\}$, setting $\mathbf{v} = \mathbf{E}_{n-1}$ (resp. $\mathbf{v} = \mathbf{E}_n$) in the n-th inequality of (2.5) (resp. the $(n-1)$-th inequality of (2.5)) and then dividing the resulting inequalities by $-\tau$, we obtain

$$\begin{aligned} &\int_{\Omega\setminus\Omega_\sigma} \varepsilon\delta\mathbf{E}_n \cdot \delta\mathbf{E}_n\,\mathrm{d}x + \int_\Omega \sigma\mathbf{E}_n \cdot \delta\mathbf{E}_n - \mathbf{H}_n \cdot \mathbf{curl}\,\delta\mathbf{E}_n\,\mathrm{d}x \\ &\qquad \leq \int_\Omega \mathbf{f}_n \cdot \delta\mathbf{E}_n\,\mathrm{d}x + \int_{\Omega\setminus\Omega_\sigma} \mathbf{z}_n^N \cdot \delta\mathbf{E}_n\,\mathrm{d}x \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &-\int_{\Omega\setminus\Omega_\sigma} \varepsilon\delta\mathbf{E}_{n-1} \cdot \delta\mathbf{E}_n\,\mathrm{d}x - \int_\Omega \sigma\mathbf{E}_{n-1} \cdot \delta\mathbf{E}_n - \mathbf{H}_{n-1} \cdot \mathbf{curl}\,\delta\mathbf{E}_n\,\mathrm{d}x \\ &\qquad \leq -\int_\Omega \mathbf{f}_{n-1} \cdot \delta\mathbf{E}_n\,\mathrm{d}x - \int_{\Omega\setminus\Omega_\sigma} \mathbf{z}_{n-1}^N \cdot \delta\mathbf{E}_n\,\mathrm{d}x. \end{aligned} \tag{2.7}$$

On the other hand, the second equation in (2.5) yields that

$$\mathbf{curl}\,\delta\mathbf{E}_n = -\tau^{-1}\mu(\delta\mathbf{H}_n - \delta\mathbf{H}_{n-1}) + \tau^{-1}(\mathbf{w}_n^N - \mathbf{w}_{n-1}^N), \quad \forall n \in \{1, \dots, N\}. \tag{2.8}$$

Adding (2.6) and (2.7) together and then utilizing (2.8), as well as the positive semi-definiteness of σ, we get

$$\begin{aligned} &\int_{\Omega\setminus\Omega_\sigma} \varepsilon(\delta\mathbf{E}_n - \delta\mathbf{E}_{n-1}) \cdot \delta\mathbf{E}_n\,\mathrm{d}x + \int_{\Omega_\sigma} \sigma(\mathbf{E}_n - \mathbf{E}_{n-1}) \cdot \delta\mathbf{E}_n\,\mathrm{d}x \\ &\qquad + \int_\Omega \mu(\delta\mathbf{H}_n - \delta\mathbf{H}_{n-1}) \cdot \delta\mathbf{H}_n\,\mathrm{d}x \end{aligned}$$

$$\leq \int_\Omega (\mathbf{f}_n - \mathbf{f}_{n-1}) \cdot \delta \mathbf{E}_n \, \mathrm{d}x + \int_{\Omega \setminus \Omega_\sigma} (\mathbf{z}_n^N - \mathbf{z}_{n-1}^N) \cdot \delta \mathbf{E}_n \, \mathrm{d}x + \int_\Omega (\mathbf{w}_n^N - \mathbf{w}_{n-1}^N) \cdot \delta \mathbf{H}_n \, \mathrm{d}x, \quad \forall n \in \{1, \ldots, N\}. \tag{2.9}$$

Now let $n_0 \in \{1, \ldots, N\}$ be arbitrarily fixed, and sum up inequality (2.9) over $\{1, \ldots, n_0\}$. Then, applying binomial formula (1.3) along with Hölder's inequality and (1.7), it follows that

$$\begin{aligned}
&\frac{1}{2}\Big(\|\delta \mathbf{E}_{n_0}\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} - \|\delta \mathbf{E}_0\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} + \sum_{n=1}^{n_0} \|\delta \mathbf{E}_n - \delta \mathbf{E}_{n-1}\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\Big) \\
&\quad + \frac{1}{2}\Big(\|\delta \mathbf{H}_{n_0}\|^2_{\mathbf{L}^2_\mu(\Omega)} - \|\delta \mathbf{H}_0\|^2_{\mathbf{L}^2_\mu(\Omega)} + \sum_{n=1}^{n_0} \|\delta \mathbf{H}_n - \delta \mathbf{H}_{n-1}\|^2_{\mathbf{L}^2_\mu(\Omega)}\Big) + \sum_{n=1}^{n_0} \tau \underline{\sigma} \|\delta \mathbf{E}_n\|^2_{\mathbf{L}^2(\Omega_\sigma)} \\
&\leq \sum_{n=1}^{n_0} \|\mathbf{f}_n - \mathbf{f}_{n-1}\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} \|\delta \mathbf{E}_n\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} + \sum_{n=1}^{n_0} \|\mathbf{f}_n - \mathbf{f}_{n-1}\|_{\mathbf{L}^2(\Omega_\sigma)} \|\delta \mathbf{E}_n\|_{\mathbf{L}^2(\Omega_\sigma)} \\
&\quad + \sum_{n=1}^{n_0} \|\mathbf{z}_n^N - \mathbf{z}_{n-1}^N\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} \|\delta \mathbf{E}_n\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} \\
&\quad + \sum_{n=1}^{n_0} \|\mathbf{w}_n^N - \mathbf{w}_{n-1}^N\|_{\mathbf{L}^2(\Omega)} \|\delta \mathbf{H}_n\|_{\mathbf{L}^2(\Omega)}.
\end{aligned} \tag{2.10}$$

Using Young's inequality together with an estimate of the type $\underline{\alpha}\|\cdot\|^2_{\mathbf{L}^2(\Omega)} \leq \|\cdot\|^2_{\mathbf{L}^2_\alpha(\Omega)}$ and the Lipschitz property of $\mathbf{f}$, the first and second terms in the right-hand side of (2.10) can be estimated by

$$\begin{aligned}
&\sum_{n=1}^{n_0} \|\mathbf{f}_n - \mathbf{f}_{n-1}\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} \|\delta \mathbf{E}_n\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} \\
&\quad \leq \sum_{n=1}^{n_0} \Big(\frac{N}{\underline{\varepsilon}} \|\mathbf{f}_n - \mathbf{f}_{n-1}\|^2_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} + \frac{1}{4N} \|\delta \mathbf{E}_n\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\Big) \\
&\quad \leq \sum_{n=1}^{n_0} \Big(\frac{N}{\underline{\varepsilon}} L^2 \tau^2 + \frac{1}{4N} \|\delta \mathbf{E}_n\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\Big) \\
&\quad \underbrace{\leq}_{\tau = \frac{T}{N}} \frac{L^2 T^2}{\underline{\varepsilon}} + \frac{1}{4} \|\delta \mathbf{E}_{n_0}\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} + \sum_{n=1}^{n_0 - 1} \frac{1}{4N} \|\delta \mathbf{E}_n\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
\sum_{n=1}^{n_0} \|\mathbf{f}_n - \mathbf{f}_{n-1}\|_{\mathbf{L}^2(\Omega_\sigma)} \|\delta \mathbf{E}_n\|_{\mathbf{L}^2(\Omega_\sigma)} &\leq \sum_{n=1}^{n_0} \Big(\frac{1}{4\tau\underline{\sigma}} \|\mathbf{f}_n - \mathbf{f}_{n-1}\|^2_{\mathbf{L}^2(\Omega_\sigma)} + \tau \underline{\sigma} \|\delta \mathbf{E}_n\|^2_{\mathbf{L}^2(\Omega_\sigma)}\Big) \\
&\leq \frac{L^2 T}{4\underline{\sigma}} + \sum_{n=1}^{n_0} \tau \underline{\sigma} \|\delta \mathbf{E}_n\|^2_{\mathbf{L}^2(\Omega_\sigma)}.
\end{aligned} \tag{2.12}$$

For the remaining terms on the right-hand side of (2.10), we find by Young's inequality and the triangle inequality that

$$\begin{aligned}\sum_{n=1}^{n_0} \|\mathbf{z}_n^N - \mathbf{z}_{n-1}^N\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} \|\delta\mathbf{E}_n\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} &\underbrace{=}_{(2.4)} \|\mathbf{z}\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} \|\delta\mathbf{E}_1\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} \\ &\leq \|\mathbf{z}\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} \|\delta\mathbf{E}_1 - \delta\mathbf{E}_0\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} + \|\mathbf{z}\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} \|\delta\mathbf{E}_0\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} \\ &\leq \frac{2}{\underline{\varepsilon}} \|\mathbf{z}\|^2_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} + \frac{1}{4} \|\delta\mathbf{E}_1 - \delta\mathbf{E}_0\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} + \frac{1}{4} \|\delta\mathbf{E}_0\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}, \end{aligned} \tag{2.13}$$

and analogously,

$$\begin{aligned}&\sum_{n=1}^{n_0} \|\mathbf{w}_n^N - \mathbf{w}_{n-1}^N\|_{\mathbf{L}^2(\Omega)} \|\delta\mathbf{H}_n\|_{\mathbf{L}^2(\Omega)} \\ &\qquad \leq \frac{2}{\underline{\mu}} \|\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{4} \|\delta\mathbf{H}_1 - \delta\mathbf{H}_0\|^2_{\mathbf{L}^2_\mu(\Omega)} + \frac{1}{4} \|\delta\mathbf{H}_0\|^2_{\mathbf{L}^2_\mu(\Omega)}. \end{aligned} \tag{2.14}$$

Applying (2.11)–(2.14) to (2.10) along with $\delta\mathbf{E}_0 = \mathbf{E}_0$ and $\delta\mathbf{H}_0 = \mathbf{H}_0$, it follows after some rearrangement that

$$\begin{aligned}&\frac{1}{4} \|\delta\mathbf{E}_{n_0}\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} + \frac{1}{2} \|\delta\mathbf{H}_{n_0}\|^2_{\mathbf{L}^2_\mu(\Omega)} \\ &\quad \leq \frac{L^2 T^2}{\underline{\varepsilon}} + \frac{L^2 T}{4\underline{\sigma}} + \frac{3}{4} \|\mathbf{E}_0\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} + \frac{3}{4} \|\mathbf{H}_0\|^2_{\mathbf{L}^2_\mu(\Omega)} + \frac{2}{\underline{\varepsilon}} \|\mathbf{z}\|^2_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} + \frac{2}{\underline{\mu}} \|\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)} \\ &\quad + \sum_{n=1}^{n_0-1} \frac{1}{4N} \|\delta\mathbf{E}_n\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}. \end{aligned}$$

By virtue of Lemma 2.2, we eventually deduce that

$$\|\delta\mathbf{E}_{n_0}\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} + \|\delta\mathbf{H}_{n_0}\|^2_{\mathbf{L}^2_\mu(\Omega)} \leq C \exp\Big(\sum_{n=1}^{n_0-1} \frac{1}{N}\Big) \leq C \exp(1),$$

with a generic constant $C > 0$ depending only on $T, L, \varepsilon, \mu, \sigma, \mathbf{E}_0, \mathbf{H}_0$. In particular, it holds that

$$\|\delta\mathbf{E}_{n_0}\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} + \|\delta\mathbf{H}_{n_0}\|_{\mathbf{L}^2(\Omega)} \leq C. \tag{2.15}$$

From (2.15) and the reversed triangle inequality, it follows by the definition of the difference quotients in (2.1) that

$$\begin{aligned}\|\mathbf{E}_{n_0}\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} + \|\mathbf{H}_{n_0}\|_{\mathbf{L}^2(\Omega)} &\leq \tau C + \|\mathbf{E}_{n_0-1}\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} + \|\mathbf{H}_{n_0-1}\|_{\mathbf{L}^2(\Omega)} \\ &\leq \cdots \leq n_0 \tau C + \|\mathbf{E}_0\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} + \|\mathbf{H}_0\|_{\mathbf{L}^2(\Omega)} \\ &\leq TC + \|\mathbf{E}_0\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} + \|\mathbf{H}_0\|_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{2.16}$$

Furthermore, the estimate

$$\|\operatorname{\mathbf{curl}} \mathbf{E}_{n_0}\|_{\mathbf{L}^2(\Omega)} \leq C \tag{2.17}$$

immediately results from (2.15) along with the discrete Faraday law in (P_N). We are left with showing the estimate for $\mathbf{E}_{n_0}$ in $\mathbf{L}^2(\Omega_\sigma)$. To do so, we test with $\mathbf{v} = 0$ in (P_N) and use the positive semi-definiteness of σ to obtain

$$\int_{\Omega\setminus\Omega_\sigma} \varepsilon\delta\mathbf{E}_{n_0}\cdot\mathbf{E}_{n_0}\,\mathrm{d}x + \int_{\Omega_\sigma} \sigma\mathbf{E}_{n_0}\cdot\mathbf{E}_{n_0}\,\mathrm{d}x - \int_{\Omega} \mathbf{H}_{n_0}\cdot\mathbf{curl}\,\mathbf{E}_{n_0}\,\mathrm{d}x \\ \leq \int_{\Omega\setminus\Omega_\sigma} \mathbf{f}_{n_0}\cdot\mathbf{E}_{n_0}\,\mathrm{d}x + \int_{\Omega_\sigma} \mathbf{f}_{n_0}\cdot\mathbf{E}_{n_0}\,\mathrm{d}x. \tag{2.18}$$

Applying the estimate

$$\int_{\Omega_\sigma} \mathbf{f}_{n_0}\cdot\mathbf{E}_{n_0}\,\mathrm{d}x \leq \frac{1}{2\underline{\sigma}}\|\mathbf{f}_{n_0}\|^2_{\mathbf{L}^2(\Omega_\sigma)} + \frac{\underline{\sigma}}{2}\|\mathbf{E}_{n_0}\|^2_{\mathbf{L}^2(\Omega_\sigma)}$$

to (2.18) together with (1.7) and Hölder's inequality, we end up with

$$\frac{\underline{\sigma}}{2}\|\mathbf{E}_{n_0}\|^2_{\mathbf{L}^2(\Omega_\sigma)} \leq \|\mathbf{f}_{n_0}\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\|\mathbf{E}_{n_0}\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} + \frac{1}{2\underline{\sigma}}\|\mathbf{f}_{n_0}\|^2_{\mathbf{L}^2(\Omega_\sigma)} \\ + \|\delta\mathbf{E}_{n_0}\|_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\|\mathbf{E}_{n_0}\|_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} \\ + \|\mathbf{H}_{n_0}\|_{\mathbf{L}^2(\Omega)}\|\,\mathbf{curl}\,\mathbf{E}_{n_0}\|_{\mathbf{L}^2(\Omega)}, \tag{2.19}$$

where all the terms on the right-hand side are bounded due to the stability shown before. Since $n_0 \in \{1,\dots,N\}$ was chosen arbitrarily, (2.15), (2.16), (2.17) and (2.19) imply that the a priori estimate (2.2) is valid. ∎

