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Weak-strong uniqueness for the mean curvature flow of double bubbles

Sebastian Hensel and Tim Laux

Abstract. We derive a weak-strong uniqueness principle for BV solutions to multiphase mean curvature flow of triple line clusters in three dimensions. Our proof is based on the explicit construction of a gradient flow calibration in the sense of the recent work of Fischer et al. (2020) for any such cluster. This extends the two-dimensional construction to the three-dimensional case of surfaces meeting along triple junctions.

Contents

1. Introduction

Multiphase mean curvature flow (MCF) arises as the L^2-gradient flow of the area functional and has been studied intensively over the last decades. Its earliest motivation comes from materials science where MCF models the slow relaxation of grain boundaries in polycrystals.

While a lot of progress has been achieved in the two-dimensional case, often referred to as network flow, in the physically most relevant case of three spatial dimensions results concerning strong solutions are just beginning to emerge. The short-time existence for the MCF of three surfaces coming together at a triple junction has been established by Freire [5] when all three surfaces can be parametrized as graphs over a single plane—a condition which then was relaxed by Depner, Garcke, and Kohsaka [3] who derived

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the local well-posedness without relying on this graphical geometry. In this work, they parametrize the surface cluster over a fixed reference surface cluster and phrase MCF as a non-local, quasilinear parabolic system of free boundary problem. Independently and as the result of an improved compactness property of Brakke flows, Schulze and White [12] established short-time existence in a similar geometric setting. Recently, Baldi, Haus, and Mantegazza [2] derived the existence of a self-similar shrinking “lens-shaped" surface cluster describing a solution to MCF just before the disappearance of the smaller bubble in the cluster. We refer the interested reader to [4] for a more detailed discussion and further relevant references. The construction of regular solutions starting from non-regular initial surface clusters has not yet been accomplished, but the recent microlocal approach [11] might give new insights. However, this approach relies on an explicit construction of gluing in self-similar expanders, which does not immediately carry over to the three-dimensional case.

Most results on weak solutions of MCF, however, are often quite general and in particular apply in our present three-dimensional case. While the theory of viscosity solutions is not available for surface clusters (even in two dimensions), Brakke's solution concept, and in particular the more refined version of Kim and Tonegawa [6] apply, and so do the conditional convergence results in [7] and [8].

In the present work, we prove the stability and weak-strong uniqueness of regular surface clusters with triple junctions moving by MCF in three dimensions. The key step is the explicit construction of a gradient flow calibration in the sense of our recent work [4] with Fischer and Simon. Therein, we constructed such a gradient flow calibration in the planar case of networks moving by curve shortening flow and proved that, in arbitrary dimension, any calibrated MCF is stable. The main contribution of the present work is the extension of the first part to the three-dimensional case, which then immediately implies the weak-strong uniqueness. The concept of gradient flow calibrations is the time-dependent counterpart of calibrations and paired calibrations for minimal surfaces and minimal surface clusters, respectively (see in particular [10]).

While our main result establishes uniqueness of BV solutions to multiphase MCF within a class of sufficiently regularly evolving double bubbles (cf. Definition 10 below), we note that this in particular implies uniqueness of (again sufficiently regular) strong solutions for MCF of double bubble geometries as constructed in the work of Depner, Garcke, and Kohsaka [3]. We emphasize that the proof of our main result does not rely on the uniqueness of such strong solutions, and that the latter was indeed left open in [3].

There are two model cases in which our new construction (essentially) reduces to the two-dimensional case: if the three-dimensional configuration is (i) rotationally symmetric or (ii) translation invariant in one direction. However, in the general case, a direct slicing argument across normal planes to the triple junction of course does not yield torsion-free tangent frames of the interfaces. By introducing suitable gauge rotations, we correct this ad hoc construction. We prove that these gauged tangent frames then give a natural extension of the respective normal vector fields to a vicinity of each interface and even across the triple line using Herring's condition. Furthermore, we show that these constructions are

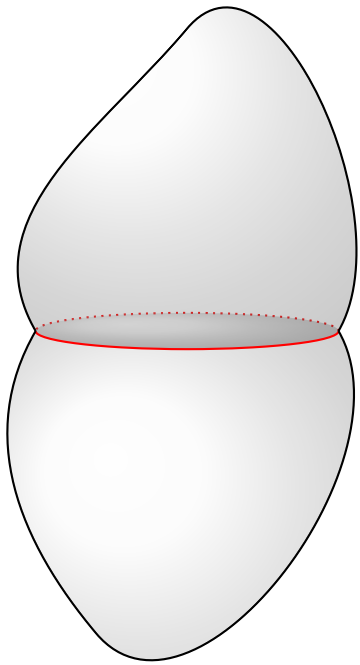

Figure 1. A simplified illustration of a double bubble in three dimensions. The triple line $\bar{\Gamma}$ along which all three interfaces meet is marked in red. We emphasize that neither flatness of an interface nor symmetry of the triple line is required for our results.

regular and first-order compatible along the triple line. Although the present method would immediately carry over to more general surface clusters only containing smooth surfaces coming together along triple lines, we restrict ourselves to the case of a "double bubble", that is, a cluster of three surfaces as displayed in Figure 1. For general surface clusters in $\mathbb{R}^3$, triple lines could meet in quadruple points, some of which will persist over time (cf. [13] for the static case). For these systems, even the short-time existence of regular solutions has not been established. It would be interesting to generalize our present work to construct a gradient flow calibration in this more general setting.

2. Main results

In our recent contribution [4] with Fischer and Simon, we developed a general approach to the question of weak-strong uniqueness of BV solutions to multiphase mean curvature flow in arbitrary ambient dimension $d \geq 2$. This approach splits into a two-step procedure.

In the first step, we introduced a novel concept of calibrated flows with respect to the gradient flow of the interface energy functional given by the (weighted) sum of the surface areas of the interfaces (cf. (8) below). This concept can be interpreted as the evolutionary analogue of the well-known notion of paired calibrations due to Lawlor and Morgan [10] from their study of the minimization problem of interfacial surface area of networks. Indeed, the main merit of a calibrated flow is that its existence (essentially) implies qualitative uniqueness and quantitative stability of BV solutions to multiphase mean curvature flow in arbitrary ambient dimension $d \geq 2$.

In the second step, we then put this theory to use by showing that any sufficiently regular network of interfaces in the plane $\mathbb{R}^2$, which in addition is subject to the correct angle condition at triple junctions, is in fact calibrated in the precise sense of [4]. The purpose of the present work is to extend this second step of the approach to the three-dimensional setting of mean curvature flow of sufficiently regular double bubbles (again with the correct angle condition along the triple line). The main contributions are summarized in the following result:

Theorem 1. *Let $T \in (0,\infty)$ be a time horizon, and let $(\bar\Omega_1, \bar\Omega_2, \bar\Omega_3)$ be a regular double bubble smoothly evolving by MCF on $[0,T]$ in the sense of Definition 10. The evolution of $(\bar\Omega_1, \bar\Omega_2, \bar\Omega_3)$ on $[0,T]$ is then calibrated in the sense that there exists an associated gradient flow calibration $((\xi_i)_{i\in\{1,2,3\}}, B)$ on $[0,T]$ (cf. Definition 2). Moreover, the smoothly evolving regular double bubble $(\bar\Omega_1, \bar\Omega_2, \bar\Omega_3)$ admits a family of transported weights $(\vartheta_i)_{i\in\{1,2,3\}}$ on $[0,T]$ in the sense of Definition 5.*

As a corollary, we obtain a weak-strong uniqueness and stability of evolutions principle for BV solutions $(\Omega_1, \Omega_2, \Omega_3)$ to multiphase MCF on $[0,T]$ (cf. Definition 12) with respect to the class of regular double bubbles smoothly evolving by MCF on $[0,T]$ in the sense of Definition 10. We refer to Theorem 6 for a more detailed statement of this corollary, and to the discussion below it for an account of the general regime of $P \geq 3$ phases on the level of the BV solution.

Proof. The existence of a gradient flow calibration $((\xi_i)_{i\in\{1,2,3\}}, B)$ on $[0,T]$ is the content of Theorem 3. Its proof occupies almost the whole paper and is carried out from Section 3 to Section 5. We emphasize in this context that the local construction at a triple line performed in Section 4 represents the core contribution of the present work. The existence of transported weights $(\vartheta_i)_{i\in\{1,2,3\}}$ on $[0,T]$ is proven in Section 6 in the form of Proposition 5.

These two existence results in turn realize the assumptions of the general conditional weak-strong uniqueness and stability of evolutions principle [4, Proposition 5] for BV solutions to multiphase mean curvature flow (with respect to the setting of $P = 3$ phases and $d = 3$ dimensions), which therefore establishes the claim of the corollary. ∎

The results of [4] together with Theorem 1 admittedly only cover two thirds of the story concerning weak-strong uniqueness for general clusters in $\mathbb{R}^3$ evolving by multiphase mean curvature flow. Indeed, one also has to allow for quadruple junctions at which four distinct phases meet (cf. the structure result on minimizers of interfacial surface energy by Taylor [13]). We expect that a suitable generalization of our ideas for the construction at a triple line should also lead to the correct construction in the case of a quadruple junction, and thus to a full-fledged weak-strong uniqueness result in $\mathbb{R}^3$.

2.1. Existence of gradient flow calibrations

For the sake of completeness, let us first restate the precise definition of the concept of a gradient flow calibration.

Definition 2 (Gradient flow calibration). Let $T \in (0,\infty)$ be a time horizon, and let $\sigma \in \mathbb{R}^{P\times P}$ be an admissible matrix of surface tensions (cf. Remark 7) for $P \geq 2$ phases. Also, let $(\bar\Omega_1,\ldots,\bar\Omega_P)$ be an evolving partition of finite interface energy on $\mathbb{R}^d \times [0,T]$ in the sense of Definition 8 in dimension $d \geq 2$, and denote by $\bigcup_{i\neq j} \bar I_{i,j}$ the associated network of evolving interfaces.

A tuple of vector fields

$$
\begin{aligned}
(\xi_i)_{i\in\{1,\ldots,P\}} &: \mathbb{R}^d \times [0,T] \to (\mathbb{R}^d)^P,\\
B &: \mathbb{R}^d \times [0,T] \to \mathbb{R}^d
\end{aligned}
$$

is called a *calibration for the L^2-gradient flow of the interface energy* (8) *on* $[0,T]$ with respect to the evolving partition $(\bar\Omega_1,\ldots,\bar\Omega_P)$—or, in short, a *gradient flow calibration*—if it is subject to the following requirements:

(i) It holds that $\xi_i, B \in C^0([0,T]; C^0_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d))$ for all $i \in \{1,\ldots,P\}$. Moreover, for each time $t \in [0,T]$, there exists an $\mathcal{H}^{d-1}$-null set $\Gamma_t \subset \mathbb{R}^d$ such that for $\Gamma := \bigcup_{t\in[0,T]} \Gamma_t \times \{t\}$ it holds that $\xi_i \in (C^0_t C^1_x \cap C^1_t C^0_x)(\mathbb{R}^d \times [0,T] \setminus \Gamma)$ for all $i \in \{1,\ldots,P\}$ and $B \in C^0_t C^1_x(\mathbb{R}^d \times [0,T] \setminus \Gamma)$. Finally, there exists $C > 0$ such that

$$
\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d\setminus\Gamma_t} |\nabla B(x,t)| + |\nabla \xi_i(x,t)| + |\partial_t \xi_i(x,t)| \leq C.
$$

(ii) For $i, j \in \{1,\ldots,P\}$ with $i \neq j$, define the vector field

$$
\xi_{i,j} := \frac{1}{\sigma_{i,j}}(\xi_i - \xi_j) \quad \text{in } \mathbb{R}^d \times [0,T]. \tag{1a}
$$

Denoting by $\bar{\mathrm{n}}_{i,j}$ the unit normal vector field along the interface $\bar I_{i,j}$ (pointing from the ith into the jth phase), it is then required that

$$
\xi_{i,j} = \bar{\mathrm{n}}_{i,j} \quad \text{along } \bar I_{i,j}. \tag{1b}
$$

Moreover, there exists $c \in (0,1)$ such that the following coercivity estimate in terms of the length of the vector field $\xi_{i,j}$ holds true:

$$
|\xi_{i,j}(x,t)| \leq 1 - c \min\big\{\mathrm{dist}^2(x, \bar I_{i,j}(t)), 1\big\}, \quad (x,t) \in \mathbb{R}^d \times [0,T]. \tag{1c}
$$

(iii) The vector field B represents a velocity field for the partition $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ in the sense that the following two approximate evolution equations hold true for the vector fields $\xi_{i,j}$, $i, j \in \{1, \ldots, P\}$ with $i \neq j$:

$$\big|\partial_t \xi_{i,j} + (B \cdot \nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}} \xi_{i,j}\big|(x,t) \leq C \min\big\{\mathrm{dist}(x, \bar{I}_{i,j}(t)), 1\big\}, \tag{1d}$$

$$\big|\partial_t |\xi_{i,j}|^2 + (B \cdot \nabla)|\xi_{i,j}|^2\big|(x,t) \leq C \min\big\{\mathrm{dist}^2(x, \bar{I}_{i,j}(t)), 1\big\}, \tag{1e}$$

for some $C > 0$ and all $(x,t) \in \mathbb{R}^d \times [0,T]$.

(iv) The velocity B represents motion by multiphase mean curvature (i.e., the L^2-gradient flow with respect to the interface energy (8)) in the sense that there exists a constant $C > 0$ such that

$$\big|\xi_{i,j} \cdot B + \nabla \cdot \xi_{i,j}\big| \leq C \min\big\{\mathrm{dist}(x, \bar{I}_{i,j}(t)), 1\big\}, \quad (x,t) \in \mathbb{R}^d \times [0,T]. \tag{1f}$$

If a gradient flow calibration exists, we say that the evolving partition $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ is *calibrated on* $[0,T]$.

Note that the required regularity from the first item of the above definition is slightly less than what is actually stated in [4, Definition 2]. However, it is easy to see that this regularity is still sufficient to ensure the validity of [4, Theorem 3].

The main result of the present work is now that any sufficiently regular and smoothly evolving double bubble admits an associated gradient flow calibration.

Theorem 3 (Existence of gradient flow calibrations). *Let $T \in (0,\infty)$, let $\sigma \in \mathbb{R}^{3\times 3}$ be an admissible matrix of surface tensions, and let $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ be a regular double bubble smoothly evolving by MCF on $[0,T]$ in the sense of Definition 10. Then, $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ is calibrated on $[0,T]$ in the sense of Definition 2.*

It turns out that the existence of a gradient flow calibration already implies a quantitative inclusion principle for the surface cluster of general BV solutions to multiphase mean curvature flow; see [4, Theorem 3]. More precisely, if at the initial time each interface of a BV solution is contained in the corresponding interface of a calibrated flow, then this inclusion property remains to be satisfied as long as the calibrated flow exists. Furthermore, this qualitative property is in fact a consequence of a quantitative stability estimate for the interface error between a general BV solution and a calibrated flow (formulated in terms of an error functional—see (3) below).

The inclusion principle, however, is of course consistent with the vanishing of a phase in the BV solution, so that weak-strong uniqueness cannot be derived by means of a gradient flow calibration alone. In order to get a control on the bulk deviations of the phases, one relies on an additional input which can be formalized as follows:

Definition 4 (Family of transported weights). Let $T \in (0,\infty)$ be a time horizon, and let $\sigma \in \mathbb{R}^{P\times P}$ be an admissible matrix of surface tensions satisfying the strict triangle inequality for $P \geq 2$ phases. Let $d \geq 2$, and let $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ be an evolving partition of finite interface energy on $\mathbb{R}^d \times [0,T]$ in the sense of Definition 8, and denote by $(\bar{\chi}_1, \ldots, \bar{\chi}_P)$ the associated family of indicator functions. We then in addition assume that the measure $\partial_t \bar{\chi}_i$ is absolutely continuous with respect to the measure $|\nabla \bar{\chi}_i|$, and that $\partial \bar{\Omega}_i(\cdot, t)$ is Lipschitz regular for all $t \in [0, T]$. Consider finally a velocity vector field $B \in C^0([0, T]; C^1_{\mathrm{cpt}}(\mathbb{R}^d; \mathbb{R}^d))$.

A map $\vartheta = (\vartheta_i)_{i \in \{1,\ldots,P\}} : \mathbb{R}^d \times [0, T] \to [-1, 1]^P$ is called a *family of transported weights for* $((\bar{\Omega}_1, \ldots, \bar{\Omega}_P), B)$ if it satisfies the following properties:

(i) In terms of regularity, we require $\vartheta_i \in (W^{1,1} \cap W^{1,\infty})(\mathbb{R}^d \times [0, T]; [-1, 1])$ for all $i \in \{1, \ldots, P\}$.

(ii) We require that $\vartheta_i(\cdot, t) = 0$ on $\partial \bar{\Omega}_i(t)$, and $\vartheta_i(\cdot, t) > 0$ in the essential exterior (resp. $\vartheta_i(\cdot, t) < 0$ in the essential interior) of $\bar{\Omega}_i(\cdot, t)$ for all $i \in \{1, \ldots, P\}$ and all $t \in [0, T]$.

(iii) Each weight is approximately advected by the velocity B in the form of

$$|\partial_t \vartheta_i + (B \cdot \nabla)\vartheta_i| \leq C|\vartheta_i| \quad \text{on } \mathbb{R}^d \times [0, T],\ i \in \{1, \ldots, P\}. \tag{2}$$

The existence of a family of transported weights is precisely what is needed to derive a quantitative stability estimate for the bulk error between a general BV solution and a calibrated flow (formulated in terms of an error functional—see (4) below), which together with the already mentioned quantitative inclusion principle then implies a weak-strong uniqueness principle for BV solutions of multiphase mean curvature flow; see [4, Proposition 5].

It is therefore of interest to extend the 2D existence result from [4] to the 3D setting of any sufficiently regular and smoothly evolving double bubble.

Proposition 5 (Existence of a family of transported weights). *Let $T \in (0, \infty)$ be a time horizon, and let $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ be a regular double bubble smoothly evolving by MCF on $[0, T]$ in the sense of Definition 10. Let B denote the velocity field from the gradient flow calibration on $[0, T]$ associated with $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$, whose existence in turn is guaranteed by Theorem 3. Then, there exists an associated family of transported weights $(\vartheta_i)_{i \in \{1,2,3\}}$ on $[0, T]$ with respect to the data $((\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3), B)$ in the precise sense of Definition 4.*

2.2. Weak-strong uniqueness and stability of evolutions

Combining Theorem 3 and Proposition 5 with the conditional stability of any calibrated MCF in arbitrary dimensions [4, Proposition 5], we obtain the following weak-strong uniqueness principle for distributional (i.e., BV) solutions to multiphase MCF in three dimensions:

Theorem 6 (Weak-strong uniqueness and quantitative stability). *Let $T \in (0, \infty)$ be a time horizon and let $d = 3$, $P = 3$, and $\sigma \in \mathbb{R}^{3\times 3}$ be a surface tension matrix satisfying the strict triangle inequality. Let $\bar{\Omega} = (\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ be a regular double bubble smoothly evolving by MCF on $[0, T]$ in the sense of Definition 10 (with respect to σ), and let $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ be a BV solution to multiphase MCF in the sense of Definition 12 (again* *with respect to σ). If the initial conditions of the regular double bubble and the BV solution coincide, then the solutions also coincide for later times on $[0,T]$. More precisely,*

$$\begin{aligned}&\mathcal{L}^3\big((\Omega_i(0)\setminus\bar\Omega_i(0))\cup(\bar\Omega_i(0)\setminus\Omega_i(0)\big)=0 \text{ for all } i\in\{1,2,3\}\\ &\quad\Longrightarrow \mathcal{L}^3\big((\Omega_i(t)\setminus\bar\Omega_i(t))\cup(\bar\Omega_i(t)\setminus\Omega_i(t))\big)=0 \text{ for a.e. } t\in[0,T] \text{ and all } i\in\{1,2,3\}.\end{aligned}$$

Moreover, we have quantitative stability estimates in the following sense: Denote by $\xi := ((\xi_i)_{i\in\{1,2,3\}}, B)$ the gradient flow calibration on $[0,T]$ from Theorem 3 with respect to $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$, and denote by $(\vartheta_i)_{i\in\{1,2,3\}}$ the corresponding family of transported weights on $[0,T]$ from Proposition 5. Let $\mathrm{n}_{i,j}(\cdot,t)$ be the measure-theoretic unit normal along the interface $\partial^*\Omega_i(t)\cap\partial^*\Omega_j(t)$ pointing from $\Omega_i(t)$ into $\Omega_j(t)$, $t\in[0,T]$. Then, the error functionals defined for all $t\in[0,T]$ by

$$E[\Omega|\xi](t) := \sum_{i,j\in\{1,2,3\},i\neq j}\sigma_{i,j}\int_{\partial^*\Omega_i(t)\cap\partial^*\Omega_j(t)} 1-\mathrm{n}_{i,j}(\cdot,t)\cdot\xi_{i,j}(\cdot,t)\,\mathrm{d}\mathcal{H}^2, \tag{3}$$

$$E[\Omega|\bar\Omega](t) := \sum_{i=1}^{3}\int_{(\Omega_i(t)\setminus\bar\Omega_i(t))\cup(\bar\Omega_i(t)\setminus\Omega_i(t))}|\vartheta_i(\cdot,t)|\,\mathrm{d}x \tag{4}$$

satisfy the stability estimates

$$E[\Omega|\xi](t) \le E[\Omega|\xi](0)e^{Ct}, \tag{5}$$

$$E[\Omega|\bar\Omega](t) \le \big(E[\Omega|\xi](0)+E[\Omega|\bar\Omega](0)\big)e^{Ct} \tag{6}$$

for almost every $t\in[0,T]$. The constant $C>0$ in these estimates depends only on the data of the smoothly evolving regular double bubble $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ on $[0,T]$ through the explicit constructions $((\xi_i)_{i\in\{1,2,3\}}, B)$ and $(\vartheta_i)_{i\in\{1,2,3\}}$.

Proof. As mentioned above, this is a straightforward application of Theorem 3, Proposition 5 and [4, Proposition 5]. ■

Remark 7 (Admissible surface tensions). Let us briefly comment on the matrix of surface tensions $\sigma\in\mathbb{R}^{P\times P}$. We say σ is admissible if it satisfies precisely the assumption in [4, Definition 9]. More concretely, we require that there exists a non-degenerate $(P-1)$-simplex $(q_1,\ldots,q_P)$ in $\mathbb{R}^{P-1}$ which represents the surface tensions in the form of $\sigma_{i,j}=|q_i-q_j|$.

In the framework of the present paper, that is, the case $P=3$, this is equivalent to the strict triangle inequality

$$\sigma_{i,j}<\sigma_{i,k}+\sigma_{k,j} \quad \text{for all choices } \{i,j,k\}=\{1,2,3\}. \tag{7}$$

In the general case $P\ge 3$, the ℓ^2-embeddability is in fact stronger than (7), and it constitutes the key ingredient to construct the missing calibration vector fields $(\xi_i)_{i\in\{4,\ldots,P\}}$, for which one may in fact argue along the same lines as in the proof of [4, Lemma 35] without requiring any additional ingredients from the constructions performed in this work.

We emphasize that only for simplicity, we considered in Theorem 6 the case of $P = 3$ phases on the level of the BV solution. Let us briefly outline the additional ingredients which are needed to establish stability estimates (5) and (6) in terms of general BV solutions $(\Omega_1, \ldots, \Omega_P)$, $P > 3$, defined on $\mathbb{R}^3 \times [0, T]$ with respect to a given ℓ^2-embeddable matrix of surface tensions $\sigma = (\sigma_{i,j})_{i,j\in\{1,\ldots,P\}} \in \mathbb{R}^{P\times P}$, and a fixed regular double bubble $(\bar\Omega_1, \bar\Omega_2, \bar\Omega_3)$ smoothly evolving by MCF with respect to the restriction $(\sigma_{i,j})_{i,j\in\{1,2,3\}}$ of the surface tension matrix $\sigma \in \mathbb{R}^{P\times P}$.

Recalling definitions (3) and (4) of the error functionals (in which one only needs to replace 3 by P in the case $P > 3$), it is clear that we have to augment the gradient flow calibration provided by Theorem 3 with additional calibrating vector fields $(\xi_i)_{i\in\{4,\ldots,P\}}$, and the family of transported weights by Proposition 5 with additional weights $(\vartheta_i)_{i\in\{4,\ldots,P\}}$, such that the resulting augmented families adhere to the requirements of Definition 2 and Definition 4, respectively, in order to allow for the desired application of [4, Proposition 5]. For consistency with our definitions, let us interpret the smoothly evolving regular double bubble as a partition $(\bar\Omega_1, \ldots, \bar\Omega_P)$ with the convention that $\bar\Omega_i := \emptyset$ for all additional phases $i \in \{4, \ldots, P\}$ in the BV solution.

Extending the family of transported weights is trivial since we may define $\vartheta_i := 1$ for all $i \in \{4, \ldots, P\}$, thus representing consistently the fact that the additional phases on the level of the smoothly evolving regular double bubble are empty.

Furthermore, the missing calibration vector fields can be constructed along the lines of the proof of [4, Lemma 35]. It is then straightforward that the associated additional vector fields

$$\sigma_{i,j}\xi_{i,j} := \xi_i - \xi_j, \quad i \in \{4, \ldots, P\} \text{ or } j \in \{4, \ldots, P\}$$

satisfy (1b)–(1f) (together with the desired regularity). Indeed, except for coercivity condition (1c), all these properties are trivially satisfied in terms of the relevant additional pairs of indices, since the associated interfaces on the level of the smoothly evolving regular double bubble are empty. With respect to (1c), the proof of [4, Lemma 37] applies verbatim without requiring any additional ingredients from the constructions of this work.

We decided to restrict ourselves to the case $P = 3$ in the formulation of Theorem 6 because we view the main contribution of this paper to be the first part of Theorem 1 (i.e., the combination of Theorem 3 and Proposition 5), and thus aim to shift the focus on the required 3D generalization of those results of [4] which are concerned with the given strong solution only (i.e., in the present work a regular double bubble smoothly evolving by MCF).

2.3. Definition of a regular double bubble smoothly evolving by MCF

This part is concerned with the formulation of a “strong solution concept” for a (topologically standard) double bubble moving by mean curvature flow, for which we are then able to show that its flow is calibrated in the precise sense of Definition 2. We start with the associated energy functional; see [4, Definition 12] for further details.

Definition 8 (Partition with finite interface energy). Consider $d \geq 2$, $P \geq 2$, and an admissible matrix of surface tensions $\sigma \in \mathbb{R}^{P \times P}$. Let $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ be a family of measurable subsets of $\mathbb{R}^d$ such that $\mathcal{L}^d(\mathbb{R}^d \setminus \bigcup_{i=1}^P \bar{\Omega}_i) = 0$ and $\mathcal{L}^d(\bar{\Omega}_i \cap \bar{\Omega}_j) = 0$ for all $i, j \in \{1, \ldots, P\}$ with $i \neq j$. We then call $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ a *partition of* $\mathbb{R}^d$ *with finite interface energy* if

$$E[(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)] := \sum_{i,j \in \{1,\ldots,P\}, i \neq j} \sigma_{i,j} \mathcal{H}^{d-1}(\partial^* \bar{\Omega}_i \cap \partial^* \bar{\Omega}_j) < \infty. \tag{8}$$

Next, let $T \in (0, \infty)$ be a time horizon, and consider a family $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ of open subsets of $\mathbb{R}^d \times [0, T]$ in the form of $\bar{\Omega}_i = \bigcup_{t \in [0,T]} \bar{\Omega}_i(t) \times \{t\}$ for all $i \in \{1, \ldots, P\}$. In this evolutionary setting, we call $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ an *evolving partition on* $\mathbb{R}^d \times [0, T]$ *with finite interface energy* if, for all $t \in [0, T]$, the family $(\bar{\Omega}_1(t), \ldots, \bar{\Omega}_P(t))$ is a partition of $\mathbb{R}^d$ with finite interface energy in the above sense and if the following holds:

$$\sup_{t \in [0,T]} E[(\bar{\Omega}_1(t), \ldots, \bar{\Omega}_P(t))] < \infty.$$

For such an evolving partition, we denote the associated evolving interfaces by $\bar{I}_{i,j} := \bigcup_{t \in [0,T]} \bar{I}_{i,j}(t) \times (t)$, where $\bar{I}_{i,j}(t) := \partial^* \bar{\Omega}_i(t) \cap \partial^* \bar{\Omega}_j(t)$ for all $t \in [0, T]$ and all pairs $i, j \in \{1, \ldots, P\}$, $i \neq j$.

In the next step, we formalize the topological setup as well as the main regularity assumptions. We also state the main compatibility condition in the form of the Herring angle condition.

Definition 9 (Regular double bubble). Let $\sigma \in \mathbb{R}^{3 \times 3}$ be an admissible matrix of surface tensions, and consider a partition $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ of $\mathbb{R}^3$ with finite interface energy in the sense of Definition 8. Assume in addition that $\bar{\Omega}_i$ is an open, non-empty and simply connected subset of $\mathbb{R}^3$ such that $\partial \bar{\Omega}_i$ is the closure of $\partial^* \bar{\Omega}_i$ for all $i \in \{1, 2, 3\}$. Then, define for each $i, j \in \{1, 2, 3\}$ with $i \neq j$ the associated interface $\bar{I}_{i,j} := \partial \bar{\Omega}_i \cap \partial \bar{\Omega}_j$, which is assumed to be non-empty.

We call $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ a *regular double bubble* if the following additional regularity conditions are satisfied:

(i) Each interface $\bar{I}_{i,j}$ is a two-dimensional, compact and simply connected manifold with boundary of class C^5. The interior of each interface is embedded.

(ii) The three interfaces $\bar{I}_{1,2}$, $\bar{I}_{2,3}$, and $\bar{I}_{3,1}$ intersect precisely along their respective boundary, which in turn is a non-empty one-dimensional, compact and connected manifold $\bar{\Gamma}$ without boundary of class C^5.

(iii) Along the triple line $\bar{\Gamma}$, the Herring angle condition

$$\sigma_{1,2} \bar{\mathrm{n}}_{1,2} + \sigma_{2,3} \bar{\mathrm{n}}_{2,3} + \sigma_{3,1} \bar{\mathrm{n}}_{3,1} = 0 \tag{9}$$

has to be satisfied, where we denote by $\bar{\mathrm{n}}_{i,j}$ the associated unit normal vector field along $\bar{I}_{i,j}$ pointing from $\bar{\Omega}_i$ into $\bar{\Omega}_j$.

With the notion of a regular double bubble in place, we finally clarify what we mean by a (sufficiently) smooth evolution of a regular double bubble with respect to mean curvature flow. It turns out that the construction of an associated gradient flow calibration in the vicinity of the evolving triple line requires two additional higher-order compatibility conditions. For a sufficiently smooth evolution of a regular double bubble, these two compatibility conditions are consequences of differentiating in time the assumed zeroth-order compatibility condition (i.e., the triple line being the common boundary of the three interfaces) or first-order compatibility condition (i.e., the Herring angle condition), respectively. Since we will require regularity down to time $t = 0$, we have to include the resulting compatibility conditions for the initial double bubble.

Definition 10 (Regular double bubble smoothly evolving by MCF). Let $\sigma \in \mathbb{R}^{3\times 3}$ be an admissible matrix of surface tensions. Let $(\bar{\Omega}_1^0, \bar{\Omega}_2^0, \bar{\Omega}_3^0)$ be an associated initial partition of $\mathbb{R}^3$ representing a regular double bubble in the sense of Definition 9. Assume in addition that $(\bar{\Omega}_1^0, \bar{\Omega}_2^0, \bar{\Omega}_3^0)$ satisfies the following two higher-order compatibility conditions:

First, we require for the scalar mean curvatures in form of $H_{i,j}^0 := -\nabla^{\tan} \cdot \bar{\mathrm{n}}_{i,j}^0$ that along the initial triple line $\bar{\Gamma}^0$ it holds that

$$\sigma_{1,2} H_{1,2}^0 + \sigma_{2,3} H_{2,3}^0 + \sigma_{3,1} H_{3,1}^0 = 0, \tag{10}$$

which by (9) is equivalent to the existence of a unique vector field $\mathrm{V}_{\bar{\Gamma}^0}$ along $\bar{\Gamma}^0$, which takes values in the normal bundle $\mathrm{Tan}^{\perp}\bar{\Gamma}^0$ such that

$$\bar{\mathrm{n}}_{i,j}^0 \cdot \mathrm{V}_{\bar{\Gamma}^0} = H_{i,j}^0 \quad \text{along } \bar{\Gamma}^0 \text{ for all } i, j \in \{1,2,3\} \text{ with } i \neq j.$$

Second, denoting by $\bar{\mathrm{t}}^0$ a unit-length tangent vector field along the initial triple line $\bar{\Gamma}^0$ and defining $\bar{\tau}_{i,j}^0 := \bar{\mathrm{n}}_{i,j}^0 \times \bar{\mathrm{t}}^0$ along $\bar{\Gamma}^0$ for all $i, j \in \{1,2,3\}$ with $i \neq j$, we require that along $\bar{\Gamma}^0$ the quantity

$$-\big(\bar{\tau}_{i,j}^0 \otimes \bar{\tau}_{i,j}^0 : \nabla^{\tan}\bar{\mathrm{n}}_{i,j}^0\big)(\bar{\tau}_{i,j}^0 \cdot \mathrm{V}_{\bar{\Gamma}^0}) + (\bar{\tau}_{i,j}^0 \cdot \nabla^{\tan}) H_{i,j}^0 \tag{11}$$

is independent of the choice of distinct $i, j \in \{1,2,3\}$ at each point on $\bar{\Gamma}^0$.

Now, let $T \in (0,\infty)$ be a time horizon and $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ be an evolving partition on $\mathbb{R}^3 \times [0,T]$ with finite interface energy in the sense of Definition 8. We call $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ a *regular double bubble smoothly evolving by MCF on* $[0,T]$ with initial data $(\bar{\Omega}_1^0, \bar{\Omega}_2^0, \bar{\Omega}_3^0)$ if it satisfies:

(i) For each $t \in [0,T]$, the family $(\bar{\Omega}_1(t), \bar{\Omega}_2(t), \bar{\Omega}_3(t))$ is a regular double bubble in the sense of Definition 9. Furthermore, the initial condition is attained in the sense that $(\bar{\Omega}_1(0), \bar{\Omega}_2(0), \bar{\Omega}_3(0)) = (\bar{\Omega}_1^0, \bar{\Omega}_2^0, \bar{\Omega}_3^0)$.

(ii) There exists a family of diffeomorphisms $\psi^t : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,T]$, such that $\psi^0(x) = x$ holds for all $x \in \mathbb{R}^3$, and $\bar{\Omega}_i(t) = \psi^t(\bar{\Omega}_i^0)$ and $\bar{I}_{i,j}(t) = \psi^t(\bar{I}_{i,j}^0)$ hold for all $t \in [0,T]$ and all $i, j \in \{1,2,3\}$, $i \neq j$. In addition, the map

$$\psi_{i,j} : \bar{I}_{i,j}^0 \times [0,T] \to \bar{I}_{i,j}, \quad (x,t) \mapsto (\psi^t(x), t)$$

is a diffeomorphism of class $(C_t^0 C_x^5 \cap C_t^1 C_x^3)(\bar{I}_{i,j}^0 \times [0,T])$.

(iii) For each $i, j \in \{1,2,3\}$ with $i \neq j$ and each $(x,t) \in \bar{I}_{i,j}$, denote by $\mathrm{V}_{\bar{I}_{i,j}}(x,t)$ the normal velocity vector of $\bar{I}_{i,j}(t)$ at $x \in \bar{I}_{i,j}(t)$. We then require motion by MCF for each interface, that is,

$$\big(\bar{\mathrm{n}}_{i,j} \cdot \mathrm{V}_{\bar{I}_{i,j}}\big)(x,t) = H_{i,j}(x,t), \quad (x,t) \in \bar{I}_{i,j},\ i, j \in \{1,2,3\} \text{ with } i \neq j. \tag{12}$$

Existence of $C^{2+2\alpha,1+\alpha}_{x,t}$ strong solutions for the MCF of double bubble geometries is contained in the work of Depner, Garcke, and Kohsaka [3] under the assumption that the initial double bubble is of class $C^{2+\alpha}_x$ and satisfies the geometric compatibility conditions up to order 2 (i.e., (9) and (10) next to the obvious zeroth-order condition that the triple line is the common boundary of the three interfaces). In order to obtain solutions with the regularity from Definition 10, one has to establish higher regularity for the strong solutions of Depner, Garcke, and Kohsaka [3], which necessitates both higher-order regularity and higher-order compatibility of the initial double bubble. However, a rigorous argument for the required higher regularity will surely be a technically demanding task and is therefore out of the scope of the present paper. In the next remark, let us at least provide a formal heuristic why the desired higher regularity result should be expected to hold true. To this end, we restrict our attention to a gain of one order of regularity within a neighborhood of the triple line.

Remark 11. The main result in the work of Depner, Garcke, and Kohsaka [3] is based on solving in a Hölder space setting a set of three non-linear non-local parabolic IBVP which are defined on the initial double bubble. The solution of each of these PDEs represents the normal component of a time-dependent diffeomorphism mapping at each time the initial double bubble to the evolved double bubble at that time. The non-locality of the associated set of PDEs just stems from the fact that the necessary tangential components of the diffeomorphisms are used only after projecting onto the triple line, so that they can be expressed up to a linear transformation by the values of the corresponding normal components at the triple line. This has the advantage of having only one unknown per interface.

Depner, Garcke, and Kohsaka [3] then solve this set of non-linear non-local parabolic PDEs by a linearization and contraction mapping argument, which in this context is by no means a trivial task, as the PDEs are defined on the curved and singular geometry of a double bubble. However, at least formally, the derivation of higher regularity of solutions should then be possible along the lines of usual regularity theory arguments which follow. For the following, we only focus on a gain of one order of regularity and only argue in the vicinity of the initial triple line:

The formal argument in principle consists of three steps. It is convenient to make use of the tangent frames $(\bar{\tau}_{i,j}, \bar{\mathrm{t}}_{i,j})$ from Construction 19 below.

(i) First, one differentiates each PDE of the system [3, (25)] in the direction of the corresponding vector field $\bar{\mathrm{t}}_{i,j}$ (a rigorous argument would proceed by the difference quotient method). By this, one obtains a non-linear non-local parabolic

IBVP for the derivative of the solution in the direction of $\bar{\mathrm{t}}_{i,j}$ which is at least of *similar structure* to the one in [3, (25)]. Of course, both the leading-order linear spatial differential operator and the non-local contribution do not simply commute with the directional spatial derivative, but the non-trivial commutators may in principle be put into the right hand side term.
Since the resulting system of PDEs is of similar structure, one may apply the arguments of Depner, Garcke, and Kohsaka [3] to deduce higher regularity for the derivative of the solution in the direction of $\bar{\mathrm{t}}_{i,j}$. Note that this requires higher regularity of the initial double bubble but not higher compatibility, because $\bar{\mathrm{t}}_{i,j}$ is tangent to the triple line.

(ii) Then, one repeats the reasoning from the first step by differentiating [3, (25)] in time. The resulting system is again of similar structure, but solving it now requires compatibility conditions up to order four for the initial double bubble. Recall that we already derived the third-order condition (cf. (11)). The fourth-order condition can be computed by differentiating the second-order one (10) in time, expressing everything in terms of spatial derivatives, and finally restricting to $t = 0$. Together with the necessary higher-order regularity of the initial double bubble, this should allow one to deduce higher regularity for the time derivative of the solution.

(iii) Finally, one aims to improve the regularity for the derivative of the solution in the direction of $\bar{\tau}_{i,j}$. This can be done by going back to the original problem [3, (25)] and extracting from the equation this directional derivative (e.g., by expressing the Laplace–Beltrami operator in local coordinates as a perturbation of the standard Euclidean Laplacian). The desired gain in regularity then follows from the previous two steps.