3. Well-posedness

This section is devoted to the well-posedness analysis for the eddy current obstacle problem (see (P_{ec})) based on the time-discrete approximation (P_N). As a preparation, for every $N \in \mathbb{N}$, we set up piecewise linear and piecewise constant (in time) interpolations out of the solution $\{(\mathbf{E}_n, \mathbf{H}_n)\}_{n=1}^N \subset (\mathbf{K}\cap\mathbf{H}_0(\mathbf{curl}))\times\mathbf{H}(\mathbf{curl})$ of (P_N) as follows:

$$\begin{aligned} &\mathbf{E}_N : [0,T] \to \mathbf{K}\cap\mathbf{H}_0(\mathbf{curl}), \quad t \mapsto \begin{cases} \mathbf{E}_0 & \text{if } t = 0, \\ \mathbf{E}_{n-1} + (t - t_{n-1})\delta\mathbf{E}_n & \text{if } t \in (t_{n-1}, t_n]; \end{cases} \\ &\mathbf{H}_N : [0,T] \to \mathbf{L}^2(\Omega), \quad t \mapsto \begin{cases} \mathbf{H}_0 & \text{if } t = 0, \\ \mathbf{H}_{n-1} + (t - t_{n-1})\delta\mathbf{H}_n & \text{if } t \in (t_{n-1}, t_n], \end{cases} \end{aligned} \tag{3.1}$$

and

$$
\begin{aligned}
\overline{\mathbf{E}}_N &: [0,T] \to \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl}), \quad t \mapsto \begin{cases} \mathbf{E}_0 & \text{if } t = 0, \\ \mathbf{E}_n & \text{if } t \in (t_{n-1}, t_n]; \end{cases} \\
\overline{\mathbf{H}}_N &: [0,T] \to \mathbf{L}^2(\Omega), \qquad\qquad t \mapsto \begin{cases} \mathbf{H}_0 & \text{if } t = 0, \\ \mathbf{H}_n & \text{if } t \in (t_{n-1}, t_n]; \end{cases} \\
\overline{\mathbf{f}}_N &: [0,T] \to \mathbf{L}^2(\Omega), \qquad\qquad t \mapsto \begin{cases} \mathbf{f}_0 & \text{if } t = 0, \\ \mathbf{f}_n & \text{if } t \in (t_{n-1}, t_n]. \end{cases}
\end{aligned}
\tag{3.2}
$$

In view of (P_N), it follows immediately that the above interpolations satisfy

$$
\begin{cases}
\displaystyle\int_{\Omega\setminus\Omega_\sigma} \varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_N(t)\cdot(\mathbf{v}-\overline{\mathbf{E}}_N(t))\,\mathrm{d}x \\
\qquad\qquad \displaystyle + \int_\Omega \sigma\overline{\mathbf{E}}_N(t)\cdot(\mathbf{v}-\overline{\mathbf{E}}_N(t)) - \overline{\mathbf{H}}_N(t)\cdot\mathbf{curl}(\mathbf{v}-\overline{\mathbf{E}}_N(t))\,\mathrm{d}x \\
\qquad \displaystyle \geq \int_\Omega \overline{\mathbf{f}}_N(t)\cdot(\mathbf{v}-\overline{\mathbf{E}}_N(t))\,\mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{K}\cap\mathbf{H}_0(\mathbf{curl}),\ \forall t \in (0,T]; \\
\displaystyle \mu\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}_N(t) + \mathbf{curl}\,\overline{\mathbf{E}}_N(t) = \mathbf{0}, \quad \forall t \in (0,T]; \\
\overline{\mathbf{E}}_N(t) \in \mathbf{K}\cap\mathbf{H}_0(\mathbf{curl}), \quad \forall t \in [0,T].
\end{cases}
\tag{$\widetilde{\mathrm{P}}_N$}
$$

Theorem 3.1. *Let Assumption 1.1 hold. Then, the eddy current obstacle problem (see (P_{ec})) admits a unique solution $(\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)\times\boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{L}^2(\Omega))$.*

Proof. Existence of a solution. By our construction (see (3.1) and (3.2)), Theorem 2.1 yields the existence of a subsequence of $\{(\mathbf{E}_N,\mathbf{H}_N)\}_{N=1}^\infty$, denoted again by the same symbol, such that

$$
\begin{aligned}
(\mathbf{E}_N,\mathbf{H}_N) &\overset{*}{\rightharpoonup} (\mathbf{E},\mathbf{H}) \quad \text{weakly-* in } L^\infty((0,T),\mathbf{H}_0(\mathbf{curl})\times\mathbf{L}^2(\Omega)) \text{ as } N\to\infty, \\
(\overline{\mathbf{E}}_N,\overline{\mathbf{H}}_N) &\overset{*}{\rightharpoonup} (\overline{\mathbf{E}},\overline{\mathbf{H}}) \quad \text{weakly-* in } L^\infty((0,T),\mathbf{H}_0(\mathbf{curl})\times\mathbf{L}^2(\Omega)) \text{ as } N\to\infty, \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{E}_N,\mathbf{H}_N) &\overset{*}{\rightharpoonup} (\boldsymbol{\xi},\boldsymbol{\zeta}) \quad \text{weakly-* in } L^\infty((0,T),\mathbf{L}^2(\Omega\setminus\Omega_\sigma)\times\mathbf{L}^2(\Omega)) \text{ as } N\to\infty,
\end{aligned}
\tag{3.3}
$$

for some $(\mathbf{E},\mathbf{H}), (\overline{\mathbf{E}},\overline{\mathbf{H}}) \in L^\infty((0,T),\mathbf{H}_0(\mathbf{curl})\times\mathbf{L}^2(\Omega))$ and $(\boldsymbol{\xi},\boldsymbol{\zeta}) \in L^\infty((0,T), \mathbf{L}^2(\Omega\setminus\Omega_\sigma)\times\mathbf{L}^2(\Omega))$. Furthermore, (3.1) and (3.2) also imply

$$
\begin{aligned}
\|\mathbf{E}_N(t)-\overline{\mathbf{E}}_N(t)\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} &\leq \tau \max_{1\leq n\leq N}\|\delta\mathbf{E}_n\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} \underbrace{\leq}_{(2.2)} \frac{TC_0}{N}, \quad \forall t\in[0,T]; \\
\|\mathbf{H}_N(t)-\overline{\mathbf{H}}_N(t)\|_{\mathbf{L}^2(\Omega)} &\leq \tau \max_{1\leq n\leq N}\|\delta\mathbf{H}_n\|_{\mathbf{L}^2(\Omega)} \underbrace{\leq}_{(2.2)} \frac{TC_0}{N}, \quad \forall t\in[0,T],
\end{aligned}
\tag{3.4}
$$

and consequently,

$$\lim_{N\to\infty} \|\mathbf{E}_N - \overline{\mathbf{E}}_N\|_{L^\infty((0,T),\mathbf{L}^2(\Omega\setminus\Omega_\sigma))} = \lim_{N\to\infty} \|\mathbf{H}_N - \overline{\mathbf{H}}_N\|_{L^\infty((0,T),\mathbf{L}^2(\Omega))} = 0. \quad (3.5)$$

By the above convergence properties together with (3.3), we obtain that

$$\mathbf{E} = \overline{\mathbf{E}} \quad \text{a.e. in } (0,T)\times(\Omega\setminus\Omega_\sigma) \qquad \text{and} \qquad \mathbf{H} = \overline{\mathbf{H}} \quad \text{a.e. in } (0,T)\times\Omega. \quad (3.6)$$

Let us now verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{E}} = \boldsymbol{\xi} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{H}} = \boldsymbol{\zeta}. \quad (3.7)$$

Indeed, the definition of the weak time derivative implies that

$$\begin{aligned}
\int_0^T (\boldsymbol{\xi}(t),\boldsymbol{\phi}(t))_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}t &\underbrace{=}_{(3.3)} \lim_{N\to\infty}\int_0^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_N(t),\boldsymbol{\phi}(t)\Big)_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}t \\
&= \lim_{N\to\infty} -\int_0^T \Big(\mathbf{E}_N(t),\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}(t)\Big)_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}t \\
&\underbrace{=}_{(3.3)\&(3.6)} -\int_0^T \Big(\overline{\mathbf{E}}(t),\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}(t)\Big)_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}t, \quad \forall \boldsymbol{\phi}\in\mathcal{C}_0^\infty((0,T),\mathbf{L}^2(\Omega\setminus\Omega_\sigma)),
\end{aligned}$$

and hence $\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{E}} = \boldsymbol{\xi}$. Analogous arguments are also valid for $\overline{\mathbf{H}}$, which concludes the proof of (3.7). Altogether, the weak-star limit $(\overline{\mathbf{E}},\overline{\mathbf{H}})$ enjoys the regularity property

$$(\overline{\mathbf{E}},\overline{\mathbf{H}}) \in W^{1,\infty}((0,T),\mathbf{L}^2(\Omega\setminus\Omega_\sigma)\times\mathbf{L}^2(\Omega)) \cap L^\infty((0,T),\mathbf{H}_0(\mathbf{curl})\times\mathbf{L}^2(\Omega)). \quad (3.8)$$

As the next step, we verify Faraday's law for $(\overline{\mathbf{E}},\overline{\mathbf{H}})$. According to ($\widetilde{\mathrm{P}}_N$), it holds that

$$\mu\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}_N(t) + \mathbf{curl}\,\overline{\mathbf{E}}_N(t) = \mathbf{0}, \quad \forall t\in(0,T],$$

from which it follows that

$$\begin{aligned}
&\int_0^T \Big(\mu\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{H}}(t) + \mathbf{curl}\,\overline{\mathbf{E}}(t),\boldsymbol{\phi}(t)\Big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t \\
&\qquad \underbrace{=}_{(3.3)\&(3.7)} \lim_{N\to\infty}\int_0^T \Big(\mu\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}_N(t) + \mathbf{curl}\,\overline{\mathbf{E}}_N(t),\boldsymbol{\phi}(t)\Big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t = 0
\end{aligned}$$

for all $\boldsymbol{\phi}\in\mathcal{C}_0^\infty((0,T),\mathbf{L}^2(\Omega))$. As a consequence, by the fundamental theorem of variational calculus, we obtain

$$\mu\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{H}}(t) + \mathbf{curl}\,\overline{\mathbf{E}}(t) = \mathbf{0} \quad \text{for a.e. } t\in(0,T). \quad (3.9)$$

Let us now prove the pointwise weak convergence

$$(\mathbf{E}_N, \mathbf{H}_N)(t) \rightharpoonup (\overline{\mathbf{E}}, \overline{\mathbf{H}})(t) \quad \text{weakly in } \mathbf{L}^2(\Omega \setminus \Omega_\sigma) \times \mathbf{L}^2(\Omega) \\ \text{as } N \to \infty \text{ for all } t \in [0,T]. \tag{3.10}$$

To this aim, let $t \in (0,T]$, $\mathbf{w} \in \mathbf{L}^2(\Omega \setminus \Omega_\sigma)$, and $\phi \in \mathcal{C}^1([0,t])$ be arbitrarily fixed. Integration by parts yields

$$\begin{aligned}
&\big(\overline{\mathbf{E}}(t), \mathbf{w}\big)_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\phi(t) - \big(\overline{\mathbf{E}}(0), \mathbf{w}\big)_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\phi(0) \\
&= \int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}s}\overline{\mathbf{E}}(s), \mathbf{w}\Big)_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\phi(s) + \big(\overline{\mathbf{E}}(s), \mathbf{w}\big)_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\frac{\mathrm{d}}{\mathrm{d}s}\phi(s)\,\mathrm{d}s \\
&\underbrace{=}_{(3.3),(3.6),(3.7)} \lim_{N\to\infty}\Big(\int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}_N(s), \mathbf{w}\Big)_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\phi(s) + \big(\mathbf{E}_N(s), \mathbf{w}\big)_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\frac{\mathrm{d}}{\mathrm{d}s}\phi(s)\,\mathrm{d}s\Big) \\
&= \lim_{N\to\infty}\Big(\big(\mathbf{E}_N(t), \mathbf{w}\big)_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\phi(t) - \big(\mathbf{E}_N(0), \mathbf{w}\big)_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\phi(0)\Big).
\end{aligned} \tag{3.11}$$

Choosing $\phi(t) \neq 0$ and $\phi(0) = 0$ (resp. $\phi(t) = 0$ and $\phi(0) \neq 0$) yields $\mathbf{E}_N(t) \rightharpoonup \overline{\mathbf{E}}(t)$ weakly in $\mathbf{L}^2(\Omega \setminus \Omega_\sigma)$ as $N \to \infty$ for all $t \in [0,T]$. By the same argumentation, we derive the pointwise weak convergence for $\{\mathbf{H}_N\}_{N=1}^\infty$. In conclusion, (3.10) is valid. As a direct consequence of (3.10) and $(\mathbf{E}_N, \mathbf{H}_N)(0) = (\mathbf{E}_0, \mathbf{H}_0)$ for all $N \in \mathbb{N}$, we have

$$\begin{aligned}
\overline{\mathbf{E}}(0) &= \mathbf{E}_0 \quad \text{a.e. in } \Omega \setminus \Omega_\sigma, \\
\overline{\mathbf{H}}(0) &= \mathbf{H}_0 \quad \text{a.e. in } \Omega,
\end{aligned} \tag{3.12}$$

which is exactly the initial value condition in (P_{ec}). Let us now introduce the subset

$$\widehat{\mathbf{K}} := \big\{\mathbf{w} \in L^2((0,T), \mathbf{L}^2(\Omega)) \mid \mathbf{w}(t) \in \mathbf{K} \text{ for a.e. } t \in (0,T)\big\}.$$

Since $\mathbf{K}$ is a closed and convex subset of $\mathbf{L}^2(\Omega)$, the subset $\widehat{\mathbf{K}} \subset L^2((0,T), \mathbf{L}^2(\Omega))$ is closed and convex. Therefore, since $\overline{\mathbf{E}}_N \in \widehat{\mathbf{K}}$ for all $N \in \mathbb{N}$, convergence property (3.3) implies that

$$\overline{\mathbf{E}} \in \widehat{\mathbf{K}} \quad \Longrightarrow \quad \overline{\mathbf{E}}(t) \in \mathbf{K} \quad \text{for a.e. } t \in (0,T). \tag{3.13}$$

By virtue of (3.8), (3.9), (3.12), and (3.13), the weak limit $(\overline{\mathbf{E}}, \overline{\mathbf{H}})$ is a solution to (P_{ec}) once we are able to show that it satisfies the variational inequality in (P_{ec}). In view of (3.2) and the Lipschitz regularity $\mathbf{f} \in W^{1,\infty}((0,T), \mathbf{L}^2(\Omega))$, it holds that

$$\lim_{N\to\infty} \overline{\mathbf{f}}_N = \mathbf{f} \quad \text{in } L^2((0,T), \mathbf{L}^2(\Omega)). \tag{3.14}$$

Now let $\mathbf{v} \in L^2((0,T), \mathbf{H}_0(\mathbf{curl}))$ be arbitrarily fixed and satisfy $\mathbf{v}(t) \in \mathbf{K}$ for a.e. $t \in (0,T)$. By standard properties of the limit superior, we deduce that

$$\int_0^T \big(\mathbf{f}(t), \mathbf{v}(t) - \overline{\mathbf{E}}(t)\big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t \underbrace{=}_{(3.14)} \lim_{N\to\infty}\int_0^T \big(\overline{\mathbf{f}}_N(t), \mathbf{v}(t) - \overline{\mathbf{E}}_N(t)\big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t$$

$$\underbrace{\leq}_{(\widetilde{\mathrm{P}}_N)} \limsup_{N\to\infty} \int_0^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_N(t), \mathbf{v}(t) - \overline{\mathbf{E}}_N(t)\Big)_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} + \big(\sigma\overline{\mathbf{E}}_N(t), \mathbf{v}(t) - \overline{\mathbf{E}}_N(t)\big)_{\mathbf{L}^2(\Omega)}$$
$$- \big(\overline{\mathbf{H}}_N(t), \mathbf{curl}(\mathbf{v}(t) - \overline{\mathbf{E}}_N(t))\big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t$$
$$\underbrace{\leq}_{(3.3)\&(3.7)} \int_0^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{E}}(t), \mathbf{v}(t)\Big)_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}t - \liminf_{N\to\infty}\int_0^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_N(t), \overline{\mathbf{E}}_N(t)\Big)_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}t$$
$$+ \int_0^T \big(\sigma\overline{\mathbf{E}}(t), \mathbf{v}(t)\big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t - \liminf_{N\to\infty}\int_0^T \big(\sigma\overline{\mathbf{E}}_N(t), \overline{\mathbf{E}}_N(t)\big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t$$
$$- \int_0^T \big(\overline{\mathbf{H}}(t), \mathbf{curl}\,\mathbf{v}(t)\big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t + \limsup_{N\to\infty}\int_0^T \big(\overline{\mathbf{H}}_N(t), \mathbf{curl}\,\overline{\mathbf{E}}_N(t)\big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t. \tag{3.15}$$

Our next step is to estimate the remaining terms on the right-hand side of (3.15). First of all, by the weak sequential lower semi-continuity of the squared norm, we infer that

$$\liminf_{N\to\infty}\int_0^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_N(t), \mathbf{E}_N(t)\Big)_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}t$$
$$= \liminf_{N\to\infty} \frac{1}{2}\big(\|\mathbf{E}_N(T)\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} - \|\mathbf{E}_0\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\big)$$
$$\underbrace{\geq}_{(3.10)} \frac{1}{2}\big(\|\overline{\mathbf{E}}(T)\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} - \|\mathbf{E}_0\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\big)$$
$$\underbrace{=}_{(3.12)} \int_0^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{E}}(t), \overline{\mathbf{E}}(t)\Big)_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}t, \tag{3.16}$$

and consequently

$$\liminf_{N\to\infty}\int_0^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_N(t), \overline{\mathbf{E}}_N(t)\Big)_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}t$$
$$\geq \liminf_{N\to\infty}\int_0^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_N(t), \overline{\mathbf{E}}_N(t) - \mathbf{E}_N(t)\Big)_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}t$$
$$+ \liminf_{N\to\infty}\int_0^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_N(t), \mathbf{E}_N(t)\Big)_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}t$$
$$\underbrace{\geq}_{(3.5)\&(3.16)} \int_0^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{E}}(t), \overline{\mathbf{E}}(t)\Big)_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}t. \tag{3.17}$$

Furthermore, the positive semi-definiteness of σ implies

$$
\begin{aligned}
&\liminf_{N\to\infty} \int_0^T \big(\sigma \overline{\mathbf{E}}_N(t), \overline{\mathbf{E}}_N(t)\big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t \\
&\quad = \liminf_{N\to\infty} \int_0^T \big(\sigma(\overline{\mathbf{E}}_N(t) - \overline{\mathbf{E}}(t)), \overline{\mathbf{E}}_N(t) - \overline{\mathbf{E}}(t)\big)_{\mathbf{L}^2(\Omega)} \\
&\qquad\qquad + \big(\sigma(\overline{\mathbf{E}}_N(t) - \overline{\mathbf{E}}(t)), \overline{\mathbf{E}}(t)\big)_{\mathbf{L}^2(\Omega)} + \big(\sigma \overline{\mathbf{E}}(t), \overline{\mathbf{E}}_N(t)\big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t \\
&\quad \geq \liminf_{N\to\infty} \int_0^T \big(\sigma(\overline{\mathbf{E}}_N(t) - \overline{\mathbf{E}}(t)), \overline{\mathbf{E}}(t)\big)_{\mathbf{L}^2(\Omega)} + \big(\sigma \overline{\mathbf{E}}(t), \overline{\mathbf{E}}_N(t)\big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t \\
&\quad \underbrace{=}_{(3.3)} \int_0^T \big(\sigma \overline{\mathbf{E}}(t), \overline{\mathbf{E}}(t)\big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t. \qquad (3.18)
\end{aligned}
$$

Using once again the weak sequential lower semi-continuity of the squared norm, we find that

$$
\begin{aligned}
&\limsup_{N\to\infty} -\int_0^T \Big(\mathbf{H}_N(t), \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}_N(t)\Big)_{\mathbf{L}^2_\mu(\Omega)}\,\mathrm{d}t = \limsup_{N\to\infty} \frac{1}{2}\big(\|\mathbf{H}_0\|^2_{\mathbf{L}^2_\mu(\Omega)} - \|\mathbf{H}_N(T)\|^2_{\mathbf{L}^2_\mu(\Omega)}\big) \\
&\quad \underbrace{\leq}_{(3.10)} \frac{1}{2}\big(\|\mathbf{H}_0\|^2_{\mathbf{L}^2_\mu(\Omega)} - \|\overline{\mathbf{H}}(T)\|^2_{\mathbf{L}^2_\mu(\Omega)}\big) \underbrace{=}_{(3.12)} -\int_0^T \Big(\overline{\mathbf{H}}(t), \frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{H}}(t)\Big)_{\mathbf{L}^2_\mu(\Omega)}\,\mathrm{d}t \\
&\quad \underbrace{=}_{(3.9)} \int_0^T \big(\overline{\mathbf{H}}(t), \mathbf{curl}\,\overline{\mathbf{E}}(t)\big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t, \qquad (3.19)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\limsup_{N\to\infty} \int_0^T \big(\overline{\mathbf{H}}_N(t), \mathbf{curl}\,\overline{\mathbf{E}}_N(t)\big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t \\
&\quad \underbrace{=}_{(\widetilde{\mathrm{P}}_N)} \limsup_{N\to\infty} -\int_0^T \Big(\overline{\mathbf{H}}_N(t), \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}_N(t)\Big)_{\mathbf{L}^2_\mu(\Omega)}\,\mathrm{d}t \\
&\quad \leq \limsup_{N\to\infty} -\int_0^T \Big(\overline{\mathbf{H}}_N(t) - \mathbf{H}_N(t), \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}_N(t)\Big)_{\mathbf{L}^2_\mu(\Omega)}\,\mathrm{d}t \\
&\qquad + \limsup_{N\to\infty} -\int_0^T \Big(\mathbf{H}_N(t), \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}_N(t)\Big)_{\mathbf{L}^2_\mu(\Omega)}\,\mathrm{d}t
\end{aligned}
$$

$$\underbrace{\leq}_{(3.5)\&(3.19)} \int_0^T \big(\overline{\mathbf{H}}(t), \mathbf{curl}\,\overline{\mathbf{E}}(t)\big)_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t. \tag{3.20}$$

Applying (3.17), (3.18), and (3.20) to (3.15) results in

$$\begin{aligned}\int_0^T \int_{\Omega\setminus\Omega_\sigma} & \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{E}}(t)\cdot(\mathbf{v}(t)-\overline{\mathbf{E}}(t))\,\mathrm{d}x \\ &+ \int_\Omega \sigma\overline{\mathbf{E}}(t)\cdot(\mathbf{v}(t)-\overline{\mathbf{E}}(t)) - \overline{\mathbf{H}}(t)\cdot\mathbf{curl}(\mathbf{v}(t)-\overline{\mathbf{E}}(t))\,\mathrm{d}x\,\mathrm{d}t \\ &\geq \int_0^T\int_\Omega \mathbf{f}(t)\cdot(\mathbf{v}(t)-\overline{\mathbf{E}}(t))\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{3.21}$$

for all $\mathbf{v} \in L^2((0,T),\mathbf{H}_0(\mathbf{curl}))$ such that $\mathbf{v} \in \mathbf{K}$ a.e. on $(0,T)$. Finally, to show that the variational inequality in (P_{ec}) holds, let us assume the contrary, that is,

$$\begin{aligned}&\exists \mathbf{q} \in \mathbf{K}\cap\mathbf{H}_0(\mathbf{curl}) \text{ and } \exists M \subset (0,T) \text{ with } |M|>0 \text{ such that} \\ &\int_{\Omega\setminus\Omega_\sigma} \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{E}}(t)\cdot(\mathbf{q}-\overline{\mathbf{E}}(t))\,\mathrm{d}x + \int_\Omega \sigma\overline{\mathbf{E}}(t)\cdot(\mathbf{q}-\overline{\mathbf{E}}(t)) - \overline{\mathbf{H}}(t)\cdot\mathbf{curl}(\mathbf{q}-\overline{\mathbf{E}}(t))\,\mathrm{d}x \\ &\qquad < \int_\Omega \mathbf{f}(t)\cdot(\mathbf{q}-\overline{\mathbf{E}}(t))\,\mathrm{d}x\end{aligned}$$

for a.e. $t \in M$, which implies

$$\begin{aligned}\int_M\int_{\Omega\setminus\Omega_\sigma} & \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{E}}(t)\cdot(\mathbf{q}-\overline{\mathbf{E}}(t))\,\mathrm{d}x + \int_\Omega \sigma\overline{\mathbf{E}}(t)\cdot(\mathbf{q}-\overline{\mathbf{E}}(t)) - \overline{\mathbf{H}}(t)\cdot\mathbf{curl}(\mathbf{q}-\overline{\mathbf{E}}(t))\,\mathrm{d}x\,\mathrm{d}t \\ &< \int_M\int_\Omega \mathbf{f}(t)\cdot(\mathbf{q}-\overline{\mathbf{E}}(t))\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.22}$$

Inserting $\mathbf{v} := \chi_M \mathbf{z} + \chi_{(0,T)\setminus M}\overline{\mathbf{E}}$ into (3.21) immediately contradicts (3.22). In conclusion, $(\overline{\mathbf{E}},\overline{\mathbf{H}})$ satisfies the variational inequality in (P_{ec}). This completes the existence proof.

Uniqueness and Lipschitz stability. Let $(\mathbf{E}_1,\mathbf{H}_1)$ and $(\mathbf{E}_2,\mathbf{H}_2)$ denote, respectively, solutions to (P_{ec}) associated with the initial data $(\mathbf{E}_0^1,\mathbf{H}_0^1)$, $(\mathbf{E}_0^2,\mathbf{H}_0^2)$ and the right-hand sides $\mathbf{f}_1,\mathbf{f}_2$ satisfying Assumption 1.1. Setting $\mathbf{v} = \mathbf{E}_2(s)$ in (P_{ec}) for $\mathbf{E} = \mathbf{E}_1$ (resp. $\mathbf{v} = \mathbf{E}_1(s)$ in (P_{ec}) for $\mathbf{E} = \mathbf{E}_2$) and multiplying with -1, we have

$$\begin{aligned}\int_{\Omega\setminus\Omega_\sigma} & \varepsilon\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}_1(s)\cdot(\mathbf{E}_1(s)-\mathbf{E}_2(s))\,\mathrm{d}x \\ &+ \int_\Omega \sigma\mathbf{E}_1(s)\cdot(\mathbf{E}_1(s)-\mathbf{E}_2(s)) - \mathbf{H}_1(s)\cdot\mathbf{curl}(\mathbf{E}_1(s)-\mathbf{E}_2(s))\,\mathrm{d}x\end{aligned}$$

$$\leq \int_\Omega \mathbf{f}_1(s) \cdot (\mathbf{E}_1(s) - \mathbf{E}_2(s))\, \mathrm{d}x \quad \text{for a.e. } s \in (0, T) \tag{3.23}$$

and

$$\begin{aligned}
&- \int_{\Omega \setminus \Omega_\sigma} \varepsilon \frac{\mathrm{d}}{\mathrm{d}s} \mathbf{E}_2(s) \cdot (\mathbf{E}_1(s) - \mathbf{E}_2(s))\, \mathrm{d}x \\
&\qquad - \int_\Omega \sigma \mathbf{E}_2(s) \cdot (\mathbf{E}_1(s) - \mathbf{E}_2(s)) - \mathbf{H}_2(s) \cdot \mathbf{curl}(\mathbf{E}_1(s) - \mathbf{E}_2(s))\, \mathrm{d}x \\
&\quad \leq - \int_\Omega \mathbf{f}_2(s) \cdot (\mathbf{E}_1(s) - \mathbf{E}_2(s))\, \mathrm{d}x \quad \text{for a.e. } s \in (0, T).
\end{aligned} \tag{3.24}$$

In addition, by the Faraday law for $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$, it holds that

$$\mathbf{curl}(\mathbf{E}_1(s) - \mathbf{E}_2(s)) = -\mu \frac{\mathrm{d}}{\mathrm{d}s} (\mathbf{H}_1(s) - \mathbf{H}_2(s)) \quad \text{for a.e. } s \in (0, T). \tag{3.25}$$

Adding (3.23) and (3.24) together and then applying (3.25) to the resulting inequality, we obtain by using the properties of σ as well as the Hölder and Young inequalities that

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s} \|\mathbf{E}_1(s) - \mathbf{E}_2(s)\|^2_{\mathbf{L}^2_\varepsilon(\Omega \setminus \Omega_\sigma)} + \underline{\sigma} \|\mathbf{E}_1(s) - \mathbf{E}_2(s)\|^2_{\mathbf{L}^2(\Omega_\sigma)} \\
&\qquad + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s} \|\mathbf{H}_1(s) - \mathbf{H}_2(s)\|^2_{\mathbf{L}^2_\mu(\Omega)} \\
&\quad \leq \|\mathbf{f}_1(s) - \mathbf{f}_2(s)\|_{\mathbf{L}^2(\Omega \setminus \Omega_\sigma)} \|\mathbf{E}_1(s) - \mathbf{E}_2(s)\|_{\mathbf{L}^2(\Omega \setminus \Omega_\sigma)} \\
&\qquad + \|\mathbf{f}_1(s) - \mathbf{f}_2(s)\|_{\mathbf{L}^2(\Omega_\sigma)} \|\mathbf{E}_1(s) - \mathbf{E}_2(s)\|_{\mathbf{L}^2(\Omega_\sigma)} \\
&\quad \leq \Big(\frac{1}{2\underline{\varepsilon}} + \frac{1}{2\underline{\sigma}} \Big) \|\mathbf{f}_1 - \mathbf{f}_2\|^2_{\mathcal{C}([0,T], \mathbf{L}^2(\Omega))} + \frac{1}{2} \|\mathbf{E}_1(s) - \mathbf{E}_2(s)\|^2_{\mathbf{L}^2_\varepsilon(\Omega \setminus \Omega_\sigma)} \\
&\qquad + \frac{\underline{\sigma}}{2} \|\mathbf{E}_1(s) - \mathbf{E}_2(s)\|^2_{\mathbf{L}^2(\Omega_\sigma)} \quad \text{for a.e. } s \in (0, T).
\end{aligned}$$