For the sake of completeness, we conclude this subsection with a definition of BV solutions to multiphase MCF. Global-in-time existence of such solutions was established by Otto and the second author [7,8] (as limit points of the thresholding scheme under an energy convergence assumption) as well as by Simon and the second author [9] (as limit points of solutions to the vector-valued Allen–Cahn equation under an energy convergence assumption).

Definition 12 (BV solutions to multiphase MCF). Let $d \geq 2$, let $P \geq 2$, let $\sigma \in \mathbb{R}^{P\times P}$ be a surface tension matrix satisfying the strict triangle inequality, and let $T \in (0,\infty)$ be a finite time horizon. Let $\Omega_0 = (\Omega_{0,1},\ldots,\Omega_{0,P})$ be an initial partition with finite interface energy (cf. Definition 8).

A tuple $\Omega = (\Omega_1,\ldots,\Omega_P)$ consisting of time-dependent partitions with finite interface energy $\Omega = (\Omega(t))_{t\in(0,T)}$, $\Omega(t) = (\Omega_1(t),\ldots,\Omega_P(t))$, $t \in (0,T)$, is called a *BV solution of multiphase MCF with initial partition* Ω_0 *and time horizon* T if:

(i) Let $\chi = (\chi_1,\ldots,\chi_P) : \mathbb{R}^d \times (0,T) \to \{0,1\}^P$ be the associated tuple of time-dependent characteristic functions with bounded variation. Then, for each phase $i \in \{1, \dots, P\}$, there exists $V_i \in L^2(0,T; L^2(\mathbb{R}^d; d|\nabla\chi_i(\cdot,t)|))$ such that

$$\int_{\mathbb{R}^d} \chi_i(\cdot,T')\zeta(\cdot,T')\,\mathrm{d}x - \int_{\mathbb{R}^d} \chi_{i,0}\zeta(\cdot,0)\,\mathrm{d}x$$
$$= \int_0^{T'} \int_{\mathbb{R}^d} \chi_i \partial_t \zeta \,\mathrm{d}x\,\mathrm{d}t - \int_0^{T'} \int_{\mathbb{R}^d} V_i \zeta \,\mathrm{d}|\nabla\chi_i|\,\mathrm{d}t$$

for a.e. $T' \in (0,T)$ and all $\zeta \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d \times [0,T))$.

(ii) Abbreviating $I_{i,j}(t) := \operatorname{supp}|\nabla\chi_i(\cdot,t)| \cap \operatorname{supp}|\nabla\chi_j(\cdot,t)|$ for $i \neq j \in \{1,\dots,P\}$ and $t \in (0,T)$, it holds that $V_i \frac{\nabla\chi_i}{|\nabla\chi_i|} = V_j \frac{\nabla\chi_j}{|\nabla\chi_j|}$ a.e. on $\bigcup_{t\in(0,T)} I_{i,j}(t) \times \{t\}$ and

$$\sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^{T'} \int_{I_{i,j}(t)} V_i \frac{\nabla\chi_i}{|\nabla\chi_i|} \cdot B \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t$$
$$= -\sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^{T'} \int_{I_{i,j}(t)} \Big(\mathrm{Id} - \frac{\nabla\chi_i}{|\nabla\chi_i|} \otimes \frac{\nabla\chi_i}{|\nabla\chi_i|}\Big) : \nabla B \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t$$

for a.e. $T' \in (0,T)$ and all $B \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d \times [0,T); \mathbb{R}^d)$.

(iii) It holds that

$$E[\Omega(T')] + \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^{T'} \int_{I_{i,j}(t)} |V_i|^2 \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \leq E[\Omega_0]$$

for a.e. $T' \in (0,T)$.

2.4. Notation

We briefly review the standard notation employed throughout the present work. The notation of geometric quantities will be introduced in the course of the paper.

We write $\mathcal{L}^d$ for the d-dimensional Lebesgue measure, $\mathcal{H}^s$ for the s-dimensional Hausdorff measure, as well as $\partial^* D$ for the reduced boundary of a set of finite perimeter. The standard Lebesgue spaces with respect to the Lebesgue measure are denoted as always by $L^p(D)$ for any $p \in [0,\infty]$ and any measurable $D \subset \mathbb{R}^d$, whereas in addition for any $k \in \mathbb{N}$ we denote by $W^{k,p}(D)$ the standard Sobolev space. We further write $C^k(D)$, $k \geq 0$, for the space of functions with bounded and continuous derivatives up to order k on $D \subset \mathbb{R}^d$. The intersection with the space $C^0_{\mathrm{cpt}}(D)$ of continuous and compactly supported functions on D is denoted by $C^k_{\mathrm{cpt}}(D)$. Vector-valued versions of these function spaces are denoted by $L^p(D;\mathbb{R}^d)$, and so on. In addition, given a differentiable function $f : D \to \mathbb{R}^m$ we write $\nabla f \in \mathbb{R}^{m\times d}$ for its Jacobian matrix, that is, $(\nabla f)_{i,j} = \partial_j f_i$ holds. If $f : M \to \mathbb{R}^m$ is a differentiable function along a given C^1 manifold M, we denote by $\nabla^{\tan}$ its tangential gradient.

For a space-time domain $D \subset \mathbb{R}^d \times [0,T]$ of the form $D = \bigcup_{t\in[0,T]} D(t) \times \{t\}$ we write $C^l_t C^k_x(D)$, $l,k \geq 0$, for the space of continuous functions f on D with continuous and bounded partial derivatives $\partial_t^{l'}\partial_x^{k'} f$ on D for any $0 \le l' \le l$ and any multi-index k' such that $0 \le |k'| \le k$. With a slight abuse of notation, the distance function $\operatorname{dist}(\cdot, D)$ with respect to such a space-time domain D is understood as the distance to the corresponding time slice, that is, $(x,t) \mapsto \operatorname{dist}(x, D(t))$ for all $(x,t) \in \mathbb{R}^d \times [0,T]$.

In terms of vector and tensor notation, we denote by $v \times w$ the cross product between two vectors $v, w \in \mathbb{R}^3$; by $v \wedge w := v \otimes w - w \otimes v$ the exterior product of $v, w \in \mathbb{R}^3$; and by $A : B := \sum_{i,j} A_{i,j} B_{i,j}$ the complete contraction of two matrices $A, B \in \mathbb{R}^{m\times n}$. Abusing notation, we will also write $a \wedge b := \min\{a,b\}$ for the minimum of two numbers $a, b \in \mathbb{R}$; however, it will always be perfectly clear from the context what the symbol $\wedge$ represents. We also occasionally use $a \vee b := \max\{a,b\}$ for the maximum of two numbers $a, b \in \mathbb{R}$.

3. Local gradient flow calibration at a smooth interface

The aim of this section is to provide the local building block of a gradient flow calibration in the vicinity of an interface present in a smoothly evolving double bubble. To this end, we introduce the following geometric setup:

Definition 13 (Localization radius for interface). Let $(\bar\Omega_1, \bar\Omega_2, \bar\Omega_3)$ be a regular double bubble smoothly evolving by MCF in the sense of Definition 10 on a time interval $[0,T]$. Fix $i, j \in \{1,2,3\}$ with $i \neq j$. We call a scale $r_{i,j} \in (0,1]$ an *admissible localization radius for the interface* $\bar I_{i,j}$ if

$$\Psi_{i,j} : \bar I_{i,j} \times (-r_{i,j}, r_{i,j}) \to \mathbb{R}^3 \times [0,T], \quad (x,t,s) \mapsto (x + s\bar{\mathrm{n}}_{i,j}(x,t), t)$$

is bijective onto its image $\operatorname{im}(\Psi_{i,j}) := \Psi_{i,j}(\bar I_{i,j} \times (-r_{i,j}, r_{i,j}))$. Moreover, it is required that the inverse Ψ^{-1} is a diffeomorphism of class $(C_t^0 C_x^4 \cap C_t^1 C_x^2)(\operatorname{im}(\Psi_{i,j}))$, and that it splits into the form

$$\Psi_{i,j}^{-1} : \operatorname{im}(\Psi_{i,j}) \to \bar I_{i,j} \times (-r_{i,j}, r_{i,j}), \quad (x,t) \mapsto (P_{i,j}(x,t), t, s_{i,j}(x,t)),$$

where the map $s_{i,j}$ represents a signed distance function (oriented by means of $\bar{\mathrm{n}}_{i,j}$, i.e., $\nabla s_{i,j} = \bar{\mathrm{n}}_{i,j}$ along the interface $\bar I_{i,j}$)

$$s_{i,j}(x,t) = \begin{cases} \operatorname{dist}(x, \bar I_{i,j}(t)) & \text{if } (x,t) \in \Psi_{i,j}(\bar I_{i,j} \times [0, r_{i,j})), \\ -\operatorname{dist}(x, \bar I_{i,j}(t)) & \text{if } (x,t) \in \Psi_{i,j}(\bar I_{i,j} \times (-r_{i,j}, 0)), \end{cases} \tag{13}$$

and the map $P_{i,j}$ being (in each time slice) the nearest-point projection onto $\bar I_{i,j}$, that is,

$$P_{i,j}(x,t) = \operatorname*{arg\,min}_{y \in \bar I_{i,j}(t)} |y - x|, \quad (x,t) \in \operatorname{im}(\Psi_{i,j}).$$

In view of Definition 10 regarding a regular double bubble smoothly evolving by MCF, it follows from the tubular neighborhood theorem that all interfaces admit an admissible localization radius in the sense of Definition 13.

We introduce some further notation and consequences with respect to Definition 13. First, the nearest-point projection onto the interface admits the representation

$$P_{i,j}(x,t) = x - s_{i,j}(x,t)\nabla s_{i,j}(x,t), \quad (x,t) \in \mathrm{im}(\Psi_{i,j}).$$

Second, it holds in terms of regularity

$$s_{i,j} \in (C_t^0 C_x^5 \cap C_t^1 C_x^3)(\mathrm{im}(\Psi_{i,j})), \quad P_{i,j} \in (C_t^0 C_x^4 \cap C_t^1 C_x^2)(\mathrm{im}(\Psi_{i,j})). \tag{14}$$

The scalar mean curvature of the interface $\bar{I}_{i,j}$ with respect to the orientation induced by $\bar{\mathrm{n}}_{i,j}$ is denoted by $H_{i,j}$. We extend these geometric quantities away from the interface, performing a slight abuse of notation, by means of

$$\bar{\mathrm{n}}_{i,j} : \mathrm{im}(\Psi_{i,j}) \to \mathbb{S}^2, \quad (x,t) \mapsto \nabla s_{i,j}(x,t), \tag{15}$$

$$H_{i,j} : \mathrm{im}(\Psi_{i,j}) \to \mathbb{R}, \quad (x,t) \mapsto -\Delta s_{i,j}(P_{i,j}(x,t),t). \tag{16}$$

Observe that these definitions immediately imply that

$$\bar{\mathrm{n}}_{i,j} \in (C_t^0 C_x^4 \cap C_t^1 C_x^2)(\mathrm{im}(\Psi_{i,j})), \quad H_{i,j} \in (C_t^0 C_x^3 \cap C_t^1 C_x^1)(\mathrm{im}(\Psi_{i,j})). \tag{17}$$

Construction 14 (Gradient flow calibration along smooth interfaces). Let the assumptions and notation of Definition 13 be in place, and let $\mathcal{Y}_{i,j} : \mathrm{im}(\Psi_{i,j}) \to \mathbb{R}^3$ be an arbitrary vector field of class $C_t^0 C_x^1(\mathrm{im}(\Psi_{i,j}))$. We then define a pair of vector fields $(\xi_{i,j}, B) : \mathrm{im}(\Psi_{i,j}) \to \mathbb{S}^2 \times \mathbb{R}^3$ as follows:

$$\xi_{i,j} := \bar{\mathrm{n}}_{i,j}, \quad B := H_{i,j}\bar{\mathrm{n}}_{i,j} + (\mathrm{Id} - \bar{\mathrm{n}}_{i,j} \otimes \bar{\mathrm{n}}_{i,j})\mathcal{Y}_{i,j}. \tag{18}$$

We call $(\xi_{i,j}, B)$ a *local gradient flow calibration for the interface* $\bar{I}_{i,j}$. ◇

We now register the properties of the pair of vector fields $(\xi_{i,j}, B)$—in particular, that it satisfies locally the requirements of Definition 2, with the exception of (1c). The latter will only be satisfied once we glue together the local constructions in Section 5 by means of a suitable family of cutoff functions.

Lemma 15. *Let the assumptions and notation of Construction 14 be in place. Then, it holds that*

$$\xi_{i,j} \in (C_t^0 C_x^4 \cap C_t^1 C_x^2)(\mathrm{im}(\Psi_{i,j})), \quad B \in C_t^0 C_x^1(\mathrm{im}(\Psi_{i,j})).$$

Moreover, there exists a constant $C > 0$ which depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0,T]$ such that the following hold throughout the space-time domain $\mathrm{im}(\Psi_{i,j})$*:*

$$|\nabla \xi_{i,j}| + |\partial_t \xi_{i,j}| \leq C, \tag{19}$$

$$|B| + |\nabla B| \leq C, \tag{20}$$

$$\partial_t \xi_{i,j} + (B \cdot \nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}}\xi_{i,j} = 0, \tag{21}$$

$$|\nabla \cdot \xi_{i,j} + B \cdot \xi_{i,j}| \leq C\,\mathrm{dist}(\cdot \bar{I}_{i,j}), \tag{22}$$

$$\partial_t |\xi_{i,j}|^2 + (B \cdot \nabla)|\xi_{i,j}|^2 = 0. \tag{23}$$

Proof. The asserted regularity follows immediately from definitions (18) and the regularity of the constituents (see (17)). Equation (21) for the time evolution of $\xi_{i,j}$ follows from differentiating in the spatial variable the PDE satisfied by the signed distance function $s_{i,j}$, that is,

$$\partial_t s_{i,j} = -H_{i,j} = -(B \cdot \nabla) s_{i,j}, \tag{24}$$

relying in the process on the product rule and $\bar{\mathrm{n}}_{i,j} = \nabla s_{i,j}$. The divergence constraint (see (22)) is a direct consequence of definitions (15), (16) and (18) in combination with the regularity of the signed distance (see (14)). Finally, equation (23) is satisfied for trivial reasons since $\xi_{i,j} \in \mathbb{S}^2$. ∎

4. Local gradient flow calibration at a triple line

This section constitutes the core of the present work. We establish the existence of a gradient flow calibration in the vicinity of the triple line for a double bubble smoothly evolving by MCF in the sense of Definition 10. The main result of this section reads as follows:

Proposition 16 (Existence of gradient flow calibration at the triple line). *Consider a regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ smoothly evolving by MCF on a time interval $[0, T]$ in the sense of Definition 10. Let $r \in (0, 1]$ be an associated admissible localization radius for the triple line in the sense of Definition 17 below. Then, there exists a potentially smaller radius $\hat{r} \in (0, r]$, only depending on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0, T]$, which gives rise to the following assertions:*

Denote by $\mathcal{N}_{\hat{r}}(\bar{\Gamma}) := \bigcup_{t \in [0,T]} B_{\hat{r}}(\bar{\Gamma}(t)) \times \{t\}$ the neighborhood of the evolving triple line. For all $i, j \in \{1, 2, 3\}$ with $i \neq j$, there exists a continuous local extension

$$\xi_{i,j} : \mathcal{N}_{\hat{r}}(\bar{\Gamma}) \to \overline{B_1(0)}$$

of the unit normal vector field $\bar{\mathrm{n}}_{i,j}|_{\bar{I}_{i,j}}$ of $\bar{I}_{i,j}$, and a continuous local extension

$$B : \mathcal{N}_{\hat{r}}(\bar{\Gamma}) \to \mathbb{R}^3$$

of the velocity vector field of the network $\mathcal{I} = \bigcup_{i,j \in \{1,2,3\}, i \neq j} \bar{I}_{i,j}$, such that the pair $((\xi_{i,j})_{i,j \in \{1,2,3\}, i \neq j}, B)$ satisfies the following list of requirements:

(i) *For all $i, j \in \{1, 2, 3\}$ with $i \neq j$, we have $\xi_{i,j} \in (C_t^0 C_x^1 \cap C_t^1 C_x^0)(\mathcal{N}_{\hat{r}}(\bar{\Gamma}) \setminus \bar{\Gamma})$ and $B \in C_t^0 C_x^1 (\mathcal{N}_{\hat{r}}(\bar{\Gamma}) \setminus \bar{\Gamma})$, with corresponding estimates throughout $\mathcal{N}_{\hat{r}}(\bar{\Gamma}) \setminus \bar{\Gamma}$ given by*

$$|\nabla \xi_{i,j}| + |\partial_t \xi_{i,j}| \leq C, \tag{25}$$

$$|B| + |\nabla B| \leq C \tag{26}$$

for some constant $C > 0$ which depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0, T]$.

(ii) *We have for all* $i, j \in \{1,2,3\}$ *with* $i \neq j$

$$\begin{aligned} \xi_{i,j}(\cdot,t) &= \bar{\mathrm{n}}_{i,j}(\cdot,t) \quad \text{along } \bar{I}_{i,j}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t)), \\ B(\cdot,t) &= \mathrm{V}_{\bar{\Gamma}}(\cdot,t) \quad \text{along } \bar{\Gamma}(t) \end{aligned} \tag{27}$$

for all $t \in [0,T]$, *where* $\mathrm{V}_{\bar{\Gamma}}$ *denotes the normal velocity of the triple line* $\bar{\Gamma}$. *Moreover, the skew-symmetry relation* $\xi_{i,j} = -\xi_{j,i}$ *holds true.*

(iii) *The Herring angle condition is satisfied in the whole space-time tubular neighborhood* $\mathcal{N}_{\hat{r}}(\bar{\Gamma})$ *of the triple line, that is,*

$$\sigma_{1,2}\xi_{1,2} + \sigma_{2,3}\xi_{2,3} + \sigma_{3,1}\xi_{3,1} = 0 \quad \text{in } \mathcal{N}_{\hat{r}}(\bar{\Gamma}).$$

(iv) *There exists a constant* $C > 0$, *depending only on the data of the smoothly evolving regular double bubble* $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ *on* $[0,T]$, *such that for all* $i, j \in \{1,2,3\}$ *with* $i \neq j$ *the estimates*

$$|\partial_t \xi_{i,j} + (B\cdot\nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}}\xi_{i,j}| \leq C\,\mathrm{dist}(\cdot, \bar{I}_{i,j}), \tag{28}$$

$$|B\cdot\xi_{i,j} + \nabla\cdot\xi_{i,j}| \leq C\,\mathrm{dist}(\cdot, \bar{I}_{i,j}), \tag{29}$$

$$\partial_t|\xi_{i,j}|^2 + (B\cdot\nabla)|\xi_{i,j}|^2 \leq C\,\mathrm{dist}^2(\cdot, \bar{I}_{i,j}) \tag{30}$$

hold true within $\mathcal{N}_{\hat{r}}(\bar{\Gamma}) \setminus \bar{\Gamma}$.

A pair $((\xi_{i,j})_{i,j\in\{1,2,3\}, i\neq j}, B)$ *subject to these conditions is called a* local gradient flow calibration at the triple line $\bar{\Gamma}$.

The remainder of this section is organized as follows: In Section 4.1, we introduce the necessary notation employed in the construction of the desired vector fields. Section 4.2 implements the construction of the main building blocks for the vector fields $((\xi_{i,j})_{i\neq j}, B)$, which will then be glued together in Section 4.3. Section 4.4 contains the proof of Proposition 16. Finally, in Section 4.5, we formalize the fact that the local gradient flow calibration at the triple line due to Proposition 16 represents an admissible perturbation of the local gradient flow calibrations at the interfaces in a suitable sense.

4.1. Local geometry at a triple line

We first provide a suitable decomposition of the space-time tubular neighborhood of the triple line of a smoothly evolving regular double bubble in the sense of Definition 10. The main ingredient is given by the following notion of an admissible localization radius for the triple line (cf. Figure 3):

Definition 17 (Localization radius for triple line). Let $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ be a regular double bubble smoothly evolving by MCF in the sense of Definition 10 on a time interval $[0,T]$. For each $i, j \in \{1,2,3\}$ with $i \neq j$, let $r_{i,j} \in (0,1]$ be an admissible localization radius for the interface $\bar{I}_{i,j}$ in the sense of Definition 13. We call $r = r_{\bar{\Gamma}} \in (0, \min\{r_{i,j} : i,j \in \{1,2,3\},\ i\neq j\}]$ an *admissible localization radius for the triple line* $\bar{\Gamma}$ if the following properties are satisfied:

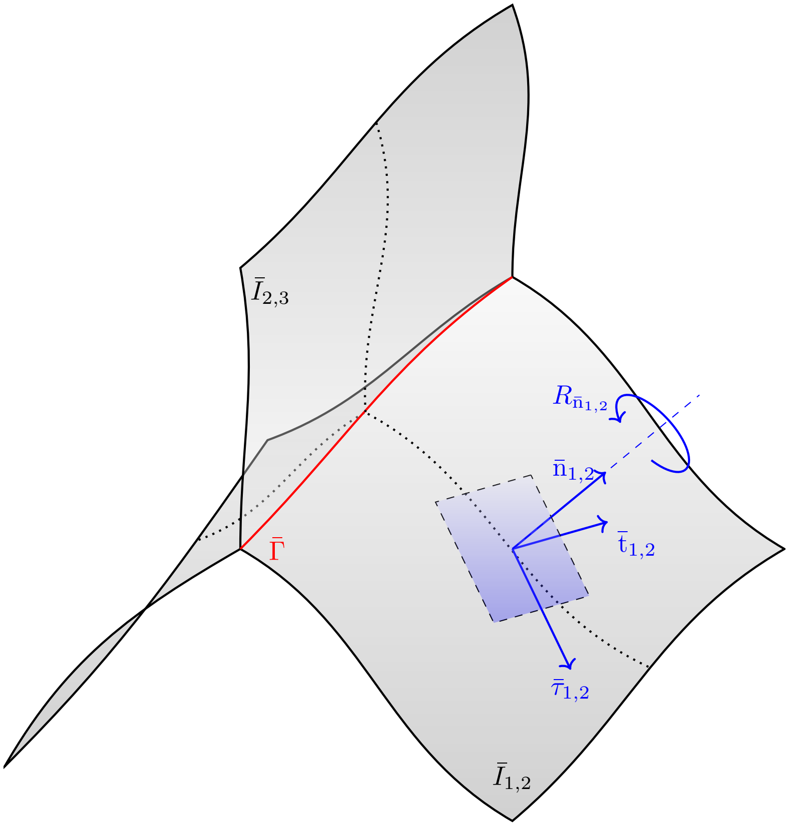

Figure 2. The smooth solution close to the triple line $\bar{\Gamma}$. Three sheets come together at fixed angles along $\bar{\Gamma}$ (here, 120°). In this general situation, one needs to introduce an additional gauge rotation field $R_{\bar{n}_{1,2}}$. At each point, this matrix is a rotation in the tangent plane spanned by $\bar{\tau}_{1,2}$ and $\bar{t}_{1,2}$, illustrated here by a shaded (blue) rectangle.

(i) *Regularity of triple line.* Define $\mathcal{N}_r(\bar{\Gamma}) := \bigcup_{t\in[0,T]} B_r(\bar{\Gamma}(t))\times\{t\}$. The squared distance to $\bar{\Gamma}$ satisfies $\operatorname{dist}^2(\cdot,\bar{\Gamma}) \in C_t^0 C_x^4(\mathcal{N}_r(\bar{\Gamma})) \cap C_t^1 C_x^2(\mathcal{N}_r(\bar{\Gamma}))$, and for the nearest-point projection onto $\bar{\Gamma}$, we have $P_{\bar{\Gamma}} \in C_t^0 C_x^4(\mathcal{N}_r(\bar{\Gamma})) \cap C_t^1 C_x^2(\mathcal{N}_r(\bar{\Gamma}))$.

(ii) *Wedge decomposition.* For each $i, j \in \{1,2,3\}$ with $i \neq j$, there exist sets $W_{\bar{I}_{i,j}} := \bigcup_{t\in[0,T]} W_{\bar{I}_{i,j}(t)}\times\{t\}$, $W_{\bar{I}_{j,i}} := W_{\bar{I}_{i,j}}$, and $W_{\bar{\Omega}_i} := \bigcup_{t\in[0,T]} W_{\bar{\Omega}_i}(t)\times\{t\}$ subject to the following conditions:

First, for each $t \in [0,T]$ the sets $(W_{\bar{I}_{i,j}}(t))_{i,j\in\{1,2,3\},i\neq j}$ and $(W_{\bar{\Omega}_i}(t))_{i\in\{1,2,3\}}$ are non-empty, pairwise disjoint open subsets of $B_r(\bar{\Gamma}(t))$. For each $x \in \bar{\Gamma}(t)$, the slice of each of these sets in the normal plane $x+\operatorname{Tan}_x^{\perp}\bar{\Gamma}(t)$ is the intersection of $B_r(\bar{\Gamma}(t))$ and a cone with apex x (cf. Figure 3). More precisely, there exist unit-length vector fields $(X^{\pm}_{\bar{I}_{i,j}})_{i,j\in\{1,2,3\},i\neq j}$ and $(X^{\pm}_{\bar{\Omega}_i})_{i\in\{1,2,3\}}$ along $\bar{\Gamma}$, taking values for each $t \in [0,T]$ in the

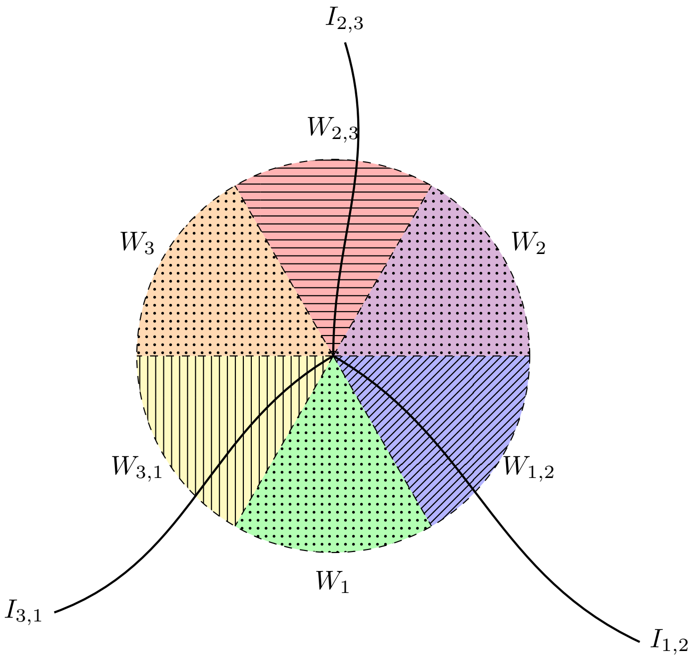

Figure 3. A cross-section orthogonal to the triple line illustrating the wedge decomposition in Definition 17. The "interpolation wedges" are marked with a dotted pattern and the "interface wedges" with striped patterns.

normal bundle $\mathrm{Tan}^{\perp}\overline{\Gamma}(t)$ and being of class $C_t^0C_x^4(\overline{\Gamma}) \cap C_t^1C_x^2(\overline{\Gamma})$, so that for all $i, j \in \{1,2,3\}$ with $i \neq j$ and all $(x,t) \in \overline{\Gamma}$, it holds that

$$
\begin{aligned}
&W_{\bar{I}_{i,j}}(t) \cap \big(x + \mathrm{Tan}_x^{\perp}\overline{\Gamma}(t)\big) \\
&\quad = \big(x + \big\{\alpha X^+_{\bar{I}_{i,j}}(x,t) + \beta X^-_{\bar{I}_{i,j}}(x,t) : \alpha, \beta \in (0,\infty)\big\}\big) \cap B_r(\overline{\Gamma}(t)),
\end{aligned} \tag{31}
$$

as well as

$$
\begin{aligned}
&W_{\bar{\Omega}_i}(t) \cap \big(x + \mathrm{Tan}_x^{\perp}\overline{\Gamma}(t)\big) \\
&\quad = \big(x + \big\{\alpha X^+_{\bar{\Omega}_i}(x,t) + \beta X^-_{\bar{\Omega}_i}(x,t) : \alpha, \beta \in (0,\infty)\big\}\big) \cap B_r(\overline{\Gamma}(t)).
\end{aligned} \tag{32}
$$

Moreover, $X^{\pm}_{\bar{I}_{i,j}} = X^{\pm}_{\bar{I}_{j,i}}$ and

$$
(X^+_{\bar{\Omega}_i}, X^-_{\bar{\Omega}_i}) \in \big\{(X^+_{\bar{I}_{i,j}}, X^-_{\bar{I}_{k,i}}), (X^+_{\bar{I}_{k,i}}, X^-_{\bar{I}_{i,j}})\big\}
$$

for all $i, j, k \in \{1,2,3\}$ such that $\{i,j,k\} = \{1,2,3\}$. The opening angles of these cones are constant along $\overline{\Gamma}$ and take values in $(0,\pi)$.

Second, for each $t \in [0,T]$ these sets provide a decomposition of the tubular neighborhood of the triple line in the sense that

$$\overline{B_r(\bar{\Gamma}(t))} = \overline{W_{\bar{I}_{1,2}}(t)} \cup \overline{W_{\bar{I}_{2,3}}(t)} \cup \overline{W_{\bar{I}_{3,1}}(t)} \cup \bigcup_{i\in\{1,2,3\}} \overline{W_{\bar{\Omega}_i}(t)}. \tag{33}$$

Third, for all $t \in [0,T]$ and all distinct $i,j \in \{1,2,3\}$, it holds that

$$\bar{I}_{i,j}(t) \cap B_r(\bar{\Gamma}(t)) \subset W_{\bar{I}_{i,j}}(t) \cup \bar{\Gamma}(t) \subset \big\{x \in \mathbb{R}^3 : (x,t) \in \mathrm{im}(\Psi_{i,j})\big\}, \tag{34}$$

$$W_{\bar{\Omega}_i}(t) \subset \bigcap_{j\in\{1,2,3\}\setminus\{i\}} \big\{x \in \mathbb{R}^3 : (x,t) \in \mathrm{im}(\Psi_{i,j})\big\}, \tag{35}$$

where we refer to Definition 13 for the diffeomorphisms $\Psi_{i,j}$.

(iii) *Comparability of distances.* There exists $C > 0$ such that for all pairwise distinct $i,j,k \in \{1,2,3\}$, it holds that (recall that $\mathcal{I} = \bigcup_{i,j\in\{1,2,3\},i\neq j} \bar{I}_{i,j}$)

$$\mathrm{dist}(\cdot,\bar{\Gamma}) + \mathrm{dist}(\cdot,\bar{I}_{i,j}) + \mathrm{dist}(\cdot,\bar{I}_{k,i}) \leq C\,\mathrm{dist}(\cdot,\mathcal{I}) \qquad \text{in } W_{\bar{\Omega}_i}, \tag{36}$$

$$\mathrm{dist}(\cdot,\bar{\Gamma}) \leq C\,\mathrm{dist}(\cdot,\bar{I}_{i,j}) \qquad \text{in } W_{\bar{I}_{j,k}} \cup W_{\bar{I}_{k,i}}, \tag{37}$$

$$\mathrm{dist}(\cdot,\bar{I}_{i,j}) \leq C\,\mathrm{dist}(\cdot,\mathcal{I}) \qquad \text{in } W_{\bar{I}_{i,j}}. \tag{38}$$

We refer from here onward to the sets $(W_{\bar{I}_{i,j}})_{i,j\in\{1,2,3\},i\neq j}$ as *interface wedges*, and to the sets $(W_{\bar{\Omega}_i})_{i\in\{1,2,3\}}$ as *interpolation wedges*.

Equations (31) and (32) simply mean that the domains $W_{\bar{\Omega}_i}(t)$ and $W_{\bar{I}_{i,j}}(t)$ are "wedges" in the sense that their respective slices across the normal space $x + \mathrm{Tan}^{\perp}\bar{\Gamma}(t)$ of the triple line have a cone structure close to $\bar{\Gamma}(t)$. The comparability (36)–(38) of distance functions in the various slices can already be guessed from Figure 3.

Let us first briefly discuss the existence of an admissible localization radius.

Lemma 18. *Let the assumptions and notation of Definition 17 be in place. Then, there exists an admissible localization radius $r = r_{\bar{\Gamma}} \in (0,1]$ for the triple line. The radius r and the associated data only depends on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1,\bar{\Omega}_2,\bar{\Omega}_3)$ on $[0,T]$.*

Proof. We provide details on how to arrange the vector fields $(X^{\pm}_{\bar{I}_{i,j}})_{i,j\in\{1,2,3\},i\neq j}$ and $(X^{\pm}_{\bar{\Omega}_i})_{i\in\{1,2,3\}}$ in order to ensure properties (31)–(33). The remaining conditions are a consequence of exploiting the uniform space-time regularity of the interfaces present in the smoothly evolving regular double bubble $(\bar{\Omega}_1,\bar{\Omega}_2,\bar{\Omega}_3)$ on $[0,T]$ (cf. Definition 10), and choosing the scale $r \in (0, r_{1,2} \wedge r_{2,3} \wedge r_{3,1}]$ to be sufficiently small.

Fix $(x,t) \in \bar{\Gamma}$, and up to a translation and rotation we may assume without loss of generality that $x = 0$ and $\mathrm{Tan}^{\perp}_x\bar{\Gamma}(t) = \{0\}\times\mathbb{R}^2 = \langle e_2,e_3\rangle$, where $\{e_1,e_2,e_3\}$ denotes the standard basis of $\mathbb{R}^3$ and $\langle e_2,e_3\rangle$ the $\mathbb{R}$-linear span of $\{e_2,e_3\}$. Denote then by $\bar{\tau}_{1,2}, \bar{\tau}_{2,3}, \bar{\tau}_{3,1} \in \langle e_2,e_3\rangle$ the tangent vectors at $x = 0$ to the interfaces $\bar{I}_{1,2}$, $\bar{I}_{2,3}$ and $\bar{I}_{3,1}$, respectively, with the orientation chosen so that along $\bar{\Gamma}$ all of them point in the direction of the associated interface (which is also described in more detail in Construction 19 below). These tangent vectors define associated half-spaces

$$\mathbb{H}_{1,2} := \big\{y \in \langle e_2, e_3\rangle : y \cdot \bar{\tau}_{1,2} > 0\big\},$$

where $\mathbb{H}_{2,3}$ and $\mathbb{H}_{3,1}$ are defined analogously.

We now construct a set of pairwise disjoint open cones $C_{\bar{\Omega}_1}, C_{\bar{\Omega}_2}, C_{\bar{\Omega}_3} \subset \langle e_2, e_3\rangle$ which will provide the cone structure of the interpolation wedges by means of the following procedure: If the cone given by $\mathbb{H}_{1,2} \cap \mathbb{H}_{3,1}$ has an opening angle strictly greater than 90°, we define $C_{\bar{\Omega}_1} := \mathbb{H}_{1,2} \cap \mathbb{H}_{3,1}$. In the other case, we define $C_{\bar{\Omega}_1}$ to be the middle third of the cone with opening vectors $\bar{\tau}_{1,2}$ and $\bar{\tau}_{3,1}$. The remaining two cones, $C_{\bar{\Omega}_2}$ and $C_{\bar{\Omega}_3}$, are defined in the same way.

Note that the opening angles of the cones $(C_{\bar{\Omega}_i})_{i\in\{1,2,3\}}$ are always strictly smaller than 180°, since the surface tensions satisfy the strict triangle inequality. We proceed by selecting cones $C_{\bar{I}_{1,2}} =: C_{\bar{I}_{2,1}}, C_{\bar{I}_{2,3}} =: C_{\bar{I}_{3,2}}, C_{\bar{I}_{3,1}} =: C_{\bar{I}_{1,3}} \subset \langle e_2, e_3\rangle$, which are uniquely determined by the requirement that together with $(C_{\bar{\Omega}_i})_{i\in\{1,2,3\}}$ they form a family of pairwise disjoint open cones in $\langle e_2, e_3\rangle$ such that

$$\langle e_2, e_3\rangle = \overline{C_{\bar{I}_{1,2}}} \cup \overline{C_{\bar{I}_{2,3}}} \cup \overline{C_{\bar{I}_{3,1}}} \cup \bigcup_{i\in\{1,2,3\}} \overline{C_{\bar{\Omega}_i}},$$
$$\bar{\tau}_{1,2} \in C_{\bar{I}_{1,2}}, \quad \bar{\tau}_{2,3} \in C_{\bar{I}_{2,3}}, \quad \bar{\tau}_{3,1} \in C_{\bar{I}_{3,1}}.$$

We finally define $(X^{\pm}_{\bar{I}_{i,j}})_{i,j\in\{1,2,3\},i\neq j}$ and $(X^{\pm}_{\bar{\Omega}_i})_{i\in\{1,2,3\}}$ by means of the unit-length opening vectors of the cones $(C_{\bar{I}_{i,j}})_{i,j\in\{1,2,3\},i\neq j}$ and $(C_{\bar{\Omega}_i})_{i\in\{1,2,3\}}$, respectively. The right hand sides of properties (31) and (32) now serve as the defining objects for the interface and interpolation wedges, respectively. ■

In the second preparatory step, we proceed with the definition of a preliminary orthonormal frame along each of the three respective interfaces in the vicinity of the triple line (cf. Figure 2).