By integration over the time interval $(0, t)$ and rearrangement, it follows that

$$\begin{aligned}
&\|\mathbf{E}_1(t) - \mathbf{E}_2(t)\|^2_{\mathbf{L}^2_\varepsilon(\Omega \setminus \Omega_\sigma)} + \underline{\sigma} \int_0^t \|\mathbf{E}_1(s) - \mathbf{E}_2(s)\|^2_{\mathbf{L}^2(\Omega_\sigma)}\, \mathrm{d}s \\
&\qquad + \|\mathbf{H}_1(t) - \mathbf{H}_2(t)\|^2_{\mathbf{L}^2_\mu(\Omega)} \\
&\quad \leq \|\mathbf{E}_0^1 - \mathbf{E}_0^2\|^2_{\mathbf{L}^2_\varepsilon(\Omega \setminus \Omega_\sigma)} + \|\mathbf{H}_0^1 - \mathbf{H}_0^2\|^2_{\mathbf{L}^2_\mu(\Omega)} + \Big(\frac{t}{\underline{\varepsilon}} + \frac{t}{\underline{\sigma}} \Big) \|\mathbf{f}_1 - \mathbf{f}_2\|^2_{\mathcal{C}([0,T], \mathbf{L}^2(\Omega))} \\
&\qquad + \int_0^t \|\mathbf{E}_1(s) - \mathbf{E}_2(s)\|^2_{\mathbf{L}^2_\varepsilon(\Omega \setminus \Omega_\sigma)}\, \mathrm{d}s
\end{aligned}$$

$$\leq \max\Big\{1, \frac{t}{\underline{\varepsilon}} + \frac{t}{\underline{\sigma}}\Big\}\big(\|(\mathbf{E}_0^1, \mathbf{H}_0^1) - (\mathbf{E}_0^2, \mathbf{H}_0^2)\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)\times\mathbf{L}^2_\mu(\Omega)} + \|\mathbf{f}_1 - \mathbf{f}_2\|^2_{\mathcal{C}([0,T],\mathbf{L}^2(\Omega))}\big)$$
$$+ \int_0^t \|\mathbf{E}_1(s) - \mathbf{E}_2(s)\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}s, \quad \forall t \in [0, T].$$

Employing the Gronwall lemma, we then arrive at

$$\begin{aligned}
&\|(\mathbf{E}_1, \mathbf{H}_1)(t) - (\mathbf{E}_2, \mathbf{H}_2)(t)\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)\times\mathbf{L}^2_\mu(\Omega)} + \underline{\sigma}\int_0^t \|\mathbf{E}_1(s) - \mathbf{E}_2(s)\|^2_{\mathbf{L}^2(\Omega_\sigma)}\,\mathrm{d}s \\
&\quad\leq \mathrm{e}^t \max\Big\{1, \frac{t}{\underline{\varepsilon}} + \frac{t}{\underline{\sigma}}\Big\}\big(\|(\mathbf{E}_0^1, \mathbf{H}_0^1) - (\mathbf{E}_0^2, \mathbf{H}_0^2)\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)\times\mathbf{L}^2_\mu(\Omega)} \\
&\qquad + \|\mathbf{f}_1 - \mathbf{f}_2\|^2_{\mathcal{C}([0,T],\mathbf{L}^2(\Omega))}\big), \quad \forall t \in [0, T].
\end{aligned} \tag{3.26}$$

In view of (3.26), we conclude that (P_{ec}) admits at most one solution. ■

Remark 3.1. Introducing the subset

$$\begin{aligned}
\mathcal{U} := \Big\{&(\mathbf{f}, \mathbf{E}_0, \mathbf{H}_0) \in W^{1,\infty}((0,T), \mathbf{L}^2(\Omega)) \times (\mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})) \times \mathbf{H}(\mathbf{curl}) \mid \\
&\int_{\Omega_\sigma} (\sigma\mathbf{E}_0 - \mathbf{curl}\,\mathbf{H}_0)\cdot(\mathbf{v} - \mathbf{E}_0)\,\mathrm{d}x \geq \int_{\Omega_\sigma} \mathbf{f}(0)\cdot(\mathbf{v} - \mathbf{E}_0)\,\mathrm{d}x \\
&\forall \mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})\Big\}
\end{aligned}$$

of $\mathcal{C}([0,T], \mathbf{L}^2(\Omega)) \times \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$, the solution operator associated to (P_{ec})

$$\begin{aligned}
&\Phi : \mathcal{U} \to \mathcal{C}([0,T], \mathbf{L}^2(\Omega\setminus\Omega_\sigma) \times \mathbf{L}^2(\Omega)) \cap L^2((0,T), \mathbf{L}^2(\Omega_\sigma) \times \mathbf{L}^2(\Omega)), \\
&(\mathbf{f}, \mathbf{E}_0, \mathbf{H}_0) \mapsto (\mathbf{E}, \mathbf{H})
\end{aligned}$$

is Lipschitz continuous, as a consequence of (3.26).

3.1. The full eddy current case in the presence of a non-conducting region

Up to this point, the displacement current $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}$ was only neglected in the region where σ is uniformly positive definite. In this section, we suppose that $\Omega \setminus \Omega_\sigma$ is of non-zero Lebesgue measure and represents an insulating region, that is,

$$\sigma = 0 \quad \text{a.e. in } \Omega \setminus \Omega_\sigma.$$

Our focus lies on the full eddy current case where the displacement current is completely removed in the whole domain containing the insulating region $\Omega \setminus \Omega_\sigma$. Here, the previously developed analysis serves as the foundation to cover this case with some additional assumptions as follows:

Assumption 3.2. We make the following assumptions:

(6) It holds that $|\Omega \setminus \Omega_\sigma| \neq 0$ and

$$\begin{aligned} \sigma &= 0 \quad \text{a.e. in } \Omega \setminus \Omega_\sigma, \\ \mathbf{f} &= 0 \quad \text{a.e. in } (0,T) \times (\Omega \setminus \Omega_\sigma), \\ \mathbf{curl}\, \mathbf{H}_0 &= 0 \quad \text{a.e. in } \Omega \setminus \Omega_\sigma. \end{aligned} \tag{3.27}$$

(7) The obstacle set $\mathbf{K}$ satisfies one of the following conditions:

$$\text{(i)} \ \exists C > 0 \quad \forall \mathbf{v} \in \mathbf{K} : \|\mathbf{v}\|_{\mathbf{L}^2(\Omega \setminus \Omega_\sigma)} \leq C; \tag{3.28a}$$

(ii) $\mathbf{K} \subset \mathbf{X}_\varepsilon(\Omega)$ and Ω is a bounded Lipschitz domain with a connected boundary, (3.28b)

where $\mathbf{X}_\varepsilon(\Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) \mid (\varepsilon \mathbf{v}, \nabla \phi)_{\mathbf{L}^2(\Omega)} = 0 \ \forall \phi \in H_0^1(\Omega)\}$.

Remark 3.3. (i) As $\Omega \setminus \Omega_\sigma$ represents an insulating region such as air, it is physically reasonable to assume that no current source is present in the insulator. Condition (3.27) on the vanishing source and vanishing initial rotational magnetic field in the insulator is indeed common in the study of the eddy current problems (see, for example, [29, page 42] or [3, page 239]).

(ii) Condition (3.28a) is obviously satisfied if the obstacle set $\mathbf{K}$ is bounded in $\mathbf{L}^2(\Omega)$. A prominent example is the set $\mathbf{K} = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \mid |\mathbf{v}(x)| \leq d(x) \text{ for a.e. } x \in \Omega\}$ for some electric obstacle $d \in L^2(\Omega)$. On the other hand, condition (3.28b) describes a physical medium with vanishing charge density, that is, the case where the electric field satisfies $\operatorname{div}(\varepsilon \mathbf{E}) \equiv 0$.

Let us now state the full eddy current problem we focus on in this section:

$$\begin{cases} \displaystyle\int_{\Omega_\sigma} \sigma \mathbf{E}(t) \cdot (\mathbf{v} - \mathbf{E}(t))\, \mathrm{d}x - \int_{\Omega} \mathbf{H}(t) \cdot \mathbf{curl}(\mathbf{v} - \mathbf{E}(t))\, \mathrm{d}x \\ \qquad \displaystyle\geq \int_{\Omega_\sigma} \mathbf{f}(t) \cdot (\mathbf{v} - \mathbf{E}(t))\, \mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl}) \text{ for a.e. } t \in (0,T); \\ \mu \dfrac{\mathrm{d}}{\mathrm{d}t} \mathbf{H}(t) + \mathbf{curl}\, \mathbf{E}(t) = \mathbf{0} \quad \text{for a.e. } t \in (0,T); \\ \mathbf{E}(t) \in \mathbf{K} \quad \text{for a.e. } t \in (0,T), \quad \mathbf{H}(0) = \mathbf{H}_0 \quad \text{a.e. in } \Omega. \end{cases} \tag{$\mathrm{P}_{\mathrm{ec}}^0$}$$

Note that in contrast to (P_{ec}), problem ($\mathrm{P}_{\mathrm{ec}}^0$) comprises an elliptic VI for the electric field $\mathbf{E}$ and an evolutionary equation for the magnetic field $\mathbf{H}$, which is why we do not impose any initial condition for $\mathbf{E}$ (cf. [23] for the case of the full eddy current equations with a constant and scalar conductivity $\sigma > 0$). The time-discrete approximation for ($\mathrm{P}_{\mathrm{ec}}^0$) reads as finding $\{(\mathbf{E}_n, \mathbf{H}_n)\}_{n=1}^N \subset (\mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})) \times \mathbf{H}(\mathbf{curl})$ such that

$$\begin{cases} \int\limits_{\Omega_\sigma} \sigma \mathbf{E}_n \cdot (\mathbf{v} - \mathbf{E}_n)\,\mathrm{d}x - \int\limits_{\Omega} \mathbf{H}_n \cdot \mathbf{curl}(\mathbf{v} - \mathbf{E}_n)\,\mathrm{d}x \\ \qquad \geq \int\limits_{\Omega_\sigma} \mathbf{f}_n \cdot (\mathbf{v} - \mathbf{E}_n)\,\mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl}),\ \forall n \in \{1, \ldots, N\}; \\ \mu\delta \mathbf{H}_n + \mathbf{curl}\,\mathbf{E}_n = \mathbf{0}, \quad \forall n \in \{1, \ldots, N\}. \end{cases} \tag{P_N^0}$$

To prove the well-posedness of (P_N^0), we reformulate it as a minimization problem in a Hilbert space as follows:

Lemma 3.4. *Let Assumption 1.1 and Assumption 3.2 be satisfied. Then, the time-discrete problem (see (P_N^0)) admits a solution $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^N \subset (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$. If (3.28b) holds true, then the solution to (P_N^0) is unique.*

Proof. First, using the discrete Faraday law, we rewrite problem (P_N^0) as

$$\begin{aligned} &\int\limits_{\Omega_\sigma} \sigma \mathbf{E}_n \cdot (\mathbf{v} - \mathbf{E}_n)\,\mathrm{d}x + \tau \int\limits_{\Omega} \mu^{-1}\,\mathbf{curl}\,\mathbf{E}_n \cdot \mathbf{curl}(\mathbf{v} - \mathbf{E}_n)\,\mathrm{d}x \\ &\quad \geq \int\limits_{\Omega_\sigma} \mathbf{f}_n \cdot (\mathbf{v} - \mathbf{E}_n)\,\mathrm{d}x \\ &\qquad + \int\limits_{\Omega} \mathbf{H}_{n-1} \cdot \mathbf{curl}(\mathbf{v} - \mathbf{E}_n)\,\mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl}),\ \forall n \in \{1, \ldots, N\}, \end{aligned}$$

which is equivalent to the minimization problem

$$\begin{aligned} \min_{\mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})} \Big(&\frac{1}{2}\|\mathbf{v}\|^2_{\mathbf{L}^2_\sigma(\Omega_\sigma)} + \frac{\tau}{2}\|\,\mathbf{curl}\,\mathbf{v}\|^2_{\mathbf{L}^2_{\mu^{-1}}(\Omega)} \\ &- \int\limits_{\Omega_\sigma} \mathbf{f}_n \cdot \mathbf{v}\,\mathrm{d}x - \int\limits_{\Omega} \mathbf{H}_{n-1} \cdot \mathbf{curl}\,\mathbf{v}\,\mathrm{d}x \Big), \quad \forall n \in \{1, \ldots, N\}. \end{aligned} \tag{3.29}$$

Next, let $n \in \{1, \ldots, N\}$ be arbitrarily fixed. For any $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl})$, it holds that

$$\begin{aligned} &\frac{1}{2}\|\mathbf{v}\|^2_{\mathbf{L}^2_\sigma(\Omega_\sigma)} + \frac{\tau}{2}\|\,\mathbf{curl}\,\mathbf{v}\|^2_{\mathbf{L}^2_{\mu^{-1}}(\Omega)} - \int\limits_{\Omega_\sigma} \mathbf{f}_n \cdot \mathbf{v}\,\mathrm{d}x - \int\limits_{\Omega} \mathbf{H}_{n-1} \cdot \mathbf{curl}\,\mathbf{v}\,\mathrm{d}x \\ &\quad \geq \frac{1}{4}\|\mathbf{v}\|^2_{\mathbf{L}^2_\sigma(\Omega_\sigma)} + \frac{\tau}{4}\|\,\mathbf{curl}\,\mathbf{v}\|^2_{\mathbf{L}^2_{\mu^{-1}}(\Omega)} - \frac{1}{\underline{\sigma}}\|\mathbf{f}_n\|^2_{\mathbf{L}^2(\Omega_\sigma)} - \frac{\overline{\mu}}{\tau}\|\mathbf{H}_{n-1}\|^2_{\mathbf{L}^2(\Omega)} \\ &\quad \geq -\frac{1}{\underline{\sigma}}\|\mathbf{f}_n\|^2_{\mathbf{L}^2(\Omega_\sigma)} - \frac{\overline{\mu}}{\tau}\|\mathbf{H}_{n-1}\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{3.30}$$

This shows that the objective functional associated with (3.29) is bounded from below. Therefore, since $\mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})$ is non-empty, we have that there exists an infimal sequence

$\{\mathbf{v}_k^n\}_{k=1}^\infty \subset \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})$ for minimization problem (3.29). Thanks to (3.30), the infimal sequence $\{\mathbf{v}_k^n\}_{k=1}^\infty \subset \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})$ satisfies

$$\|\mathbf{v}_k^n\|_{\mathbf{L}^2(\Omega_\sigma)} + \|\mathbf{curl}\,\mathbf{v}_k^n\|_{\mathbf{L}^2(\Omega)} \leq C, \quad k \in \mathbb{N}, \tag{3.31}$$

for some constant $C > 0$, independent of k. Now, if (3.28a) is satisfied, then in view of (3.31) it follows that the infimal sequence $\{\mathbf{v}_k^n\}_{k=1}^\infty$ is bounded in $\mathbf{H}_0(\mathbf{curl})$. On the other hand, if (3.28b) is satisfied, then it implies the Poincaré–Friedrichs-type inequality [2, Lemma 3.1]

$$\exists C_\mathrm{p} > 0 \quad \forall \mathbf{v} \in \mathbf{X}_\varepsilon(\Omega) \cap \mathbf{H}_0(\mathbf{curl}) \ : \ \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \leq C_\mathrm{p}\|\mathbf{curl}\,\mathbf{v}\|_{\mathbf{L}^2(\Omega)}, \tag{3.32}$$

which yields due to (3.31) the boundedness of $\{\mathbf{v}_k^n\}_{k=1}^\infty$ in $\mathbf{H}_0(\mathbf{curl})$. In conclusion, for every $n \in \{1, \ldots, N\}$, the existence of a minimizer to (3.29) follows by standard arguments as in the proof of the direct method of variational calculus. Finally, if (3.28b) holds true, then due to (3.32) the objective functional associated with (3.29) is strictly convex, and so minimization problem (3.29) admits a unique solution. ∎

Lemma 3.5. *Let Assumption 1.1 and Assumption 3.2 hold. Then, there exists a positive real constant C_0, depending only on $T, \mu, \sigma, \boldsymbol{f}, \boldsymbol{H}_0$ such that, for any $N \in \mathbb{N}$, every solution $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^N \subset (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ of (P_N^0) satisfies*

$$\max_{1\leq n\leq N}\left[\|\boldsymbol{E}_n\|_{L^2(\Omega)} + \|\boldsymbol{H}_n\|_{L^2(\Omega)} + \|\delta\boldsymbol{H}_n\|_{L^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{E}_n\|_{L^2(\Omega)}\right] \leq C_0. \tag{3.33}$$

Proof. Let $N \in \mathbb{N}$ be arbitrarily fixed and let $\{(\mathbf{E}_n, \mathbf{H}_n)\}_{n=1}^N \subset (\mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})) \times \mathbf{H}(\mathbf{curl})$ denote a solution to (P_N^0). Further, let $\mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl})$ and $n_0 \in \{1, \ldots, N\}$ be arbitrarily fixed. The lines of argumentation are similar to Theorem 2.1, where we simply set $\mathbf{z}$ to be zero, thanks to (3.27) and since ε does not appear in (P_N^0). Then, together with the fact that $\mathbf{f} = 0$ a.e. in $(0, T) \times (\Omega \setminus \Omega_\sigma)$, by analogous argumentation to the proof of Theorem 2.1, it follows that

$$\begin{aligned}
&\frac{1}{2}\Big(\|\delta\mathbf{H}_{n_0}\|^2_{\mathbf{L}^2_\mu(\Omega)} - \|\delta\mathbf{H}_0\|^2_{\mathbf{L}^2_\mu(\Omega)} + \sum_{n=1}^{n_0}\|\delta\mathbf{H}_n - \delta\mathbf{H}_{n-1}\|^2_{\mathbf{L}^2_\mu(\Omega)}\Big) + \sum_{n=1}^{n_0}\tau\underline{\sigma}\|\delta\mathbf{E}_n\|^2_{\mathbf{L}^2(\Omega_\sigma)} \\
&\quad \leq \sum_{n=1}^{n_0}\|\mathbf{f}_n - \mathbf{f}_{n-1}\|_{\mathbf{L}^2(\Omega_\sigma)}\|\delta\mathbf{E}_n\|_{\mathbf{L}^2(\Omega_\sigma)} + \sum_{n=1}^{n_0}\|\mathbf{w}_n^N - \mathbf{w}_{n-1}^N\|_{\mathbf{L}^2(\Omega)}\|\delta\mathbf{H}_n\|_{\mathbf{L}^2(\Omega)}.
\end{aligned} \tag{3.34}$$

In turn, this implies the estimate

$$\frac{1}{2}\|\delta\mathbf{H}_{n_0}\|^2_{\mathbf{L}^2_\mu(\Omega)} \leq \frac{L^2 T}{4\underline{\sigma}} + \frac{3}{4}\|\mathbf{H}_0\|^2_{\mathbf{L}^2_\mu(\Omega)} + \frac{2}{\underline{\mu}}\|\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)}.$$

In view of the above estimate, we obtain $\|\mathbf{H}_{n_0}\|^2_{\mathbf{L}^2_\mu(\Omega)} \leq C$ and $\|\mathbf{curl}\,\mathbf{E}_{n_0}\|_{\mathbf{L}^2(\Omega)} \leq C$, due to (2.16) and the discrete Faraday law (P_N^0). The bound on $\|\mathbf{E}_{n_0}\|_{\mathbf{L}^2(\Omega_\sigma)}$ is obtained by testing with $\mathbf{v} = 0$ in (P_N^0) and proceeding as in (2.18) and (2.19). The bound on $\|\mathbf{E}_{n_0}\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}$ is an immediate result of (3.28a) or (3.28b) along with the Poincaré–Friedrichs-type inequality (3.32) and the estimate $\|\mathbf{curl}\,\mathbf{E}_{n_0}\|_{\mathbf{L}^2(\Omega)} \leq C$. ∎

In view of (P_N^0), invoking again constructions (3.1) and (3.2), it follows that the interpolations satisfy

$$\begin{cases} \int_{\Omega_\sigma} \sigma \overline{\mathbf{E}}_N(t) \cdot (\mathbf{v} - \overline{\mathbf{E}}_N(t)) \, \mathrm{d}x - \int_{\Omega} \overline{\mathbf{H}}_N(t) \cdot \mathbf{curl}(\mathbf{v} - \overline{\mathbf{E}}_N(t)) \, \mathrm{d}x \\ \qquad \geq \int_{\Omega_\sigma} \bar{\mathbf{f}}_N(t) \cdot (\mathbf{v} - \overline{\mathbf{E}}_N(t)) \, \mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl}) \text{ for all } t \in (0, T]; \\ \mu \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{H}_N(t) + \mathbf{curl}\, \overline{\mathbf{E}}_N(t) = \mathbf{0} \quad \text{for all } t \in (0, T]; \\ \overline{\mathbf{E}}_N(t) \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl}) \quad \text{for all } t \in [0, T]. \end{cases} \tag{$\widetilde{\mathrm{P}}_N^0$}$$

Theorem 3.2. *Let Assumption 1.1 and Assumption 3.2 hold. Then, the eddy current obstacle problem* ($\mathrm{P}_{\mathrm{ec}}^0$) *admits a solution* $(\boldsymbol{E}, \boldsymbol{H}) \in L^\infty((0, T), \boldsymbol{H}_0(\mathbf{curl})) \times W^{1,\infty}((0, T), \boldsymbol{L}^2(\Omega))$. *If* (3.28b) *holds true, then the solution to* ($\mathrm{P}_{\mathrm{ec}}^0$) *is unique.*

Proof. First, as in the proof of Theorem 3.1, the a priori estimate from Theorem 3.5 yields the existence of a subsequence of $\{(\mathbf{E}_N, \mathbf{H}_N)\}_{N=1}^\infty$, denoted again by the same symbol, such that

$$\left(\overline{\mathbf{E}}_N, \mathbf{H}_N, \overline{\mathbf{H}}_N, \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}_N\right) \overset{*}{\rightharpoonup} \left(\overline{\mathbf{E}}, \overline{\mathbf{H}}, \overline{\mathbf{H}}, \frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{H}}\right)$$

weakly-* in $L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega))$ as $N \to \infty$ for some $(\overline{\mathbf{E}}, \overline{\mathbf{H}}) \in L^\infty((0, T), \mathbf{H}_0(\mathbf{curl})) \times W^{1,\infty}((0, T), \mathbf{L}^2(\Omega))$. Passing to the limit in the discrete Faraday law as in (3.9), we then obtain

$$\mu \frac{\mathrm{d}}{\mathrm{d}t} \overline{\mathbf{H}}(t) + \mathbf{curl}\, \overline{\mathbf{E}}(t) = \mathbf{0} \quad \text{for a.e. } t \in (0, T).$$

Analogously to (3.11), we obtain the pointwise weak convergence

$$\mathbf{H}_N(t) \rightharpoonup \overline{\mathbf{H}}(t) \quad \text{weakly in } \mathbf{L}^2(\Omega) \text{ as } N \to \infty \text{ for all } t \in [0, T],$$

which implies the initial condition $\overline{\mathbf{H}}(0) = \mathbf{H}_0$ a.e. in Ω. Also, as in the proof of Theorem 3.1, the above weak-star convergence yields the feasibility $\overline{\mathbf{E}}(t) \in \mathbf{K}$ for a.e. $t \in (0, T)$. Ultimately, the final passage to the limit in ($\widetilde{\mathrm{P}}_N^0$) follows again the same arguments as in the proof of Theorem 3.1. In conclusion, the weak-star limit $(\overline{\mathbf{E}}, \overline{\mathbf{H}})$ satisfies ($\mathrm{P}_{\mathrm{ec}}^0$). Let us now assume that (3.28b) is valid and let $(\mathbf{E}_1, \mathbf{H}_1)$, and $(\mathbf{E}_2, \mathbf{H}_2)$ denote, respectively, solutions to ($\mathrm{P}_{\mathrm{ec}}^0$). Setting $\mathbf{v} = \mathbf{E}_2(s)$ in ($\mathrm{P}_{\mathrm{ec}}^0$) for $\mathbf{E} = \mathbf{E}_1$ (resp. $\mathbf{v} = \mathbf{E}_1(s)$ in ($\mathrm{P}_{\mathrm{ec}}^0$) for $\mathbf{E} = \mathbf{E}_2$) we can proceed as in (3.23), (3.24) and (3.25) to obtain the estimate

$$\|\mathbf{E}_1(s) - \mathbf{E}_2(s)\|_{\mathbf{L}_\sigma^2(\Omega_\sigma)}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\mathbf{H}_1(s) - \mathbf{H}_2(s)\|_{\mathbf{L}_\mu^2(\Omega)}^2 \leq 0 \quad \text{for a.e. } s \in (0, T).$$

As $\mathbf{H}_1(0) = \mathbf{H}_0 = \mathbf{H}_2(0)$, the above inequality implies that $\mathbf{H}_1 = \mathbf{H}_2$, which yields due to the Faraday law in ($\mathrm{P}_{\mathrm{ec}}^0$) that $\mathbf{curl}(\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{0}$. As a result of the Poincaré–Friedrichs-type inequality (3.32), it then follows that $\mathbf{E}_1 = \mathbf{E}_2$. This completes the proof. ■

Remark 3.6. Let us mention that even in the case of the eddy current equations, without assuming additional conditions such as (3.28b), uniqueness of the solution cannot be expected in general.

Remark 3.7. The analysis in this section with respect to condition (3.28b) reveals that a local Poincaré–Friedrichs-type inequality in the insulator $\Omega \setminus \Omega_\sigma$ is sufficient to obtain an existence and uniqueness result for ($\mathrm{P}^0_{\mathrm{ec}}$). This allows us to work with another obstacle set $\mathbf{K}$ as follows: Suppose again that Ω is a bounded Lipschitz domain such that $\overline{\Omega_\sigma} \subset \Omega$ and $\Omega \setminus \Omega_\sigma$ is connected. Then, for the obstacle set $\mathbf{K}$, an alternative assumption to (3.28b) reads

$$\begin{aligned}\mathbf{K} \subset \widetilde{\mathbf{X}_\varepsilon}(\Omega) := \big\{\mathbf{v} \in \mathbf{L}^2(\Omega) \mid (\varepsilon\mathbf{v}, \nabla\phi)_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} &= 0 \quad \forall \phi \in H^1(\Omega \setminus \Omega_\sigma), \\ (\varepsilon\mathbf{v}, \mathbf{h})_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} &= 0 \quad \forall\, \mathbf{h} \in \mathscr{H}\big\},\end{aligned}$$

where $\mathscr{H}$ denotes the finite-dimensional vector space of Neumann fields related to topological quantities of the physical domain Ω and the insulating region $\Omega \setminus \Omega_\sigma$ (see [3, page 13] for its definition). As proven in [3, Lemma 2.2], the Poincaré–Friedrichs-type inequality below holds true:

$$\begin{aligned}&\exists C_{\mathrm{p}} > 0 \quad \forall \mathbf{v} \in \widetilde{\mathbf{X}_\varepsilon}(\Omega) \cap \mathbf{H}_0(\mathbf{curl}) : \\ &\qquad \|\mathbf{v}\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} \le C_{\mathrm{p}}\big(\|\,\mathbf{curl}\,\mathbf{v}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{v}\|_{\mathbf{L}^2(\Omega_\sigma)}\big).\end{aligned} \tag{3.35}$$

With (3.35) at hand, the existence of a unique solution to ($\mathrm{P}^0_{\mathrm{ec}}$) is obtained under minor and obvious changes of this section.

4. Justification of the eddy current model

Theorem 3.1 implies that both the Maxwell obstacle problem (P) (by choosing $\Omega_\sigma = \emptyset$) and the eddy current model (P_{ec}) admit unique solutions, which we denote in the following, respectively, by

$$(\mathbf{E}, \mathbf{H}) \in W^{1,\infty}((0,T), \mathbf{L}^2(\Omega \setminus \Omega_\sigma) \times \mathbf{L}^2(\Omega)) \cap L^\infty((0,T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega)) \tag{4.1}$$

and

$$(\mathbf{E}_{\mathrm{ec}}, \mathbf{H}_{\mathrm{ec}}) \in W^{1,\infty}((0,T), \mathbf{L}^2(\Omega \setminus \Omega_\sigma) \times \mathbf{L}^2(\Omega)) \cap L^\infty((0,T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega)). \tag{4.2}$$

Our goal now is to justify the eddy current model (P_{ec}) in the sense that its unique solution $(\mathbf{E}_{\mathrm{ec}}, \mathbf{H}_{\mathrm{ec}})$ is close to $(\mathbf{E}, \mathbf{H})$ under a reasonable smallness condition on $\|\varepsilon/\underline{\sigma}\|_{L^\infty(\Omega_\sigma)^{3\times3}}$. By proposing an additional assumption on the initial data (see Assumption 4.1), we are able to not only justify the eddy current model but also prove an a priori error estimate for the eddy current approximation with a linear convergence rate in terms of $\|\varepsilon\|_{L^\infty(\Omega_\sigma)^{3\times3}}$.