Construction 19 (Preliminary choice of tangent frame). Let the assumptions and notation of Definition 17 be in place. In particular, let $r \in (0, r_{1,2} \wedge r_{2,3} \wedge r_{3,1}]$ be an associated admissible localization radius for the triple line $\bar{\Gamma}$. We then provide for all $t \in [0,T]$ and all distinct phases $i,j \in \{1,2,3\}$ two tangent vector fields $\bar{\tau}_{i,j}(\cdot,t), \bar{\mathfrak{t}}_{i,j}(\cdot,t) \in \mathbb{S}^2$ along the local interface patch $\bar{I}_{i,j}(t) \cap B_r(\bar{\Gamma}(t))$ by means of the following procedure:

First, slicing the interface $\bar{I}_{i,j}(t)$ along the planes $y+\mathrm{Tan}_y^{\perp}\bar{\Gamma}(t)$ produces a family of curves

$$\bar{I}^y_{i,j}(t) := \bar{I}_{i,j}(t) \cap (y+\mathrm{Tan}_y^{\perp}\bar{\Gamma}(t)) \cap B_r(\bar{\Gamma}(t))$$

for all $y \in \bar{\Gamma}(t)$. Second, for each $t \in [0,T]$ and each $y \in \bar{\Gamma}(t)$, denote by $\bar{\tau}^y_{i,j}(\cdot,t) \in \mathbb{S}^2$ the tangent vector field along the curve $\bar{I}^y_{i,j}(t)$ which is oriented by $y+\frac{r}{2}\bar{\tau}^y_{i,j}(y,t) \in W_{\bar{I}_{i,j}}(t) \cap (y+\mathrm{Tan}_y^{\perp}\bar{\Gamma}(t))$. We then define two tangent vector fields on the local interface patch $\bar{I}_{i,j} \cap \mathcal{N}_r(\bar{\Gamma})$ by

$$\bar{\tau}_{i,j}(x,t) := \bar{\tau}^y_{i,j}(x,t)\big|_{y=P_{\bar{\Gamma}}(x,t)} \in \mathbb{S}^2, \quad (x,t) \in \bar{I}_{i,j} \cap \mathcal{N}_r(\bar{\Gamma}),$$
$$\bar{\mathrm{t}}_{i,j}(x,t) := (\bar{\mathrm{n}}_{i,j} \times \bar{\tau}_{i,j})(x,t) \in \mathbb{S}^2, \quad (x,t) \in \bar{I}_{i,j} \cap \mathcal{N}_r(\bar{\Gamma}).$$

This yields an orthonormal frame $(\bar{\mathrm{n}}_{i,j}, \bar{\tau}_{i,j}, \bar{\mathrm{t}}_{i,j})$ on $\bar{I}_{i,j} \cap \mathcal{N}_r(\bar{\Gamma})$. Observe also that

$$\bar{\mathrm{t}}_{1,2}|_{\bar{\Gamma}} = \bar{\mathrm{t}}_{2,3}|_{\bar{\Gamma}} = \bar{\mathrm{t}}_{3,1}|_{\bar{\Gamma}}. \tag{39}$$

By a minor abuse of notation, we finally introduce extensions of these tangential vector fields away from the interfaces, namely

$$(\bar{\tau}_{i,j}, \bar{\mathrm{t}}_{i,j})(x,t) := (\bar{\tau}_{i,j}, \bar{\mathrm{t}}_{i,j})(y,t)\big|_{y=P_{i,j}(x,t)}, \quad (x,t) \in \mathrm{im}(\Psi_{i,j}) \cap \mathcal{N}_r(\bar{\Gamma}). \tag{40}$$

We refer to Definition 13 for the diffeomorphism $\Psi_{i,j}$ and the projection $P_{i,j}$ onto the interface $\bar{I}_{i,j}$. We register in terms of regularity that

$$\bar{\tau}_{i,j}, \bar{\mathrm{t}}_{i,j} \in (C^0_t C^4_x \cap C^1_t C^2_x)(\mathrm{im}(\Psi_{i,j}) \cap \mathcal{N}_r(\bar{\Gamma})). \tag{41}$$

This concludes our construction of *orthonormal frames* $(\bar{\mathrm{n}}_{i,j}, \bar{\tau}_{i,j}, \bar{\mathrm{t}}_{i,j})$. ◇

In the rest of the paper, we will repeatedly rely on an explicit representation of the gradients for the normal and tangential vector fields. These formulas are the content of the following result:

Lemma 20. *Let the assumptions and notation of Definition 17 and Construction 19 be in place. To ease notation, let $\bar{I} := \bar{I}_{1,2}$, $\bar{I}' := \bar{I}_{2,3}$ and $\bar{I}'' := \bar{I}_{3,1}$ for the three interfaces present in the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$. We proceed accordingly for the associated orthonormal frames $(\bar{\mathrm{n}}, \bar{\tau}, \bar{\mathrm{t}})$, $(\bar{\mathrm{n}}', \bar{\tau}', \bar{\mathrm{t}}')$ and $(\bar{\mathrm{n}}'', \bar{\tau}'', \bar{\mathrm{t}}'')$, respectively.*

Using also the abbreviations $\kappa_{\bar{\tau}\bar{\tau}} := -\bar{\tau} \otimes \bar{\tau} : \nabla\bar{\mathrm{n}}$, $\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}} := -\bar{\mathrm{t}} \otimes \bar{\mathrm{t}} : \nabla\bar{\mathrm{n}}$ as well as $\kappa_{\bar{\tau}\bar{\mathrm{t}}} := -\bar{\tau} \otimes \bar{\mathrm{t}} : \nabla\bar{\mathrm{n}}$, it holds that $\kappa_{\bar{\tau}\bar{\mathrm{t}}} = -\bar{\mathrm{t}} \otimes \bar{\tau} : \nabla\bar{\mathrm{n}}$ and

$$\nabla\bar{\mathrm{n}} = -\kappa_{\bar{\tau}\bar{\tau}}\,\bar{\tau} \otimes \bar{\tau} - \kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\,\bar{\mathrm{t}} \otimes \bar{\mathrm{t}} - \kappa_{\bar{\tau}\bar{\mathrm{t}}}\,(\bar{\mathrm{t}} \otimes \bar{\tau} + \bar{\tau} \otimes \bar{\mathrm{t}}), \tag{42}$$
$$\nabla\bar{\tau} = \kappa_{\bar{\tau}\bar{\tau}}\,\bar{\mathrm{n}} \otimes \bar{\tau} - (\nabla \cdot \bar{\mathrm{t}})\,\bar{\mathrm{t}} \otimes \bar{\tau} + \kappa_{\bar{\tau}\bar{\mathrm{t}}}\,\bar{\mathrm{n}} \otimes \bar{\mathrm{t}} + (\nabla \cdot \bar{\tau})\,\bar{\mathrm{t}} \otimes \bar{\mathrm{t}}, \tag{43}$$
$$\nabla\bar{\mathrm{t}} = \kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\,\bar{\mathrm{n}} \otimes \bar{\mathrm{t}} + \kappa_{\bar{\tau}\bar{\mathrm{t}}}\,\bar{\mathrm{n}} \otimes \bar{\tau} + (\nabla \cdot \bar{\mathrm{t}})\,\bar{\tau} \otimes \bar{\tau} - (\nabla \cdot \bar{\tau})\,\bar{\tau} \otimes \bar{\mathrm{t}} \tag{44}$$

along the local interface patch $\bar{I} \cap \mathcal{N}_r(\bar{\Gamma})$. Analogous formulas of course hold true for $(\bar{\mathrm{n}}', \bar{\tau}', \bar{\mathrm{t}}')$ along $\bar{I}' \cap \mathcal{N}_r(\bar{\Gamma})$ in terms of $(\kappa'_{\bar{\tau}'\bar{\tau}'}, \kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'}, \kappa'_{\bar{\tau}'\bar{\mathrm{t}}'})$, and for $(\bar{\mathrm{n}}'', \bar{\tau}'', \bar{\mathrm{t}}'')$ along $\bar{I}'' \cap \mathcal{N}_r(\bar{\Gamma})$ in terms of $(\kappa''_{\bar{\tau}''\bar{\tau}''}, \kappa''_{\bar{\mathrm{t}}''\bar{\mathrm{t}}''}, \kappa''_{\bar{\tau}''\bar{\mathrm{t}}''})$.

Proof. Representation (42) is essentially just a rephrasing of the definition of the coefficients $\kappa_{\bar{\tau}\bar{\tau}}$, $\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}$ and $\kappa_{\bar{\tau}\bar{\mathrm{t}}}$. The only additional ingredients needed for the validity of (42) are $(\nabla\bar{\mathrm{n}})^{\mathsf{T}}\bar{\mathrm{n}} = \frac{1}{2}\nabla|\bar{\mathrm{n}}|^2 = 0$ and the symmetry of $\nabla\bar{\mathrm{n}} = \nabla^2 s_{1,2}$ (cf. (15)).

For a proof of (43), we write $\bar\tau = J\bar{\mathrm{n}}$ where $J = \bar\tau \wedge \bar{\mathrm{n}} + \bar{\mathrm{t}} \otimes \bar{\mathrm{t}}$ denotes the associated rotation matrix around the $\bar{\mathrm{t}}$-axis. Based on $(\bar{\mathrm{n}}\cdot\nabla)\bar\tau = 0$, $(\nabla\bar\tau)^{\mathsf{T}}\bar\tau = \frac{1}{2}\nabla|\bar\tau|^2 = 0$ and (42), we then obtain

$$
\begin{aligned}
\nabla\bar\tau &= \kappa_{\bar\tau\bar\tau}\bar{\mathrm{n}}\otimes\bar\tau - \kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\bar{\mathrm{t}}\otimes\bar{\mathrm{t}} - \kappa_{\bar\tau\bar{\mathrm{t}}}(\bar{\mathrm{t}}\otimes\bar\tau - \bar{\mathrm{n}}\otimes\bar{\mathrm{t}}) \\
&\quad + \big((\bar\tau\cdot\nabla)J\big)\bar{\mathrm{n}}\otimes\bar\tau + \big((\bar{\mathrm{t}}\cdot\nabla)J\big)\bar{\mathrm{n}}\otimes\bar{\mathrm{t}} \\
&= \kappa_{\bar\tau\bar\tau}\bar{\mathrm{n}}\otimes\bar\tau - \kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\bar{\mathrm{t}}\otimes\bar{\mathrm{t}} - \kappa_{\bar\tau\bar{\mathrm{t}}}(\bar{\mathrm{t}}\otimes\bar\tau - \bar{\mathrm{n}}\otimes\bar{\mathrm{t}}) \\
&\quad + \big(\bar{\mathrm{n}}\otimes\bar{\mathrm{n}} : (\bar\tau\cdot\nabla)J\big)\bar{\mathrm{n}}\otimes\bar\tau + \big(\bar{\mathrm{t}}\otimes\bar{\mathrm{n}} : (\bar\tau\cdot\nabla)J\big)\bar{\mathrm{t}}\otimes\bar\tau \\
&\quad + \big(\bar{\mathrm{n}}\otimes\bar{\mathrm{n}} : (\bar{\mathrm{t}}\cdot\nabla)J\big)\bar{\mathrm{n}}\otimes\bar{\mathrm{t}} + \big(\bar{\mathrm{t}}\otimes\bar{\mathrm{n}} : (\bar{\mathrm{t}}\cdot\nabla)J\big)\bar{\mathrm{t}}\otimes\bar{\mathrm{t}}.
\end{aligned}
$$

For the two appearing $(\bar{\mathrm{n}}\otimes\bar{\mathrm{n}})$-components of ∇J, it suffices to take the symmetric part of J into account, which is $\bar{\mathrm{t}}\otimes\bar{\mathrm{t}}$. It then follows from $\bar{\mathrm{t}}\cdot\bar{\mathrm{n}} = 0$ that

$$
\bar{\mathrm{n}}\otimes\bar{\mathrm{n}} : (\bar\tau\cdot\nabla)J = \bar{\mathrm{n}}\otimes\bar{\mathrm{n}} : (\bar{\mathrm{t}}\cdot\nabla)J = 0.
$$

Based on (42), $\bar{\mathrm{t}}\cdot(\bar{\mathrm{t}}\cdot\nabla)\bar{\mathrm{t}} = \frac{1}{2}(\bar{\mathrm{t}}\cdot\nabla)|\bar{\mathrm{t}}|^2 = 0$ and $\bar{\mathrm{t}} = \bar{\mathrm{n}}\times\bar\tau$, we may further compute

$$
\begin{aligned}
\bar{\mathrm{t}}\otimes\bar{\mathrm{n}} : (\bar{\mathrm{t}}\cdot\nabla)J &= (\bar{\mathrm{t}}\otimes\bar{\mathrm{n}}) : (\bar{\mathrm{t}}\cdot\nabla)(\bar\tau\wedge\bar{\mathrm{n}}) + \bar{\mathrm{n}}\cdot(\bar{\mathrm{t}}\cdot\nabla)\bar{\mathrm{t}} \\
&= (\bar{\mathrm{t}}\otimes\bar{\mathrm{n}}) : (\bar{\mathrm{t}}\cdot\nabla)(\bar\tau\wedge\bar{\mathrm{n}}) - \bar{\mathrm{t}}\otimes\bar{\mathrm{t}} : \nabla\bar{\mathrm{n}} \\
&= (\bar{\mathrm{t}}\otimes\bar{\mathrm{n}}) : (\bar{\mathrm{t}}\cdot\nabla)(\bar\tau\wedge\bar{\mathrm{n}}) + \kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}.
\end{aligned}
$$

Based on (42), $\bar{\mathrm{t}}\cdot\bar{\mathrm{n}} = 0$, $(\bar{\mathrm{t}}\otimes\bar{\mathrm{n}}) : (\bar\tau\cdot\nabla)(\bar\tau\wedge\bar{\mathrm{n}}) = (\bar{\mathrm{t}}\otimes\bar\tau) : \nabla\bar\tau$ and $\bar{\mathrm{t}} = \bar{\mathrm{n}}\times\bar\tau$, we in addition have

$$
\begin{aligned}
\bar{\mathrm{t}}\otimes\bar{\mathrm{n}} : (\bar\tau\cdot\nabla)J &= (\bar{\mathrm{t}}\otimes\bar\tau) : \nabla\bar\tau + \bar{\mathrm{n}}\cdot(\bar\tau\cdot\nabla)\bar{\mathrm{t}} \\
&= (\bar{\mathrm{t}}\otimes\bar\tau) : \nabla\bar\tau - \bar{\mathrm{t}}\otimes\bar\tau : \nabla\bar{\mathrm{n}} \\
&= (\bar{\mathrm{t}}\otimes\bar\tau) : \nabla\bar\tau + \kappa_{\bar\tau\bar{\mathrm{t}}}.
\end{aligned}
$$

The combination of the previous four displays yields

$$
\begin{aligned}
\nabla\bar\tau &= \kappa_{\bar\tau\bar\tau}\bar{\mathrm{n}}\otimes\bar\tau + ((\bar{\mathrm{t}}\otimes\bar\tau) : \nabla\bar\tau)\bar{\mathrm{t}}\otimes\bar\tau + \kappa_{\bar\tau\bar{\mathrm{t}}}\bar{\mathrm{n}}\otimes\bar{\mathrm{t}} + (\nabla\cdot\bar\tau)\bar{\mathrm{t}}\otimes\bar{\mathrm{t}}, \\
\nabla\cdot\bar\tau &= (\bar{\mathrm{t}}\otimes\bar{\mathrm{n}}) : (\bar{\mathrm{t}}\cdot\nabla)(\bar\tau\wedge\bar{\mathrm{n}}).
\end{aligned}
$$

Moreover, exploiting $\bar{\mathrm{t}} = \bar{\mathrm{n}}\times\bar\tau$ and employing the product rule, (42), and the previous display yields

$$
\begin{aligned}
\nabla\bar{\mathrm{t}} &= \kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\bar{\mathrm{n}}\otimes\bar{\mathrm{t}} + (\nabla\cdot\bar{\mathrm{t}})\,\bar\tau\otimes\bar\tau + \kappa_{\bar\tau\bar{\mathrm{t}}}\bar{\mathrm{n}}\otimes\bar\tau - (\nabla\cdot\bar\tau)\,\bar\tau\otimes\bar{\mathrm{t}}, \\
\nabla\cdot\bar{\mathrm{t}} &= -(\bar{\mathrm{t}}\otimes\bar\tau) : \nabla\bar\tau.
\end{aligned}
$$

The previous two displays in turn directly imply (43) and (44). ∎

The orthonormal frames provided by Construction 19 together with the signed distance functions (see (13)) constitute all ingredients for the construction of a suitable building

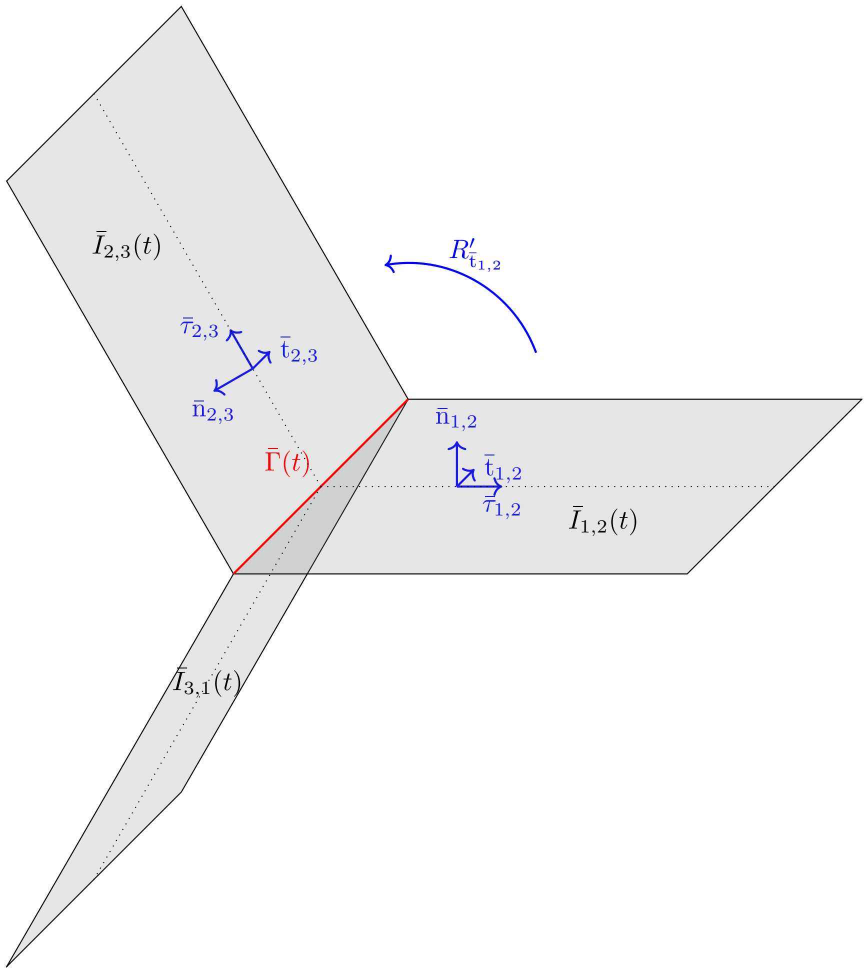

Figure 4. Local geometry at the triple line and preliminary construction of tangent frame. For simplicity, we illustrate here the case of three flat sheets coming together at equal angles of 120° along a straight triple line $\overline{\Gamma}(t)$. In this case, the "Herring" rotation $R'(y,t)$ is a rotation by 120° about the axis given by the tangent vector $\bar{t} = \bar{t}_{1,2}(y,t)$ of $\overline{\Gamma}(t)$. The dotted lines represent the three slices $\overline{I}^y_{i,j}(t)$ of the interfaces $\overline{I}_{i,j}$.

block $\widetilde{\xi}_{i,j}$ for the vector field $\xi_{i,j}$, at least in $\mathcal{N}_r(\overline{\Gamma}) \cap \mathrm{im}(\Psi_{i,j})$—see Construction 23 below. However, we also have to provide a construction of the vector field $\xi_{i,j}$ outside of the domain $\mathcal{N}_r(\overline{\Gamma}) \cap \mathrm{im}(\Psi_{i,j})$, that is, where this vector field a priori does not have a "natural" definition. The guiding principle is to mimic the Herring angle condition valid on the triple line:

$$\sigma_{1,2}\bar{n}_{1,2} + \sigma_{2,3}\bar{n}_{2,3} + \sigma_{3,1}\bar{n}_{3,1} = 0.$$

This condition motivates to appropriately rotate the already defined candidate vector fields $\widetilde{\xi}_{j,k}$ and $\widetilde{\xi}_{k,i}$ to provide the building blocks for the vector field $\xi_{i,j}$ throughout $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi_{j,k})$ and $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi_{k,i})$, respectively.

The rotations used in this procedure have to be chosen carefully so that our constructions will satisfy the requirements of a local gradient flow calibration at the triple line, such as sufficiently high regularity (in particular, adequate compatibility along the triple line) and the validity of the required evolution equations (up to a desired error in the distance to the interface).

Construction 21 (Gauged Herring rotation fields). Let the assumptions and notation of Definition 17, Construction 19 and Lemma 20 be in place. Consistent with the notational conventions of the latter, denote by Ψ, Ψ' and Ψ'' the diffeomorphisms from Definition 13 with respect to the interfaces $\bar{I}$, $\bar{I}'$ and $\bar{I}''$. We proceed accordingly for the surface tensions $(\sigma, \sigma', \sigma'')$ and the projections (P, P', P'').

We now define a pair of *Herring rotation fields*

$$R'_{\bar{\mathrm{t}}}, R''_{\bar{\mathrm{t}}} : \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \to SO(3) \subset \mathbb{R}^{3\times 3}$$

around the $\bar{\mathrm{t}}$-axis by

$$R'_{\bar{\mathrm{t}}}(x,t) := \cos\theta' \,\mathrm{Id} + \sin\theta' \,(\bar{\tau} \wedge \bar{\mathrm{n}})(x,t) + (1-\cos\theta')\,(\bar{\mathrm{t}} \otimes \bar{\mathrm{t}})(x,t), \tag{45}$$
$$R''_{\bar{\mathrm{t}}}(x,t) := \cos\theta'' \,\mathrm{Id} + \sin\theta'' \,(\bar{\tau} \wedge \bar{\mathrm{n}})(x,t) + (1-\cos\theta'')\,(\bar{\mathrm{t}} \otimes \bar{\mathrm{t}})(x,t)$$

for all $(x,t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi)$ (cf. Figure 4). The associated angles $\theta', \theta'' \in (0,\pi)$ are independent of $(x,t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi)$ and chosen based on the triple of surface tensions $(\sigma, \sigma', \sigma'')$ such that the relations

$$R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} = \bar{\mathrm{n}}', \tag{46}$$
$$R''_{\bar{\mathrm{t}}}\bar{\mathrm{n}} = \bar{\mathrm{n}}'' \tag{47}$$

hold true along the triple line $\bar{\Gamma}$. Hence, the Herring condition (9) implies that for all $(x,t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi)$ and all $v \in \mathbb{R}^3$ such that $v \cdot \bar{\mathrm{t}}(x,t) = 0$, it holds that

$$\sigma v + \sigma' R'_{\bar{\mathrm{t}}}(x,t)v + \sigma'' R''_{\bar{\mathrm{t}}}(x,t)v = 0. \tag{48}$$

Analogously, one defines a pair of rotations $(R_{\bar{\mathrm{t}}'}, R''_{\bar{\mathrm{t}}'})$ (resp. $(R_{\bar{\mathrm{t}}''}, R'_{\bar{\mathrm{t}}''})$) throughout the region $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi')$ (resp. $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi'')$).

Apart from the Herring rotation fields, we also introduce the *gauge rotation field*

$$R_{\bar{\mathrm{n}}} := R^{(2)}_{\bar{\mathrm{n}}} R^{(1)}_{\bar{\mathrm{n}}} : \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \to SO(3) \subset \mathbb{R}^{3\times 3} \tag{49}$$

around the $\bar{\mathrm{n}}$-axis (cf. Figure 2). The auxiliary rotation fields $R^{(1)}_{\bar{\mathrm{n}}}$ and $R^{(2)}_{\bar{\mathrm{n}}}$ around the

$\bar{\mathrm{n}}$-axis are defined via

$$R^{(1)}_{\bar{\mathrm{n}}}(x,t) := \cos\delta(x,t)\,\mathrm{Id} + \sin\delta(x,t)\,(\bar{\mathrm{t}}\wedge\bar{\tau})(x,t) + (1-\cos\delta(x,t))\,(\bar{\mathrm{n}}\otimes\bar{\mathrm{n}})(x,t), \tag{50}$$

$$R^{(2)}_{\bar{\mathrm{n}}}(x,t) := \cos\omega(x,t)\,\mathrm{Id} + \sin\omega(x,t)\,(\bar{\mathrm{t}}\wedge\bar{\tau})(x,t) + (1-\cos\omega(x,t))\,(\bar{\mathrm{n}}\otimes\bar{\mathrm{n}})(x,t). \tag{51}$$

Here, the rotation angle $\delta(x,t)$ is given explicitly by

$$\delta(x,t) := s(x,t)\kappa_{\bar{\tau}\bar{\mathrm{t}}}(x,t), \quad (x,t)\in\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi), \tag{52}$$

and the angle $\omega(x,t)$ is given by the extension

$$\omega(x,t) := \widehat{\omega}(P(x,t),t), \quad (x,t)\in\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi) \tag{53}$$

of $\widehat{\omega}(x,t)$, which in turn is defined by the one-parameter family of ODEs

$$\begin{cases} \widehat{\omega}(x,t) = 0, & (x,t)\in\bar{\Gamma}, \\ \big(\bar{\tau}(x,t)\cdot\nabla\big)\widehat{\omega}(x,t) = (\nabla\cdot\bar{\mathrm{t}})(x,t), & (x,t)\in\bar{I}\cap\mathcal{N}_r(\bar{\Gamma}). \end{cases} \tag{54}$$

Analogously, one defines a gauge rotation $R_{\bar{\mathrm{n}}'} := R^{(2)}_{\bar{\mathrm{n}}'}R^{(1)}_{\bar{\mathrm{n}}'}$ (resp. $R_{\bar{\mathrm{n}}''} := R^{(2)}_{\bar{\mathrm{n}}''}R^{(1)}_{\bar{\mathrm{n}}''}$) throughout the region $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi')$ (resp. $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi'')$).

We finally define via conjugation a pair of *gauged Herring rotation fields*

$$\widetilde{R}'_{\bar{I}} := R_{\bar{\mathrm{n}}}R'_{\bar{\mathrm{t}}}R^{\mathsf{T}}_{\bar{\mathrm{n}}} : \mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi) \to SO(3)\subset\mathbb{R}^{3\times3}, \tag{55}$$

$$\widetilde{R}''_{\bar{I}} := R_{\bar{\mathrm{n}}}R''_{\bar{\mathrm{t}}}R^{\mathsf{T}}_{\bar{\mathrm{n}}} : \mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi) \to SO(3)\subset\mathbb{R}^{3\times3}, \tag{56}$$

and analogously, a pair $(\widetilde{R}_{\bar{I}'}, \widetilde{R}''_{\bar{I}'})$ (resp. $(\widetilde{R}_{\bar{I}''}, \widetilde{R}'_{\bar{I}''})$) of gauged Herring rotation fields throughout the region $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi')$ (resp. $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi'')$). ◇

In a symmetric setting with either rotational or translational symmetry (cf. Figure 4), the gauge rotations $R_{\bar{\mathrm{n}}}$, $R_{\bar{\mathrm{n}}'}$, and $R_{\bar{\mathrm{n}}''}$ are not needed and, in fact, reduce to the identity matrix. In the general case (cf. Figure 2), they account for the fact that, for instance, the normal vector field $\bar{\mathrm{n}}(\cdot,t)$ evaluated along a slice $\bar{I}(t)\cap(x+\mathrm{Tan}^{\perp}_x\bar{\Gamma}(t))$ for some $x\in\bar{\Gamma}(t)$ will in general rotate out of the plane $x+\mathrm{Tan}^{\perp}_x\bar{\Gamma}(t)$ as one moves away from the triple line point x.

We conclude this section with the derivation of compatibility conditions along the triple line. These represent the last missing ingredients to ensure compatibility of the main building blocks $\widetilde{\xi}_{i,j}$ (cf. Construction 23 below) for the vector field $\xi_{i,j}$ and its rotated counterparts along the triple line (see Lemma 24 below).

Lemma 22. *Let the assumptions and notation of Definition 17, Construction 19, Lemma 20, and Construction 21 be in place. Consistent with the notational conventions of the* *latter two, denote by H, H' and H'' the extended scalar mean curvatures defined by (16) with respect to the interfaces $\bar{I}$, $\bar{I}'$ and $\bar{I}''$. Denote by $\mathrm{V}_{\bar{\Gamma}}$ the normal velocity vector field of the triple line.*

Then, the following compatibility conditions are satisfied along the triple line $\bar{\Gamma}$:

$$\bar{\tau}' = R'_{\bar{\mathrm{t}}}\bar{\tau}, \quad \bar{\tau}'' = R''_{\bar{\mathrm{t}}}\bar{\tau}, \tag{57}$$

$$\kappa'_{\bar{\tau}'\bar{\mathrm{t}}'} = \kappa_{\bar{\tau}\bar{\mathrm{t}}}, \quad \kappa''_{\bar{\tau}''\bar{\mathrm{t}}''} = \kappa_{\bar{\tau}\bar{\mathrm{t}}}, \tag{58}$$

$$\kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'} = (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}} - (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})\nabla\cdot\bar{\tau}, \tag{59}$$

$$\kappa''_{\bar{\mathrm{t}}''\bar{\mathrm{t}}''} = (R''_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}} - (R''_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})\nabla\cdot\bar{\tau}, \tag{60}$$

$$\nabla\cdot\bar{\tau}' = (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\nabla\cdot\bar{\tau}, \tag{61}$$

$$\nabla\cdot\bar{\tau}'' = (R''_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}} + (R''_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\nabla\cdot\bar{\tau}, \tag{62}$$

$$\sigma H + \sigma' H' + \sigma'' H'' = 0, \tag{63}$$

$$\begin{aligned}\kappa''_{\bar{\tau}'',\bar{\tau}''}(\bar{\tau}''\cdot\mathrm{V}_{\bar{\Gamma}}) + (\bar{\tau}''\cdot\nabla)H'' &= \kappa'_{\bar{\tau}'\bar{\tau}'}(\bar{\tau}'\cdot\mathrm{V}_{\bar{\Gamma}}) + (\bar{\tau}'\cdot\nabla)H' \\ &= \kappa_{\bar{\tau}\bar{\tau}}(\bar{\tau}\cdot\mathrm{V}_{\bar{\Gamma}}) + (\bar{\tau}\cdot\nabla)H.\end{aligned} \tag{64}$$

Of course, the analogues of (57) as well as (59)–(62) hold true for the appropriate relabelings of the associated data.

Next, we introduce a gauged orthonormal frame on $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi)$ by means of

$$(\bar{\mathrm{n}}, \bar{\tau}_*, \bar{\mathrm{t}}_*) := (\bar{\mathrm{n}}, R_{\bar{\mathrm{n}}}\bar{\tau}, R_{\bar{\mathrm{n}}}\bar{\mathrm{t}}). \tag{65}$$

Then, the following compatibility condition holds true:

$$(\bar{\mathrm{n}}, \bar{\tau}_*, \bar{\mathrm{t}}_*) = (\bar{\mathrm{n}}, \bar{\tau}, \bar{\mathrm{t}}) \quad \text{along the triple line } \bar{\Gamma}. \tag{66}$$

The analogue of (66) with respect to the gauged frame $(\bar{\mathrm{n}}', \bar{\tau}'_*, \bar{\mathrm{t}}'_*) := (\bar{\mathrm{n}}', R_{\bar{\mathrm{n}}'}\bar{\tau}', R_{\bar{\mathrm{n}}'}\bar{\mathrm{t}}')$ on $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi')$ (resp. $(\bar{\mathrm{n}}'', \bar{\tau}''_*, \bar{\mathrm{t}}''_*) := (\bar{\mathrm{n}}'', R_{\bar{\mathrm{n}}''}\bar{\tau}'', R_{\bar{\mathrm{n}}''}\bar{\mathrm{t}}'')$ on $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi'')$) is also satisfied.

Proof. Except for conditions (57) and (66), the asserted compatibility conditions are consequences of differentiating the existing zeroth- and first-order compatibility conditions along the triple line.

Step 1: Proof of (57). By (39) and the choice of the orientation for the tangent fields $(\bar{\tau}, \bar{\tau}', \bar{\tau}'')$ along the triple line (cf. Construction 19), it holds that

$$\bar{\tau} = J\bar{\mathrm{n}}, \quad \bar{\tau}' = J\bar{\mathrm{n}}', \quad \bar{\tau}'' = J\bar{\mathrm{n}}'' \qquad \text{on } \bar{\Gamma}. \tag{67}$$

In terms of a single 90° rotation field around the $\bar{\mathrm{t}}$-axis, we have

$$J = (\bar{\tau}\wedge\bar{\mathrm{n}}) + \bar{\mathrm{t}}\otimes\bar{\mathrm{t}} = (\bar{\tau}'\wedge\bar{\mathrm{n}}') + \bar{\mathrm{t}}'\otimes\bar{\mathrm{t}}' = (\bar{\tau}''\wedge\bar{\mathrm{n}}'') + \bar{\mathrm{t}}''\otimes\bar{\mathrm{t}}'' \quad \text{on } \bar{\Gamma}. \tag{68}$$

Hence, it follows from (46) and the fact that the Herring rotation $R'_{\bar{\mathrm{t}}}$ is a rotation around the same axis that

$$R'_{\bar{\mathrm{t}}}\bar{\tau} = R'_{\bar{\mathrm{t}}}J\bar{\mathrm{n}} = JR'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} = J\bar{\mathrm{n}}' = \bar{\tau}' \quad \text{on } \bar{\Gamma}.$$

This proves the first asserted identity of (57); the second of course follows analogously based on (47).

Step 2: Proof of (58)–(60). Since the Herring rotation $R'_{\bar{\mathrm{t}}}$ defined by (45) is a rotation around the $\bar{\mathrm{t}}$-axis with constant angle, the coefficients in the representation $R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} = (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\bar{\mathrm{n}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})\bar{\tau}$ are constant. Hence, together with formulas (42) and (43), we may compute along $\bar{\Gamma}$

$$
\begin{aligned}
(\bar{\mathrm{t}}\cdot\nabla)R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} &= (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})(\bar{\mathrm{t}}\cdot\nabla)\bar{\mathrm{n}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})(\bar{\mathrm{t}}\cdot\nabla)\bar{\tau} \\
&= \big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})(\nabla\cdot\bar{\tau}) - (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\big)\bar{\mathrm{t}} - (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\kappa_{\bar{\tau}\bar{\mathrm{t}}}\bar{\tau} \\
&\qquad + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})\kappa_{\bar{\tau}\bar{\mathrm{t}}}\bar{\mathrm{n}}.
\end{aligned}
$$

Furthermore, by the analogue of (43) for the tangent field $\bar{\tau}'$ as well as identities (39) and (57), and again the fact that $R'_{\bar{\mathrm{t}}}$ and J commute, we obtain along the triple line $\bar{\Gamma}$

$$
\begin{aligned}
(\bar{\mathrm{t}}'\cdot\nabla)\bar{\mathrm{n}}' &= -\kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'}\bar{\mathrm{t}}' - \kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\bar{\tau}' = -\kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'}\bar{\mathrm{t}} - (R'_{\bar{\mathrm{t}}}\bar{\tau}\cdot\bar{\tau})\kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\bar{\tau} - (R'_{\bar{\mathrm{t}}}\bar{\tau}\cdot\bar{\mathrm{n}})\kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\bar{\mathrm{n}} \\
&= -\kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'}\bar{\mathrm{t}} - (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\bar{\tau} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})\kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\bar{\mathrm{n}}.
\end{aligned}
$$

Hence, the defining condition (46) of the Herring rotation $R'_{\bar{\mathrm{t}}}$ and matching coefficients in the previous two displays implies the first identity of (58) as well as (59) (note that of course, either $(R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})$ or $(R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})$ is non-zero). The second identity of (58) as well as (60) in turn follow from an analogous computation based on (47).

Step 3: Proof of (61)–(62). These two compatibility conditions are derived as in the previous step, this time computing the tangential derivative along the triple line for both sides of the identities from (57), respectively.

Step 4: Proof of (63)–(64). By (12), the normal velocity $\mathrm{V}_{\bar{\Gamma}}$ of the triple line satisfies along $\bar{\Gamma}$

$$
\mathrm{V}_{\bar{\Gamma}}\cdot\sigma\bar{\mathrm{n}} = \sigma H, \quad \mathrm{V}_{\bar{\Gamma}}\cdot\sigma'\bar{\mathrm{n}}' = \sigma' H', \quad \mathrm{V}_{\bar{\Gamma}}\cdot\sigma''\bar{\mathrm{n}}'' = \sigma'' H''. \tag{69}
$$

Summing these identities results in (63), thanks to the Herring angle condition (9) being satisfied at each time.

To derive compatibility condition (64), we differentiate the Herring angle condition and obtain

$$
(\partial_t + \mathrm{V}_{\bar{\Gamma}}\cdot\nabla)(\sigma\bar{\mathrm{n}} + \sigma'\bar{\mathrm{n}}' + \sigma''\bar{\mathrm{n}}'') = 0.
$$

Now we compute, using (15) and (24) for the first term and (42) for the second one,

$$
\partial_t\bar{\mathrm{n}} + (\mathrm{V}_{\bar{\Gamma}}\cdot\nabla)\bar{\mathrm{n}} = -(\bar{\mathrm{t}}\cdot\nabla H)\bar{\mathrm{t}} - (\bar{\tau}\cdot\nabla H)\bar{\tau} - (\mathrm{V}_{\bar{\Gamma}}\cdot\bar{\tau})(\kappa_{\bar{\tau}\bar{\tau}}\bar{\tau}+\kappa_{\bar{\tau}\bar{\mathrm{t}}}\bar{\mathrm{t}}) \tag{70}
$$

on $\bar{\Gamma}$. The analogous equations hold for $\bar{\mathrm{n}}'$ and $\bar{\mathrm{n}}''$. Plugging those into (70) and using (39) and (58), we obtain

$$
\begin{aligned}
0 = \big(\bar{\mathrm{t}}\cdot\nabla(\sigma H + \sigma' H' + \sigma'' H'')\big)\bar{\mathrm{t}} &+ (\bar{\tau}\cdot\nabla H)\sigma\bar{\tau} + (\bar{\tau}'\cdot\nabla H')\sigma'\bar{\tau}' + (\bar{\tau}''\cdot\nabla H'')\sigma''\bar{\tau}'' \\
&+ \kappa_{\bar{\tau}\bar{\tau}}(\mathrm{V}_{\bar{\Gamma}}\cdot\bar{\tau})\sigma\bar{\tau} + \kappa'_{\bar{\tau}'\bar{\tau}'}(\mathrm{V}_{\bar{\Gamma}}\cdot\bar{\tau}')\sigma'\bar{\tau}' + \kappa''_{\bar{\tau}''\bar{\tau}''}(\mathrm{V}_{\bar{\Gamma}}\cdot\bar{\tau}'')\sigma''\bar{\tau}'' \\
&+ \mathrm{V}_{\bar{\Gamma}}\cdot(\sigma\bar{\tau}+\sigma'\bar{\tau}'+\sigma''\bar{\tau}'')(\kappa_{\bar{\tau}\bar{\mathrm{t}}}\bar{\mathrm{t}})
\end{aligned}
$$

on $\bar{\Gamma}$. Differentiating (63) along $\bar{\Gamma}$, we see that the first term vanishes. The last term vanishes by applying the fixed rotation J to the Herring condition (9). Thus, since the three vectors $\bar{\tau}$, $\bar{\tau}'$, and $\bar{\tau}''$ lie in one plane, we deduce (64) from the previous display.

Step 5: Proof of (66). Requirement (66) is immediate from definitions (49)–(54) in the form of

$$R_{\bar{\mathrm{n}}} = \mathrm{Id} \tag{71}$$

along the triple line $\bar{\Gamma}$. ■

With all of these ingredients in place, we may soon move on with the construction of a local gradient flow calibration at a triple line.

4.2. Extension of vector fields close to each interface

The aim of this section is to provide auxiliary extensions of the unit normal vector fields and an auxiliary extension of the normal velocity vector field which are defined in the neighborhood $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi_{i,j})$ for each interface $\bar{I}_{i,j}$, respectively. These extensions constitute the main building blocks for the desired extensions from Proposition 16.

Throughout this whole subsection, let the assumptions of Proposition 16 and the notation of Section 3 and Section 4.1 be in place. In particular, let us again make use of the following notational conventions which basically aim to drop the indices $i, j \in \{1,2,3\}$: we denote by $\bar{I} := \bar{I}_{1,2}$, $\bar{I}' := \bar{I}_{2,3}$, $\bar{I}'' := \bar{I}_{3,1}$ the three interfaces present in the given smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0, T]$. We proceed accordingly for the associated orthonormal frames $(\bar{\mathrm{n}}, \bar{\tau}, \bar{\mathrm{t}})$, $(\bar{\mathrm{n}}', \bar{\tau}', \bar{\mathrm{t}}')$, $(\bar{\mathrm{n}}'', \bar{\tau}'', \bar{\mathrm{t}}'')$ according to Construction 19, the surface tensions $(\sigma, \sigma', \sigma'')$, the signed distances (s, s', s''), the projections (P, P', P''), the scalar mean curvatures (H, H', H'') and the diffeomorphisms (Ψ, Ψ', Ψ'') from Definition 13.

Construction 23 (Extension of normal vector fields close to their associated interfaces). Define a coefficient function $\alpha : \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \to \mathbb{R}$ by

$$\alpha(x,t) := \alpha_{\mathrm{vel}}(x,t) + (\nabla \cdot \bar{\tau})(x,t), \quad (x,t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi), \tag{72}$$

where $\alpha_{\mathrm{vel}} : \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \to \mathbb{R}$ denotes, for the time being, an arbitrary coefficient function of class $C^0_t C^2_x(\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi))$ such that along the triple line it holds that

$$\alpha_{\mathrm{vel}}(x,t) = \bar{\tau}(x,t) \cdot \mathrm{V}_{\bar{\Gamma}}(x,t), \quad (x,t) \in \bar{\Gamma}. \tag{73}$$

Here, $\mathrm{V}_{\bar{\Gamma}}$ denotes again the normal velocity vector field of the triple line $\bar{\Gamma}$. Recall finally definition (65) of the gauged orthonormal frame $(\bar{\mathrm{n}}, \bar{\tau}_*, \bar{\mathrm{t}}_*)$.