Assumption 4.1. The initial value $(\mathbf{E}_0, \mathbf{H}_0)$ satisfies

$$\int\limits_{\Omega\setminus\Omega_\sigma} (\sigma \mathbf{E}_0 - \mathbf{curl}\, \mathbf{H}_0) \cdot (\mathbf{v} - \mathbf{E}_0)\, \mathrm{d}x \geq \int\limits_{\Omega\setminus\Omega_\sigma} \mathbf{f}(0) \cdot (\mathbf{v} - \mathbf{E}_0)\, \mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{K} \cap \mathbf{H}_0(\mathbf{curl}) \tag{4.3}$$

and

$$\mathbf{curl}\, \mathbf{E}_0 = 0 \quad \text{a.e. on } \Omega. \tag{4.4}$$

Remark 4.2. The inequality in (4.3) is of technical importance and is trivially satisfied if $\sigma \mathbf{E}_0 - \mathbf{curl}\, \mathbf{H}_0 = \mathbf{f}(0)$ a.e. in $\Omega \setminus \Omega_\sigma$. Note that in real applications $\Omega \setminus \Omega_\sigma$ typically represents a non-conducting medium such that the conductivity $\sigma_{|\Omega\setminus\Omega_\sigma}$ is zero. In this case (4.3) is satisfied if $-\mathbf{curl}\, \mathbf{H}_0 = \mathbf{f}(0)$ a.e. in $\Omega\setminus\Omega_\sigma$.

In what follows, if $|\Omega \setminus \Omega_\sigma| \neq 0$, the constant $\underline{\varepsilon}(\Omega \setminus \Omega_\sigma)$ denotes a uniform lower bound for the lowest eigenvalues of ε in $\Omega \setminus \Omega_\sigma$, that is, it satisfies

$$\varepsilon(x)\xi \cdot \xi \geq \underline{\varepsilon}(\Omega \setminus \Omega_\sigma)|\xi|^2 \quad \text{for a.e. } x \in \Omega \setminus \Omega_\sigma \text{ and all } \xi \in \mathbb{R}^3. \tag{4.5}$$

Furthermore, let $L(\Omega_\sigma)$ and $L(\Omega \setminus \Omega_\sigma)$ denote, respectively, the Lipschitz constants of $\mathbf{f}$ in Ω_σ and $\Omega \setminus \Omega_\sigma$, that is,

$$\begin{aligned} \|\mathbf{f}(t_1) - \mathbf{f}(t_2)\|_{\mathbf{L}^2(\Omega_\sigma)} &\leq L(\Omega_\sigma)|t_1 - t_2| \qquad \forall t_1, t_2 \in [0, T], \\ \|\mathbf{f}(t_1) - \mathbf{f}(t_2)\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)} &\leq L(\Omega \setminus \Omega_\sigma)|t_1 - t_2| \quad \forall t_1, t_2 \in [0, T]. \end{aligned} \tag{4.6}$$

Theorem 4.1. *Let Assumption 1.1 and Assumption 4.1 be satisfied. If $|\Omega \setminus \Omega_\sigma| \neq 0$, then it holds that*

$$\begin{aligned} &\|(\boldsymbol{E}, \boldsymbol{H}) - (\boldsymbol{E}_{\mathrm{ec}}, \boldsymbol{H}_{\mathrm{ec}})\|_{\mathcal{C}([0,T], L^2_\varepsilon(\Omega\setminus\Omega_\sigma)\times L^2_\mu(\Omega))} + \|\boldsymbol{E} - \boldsymbol{E}_{\mathrm{ec}}\|_{L^2((0,T), L^2_\sigma(\Omega_\sigma))} \\ &\quad \leq 2\Bigg(\frac{L(\Omega_\sigma)^2 T}{\underline{\sigma}} + \frac{2L(\Omega \setminus \Omega_\sigma)T}{\sqrt{\underline{\varepsilon}(\Omega \setminus \Omega_\sigma)}} \\ &\qquad \cdot \sqrt{\frac{4L(\Omega \setminus \Omega_\sigma)^2 T^2}{\underline{\varepsilon}(\Omega \setminus \Omega_\sigma)} + \frac{2L(\Omega_\sigma)^2 T}{\underline{\sigma}}}\Bigg)^{1/2} \left\| \frac{\varepsilon}{\underline{\sigma}} \right\|_{L^\infty(\Omega_\sigma)^{3\times3}}. \end{aligned} \tag{4.7}$$

If $\Omega_\sigma = \Omega$, then

$$\|\boldsymbol{H} - \boldsymbol{H}_{\mathrm{ec}}\|_{\mathcal{C}([0,T], L^2_\mu(\Omega))} + \|\boldsymbol{E} - \boldsymbol{E}_{\mathrm{ec}}\|_{L^2((0,T), L^2_\sigma(\Omega))} \leq 2\frac{L\sqrt{T}}{\sqrt{\underline{\sigma}}} \left\| \frac{\varepsilon}{\underline{\sigma}} \right\|_{L^\infty(\Omega)^{3\times3}}. \tag{4.8}$$

Remark 4.3. If the applied current source $\mathbf{f}$ is only acting in the conducting region Ω_σ, we have $L(\Omega \setminus \Omega_\sigma) = 0$ so that the upper bound for (4.7) precisely coincides with the one in (4.8) given by $2L\sqrt{T}/\sqrt{\underline{\sigma}}\|\varepsilon/\underline{\sigma}\|_{L^\infty(\Omega)^{3\times3}}$.

Proof. We split the proof into three parts.

Step 1: Boundedness of $t \mapsto \int_0^t \|\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{E}(s)\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\,\mathrm{d}s$ *with an upper bound being independent of* $\varepsilon_{|\Omega_\sigma}$. Setting $\mathbf{v} = \mathbf{E}(s+h)$ (resp. $\mathbf{v} = \mathbf{E}(s)$) in (P) for $t = s$ (resp. $t = s+h$) and then adding the resulting inequalities, we obtain (similarly to the uniqueness proof for Theorem 3.1) by employing the Faraday law, the properties of σ, and Hölder's inequality that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\mathbf{E}(s+h)-\mathbf{E}(s)\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\mathbf{E}(s+h)-\mathbf{E}(s)\|^2_{\mathbf{L}^2_\varepsilon(\Omega_\sigma)}\\
&\qquad + \underline{\sigma}\|\mathbf{E}(s+h)-\mathbf{E}(s)\|^2_{\mathbf{L}^2(\Omega_\sigma)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\mathbf{H}(s+h)-\mathbf{H}(s)\|^2_{\mathbf{L}^2_\mu(\Omega)}\\
&\quad \le \|\mathbf{f}(s+h)-\mathbf{f}(s)\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\|\mathbf{E}(s+h)-\mathbf{E}(s)\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\\
&\qquad + \|\mathbf{f}(s+h)-\mathbf{f}(s)\|_{\mathbf{L}^2(\Omega_\sigma)}\|\mathbf{E}(s+h)-\mathbf{E}(s)\|_{\mathbf{L}^2(\Omega_\sigma)}
\end{aligned}$$

for a.e. $s \in (0,T)$ and a.e. $h \in (0, T-s)$. Integrating the above inequality over $(0,t)$ and dividing by h^2, we obtain that

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\mathbf{E}(t+h)-\mathbf{E}(t)}{h}\right\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} - \frac{1}{2}\left\|\frac{\mathbf{E}(h)-\mathbf{E}_0}{h}\right\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\\
&\quad + \frac{1}{2}\left\|\frac{\mathbf{E}(t+h)-\mathbf{E}(t)}{h}\right\|^2_{\mathbf{L}^2_\varepsilon(\Omega_\sigma)} - \frac{1}{2}\left\|\frac{\mathbf{E}(h)-\mathbf{E}_0}{h}\right\|^2_{\mathbf{L}^2_\varepsilon(\Omega_\sigma)}\\
&\quad + \underline{\sigma}\int_0^t\left\|\frac{\mathbf{E}(s+h)-\mathbf{E}(s)}{h}\right\|^2_{\mathbf{L}^2(\Omega_\sigma)}\mathrm{d}s + \frac{1}{2}\left\|\frac{\mathbf{H}(t+h)-\mathbf{H}(t)}{h}\right\|^2_{\mathbf{L}^2_\mu(\Omega)}\\
&\quad - \frac{1}{2}\left\|\frac{\mathbf{H}(h)-\mathbf{H}_0}{h}\right\|^2_{\mathbf{L}^2_\mu(\Omega)}\\
&\le \int_0^t\left\|\frac{\mathbf{f}(s+h)-\mathbf{f}(s)}{h}\right\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\left\|\frac{\mathbf{E}(s+h)-\mathbf{E}(s)}{h}\right\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\mathrm{d}s\\
&\quad + \int_0^t\left\|\frac{\mathbf{f}(s+h)-\mathbf{f}(s)}{h}\right\|_{\mathbf{L}^2(\Omega_\sigma)}\left\|\frac{\mathbf{E}(s+h)-\mathbf{E}(s)}{h}\right\|_{\mathbf{L}^2(\Omega_\sigma)}\mathrm{d}s\\
&\underbrace{\le}_{(4.5)} \frac{1}{\sqrt{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)}}\int_0^t\left\|\frac{\mathbf{f}(s+h)-\mathbf{f}(s)}{h}\right\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\left\|\frac{\mathbf{E}(s+h)-\mathbf{E}(s)}{h}\right\|_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\mathrm{d}s\\
&\quad + \int_0^t\frac{1}{2\underline{\sigma}}\left\|\frac{\mathbf{f}(s+h)-\mathbf{f}(s)}{h}\right\|^2_{\mathbf{L}^2(\Omega_\sigma)}\mathrm{d}s + \frac{\underline{\sigma}}{2}\int_0^t\left\|\frac{\mathbf{E}(s+h)-\mathbf{E}(s)}{h}\right\|^2_{\mathbf{L}^2(\Omega_\sigma)}\mathrm{d}s
\end{aligned} \tag{4.9}$$

for any $t \in (0,T)$ and $h \in (0, T-t)$. Note that if $|\Omega\setminus\Omega_\sigma| = 0$ then all integrals over $\Omega\setminus\Omega_\sigma$ vanish, and we may simply set $\underline{\varepsilon}(\Omega\setminus\Omega_\sigma) = 1$ in the case of $|\Omega\setminus\Omega_\sigma| = 0$. Now, by Lipschitz property (4.6) and the regularity property $\mathbf{E} \in W^{1,\infty}((0,T), \mathbf{L}^2(\Omega))$, it follows that

$$\left\|\frac{\mathbf{E}(t+h)-\mathbf{E}(t)}{h}\right\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} + \underline{\sigma}\int_0^t \left\|\frac{\mathbf{E}(s+h)-\mathbf{E}(s)}{h}\right\|^2_{\mathbf{L}^2(\Omega_\sigma)} \mathrm{d}s$$
$$\leq \left\|\frac{\mathbf{E}(h)-\mathbf{E}_0}{h}\right\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} + \left\|\frac{\mathbf{E}(h)-\mathbf{E}_0}{h}\right\|^2_{\mathbf{L}^2_\varepsilon(\Omega_\sigma)} + \left\|\frac{\mathbf{H}(h)-\mathbf{H}_0}{h}\right\|^2_{\mathbf{L}^2_\mu(\Omega)}$$
$$+ \frac{2L(\Omega\setminus\Omega_\sigma)t}{\sqrt{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)}}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\right\|_{L^\infty((0,t),\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma))}$$
$$+ \frac{L(\Omega_\sigma)^2 t}{\underline{\sigma}}, \quad \forall t\in(0,T),\ h\in(0,T-t). \tag{4.10}$$

Our goal now is to show the boundedness of the difference quotients at the point 0 appearing on the right-hand side of (4.10). Setting $\mathbf{v}=\mathbf{E}_0$ in (P) yields

$$\int_\Omega \varepsilon\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}(s)\cdot(\mathbf{E}(s)-\mathbf{E}_0) + \sigma\mathbf{E}(s)\cdot(\mathbf{E}(s)-\mathbf{E}_0) - \mathbf{H}(s)\cdot\mathbf{curl}(\mathbf{E}(s)-\mathbf{E}_0)\,\mathrm{d}x$$
$$\leq \int_\Omega \mathbf{f}(s)\cdot(\mathbf{E}(s)-\mathbf{E}_0)\,\mathrm{d}x \tag{4.11}$$

for a.e. $s\in(0,T)$. On the other hand, since $\mathbf{E}(s)\in\mathbf{K}\cap\mathbf{H}_0(\mathbf{curl})$ holds for a.e. $s\in(0,T)$, a combination of (1.8) and (4.3) ensures that

$$\int_\Omega -\varepsilon\underbrace{\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}_0}_{=0}\cdot(\mathbf{E}(s)-\mathbf{E}_0) - \sigma\mathbf{E}_0\cdot(\mathbf{E}(s)-\mathbf{E}_0) + \mathbf{H}_0\cdot\mathbf{curl}(\mathbf{E}(s)-\mathbf{E}_0)\,\mathrm{d}x$$
$$\leq \int_\Omega -\mathbf{f}(0)\cdot(\mathbf{E}(s)-\mathbf{E}_0)\,\mathrm{d}x \tag{4.12}$$

for a.e. $s\in(0,T)$. Therefore, adding (4.11) and (4.12) together results in

$$\int_\Omega \varepsilon\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{E}(s)-\mathbf{E}_0)\cdot(\mathbf{E}(s)-\mathbf{E}_0) + \sigma(\mathbf{E}(s)-\mathbf{E}_0)\cdot(\mathbf{E}(s)-\mathbf{E}_0)$$
$$-(\mathbf{H}(s)-\mathbf{H}_0)\cdot\mathbf{curl}(\mathbf{E}(s)-\mathbf{E}_0)\,\mathrm{d}x$$
$$\leq \int_\Omega (\mathbf{f}(s)-\mathbf{f}(0))\cdot(\mathbf{E}(s)-\mathbf{E}_0)\,\mathrm{d}x \quad \text{for a.e. } s\in(0,T). \tag{4.13}$$

In addition, the Faraday law in (P) along with (4.4) yields

$$\mathbf{curl}(\mathbf{E}(s)-\mathbf{E}_0) = -\mu\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{H}(s)-\mathbf{H}_0) \quad \text{for a.e. } s\in(0,T). \tag{4.14}$$