We then define an initial extension $\tilde{\xi} : \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \to \mathbb{R}^3$ for the normal vector field $\bar{\mathrm{n}}|_{\bar{I}}$ of the interface $\bar{I}$ by means of the *gauged expansion ansatz*

$$\tilde{\xi}(x,t) := \bar{\mathrm{n}}(x,t) + \alpha(P_{\bar{\Gamma}}(x,t),t)s(x,t)\bar{\tau}_*(x,t) - \frac{1}{2}\alpha^2(P_{\bar{\Gamma}}(x,t),t)s^2(x,t)\bar{\mathrm{n}}(x,t) \tag{74}$$

for all $(x,t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi)$.

Analogously, one defines initial extensions $\widetilde{\xi}' : \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi') \to \mathbb{R}^3$ as well as $\widetilde{\xi}'' : \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi'') \to \mathbb{R}^3$ of the normal vector fields $\bar{\mathrm{n}}'|_{\bar{I}'}$ and $\bar{\mathrm{n}}''|_{\bar{I}''}$. ◇

The following result shows that, after applying the correct gauged Herring rotation as provided by Construction 21, the initial extensions of our normal vector fields are regular and first-order compatible along the triple line $\bar{\Gamma}$:

Lemma 24. *Let $(\widetilde{\xi}, \widetilde{\xi}', \widetilde{\xi}'')$ be the initial extensions from Construction 23 of the normal vector fields $(\bar{\mathrm{n}}|_{\bar{I}}, \bar{\mathrm{n}}'|_{\bar{I}'}, \bar{\mathrm{n}}''|_{\bar{I}''})$. Moreover, let $(\widetilde{R}'_{\bar{I}}, \widetilde{R}''_{\bar{I}})$, $(\widetilde{R}_{\bar{I}'}, \widetilde{R}''_{\bar{I}'})$ and $(\widetilde{R}_{\bar{I}''}, \widetilde{R}'_{\bar{I}''})$ be the gauged Herring rotations as provided by Construction 21.*

Then, it holds that $(\widetilde{\xi}, \widetilde{R}'_{\bar{I}}\widetilde{\xi}, \widetilde{R}''_{\bar{I}}\widetilde{\xi}) \in (C^0_t C^2_x \cap C^1_t C^0_x)(\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi))$ with corresponding estimates

$$|(\nabla, \nabla^2, \partial_t)(\widetilde{\xi}, \widetilde{R}'_{\bar{I}}\widetilde{\xi}, \widetilde{R}''_{\bar{I}}\widetilde{\xi})| \leq C \quad \text{in } \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi), \tag{75}$$

where the constant $C > 0$ only depends on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0, T]$. Moreover, the constructions are first-order compatible in the sense that along the triple line $\bar{\Gamma}$,

$$\widetilde{R}'_{\bar{I}}\widetilde{\xi} = \widetilde{\xi}', \quad \widetilde{R}''_{\bar{I}}\widetilde{\xi} = \widetilde{\xi}'', \tag{76}$$

$$\nabla\big(\widetilde{R}'_{\bar{I}}\widetilde{\xi}\big) = \nabla\widetilde{\xi}', \quad \nabla\big(\widetilde{R}''_{\bar{I}}\widetilde{\xi}\big) = \nabla\widetilde{\xi}''. \tag{77}$$

Analogous claims are satisfied in terms of the vector fields $(\widetilde{R}_{\bar{I}'}\widetilde{\xi}', \widetilde{\xi}', \widetilde{R}''_{\bar{I}'}\widetilde{\xi}')$ (resp. the vector fields $(\widetilde{R}_{\bar{I}''}\widetilde{\xi}'', \widetilde{R}'_{\bar{I}''}\widetilde{\xi}'', \widetilde{\xi}'')$) throughout the region $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi')$ (resp. the region $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi'')$).

Proof. We split the proof into two steps.

Step 1: Regularity estimates. We first claim that for each $\mathcal{R} \in \{R'_{\bar{\mathrm{t}}}, R''_{\bar{\mathrm{t}}}, R_{\bar{\mathrm{n}}}\}$,

$$|(\nabla, \nabla^2, \partial_t)\mathcal{R}| \leq C \quad \text{in } \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \tag{78}$$

for some constant $C > 0$ which depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0, T]$, and that analogous estimates hold true for $\mathcal{R} \in \{R_{\bar{\mathrm{t}}'}, R''_{\bar{\mathrm{t}}'}, R_{\bar{\mathrm{n}}'}\}$ in $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi')$, or for $\mathcal{R} \in \{R_{\bar{\mathrm{t}}''}, R'_{\bar{\mathrm{t}}''}, R_{\bar{\mathrm{n}}''}\}$ in $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi'')$.

For a Herring rotation $\mathcal{R} \in \{R'_{\bar{\mathrm{t}}}, R''_{\bar{\mathrm{t}}}\}$, claim (78) follows directly from the regularity of the frame $(\bar{\mathrm{n}}, \bar{\tau}, \bar{\mathrm{t}})$ (see (17) and (41)), since the associated angles θ', θ'' are independent of $(x, t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi)$; see Construction 21. In terms of the gauge rotation $\mathcal{R} = R_{\bar{\mathrm{n}}}$, it suffices to show that

$$|(\nabla, \nabla^2, \partial_t)(\delta, \omega)| \leq C \quad \text{in } \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi)$$

for the associated angles (δ, ω) defined in (52) and (53), respectively. For the angle δ, the regularity estimate from the previous display can be deduced from the regularity of the normal $\bar{\mathrm{n}}$ (see (17)). The regularity estimate for the angle ω in turn follows from the regularity of the projection onto the interface $\bar{I}$ (see (14)), the regularity of the tangent vector fields $(\bar{\tau}, \bar{\mathrm{t}})$ (see (41)), and from explicitly integrating (in each time slice) ODE (54) along the integral lines of the tangent vector field $\bar{\tau}$.

We next claim that there exist constants $c_1, c_2 \in (-1, 1)$ only depending on the surface tensions such that

$$\alpha_{\mathrm{vel}}(x,t) = (1-c_1^2)^{-1} c_2 \big(H'(x,t) - c_1 H(x,t)\big) \tag{79}$$

for all $(x,t) \in \bar{\Gamma}$. For a proof of (79), we define $c_1 := \bar{\tau}(x,t) \cdot \bar{\tau}'(x,t)$ and $c_2 := \bar{\mathrm{n}}'(x,t) \cdot \bar{\tau}(x,t) = -\bar{\mathrm{n}}(x,t) \cdot \bar{\tau}'(x,t)$, and then simply observe from (69) and (73) that

$$\begin{aligned} \alpha_{\mathrm{vel}}(x,t) &= c_2 H'(x,t) + c_1 \alpha'_{\mathrm{vel}}(x,t), \\ \alpha'_{\mathrm{vel}}(P_{\bar{\Gamma}}(x,t),t) &= -c_2 H(x,t) + c_1 \alpha_{\mathrm{vel}}(x,t) \quad \text{on } \bar{\Gamma}. \end{aligned}$$

Inserting the second identity of the previous display into the first one then directly yields claim (79).

The upshot of (78) and (79) is now the following: First, it follows from (72), the regularity of the projection onto the triple line $\bar{\Gamma}$ (cf. Definition 17 (i)), the regularity of the tangent $\bar{\tau}$ (see (41)), representation (79) and finally, the regularity of the extended scalar mean curvatures (see (17)) that $\alpha_{\bar{\Gamma}}(x,t) := \alpha(P_{\bar{\Gamma}}(x,t),t)$ satisfies

$$|\alpha_{\bar{\Gamma}}| + |(\nabla, \nabla^2, \partial_t)\alpha_{\bar{\Gamma}}| \leq C \quad \text{in } \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi).$$

The previous display in combination with (78) and expansion ansatz (74) finally implies the asserted regularity estimate (75).

Step 2: First-order compatibility along triple line. Zeroth-order conditions (76) are immediate from definitions (74) as well as identities (46) and (47), respectively. For a proof of the first-order condition, we focus on deriving the first identity of (77). The second follows along the same lines.

Recalling the definition of the gauged Herring rotation (see (55)) and the gauged expansion ansatz (see (74)), we compute on the interface $\bar{I}$ (using the abbreviation $\alpha_{\bar{\Gamma}}(\cdot,t) := \alpha(P_{\bar{\Gamma}}(\cdot,t),t)$ for $t \in [0,T]$)

$$\nabla\big(\widetilde{R}'_{\bar{I}}\,\widetilde{\xi}\big) = (\nabla R_{\bar{\mathrm{n}}}) R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} + R_{\bar{\mathrm{n}}} \nabla\big(R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\big) + \alpha_{\bar{\Gamma}}\big(R_{\bar{\mathrm{n}}} R'_{\bar{\mathrm{t}}}\bar{\tau}\big) \otimes \bar{\mathrm{n}}. \tag{80}$$

Let us now first compute $\nabla(R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}})$ and neglect the gauge rotations for a while. Recalling the fact that $R'_{\bar{\mathrm{t}}}$ is a rotation around the $\bar{\mathrm{t}}$-axis with constant angle (see (45)), we obtain on the interface $\bar{I}$

$$\nabla(R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}) = \nabla\big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}})\bar{\mathrm{n}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\tau})\bar{\tau}\big) = (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}})\,\nabla\bar{\mathrm{n}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\tau})\,\nabla\bar{\tau}.$$

Plugging in identities (42) and (43), and using in a second step that $R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}} = R'_{\bar{\mathrm{t}}}\bar{\tau} \cdot \bar{\tau}$ as well as $R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\tau} = -R'_{\bar{\mathrm{t}}}\bar{\tau} \cdot \bar{\mathrm{n}}$, we further compute

$$\begin{aligned}
\nabla(R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}) &= -\kappa_{\bar{\tau}\bar{\tau}}\big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}})\,\bar{\tau} \otimes \bar{\tau} - (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\tau})\,\bar{\mathrm{n}} \otimes \bar{\tau}\big) \\
&\quad - \big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}})\kappa_{\bar{\tau}\bar{\mathrm{t}}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\tau})(\nabla \cdot \bar{\mathrm{t}})\big)\,\bar{\mathrm{t}} \otimes \bar{\tau} \\
&\quad - \kappa_{\bar{\tau}\bar{\mathrm{t}}}\big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}})\,\bar{\tau} \otimes \bar{\mathrm{t}} - (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\tau})\,\bar{\mathrm{n}} \otimes \bar{\mathrm{t}}\big) \\
&\quad - \big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}})\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}} - (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\tau})(\nabla \cdot \bar{\tau})\big)\,\bar{\mathrm{t}} \otimes \bar{\mathrm{t}} \\
&= -\kappa_{\bar{\tau}\bar{\tau}}R'_{\bar{\mathrm{t}}}\bar{\tau} \otimes \bar{\tau} - \big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}})\kappa_{\bar{\tau}\bar{\mathrm{t}}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\tau})(\nabla \cdot \bar{\mathrm{t}})\big)\,\bar{\mathrm{t}} \otimes \bar{\tau} \\
&\quad - \kappa_{\bar{\tau}\bar{\mathrm{t}}}R'_{\bar{\mathrm{t}}}\bar{\tau} \otimes \bar{\mathrm{t}} - \big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}})\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}} - (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\tau})(\nabla \cdot \bar{\tau})\big)\,\bar{\mathrm{t}} \otimes \bar{\mathrm{t}},
\end{aligned} \tag{81}$$

which holds true on the interface $\bar{I}$.

Recalling the choice for α (see (72)), we may infer from formula (81) for $\nabla(R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}})$, substituting $\kappa_{\bar{\tau}\bar{\tau}} = H - \kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}$ along $\bar{I}$, identity (71), and formula (80) the following representation for the gradient of $\widetilde{R}'_{\bar{I}}\,\widetilde{\xi}$ along the triple line $\bar{\Gamma}$:

$$\begin{aligned}
\nabla(\widetilde{R}'_{\bar{I}}\,\widetilde{\xi}) &= R'_{\bar{\mathrm{t}}}\bar{\tau} \otimes (-H\bar{\tau} + \alpha_{\mathrm{vel}}\bar{\mathrm{n}}) + R'_{\bar{\mathrm{t}}}\bar{\tau} \otimes (\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\bar{\tau} + (\nabla \cdot \bar{\tau})\bar{\mathrm{n}}) \\
&\quad + ((\bar{\mathrm{t}} \cdot \nabla)\widetilde{R}'_{\bar{I}}\,\widetilde{\xi}\,) \otimes \bar{\mathrm{t}} - ((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}})\kappa_{\bar{\tau}\bar{\mathrm{t}}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\tau})(\nabla \cdot \bar{\mathrm{t}})\big)\,\bar{\mathrm{t}} \otimes \bar{\tau} \\
&\quad + (\nabla R_{\bar{\mathrm{n}}})R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}.
\end{aligned} \tag{82}$$

A direct computation based on ansatz (74), identities (42), (43), and (71), and substituting $\kappa'_{\bar{\tau}'\bar{\tau}'} = H' - \kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'}$ also yields along the triple line $\bar{\Gamma}$

$$\begin{aligned}
\nabla\widetilde{\xi}' &= \bar{\tau}' \otimes (-H'\bar{\tau}' + \alpha'_{\mathrm{vel}}\bar{\mathrm{n}}') + \bar{\tau}' \otimes (\kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'}\bar{\tau}' + (\nabla \cdot \bar{\tau}')\bar{\mathrm{n}}') \\
&\quad + (\bar{\mathrm{t}}' \cdot \nabla)\widetilde{\xi}' \otimes \bar{\mathrm{t}}' - \kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\,\bar{\mathrm{t}}' \otimes \bar{\tau}' + (\nabla R_{\bar{\mathrm{n}}'})\bar{\mathrm{n}}'.
\end{aligned} \tag{83}$$

We proceed by comparing formulas (82) and (83). Recalling that we denoted by $\mathrm{V}_{\bar{\Gamma}}$ the normal velocity vector field of the triple line, we obtain from (69), the choice of α_{vel} (see (73)), identities (67) and (68), and the zeroth-order compatibility along the triple line (see (57)), that the first terms in (82) and (83) are identical:

$$R'_{\bar{\mathrm{t}}}\bar{\tau} \otimes (-H\bar{\tau} + \alpha_{\mathrm{vel}}\bar{\mathrm{n}}) = -\bar{\tau}' \otimes J\mathrm{V}_{\bar{\Gamma}} = \bar{\tau}' \otimes (-H'\bar{\tau}' + \alpha'_{\mathrm{vel}}\bar{\mathrm{n}}') \quad \text{along } \bar{\Gamma}.$$

Moreover, by compatibility conditions (57), (59) and (61) along the triple line, as well as $R'_{\bar{\mathrm{t}}}\bar{\tau} \cdot \bar{\tau} = R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}}$ and $R'_{\bar{\mathrm{t}}}\bar{\tau} \cdot \bar{\mathrm{n}} = -R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\tau}$, we may infer that the second terms agree, too:

$$R'_{\bar{\mathrm{t}}}\bar{\tau} \otimes (\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\bar{\tau} + (\nabla \cdot \bar{\tau})\bar{\mathrm{n}}) = \bar{\tau}' \otimes (\kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'}\bar{\tau}' + (\nabla \cdot \bar{\tau}')\bar{\mathrm{n}}') \quad \text{along } \bar{\Gamma}.$$

From the last two identities together with (82), (83), (76), and (39) we therefore obtain along the triple line $\bar{\Gamma}$

$$\begin{aligned}
\nabla\big(\widetilde{R}_{\bar{I}}\,\widetilde{\xi}\big) - \nabla\widetilde{\xi}' &= -\big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}})\kappa_{\bar{\tau}\bar{\mathrm{t}}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} \cdot \bar{\tau})(\nabla \cdot \bar{\mathrm{t}})\big)\,\bar{\mathrm{t}} \otimes \bar{\tau} + \kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\,\bar{\mathrm{t}} \otimes \bar{\tau}' \\
&\quad + (\nabla R_{\bar{\mathrm{n}}})R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} - (\nabla R_{\bar{\mathrm{n}}'})\bar{\mathrm{n}}'.
\end{aligned} \tag{84}$$

In the rotationally symmetric case, the right hand side terms in the first line of (84) actually vanish. However, there is no reason in general why these terms should vanish without assuming additional symmetry. This is the motivation for the introduction of the additional gauge rotation matrices around the normal axis. Their definition is arranged in such a way so that their contribution in (84) exactly cancels the right hand side terms of the first line.

First, we obtain from definitions (49)–(54) along the triple line

$$(\nabla R_{\bar{\mathrm{n}}})R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} = \big((\bar{\tau}\cdot\nabla)R^{(2)}_{\bar{\mathrm{n}}}\big)R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\otimes\bar{\tau} + \big((\bar{\mathrm{n}}\cdot\nabla)R^{(1)}_{\bar{\mathrm{n}}}\big)R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\otimes\bar{\mathrm{n}}. \tag{85}$$

Let us next compute the two relevant directional derivatives of the gauge rotation matrices. We first observe that due to (50) and (52),

$$(\bar{\mathrm{n}}\cdot\nabla)R^{(1)}_{\bar{\mathrm{n}}} = \kappa_{\bar{\tau}\bar{\mathrm{t}}}\,\bar{\mathrm{t}}\wedge\bar{\tau} \tag{86}$$

along the interface $\bar{I}$. This in turn entails by $R'_{\bar{\mathrm{t}}}\bar{\tau}\cdot\bar{\mathrm{n}} = -R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau}$

$$\big((\bar{\mathrm{n}}\cdot\nabla)R^{(1)}_{\bar{\mathrm{n}}}\big)R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\otimes\bar{\mathrm{n}} = -\kappa_{\bar{\tau}\bar{\mathrm{t}}}(R'_{\bar{\mathrm{t}}}\bar{\tau}\cdot\bar{\mathrm{n}})\,\bar{\mathrm{t}}\otimes\bar{\mathrm{n}} \quad \text{along } \bar{\Gamma}. \tag{87}$$

Moreover, we may compute based on (51), (53), and (54) on the triple line $\bar{\Gamma}$

$$(\bar{\tau}\cdot\nabla)R^{(2)}_{\bar{\mathrm{n}}} = (\nabla\cdot\bar{\mathrm{t}})\,\bar{\mathrm{t}}\wedge\bar{\tau},$$

from which we deduce

$$\big((\bar{\tau}\cdot\nabla)R^{(2)}_{\bar{\mathrm{n}}}\big)R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\otimes\bar{\tau} = \big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})(\nabla\cdot\bar{\mathrm{t}})\big)\,\bar{\mathrm{t}}\otimes\bar{\tau} \quad \text{along } \bar{\Gamma}. \tag{88}$$

A straightforward computation shows that along the triple line $\bar{\Gamma}$, it holds that

$$(\nabla R_{\bar{\mathrm{n}}'})\bar{\mathrm{n}}' = \nabla\big(R_{\bar{\mathrm{n}}'}\bar{\mathrm{n}}'\big) - R_{\bar{\mathrm{n}}'}\nabla\bar{\mathrm{n}}' = \nabla\bar{\mathrm{n}}' - \nabla\bar{\mathrm{n}}' = 0. \tag{89}$$

Combining (85), (87), (88), and (89) with compatibility conditions (57) and (58) finally yields the desired cancellation

$$\begin{aligned} &-\big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\kappa_{\bar{\tau}\bar{\mathrm{t}}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})(\nabla\cdot\bar{\mathrm{t}})\big)\,\bar{\mathrm{t}}\otimes\bar{\tau} + \kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\,\bar{\mathrm{t}}\otimes\bar{\tau}' \\ &\quad + (\nabla R_{\bar{\mathrm{n}}})R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} - (\nabla R_{\bar{\mathrm{n}}'})\bar{\mathrm{n}}' = 0 \end{aligned}$$

along the triple line $\bar{\Gamma}$. By (84), this in turn concludes the proof of Lemma 24. ■

We proceed with the construction of suitable candidate velocity fields.

Construction 25 (Extension of velocity fields close to their associated interfaces). Recall that $\mathrm{V}_{\bar{\Gamma}}$ denotes the normal velocity of the triple line $\bar{\Gamma}$, and recall definition (65) of the gauged orthonormal frame $(\bar{\mathrm{n}}, \bar{\tau}_*, \bar{\mathrm{t}}_*)$. We then define a coefficient function

$$\alpha_{\mathrm{vel}} : \mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi) \to \mathbb{R}, \quad (x,t)\mapsto\hat{\alpha}_{\mathrm{vel}}(P(x,t),t), \tag{90}$$

where the coefficient α_{vel} is defined by projection onto the interface $\bar{I}$ in terms of the solution of the following family of ODEs, solved along the integral lines of the tangent vector field $\bar{\tau}_*$ with initial condition posed on the triple line $\bar{\Gamma}$:

$$\begin{cases} \hat{\alpha}_{\mathrm{vel}}(x,t) = (\bar{\tau}_* \cdot \mathrm{V}_{\bar{\Gamma}})(x,t), & (x,t) \in \bar{\Gamma}, \\ (\bar{\tau}_* \cdot \nabla)\hat{\alpha}_{\mathrm{vel}}(x,t) = (H\kappa_{\bar{\tau}_*\bar{\tau}_*})(x,t), & (x,t) \in \bar{I} \cap \mathcal{N}_r(\bar{\Gamma}). \end{cases} \tag{91}$$

Note that the choice of the initial value in (91) is consistent with (73). Next, we define another coefficient function

$$\beta : \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \to \mathbb{R}, \quad (x,t) \mapsto -\big((\bar{\tau}_* \cdot \nabla)H\big)(x,t) - (\alpha_{\mathrm{vel}}\kappa_{\bar{\tau}_*\bar{\tau}_*})(x,t). \tag{92}$$

We now define a preliminary extension $\widetilde{B} : \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \to \mathbb{R}^3$ of the normal velocity vector field $(H\bar{\mathrm{n}})|_{\bar{I}}$ for the interface $\bar{I}$ in terms of the *gauged expansion ansatz*

$$\widetilde{B}(x,t) := H(x,t)\,\bar{\mathrm{n}}(x,t) + \alpha_{\mathrm{vel}}(x,t)\,\bar{\tau}_*(x,t) + \beta(x,t)s(x,t)\,\bar{\tau}_*(x,t) \tag{93}$$

for all $(x,t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi)$.

Analogously, one defines preliminary extensions $\widetilde{B}' : \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi') \to \mathbb{R}^3$ as well as $\widetilde{B}'' : \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi'') \to \mathbb{R}^3$ of the normal velocity vector fields $(H'\bar{\mathrm{n}}')|_{\bar{I}'}$ and $(H''\bar{\mathrm{n}}'')|_{\bar{I}''}$, respectively. ◇

Note carefully that even away from the triple line we do not introduce a tangential velocity in $\bar{\mathrm{t}}_*$-direction. As the proof of the next result shows, this will entail that the gradients of the auxiliary velocities $\widetilde{B}$, $\widetilde{B}'$ and $\widetilde{B}''$ do not fully match along the triple line. However, the only mismatch appears in, at least for our purposes, inessential components. More precisely, in terms of, say, $\nabla\widetilde{B}$ the only non-matching terms result from its $\bar{\mathrm{t}}_* \otimes \bar{\tau}_*$- (resp. $\bar{\mathrm{t}}_* \otimes \bar{\mathrm{n}}$-) component. In view of the desired evolution equation (1d) and the fact that $\widetilde{\xi} \perp \bar{\mathrm{t}}_*$ due to (74), this specific component of $\nabla\widetilde{B}$ is intrinsically irrelevant for a gradient flow calibration (this argument turns out to be robust even with respect to the interpolation construction from Section 4.3).

Lemma 26. *Let $(\widetilde{B}, \widetilde{B}', \widetilde{B}'')$ be the preliminary extensions from Construction 25 of the normal velocity vector fields $((H\bar{\mathrm{n}})|_{\bar{I}}, (H'\bar{\mathrm{n}}')|_{\bar{I}'}, (H''\bar{\mathrm{n}}'')|_{\bar{I}''})$.*

Then, it holds that $\widetilde{B} \in C^0_t C^2_x(\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi))$ with corresponding estimate

$$|\widetilde{B}| + |\nabla\widetilde{B}| + |\nabla^2\widetilde{B}| \leq C \quad \text{in } \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi), \tag{94}$$

where the constant $C > 0$ only depends on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0,T]$. Analogous claims hold true for $\widetilde{B}'$ (resp. $\widetilde{B}''$) throughout $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi')$ (resp. $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi'')$).

Moreover, the constructions are essentially first-order compatible in the sense that *along the triple line $\bar{\Gamma}$, it holds that*

$$\widetilde{B} = \widetilde{B}' = \widetilde{B}'' = \mathrm{V}_{\bar{\Gamma}}, \tag{95}$$

$$(\mathrm{Id} - \bar{\mathrm{t}} \otimes \bar{\mathrm{t}})(\nabla \widetilde{B}) = (\mathrm{Id} - \bar{\mathrm{t}}' \otimes \bar{\mathrm{t}}')(\nabla \widetilde{B}') = (\mathrm{Id} - \bar{\mathrm{t}}'' \otimes \bar{\mathrm{t}}'')(\nabla \widetilde{B}''), \tag{96}$$

for which one should also recall that $\bar{\mathrm{t}} = \bar{\mathrm{t}}' = \bar{\mathrm{t}}''$ along $\bar{\Gamma}$ (cf. (39)).

Note that here the projection $\mathrm{Id} - \bar{\mathrm{t}} \otimes \bar{\mathrm{t}}$ acts on the components of $\widetilde{B}$, not ∇.

Proof. Step 1: Regularity estimates. Due to definition (92), regularity estimates (78) for the gauge rotations, the regularity of the frame $(\bar{\mathrm{n}}, \bar{\tau}, \bar{\mathrm{t}})$ (see (17) and (41)), the regularity of the extended scalar mean curvatures (see (17)), and finally expansion ansatz (93), it suffices to prove that

$$|\alpha_{\mathrm{vel}}| + |(\nabla, \nabla^2)\alpha_{\mathrm{vel}}| \leq C \quad \text{in } \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi), \tag{97}$$

where $C > 0$ is a constant which depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0, T]$.

Estimate (97) in turn follows directly from explicitly integrating (in each time slice) ODEs (91) along the integral lines of the tangent field $\bar{\tau}_*$, and exploiting as before the regularity of the associated geometric quantities.

Step 2: Zeroth-order compatibility at the triple line. Condition (95) is immediate from definition (93), identities (69), and the specific choices (see (90)–(91)).

Step 3: First-order compatibility at the triple line. We proceed with the proof of (96). Observe that we have on the interface $\bar{I}$ by direct analogy to the proofs of (42) and (43) that

$$\nabla \bar{\mathrm{n}} = -\kappa_{\bar{\tau}_*\bar{\tau}_*}\, \bar{\tau}_* \otimes \bar{\tau}_* - \kappa_{\bar{\mathrm{t}}_*\bar{\mathrm{t}}_*}\, \bar{\mathrm{t}}_* \otimes \bar{\mathrm{t}}_* - \kappa_{\bar{\tau}_*\bar{\mathrm{t}}_*}\, (\bar{\mathrm{t}}_* \otimes \bar{\tau}_* + \bar{\tau}_* \otimes \bar{\mathrm{t}}_*), \tag{98}$$

$$\begin{aligned} \nabla \bar{\tau}_* &= \kappa_{\bar{\tau}_*\bar{\tau}_*}\, \bar{\mathrm{n}} \otimes \bar{\tau}_* - (\nabla \cdot \bar{\mathrm{t}}_*)\, \bar{\mathrm{t}}_* \otimes \bar{\tau}_* + \kappa_{\bar{\tau}_*\bar{\mathrm{t}}_*}\, \bar{\mathrm{n}} \otimes \bar{\mathrm{t}}_* + (\nabla \cdot \bar{\tau}_*)\, \bar{\mathrm{t}}_* \otimes \bar{\mathrm{t}}_* \\ &\quad + (\bar{\mathrm{n}} \cdot \nabla)\bar{\tau}_* \otimes \bar{\mathrm{n}}. \end{aligned} \tag{99}$$

It follows directly from definitions (40) and (65) of our orthonormal frames, definitions (49)–(54) of the gauge rotations, as well as formula (86) being valid along the interface $\bar{I}$ that

$$(\bar{\mathrm{n}} \cdot \nabla)\bar{\tau}_* = R^{(2)}_{\bar{\mathrm{n}}}\big((\bar{\mathrm{n}} \cdot \nabla) R^{(1)}_{\bar{\mathrm{n}}}\big)\bar{\tau} = \kappa_{\bar{\tau}\bar{\mathrm{t}}} R_{\bar{\mathrm{n}}}\bar{\mathrm{t}} = \kappa_{\bar{\tau}\bar{\mathrm{t}}}\bar{\mathrm{t}}_* \quad \text{along } \bar{I}.$$

Starting now from definition (93), the previous display, the choices of the coefficient functions (see (90)–(92)), as well as formulas (98) and (99) directly entail along the interface $\bar{I}$

$$\begin{aligned} \nabla \widetilde{B} &= \beta\, \bar{\tau}_* \otimes \bar{\mathrm{n}} + \big((\bar{\tau}_* \cdot \nabla) H + \hat{\alpha}_{\mathrm{vel}} \kappa_{\bar{\tau}_*\bar{\tau}_*}\big)\, \bar{\mathrm{n}} \otimes \bar{\tau}_* \\ &\quad + \big((\bar{\tau}_* \cdot \nabla)\hat{\alpha}_{\mathrm{vel}} - H \kappa_{\bar{\tau}_*\bar{\tau}_*}\big)\, \bar{\tau}_* \otimes \bar{\tau}_* + \big((\bar{\mathrm{t}}_* \cdot \nabla)\widetilde{B} \otimes \bar{\mathrm{t}}_*\big) \\ &\quad - \big(H \kappa_{\bar{\tau}_*\bar{\mathrm{t}}_*} + \hat{\alpha}_{\mathrm{vel}} (\nabla \cdot \bar{\mathrm{t}}_*)\big)\, \bar{\mathrm{t}}_* \otimes \bar{\tau}_* + \kappa_{\bar{\tau}\bar{\mathrm{t}}} \bar{\mathrm{t}}_* \otimes \bar{\mathrm{n}} \end{aligned}$$

$$
\begin{aligned}
&= \beta\, \bar{\tau}_* \wedge \bar{\mathrm{n}} + (\bar{\mathrm{t}}_* \cdot \nabla)\widetilde{B} \otimes \bar{\mathrm{t}}_* \\
&\qquad - \big(H\kappa_{\bar{\tau}_*\bar{\mathrm{t}}_*} + \widehat{\alpha}_{\mathrm{vel}}(\nabla \cdot \bar{\mathrm{t}}_*)\big)\, \bar{\mathrm{t}}_* \otimes \bar{\tau}_* + \kappa_{\bar{\tau}\bar{\mathrm{t}}}\bar{\mathrm{t}}_* \otimes \bar{\mathrm{n}}.
\end{aligned} \tag{100}
$$

Hence, the already established zeroth-order condition (95) together with compatibility conditions (64) and (66) in the form of $\beta = \beta' = \beta''$ along $\bar{\Gamma}$ imply (96). ■

The following result provides the approximate evolution equations for our auxiliary constructions $(\widetilde{\xi}, \widetilde{R}'_{\bar{I}}\,\widetilde{\xi}, \widetilde{R}''_{\bar{I}}\,\widetilde{\xi}\,)$ in terms of the associated auxiliary velocity $\widetilde{B}$, which will eventually lead us to (1d)–(1f):

Lemma 27. *Let $(\widetilde{\xi}, \widetilde{\xi}', \widetilde{\xi}'')$ be the initial extensions from Construction 23 of the normal vector fields $(\bar{\mathrm{n}}|_{\bar{I}}, \bar{\mathrm{n}}'|_{\bar{I}'}, \bar{\mathrm{n}}''|_{\bar{I}''})$. Moreover, let $(\widetilde{R}'_{\bar{I}}, \widetilde{R}''_{\bar{I}})$, $(\widetilde{R}_{\bar{I}'}, \widetilde{R}''_{\bar{I}'})$ and $(\widetilde{R}_{\bar{I}''}, \widetilde{R}'_{\bar{I}''})$ be the gauged Herring rotations as provided by Construction 21, respectively. Finally, let $(\widetilde{B}, \widetilde{B}', \widetilde{B}'')$ be the initial extensions from Construction 25 of the normal velocity vector fields $((H\bar{\mathrm{n}})|_{\bar{I}}, (H'\bar{\mathrm{n}}')|_{\bar{I}'}, (H''\bar{\mathrm{n}}'')|_{\bar{I}''})$.*

Then, there exists a constant $C > 0$, which depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0, T]$, such that for each rotation $\mathcal{R} \in \{\mathrm{Id}, \widetilde{R}'_{\bar{I}}, \widetilde{R}''_{\bar{I}}\}$ it holds that

$$
\big|1 - |\mathcal{R}\widetilde{\xi}|^2\big| \leq C \operatorname{dist}^4(\cdot, \bar{I}), \tag{101}
$$

$$
\big|\nabla|\mathcal{R}\widetilde{\xi}|^2\big| \leq C \operatorname{dist}^3(\cdot, \bar{I}), \tag{102}
$$

$$
\big|\partial_t|\mathcal{R}\widetilde{\xi}|^2\big| \leq C \operatorname{dist}^3(\cdot, \bar{I}), \tag{103}
$$

$$
|\partial_t \mathcal{R}\widetilde{\xi} + (\widetilde{B} \cdot \nabla)\mathcal{R}\widetilde{\xi} + (\nabla \widetilde{B})^{\mathsf{T}}\mathcal{R}\widetilde{\xi}| \leq C \begin{cases} \operatorname{dist}(\cdot, \bar{I}) & \text{if } \mathcal{R} = \mathrm{Id}, \\ \operatorname{dist}(\cdot, \bar{\Gamma}) & \text{else}, \end{cases} \tag{104}
$$

$$
|\nabla \cdot \mathcal{R}\widetilde{\xi} + \widetilde{B} \cdot \mathcal{R}\widetilde{\xi}| \leq C \begin{cases} \operatorname{dist}(\cdot, \bar{I}) & \text{if } \mathcal{R} = \mathrm{Id}, \\ \operatorname{dist}(\cdot, \bar{\Gamma}) & \text{else} \end{cases} \tag{105}
$$

throughout the domain $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi)$.

Analogous estimates hold true throughout the domain $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi')$ in terms of the vector fields $(\mathcal{R}\widetilde{\xi}', \widetilde{B}')$ for each rotation $\mathcal{R} \in \{\widetilde{R}_{\bar{I}'}, \mathrm{Id}, \widetilde{R}''_{\bar{I}'}\}$, as well as throughout $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi'')$ in terms of $(\mathcal{R}\widetilde{\xi}'', \widetilde{B}'')$ for each $\mathcal{R} \in \{\widetilde{R}_{\bar{I}''}, \widetilde{R}'_{\bar{I}''}, \mathrm{Id}\}$.

Proof. Fix a rotation $\mathcal{R} \in \{\mathrm{Id}, \widetilde{R}'_{\bar{I}}, \widetilde{R}''_{\bar{I}}\}$, and for the purposes of the proof abbreviate $\alpha_{\bar{\Gamma}}(\cdot, t) := \alpha(P_{\bar{\Gamma}}(\cdot, t), t)$, $t \in [0, T]$.

Step 1: Proof of (101)–(103). It follows immediately from ansatz (74) and the orthogonality $\bar{\tau}_* \cdot \bar{\mathrm{n}} = 0$ that

$$
|\mathcal{R}\widetilde{\xi}|^2 = |\widetilde{\xi}|^2 = \Big(1 - \frac{1}{2}\alpha_{\bar{\Gamma}}^2 s^2\Big)^2 + \alpha_{\bar{\Gamma}}^2 s^2 = 1 + \frac{1}{4}\alpha_{\bar{\Gamma}}^4 s^4.
$$

The previous display of course immediately implies estimates (101)–(103).

Step 2: Proof of (105). By regularity estimates (75) and (94), it suffices to show that (105) is exact *on* the interface $\bar{I}$ if $\mathcal{R} = \mathrm{Id}$, or otherwise that (105) is exact *on* the triple line $\bar{\Gamma}$. To this end, let us first assume that $\mathcal{R}_{\bar{t}} = \mathrm{Id}$. Then, we also have $\mathrm{R} = \mathrm{Id}$ and hence, we may directly infer from definitions (74) and (93) of $\widetilde{\xi}$ and $\widetilde{B}$, respectively, that $\nabla \cdot \widetilde{\xi} = H = \widetilde{\xi} \cdot \widetilde{B}$ on the interface $\bar{I}$. In the remaining cases, we express $\mathcal{R} = R_{\bar{\mathrm{n}}}\mathcal{R}_{\bar{t}}R_{\bar{\mathrm{n}}}^{\mathsf{T}}$ in terms of the associated Herring rotation $\mathcal{R}_{\bar{t}} \in \{R'_{\bar{t}}, R''_{\bar{t}}\}$, and then simply read off from (80), (81), (85), (87) and (88) that

$$\nabla \cdot \mathcal{R}\widetilde{\xi} = -H(\mathcal{R}_{\bar{t}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}}) + (\nabla \cdot \bar{\tau})(\mathcal{R}_{\bar{t}}\bar{\mathrm{n}} \cdot \bar{\tau}) - \alpha_{\bar{\Gamma}}(\mathcal{R}_{\bar{t}}\bar{\mathrm{n}} \cdot \bar{\tau})$$

along the triple line $\bar{\Gamma}$. Moreover, definitions (74) and (93) directly imply that

$$\widetilde{B} \cdot \mathcal{R}\widetilde{\xi} = H(\mathcal{R}_{\bar{t}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}}) + \alpha_{\mathrm{vel}}(\mathcal{R}_{\bar{t}}\bar{\mathrm{n}} \cdot \bar{\tau})$$

holds true on the interface $\bar{I}$. Hence, estimate (105) follows from the previous two displays in combination with the choice shown in (72).

Step 3: Proof of (104). It suffices again to check that (104) is exact on the interface $\bar{I}$ if $\mathcal{R} = \mathrm{Id}$, or otherwise that (104) is exact on the triple line $\bar{\Gamma}$. Let us also again express $\mathcal{R} = R_{\bar{\mathrm{n}}}\mathcal{R}_{\bar{t}}R_{\bar{\mathrm{n}}}^{\mathsf{T}}$ in terms of the associated Herring rotation $\mathcal{R}_{\bar{t}} \in \{\mathrm{Id}, R'_{\bar{t}}, R''_{\bar{t}}\}$.