Applying (4.14) to (4.13), integrating the resulting inequality over $(0,h)$ and dividing by h^2, we follow the same argumentation as before to deduce by the Hölder and Young inequalities as well as the properties of σ that

$$\begin{aligned}
&\frac{1}{2}\Big\|\frac{\mathbf{E}(h)-\mathbf{E}_0}{h}\Big\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}+\frac{1}{2}\Big\|\frac{\mathbf{E}(h)-\mathbf{E}_0}{h}\Big\|^2_{\mathbf{L}^2_\varepsilon(\Omega_\sigma)}+\underline{\sigma}\int_0^h\Big\|\frac{\mathbf{E}(s)-\mathbf{E}_0}{h}\Big\|^2_{\mathbf{L}^2(\Omega_\sigma)}\mathrm{d}s\\
&\qquad+\frac{1}{2}\Big\|\frac{\mathbf{H}(h)-\mathbf{H}_0}{h}\Big\|^2_{\mathbf{L}^2_\mu(\Omega)}\\
&\quad\le\int_0^h\Big\|\frac{\mathbf{f}(s)-\mathbf{f}(0)}{h}\Big\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\Big\|\frac{\mathbf{E}(s)-\mathbf{E}_0}{h}\Big\|_{\mathbf{L}^2(\Omega\setminus\Omega_\sigma)}\mathrm{d}s\\
&\qquad+\int_0^h\Big\|\frac{\mathbf{f}(s)-\mathbf{f}(0)}{h}\Big\|_{\mathbf{L}^2(\Omega_\sigma)}\Big\|\frac{\mathbf{E}(s)-\mathbf{E}_0}{h}\Big\|_{\mathbf{L}^2(\Omega_\sigma)}\mathrm{d}s\\
&\quad\le\int_0^h\Big(\frac{1}{2\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)}+\frac{1}{2\underline{\sigma}}\Big)\Big\|\frac{\mathbf{f}(s)-\mathbf{f}(0)}{h}\Big\|^2_{\mathbf{L}^2(\Omega)}\mathrm{d}s+\frac{1}{2}\int_0^h\Big\|\frac{\mathbf{E}(s)-\mathbf{E}_0}{h}\Big\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\mathrm{d}s\\
&\qquad+\frac{\underline{\sigma}}{2}\int_0^h\Big\|\frac{\mathbf{E}(s)-\mathbf{E}_0}{h}\Big\|^2_{\mathbf{L}^2(\Omega_\sigma)}\mathrm{d}s,\quad\forall h\in(0,T),
\end{aligned}$$

and consequently, by the Lipschitz continuity of $\mathbf{f}$ as well as rearrangement, we arrive at

$$\begin{aligned}
&\Big\|\frac{\mathbf{E}(h)-\mathbf{E}_0}{h}\Big\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}+\Big\|\frac{\mathbf{E}(h)-\mathbf{E}_0}{h}\Big\|^2_{\mathbf{L}^2_\varepsilon(\Omega_\sigma)}+\Big\|\frac{\mathbf{H}(h)-\mathbf{H}_0}{h}\Big\|^2_{\mathbf{L}^2_\mu(\Omega)}\\
&\quad\le\frac{1}{3}hL^2\Big(\frac{1}{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)}+\frac{1}{\underline{\sigma}}\Big)+\int_0^h\Big\|\frac{\mathbf{E}(s)-\mathbf{E}_0}{h}\Big\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}\mathrm{d}s,\quad\forall h\in(0,T).
\end{aligned}$$

In conclusion, Gronwall's lemma delivers

$$\begin{aligned}
&\Big\|\frac{\mathbf{E}(h)-\mathbf{E}_0}{h}\Big\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}+\Big\|\frac{\mathbf{E}(h)-\mathbf{E}_0}{h}\Big\|^2_{\mathbf{L}^2_\varepsilon(\Omega_\sigma)}+\Big\|\frac{\mathbf{H}(h)-\mathbf{H}_0}{h}\Big\|^2_{\mathbf{L}^2_\mu(\Omega)}\\
&\quad\le\frac{1}{3}hL^2\Big(\frac{1}{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)}+\frac{1}{\underline{\sigma}}\Big)\,\mathrm{e}^h
\end{aligned}\tag{4.15}$$

for all $h\in(0,T)$. Going back to (4.10) and on account of (4.15), we attain

$$\begin{aligned}
&\Big\|\frac{\mathbf{E}(t+h)-\mathbf{E}(t)}{h}\Big\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)}+\underline{\sigma}\int_0^t\Big\|\frac{\mathbf{E}(s+h)-\mathbf{E}(s)}{h}\Big\|^2_{\mathbf{L}^2(\Omega_\sigma)}\mathrm{d}s\\
&\quad\le\frac{1}{3}hL^2\Big(\frac{1}{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)}+\frac{1}{\underline{\sigma}}\Big)\,\mathrm{e}^h+\frac{2L(\Omega\setminus\Omega_\sigma)t}{\sqrt{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)}}\Big\|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\Big\|_{L^\infty((0,t),\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma))}\\
&\qquad+\frac{L(\Omega_\sigma)^2t}{\underline{\sigma}},\quad\forall t\in(0,T),\ h\in(0,T-t).
\end{aligned}\tag{4.16}$$

By passing to the limit $h \to 0$ in the first term of the left-hand side of (4.16), we obtain that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(t)\right\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} \le \frac{2L(\Omega\setminus\Omega_\sigma)T}{\sqrt{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)}}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\right\|_{L^\infty((0,T),\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma))} + \frac{L(\Omega_\sigma)^2T}{\underline{\sigma}}$$
$$\le \frac{2L(\Omega\setminus\Omega_\sigma)^2T^2}{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)} + \frac{1}{2}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\right\|^2_{L^\infty((0,T),\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma))} + \frac{L(\Omega_\sigma)^2T}{\underline{\sigma}}$$

for a.e. $t \in (0,T)$, from which it follows that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\right\|^2_{L^\infty((0,T),\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma))} \le \frac{4L(\Omega\setminus\Omega_\sigma)^2T^2}{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)} + \frac{2L(\Omega_\sigma)^2T}{\underline{\sigma}}. \tag{4.17}$$

Finally, using again the regularity property $\mathbf{E} \in W^{1,\infty}((0,T),\mathbf{L}^2(\Omega))$, Fatou's lemma yields

$$\int_0^t \left\|\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}(s)\right\|^2_{\mathbf{L}^2(\Omega_\sigma)}\mathrm{d}s \le \liminf_{h\to 0}\int_0^t \left\|\frac{\mathbf{E}(s+h)-\mathbf{E}(s)}{h}\right\|^2_{\mathbf{L}^2(\Omega_\sigma)}\mathrm{d}s$$
$$\underbrace{\le}_{(4.16)\&(4.17)} \frac{L(\Omega_\sigma)^2T}{\underline{\sigma}^2} + \frac{2L(\Omega\setminus\Omega_\sigma)T}{\underline{\sigma}\sqrt{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)}}$$
$$\cdot\sqrt{\frac{4L(\Omega\setminus\Omega_\sigma)^2T^2}{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)} + \frac{2L(\Omega_\sigma)^2T}{\underline{\sigma}}}, \quad \forall t \in (0,T). \tag{4.18}$$

Step 2: The proof of (4.7) *for* $|\Omega\setminus\Omega_\sigma| \ne 0$. We start by inserting $\mathbf{v} = \mathbf{E}(s)$ in (P_{ec}) and $\mathbf{v} = \mathbf{E}_{\mathrm{ec}}(s)$ in (P) to obtain that

$$\int_{\Omega\setminus\Omega_\sigma} \varepsilon\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}_{\mathrm{ec}}(s)\cdot(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\,\mathrm{d}x$$
$$+\int_\Omega \sigma\mathbf{E}_{\mathrm{ec}}(s)\cdot(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s)) - \mathbf{H}_{\mathrm{ec}}(s)\cdot\mathbf{curl}(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\,\mathrm{d}x$$
$$\le \int_\Omega \mathbf{f}(s)\cdot(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\,\mathrm{d}x \quad \text{for a.e. } s \in (0,T) \tag{4.19}$$

and

$$-\int_\Omega \varepsilon\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}(s)\cdot(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\,\mathrm{d}x$$
$$-\int_\Omega \sigma\mathbf{E}(s)\cdot(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s)) - \mathbf{H}(s)\cdot\mathbf{curl}(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\,\mathrm{d}x$$
$$\le -\int_\Omega \mathbf{f}(s)\cdot(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\,\mathrm{d}x \quad \text{for a.e. } s \in (0,T). \tag{4.20}$$

Adding inequalities (4.19) and (4.20) together results in

$$\int_{\Omega\setminus\Omega_\sigma} \varepsilon \frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\cdot(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\,\mathrm{d}x - \int_{\Omega_\sigma} \varepsilon \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}(s)\cdot(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\,\mathrm{d}x$$
$$+\int_{\Omega} \sigma(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\cdot(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))$$
$$-(\mathbf{H}_{\mathrm{ec}}(s)-\mathbf{H}(s))\cdot\mathbf{curl}(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\,\mathrm{d}x \le 0 \quad \text{for a.e. } s\in(0,T). \tag{4.21}$$

By the Faraday law for the solutions of (P_{ec}) and (P), we have that

$$\mathbf{curl}(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s)) = -\mu\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{H}_{\mathrm{ec}}(s)-\mathbf{H}(s)) \quad \text{for a.e. } s\in(0,T), \tag{4.22}$$

and thus, applying (4.22) to (4.21) leads to

$$\int_{\Omega\setminus\Omega_\sigma} \varepsilon \frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\cdot(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\,\mathrm{d}x$$
$$+\int_{\Omega} \mu\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{H}_{\mathrm{ec}}(s)-\mathbf{H}(s))\cdot(\mathbf{H}_{\mathrm{ec}}(s)-\mathbf{H}(s))\,\mathrm{d}x$$
$$+\int_{\Omega} \sigma(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\cdot(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\,\mathrm{d}x$$
$$\le \int_{\Omega_\sigma} \varepsilon\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}(s)\cdot(\mathbf{E}_{\mathrm{ec}}(s)-\mathbf{E}(s))\,\mathrm{d}x \tag{4.23}$$

for a.e. $s\in(0,T)$. Since $\mathbf{E}(0)=\mathbf{E}_{\mathrm{ec}}(0)=\mathbf{E}_0$ in $\Omega\setminus\Omega_\sigma$ and $\mathbf{H}(0)=\mathbf{H}_{\mathrm{ec}}(0)=\mathbf{H}_0$, we find after integrating (4.23) over $(0,t)$ that

$$\frac{1}{2}\left\|\mathbf{E}(t)-\mathbf{E}_{\mathrm{ec}}(t)\right\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} + \frac{1}{2}\left\|\mathbf{H}(t)-\mathbf{H}_{\mathrm{ec}}(t)\right\|^2_{\mathbf{L}^2_\mu(\Omega)} + \int_0^t \left\|\mathbf{E}(s)-\mathbf{E}_{\mathrm{ec}}(s)\right\|^2_{\mathbf{L}^2_\sigma(\Omega_\sigma)}\,\mathrm{d}s$$
$$\le \int_0^t \left\|\varepsilon\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}(s)\right\|_{\mathbf{L}^2(\Omega_\sigma)}\left\|\mathbf{E}(s)-\mathbf{E}_{\mathrm{ec}}(s)\right\|_{\mathbf{L}^2(\Omega_\sigma)}\,\mathrm{d}s \le \frac{1}{2\underline{\sigma}}\int_0^t \left\|\varepsilon\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}(s)\right\|^2_{\mathbf{L}^2(\Omega_\sigma)}\,\mathrm{d}s$$
$$+\frac{1}{2}\int_0^t \|\mathbf{E}(s)-\mathbf{E}_{\mathrm{ec}}(s)\|^2_{\mathbf{L}^2_\sigma(\Omega_\sigma)}\,\mathrm{d}s, \quad \forall t\in(0,T), \tag{4.24}$$

and consequently,

$$\|\mathbf{E}(t)-\mathbf{E}_{\mathrm{ec}}(t)\|^2_{\mathbf{L}^2_\varepsilon(\Omega\setminus\Omega_\sigma)} + \|\mathbf{H}(t)-\mathbf{H}_{\mathrm{ec}}(t)\|^2_{\mathbf{L}^2_\mu(\Omega)} + \int_0^t \|\mathbf{E}(s)-\mathbf{E}_{\mathrm{ec}}(s)\|^2_{\mathbf{L}^2_\sigma(\Omega_\sigma)}\,\mathrm{d}s$$
$$\le \frac{1}{\underline{\sigma}}\int_0^t \left\|\varepsilon\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}(s)\right\|^2_{\mathbf{L}^2(\Omega_\sigma)}\,\mathrm{d}s \underbrace{\le}_{(1.5)} \frac{1}{\underline{\sigma}}\|\varepsilon\|^2_{L^\infty(\Omega_\sigma)^{3\times 3}}\int_0^t \left\|\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}(s)\right\|^2_{\mathbf{L}^2(\Omega_\sigma)}\,\mathrm{d}s \tag{4.25}$$

for all $t \in (0,T)$. Eventually, applying (4.18) to (4.25) yields

$$\|\mathbf{E}(t)-\mathbf{E}_{\mathrm{ec}}(t)\|^2_{\mathbf{L}^2_{\varepsilon}(\Omega\setminus\Omega_\sigma)}+\|\mathbf{H}(t)-\mathbf{H}_{\mathrm{ec}}(t)\|^2_{\mathbf{L}^2_{\mu}(\Omega)}+\|\mathbf{E}-\mathbf{E}_{\mathrm{ec}}\|^2_{L^2((0,t),\mathbf{L}^2_{\sigma}(\Omega_\sigma))}$$
$$\le\Big(\frac{L(\Omega_\sigma)^2T}{\underline{\sigma}}+\frac{2L(\Omega\setminus\Omega_\sigma)T}{\sqrt{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)}}\sqrt{\frac{4L(\Omega\setminus\Omega_\sigma)^2T^2}{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)}+\frac{2L(\Omega_\sigma)^2T}{\underline{\sigma}}}\Big)\Big\|\frac{\varepsilon}{\sigma}\Big\|^2_{L^\infty(\Omega_\sigma)^{3\times3}}$$

for all $t \in (0,T)$. In view of regularity properties (4.1) and (4.2), the above pointwise estimate leads immediately to uniform estimate (4.7).