Using that the vector field $\mathcal{R}\bar{\mathrm{n}} = R_{\bar{\mathrm{n}}}\mathcal{R}_{\bar{t}}\bar{\mathrm{n}}$ lies in the $(\bar{\mathrm{n}}, R_{\bar{\mathrm{n}}}\bar{\tau})$-plane and has constant coefficients in this frame, we compute along the interface $\bar{I}$, relying also on (74),

$$\begin{aligned}\partial_t \mathcal{R}\widetilde{\xi} + (\widetilde{B} \cdot \nabla)\mathcal{R}\widetilde{\xi} + (\nabla\widetilde{B})^{\mathsf{T}}\mathcal{R}\widetilde{\xi} &= (R_{\bar{\mathrm{n}}}\mathcal{R}_{\bar{t}}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}})\big(\partial_t\bar{\mathrm{n}} + (\widetilde{B} \cdot \nabla)\bar{\mathrm{n}} + (\nabla\widetilde{B})^{\mathsf{T}}\bar{\mathrm{n}}\big) \\ &\quad + (R_{\bar{\mathrm{n}}}\mathcal{R}_{\bar{t}}\bar{\mathrm{n}} \cdot R_{\bar{\mathrm{n}}}\bar{\tau})\big(\partial_t\bar{\tau}_* + (\widetilde{B} \cdot \nabla)\bar{\tau}_* + (\nabla\widetilde{B})^{\mathsf{T}}\bar{\tau}_*\big) \\ &\quad + \alpha_{\bar{\Gamma}}(\partial_t s + (\widetilde{B} \cdot \nabla)s)\,\bar{\tau}_*. \end{aligned} \tag{106}$$

The last right hand side term of (106) vanishes due to $(\widetilde{B} \cdot \nabla)s = H$ and (24). Differentiating this equation in space yields, because of $\nabla s = \bar{\mathrm{n}}$,

$$0 = \nabla\big(\partial_t s + (\widetilde{B} \cdot \nabla)s\big) = \partial_t\bar{\mathrm{n}} + (\widetilde{B} \cdot \nabla)\bar{\mathrm{n}} + (\nabla\widetilde{B})^{\mathsf{T}}\bar{\mathrm{n}}.$$

Hence, also the first right hand side term of (106) vanishes. Since $\mathcal{R}_{\bar{t}} = \mathrm{Id}$ if and only if $\mathcal{R} = \mathrm{Id}$, estimate (104) already follows from these arguments in the case $\mathcal{R} = \mathrm{Id}$. Hence, let us restrict to the case $\mathcal{R} \neq \mathrm{Id}$ in what follows. Recall from claim (104) that it then suffices to estimate in terms of the distance to the triple line.

It follows from $|\bar{\tau}_*| = 1$ that $\bar{\tau}_* \cdot (\partial_t\bar{\tau}_* + (\widetilde{B} \cdot \nabla)\bar{\tau}_*) = 0$. Furthermore, the ansatz for the velocity field $\widetilde{B}$ is arranged such that $\bar{\tau}_* \otimes \bar{\tau}_* : \nabla\widetilde{B} = 0$ (cf. identity (100)). Hence, in the evolution equation for the tangent vector field $\bar{\tau}_*$ we may neglect the $\bar{\tau}_*$-component. The $\bar{\mathrm{n}}$-component also vanishes as a consequence of the orthogonality given by $\bar{\tau}_* \cdot \bar{\mathrm{n}} = 0$, the skew-symmetry $\bar{\tau}_* \otimes \bar{\mathrm{n}} : \nabla\widetilde{B} = -\bar{\mathrm{n}} \otimes \bar{\tau}_* : \nabla\widetilde{B}$ (cf. again (100)), and the already established evolution equation for the unit normal vector field $\bar{\mathrm{n}}$

$$\bar{\mathrm{n}} \cdot \big(\partial_t\bar{\tau}_* + (\widetilde{B} \cdot \nabla)\bar{\tau}_* + (\nabla\widetilde{B})^{\mathsf{T}}\bar{\tau}_*\big) = -\bar{\tau}_* \cdot \big(\partial_t\bar{\mathrm{n}} + (\widetilde{B} \cdot \nabla)\bar{\mathrm{n}} + (\nabla\widetilde{B})^{\mathsf{T}}\bar{\mathrm{n}}\big) = 0.$$

It therefore suffices to check that the velocity field $\widetilde{B}$ correctly captures the translation and rotation of the tangent vector field $\bar{\tau}_*$ in $\bar{t}_*$-direction *on* the triple line $\bar{\Gamma}$, that is,

$\bar{\mathrm{t}}_* \cdot (\partial_t \bar{\tau}_* + (\widetilde{B} \cdot \nabla)\bar{\tau}_* + (\nabla \widetilde{B})^{\mathsf{T}} \bar{\tau}_*) = 0$, or equivalently, by exploiting the orthogonality $\bar{\tau}_* \cdot \bar{\mathrm{t}}_* = 0$ that

$$\bar{\tau}_* \cdot \big(\partial_t \bar{\mathrm{t}}_* + (\widetilde{B} \cdot \nabla)\bar{\mathrm{t}}_*\big) = \bar{\mathrm{t}}_* \cdot (\nabla \widetilde{B})^{\mathsf{T}} \bar{\tau}_* \tag{107}$$

along the triple line $\bar{\Gamma}$.

In order to prove (107), we start by noticing that as a consequence of definition (93), as well as formulas (98) and (99), we have

$$\bar{\mathrm{t}}_* \cdot (\nabla \widetilde{B})^{\mathsf{T}} \bar{\tau}_* = \bar{\tau}_* \cdot (\bar{\mathrm{t}}_* \cdot \nabla)\widetilde{B} = -H\kappa_{\bar{\tau}_* \bar{\mathrm{t}}_*} + (\bar{\mathrm{t}}_* \cdot \nabla)\alpha_{\mathrm{vel}} \quad \text{on } \bar{I}. \tag{108}$$

That this expression equals $\bar{\tau}_* \cdot (\partial_t \bar{\mathrm{t}}_* + (\widetilde{B} \cdot \nabla)\bar{\mathrm{t}}_*)$ on the triple line $\bar{\Gamma}$ is a consequence of the following considerations: Let $\psi_{\bar{\Gamma}}(\cdot, t) : \bar{\Gamma}^0 \times [0, T] \to \bar{\Gamma}(t)$, $t \in [0, T]$, be a normal parametrization of the triple line, that is, we have $\partial_t \psi_{\bar{\Gamma}}(x_0, t) = \mathrm{V}_{\bar{\Gamma}}(\psi_{\bar{\Gamma}}(x_0, t), t)$ for all $(x_0, t) \in \bar{\Gamma}^0 \times [0, T]$. Choose, moreover, a C^5 diffeomorphic parametrization $\varphi_0 : [0, 1] \to \bar{\Gamma}^0$ of the initial triple line, and define for all $t \in [0, T]$ the dynamic parametrizations

$$\varphi : [0, 1] \times [0, T] \to \bar{\Gamma}(t), \quad (s, t) \mapsto \psi_{\bar{\Gamma}}(\varphi_0(s), t).$$

Observe then that due to the zeroth-order compatibility condition (95) and the definition of $\widetilde{B}$ (see (93)), it holds for all $(s, t) \in [0, 1] \times [0, T]$ that

$$\partial_t \varphi(s, t) = \widetilde{B}(\varphi(s, t), t) = (H\bar{\mathrm{n}})(\varphi(s, t), t) + (\alpha_{\mathrm{vel}} \bar{\tau}_*)(\varphi(s, t), t). \tag{109}$$

Define finally the differential operator $\partial_v := \frac{\partial_s}{|\partial_s \varphi|}$. Note that $\partial_v \varphi(\cdot, t)$ is a unit tangent vector field along the triple line $\bar{\Gamma}(t)$ for all $t \in [0, T]$, and we may choose the orientation such that $\partial_v \varphi(\cdot, t) = \bar{\mathrm{t}}_*(\varphi(\cdot, t), t)$ for all $t \in [0, T]$. A straightforward computation now yields

$$\partial_t \partial_v \varphi = \partial_v \partial_t \varphi - (\partial_v \partial_t \varphi \cdot \partial_v \varphi)\partial_v \varphi.$$

In particular, the commutator $[\partial_t \partial_v, \partial_v \partial_t]\varphi$ vanishes in $\bar{\tau}_*$-direction along the triple line. Using the chain rule and the first identity in formula (109), we thus obtain for all $(s, t) \in [0, 1] \times [0, T]$, by the orthogonality of the frame $(\bar{\mathrm{n}}, \bar{\tau}_*, \bar{\mathrm{t}}_*)$, the second identity in (109), as well as (98) and (99)

$$\begin{aligned}
\big(\bar{\tau}_* \cdot \big(\partial_t \bar{\mathrm{t}}_* + (\widetilde{B} \cdot \nabla)\bar{\mathrm{t}}_*\big)\big)(\varphi(s, t), t) &= \bar{\tau}_*(\varphi(s, t), t) \cdot \partial_t \partial_v \varphi(s, t) \\
&= \bar{\tau}_*(\varphi(s, t), t) \cdot \partial_v \partial_t \varphi(s, t) \\
&= \big(\bar{\tau}_* \cdot (\bar{\mathrm{t}}_* \cdot \nabla)(H\bar{\mathrm{n}} + \alpha_{\mathrm{vel}} \bar{\tau}_*)\big)(\varphi(s, t), t) \\
&= -(H\kappa_{\bar{\tau}_* \bar{\mathrm{t}}_*})(\varphi(s, t), t) + \big((\bar{\mathrm{t}}_* \cdot \nabla)\alpha_{\mathrm{vel}}\big)(\varphi(s, t), t).
\end{aligned}$$

Hence, we may obtain (107) by (108), which concludes the proof. ■

4.3. Global construction by interpolation

Throughout this whole subsection, let the assumptions of Proposition 16 and the notation of Sections 3, 4.1 and 4.2 be in place. The next results provide the last missing ingredient for the construction of a local gradient flow calibration at the triple line. We refer to Definition 17 and Figure 3 to recall the geometric setup.

Lemma 28. *Let $i, j, k \in \{1, 2, 3\}$ be such that $\{i, j, k\} = \{1, 2, 3\}$. For each interpolation wedge $W_{\bar{\Omega}_i}$, there exists a pair of associated interpolation functions*

$$\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}, \lambda^{\bar{I}_{k,i}}_{\bar{\Omega}_i} : \bigcup_{t\in[0,T]} \big(\overline{W_{\bar{\Omega}_i}(t)} \setminus \bar{\Gamma}(t)\big) \times \{t\} \to [0, 1]$$

of class $(C^0_t C^1_x \cap C^1_t C^0_x)(\bigcup_{t\in[0,T]}(\overline{W_{\bar{\Omega}_i}(t)} \setminus \bar{\Gamma}(t)) \times \{t\})$ such that

$$\lambda^{\bar{I}_{k,i}}_{\bar{\Omega}_i} = 1 - \lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i},$$

and where $\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}$ is subject to the following additional requirements:

(i) *On the boundary of the interpolation wedge $W_{\bar{\Omega}_i}$, the values of $\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}$ and its derivatives are given by*

$$\begin{aligned} &\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}(\cdot, t) = 0, && \text{on } \big(\partial W_{\bar{\Omega}_i}(t) \cap \partial W_{\bar{I}_{k,i}}(t)\big) \setminus \bar{\Gamma}(t),\\ &\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}(\cdot, t) = 1, && \text{on } \big(\partial W_{\bar{\Omega}_i}(t) \cap \partial W_{\bar{I}_{i,j}}(t)\big) \setminus \bar{\Gamma}(t),\\ \nabla&\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}(\cdot, t) = 0, \quad \partial_t\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}(\cdot, t) = 0, && \text{on } \big(B_r(\bar{\Gamma}(t)) \cap \partial W_{\bar{\Omega}_i}(t)\big) \setminus \bar{\Gamma}(t), \end{aligned}$$

for all $t \in [0, T]$.

(ii) *There exists a constant $C > 0$, which depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0, T]$, such that the estimate*

$$|\partial_t\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}| + |\nabla\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}| \leq C \operatorname{dist}^{-1}(\cdot, \bar{\Gamma}) \tag{110}$$

holds true on $\bigcup_{t\in[0,T]}(\overline{W_{\bar{\Omega}_i}(t)} \setminus \bar{\Gamma}(t)) \times \{t\}$.

(iii) *Denoting again by $\mathrm{V}_{\bar{\Gamma}}$ the normal velocity vector field of the triple line $\bar{\Gamma}$, we have an improved estimated on the advective derivative*

$$\big|\partial_t\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}(\cdot, t) + \big(\mathrm{V}_{\bar{\Gamma}}(P_{\bar{\Gamma}}(\cdot, t), t) \cdot \nabla\big)\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}(\cdot, t)\big| \leq C \tag{111}$$

on $\overline{W_{\bar{\Omega}_i}(t)} \setminus \bar{\Gamma}(t)$ for all $t \in [0, T]$. The constant $C > 0$ depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0, T]$.

Proof. Let $i, j, k \in \{1, 2, 3\}$ be such that $\{i, j, k\} = \{1, 2, 3\}$. For the construction of the interpolation function

$$\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i} =: 1 - \lambda^{\bar{I}_{k,i}}_{\bar{\Omega}_i},$$

we first choose a smooth function $\tilde{\lambda} : \mathbb{R} \to [0, 1]$ such that $\tilde{\lambda} \equiv 0$ on $[\frac{2}{3}, \infty)$ and $\tilde{\lambda} \equiv 1$ on $(-\infty, \frac{1}{3}]$. Denote next by $\theta_i \in (0, \pi)$ the constant opening angle of the interpolation wedge W_i (cf. representation (32)). We then define $\lambda_i : [-1, 1] \to [0, 1]$ by $\lambda_i(u) := \widetilde{\lambda}(\frac{1-u}{1-\cos(\theta_i)})$, and based on this auxiliary map, an interpolation function

$$\lambda_i^+(x,t) := \lambda_i\Big(X^+_{\bar{\Omega}_i}\big(P_{\bar{\Gamma}}(x,t),t\big) \cdot \frac{x - P_{\bar{\Gamma}}(x,t)}{|x - P_{\bar{\Gamma}}(x,t)|}\Big), \quad t \in [0,T],\ x \in \overline{W_{\bar{\Omega}_i}(t)} \setminus \bar{\Gamma}(t).$$

The interpolation function $\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}$ is then either defined by λ_i^+ or by $1 - \lambda_i^+$, depending on the right choice of "orientation" to satisfy the first item of (28), which in turn is then an immediate consequence of the definitions. For the proof of (110) and (111), it anyhow suffices to work on the level of the interpolation function λ_i^+.

The qualitative regularity of λ_i^+ and the corresponding regularity estimate (110) follow directly from the chain rule, the definition of λ_i^+, and the regularity requirements of Definition 17. For the improved estimate (111) on the advective derivative, we need an appropriate representation of $\partial_t P_{\bar{\Gamma}}$ in $\mathcal{N}_r(\bar{\Gamma})$. Abbreviating $g(x,t) := \frac{1}{2}\mathrm{dist}^2(x, \bar{\Gamma}(t))$ as well as $g_{\bar{\Gamma}}(x,t) := g(P_{\bar{\Gamma}}(x,t),t)$ for all $(x,t) \in \mathcal{N}_r(\bar{\Gamma})$, we obtain by the chain rule

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\nabla g_{\bar{\Gamma}}(x,t)\big) \\ &= (\nabla \partial_t g)(y,t)|_{y=P_{\bar{\Gamma}}(x,t)} + (\nabla^2 g)(y,t)|_{y=P_{\bar{\Gamma}}(x,t)} \partial_t P_{\bar{\Gamma}}(x,t), \quad (x,t) \in \mathcal{N}_r(\bar{\Gamma}). \end{aligned}$$

However, it is a well-known fact that $-\nabla \partial_t g$ evaluated along $\bar{\Gamma}$ precisely represents the normal velocity of $\bar{\Gamma}$ (cf. [1, Theorem 7 *ii)*, p. 18]). Hence, the previous display updates to

$$\mathrm{V}_{\bar{\Gamma}}\big(P_{\bar{\Gamma}}(x,t),t\big) = \nabla^2 g(y,t)|_{y=P_{\bar{\Gamma}}(x,t)} \partial_t P_{\bar{\Gamma}}(x,t)$$

for all $(x,t) \in \mathcal{N}_r(\bar{\Gamma})$. Moreover, $\nabla^2 g(\cdot,t)$ evaluated along the triple line $\bar{\Gamma}(t)$ represents for all $t \in [0,T]$ the projection onto the normal bundle $\mathrm{Tan}^\perp \bar{\Gamma}(t)$ for all $t \in [0,T]$ (cf. [1, Theorem 2 *ii)*, p. 12]). In other words,

$$\mathrm{V}_{\bar{\Gamma}}\big(P_{\bar{\Gamma}}(x,t),t\big) = (\mathrm{Id} - \bar{\mathrm{t}} \otimes \bar{\mathrm{t}})(y,t)|_{y=P_{\bar{\Gamma}}(x,t)} \partial_t P_{\bar{\Gamma}}(x,t) \tag{112}$$

for all $(x,t) \in \mathcal{N}_r(\bar{\Gamma})$.

Abbreviating

$$u_i^+ := u_i^+(x,t) := X^+_{\bar{\Omega}_i}(P_{\bar{\Gamma}}(x,t),t) \cdot \frac{x - P_{\bar{\Gamma}}(x,t)}{|x - P_{\bar{\Gamma}}(x,t)|},$$

we may now compute by an application of the chain rule

$$\begin{aligned} \partial_t \lambda_i^+(x,t) &= \lambda_i'(u_i^+) X^+_{\bar{\Omega}_i}\big(P_{\bar{\Gamma}}(x,t),t\big) \cdot \partial_t \frac{x - P_{\bar{\Gamma}}(x,t)}{|x - P_{\bar{\Gamma}}(x,t)|} \\ &\quad + \lambda_i'(u_i^+) \frac{x - P_{\bar{\Gamma}}(x,t)}{|x - P_{\bar{\Gamma}}(x,t)|} \cdot \big((\bar{\mathrm{t}} \cdot \nabla) X^+_{\bar{\Omega}_i}\big)(y,t)\big|_{y=P_{\bar{\Gamma}}(x,t)} \big(\bar{\mathrm{t}}(y,t)|_{y=P_{\bar{\Gamma}}(x,t)} \cdot \partial_t P_{\bar{\Gamma}}(x,t)\big) \\ &\quad + \lambda_i'(u_i^+) \frac{x - P_{\bar{\Gamma}}(x,t)}{|x - P_{\bar{\Gamma}}(x,t)|} \cdot \big(\partial_t X^+_{\bar{\Omega}_i}\big)(y,t)\big|_{y=P_{\bar{\Gamma}}(x,t)} \end{aligned}$$

for all $(x,t) \in \bigcup_{t\in[0,T]} (\overline{W_{\bar\Omega_i}(t) \setminus \bar\Gamma(t))} \times \{t\}$. Observe that the last two right hand side terms in the previous display are bounded by the regularity of the projection $P_{\bar\Gamma}$ and the regularity of the vector field $X^{+}_{\bar\Omega_i}$ (cf. Definition 17). Next, for all $(x,t) \in \mathcal{N}_r(\bar\Gamma) \setminus \bar\Gamma$,

$$\partial_t \frac{x - P_{\bar\Gamma}(x,t)}{|x - P_{\bar\Gamma}(x,t)|} = -\frac{1}{|x-y|}\Big(\mathrm{Id} - \frac{x-y}{|x-y|} \otimes \frac{x-y}{|x-y|}\Big)\Big|_{y=P_{\bar\Gamma}(x,t)} \partial_t P_{\bar\Gamma}(x,t),$$

so that together with (112), $X^{+}_{\bar\Omega_i}(y,t), \mathrm{V}_{\bar\Gamma}(y,t) \in \mathrm{Tan}^{\perp}_y \bar\Gamma(t)$ for all $(y,t) \in \bar\Gamma$, as well as $\nabla P_{\bar\Gamma}(x,t) = (\bar{\mathrm{t}}(y,t)|_{y=P_{\bar\Gamma}(x,t)} \cdot \nabla) P_{\bar\Gamma}(x,t) \otimes \bar{\mathrm{t}}(y,t)|_{y=P_{\bar\Gamma}(x,t)}$ for all $(x,t) \in \mathcal{N}_r(\bar\Gamma)$,

$$\begin{aligned}
&\partial_t \lambda_i^{+}(x,t) \\
&= -\lambda_i'(u_i^{+}) \frac{1}{|x-y|}\Big(\mathrm{Id} - \frac{x-y}{|x-y|} \otimes \frac{x-y}{|x-y|}\Big) X^{+}_{\bar\Omega_i}(y,t)\Big|_{y=P_{\bar\Gamma}(x,t)} \cdot \partial_t P_{\bar\Gamma}(x,t) + O(1) \\
&= -\lambda_i'(u_i^{+}) \frac{1}{|x-y|}\Big(\mathrm{Id} - \frac{x-y}{|x-y|} \otimes \frac{x-y}{|x-y|}\Big) X^{+}_{\bar\Omega_i}(y,t) \cdot \mathrm{V}_{\bar\Gamma}(y,t)\Big|_{y=P_{\bar\Gamma}(x,t)} + O(1) \\
&= -\big(\mathrm{V}_{\bar\Gamma}\big(P_{\bar\Gamma}(x,t),t\big) \cdot \nabla\big)\lambda_i^{+}(x,t) + O(1)
\end{aligned}$$

for all $(x,t) \in \bigcup_{t\in[0,T]} (\overline{W_{\bar\Omega_i}(t) \setminus \bar\Gamma(t))} \times \{t\}$, as asserted. ∎

We may now provide the desired extensions $(\xi_{i,j})_{i,j\in\{1,2,3\},i\neq j}$ for the unit normal vector fields as well as the desired extension B of the velocity vector field within a space-time tubular neighborhood $\mathcal{N}_{\hat r}(\bar\Gamma)$ of the evolving triple line $\bar\Gamma$, where the radius $\hat r > 0$ has to be chosen suitably and is potentially smaller than the admissible localization radius r.

Construction 29 (Gradient flow calibration at the triple line). Let $(\widetilde\xi, \widetilde\xi', \widetilde\xi'')$ be the preliminary extensions from Construction 23 of the normal vector fields $(\bar{\mathrm{n}}|_{\bar I}, \bar{\mathrm{n}}'|_{\bar I'}, \bar{\mathrm{n}}''|_{\bar I''})$. Let $(\widetilde R'_{\bar I}, \widetilde R''_{\bar I})$, $(\widetilde R_{\bar I'}, \widetilde R''_{\bar I'})$ and $(\widetilde R_{\bar I''}, \widetilde R'_{\bar I''})$ be the gauged Herring rotations as provided by Construction 21, and let $(\widetilde B, \widetilde B', \widetilde B'')$ be the preliminary extensions of the normal velocity vector fields from Construction 25. We also introduce the abbreviations $\bar\Omega := \bar\Omega_1$, $\bar\Omega' := \bar\Omega_2$ and $\bar\Omega'' := \bar\Omega_3$.

With these ingredients in place, we first define a scale

Displayed formula to improve page breaks.

$$\hat r := r \wedge (2C)^{-\frac{1}{4}},$$

where $C > 0$ denotes the (maximum of the) constant(s) from estimate(s) (101). This choice of $\hat r \in (0, r]$ then entails, due to (101), that

$$|\widetilde\xi|^2 \in \big[\tfrac{1}{2}, \tfrac{3}{2}\big] \quad \text{in } \mathcal{N}_{\hat r}(\bar\Gamma) \cap \mathrm{im}(\Psi), \tag{113}$$

$$|\widetilde\xi'|^2 \in \big[\tfrac{1}{2}, \tfrac{3}{2}\big] \quad \text{in } \mathcal{N}_{\hat r}(\bar\Gamma) \cap \mathrm{im}(\Psi'), \tag{114}$$

$$|\widetilde\xi''|^2 \in \big[\tfrac{1}{2}, \tfrac{3}{2}\big] \quad \text{in } \mathcal{N}_{\hat r}(\bar\Gamma) \cap \mathrm{im}(\Psi''). \tag{115}$$

Based on these non-degeneracy conditions and properties (33)–(35) from the wedge decomposition of $\mathcal{N}_r(\bar{\Gamma})$, we construct a well-defined set of vector fields

$$\xi, \xi', \xi'' : \mathcal{N}_{\hat{r}}(\bar{\Gamma}) \to \overline{B_1(0)},$$
$$B : \mathcal{N}_{\hat{r}}(\bar{\Gamma}) \to \mathbb{R}^3$$

by the following procedure: On the closure of the interface wedges we define

$$(\xi, \xi', \xi'') := \big|\widetilde{\xi}\big|^{-1}\big(\widetilde{\xi},\ \widetilde{R}'_{\bar{I}}\widetilde{\xi},\ \widetilde{R}''_{\bar{I}}\widetilde{\xi}\big) \quad \text{on } \overline{W_{\bar{I}}}, \tag{116}$$
$$(\xi, \xi', \xi'') := \big|\widetilde{\xi}'\big|^{-1}\big(\widetilde{R}_{\bar{I}'}\widetilde{\xi}',\ \widetilde{\xi}',\ \widetilde{R}''_{\bar{I}'}\widetilde{\xi}'\big) \quad \text{on } \overline{W_{\bar{I}'}}, \tag{117}$$
$$(\xi, \xi', \xi'') := \big|\widetilde{\xi}''\big|^{-1}\big(\widetilde{R}_{\bar{I}''}\widetilde{\xi}'',\ \widetilde{R}'_{\bar{I}''}\widetilde{\xi}'',\ \widetilde{\xi}''\big) \quad \text{on } \overline{W_{\bar{I}''}}, \tag{118}$$

as well as

$$B := \widetilde{B} \quad \text{on } \overline{W_{\bar{I}}}, \qquad B := \widetilde{B}' \quad \text{on } \overline{W_{\bar{I}'}}, \qquad B := \widetilde{B}'' \quad \text{on } \overline{W_{\bar{I}''}}. \tag{119}$$

On the interpolation wedges, say $W_{\bar{\Omega}}$, we define

$$\xi := \lambda_{\bar{\Omega}}^{\bar{I}}|\widetilde{\xi}|^{-1}\widetilde{\xi} + \lambda_{\bar{\Omega}}^{\bar{I}''}|\widetilde{\xi}''|^{-1}\widetilde{R}_{\bar{I}''}\widetilde{\xi}'', \tag{120}$$
$$\xi' := \lambda_{\bar{\Omega}}^{\bar{I}}|\widetilde{\xi}|^{-1}\widetilde{R}'_{\bar{I}}\widetilde{\xi} + \lambda_{\bar{\Omega}}^{\bar{I}''}|\widetilde{\xi}''|^{-1}\widetilde{R}'_{\bar{I}''}\widetilde{\xi}'', \tag{121}$$
$$\xi'' := \lambda_{\bar{\Omega}}^{\bar{I}}|\widetilde{\xi}|^{-1}\widetilde{R}''_{\bar{I}}\widetilde{\xi} + \lambda_{\bar{\Omega}}^{\bar{I}''}|\widetilde{\xi}''|^{-1}\widetilde{\xi}'', \tag{122}$$
$$B := \lambda_{\bar{\Omega}}^{\bar{I}}\widetilde{B} + \lambda_{\bar{\Omega}}^{\bar{I}''}\widetilde{B}''. \tag{123}$$

On the remaining two interpolation wedges, $W_{\bar{\Omega}'}$ and $W_{\bar{\Omega}''}$, one proceeds analogously for the definition of these vector fields. ◇

4.4. Proof of Proposition 16

Let (ξ, ξ', ξ'', B) be the vector fields from Construction 29. We aim to show that this tuple of vector fields gives rise to a local gradient flow calibration at the triple line $\bar{\Gamma}$ in the sense of Proposition 16 after defining

$$\xi_{1,2} := \xi, \quad \xi_{2,3} := \xi', \quad \xi_{3,1} := \xi'' \qquad \text{in } \mathcal{N}_{\hat{r}}(\bar{\Gamma}),$$

as well as

$$\xi_{j,i} := -\xi_{i,j}$$

for the remaining set of distinct phases $i, j \in \{1, 2, 3\}$. The proof is now split into several steps.

In *Step 1* of the proof, we will derive the following useful compatibility estimates valid throughout interpolation wedges and which are needed in all subsequent steps:

$$\left|\frac{\widetilde{\xi}}{|\widetilde{\xi}|} - \frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}\right| + \left|\frac{\widetilde{R}'_{\bar{I}}\widetilde{\xi}}{|\widetilde{\xi}|} - \frac{\widetilde{R}'_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}\right| + \left|\frac{\widetilde{R}''_{\bar{I}}\widetilde{\xi}}{|\widetilde{\xi}|} - \frac{\widetilde{\xi}''}{|\widetilde{\xi}''|}\right| \leq C\,\mathrm{dist}^2(\cdot, \bar{\Gamma}) \tag{124}$$

in $W_{\bar{\Omega}} \cap \mathcal{N}_{\hat{r}}(\bar{\Gamma})$, with analogous estimates being satisfied in the other two interpolation wedges. Moreover, the constant $C > 0$ only depends on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0, T]$.

Then, in *Step 2*, we will verify that (ξ, ξ', ξ'', B) are continuous vector fields throughout $\mathcal{N}_{\hat{r}}(\bar{\Gamma})$; as well as that the constructed extensions of the unit normals (ξ, ξ', ξ'') are of class $(C_t^0 C_x^1 \cap C_t^1 C_x^0)(\mathcal{N}_{\hat{r}}(\bar{\Gamma}) \setminus \bar{\Gamma})$, whereas the extended velocity B is of class $C_t^0 C_x^1(\mathcal{N}_{\hat{r}}(\bar{\Gamma}) \setminus \bar{\Gamma})$; and that there exists a constant $C > 0$ depending only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0, T]$ such that the estimate

$$|(\partial_t, \nabla)(\xi, \xi', \xi'')| + |B| + |\nabla B| \leq C \tag{125}$$

holds true throughout $\mathcal{N}_{\hat{r}}(\bar{\Gamma}) \setminus \bar{\Gamma}$. Moreover, we will show that

$$\xi = \bar{\mathrm{n}}|_{\bar{I}} \quad \text{along } \bar{I} \cap \mathcal{N}_{\hat{r}}(\bar{\Gamma}), \tag{126}$$

$$B = \mathrm{V}_{\bar{\Gamma}} \quad \text{along } \bar{\Gamma}, \tag{127}$$

$$\sigma\xi + \sigma'\xi' + \sigma''\xi'' = 0 \quad \text{in } \mathcal{N}_{\hat{r}}(\bar{\Gamma}), \tag{128}$$

where property (126) is also satisfied in terms of $(\xi', \bar{\mathrm{n}}'|_{\bar{I}'})$ along $\bar{I}' \cap \mathcal{N}_{\hat{r}}(\bar{\Gamma})$, or in terms of $(\xi'', \bar{\mathrm{n}}''|_{\bar{I}''})$ along $\bar{I}'' \cap \mathcal{N}_{\hat{r}}(\bar{\Gamma})$.

Step 3 of the proof is then devoted to the verification of the approximate evolution equation

$$|\partial_t \xi + (B \cdot \nabla)\xi + (\nabla B)^{\mathsf{T}}\xi| \leq C \operatorname{dist}(\cdot, \bar{I}) \quad \text{in } \mathcal{N}_{\hat{r}}(\bar{\Gamma}) \setminus \bar{\Gamma}, \tag{129}$$

then in *Step 4* we will prove the estimate

$$|\nabla \cdot \xi + B \cdot \xi| \leq C \operatorname{dist}(\cdot, \bar{I}) \quad \text{in } \mathcal{N}_{\hat{r}}(\bar{\Gamma}) \setminus \bar{\Gamma}. \tag{130}$$

We finally conclude in *Step 5* by deducing the estimate

$$(\partial_t + B \cdot \nabla)|\xi|^2 \leq C \operatorname{dist}^2(\cdot, \bar{I}) \quad \text{in } \mathcal{N}_{\hat{r}}(\bar{\Gamma}). \tag{131}$$

We record for completeness that analogous estimates with respect to (129)–(131) are satisfied for (ξ', B) (resp. (ξ'', B)) in terms of $\operatorname{dist}(\cdot, \bar{I}')$ (resp. $\operatorname{dist}(\cdot, \bar{I}'')$) and that the constant $C > 0$ again only depends on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0, T]$.

Step 1: Proof of (124). Adding zero, making use of the reverse triangle inequality and recalling non-degeneracy conditions (113)–(115), we may estimate

$$\begin{aligned}
\left|\frac{\tilde{\xi}}{|\tilde{\xi}|} - \frac{\tilde{R}_{\bar{I}''}\tilde{\xi}''}{|\tilde{\xi}''|}\right| &\leq \frac{1}{|\tilde{\xi}|}|\tilde{\xi} - \tilde{R}_{\bar{I}''}\tilde{\xi}''| + \left|\frac{1}{|\tilde{\xi}|} - \frac{1}{|\tilde{R}_{\bar{I}''}\tilde{\xi}''|}\right| |\tilde{R}_{\bar{I}''}\tilde{\xi}''| \\
&\leq \frac{1}{|\tilde{\xi}|}|\tilde{\xi} - \tilde{R}_{\bar{I}''}\tilde{\xi}''| + \frac{1}{|\tilde{\xi}|}\Big||\tilde{\xi}| - |\tilde{R}_{\bar{I}''}\tilde{\xi}''|\Big| \leq 2\sqrt{2}|\tilde{\xi} - \tilde{R}_{\bar{I}''}\tilde{\xi}''|.
\end{aligned}$$

Due to compatibility conditions (76) and (77) as well as regularity estimates (75), the previous estimate then easily upgrades to

$$\left|\frac{\widetilde{\xi}}{|\widetilde{\xi}|}-\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}\right|\leq C\,\mathrm{dist}^2(\cdot,\bar{\Gamma})\quad\text{in } W_{\bar{\Omega}}\cap\mathcal{N}_{\hat{r}}(\bar{\Gamma})$$

by inserting a second-order Taylor expansion with base point located at the unique nearest point on the triple line $\bar{\Gamma}$. The other two terms on the left hand side of (124) are treated analogously.

Step 2: Proof of (125)–(128). In terms of the asserted qualitative regularity, we observe that the first item of Lemma 28 together with the definitions from Construction 29 ensure that the vector fields (ξ,ξ',ξ'',B) and their required derivatives are continuous across the boundaries of the interpolation wedges (away from the triple line). Continuity of B throughout the whole space-time neighborhood $\mathcal{N}_r(\bar{\Gamma})$ with the asserted representation (127) along the triple line $\bar{\Gamma}$ follows from compatibility condition (95). The unit normal extensions (ξ,ξ',ξ'') are continuous throughout $\mathcal{N}_r(\bar{\Gamma})$ due to compatibility estimates (124). Representation (126) along the associated interface in turn is a consequence of expansion ansatz (74) and inclusion (34).

Next, on interface wedges, regularity estimate (125) follows directly from estimates (75) and (94). For the derivation of (125) throughout an interpolation wedge, say $W_{\bar{\Omega}}\cap\mathcal{N}_{\hat{r}}(\bar{\Gamma})$, we simply compute by plugging in the definitions from Construction 29 and recalling from Lemma 28 that $\lambda_{\bar{\Omega}}^{\bar{I}''}=1-\lambda_{\bar{\Omega}}^{\bar{I}}$:

$$\begin{aligned}(\partial_t,\nabla)\xi&=\lambda_{\bar{\Omega}}^{\bar{I}}(\partial_t,\nabla)\frac{\widetilde{\xi}}{|\widetilde{\xi}|}+\lambda_{\bar{\Omega}}^{\bar{I}''}(\partial_t,\nabla)\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}+\Big(\frac{\widetilde{\xi}}{|\widetilde{\xi}|}-\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}\Big)\otimes(\partial_t,\nabla)\lambda_{\bar{\Omega}}^{\bar{I}},\\ \nabla B&=\lambda_{\bar{\Omega}}^{\bar{I}}\nabla\widetilde{B}+\lambda_{\bar{\Omega}}^{\bar{I}''}\nabla\widetilde{B}''+(\widetilde{B}-\widetilde{B}'')\otimes\nabla\lambda_{\bar{\Omega}}^{\bar{I}}.\end{aligned}$$

We thus infer (125) from the chain rule in the form of $\nabla\frac{1}{|f|}=-\frac{(\nabla f)^{\mathsf{T}}f}{|f|^3}$, regularity estimates (75), (94) and (110), and compatibility conditions (124) and (95).

We turn to the proof of (128). Recalling expansion ansatz (74) and definitions (55) and (56)) of the gauged Herring rotations, we deduce from (48) that

$$\sigma\widetilde{\xi}+\sigma'\widetilde{R}'_{\bar{I}}\widetilde{\xi}+\sigma''\widetilde{R}''_{\bar{I}}\widetilde{\xi}=0\quad\text{throughout } \mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi),\tag{132}$$

and analogously, throughout $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi')$ in terms of $(\widetilde{R}_{\bar{I}'}\widetilde{\xi}',\widetilde{\xi}',\widetilde{R}''_{\bar{I}'}\widetilde{\xi}')$, or throughout $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi'')$ in terms of $(\widetilde{R}_{\bar{I}''}\widetilde{\xi}'',\widetilde{R}'_{\bar{I}''}\widetilde{\xi}'',\widetilde{\xi}'')$. Due to inclusion (34) and the definitions from Construction 29, we thus obtain from (132)

$$\sigma\xi+\sigma'\xi'+\sigma''\xi''=|\widetilde{\xi}|^{-1}\big(\sigma\widetilde{\xi}+\sigma'\widetilde{R}'_{\bar{I}}\widetilde{\xi}+\sigma''\widetilde{R}''_{\bar{I}}\widetilde{\xi}\big)=0\quad\text{in } W_{\bar{I}}\cap\mathcal{N}_{\hat{r}}(\bar{\Gamma}).$$

An analogous argument works in the case of the other two interface wedges.

On interpolation wedges, say $W_{\overline{\Omega}}$, the extended Herring angle condition (128) follows from a linear combination of the previous ingredients. More precisely, the definitions from Construction 29 and cancellations (132) directly imply

$$\begin{aligned}&\sigma\xi + \sigma'\xi' + \sigma''\xi'' \\ &\quad = \lambda_{\overline{\Omega}}^{\overline{I}}|\widetilde{\xi}|^{-1}\big(\sigma\widetilde{\xi} + \sigma'\widetilde{R}'_{\overline{I}}\widetilde{\xi} + \sigma''\widetilde{R}''_{\overline{I}}\widetilde{\xi}\big) + \lambda_{\overline{\Omega}}^{\overline{I}''}|\widetilde{\xi}''|^{-1}\big(\sigma\widetilde{R}_{\overline{I}''}\widetilde{\xi}'' + \sigma'\widetilde{R}'_{\overline{I}''}\widetilde{\xi}'' + \sigma''\widetilde{\xi}''\big) = 0\end{aligned}$$

throughout $W_{\overline{\Omega}} \cap \mathcal{N}_{\hat{r}}(\overline{\Gamma})$, as desired. This concludes the proof of (128), and thus *Step 2* of the proof, since on the other interpolation wedges, (128) follows analogously.

Step 3: Proof of (129). We first claim that for each rotation $\mathcal{R} \in \{\mathrm{Id}, \widetilde{R}'_{\overline{I}}, \widetilde{R}''_{\overline{I}}\}$, it holds throughout $\mathcal{N}_{\hat{r}}(\overline{\Gamma}) \cap \mathrm{im}(\Psi)$ that

$$\Big|\big(\partial_t + (\widetilde{B}\cdot\nabla) + (\nabla\widetilde{B})^{\mathsf{T}}\big)\frac{\mathcal{R}\widetilde{\xi}}{|\mathcal{R}\widetilde{\xi}|}\Big| \leq C\begin{cases}\mathrm{dist}(\cdot,\overline{I}) & \text{if } \mathcal{R} = \mathrm{Id},\\ \mathrm{dist}(\cdot,\overline{\Gamma}) & \text{else,}\end{cases} \tag{133}$$

$$\Big|\widetilde{B}\cdot\frac{\mathcal{R}\widetilde{\xi}}{|\mathcal{R}\widetilde{\xi}|} + \nabla\cdot\frac{\mathcal{R}\widetilde{\xi}}{|\mathcal{R}\widetilde{\xi}|}\Big| \leq C\begin{cases}\mathrm{dist}(\cdot,\overline{I}) & \text{if } \mathcal{R} = \mathrm{Id},\\ \mathrm{dist}(\cdot,\overline{\Gamma}) & \text{else,}\end{cases} \tag{134}$$

for some constant $C > 0$ which depends only on the smoothly evolving regular double bubble $(\overline{\Omega}_1, \overline{\Omega}_2, \overline{\Omega}_3)$ on $[0,T]$. Moreover, analogous estimates hold true throughout the domain $\mathcal{N}_{\hat{r}}(\overline{\Gamma}) \cap \mathrm{im}(\Psi')$ in terms of the vector fields $(\mathcal{R}\widetilde{\xi}', \widetilde{B}')$ for each rotation $\mathcal{R} \in \{\widetilde{R}_{\overline{I}'}, \mathrm{Id}, \widetilde{R}''_{\overline{I}'}\}$, as well as throughout $\mathcal{N}_{\hat{r}}(\overline{\Gamma}) \cap \mathrm{im}(\Psi'')$ in terms of $(\mathcal{R}\widetilde{\xi}'', \widetilde{B}'')$ for each $\mathcal{R} \in \{\widetilde{R}_{\overline{I}''}, \widetilde{R}'_{\overline{I}''}, \mathrm{Id}\}$.

Estimate (133) follows from the straightforward computation

$$\begin{aligned}\big(\partial_t + (\widetilde{B}\cdot\nabla) + (\nabla\widetilde{B})^{\mathsf{T}}\big)\frac{\mathcal{R}\widetilde{\xi}}{|\mathcal{R}\widetilde{\xi}|} &= \frac{1}{|\mathcal{R}\widetilde{\xi}|}\big(\partial_t + (\widetilde{B}\cdot\nabla) + (\nabla\widetilde{B})^{\mathsf{T}}\big)\mathcal{R}\widetilde{\xi} \\ &\quad - \frac{\partial_t|\mathcal{R}\widetilde{\xi}|^2 + (\widetilde{B}\cdot\nabla)|\mathcal{R}\widetilde{\xi}|^2}{2|\mathcal{R}\widetilde{\xi}|^3}\mathcal{R}\widetilde{\xi}\end{aligned}$$

together with condition (113) and estimates (102), (103) and (104). Estimate (134) in turn can be deduced from the same ingredients as well as

$$\widetilde{B}\cdot\frac{\mathcal{R}\widetilde{\xi}}{|\mathcal{R}\widetilde{\xi}|} + \nabla\cdot\frac{\mathcal{R}\widetilde{\xi}}{|\mathcal{R}\widetilde{\xi}|} = \frac{1}{|\mathcal{R}\widetilde{\xi}|}\big(B\cdot\mathcal{R}\widetilde{\xi} + \nabla\cdot\mathcal{R}\widetilde{\xi}\big) - \frac{(\mathcal{R}\widetilde{\xi}\cdot\nabla)|\mathcal{R}\widetilde{\xi}|^2}{2|\mathcal{R}\widetilde{\xi}|^3}.$$

On interface wedges, facilitated by inclusion (34), claim (129) now follows from an application of estimate (133) and, if needed, a simple post-processing by means of (37). So, let us directly move on with the verification of (129) throughout interpolation wedges, say $W_{\overline{\Omega}} \cap \mathcal{N}_{\hat{r}}(\overline{\Gamma})$. Plugging in definitions (120)–(123) from Construction 29, we may compute, based on the product rule, adding zero, and recalling from Lemma 28 that $\lambda_{\bar\Omega}^{\bar I''} = 1 - \lambda_{\bar\Omega}^{\bar I}$,

$$
\begin{aligned}
&\big(\partial_t+(B\cdot\nabla)+(\nabla B)^{\mathsf T}\big)\xi \\
&\quad= \lambda_{\bar\Omega}^{\bar I}\big(\partial_t+(\widetilde B\cdot\nabla)+(\nabla\widetilde B)^{\mathsf T}\big)\frac{\widetilde\xi}{|\widetilde\xi|} + \big(1-\lambda_{\bar\Omega}^{\bar I}\big)\big(\partial_t+(\widetilde B''\cdot\nabla)+(\nabla\widetilde B'')^{\mathsf T}\big)\frac{\widetilde R_{\bar I''}\widetilde\xi''}{|\widetilde\xi''|} \\
&\qquad+ \Big(\frac{\widetilde\xi}{|\widetilde\xi|} - \frac{\widetilde R_{\bar I''}\widetilde\xi''}{|\widetilde\xi''|}\Big)\big(\partial_t+(B\cdot\nabla)\big)\lambda_{\bar\Omega}^{\bar I} \\
&\qquad+ \lambda_{\bar\Omega}^{\bar I}\big((B-\widetilde B)\cdot\nabla\big)\frac{\widetilde\xi}{|\widetilde\xi|} + \big(1-\lambda_{\bar\Omega}^{\bar I}\big)\big((B-\widetilde B'')\cdot\nabla\big)\frac{\widetilde R_{\bar I''}\widetilde\xi''}{|\widetilde\xi''|} \\
&\qquad+ \lambda_{\bar\Omega}^{\bar I}\big(\nabla B-\nabla\widetilde B\big)^{\mathsf T}\frac{\widetilde\xi}{|\widetilde\xi|} + \big(1-\lambda_{\bar\Omega}^{\bar I}\big)\big(\nabla B-\nabla\widetilde B''\big)^{\mathsf T}\frac{\widetilde R_{\bar I''}\widetilde\xi''}{|\widetilde\xi''|}.
\end{aligned} \tag{135}
$$

The first two right hand side terms of the previous display are at least of order $O(\mathrm{dist}(\cdot,\bar\Gamma))$ due to estimates (133), which in turn are available this time due to inclusion (35). The third, fourth and fifth right hand side terms are of the same order thanks to compatibility conditions (124) and (95), regularity estimates (75), (94) and (125), estimate (111) on the advective derivative of an interpolation function, and non-degeneracy conditions (113)–(115).

Regarding the two right hand side terms from the last line of the previous display, we may argue as follows: Plugging in the definition of B from Construction 29, we compute by the product rule, the identity $\lambda_{\bar\Omega}^{\bar I} + \lambda_{\bar\Omega}^{\bar I''} = 1$ and by carefully noting that $\widetilde\xi \perp \bar{\mathrm t}_*$ throughout $\mathcal N_r(\bar\Gamma)\cap \mathrm{im}(\Psi)$ due to expansion ansatz (74),

$$
\begin{aligned}
(\nabla B-\nabla\widetilde B)^{\mathsf T}\widetilde\xi &= (1-\lambda_{\bar\Omega}^{\bar I})(\nabla\widetilde B''-\nabla\widetilde B)^{\mathsf T}(\mathrm{Id}-\bar{\mathrm t}_*\otimes\bar{\mathrm t}_*)\widetilde\xi \\
&\quad+ \big((\widetilde B-\widetilde B'')\cdot(\mathrm{Id}-\bar{\mathrm t}_*\otimes\bar{\mathrm t}_*)\widetilde\xi\big)\nabla\lambda_{\bar\Omega}^{\bar I}.
\end{aligned}
$$

Abbreviating $\bar{\mathrm t}_{\bar\Gamma}(x,t) := \bar{\mathrm t}(P_{\bar\Gamma}(x,t),t)$ for all $(x,t)\in\mathcal N_r(\bar\Gamma)\cap\mathrm{im}(\Psi)$ and recalling compatibility conditions (65) and (95), as well as regularity estimate (110) for the interpolation function, we may switch from $\bar{\mathrm t}_*$ to $\bar{\mathrm t}_{\bar\Gamma}$ in the previous display at the cost of an admissible error:

$$
\begin{aligned}
(\nabla B-\nabla\widetilde B)^{\mathsf T}\widetilde\xi &= (1-\lambda_{\bar\Omega}^{\bar I})(\nabla\widetilde B''-\nabla\widetilde B)^{\mathsf T}(\mathrm{Id}-\bar{\mathrm t}_{\bar\Gamma}\otimes\bar{\mathrm t}_{\bar\Gamma})\widetilde\xi \\
&\quad+ \big((\widetilde B-\widetilde B'')\cdot(\mathrm{Id}-\bar{\mathrm t}_{\bar\Gamma}\otimes\bar{\mathrm t}_{\bar\Gamma})\widetilde\xi\big)\nabla\lambda_{\bar\Omega}^{\bar I} + O(\mathrm{dist}(\cdot,\bar\Gamma)).
\end{aligned}
$$

It then follows from compatibility conditions (39), (95) and (96), and again regularity estimate (110) for the interpolation function that

$$
(\nabla B-\nabla\widetilde B)^{\mathsf T}\widetilde\xi = O(\mathrm{dist}(\cdot,\bar\Gamma)).
$$

One may argue similarly for the second term after replacing $|\widetilde\xi''|^{-1}\widetilde R_{\bar I''}\widetilde\xi''$ by $|\widetilde\xi|^{-1}\widetilde\xi$ using compatibility estimate (124).

In summary, the asserted estimate (129) in terms of ξ now follows from the previously derived estimates for the right hand side terms of (135) and a subsequent post-processing of them by means of (36). We finally remark that the argument proceeds analogously for the other two vector fields ξ' and ξ'', respectively.

Step 4: Proof of (130). Thanks to inclusion (34), estimate (134), and, if needed, estimate (37), it again suffices to provide additional details only for the argument for (130) on interpolation wedges, say $W_{\bar{\Omega}} \cap \mathcal{N}_{\hat{r}}(\bar{\Gamma})$. Plugging in definitions (120)–(123) from Construction 29, applying the product rule, recalling from Lemma 28 that $\lambda^{\bar{I}''}_{\bar{\Omega}} = 1 - \lambda^{\bar{I}}_{\bar{\Omega}}$, and adding zero yields

$$
\begin{aligned}
B \cdot \xi + \nabla \cdot \xi = {} & \lambda^{\bar{I}}_{\bar{\Omega}} \Big(\widetilde{B} \cdot \frac{\widetilde{\xi}}{|\widetilde{\xi}|} + \nabla \cdot \frac{\widetilde{\xi}}{|\widetilde{\xi}|} \Big) + \big(1 - \lambda^{\bar{I}}_{\bar{\Omega}}\big) \Big(\widetilde{B}'' \cdot \frac{\widetilde{R}_{\bar{I}''} \widetilde{\xi}''}{|\widetilde{\xi}''|} + \nabla \cdot \frac{\widetilde{R}_{\bar{I}''} \widetilde{\xi}''}{|\widetilde{\xi}''|} \Big) \\
& + \lambda^{\bar{I}}_{\bar{\Omega}} (B - \widetilde{B}) \cdot \frac{\widetilde{\xi}}{|\widetilde{\xi}|} + \big(1 - \lambda^{\bar{I}}_{\bar{\Omega}}\big)(B - \widetilde{B}'') \cdot \frac{\widetilde{R}_{\bar{I}''} \widetilde{\xi}''}{|\widetilde{\xi}''|} \\
& + \Big(\frac{\widetilde{\xi}}{|\widetilde{\xi}|} - \frac{\widetilde{R}_{\bar{I}''} \widetilde{\xi}''}{|\widetilde{\xi}''|} \Big) \cdot \nabla \lambda^{\bar{I}}_{\bar{\Omega}}.
\end{aligned}
$$

The right hand side terms of the previous display are all at least of order $O(\mathrm{dist}(\cdot, \bar{\Gamma}))$—and thus of required order, due to (36)—by an application of inclusion (35), estimates (134), compatibility conditions (95) and (124), as well as regularity estimate (110) for the interpolation function.

This proves (130) in terms of ξ. The argument proceeds again analogously for the other two vector fields ξ' and ξ''.

Step 5: Proof of (131). There is nothing to prove throughout interface wedges since the unit normal extensions (ξ, ξ', ξ'') are unit-length vectors (cf. the definitions from Construction 29). On interpolation wedges, say $W_{\bar{\Omega}} \cap \mathcal{N}_{\hat{r}}(\bar{\Gamma})$, we may compute by definition (120) from Construction 29

$$
|\xi|^2 = 1 - \lambda^{\bar{I}}_{\bar{\Omega}} \lambda^{\bar{I}''}_{\bar{\Omega}} \Big| \frac{\widetilde{\xi}}{|\widetilde{\xi}|} - \frac{\widetilde{R}_{\bar{I}''} \widetilde{\xi}''}{|\widetilde{\xi}''|} \Big|^2. \tag{136}
$$

Estimate (131) is thus a consequence of (124), (110), (125) and (36). One may argue analogously for the other two vector fields ξ' and ξ''. ■

4.5. Compatibility of local gradient flow calibrations

A regular double bubble is built out of two distinct topological features: the three two-phase interfaces and the triple line. For each of these topological features, we so far constructed a tuple of vector fields living in a space-time neighborhood of the feature and locally mimicking the requirements of a gradient flow calibration. The remaining step in the construction consists of pasting together these local vector fields into globally defined ones. This task will be carried out in Section 5. The key issue is to transfer properties from the local to the global level, which turns out to be possible because, among other things, the local constructions for the two distinct topological features can be arranged to be sufficiently compatible. We formalize this as follows:

Proposition 30. *Let $(\bar\Omega_1, \bar\Omega_2, \bar\Omega_3)$ be a regular double bubble smoothly evolving by MCF in the sense of Definition 10 on a time interval $[0, T]$. Let $\hat r \in (0, 1]$ be the localization scale of Proposition 16, and for each pair of distinct phases $i, j \in \{1, 2, 3\}$, denote by $(\xi^{\bar I_{i,j}}_{i,j}, B^{\bar I_{i,j}})$ the local gradient flow calibration for the interface $\bar I_{i,j}$ from Construction 14.*

For all $i, j \in \{1, 2, 3\}$ with $i \neq j$, there exist a choice of the tangential component $\mathcal{Y}_{i,j}$ of $B^{\bar I_{i,j}}$ and a local gradient flow calibration $((\xi^{\bar\Gamma}_{i,j})_{i,j\in\{1,2,3\},i\neq j}, B^{\bar\Gamma})$ at the triple line in the sense of Proposition 16 such that in addition the following compatibility estimates hold true:

$$\big|\xi^{\bar I_{i,j}}_{i,j} - \xi^{\bar\Gamma}_{i,j}\big| + \big|(\nabla\xi^{\bar I_{i,j}}_{i,j})^\mathsf{T}\xi^{\bar\Gamma}_{i,j}\big| + \big|(\nabla\xi^{\bar\Gamma}_{i,j})^\mathsf{T}\xi^{\bar I_{i,j}}_{i,j}\big| \leq C\,\mathrm{dist}(\cdot, \bar I_{i,j}), \tag{137}$$

$$\big|(\xi^{\bar I_{i,j}}_{i,j} - \xi^{\bar\Gamma}_{i,j}) \cdot \xi^{\bar I_{i,j}}_{i,j}\big| \leq C\,\mathrm{dist}^2(\cdot, \bar I_{i,j}), \tag{138}$$

$$\big|B^{\bar I_{i,j}} - B^{\bar\Gamma}\big| \leq C\,\mathrm{dist}(\cdot, \bar I_{i,j}), \tag{139}$$

$$\big|(\nabla B^{\bar I_{i,j}} - \nabla B^{\bar\Gamma})^\mathsf{T}\xi^{\bar I_{i,j}}_{i,j}\big| \leq C\,\mathrm{dist}(\cdot, \bar I_{i,j}) \tag{140}$$

throughout $\mathcal{N}_{\frac{\hat r}{2}}(\bar\Gamma) \cap (W_{\bar I_{i,j}} \cup W_{\bar\Omega_i} \cup W_{\bar\Omega_j})$, where $C > 0$ is a constant which depends only on the smoothly evolving regular double bubble $(\bar\Omega_1, \bar\Omega_2, \bar\Omega_3)$ on $[0, T]$.

Proof. Let $((\xi^{\bar\Gamma}_{i,j})_{i,j\in\{1,2,3\},i\neq j}, B^{\bar\Gamma})$ be the local gradient flow calibration at the triple line $\bar\Gamma$ as constructed in the proof of Proposition 16, and let $i, j \in \{1, 2, 3\}$ be distinct phases.

Step 1: Proof of (137). The estimate $|\xi^{\bar I_{i,j}}_{i,j} - \xi^{\bar\Gamma}_{i,j}| \leq C\,\mathrm{dist}(\cdot, \bar I_{i,j})$ is an immediate consequence of regularity estimates (19) and (25), inclusions (34)–(35), as well as the extension property $\xi^{\bar I_{i,j}}_{i,j} = \bar{\mathrm{n}}_{i,j} = \xi^{\bar\Gamma}_{i,j}$ along $\bar I_{i,j} \cap \mathcal{N}_{\hat r}(\bar\Gamma)$.

The estimate $|(\nabla\xi^{\bar I_{i,j}}_{i,j})^\mathsf{T}\xi^{\bar\Gamma}_{i,j}| \leq C\,\mathrm{dist}(\cdot, \bar I_{i,j})$ follows from adding zero, the already established estimate for the first left hand side term of (137), and $\xi^{\bar I_{i,j}}_{i,j}$ being a unit-length vector field due to (18).

For the remaining part of (137), it suffices to estimate $\frac{1}{2}\nabla|\xi^{\bar\Gamma}_{i,j}|^2$ due to (25) and the already established estimate for the first left hand side term of (137). Throughout the interpolation wedge $W_{\bar I_{i,j}} \cap \mathcal{N}_{\hat r}(\bar\Gamma)$, we have $|\xi^{\bar\Gamma}_{i,j}| \equiv 1$ in view of definitions (116)–(118), so that the desired estimate is satisfied for trivial reasons. Within the relevant interpolation wedges, one may use representation (136) and then deduce $|\frac{1}{2}\nabla|\xi^{\bar\Gamma}_{i,j}|^2| \leq C\,\mathrm{dist}(\cdot, \bar I_{i,j})$ from (124), (110) and (36).

Step 2: Proof of (138). Denote by $\tilde\xi^{\bar I_{i,j}}$ the auxiliary extension of the unit normal $\bar{\mathrm{n}}_{i,j}|_{\bar I_{i,j}}$ from Construction 23. Due to (116)–(118), (120)–(122), and compatibility estimates (124), it holds that

$$\xi^{\bar\Gamma}_{i,j} = |\tilde\xi^{\bar I_{i,j}}|^{-1}\tilde\xi^{\bar I_{i,j}} + O(\mathrm{dist}^2(\cdot, \bar I_{i,j})) \quad \text{in } \mathcal{N}_{\hat r}(\bar\Gamma) \cap (W_{\bar I_{i,j}} \cup W_{\bar\Omega_i} \cup W_{\bar\Omega_j}).$$

Making use of non-degeneracy conditions (113)–(115) and estimate (101), we also obtain

$$|\widetilde{\xi}^{\bar{I}_{i,j}}|^{-1} - 1 = \frac{1-|\widetilde{\xi}^{\bar{I}_{i,j}}|^2}{|\widetilde{\xi}^{\bar{I}_{i,j}}|\big(1+|\widetilde{\xi}^{\bar{I}_{i,j}}|\big)} = O(\mathrm{dist}^2(\cdot,\bar{I}_{i,j})) \quad \text{in } \mathcal{N}_{\hat{r}}(\bar{\Gamma}) \cap \mathrm{im}(\Psi_{i,j}).$$

Recalling precise representations (18) and (74), we thus infer from the previous two displays that

$$\big|(\xi^{\bar{I}_{i,j}}_{i,j} - \xi^{\bar{\Gamma}}_{i,j}) \cdot \xi^{\bar{I}_{i,j}}_{i,j}\big| \leq \big|1 - |\widetilde{\xi}^{\bar{I}_{i,j}}|^{-1}\big| + O(\mathrm{dist}^2(\cdot,\bar{I}_{i,j})) = O(\mathrm{dist}^2(\cdot,\bar{I}_{i,j}))$$

throughout $\mathcal{N}_{\hat{r}}(\bar{\Gamma}) \cap (W_{\bar{I}_{i,j}} \cup W_{\bar{\Omega}_i} \cup W_{\bar{\Omega}_j})$, as asserted.

Step 3: Construction of the tangential component $\mathcal{Y}_{i,j}$ of $B^{\bar{I}_{i,j}}$. Let θ be a smooth and even cutoff function with $\theta(r)=1$ for $|r|\leq\frac{1}{2}$ and $\theta(r)=0$ for $|r|\geq 1$. Denote by $\widetilde{B}^{\bar{I}_{i,j}}$ the auxiliary local velocity field from Construction 25 with respect to the interface $\bar{I}_{i,j}$. The tangential component $\mathcal{Y}_{i,j}$ of $B^{\bar{I}_{i,j}}$ is then simply defined by

$$\mathcal{Y}_{i,j} := \theta\Big(\frac{\mathrm{dist}(\cdot,\bar{\Gamma})}{\hat{r}}\Big)(\mathrm{Id} - \bar{\mathrm{n}}_{i,j}\otimes\bar{\mathrm{n}}_{i,j})\widetilde{B}^{\bar{I}_{i,j}} \quad \text{in } \mathrm{im}(\Psi_{i,j}). \tag{141}$$

Note that $\mathcal{Y}_{i,j} \in C^0_tC^1_x(\mathrm{im}(\Psi_{i,j}))$ as required by Construction 14 due to the regularity of the normal $\bar{\mathrm{n}}_{i,j}$ (see (17)) and regularity estimate (94) for $\widetilde{B}^{\bar{I}_{i,j}}$.

Step 4: Proof of (139)–(140). It follows from expansion ansatz (93), definitions (119) and (18), the choice of the tangential component (see (141)), as well as the choice of the cutoff θ from the previous step that

$$B^{\bar{I}_{i,j}} = B^{\bar{\Gamma}} \quad \text{throughout } W_{\bar{I}_{i,j}} \cap \mathcal{N}_{\frac{\hat{r}}{2}}(\bar{\Gamma}). \tag{142}$$

More precisely, denoting by $\widetilde{B}^{\bar{I}_{i,j}}$ the auxiliary local velocity field from Construction 25 with respect to the interface $\bar{I}_{i,j}$, we in fact have

$$B^{\bar{I}_{i,j}} = \widetilde{B}^{\bar{I}_{i,j}} \quad \text{throughout } (W_{\bar{I}_{i,j}} \cup W_{\bar{\Omega}_i} \cup W_{\bar{\Omega}_j}) \cap \mathcal{N}_{\frac{\hat{r}}{2}}(\bar{\Gamma}). \tag{143}$$

Now, (139) follows directly from a Taylor expansion argument exploiting regularity estimates (20) and (26), as well as inclusions (34)–(35).

In view of (142), estimate (140) is satisfied for trivial reasons throughout the interface wedge $\mathcal{N}_{\frac{\hat{r}}{2}}(\bar{\Gamma}) \cap W_{\bar{I}_{i,j}}$. Within the relevant interpolation wedges, say for concreteness $\mathcal{N}_{\frac{\hat{r}}{2}}(\bar{\Gamma}) \cap W_{\bar{\Omega}_i}$, we make use of (143). Let $k \in \{1,2,3\} \setminus \{i,j\}$ denote the third phase. It then follows from (143) and expressing definition (123) in the form of

$$B^{\bar{\Gamma}} = \lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}\widetilde{B}^{\bar{I}_{i,j}} + \big(1-\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}\big)\widetilde{B}^{\bar{I}_{k,i}}$$

that

$$\nabla B^{\bar{I}_{i,j}} - \nabla B^{\bar{\Gamma}} = \big(1-\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}\big)(\nabla\widetilde{B}^{\bar{I}_{i,j}} - \nabla\widetilde{B}^{\bar{I}_{k,i}}) - (\widetilde{B}^{\bar{I}_{i,j}} - \widetilde{B}^{\bar{I}_{k,i}}) \otimes \nabla\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}.$$

Since $\xi_{i,j}^{\bar I_{i,j}} = \bar{\mathrm{n}}_{i,j}$ due to (18), estimate (140) now follows throughout the interpolation wedge $\mathcal{N}_{\frac{\hat r}{2}}(\bar\Gamma) \cap W_{\bar\Omega_i}$ by the same argument which deals with estimating the last two right hand side terms of (135). We recall for convenience that the essential input for the latter is given by compatibility conditions (95) and (96) for the auxiliary velocity fields $\widetilde{B}^{\bar I_{i,j}}$ and $\widetilde{B}^{\bar I_{k,i}}$. ∎

5. Gradient flow calibrations for double bubbles

5.1. Localization of topological features

We start by introducing a family of suitable cutoff functions localizing around the interfaces and the triple line in a smoothly evolving regular double bubble. This family will be used to provide the construction of a gradient flow calibration by means of gluing together the local constructions from the previous two sections.

Lemma 31. *Let $(\bar\Omega_1, \bar\Omega_2, \bar\Omega_3)$ be a regular double bubble smoothly evolving by MCF in the sense of Definition 10 on a time interval $[0,T]$. Let the notation of Definition 13 (and Definition 17) be in place, and let $\hat r \in (0,1]$ be the radius of Proposition 16. In particular, let $(r_{i,j})_{i,j\in\{1,2,3\},i\neq j}$ be admissible localization radii for the interfaces in the sense of Definition 13 such that $\hat r \leq r_{1,2} \wedge r_{2,3} \wedge r_{3,1}$. We next define for each pair $i,j \in \{1,2,3\}$ with $i \neq j$ a scale*

$$3\ell_{i,j} := \min_{t\in[0,T]} \min_{\substack{k,l\in\{1,2,3\},k\neq l,\\ (k,l)\notin\{(i,j),(j,i)\}}} \mathrm{dist}\big(\bar I_{i,j}(t) \setminus B_{\hat r}(\bar\Gamma(t)), \bar I_{k,l}(t)\big) > 0,$$

and based on these a localization scale $\bar r \in (0, r_{1,2} \wedge r_{2,3} \wedge r_{3,1}]$ by

$$2\bar r := \hat r \wedge \min_{i,j\in\{1,2,3\},i\neq j} \ell_{i,j}. \tag{144}$$

There then exists a collection of continuous cutoff functions

$$\eta_{\bar\Gamma}, \eta_{\bar I_{1,2}}, \eta_{\bar I_{2,3}}, \eta_{\bar I_{3,1}} : \mathbb{R}^3 \times [0,T] \to [0,1]$$

satisfying the following properties:

(i) *The cutoff functions are of class $(C_t^0 C_x^1 \cap C_t^1 C_x^0)(\mathbb{R}^3 \times [0,T] \setminus \bar\Gamma)$ with corresponding regularity estimates*

$$|(\partial_t, \nabla)(\eta_{\bar\Gamma}, \eta_{\bar I_{1,2}}, \eta_{\bar I_{2,3}}, \eta_{\bar I_{3,1}})| \leq C \quad \text{in } \mathbb{R}^3 \times [0,T] \setminus \bar\Gamma \tag{145}$$

for some constant $C > 0$ depending only on the data of the smoothly evolving regular double bubble $(\bar\Omega_1, \bar\Omega_2, \bar\Omega_3)$ on $[0,T]$.

(ii) *The family $(\eta_{\bar\Gamma}, (\eta_{\bar I_{i,j}})_{i,j\in\{1,2,3\},i\neq j})$ is a partition of unity for the evolving surface cluster in the sense that $\eta_{\bar\Gamma} + \eta_{\bar I_{1,2}} + \eta_{\bar I_{2,3}} + \eta_{\bar I_{3,1}} \equiv 1$ holds true on the surface cluster $\mathcal{I} := \bigcup_{i,j\in\{1,2,3\},i\neq j} \bar I_{i,j}$.*

Moreover, for all pairwise distinct $i, j, k \in \{1,2,3\}$*, it holds that*

$$\eta_{\bar{I}_{k,i}} \leq C(\operatorname{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1) \quad \text{in } \mathbb{R}^3 \times [0,T], \tag{146}$$

$$|\nabla \eta_{\bar{I}_{k,i}}| \leq C(\operatorname{dist}(\cdot, \bar{I}_{i,j}) \wedge 1) \quad \text{in } \mathbb{R}^3 \times [0,T] \setminus \bar{\Gamma}, \tag{147}$$

$$|\partial_t \eta_{\bar{I}_{k,i}}| \leq C(\operatorname{dist}(\cdot, \bar{I}_{i,j}) \wedge 1) \quad \text{in } \mathbb{R}^3 \times [0,T] \setminus \bar{\Gamma}. \tag{148}$$

Defining

$$\eta_{\text{bulk}} := 1 - \eta_{\bar{\Gamma}} - \eta_{\bar{I}_{1,2}} - \eta_{\bar{I}_{2,3}} - \eta_{\bar{I}_{3,1}},$$

we have $\eta_{\text{bulk}} \in [0,1]$ *on* $\mathbb{R}^3 \times [0,T]$*, and the bulk cutoff is subject to the estimates*

$$\frac{1}{C}(\operatorname{dist}^2(\cdot, \mathcal{I}) \wedge 1) \leq \eta_{\text{bulk}} \leq C(\operatorname{dist}^2(\cdot, \mathcal{I}) \wedge 1) \quad \text{in } \mathbb{R}^3 \times [0,T], \tag{149}$$

$$|\nabla \eta_{\text{bulk}}| \leq C(\operatorname{dist}(\cdot, \mathcal{I}) \wedge 1) \quad \text{in } \mathbb{R}^3 \times [0,T] \setminus \bar{\Gamma}, \tag{150}$$

$$|\partial_t \eta_{\text{bulk}}| \leq C(\operatorname{dist}(\cdot, \mathcal{I}) \wedge 1) \quad \text{in } \mathbb{R}^3 \times [0,T] \setminus \bar{\Gamma}. \tag{151}$$

The constant $C \geq 1$ *in estimates* (146)–(151) *depends only on the data of the smoothly evolving regular double bubble* $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ *on* $[0,T]$.

(iii) *For all pairwise distinct* $i, j, k \in \{1,2,3\}$ *and all* $t \in [0,T]$*, it holds that*

$$\operatorname{supp} \eta_{\bar{I}_{i,j}}(\cdot, t) \subset \Psi_{i,j}(\bar{I}_{i,j}(t) \times \{t\} \times [-\bar{r}, \bar{r}]), \tag{152}$$

$$B_{\hat{r}}(\bar{\Gamma}(t)) \cap \operatorname{supp} \eta_{\bar{I}_{i,j}}(\cdot, t) \subset B_{\hat{r}}(\bar{\Gamma}(t)) \cap \big(W_{\bar{I}_{i,j}}(t) \cup W_{\bar{\Omega}_i}(t) \cup W_{\bar{\Omega}_j}(t)\big), \tag{153}$$

$$\operatorname{supp} \eta_{\bar{I}_{i,j}}(\cdot, t) \cap \operatorname{supp} \eta_{\bar{I}_{j,k}}(\cdot, t) \subset B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_j}(t), \tag{154}$$

$$\operatorname{supp} \eta_{\bar{\Gamma}}(\cdot, t) \subset B_{\hat{r}/2}(\bar{\Gamma}(t)). \tag{155}$$

Proof. The proof is split into several steps.

Step 1: Definition of building blocks. Let θ be a smooth and even cutoff function with $\theta(r) = 1$ for $|r| \leq \frac{1}{2}$ and $\theta(r) = 0$ for $|r| \geq 1$. Then, define a smooth quadratic profile $\zeta : \mathbb{R} \to [0,1]$ by

$$\zeta(r) = (1 - r^2)\theta(r^2), \quad r \in \mathbb{R}. \tag{156}$$

Let $\delta \in (0,1]$ be a constant whose value will be determined in subsequent steps of the proof. For all distinct $i, j \in \{1,2,3\}$, we define auxiliary cutoff functions

$$\zeta_{\bar{I}_{i,j}} := \zeta\Big(\frac{s_{i,j}}{\delta \bar{r}}\Big) \quad \text{in } \operatorname{im}(\Psi_{i,j}), \tag{157}$$

$$\zeta_{\bar{\Gamma}} := \zeta\Big(\frac{\operatorname{dist}(\cdot, \bar{\Gamma})}{\hat{r}/2}\Big) \quad \text{in } \mathbb{R}^3 \times [0,T]. \tag{158}$$

Note that as a consequence of the regularity of the signed distance (see (14)), expressing $\operatorname{dist}(x, \bar{\Gamma}(t)) = |x - P_{\bar{\Gamma}}(x,t)|$ for all $x \in B_{\hat{r}}(\bar{\Gamma}(t))$ and all $t \in [0,T]$, the regularity of the projection $P_{\bar{\Gamma}}$ onto the triple line $\bar{\Gamma}$ from Definition 17, and (156), it holds that

$$|(\partial_t, \nabla)\zeta_{\bar{I}_{i,j}}| \leq C \operatorname{dist}(\cdot, \bar{I}_{i,j}) \quad \text{in } \operatorname{im}(\Psi_{i,j}), \tag{159}$$

$$|(\partial_t, \nabla)\zeta_{\bar{\Gamma}}| \leq C \operatorname{dist}(\cdot, \bar{\Gamma}) \quad \text{in } \mathbb{R}^3 \times [0, T]. \tag{160}$$

Step 2: Definition of interface cutoffs. Fix distinct $i, j \in \{1, 2, 3\}$. We define the cutoff $\eta_{\bar{I}_{i,j}} : \mathbb{R}^3 \times [0, T] \to [0, 1]$ for the two-phase interface $\bar{I}_{i,j}$ by

$$\eta_{\bar{I}_{i,j}}(\cdot, t) := \zeta_{\bar{I}_{i,j}}(\cdot, t) \quad \text{in } \operatorname{im}(\Psi_{i,j}(t)) \setminus B_{\hat{r}}(\bar{\Gamma}(t)), \tag{161}$$

$$\eta_{\bar{I}_{i,j}}(\cdot, t) := (1 - \zeta_{\bar{\Gamma}}(\cdot, t))\zeta_{\bar{I}_{i,j}}(\cdot, t) \quad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{i,j}}(t), \tag{162}$$

$$\eta_{\bar{I}_{i,j}}(\cdot, t) := \lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}(\cdot, t)(1 - \zeta_{\bar{\Gamma}}(\cdot, t))\zeta_{\bar{I}_{i,j}}(\cdot, t) \quad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t), \tag{163}$$

$$\eta_{\bar{I}_{i,j}}(\cdot, t) := \lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_j}(\cdot, t)(1 - \zeta_{\bar{\Gamma}}(\cdot, t))\zeta_{\bar{I}_{i,j}}(\cdot, t) \quad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_j}(t), \tag{164}$$

$$\eta_{\bar{I}_{i,j}}(\cdot, t) := 0 \quad \text{else} \tag{165}$$

for all $t \in [0, T]$. Here, the maps $\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}$ (resp. $\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_j}$) are the interpolation functions of Lemma 28 on the interpolation wedges $W_{\bar{\Omega}_i}$ (resp. $W_{\bar{\Omega}_j}$). Observe that (162) is well-defined because of (34), and that (163) (resp. (164)) are well-defined as a consequence of (35). In particular, properties (152)–(154) are immediate consequences of definitions (161)–(165) and the choice of the localization scale $\bar{r}$ (see (144)). Finally, in order to ensure continuity of $\eta_{\bar{I}_{i,j}}$ throughout $\mathbb{R}^3 \times [0, T]$ (i.e., compatibility of definition (161) resp. definition (165) with definitions (162)–(164)), we choose the constant $\delta \in (0, \frac{1}{2}]$ small enough such that for all $t \in [0, T]$ and all distinct $i, j \in \{1, 2, 3\}$, it holds that

$$\partial B_{\hat{r}}(\bar{\Gamma}(t)) \cap \overline{\Psi_{i,j}(\bar{I}_{i,j}(t) \times \{t\} \times [-\delta\hat{r}, \delta\hat{r}])} \subset\subset W_{\bar{I}_{i,j}}(t). \tag{166}$$

Step 3: Definition of triple line cutoff. We construct a cutoff for the triple line $\eta_{\bar{\Gamma}} : \mathbb{R}^3 \times [0, T] \to [0, 1]$ as follows: for all distinct $i, j, k \in \{1, 2, 3\}$ and all $t \in [0, T]$, we define

$$\eta_{\bar{\Gamma}}(\cdot, t) := \zeta_{\bar{\Gamma}}(\cdot, t)\zeta_{\bar{I}_{i,j}}(\cdot, t) \quad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{i,j}}(t), \tag{167}$$

$$\begin{aligned} \eta_{\bar{\Gamma}}(\cdot, t) := {} & \lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}(\cdot, t)\zeta_{\bar{\Gamma}}(\cdot, t)\zeta_{\bar{I}_{i,j}}(\cdot, t) \\ & + \lambda^{\bar{I}_{k,i}}_{\bar{\Omega}_i}(\cdot, t)\zeta_{\bar{\Gamma}}(\cdot, t)\zeta_{\bar{I}_{k,i}}(\cdot, t) \quad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t), \end{aligned} \tag{168}$$

$$\eta_{\bar{\Gamma}}(\cdot, t) := 0 \quad \text{in } \mathbb{R}^3 \setminus B_{\hat{r}}(\bar{\Gamma}(t)). \tag{169}$$

Because of (33), definitions (167)–(169) provide a definition of $\eta_{\bar{\Gamma}}$ on the whole space-time domain $\mathbb{R}^3 \times [0, T]$. Property (155) is obviously satisfied in view of (169). Since

$$\lambda^{\bar{I}_{i,k}}_{\bar{\Omega}_i} = 1 - \lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}$$

on interpolation wedges $W_{\bar{\Omega}_i}$, we indeed have $\eta_{\bar{\Gamma}}(x, t) \in [0, 1]$ for all $(x, t) \in \mathbb{R}^3 \times [0, T]$.

Step 4: Partition of unity property along the surface cluster. Define the bulk cutoff $\eta_{\mathrm{bulk}} := 1 - \eta_{\bar{\Gamma}} - \eta_{\bar{I}_{1,2}} - \eta_{\bar{I}_{2,3}} - \eta_{\bar{I}_{3,1}}$. We claim that

$$\eta_{\mathrm{bulk}} = 0 \quad \text{along } \mathcal{I} = \bigcup_{i,j \in \{1,2,3\}, i \neq j} \bar{I}_{i,j}. \tag{170}$$

Fix $t \in [0, T]$ and a point $x \in \mathcal{I}(t) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$. There exists a unique pair of distinct phases $i, j \in \{1,2,3\}$ such that $x \in \bar{I}_{i,j}(t)$ and, because of localization properties (154) and (155), $\eta_{\mathrm{bulk}}(x,t) = 1 - \eta_{\bar{I}_{i,j}}(x,t)$. It then follows from definitions (161) and (157) that $\eta_{\mathrm{bulk}}(x,t) = 0$.

Now fix $t \in [0,T]$ and consider a point $x \in \mathcal{I}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t))$. Let $i, j \in \{1,2,3\}$ be the unique pair of distinct phases such that $x \in \bar{I}_{i,j}(t)$. As a consequence of (34), localization properties (153)–(155), and definitions (162) and (167), we obtain that $\eta_{\mathrm{bulk}}(x,t) = 1 - \eta_{\bar{\Gamma}}(x,t) - \eta_{\bar{I}_{i,j}}(x,t) = 1 - \zeta_{\bar{I}_{i,j}}(x,t)$. Hence, $\eta_{\mathrm{bulk}}(x,t) = 0$ due to definition (157). This concludes the proof of (170).

Step 5: Regularity of cutoff functions. Fix $i, j \in \{1,2,3\}$ such that $i \neq j$. The required derivatives of $\eta_{\bar{I}_{i,j}}$ exist in $\mathbb{R}^3 \setminus \overline{B_{\hat{r}}(\bar{\Gamma}(t))}$ (resp. in $B_{\hat{r}}(\bar{\Gamma}(t)) \setminus \bar{\Gamma}(t)$) in a pointwise sense for all $t \in [0,T]$ due to the definition of $\eta_{\bar{I}_{i,j}}$ from *Step 2* of this proof, definitions (157) and (158), the properties of the interpolation functions from Lemma 28, and the regularity of the auxiliary cutoff functions (see (159) and (160)). By the choice of the scale $\delta \in (0,1]$ (see (166)), these derivatives do not jump across the boundary of $B_{\hat{r}}(\bar{\Gamma}(t))$. Hence, $\partial_t \eta_{\bar{I}_{i,j}}$ and $\nabla \eta_{\bar{I}_{i,j}}$ exist in a pointwise sense in $\mathbb{R}^3 \times [0,T] \setminus \bar{\Gamma}$.

In terms of the required bounds for these derivatives (see (145)), the only possibly critical cases are those for which at least one derivative hits an interpolation function present in definitions (163) and (164). The blow-up of these derivatives (see Lemma 28), however, is always cured by the presence of the term $1 - \zeta_{\bar{\Gamma}}$. In summary, $\eta_{\bar{I}_{i,j}} \in (C_t^0 C_x^1 \cap C_t^1 C_x^0)(\mathbb{R}^3 \times [0,T] \setminus \bar{\Gamma})$ and (145) holds true.