Step 3: The proof of (4.8) *for* $\Omega_\sigma=\Omega$. In this case, inequality (4.16) turns out to be

$$\underline{\sigma}\int_0^t\Big\|\frac{\mathbf{E}(s+h)-\mathbf{E}(s)}{h}\Big\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s\le\frac{1}{3}hL^2\Big(\frac{1}{\underline{\varepsilon}(\Omega\setminus\Omega_\sigma)}+\frac{1}{\underline{\sigma}}\Big)\mathrm{e}^h+\frac{L^2t}{\underline{\sigma}},\qquad\forall t\in(0,T),$$

and so, by Fatou's lemma

$$\int_0^t\Big\|\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}(s)\Big\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s\le\liminf_{h\to0}\int_0^t\Big\|\frac{\mathbf{E}(s+h)-\mathbf{E}(s)}{h}\Big\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s\le\frac{L^2t}{\underline{\sigma}^2}\tag{4.26}$$

for all $t \in (0,T)$. Now, in the case of $\Omega_\sigma=\Omega$, inequality (4.25) reads as

$$\|\mathbf{H}(t)-\mathbf{H}_{\mathrm{ec}}(t)\|^2_{\mathbf{L}^2_{\mu}(\Omega)}+\|\mathbf{E}-\mathbf{E}_{\mathrm{ec}}\|^2_{L^2((0,t),\mathbf{L}^2_{\sigma}(\Omega))}$$
$$\le\frac{1}{\underline{\sigma}}\|\varepsilon\|^2_{L^\infty(\Omega)^{3\times3}}\int_0^t\Big\|\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}(s)\Big\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s,\qquad\forall t\in(0,T).\tag{4.27}$$

The final claim (see (4.8)) follows therefore by applying (4.26) to (4.27). ■

5. Numerical verification

We close this paper with a brief numerical verification of our theoretical findings. In particular, our numerical test confirms the linear convergence rate with respect to $\varepsilon_{|\Omega_\sigma}$ for the eddy current approximation (see Theorem 4.1). Note that the following example is of merely academic nature as the conducting domain is chosen to be equal to the whole domain. So, for the test, we consider $\Omega=(-1,1)^3$, $T=1$, $\mu\equiv1$, $\sigma\equiv1$ and $\Omega_\sigma=\Omega$, with $(\mathbf{0},\mathbf{0})$ as an initial value. For the applied current source, we choose $\mathbf{f}:[0,1]\times\Omega\to\mathbb{R}^3$ defined by

$$\mathbf{f}(t,x_1,x_2,x_3):=\begin{cases}\Big(0,\dfrac{-tx_3}{\sqrt{x_2^2+x_3^2}},\dfrac{tx_2}{\sqrt{x_2^2+x_3^2}}\Big) & \text{if } (x_1,x_2,x_3)\in P,\\ 0 & \text{if } (x_1,x_2,x_3)\notin P,\end{cases}$$

where $P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_1 \leq 0.5, 0.3 \leq \sqrt{x_2^2 + x_3^2} \leq 0.5\}$ models a cylindrical pipe coil. Furthermore, the feasible set is set to be

$$\mathbf{K} = \left\{\mathbf{v} \in \mathbf{L}^2(\Omega) \mid |\mathbf{v}(x)|_\infty \leq 5 \cdot 10^{-4} \text{ for a.e. } x \in \omega\right\},$$

with the obstacle region

$$\omega := \left\{(x_1, x_2, x_3) \in \Omega \mid -0.25 \leq x_1 \leq -0.125, |x_2| \leq 0.5, |x_3| \leq 0.5\right\}.$$

Note that the choice of the bound $5 \cdot 10^{-4}$ in the obstacle set $\mathbf{K}$ is of no particular importance. With the choice of our bound, we strive to model the effects of electric shielding. Our numerical computation is based on the time-discrete (implicit Euler) scheme (P_N) along with the space discretization consisting of Nédélec's edge elements [17] for $\mathbf{E}$ and piecewise constant elements for $\mathbf{H}$. The corresponding finite element approximation of the time-discrete problems in (P_N) (with roughly 829, 000 degrees of freedom) were solved by the primal dual active set algorithm (see [16]) implemented on the open-source platform FEniCS [19] (see Figures 1 and 2 for a visualization). We note that the primal dual method approximates the elliptic variational inequalities in (P_N) by equalities on the corresponding active and inactive sets that are iteratively updated. To verify the convergence of the eddy current approximation, we use the quantity

$$\mathrm{Error}_k = \|(\mathbf{E}, \mathbf{H})_k - (\mathbf{E}_{\mathrm{ec}}, \mathbf{H}_{\mathrm{ec}})\|_{L^2((0,T),\mathbf{L}^2(\Omega)) \times \mathscr{C}([0,T],\mathbf{L}^2(\Omega))}$$

with $(\mathbf{E}, \mathbf{H})_k$ being the numerical solution to (P) for $\varepsilon = \frac{1}{2^k}$ and $(\mathbf{E}_{\mathrm{ec}}, \mathbf{H}_{\mathrm{ec}})$ being the numerical solution to the eddy current model (P_{ec}). Furthermore, to check the experimental order of convergence with respect to ε, we make use of the following quantity:

$$\mathrm{EOC}_k = \frac{\log(\mathrm{Error}_{k+1}) - \log(\mathrm{Error}_k)}{\log(2^{-(k+1)}) - \log(2^{-k})}.$$

Table 1 depicts the computed error and experimental order of convergence for the values $k = 4, \ldots, 14$. In agreement with our theoretical finding (Theorem 4.1), we observe that the eddy current approximation (P_{ec}) becomes closer and closer to (P) as ε decreases. More importantly, the experimental order of convergence is readily very close to 1, which exactly confirms the linear convergence rate in a priori error estimate (4.8).

$\frac{\varepsilon}{\sigma}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$	$\frac{1}{2^6}$	$\frac{1}{2^7}$	$\frac{1}{2^8}$	$\frac{1}{2^9}$	$\frac{1}{2^{10}}$	$\frac{1}{2^{11}}$	$\frac{1}{2^{12}}$	$\frac{1}{2^{13}}$	$\frac{1}{2^{14}}$
Error_k	$1.9 \cdot 10^{-3}$	$9.9 \cdot 10^{-4}$	$5.1 \cdot 10^{-4}$	$2.6 \cdot 10^{-4}$	$1.3 \cdot 10^{-4}$	$6.7 \cdot 10^{-5}$	$3.4 \cdot 10^{-5}$	$1.7 \cdot 10^{-5}$	$8.5 \cdot 10^{-6}$	$4.2 \cdot 10^{-6}$	$2.1 \cdot 10^{-6}$
EOC_k	0.9487	0.9512	0.9547	0.9749	0.9866	0.9933	0.9972	0.9984	0.9985	0.9991	0.9999

Table 1. Convergence behavior of the eddy current approximation.

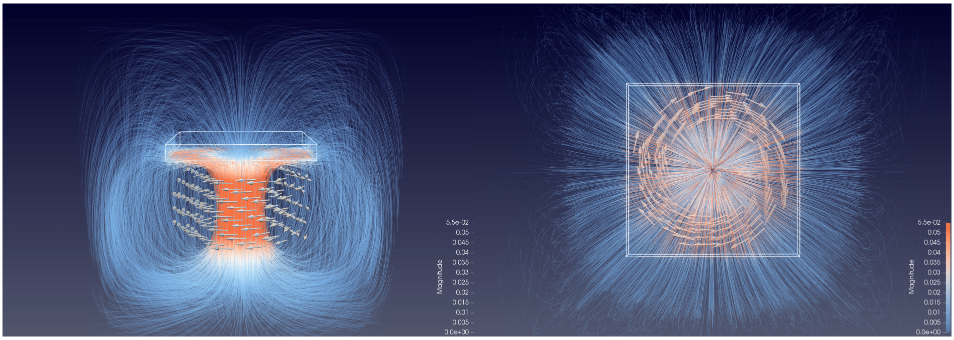

Figure 1. Computed magnetic field from two different views at the last time step together with the applied circular current and the outlined obstacle.

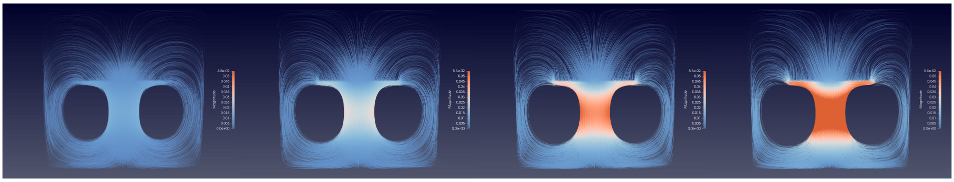

Figure 2. Evolution of the magnetic field at the time steps $t_n = \frac{n}{4}$ with $n \in \{1, 2, 3, 4\}$.

Funding. This work was partially supported by the German Research Foundation (DFG Grants YO 159/4-1 and YO 159/2-2).

References

[1] A. Alonso, A mathematical justification of the low-frequency heterogeneous time-harmonic Maxwell equations. *Math. Models Methods Appl. Sci.* **9** (1999), no. 3, 475–489 Zbl 0932.35192 MR 1686609

[2] A. Alonso and A. Valli, Unique solvability for high-frequency heterogeneous time-harmonic Maxwell equations via the Fredholm alternative theory. *Math. Methods Appl. Sci.* **21** (1998), no. 6, 463–477 Zbl 0923.35178 MR 1615410

[3] A. Alonso Rodríguez and A. Valli, *Eddy current approximation of Maxwell equations*. MS&A. Modeling, Simulation and Applications 4, Springer, Milan, 2010 Zbl 1204.78001 MR 2680968

[4] H. Ammari, A. Buffa, and J.-C. Nédélec, A justification of eddy currents model for the Maxwell equations. *SIAM J. Appl. Math.* **60** (2000), no. 5, 1805–1823 Zbl 0978.35070 MR 1761772

[5] L. Arnold and B. Harrach, A unified variational formulation for the parabolic-elliptic eddy current equations. *SIAM J. Appl. Math.* **72** (2012), no. 2, 558–576 Zbl 1252.35256 MR 2914339

[6] A. Bossavit, *Électromagnétisme, en vue de la modélisation*. Math. Appl. (Berl.) 14, Springer, Paris, 1993 Zbl 0787.65090 MR 1616583

[7] A. Bossavit, Numerical modelling of superconductors in three dimensions: A model and a finite element method. *IEEE Trans. Magn.* **30** (1994), 3363–3366

[8] A. Bossavit, *Computational electromagnetism*. Electromagnetism, Academic Press, San Diego, CA, 1998 Zbl 0945.78001 MR 1488417

[9] D. S. Clark, Short proof of a discrete Gronwall inequality. *Discrete Appl. Math.* **16** (1987), no. 3, 279–281 Zbl 0612.39004 MR 878027

[10] M. Costabel, M. Dauge, and S. Nicaise, Singularities of eddy current problems. *M2AN, Math. Model. Numer. Anal.* **37** (2003), no. 5, 807–831 Zbl 1170.35353 MR 2020865

[11] G. Duvaut and J.-L. Lions, *Inequalities in mechanics and physics*. Grundlehren Math. Wiss. 219, Springer, Berlin-New York, 1976 Zbl 0331.35002 MR 0521262

[12] C. M. Elliott and Y. Kashima, A finite-element analysis of critical-state models for type-II superconductivity in 3D. *IMA J. Numer. Anal.* **27** (2007), no. 2, 293–331 Zbl 1119.82046 MR 2317006

[13] R. Glowinski, *Numerical methods for nonlinear variational problems*. Sci. Comput., Springer, Berlin, 2008 Zbl 1139.65050 MR 2423313

[14] M. Hintermüller, A. Laurain, and I. Yousept, Shape sensitivities for an inverse problem in magnetic induction tomography based on the eddy current model. *Inverse Probl.* **31** (2015), no. 6, article ID 065006 Zbl 1335.35297 MR 3350625

[15] R. H. W. Hoppe and J. Schöberl, Convergence of adaptive edge element methods for the 3D eddy currents equations. *J. Comput. Math.* **27** (2009), no. 5, 657–676 Zbl 1212.65126 MR 2536907

[16] K. Ito and K. Kunisch, Semi-smooth Newton methods for variational inequalities of the first kind. *M2AN, Math. Model. Numer. Anal.* **37** (2003), no. 1, 41–62 Zbl 1027.49007 MR 1972649

[17] J.-C. Nédélec, Mixed finite elements in $\mathbf{R}^3$. *Numer. Math.* **35** (1980), no. 3, 315–341 Zbl 0419.65069 MR 592160

[18] J.-L. Lions and G. Stampacchia, Variational inequalities. *Comm. Pure Appl. Math.* **20** (1967), 493–519 Zbl 0152.34601 MR 216344

[19] A. Logg, K.-A. Mardal, and G. N. Wells (eds.), *Automated solution of differential equations by the finite element method*. Lect. Notes Comput. Sci. Eng. 84, Springer, Heidelberg, 2012 Zbl 1247.65105 MR 3075806

[20] P. Lorrain and D.-R. Corson, *Electromagnetic fields and waves*. Second ed. A Series of Books in Physics, W. H. Freeman, 1970

[21] A. Milani, On a variational inequality with time dependent convex constraints for the Maxwell equations. *Rend. Sem. Mat. Univ. Politec. Torino* **36** (1977/78), 389–401 (1979) MR 530996

[22] A. Milani, On a variational inequality with time dependent convex constraint for the Maxwell equations. II. *Rend. Sem. Mat. Univ. Politec. Torino* **43** (1985), no. 1, 171–183 Zbl 0611.49002 MR 859855

[23] A. Milani and R. Picard, Weak solution theory for Maxwell's equations in the semistatic limit case. *J. Math. Anal. Appl.* **191** (1995), no. 1, 77–100 Zbl 0826.35122 MR 1323765

[24] F. Miranda, J.-F. Rodrigues, and L. Santos, On a p-curl system arising in electromagnetism. *Discrete Contin. Dyn. Syst. Ser. S* **5** (2012), no. 3, 605–629 Zbl 1252.35257 MR 2861829

[25] F. Miranda, J. F. Rodrigues, and L. Santos, Evolutionary quasi-variational and variational inequalities with constraints on the derivatives. *Adv. Nonlinear Anal.* **9** (2020), no. 1, 250–277 Zbl 1417.35233 MR 3935872

[26] S. Nicaise and F. Tröltzsch, A coupled Maxwell integrodifferential model for magnetization processes. *Math. Nachr.* **287** (2014), no. 4, 432–452 Zbl 1286.35140 MR 3179672

[27] J. F. Rodrigues and L. Santos, A parabolic quasi-variational inequality arising in a superconductivity model. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **29** (2000), no. 1, 153–169 Zbl 0953.35079 MR 1765540

[28] F. Tröltzsch and A. Valli, Optimal control of low-frequency electromagnetic fields in multiply connected conductors. *Optimization* **65** (2016), no. 9, 1651–1673 Zbl 1345.49027 MR 3515109

[29] F. Tröltzsch and A. Valli, Optimal voltage control of non-stationary eddy current problems. *Math. Control Relat. Fields* **8** (2018), no. 1, 35–56 Zbl 1406.35384 MR 3810867

[30] M. Winckler and I. Yousept, Fully discrete scheme for Bean's critical-state model with temperature effects in superconductivity. *SIAM J. Numer. Anal.* **57** (2019), no. 6, 2685–2706 Zbl 1427.35269 MR 4031469

[31] I. Yousept, Finite element analysis of an optimal control problem in the coefficients of time-harmonic eddy current equations. *J. Optim. Theory Appl.* **154** (2012), no. 3, 879–903 Zbl 1261.49001 MR 2957021

[32] I. Yousept, Hyperbolic Maxwell variational inequalities for Bean's critical-state model in type-II superconductivity. *SIAM J. Numer. Anal.* **55** (2017), no. 5, 2444–2464 Zbl 1377.35201 MR 3711590

[33] I. Yousept, Optimal control of non-smooth hyperbolic evolution Maxwell equations in type-II superconductivity. *SIAM J. Control Optim.* **55** (2017), no. 4, 2305–2332 Zbl 1377.35236 MR 3679913

[34] I. Yousept, Hyperbolic Maxwell variational inequalities of the second kind. *ESAIM Control Optim. Calc. Var.* **26** (2020), paper no. 34 Zbl 1446.35079 MR 4116682

[35] I. Yousept, Well-posedness theory for electromagnetic obstacle problems. *J. Differential Equations* **269** (2020), no. 10, 8855–8881 Zbl 1442.35271 MR 4114955

Received 27 January 2021; revised 12 October 2021.

Maurice Hensel
Fakultät für Mathematik, University of Duisburg-Essen, Thea-Leymann-Straße 9, 45127 Essen, Germany; maurice.hensel@uni-due.de

Irwin Yousept
Fakultät für Mathematik, University of Duisburg-Essen, Thea-Leymann-Straße 9, 45127 Essen, Germany; irwin.yousept@uni-due.de