Along similar lines, one checks that $\partial_t \eta_{\bar{\Gamma}}$ and $\nabla \eta_{\bar{\Gamma}}$ exist in a pointwise sense in $\mathbb{R}^3 \times [0,T] \setminus \bar{\Gamma}$. The required cancellations to counteract the blow-up of derivatives of the interpolation parameter in interpolation wedges this time comes from recalling

$$\lambda_{\bar{\Omega}_i}^{\bar{I}_{k,i}} = 1 - \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}},$$

which in turn ensures that potentially critical terms always involve the term $\zeta_{\bar{I}_{i,j}} - \zeta_{\bar{I}_{k,i}}$. As the latter vanishes to first-order at the triple line and has a bounded second-order spatial derivative within interpolation wedges, it follows that $\eta_{\bar{\Gamma}} \in (C_t^0 C_x^1 \cap C_t^1 C_x^0)(\mathbb{R}^3 \times [0,T] \setminus \bar{\Gamma})$, and that (145) holds true.

Step 6: Estimates for the bulk cutoff. By construction, it holds that $\eta_{\mathrm{bulk}}(\cdot,t) \equiv 1$ outside of the space-time domain $B_{\hat{r}}(\bar{\Gamma}(t)) \cup \bigcup_{i,j \in \{1,2,3\}, i \neq j} \Psi_{i,j}(\bar{I}_{i,j}(t) \times \{t\} \times [-\bar{r}, \bar{r}])$ for all $t \in [0,T]$. Hence, for a proof of $\eta_{\mathrm{bulk}} \in [0,1]$ and estimates (149)–(151), we may restrict our attention to $\bigcup_{i,j \in \{1,2,3\}, i \neq j} \Psi_{i,j}(\bar{I}_{i,j}(t) \times \{t\} \times [-\bar{r}, \bar{r}]) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$ and $B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$.

In view of the choice of the localization scale $\bar{r}$ (see (144)), one may argue separately on $\Psi_{i,j}(\bar{I}_{i,j}(t) \times \{t\} \times [-\bar{r}, \bar{r}]) \setminus B_{\widehat{r}}(\bar{\Gamma}(t))$ for each pair of distinct phases $i, j \in \{1, 2, 3\}$ and all $t \in [0, T]$. Because of localization properties (154) and (155), it holds that

$$\begin{aligned}\eta_{\mathrm{bulk}}(\cdot, t) &= 1 - \eta_{\bar{I}_{i,j}}(\cdot, t) \\ &= 1 - \zeta_{\bar{I}_{i,j}}(\cdot, t) \quad \text{in } \Psi_{i,j}(\bar{I}_{i,j}(t) \times \{t\} \times [-\bar{r}, \bar{r}]) \setminus B_{\widehat{r}}(\bar{\Gamma}(t))\end{aligned} \tag{171}$$

for all $t \in [0, T]$. Hence, $\eta_{\mathrm{bulk}} \in [0, 1]$ and estimates (149)–(151) follow from definitions (161) and (157) in combination with the quadratic behavior around the origin of profile (156). Note in this context that (144) precisely ensures that the error can be expressed in terms of $\mathrm{dist}(\cdot, \mathcal{I})$, as required.

We move on to the argument in the ball $B_{\widehat{r}}(\bar{\Gamma}(t))$ for all $t \in [0, T]$. On interface wedges, we infer from localization properties (153) and (154) as well as definitions (162) and (167) that

$$\begin{aligned}\eta_{\mathrm{bulk}}(\cdot, t) &= 1 - \eta_{\bar{\Gamma}}(\cdot, t) - \eta_{\bar{I}_{i,j}}(\cdot, t) \\ &= 1 - \zeta_{\bar{I}_{i,j}}(\cdot, t) \quad \text{in } B_{\widehat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{i,j}}(t)\end{aligned} \tag{172}$$

for all $t \in [0, T]$, so that the asserted bounds follow as in the previous case together with bound (38) to express the error in terms of $\mathrm{dist}(\cdot, \mathcal{I})$.

On interpolation wedges, we may compute based on (153) and (154) as well as (163) and (168) that (recall the relation $\lambda_{\bar{\Omega}_i}^{\bar{I}_{k,i}} = 1 - \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}$)

$$\begin{aligned}\eta_{\mathrm{bulk}}(\cdot, t) &= 1 - \eta_{\bar{\Gamma}}(\cdot, t) - \eta_{\bar{I}_{i,j}}(\cdot, t) - \eta_{\bar{I}_{k,i}}(\cdot, t) \\ &= \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}(1 - \zeta_{\bar{I}_{i,j}})(\cdot, t) + (1 - \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}})(1 - \zeta_{\bar{I}_{k,i}})(\cdot, t) \quad \text{in } B_{\widehat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t)\end{aligned} \tag{173}$$

for all $t \in [0, T]$. It follows immediately that $\eta_{\mathrm{bulk}}(\cdot, t) \in [0, 1]$. Moreover, definition (157), the quadratic behavior around the origin of profile (156), and estimate (36) directly imply (149). Finally, since

$$\begin{aligned}\nabla \eta_{\mathrm{bulk}}(\cdot, t) &= -\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}(\cdot, t) \nabla \zeta_{\bar{I}_{i,j}}(\cdot, t) - (1 - \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}})(\cdot, t) \nabla \zeta_{\bar{I}_{k,i}}(\cdot, t) \\ &\quad - (\zeta_{\bar{I}_{i,j}} - \zeta_{\bar{I}_{k,i}})(\cdot, t) \nabla \lambda_i^{\bar{I}_{i,j}}(\cdot, t) \quad \text{in } B_{\widehat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t)\end{aligned}$$

for all $t \in [0, T]$, we obtain (150) and (151) because the blow-up of $\nabla \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}$ (see Lemma 28) is canceled to the required order by the term $\zeta_{\bar{I}_{i,j}} - \zeta_{\bar{I}_{k,i}}$. Indeed, the latter vanishes to first-order at the triple line and has a bounded second-order spatial derivative within interpolation wedges.

Step 7: Error estimates for interface cutoffs. Bounds (146)–(148) are trivially fulfilled outside of $B_{\widehat{r}}(\bar{\Gamma}(t))$ for all $t \in [0, T]$ by construction and the choice of the localization scale $\bar{r}$ (see (144)). In view of definitions (162)–(164) and definition (158), we also have

$\eta_{\bar{I}_{k,i}}(\cdot,t) \leq 1 - \zeta_{\bar{\Gamma}}(\cdot,t) \leq C\,\mathrm{dist}^2(\cdot,\bar{\Gamma}(t))$ in $B_{\hat{r}}(\bar{\Gamma}(t)) \cap (W_{\bar{I}_{k,i}}(t) \cup W_{\bar{\Omega}_i}(t) \cup W_{\bar{\Omega}_k}(t))$ for all $t \in [0,T]$. Recalling bounds (36) and (37), this in turn implies (146) throughout $B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$.

For a proof of (147) and (148), note that

$$|(\partial_t,\nabla)\eta_{\bar{I}_{k,i}}(\cdot,t)| \leq C(1-\zeta_{\bar{\Gamma}}(\cdot,t)) + C|(\partial_t,\nabla)\,\mathrm{dist}(\cdot,\bar{\Gamma}(t))|\,\mathrm{dist}(\cdot,\bar{\Gamma}(t))$$

in $B_{\hat{r}}(\bar{\Gamma}(t)) \cap (W_{\bar{I}_{k,i}}(t) \cup W_{\bar{\Omega}_i}(t) \cup W_{\bar{\Omega}_k}(t))$ for all $t \in [0,T]$. The first right hand side term is estimated as before, while the second one is of required order due to bounds (36) and (37) and the regularity of the projection onto the triple line $\bar{\Gamma}$ (see Definition 17), which in turn one may employ throughout $B_{\hat{r}}(\bar{\Gamma}(t))$ based on the representation $|x - P_{\bar{\Gamma}}(x,t)| = \mathrm{dist}(x,\bar{\Gamma}(t))$. ∎

5.2. Construction of a gradient flow calibration

We have everything in place to provide the construction of a gradient flow calibration for a regular double bubble smoothly evolving by MCF. We first introduce a global definition for the vector fields $\xi_{i,j}$ extending the unit normal vector fields $\bar{\mathrm{n}}_{i,j}|_{\bar{I}_{i,j}}$ of the interfaces $\bar{I}_{i,j}$.

Construction 32 (Global extensions of the unit normal vector fields). Let $(\bar{\Omega}_1,\bar{\Omega}_2,\bar{\Omega}_3)$ be a regular double bubble smoothly evolving by MCF in the sense of Definition 10 on a time interval $[0,T]$. Let $(\eta_{\bar{\Gamma}}, (\eta_{\bar{I}_{i,j}})_{i,j\in\{1,2,3\},i\neq j})$ be the partition of unity from the proof of Lemma 31. Fix $i,j \in \{1,2,3\}$ with $i \neq j$. We then define a family of vector fields

$$\xi_{i,j}^{\bar{I}_{k,l}} : \bigcup_{t\in[0,T]} \mathrm{supp}\,\eta_{\bar{I}_{k,l}}(\cdot,t)\times\{t\} \to \overline{B_1(0)}, \quad k,l\in\{1,2,3\},\ k\neq l, \tag{174}$$

$$\xi_{i,j}^{\bar{\Gamma}} : \bigcup_{t\in[0,T]} \mathrm{supp}\,\eta_{\bar{\Gamma}}(\cdot,t)\times\{t\} \to \overline{B_1(0)} \tag{175}$$

by means of the following procedure:

For $k,l \in \{1,2,3\}$ with $(k,l) \in \{(i,j),(j,i)\}$, we let $\xi_{i,j}^{\bar{I}_{k,l}}$ be the corresponding vector field from Construction 14 for the interface $\bar{I}_{k,l}$. For $k,l \in \{1,2,3\}$ with $(k,l) \notin \{(i,j),(j,i)\}$ and $k \neq l$, we define

$$\xi_{i,j}^{\bar{I}_{k,l}} := \frac{1}{2}\Big(\frac{\sigma_{l,i}-\sigma_{l,j}}{\sigma_{i,j}}\xi_{k,l}^{\bar{I}_{k,l}} + \frac{\sigma_{k,i}-\sigma_{k,j}}{\sigma_{i,j}}\xi_{l,k}^{\bar{I}_{k,l}}\Big),$$

which is well-defined when reversing the roles of i,j and k,l in the previous step. Finally, we denote by $\xi_{i,j}^{\bar{\Gamma}}$ the corresponding vector field from the proof of Proposition 30.

With this family of local vector fields in place, we now define a global vector field $\xi_{i,j} : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ by

$$\xi_{i,j} := \eta_{\bar{\Gamma}}\xi_{i,j}^{\bar{\Gamma}} + \eta_{\bar{I}_{1,2}}\xi_{i,j}^{\bar{I}_{1,2}} + \eta_{\bar{I}_{2,3}}\xi_{i,j}^{\bar{I}_{2,3}} + \eta_{\bar{I}_{3,1}}\xi_{i,j}^{\bar{I}_{3,1}} \tag{176}$$

for all distinct pairs of phases $i,j \in \{1,2,3\}$. ◇

We proceed by showing that the vector fields from the previous construction satisfy structural assumption (1a) and coercivity estimate (1c) of a gradient flow calibration.

Lemma 33. *Let the assumptions and notation of Construction 32 be in place. Fix $i,j\in\{1,2,3\}$ such that $i\neq j$. The vector field $\xi_{i,j}$ is then subject to the following list of properties:*

(i) *It holds that $\xi_{i,j}\in(C^0_tC^1_x\cap C^1_tC^0_x)(\mathbb{R}^3\times[0,T]\setminus\bar\Gamma)$, and there exists a constant $C>0$ which depends only on the data of the smoothly evolving regular double bubble $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ on $[0,T]$ such that*

$$|(\partial_t,\nabla)\xi_{i,j}|\leq C\quad\text{in }\mathbb{R}^3\times[0,T]\setminus\bar\Gamma. \tag{177}$$

Moreover, it holds that $\xi_{i,j}=\bar{\mathrm{n}}_{i,j}$ along $\bar I_{i,j}$.

(ii) *For each phase $i\in\{1,2,3\}$, there exists a vector field $\xi_i:\mathbb{R}^3\times[0,T]\to\mathbb{R}^3$ of class $(C^0_tC^1_x\cap C^1_tC^0_x)(\mathbb{R}^3\times[0,T]\setminus\bar\Gamma)$ such that $\sigma_{i,j}\xi_{i,j}=\xi_i-\xi_j$ holds true on $\mathbb{R}^3\times[0,T]$.*

(iii) *There exists a constant $c\in(0,1)$, which depends only on the data of the smoothly evolving regular double bubble $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ on $[0,T]$, such that*

$$c(\mathrm{dist}^2(\cdot,\bar I_{i,j})\wedge 1)\leq 1-|\xi_{i,j}|\quad\text{in }\mathbb{R}^3\times[0,T]. \tag{178}$$

Proof. The proof is performed in three steps.

Step 1: Regularity and structural properties. The asserted qualitative regularity of the vector fields $\xi_{i,j}$ together with estimate (177) follows from definition (176), the regularity of the cutoff functions (see (145)), as well as the regularity of the local building blocks (see (174) and (175)) in the form of

$$\Big|(\partial_t,\nabla)(\xi^{\bar I_{k,l}}_{i,j},\xi^{\bar\Gamma}_{i,j})\Big|\leq C\quad\text{in }\mathbb{R}^3\times[0,T], \tag{179}$$

which in turn is a consequence of the definitions from Construction 32 and regularity estimates (19) and (25). The property $\xi_{i,j}|_{\bar I_{i,j}}\equiv\bar{\mathrm{n}}_{i,j}$ is immediate from definition (176), the fact that $(\eta_{\bar\Gamma},(\eta_{\bar I_{i,j}})_{i,j\in\{1,2,3\},i\neq j})$ constitutes a partition of unity along the network $\mathcal{I}$, and the corresponding property in terms of the local constructions from Lemma 15 and Proposition 16.

The existence of vector fields $(\xi_i)_{i\in\{1,2,3\}}$ of class $(C^0_tC^1_x\cap C^1_tC^0_x)(\mathbb{R}^3\times[0,T]\setminus\bar\Gamma)$ such that $\sigma_{i,j}\xi_{i,j}=\xi_i-\xi_j$ holds true on $\mathbb{R}^3\times[0,T]$ follows from the following considerations: Let $i,j,k\in\{1,2,3\}$ be pairwise distinct. We define $\xi^{\bar\Gamma}_i:=\frac{1}{3}(\sigma_{i,j}\xi^{\bar\Gamma}_{i,j}+\sigma_{i,k}\xi^{\bar\Gamma}_{i,k})$. Since $\sigma_{1,2}\xi^{\bar\Gamma}_{1,2}+\sigma_{2,3}\xi^{\bar\Gamma}_{2,3}+\sigma_{3,1}\xi^{\bar\Gamma}_{3,1}=0$ holds true in the support of $\eta_{\bar\Gamma}$ (see Proposition 16), we indeed obtain $\sigma_{i,j}\xi^{\bar\Gamma}_{i,j}=\xi^{\bar\Gamma}_i-\xi^{\bar\Gamma}_j$. Next, fix $k,l\in\{1,2,3\}$ with $k\neq l$, and let $i\in\{1,2,3\}$. We may then define

$$\xi^{\bar I_{k,l}}_i:=\frac{1}{2}(\sigma_{l,i}\xi^{\bar I_{k,l}}_{k,l}+\sigma_{k,i}\xi^{\bar I_{k,l}}_{l,k}).$$

Again, plugging in the definitions immediately shows $\sigma_{i,j}\xi_{i,j}^{\bar{I}_{k,l}} = \xi_i^{\bar{I}_{k,l}} - \xi_j^{\bar{I}_{k,l}}$ for all $i, j \in \{1,2,3\}$ such that $i \neq j$. Defining

$$\xi_i := \eta_{\bar{\Gamma}}\xi_i^{\bar{\Gamma}} + \eta_{\bar{I}_{1,2}}\xi_i^{\bar{I}_{1,2}} + \eta_{\bar{I}_{2,3}}\xi_i^{\bar{I}_{2,3}} + \eta_{\bar{I}_{3,1}}\xi_i^{\bar{I}_{3,1}}$$

therefore entails the desired conclusion.

Step 2: A coercivity condition. As a preparation for the proof of (178), we claim that there exists a constant $\varepsilon = \varepsilon(\sigma) \in (0,1)$ such that for all $i, j \in \{1,2,3\}$ with $i \neq j$, as well as all $k, l \in \{1,2,3\}$ with $(k,l) \notin \{(i,j),(j,i)\}$ and $k \neq l$, it holds that

$$\big|\xi_{i,j}^{\bar{I}_{k,l}}\big| \leq \varepsilon < 1. \tag{180}$$

Indeed, estimate (180) is an immediate consequence of the definition of the vector field

$$\xi_{i,j}^{\bar{I}_{k,l}} = \frac{1}{2}\Big(\frac{\sigma_{l,i} - \sigma_{l,j}}{\sigma_{i,j}}\xi_{k,l}^{\bar{I}_{k,l}} + \frac{\sigma_{k,i} - \sigma_{k,j}}{\sigma_{i,j}}\xi_{l,k}^{\bar{I}_{k,l}}\Big)$$

(see Construction 32), and the facts that $|\frac{\sigma_{l,i}-\sigma_{l,j}}{\sigma_{i,j}}| < 1$ and $|\frac{\sigma_{k,i}-\sigma_{k,j}}{\sigma_{i,j}}| < 1$, which in turn are true since the matrix of surface tensions satisfies the strict triangle inequality by assumption.

Step 3: Proof of estimate (178). Fix $i, j \in \{1,2,3\}$ such that $i \neq j$. By localization properties (152)–(155) and the choice of the localization scale $\bar{r}$ (see (144)), it suffices to establish the desired estimate throughout $\operatorname{supp}\eta_{\bar{I}_{k,l}}(\cdot,t) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$, $B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{k,l}}(t)$ or $B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_l}(t)$ for all distinct phases $k, l \in \{1,2,3\}$ and all $t \in [0,T]$. Hence, fix such $k, l \in \{1,2,3\}$ with $k \neq l$ and $t \in [0,T]$, and then observe that due to definition (176) and localization properties (152)–(155), it holds that

$$\xi_{i,j} = \begin{cases} \eta_{\bar{I}_{k,l}}\xi_{i,j}^{\bar{I}_{k,l}} & \text{on } \operatorname{supp}\eta_{\bar{I}_{k,l}}(\cdot,t) \setminus B_{\hat{r}}(\bar{\Gamma}(t)), \\ \eta_{\bar{\Gamma}}\xi_{i,j}^{\bar{\Gamma}} + \eta_{\bar{I}_{k,l}}\xi_{i,j}^{\bar{I}_{k,l}} & \text{on } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{k,l}}(t), \\ \eta_{\bar{\Gamma}}\xi_{i,j}^{\bar{\Gamma}} + \eta_{\bar{I}_{k,l}}\xi_{i,j}^{\bar{I}_{k,l}} + \eta_{\bar{I}_{l,m}}\xi_{i,j}^{\bar{I}_{l,m}} & \text{on } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_l}(t),\ m \in \{1,2,3\} \setminus \{k,l\}. \end{cases} \tag{181}$$

Based on (181), we now distinguish between two cases.

Substep 3.1: Assume that $(k,l) \in \{(i,j),(j,i)\}$. In other words, both the phases k and l are present at the interface $\bar{I}_{i,j}$. In this case, observe first that throughout the three domains represented in (181) it holds due to (36), (38) and (144) that the distance to $\mathcal{I}$ is comparable to the distance to $\bar{I}_{i,j}$: $\frac{1}{C}\operatorname{dist}(\cdot,\bar{I}_{i,j}) \leq \operatorname{dist}(\cdot,\mathcal{I}) \leq C\operatorname{dist}(\cdot,\bar{I}_{i,j})$ for some constant $C \geq 1$. Furthermore, it follows from (181) and the triangle inequality that $|\xi_{i,j}| \leq 1 - \eta_{\mathrm{bulk}}$ throughout the three domains represented in (181). Hence, bound (178) follows from the lower bound in (149).

Substep 3.2: Assume that $(k,l) \notin \{(i,j),(j,i)\}$. In the first case of equation (181), estimate (178) follows immediately from coercivity condition (180). In the third case of (181), we may additionally assume that $(l, m) \notin \{(i, j), (j, i)\}$; otherwise, we are again in the setting of the argument from *Substep 3.1* above. Plugging in definitions (163), (164) and (168), as well as exploiting coercivity condition (180) for both the vector fields $\xi_{i,j}^{\bar{I}_{k,l}}$ and $\xi_{i,j}^{\bar{I}_{l,m}}$ (which is applicable due to our assumptions), we may estimate from below

$$1 - |\xi_{i,j}| \geq 1 - (\eta_{\bar{\Gamma}} + \varepsilon\eta_{\bar{I}_{k,l}} + \varepsilon\eta_{\bar{I}_{l,m}}) \geq (1-\varepsilon)(1-\zeta_{\bar{\Gamma}}) \geq (1-\varepsilon)(\operatorname{dist}^2(\cdot, \bar{\Gamma}) \wedge 1)$$

on $B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_l}(t)$ for all $t \in [0, T]$, so that (178) follows again. Since the argument proceeds similarly in the second case of (181), we may conclude the proof. ■

The next step consists of providing the global definition of a suitable velocity field along which a smoothly evolving regular double bubble and our associated constructions are transported.

Construction 34 (Global extension of velocity vector field). Let $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ be a regular double bubble smoothly evolving by MCF in the sense of Definition 10 on a time interval $[0, T]$. Let $(\eta_{\bar{\Gamma}}, (\eta_{\bar{I}_{i,j}})_{i,j\in\{1,2,3\},i\neq j})$ be the partition of unity from the proof of Lemma 31. We then introduce a family of vector fields

$$B^{\bar{I}_{i,j}} : \bigcup_{t\in[0,T]} \operatorname{supp} \eta_{\bar{I}_{i,j}}(\cdot, t) \times \{t\} \to \mathbb{R}^3 \quad \text{for all } i, j \in \{1, 2, 3\},\ i \neq j, \tag{182}$$

$$B^{\bar{\Gamma}} : \bigcup_{t\in[0,T]} \operatorname{supp} \eta_{\bar{\Gamma}}(\cdot, t) \times \{t\} \to \mathbb{R}^3 \tag{183}$$

as follows: the velocity field $B^{\bar{\Gamma}}$ denotes the corresponding vector field from the proof of Proposition 30, whereas $B^{\bar{I}_{i,j}}$ is the velocity field from Construction 14 with tangential component chosen as in the proof of Proposition 30.

With this family of local vector fields in place, we now define a global velocity field

$$B := \eta_{\bar{\Gamma}} B^{\bar{\Gamma}} + \eta_{\bar{I}_{1,2}} B^{\bar{I}_{1,2}} + \eta_{\bar{I}_{2,3}} B^{\bar{I}_{2,3}} + \eta_{\bar{I}_{3,1}} B^{\bar{I}_{3,1}} \tag{184}$$

throughout $\mathbb{R}^3 \times [0, T]$. ◇

A crucial ingredient for the proof of estimates (1d) and (1e) are the following bounds on the advective derivatives of the partition of unity from Lemma 31:

Lemma 35. *Let the assumptions and notation of Construction 34 be in place. In particular, $(\eta_{\bar{\Gamma}}, (\eta_{\bar{I}_{i,j}})_{i,j\in\{1,2,3\},i\neq j})$ denotes the partition of unity from the proof of Lemma 31. Then, $B \in C_t^0 C_x^1(\mathbb{R}^3 \times [0, T] \setminus \bar{\Gamma})$ with corresponding estimate*

$$|B| + |\nabla B| \leq C \quad \text{in } \mathbb{R}^3 \times [0, T] \setminus \bar{\Gamma}. \tag{185}$$

Moreover, the velocity field B gives rise to an improved estimate on the advective derivative of the bulk cutoff in the form of

$$|\partial_t \eta_{\text{bulk}} + (B \cdot \nabla)\eta_{\text{bulk}}| \leq C(\operatorname{dist}^2(\cdot, \mathcal{I}) \wedge 1) \quad \text{in } \mathbb{R}^3 \times [0, T], \tag{186}$$

and similarly, for all pairwise distinct phases $i, j, k \in \{1,2,3\}$ *we have*

$$|\partial_t \eta_{\bar{I}_{k,i}} + (B\cdot\nabla)\eta_{\bar{I}_{k,i}}| \leq C(\operatorname{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1) \quad \text{in } \mathbb{R}^3 \times [0,T]. \tag{187}$$

The constant $C > 0$ *in estimates* (185)–(187) *depends only on the data of the smoothly evolving regular double bubble* $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ *on* $[0,T]$.

Proof. The proof is decomposed into three steps.

Step 1: Regularity estimates. The asserted qualitative regularity of the velocity field B together with the associated estimate (185) follow from its definition (see (184)), the regularity of the cutoff functions (see (145)), as well as the regularity of the local building blocks (see (182) and (183)) in the form of

$$\big|(B^{\bar{\Gamma}}, B^{\bar{I}_{i,j}})\big| + \big|\nabla(B^{\bar{\Gamma}}, B^{\bar{I}_{i,j}})\big| \leq C \tag{188}$$

which is a consequence of (20) and (26).

Step 2: Proof of (186). It holds that $\eta_{\mathrm{bulk}}(\cdot,t) \equiv 1$ outside of the space-time domain $B_{\hat{r}}(\bar{\Gamma}(t)) \cup \bigcup_{i,j\in\{1,2,3\}, i\neq j} \Psi_{i,j}(\bar{I}_{i,j}(t) \times \{t\} \times [-\bar{r},\bar{r}])$ for all $t \in [0,T]$, by construction. Hence, for a proof of estimate (186), we may restrict our attention to $\bigcup_{i,j\in\{1,2,3\}, i\neq j} \Psi_{i,j}(\bar{I}_{i,j}(t) \times \{t\} \times [-\bar{r},\bar{r}]) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$ and $B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$. By the choice of the localization scale $\bar{r}$ (see (144)), one may even argue separately on $\Psi_{i,j}(\bar{I}_{i,j}(t) \times \{t\} \times [-\bar{r},\bar{r}]) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$ for each pair of distinct phases $i, j \in \{1,2,3\}$ and all $t \in [0,T]$.

Substep 2.1: Proof of (186) *on* $\Psi_{i,j}(\bar{I}_{i,j}(t) \times \{t\} \times [-\bar{r},\bar{r}]) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$. It follows from representation (171) and definition (184) that $B = \eta_{\bar{I}_{i,j}} B^{\bar{I}_{i,j}}$ and

$$|\partial_t \eta_{\mathrm{bulk}} + (B\cdot\nabla)\eta_{\mathrm{bulk}}| \leq \big|\partial_t \zeta_{\bar{I}_{i,j}} + (B^{\bar{I}_{i,j}}\cdot\nabla)\zeta_{\bar{I}_{i,j}}\big| + \eta_{\mathrm{bulk}}\big|(B^{\bar{I}_{i,j}}\cdot\nabla)\zeta_{\bar{I}_{i,j}}\big| \tag{189}$$

throughout $\Psi_{i,j}(\bar{I}_{i,j}(t) \times \{t\} \times [-\bar{r},\bar{r}]) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$.

Recall that the signed distance $s_{i,j}$ satisfies

$$\partial_t s_{i,j} + (B^{\bar{I}_{i,j}}\cdot\nabla)s_{i,j} = 0 \quad \text{in } \operatorname{im}(\Psi_{i,j}), \tag{190}$$

as a consequence of the choice of the local velocity $B^{\bar{I}_{i,j}}$ (cf. Construction 34, Construction 14 and (24)). Hence, we infer from definition (157) and an application of the chain rule that

$$\partial_t \zeta_{\bar{I}_{i,j}} + (B^{\bar{I}_{i,j}}\cdot\nabla)\zeta_{\bar{I}_{i,j}} = 0 \quad \text{in } \operatorname{im}(\Psi_{i,j}). \tag{191}$$

For an estimate of the second right hand side term of (189), we simply make use of the upper bound for the bulk cutoff (see (149)) as well as regularity estimates (188) and (159) of $B^{\bar{I}_{i,j}}$ and $\zeta_{\bar{I}_{i,j}}$, respectively.

Substep 2.2: Proof of (186) *on* $B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{i,j}}(t)$. Throughout the interface wedge $W_{\bar{I}_{i,j}}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t))$, it holds that $B = \eta_{\bar{\Gamma}} B^{\bar{\Gamma}} + \eta_{\bar{I}_{i,j}} B^{\bar{I}_{i,j}}$, thanks to representation (172) and definition (184). We may then estimate, making use again of (172),

$$
\begin{aligned}
|\partial_t \eta_{\mathrm{bulk}} + (B\cdot\nabla)\eta_{\mathrm{bulk}}| \leq \big|\partial_t \zeta_{\bar{I}_{i,j}} + (B^{\bar{I}_{i,j}}\cdot\nabla)\zeta_{\bar{I}_{i,j}}\big| \\
+ \eta_{\mathrm{bulk}}\big|(B^{\bar{I}_{i,j}}\cdot\nabla)\zeta_{\bar{I}_{i,j}}\big| + \eta_{\bar{\Gamma}}\big|B^{\bar{\Gamma}} - B^{\bar{I}_{i,j}}\big||\nabla\zeta_{\bar{I}_{i,j}}| \qquad (192)
\end{aligned}
$$

on $W_{\bar{I}_{i,j}}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$. Thanks to (34), identity (191) is still applicable on an interface wedge. In particular, the first two right hand side terms of (192) can be estimated along the same lines as in *Substep 2.1*. The third right hand side term is of required order due to compatibility estimate (139), bound (38), and regularity estimate (159).

Substep 2.3: Proof of (186) *on* $B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t)$. Throughout $W_{\bar{\Omega}_i}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t))$, we may represent, as a consequence of identity (173), the global velocity defined by (184) in the form of $B = \eta_{\bar{\Gamma}} B^{\bar{\Gamma}} + \eta_{\bar{I}_{i,j}} B^{\bar{I}_{i,j}} + \eta_{\bar{I}_{k,i}} B^{\bar{I}_{k,i}}$. Plugging in (173) and adding zero twice then entails

$$
\begin{aligned}
|\partial_t \eta_{\mathrm{bulk}} + (B\cdot\nabla)\eta_{\mathrm{bulk}}| &\leq \big|\partial_t \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}} + (B\cdot\nabla)\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}\big||\zeta_{\bar{I}_{i,j}} - \zeta_{\bar{I}_{k,i}}| \\
&+ \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}\big|\partial_t\zeta_{\bar{I}_{i,j}} + (B^{\bar{I}_{i,j}}\cdot\nabla)\zeta_{\bar{I}_{i,j}}\big| + (1-\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}})\big|\partial_t\zeta_{\bar{I}_{i,k}} + (B^{\bar{I}_{k,i}}\cdot\nabla)\zeta_{\bar{I}_{k,i}}\big| \\
&+ \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}\eta_{\bar{\Gamma}}\big|B^{\bar{I}_{i,j}} - B^{\bar{\Gamma}}\big||\nabla\zeta_{\bar{I}_{i,j}}| + (1-\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}})\eta_{\bar{\Gamma}}\big|B^{\bar{I}_{k,i}} - B^{\bar{\Gamma}}\big||\nabla\zeta_{\bar{I}_{k,i}}| \\
&+ \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}\eta_{\bar{I}_{k,i}}\big|B^{\bar{I}_{i,j}} - B^{\bar{I}_{k,i}}\big||\nabla\zeta_{\bar{I}_{i,j}}| + (1-\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}})\eta_{\bar{I}_{i,j}}\big|B^{\bar{I}_{k,i}} - B^{\bar{I}_{i,j}}\big||\nabla\zeta_{\bar{I}_{k,i}}| \\
&+ \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}\eta_{\mathrm{bulk}}\big|(B^{\bar{I}_{i,j}}\cdot\nabla)\zeta_{\bar{I}_{i,j}}\big| + (1-\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}})\eta_{\mathrm{bulk}}\big|(B^{\bar{I}_{k,i}}\cdot\nabla)\zeta_{\bar{I}_{k,i}}\big|. \qquad (193)
\end{aligned}
$$

The last eight right hand side terms of (193) can be estimated by means of the same ingredients as in the previous two substeps, relying in the process also on (35) and (36). Hence, we focus only on the first right hand side term of (193). Since the difference $\zeta_{\bar{I}_{i,j}} - \zeta_{\bar{I}_{k,i}}$ vanishes to first-order at the triple line and has a bounded second-order spatial derivative within interpolation wedges, we have the bound

$$
|\zeta_{\bar{I}_{i,j}} - \zeta_{\bar{I}_{k,i}}| \leq C\,\mathrm{dist}^2(\cdot,\bar{\Gamma}) \qquad (194)
$$

on $W_{\bar{\Omega}_i}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$. Since the advective derivative of the interpolation parameter is bounded within interpolation wedges in the form of (111), we may add zero and exploit property (27) as well as regularity estimates (185) and (110) to obtain

$$
\big|\partial_t \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}} + (B\cdot\nabla)\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}\big| \leq C \qquad (195)
$$

throughout $W_{\bar{\Omega}_i}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$. Post-processing (194) by means of (36) thus entails (186) on $W_{\bar{\Omega}_i}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$.

Step 3: Proof of (187). Fix $i,j,k \in \{1,2,3\}$ such that $\{i,j,k\} = \{1,2,3\}$. Due to localization properties (152)–(154), the choice of the localization scale $\bar{r}$ (see (144)), and regularity estimates (145) and (185), estimate (187) is satisfied for trivial reasons outside of $B_{\hat{r}}(\bar{\Gamma}(t)) \cap (W_{\bar{\Omega}_k}(t) \cup W_{\bar{\Omega}_i}(t) \cup W_{\bar{I}_{k,i}}(t))$ for all $t \in [0,T]$.

Substep 3.1: Proof of (187) *on* $B_{\widehat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{k,i}}(t)$. Based on representation (172) as well as definition (162), it holds that $\eta_{\bar{I}_{k,i}} = (1-\zeta_{\bar{\Gamma}})(1-\eta_{\mathrm{bulk}})$ on $B_{\widehat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{k,i}}(t)$ for all $t \in [0,T]$. By an application of the product rule and the already established estimate (186) for the advective derivative of the bulk cutoff, we thus infer

$$\left|\partial_t \eta_{\bar{I}_{k,i}} + (B\cdot\nabla)\eta_{\bar{I}_{k,i}}\right| \leq |\partial_t \zeta_{\bar{\Gamma}} + (B\cdot\nabla)\zeta_{\bar{\Gamma}}| + C(\mathrm{dist}^2(\cdot,\bar{I}_{i,j}) \wedge 1)$$

on $B_{\widehat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{k,i}}(t)$ for all $t \in [0,T]$. Expressing $\mathrm{dist}(x,\bar{\Gamma}(t)) = |x - P_{\bar{\Gamma}}(x,t)|$ for all $x \in B_{\widehat{r}}(\bar{\Gamma}(t))$ and all $t \in [0,T]$, as well as recalling relations (112) and (27), we may compute

$$\begin{aligned}\partial_t \,\mathrm{dist}(x,\bar{\Gamma}(t)) &= -\frac{x - P_{\bar{\Gamma}}(x,t)}{|x - P_{\bar{\Gamma}}(x,t)|}\cdot B(P_{\bar{\Gamma}}(x,t),t) \\ &= -\big(B(P_{\bar{\Gamma}}(x,t),t)\cdot\nabla\big)\,\mathrm{dist}(x,\bar{\Gamma}(t)) \end{aligned} \tag{196}$$

for all $x \in B_{\widehat{r}}(\bar{\Gamma}(t)) \setminus \bar{\Gamma}(t)$ and all $t \in [0,T]$. It is now a consequence of the chain rule and regularity estimates (185) and (160) that

$$|\partial_t \zeta_{\bar{\Gamma}} + (B\cdot\nabla)\zeta_{\bar{\Gamma}}| \leq C(\mathrm{dist}^2(\cdot,\bar{\Gamma}) \wedge 1) \tag{197}$$

throughout $B_{\widehat{r}}(\bar{\Gamma}(t)) \setminus \bar{\Gamma}(t)$ for all $t \in [0,T]$. Post-processing the previous display by means of (37) then yields (187) on $B_{\widehat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{k,i}}(t)$ for all $t \in [0,T]$.

Substep 3.2: Proof of (187) *on* $B_{\widehat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t)$. Recall (163)–(164), that is, $\eta_{\bar{I}_{k,i}} = \lambda_{\bar{\Omega}_i}^{\bar{I}_{k,i}}(1-\zeta_{\bar{\Gamma}})\zeta_{\bar{I}_{k,i}}$ on $B_{\widehat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t)$ for all $t \in [0,T]$. It then directly follows from the product rule, the trivial estimate $1-\zeta_{\bar{\Gamma}} \leq C(\mathrm{dist}^2(\cdot,\bar{\Gamma}) \wedge 1)$, estimate (195) on the advective derivative of the interpolation function

$$\lambda_{\bar{\Omega}_i}^{\bar{I}_{k,i}} = 1 - \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}},$$

regularity estimates (159) and (185), estimate (197), and finally bound (36) that (187) holds true on $B_{\widehat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t)$ for all $t \in [0,T]$.

This concludes the proof of Lemma 35, since the argument on the other relevant interpolation wedge proceeds analogously. ∎

5.3. Approximate transport equations and motion by mean curvature

We shall now establish the validity of estimates (1d)–(1f) in terms of the global extensions $(\xi_{i,j})_{i,j\in\{1,2,3\},i\neq j}$ of the unit normal vector fields from Construction 32 and the global extension B of the velocity field from Construction 34.

Lemma 36. *Let the assumptions and notation from Construction 32 and Construction 34 be in place. There exists a constant $C>0$, which depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1,\bar{\Omega}_2,\bar{\Omega}_3)$ on $[0,T]$, such that for all $i,j \in \{1,2,3\}$*

with $i \neq j$, it holds throughout $\mathbb{R}^3 \times [0,T]$ that

$$|\partial_t \xi_{i,j} + (B\cdot\nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}}\xi_{i,j}| \leq C(\mathrm{dist}(\cdot,\bar{I}_{i,j}) \wedge 1), \tag{198}$$

$$|B\cdot\xi_{i,j} + \nabla\cdot\xi_{i,j}| \leq C(\mathrm{dist}(\cdot,\bar{I}_{i,j}) \wedge 1), \tag{199}$$

$$|\xi_{i,j}\cdot(\partial_t \xi_{i,j} + (B\cdot\nabla)\xi_{i,j})| \leq C(\mathrm{dist}^2(\cdot,\bar{I}_{i,j}) \wedge 1). \tag{200}$$

Proof. The main point of the proof is the reduction to the corresponding assertions on the level of the local constructions $(\xi_{i,j}^{\bar{I}_{i,j}}, B^{\bar{I}_{i,j}})$ at two-phase interfaces (see Lemma 15) and the local construction $(\xi^{\bar{\Gamma}}, B^{\bar{\Gamma}})$ at a triple line (see Proposition 16). The reduction argument is facilitated by an interplay of estimates (146)–(151) (resp. (186) and (187)) with sufficient compatibility of the local and global constructions. We list and prove the required compatibility estimates in a first step before starting with the proof of bounds (198)–(200).

Step 1: Compatibility estimates. We claim that for all $i,j \in \{1,2,3\}$ with $i \neq j$ it holds on $\mathbb{R}^3 \times [0,T]$ that

$$\mathbb{1}_{\mathrm{supp}\,\eta_{\bar{I}_{i,j}}}\big|\xi_{i,j} - \xi_{i,j}^{\bar{I}_{i,j}}\big| + \mathbb{1}_{\mathrm{supp}\,\eta_{\bar{\Gamma}}}\big|\xi_{i,j} - \xi_{i,j}^{\bar{\Gamma}}\big| \leq C(\mathrm{dist}(\cdot,\bar{I}_{i,j}) \wedge 1), \tag{201}$$

$$\mathbb{1}_{\mathrm{supp}\,\eta_{\bar{I}_{i,j}}}\big|B - B^{\bar{I}_{i,j}}\big| + \mathbb{1}_{\mathrm{supp}\,\eta_{\bar{\Gamma}}}\big|B - B^{\bar{\Gamma}}\big| \leq C(\mathrm{dist}(\cdot,\bar{I}_{i,j}) \wedge 1), \tag{202}$$

$$\begin{aligned}\mathbb{1}_{\mathrm{supp}\,\eta_{\bar{I}_{i,j}}}\big|(\nabla B - \nabla B^{\bar{I}_{i,j}})^{\mathsf{T}}\xi_{i,j}^{\bar{I}_{i,j}}\big| + \mathbb{1}_{\mathrm{supp}\,\eta_{\bar{\Gamma}}}\big|(\nabla B - \nabla B^{\bar{\Gamma}})^{\mathsf{T}}\xi_{i,j}^{\bar{\Gamma}}\big| \\ \leq C(\mathrm{dist}(\cdot,\bar{I}_{i,j}) \wedge 1),\end{aligned} \tag{203}$$

$$\begin{aligned}\mathbb{1}_{\mathrm{supp}\,\eta_{\bar{I}_{i,j}}}\big|\big(\xi_{i,j} - \xi_{i,j}^{\bar{I}_{i,j}}\big)\cdot\xi_{i,j}^{\bar{I}_{i,j}}\big| + \mathbb{1}_{\mathrm{supp}\,\eta_{\bar{\Gamma}}}\big|\big(\xi_{i,j} - \xi_{i,j}^{\bar{\Gamma}}\big)\cdot\xi_{i,j}^{\bar{\Gamma}}\big| \\ \leq C(\mathrm{dist}^2(\cdot,\bar{I}_{i,j}) \wedge 1),\end{aligned} \tag{204}$$

$$\begin{aligned}\mathbb{1}_{\mathrm{supp}\,\eta_{\bar{I}_{i,j}}}\big|\xi_{i,j}^{\bar{I}_{i,j}}\cdot\big((B - B^{\bar{I}_{i,j}})\cdot\nabla\big)\xi_{i,j}^{\bar{I}_{i,j}}\big| + \mathbb{1}_{\mathrm{supp}\,\eta_{\bar{\Gamma}}}\big|\xi_{i,j}^{\bar{\Gamma}}\cdot\big((B - B^{\bar{\Gamma}})\cdot\nabla\big)\xi_{i,j}^{\bar{\Gamma}}\big| \\ \leq C(\mathrm{dist}^2(\cdot,\bar{I}_{i,j}) \wedge 1).\end{aligned} \tag{205}$$

For a proof of these compatibility estimates, we only focus on the respective first left hand side terms. The proof for the second left hand side terms follows along the same lines while switching the roles of $\bar{I}_{i,j}$ and $\bar{\Gamma}$ in the process.

Inserting definition (176) and exploiting estimate (146) yields

$$\xi_{i,j} - \xi_{i,j}^{\bar{I}_{i,j}} = \eta_{\bar{\Gamma}}(\xi_{i,j}^{\bar{\Gamma}} - \xi_{i,j}^{\bar{I}_{i,j}}) - \eta_{\mathrm{bulk}}\xi_{i,j}^{\bar{I}_{i,j}} + O(\mathrm{dist}^2(\cdot,\bar{I}_{i,j}) \wedge 1)$$

on $\mathrm{supp}\,\eta_{\bar{I}_{i,j}}$. Hence, we obtain the asserted bound (201) thanks to estimates (137) and (149).

Next, definition (184) together with estimates (139), (146), (149) and (188) implies

$$B - B^{\bar{I}_{i,j}} = \eta_{\bar{\Gamma}}(B^{\bar{\Gamma}} - B^{\bar{I}_{i,j}}) - \eta_{\mathrm{bulk}}B^{\bar{I}_{i,j}} + O(\mathrm{dist}(\cdot,\bar{I}_{i,j}) \wedge 1) = O(\mathrm{dist}(\cdot,\bar{I}_{i,j}) \wedge 1)$$

on $\mathrm{supp}\,\eta_{\bar{I}_{i,j}}$, as required.

Moreover, it holds on supp $\eta_{\bar{I}_{i,j}}$ as a consequence of definition (184), the product rule, the already established compatibility estimate (202), as well as estimates (146), (147) and (188) that

$$\begin{aligned}(\nabla B - \nabla B^{\bar{I}_{i,j}})^{\mathsf{T}} \xi^{\bar{I}_{i,j}}_{i,j} &= \eta_{\bar{\Gamma}} (\nabla B^{\bar{\Gamma}} - \nabla B^{\bar{I}_{i,j}})^{\mathsf{T}} \xi^{\bar{I}_{i,j}}_{i,j} - \eta_{\mathrm{bulk}} (\nabla B^{\bar{I}_{i,j}})^{\mathsf{T}} \xi^{\bar{I}_{i,j}}_{i,j} \\ &\quad + (B^{\bar{\Gamma}} \cdot \xi^{\bar{I}_{i,j}}_{i,j}) \nabla \eta_{\bar{\Gamma}} + (B^{\bar{I}_{i,j}} \cdot \xi^{\bar{I}_{i,j}}_{i,j}) \nabla \eta_{\bar{I}_{i,j}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1) \\ &= \eta_{\bar{\Gamma}} (\nabla B^{\bar{\Gamma}} - \nabla B^{\bar{I}_{i,j}})^{\mathsf{T}} \xi^{\bar{I}_{i,j}}_{i,j} \\ &\quad - (B \cdot \xi^{\bar{I}_{i,j}}_{i,j}) \nabla \eta_{\mathrm{bulk}} - \eta_{\mathrm{bulk}} (\nabla B^{\bar{I}_{i,j}})^{\mathsf{T}} \xi^{\bar{I}_{i,j}}_{i,j} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1).\end{aligned}$$

The previous display in turn implies (203) in view of bounds (140), (149), (150), (188) and (185).

By the argument for (201), we also have

$$(\xi_{i,j} - \xi^{\bar{I}_{i,j}}_{i,j}) \cdot \xi^{\bar{I}_{i,j}}_{i,j} = \eta_{\bar{\Gamma}} (\xi^{\bar{\Gamma}}_{i,j} - \xi^{\bar{I}_{i,j}}_{i,j}) \cdot \xi^{\bar{I}_{i,j}}_{i,j} - \eta_{\mathrm{bulk}} |\xi^{\bar{I}_{i,j}}_{i,j}|^2 + O(\mathrm{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1)$$

on supp $\eta_{\bar{I}_{i,j}}$. Hence, we deduce from (138) and (149) that (204) holds true.

Finally, based on definition (184) and estimates (146), (179) and (188), we may bound on supp $\eta_{\bar{I}_{i,j}}$

$$\begin{aligned}&\xi^{\bar{I}_{i,j}}_{i,j} \cdot \big((B - B^{\bar{I}_{i,j}}) \cdot \nabla\big) \xi^{\bar{I}_{i,j}}_{i,j} \\ &\quad = \eta_{\bar{\Gamma}} (\xi^{\bar{I}_{i,j}}_{i,j} - \xi^{\bar{\Gamma}}_{i,j}) \cdot \big((B^{\bar{\Gamma}} - B^{\bar{I}_{i,j}}) \cdot \nabla\big) \xi^{\bar{I}_{i,j}}_{i,j} + \eta_{\bar{\Gamma}} \xi^{\bar{\Gamma}}_{i,j} \cdot \big((B^{\bar{\Gamma}} - B^{\bar{I}_{i,j}}) \cdot \nabla\big) \xi^{\bar{I}_{i,j}}_{i,j} \\ &\qquad - \eta_{\mathrm{bulk}} \xi^{\bar{I}_{i,j}}_{i,j} \cdot (B^{\bar{I}_{i,j}} \cdot \nabla) \xi^{\bar{I}_{i,j}}_{i,j} + O(\mathrm{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1),\end{aligned}$$

so that (137), (139), (149), (179) and (188) entail the desired estimate (205).

Step 2: Proof of (198). For the sake of brevity, from now on we refrain from explicitly spelling out the application of regularity estimates (177), (179), (185) or (188), and thus solely concentrate on the error contributions in terms of the distance to the interface $\bar{I}_{i,j}$.

We start estimating based on definition (176), the product rule, as well as bounds (146) and (148)

$$\partial_t \xi_{i,j} = \eta_{\bar{\Gamma}} \partial_t \xi^{\bar{\Gamma}}_{i,j} + \eta_{\bar{I}_{i,j}} \partial_t \xi^{\bar{I}_{i,j}}_{i,j} + \xi^{\bar{\Gamma}}_{i,j} \partial_t \eta_{\bar{\Gamma}} + \xi^{\bar{I}_{i,j}}_{i,j} \partial_t \eta_{\bar{I}_{i,j}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1).$$

As a consequence of compatibility estimate (201) and bounds (148), we may add zero twice and obtain

$$\begin{aligned}\xi^{\bar{\Gamma}}_{i,j} \partial_t \eta_{\bar{\Gamma}} + \xi^{\bar{I}_{i,j}}_{i,j} \partial_t \eta_{\bar{I}_{i,j}} &= \xi_{i,j} (\partial_t \eta_{\bar{\Gamma}} + \partial_t \eta_{\bar{I}_{i,j}}) + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1) \\ &= -\xi_{i,j} \partial_t \eta_{\mathrm{bulk}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1).\end{aligned}$$

The previous two displays combine to give

$$\partial_t \xi_{i,j} = \eta_{\bar{\Gamma}} \partial_t \xi^{\bar{\Gamma}}_{i,j} + \eta_{\bar{I}_{i,j}} \partial_t \xi^{\bar{I}_{i,j}}_{i,j} - \xi_{i,j} \partial_t \eta_{\mathrm{bulk}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1). \tag{206}$$

Replacing the differential operator ∂_t by $(B\cdot\nabla)$ in the previous argument entails

$$\begin{aligned}(B\cdot\nabla)\xi_{i,j} &= \eta_{\overline{\Gamma}}(B\cdot\nabla)\xi^{\overline{\Gamma}}_{i,j} + \eta_{\overline{I}_{i,j}}(B\cdot\nabla)\xi^{\overline{I}_{i,j}}_{i,j}\\ &\quad - \xi_{i,j}(B\cdot\nabla)\eta_{\mathrm{bulk}} + O(\mathrm{dist}(\cdot,\overline{I}_{i,j})\wedge 1).\end{aligned}$$

Making use of compatibility estimate (202) updates the previous display to

$$\begin{aligned}(B\cdot\nabla)\xi_{i,j} &= \eta_{\overline{\Gamma}}(B^{\overline{\Gamma}}\cdot\nabla)\xi^{\overline{\Gamma}}_{i,j} + \eta_{\overline{I}_{i,j}}(B^{\overline{I}_{i,j}}\cdot\nabla)\xi^{\overline{I}_{i,j}}_{i,j}\\ &\quad - \xi_{i,j}(B\cdot\nabla)\eta_{\mathrm{bulk}} + O(\mathrm{dist}(\cdot,\overline{I}_{i,j})\wedge 1).\end{aligned}\tag{207}$$

Furthermore, inserting definition (176), recalling estimate (146), and adding zero based on compatibility estimate (203) allows to estimate

$$\begin{aligned}(\nabla B)^{\mathsf{T}}\xi_{i,j} &= \eta_{\overline{\Gamma}}(\nabla B)^{\mathsf{T}}\xi^{\overline{\Gamma}}_{i,j} + \eta_{\overline{I}_{i,j}}(\nabla B)^{\mathsf{T}}\xi^{\overline{I}_{i,j}}_{i,j} + O(\mathrm{dist}(\cdot,\overline{I}_{i,j})\wedge 1)\\ &= \eta_{\overline{\Gamma}}(\nabla B^{\overline{\Gamma}})^{\mathsf{T}}\xi^{\overline{\Gamma}}_{i,j} + \eta_{\overline{I}_{i,j}}(\nabla B^{\overline{I}_{i,j}})^{\mathsf{T}}\xi^{\overline{I}_{i,j}}_{i,j} + O(\mathrm{dist}(\cdot,\overline{I}_{i,j})\wedge 1).\end{aligned}\tag{208}$$

The desired estimate (198) thus follows from (206)–(208), estimate (186) of the advective derivative of the bulk cutoff, as well as the local versions (21) and (28) of (198), respectively.

Step 3: Proof of (199). We compute as a consequence of definition (176), estimate (146), and compatibility estimate (202)

$$\begin{aligned}B\cdot\xi_{i,j} &= \eta_{\overline{\Gamma}}B\cdot\xi^{\overline{\Gamma}}_{i,j} + \eta_{\overline{I}_{i,j}}B\cdot\xi^{\overline{I}_{i,j}}_{i,j} + O(\mathrm{dist}(\cdot,\overline{I}_{i,j})\wedge 1)\\ &= \eta_{\overline{\Gamma}}B^{\overline{\Gamma}}\cdot\xi^{\overline{\Gamma}}_{i,j} + \eta_{\overline{I}_{i,j}}B^{\overline{I}_{i,j}}\cdot\xi^{\overline{I}_{i,j}}_{i,j} + O(\mathrm{dist}(\cdot,\overline{I}_{i,j})\wedge 1).\end{aligned}\tag{209}$$

We also directly estimate by means of definition (176), estimate (147), as well as compatibility estimate (201)

$$\begin{aligned}\nabla\cdot\xi_{i,j} &= \eta_{\overline{\Gamma}}\nabla\cdot\xi^{\overline{\Gamma}}_{i,j} + \eta_{\overline{I}_{i,j}}\nabla\cdot\xi^{\overline{I}_{i,j}}_{i,j} + (\xi^{\overline{\Gamma}}_{i,j}\cdot\nabla)\eta_{\overline{\Gamma}} + (\xi^{\overline{I}_{i,j}}_{i,j}\cdot\nabla)\eta_{\overline{I}_{i,j}} + O(\mathrm{dist}(\cdot,\overline{I}_{i,j})\wedge 1)\\ &= \eta_{\overline{\Gamma}}\nabla\cdot\xi^{\overline{\Gamma}}_{i,j} + \eta_{\overline{I}_{i,j}}\nabla\cdot\xi^{\overline{I}_{i,j}}_{i,j} - (\xi_{i,j}\cdot\nabla)\eta_{\mathrm{bulk}} + O(\mathrm{dist}(\cdot,\overline{I}_{i,j})\wedge 1).\end{aligned}\tag{210}$$

Hence, estimate (199) follows by combining (209)–(210), estimate (150) for the bulk cutoff, and the local versions of (199) given by (22) and (29), respectively.

Step 4: Proof of (200). Plugging in definition (176), recalling estimate (146), and denoting by $k\in\{1,2,3\}\setminus\{i,j\}$ the remaining phase yields

$$\begin{aligned}\xi_{i,j}\cdot\partial_t\xi_{i,j} &= \eta_{\overline{\Gamma}}\xi^{\overline{\Gamma}}_{i,j}\cdot\partial_t\xi_{i,j} + \eta_{\overline{I}_{i,j}}\xi^{\overline{I}_{i,j}}_{i,j}\cdot\partial_t\xi_{i,j} + O(\mathrm{dist}^2(\cdot,\overline{I}_{i,j})\wedge 1)\\ &= \eta^2_{\overline{\Gamma}}\xi^{\overline{\Gamma}}_{i,j}\cdot\partial_t\xi^{\overline{\Gamma}}_{i,j} + \eta^2_{\overline{I}_{i,j}}\xi^{\overline{I}_{i,j}}_{i,j}\cdot\partial_t\xi^{\overline{I}_{i,j}}_{i,j}\\ &\qquad + \eta_{\overline{\Gamma}}\eta_{\overline{I}_{i,j}}\xi^{\overline{\Gamma}}_{i,j}\cdot\partial_t\xi^{\overline{I}_{i,j}}_{i,j} + \eta_{\overline{\Gamma}}\eta_{\overline{I}_{i,j}}\xi^{\overline{I}_{i,j}}_{i,j}\cdot\partial_t\xi^{\overline{\Gamma}}_{i,j}\end{aligned}$$

$$
\begin{aligned}
&+ \eta_{\bar\Gamma}\xi^{\bar\Gamma}_{i,j} \cdot \big(\xi^{\bar\Gamma}_{i,j}\partial_t\eta_{\bar\Gamma} + \xi^{\bar I_{i,j}}_{i,j}\partial_t\eta_{\bar I_{i,j}} + \xi^{\bar I_{j,k}}_{j,k}\partial_t\eta_{\bar I_{j,k}} + \xi^{\bar I_{k,i}}_{k,i}\partial_t\eta_{\bar I_{k,i}}\big) \\
&+ \eta_{\bar I_{i,j}}\xi^{\bar I_{i,j}}_{i,j} \cdot \big(\xi^{\bar\Gamma}_{i,j}\partial_t\eta_{\bar\Gamma} + \xi^{\bar I_{i,j}}_{i,j}\partial_t\eta_{\bar I_{i,j}} + \xi^{\bar I_{j,k}}_{j,k}\partial_t\eta_{\bar I_{j,k}} + \xi^{\bar I_{k,i}}_{k,i}\partial_t\eta_{\bar I_{k,i}}\big) \\
&+ O(\mathrm{dist}^2(\cdot,\bar I_{i,j}) \wedge 1).
\end{aligned}
$$

Compatibility estimates (201) and (204) in combination with bounds (146), and (149) provide an upgrade of the previous display in the form of

$$
\begin{aligned}
\xi_{i,j} \cdot \partial_t\xi_{i,j} = {}& \eta^2_{\bar\Gamma}\xi^{\bar\Gamma}_{i,j} \cdot \partial_t\xi^{\bar\Gamma}_{i,j} + \eta^2_{\bar I_{i,j}}\xi^{\bar I_{i,j}}_{i,j} \cdot \partial_t\xi^{\bar I_{i,j}}_{i,j} \\
&+ \eta_{\bar\Gamma}\eta_{\bar I_{i,j}}\xi^{\bar\Gamma}_{i,j} \cdot \partial_t\xi^{\bar I_{i,j}}_{i,j} + \eta_{\bar\Gamma}\eta_{\bar I_{i,j}}\xi^{\bar I_{i,j}}_{i,j} \cdot \partial_t\xi^{\bar\Gamma}_{i,j} \\
&+ \eta_{\bar\Gamma}(\xi^{\bar\Gamma}_{i,j} \cdot \xi_{i,j})\partial_t(\eta_{\bar\Gamma}+\eta_{\bar I_{i,j}}) \\
&+ \eta_{\bar I_{i,j}}(\xi^{\bar I_{i,j}}_{i,j} \cdot \xi_{i,j})\partial_t(\eta_{\bar\Gamma}+\eta_{\bar I_{i,j}}) \\
&+ \eta_{\bar\Gamma}\xi^{\bar\Gamma}_{i,j} \cdot \big(\xi^{\bar I_{j,k}}_{j,k}\partial_t\eta_{\bar I_{j,k}} + \xi^{\bar I_{k,i}}_{k,i}\partial_t\eta_{\bar I_{k,i}}\big) \\
&+ \eta_{\bar I_{i,j}}\xi^{\bar I_{i,j}}_{i,j} \cdot \big(\xi^{\bar I_{j,k}}_{j,k}\partial_t\eta_{\bar I_{j,k}} + \xi^{\bar I_{k,i}}_{k,i}\partial_t\eta_{\bar I_{k,i}}\big) \\
&+ O(\mathrm{dist}^2(\cdot,\bar I_{i,j}) \wedge 1).
\end{aligned}
$$

Substituting the differential operator $(B\cdot\nabla)$ for ∂_t in the previous argument, making use of compatibility estimates (205), (201) and (202), and exploiting twice estimate (187) then shows that

$$
\begin{aligned}
\xi_{i,j} \cdot (\partial_t+B\cdot\nabla)\xi_{i,j} = {}& \eta^2_{\bar\Gamma}\xi^{\bar\Gamma}_{i,j} \cdot (\partial_t+B^{\bar\Gamma}\cdot\nabla)\xi^{\bar\Gamma}_{i,j} + \eta^2_{\bar I_{i,j}}\xi^{\bar I_{i,j}}_{i,j} \cdot (\partial_t+B^{\bar I_{i,j}}\cdot\nabla)\xi^{\bar I_{i,j}}_{i,j} \\
&+ \eta_{\bar\Gamma}\eta_{\bar I_{i,j}}\xi^{\bar\Gamma}_{i,j} \cdot (\partial_t+B^{\bar I_{i,j}}\cdot\nabla)\xi^{\bar I_{i,j}}_{i,j} + \eta_{\bar\Gamma}\eta_{\bar I_{i,j}}\xi^{\bar I_{i,j}}_{i,j} \cdot (\partial_t+B^{\bar\Gamma}\cdot\nabla)\xi^{\bar\Gamma}_{i,j} \\
&- \eta_{\bar\Gamma}(\xi^{\bar\Gamma}_{i,j} \cdot \xi_{i,j})(\partial_t+B\cdot\nabla)\eta_{\mathrm{bulk}} - \eta_{\bar I_{i,j}}(\xi^{\bar I_{i,j}}_{i,j} \cdot \xi_{i,j})(\partial_t+B\cdot\nabla)\eta_{\mathrm{bulk}} \\
&+ O(\mathrm{dist}^2(\cdot,\bar I_{i,j}) \wedge 1).
\end{aligned}
$$

Hence, employing the local versions (23) and (30) of (200) and making use of estimate (186) for the bulk cutoff shows that

$$
\begin{aligned}
\xi_{i,j} \cdot (\partial_t+B\cdot\nabla)\xi_{i,j} = {}& \eta_{\bar\Gamma}\eta_{\bar I_{i,j}}\xi^{\bar\Gamma}_{i,j} \cdot (\partial_t+B^{\bar I_{i,j}}\cdot\nabla)\xi^{\bar I_{i,j}}_{i,j} + \eta_{\bar\Gamma}\eta_{\bar I_{i,j}}\xi^{\bar I_{i,j}}_{i,j} \cdot (\partial_t+B^{\bar\Gamma}\cdot\nabla)\xi^{\bar\Gamma}_{i,j} \\
&+ O(\mathrm{dist}^2(\cdot,\bar I_{i,j}) \wedge 1).
\end{aligned} \tag{211}
$$

Adding zero, making use of the local evolution equations (21) and (23), and exploiting compatibility estimates (201) and (203) further implies that

$$
\begin{aligned}
&\eta_{\bar\Gamma}\eta_{\bar I_{i,j}}\xi^{\bar\Gamma}_{i,j} \cdot \big(\partial_t\xi^{\bar I_{i,j}}_{i,j} + (B^{\bar I_{i,j}}\cdot\nabla)\xi^{\bar I_{i,j}}_{i,j}\big) \\
&\quad = \eta_{\bar\Gamma}\eta_{\bar I_{i,j}}\xi^{\bar\Gamma}_{i,j} \cdot \big(\partial_t\xi^{\bar I_{i,j}}_{i,j} + (B^{\bar I_{i,j}}\cdot\nabla)\xi^{\bar I_{i,j}}_{i,j} + (\nabla B^{\bar I_{i,j}})^{\mathsf T}\xi^{\bar I_{i,j}}_{i,j}\big) - \eta_{\bar\Gamma}\eta_{\bar I_{i,j}}\xi^{\bar\Gamma}_{i,j} \cdot (\nabla B^{\bar I_{i,j}})^{\mathsf T}\xi^{\bar I_{i,j}}_{i,j} \\
&\quad = -\eta_{\bar\Gamma}\eta_{\bar I_{i,j}}\big(\xi^{\bar I_{i,j}}_{i,j} - \xi^{\bar\Gamma}_{i,j}\big)(\nabla B)^{\mathsf T}\xi^{\bar I_{i,j}}_{i,j} + O(\mathrm{dist}^2(\cdot,\bar I_{i,j}) \wedge 1).
\end{aligned}
$$

Switching the roles of $\bar{\Gamma}$ and $\bar{I}_{i,j}$ in the argument leading to the previous display, relying in the process on the local evolution equations (28) and (30), we then in summary obtain together with (201) that

$$\begin{aligned}
&\eta_{\bar{\Gamma}}\eta_{\bar{I}_{i,j}}\xi^{\bar{\Gamma}}_{i,j}\cdot\big(\partial_t\xi^{\bar{I}_{i,j}}_{i,j}+(B^{\bar{I}_{i,j}}\cdot\nabla)\xi^{\bar{I}_{i,j}}_{i,j}\big)+\eta_{\bar{\Gamma}}\eta_{\bar{I}_{i,j}}\xi^{\bar{I}_{i,j}}_{i,j}\cdot\big(\partial_t\xi^{\bar{\Gamma}}_{i,j}+(B^{\bar{\Gamma}}\cdot\nabla)\xi^{\bar{\Gamma}}_{i,j}\big)\\
&\quad=-\eta_{\bar{\Gamma}}\eta_{\bar{I}_{i,j}}\big(\xi^{\bar{I}_{i,j}}_{i,j}-\xi^{\bar{\Gamma}}_{i,j}\big)(\nabla B)^{\mathsf{T}}\big(\xi^{\bar{I}_{i,j}}_{i,j}-\xi^{\bar{\Gamma}}_{i,j}\big)+O(\mathrm{dist}^2(\cdot,\bar{I}_{i,j})\wedge 1)\\
&\quad=O(\mathrm{dist}^2(\cdot,\bar{I}_{i,j})\wedge 1).
\end{aligned}\tag{212}$$

The combination of estimates (211) and (212) thus entails bound (200). ∎

5.4. Existence of a gradient flow calibration: Proof of Theorem 3

This is only a matter of collecting already established facts. More precisely, the required regularity for $((\xi_{i,j})_{i,j\in\{1,2,3\},i\neq j}, B)$ is part of Lemma 33 and Lemma 35, respectively. The calibration (resp. extension) property (1a) as well as the coercivity estimate (1c) for the extensions of the unit normal vector fields follow from Lemma 33. Finally, we note that estimates (1d)–(1f) are the content of Lemma 36. ∎

6. Existence of transported weights: Proof of Proposition 5

Proof of Proposition 5. The proof proceeds in several steps.

Step 1: Construction of an auxiliary family of transported weights. We first fix a smooth truncation of the identity. More precisely, let $\vartheta:\mathbb{R}\to\mathbb{R}$ be a smooth and non-decreasing map such that $\vartheta(r)=r$ for $|r|\leq\frac{1}{2}$, $\vartheta(r)=1$ for $r\geq 1$ and $\vartheta(r)=-1$ for $r\leq -1$. Let $\hat{r}\in(0,1]$ be the localization scale of Proposition 16, let $\bar{r}\in(0,1]$ be the localization scale defined by (144), and let finally $\delta\in(0,1]$ be the constant from *Step 2* of the proof of Lemma 31 (cf. the defining property (166) for all $i,j\in\{1,2,3\}$, $i\neq j$). We then define building blocks

$$\vartheta_{i,j}:=\vartheta\Big(\frac{s_{i,j}}{\delta\bar{r}}\Big)\qquad\text{in } \mathrm{im}(\Psi_{i,j}),\tag{213}$$

$$\vartheta_{\mathrm{ext}}:=\vartheta\Big(\frac{\mathrm{dist}(\cdot,\bar{\Gamma})}{\hat{r}}\Big)\qquad\text{in } \mathbb{R}^3\times[0,T].\tag{214}$$

Note that by definition (144) of the localization scale $\bar{r}$, we have for all phases $i\in\{1,2,3\}$ a covering of $\partial\bar{\Omega}_i$ in the form of

$$\partial\bar{\Omega}_i\subset B_{\hat{r}}(\bar{\Gamma}(t))\cup\bigcup_{j\in\{1,2,3\},j\neq i}\mathrm{im}_{\bar{r}}(\Psi_{i,j})(t)\setminus B_{\hat{r}}(\bar{\Gamma}(t))=:\mathcal{N}^{\partial\bar{\Omega}_i}_{\hat{r},\bar{r}}(t),\tag{215}$$

for all $t\in[0,T]$, and where we abbreviated

$$\mathrm{im}_{\bar{r}}(\Psi_{i,j})(t):=\Psi_{i,j}(\bar{I}_{i,j}(t)\times\{t\}\times[-\bar{r},\bar{r}]),\quad t\in[0,T].$$

Note that this also implies a disjoint covering of $\mathbb{R}^3$ by means of

$$\mathbb{R}^3 = \mathcal{N}^{\partial\bar{\Omega}_i}_{\hat{r},\bar{r}}(t) \cup \big(\bar{\Omega}_i(t) \setminus \mathcal{N}^{\partial\bar{\Omega}_i}_{\hat{r},\bar{r}}(t)\big) \cup \big((\mathbb{R}^3 \setminus \bar{\Omega}_i(t)) \setminus \mathcal{N}^{\partial\bar{\Omega}_i}_{\hat{r},\bar{r}}(t)\big) \tag{216}$$

for all $t \in [0,T]$.

For each phase $i \in \{1,2,3\}$, denote by $j,k \in \{1,2,3\} \setminus \{i\}$ the remaining two phases. We then define, based on building blocks (213) and (214), an auxiliary weight $\widehat{\vartheta}_i : \mathbb{R}^3 \times [0,T] \to [-1,1]$ by

$$\widehat{\vartheta}_i(\cdot,t) := \vartheta_{i,\ell}(\cdot,t) \qquad \text{in } \mathrm{im}_{\bar{r}}(\Psi_{i,j})(t) \setminus B_{\hat{r}}(\bar{\Gamma}(t)),\ \ell \neq i, \tag{217}$$

$$\widehat{\vartheta}_i(\cdot,t) := \vartheta_{i,\ell}(\cdot,t) \qquad \text{in } \overline{W_{\bar{I}_{i,j}}(t)} \cap B_{\hat{r}}(\bar{\Gamma}(t)),\ \ell \neq i, \tag{218}$$

$$\widehat{\vartheta}_i(\cdot,t) := \lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}(\cdot,t)\vartheta_{i,j}(\cdot,t) + \lambda^{\bar{I}_{k,i}}_{\bar{\Omega}_i}(\cdot,t)\vartheta_{i,k}(\cdot,t) \qquad \text{in } W_{\bar{\Omega}_i}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t)), \tag{219}$$

$$\widehat{\vartheta}_i(\cdot,t) := \vartheta_{\mathrm{ext}}(\cdot,t) \qquad \text{in } \overline{W_{\bar{I}_{j,k}}(t)} \cap B_{\hat{r}}(\bar{\Gamma}(t)), \tag{220}$$

$$\widehat{\vartheta}_i(\cdot,t) := \lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_j}(\cdot,t)\vartheta_{i,j}(\cdot,t) + \lambda^{\bar{I}_{j,k}}_{\bar{\Omega}_j}(\cdot,t)\vartheta_{\mathrm{ext}}(\cdot,t) \qquad \text{in } W_{\bar{\Omega}_j}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t)), \tag{221}$$

$$\widehat{\vartheta}_i(\cdot,t) := \lambda^{\bar{I}_{k,i}}_{\bar{\Omega}_k}(\cdot,t)\vartheta_{i,k}(\cdot,t) + \lambda^{\bar{I}_{j,k}}_{\bar{\Omega}_k}(\cdot,t)\vartheta_{\mathrm{ext}}(\cdot,t) \qquad \text{in } W_{\bar{\Omega}_k}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t)), \tag{222}$$

$$\widehat{\vartheta}_i(\cdot,t) := -1 \qquad \text{in } \bar{\Omega}_i(t) \setminus \mathcal{N}^{\partial\bar{\Omega}_i}_{\hat{r},\bar{r}}(t), \tag{223}$$

$$\widehat{\vartheta}_i(\cdot,t) := 1 \qquad \text{else} \tag{224}$$

for all $t \in [0,T]$. For the construction and properties of the interpolation functions, we refer to Lemma 31. Note that $\widehat{\vartheta}_i$ is well-defined in view of (215), (216) and (33). Moreover, due to the defining property (166) of the constant $\delta \in (0,1]$, we infer that $\widehat{\vartheta}_i$ is continuous throughout $\mathbb{R}^3 \times [0,T]$.

Step 2: Properties of the auxiliary family of transported weights. In this step, we verify that the auxiliary family $\widehat{\vartheta} = (\widehat{\vartheta}_i)_{i\in\{1,2,3\}}$ satisfies all the requirements of Definition 4 with the (obvious) exception that $\widehat{\vartheta}_i \in L^1(\mathbb{R}^3 \times [0,T])$. The $W^{1,\infty}$-regularity on $\mathbb{R}^3 \times [0,T]$ as well as the required conditions from item (ii) of Definition 4 are immediate from definitions (217)–(224). Hence, we focus in the following on the deduction of advection estimate (2):

Substep 2.1: Preliminary estimates. We first claim that for all $i,j \in \{1,2,3\}$ with $i \neq j$ and all $t \in [0,T]$, it holds that

$$|\partial_t\vartheta_{i,j} + (B\cdot\nabla)\vartheta_{i,j}|(\cdot,t) \leq C\,\mathrm{dist}(\cdot,\partial\bar{\Omega}_i(t)) \quad \text{in } \mathrm{im}_{\bar{r}}(\Psi_{i,j})(t) \setminus B_{\hat{r}}(\bar{\Gamma}(t)), \tag{225}$$

$$|\partial_t\vartheta_{i,j} + (B\cdot\nabla)\vartheta_{i,j}|(\cdot,t) \leq C\,\mathrm{dist}(\cdot,\partial\bar{\Omega}_i(t)) \quad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap \big(W_{\bar{I}_{i,j}}(t) \cup W_{\bar{\Omega}_i}(t) \cup W_{\bar{\Omega}_j}(t)\big), \tag{226}$$

$$|\partial_t \vartheta_{\mathrm{ext}} + (B\cdot\nabla)\vartheta_{\mathrm{ext}}|(\cdot,t) \leq C\,\mathrm{dist}(\cdot,\bar{\Gamma}(t)) \quad \text{in } B_{\hat{r}}(\bar{\Gamma}(t))\setminus\bar{\Gamma}(t). \tag{227}$$

We start with a proof of (225). It follows from representation (171) and definition (184) that $B = \eta_{\bar{I}_{i,j}} B^{\bar{I}_{i,j}}$ in $\mathrm{im}_{\bar{r}}(\Psi_{i,j})(t)\setminus B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t\in[0,T]$. We may then estimate by the chain rule, definition (213), identity (190), representation (171), as well as estimate (149)

$$|\partial_t \vartheta_{i,j} + (B\cdot\nabla)\vartheta_{i,j}| \leq \eta_{\mathrm{bulk}}|(B^{\bar{I}_{i,j}}\cdot\nabla)\vartheta_{i,j}| \leq C\,\mathrm{dist}(\cdot,\bar{\Omega}_i)$$

throughout $\mathrm{im}_{\bar{r}}(\Psi_{i,j})(t)\setminus B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t\in[0,T]$.

We next prove (226). Throughout the interface wedge $W_{\bar{I}_{i,j}}(t)\cap B_{\hat{r}}(\bar{\Gamma}(t))$, it holds that $B = \eta_{\bar{\Gamma}}B^{\bar{\Gamma}} + \eta_{\bar{I}_{i,j}}B^{\bar{I}_{i,j}}$ thanks to representation (172) and definition (184). Employing (172) once more, we then estimate making also use of the chain rule, definition (213) and identity (190)

$$|\partial_t \vartheta_{i,j} + (B\cdot\nabla)\vartheta_{i,j}| \leq \eta_{\mathrm{bulk}}|(B^{\bar{I}_{i,j}}\cdot\nabla)\vartheta_{i,j}| + \eta_{\bar{\Gamma}}\big|\big((B^{\bar{\Gamma}} - B^{\bar{I}_{i,j}})\cdot\nabla\big)\vartheta_{i,j}\big|$$

on $W_{\bar{I}_{i,j}}(t)\cap B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t\in[0,T]$. Post-processing the previous display by means of (149), (139) and (38) thus yields (226) on $W_{\bar{I}_{i,j}}(t)\cap B_{\hat{r}}(\bar{\Gamma}(t))$, $t\in[0,T]$.

Throughout $W_{\bar{\Omega}_i}(t)\cap B_{\hat{r}}(\bar{\Gamma}(t))$, we may write, as a consequence of representation (173), the global velocity defined by (184) in the form of $B = \eta_{\bar{\Gamma}}B^{\bar{\Gamma}} + \eta_{\bar{I}_{i,j}}B^{\bar{I}_{i,j}} + \eta_{\bar{I}_{k,i}}B^{\bar{I}_{k,i}}$. Hence, based on the same ingredients as in the case of interface wedges, we may estimate

$$\begin{aligned}
&|\partial_t \vartheta_{i,j} + (B\cdot\nabla)\vartheta_{i,j}| \\
&\quad\leq \eta_{\mathrm{bulk}}|(B^{\bar{I}_{i,j}}\cdot\nabla)\vartheta_{i,j}| + \eta_{\bar{\Gamma}}\big|\big((B^{\bar{\Gamma}} - B^{\bar{I}_{i,j}})\cdot\nabla\big)\vartheta_{i,j}\big| + \eta_{\bar{I}_{k,i}}\big|\big((B^{\bar{I}_{k,i}} - B^{\bar{I}_{i,j}})\cdot\nabla\big)\vartheta_{i,j}\big|
\end{aligned}$$

on $W_{\bar{\Omega}_i}(t)\cap B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t\in[0,T]$. The previous display in turn upgrades to the desired estimate (226), thanks to (149), (139) and (36).

Finally, estimate (227) is a direct consequence of the chain rule, definition (214), identity (196) and regularity estimate (185).

Substep 2.2: Proof of (2) *in terms of* $(\hat{\vartheta}_i)_{i\in\{1,2,3\}}$. We first observe that as a consequence of definitions (217)–(224), there exists $C\geq 1$ such that

$$\frac{1}{C}|\hat{\vartheta}_i| \leq \mathrm{dist}(\cdot,\partial\bar{\Omega}_i) \leq C|\hat{\vartheta}_i| \quad \text{in } \mathbb{R}^3\times[0,T].$$

Modulo this post-processing, claim (2) in terms of $\hat{\vartheta}_i$ is then directly implied for regions (217), (218) and (220) by estimates (225)–(227) and (37). Furthermore, the only additional ingredients needed in the interpolation regions (219), (221) and (222) are given by estimate (195) for the interpolation functions as well as bound (36). Since there is nothing to prove for the regions (223) and (224), this in turn concludes the proof of (2) in terms of $(\hat{\vartheta}_i)_{i\in\{1,2,3\}}$.

Step 3: Enforcing integrability of the weights. We slightly modify the construction from the previous step to take care of the integrability issue. To this end, we first choose a smooth and concave function $\kappa : [0,\infty) \to [0,1]$ such that $\kappa(0) = 0$ as well as $\kappa(r) = 1$ for $r \geq 1$, which we think of as an upper concave approximation of the map $r \mapsto r \wedge 1$ on the interval $[0,\infty)$. Choose a sufficiently large radius $R > 0$ such that

$$\bigcup_{t\in[0,T]} \bigcup_{i,j\in\{1,2,3\}, i\neq j} B_{\hat{r}}(\bar{I}_{i,j}(t)) \times \{t\} \subset\subset B_R(0).$$

We then define a weight $\eta_R \in W^{1,\infty}(\mathbb{R}^3) \cap W^{1,1}(\mathbb{R}^3)$ by

$$\eta_R(x) := \kappa(\exp(R - |x|)), \quad x \in \mathbb{R}^3,$$

with its spatial gradient being bounded in the form of

$$|\nabla \eta_R| \leq C|\eta_R| \quad \text{in } \mathbb{R}^3.$$

With all of these ingredients in place, we may finally define $\vartheta_i := \eta_R \widehat{\vartheta}_i$ for all phases $i \in \{1,2,3\}$. Note that $\vartheta_i \in W^{1,1}(\mathbb{R}^3 \times [0,T]; [-1,1])$, as desired. Moreover, the weights ϑ_i directly inherit all the other required properties of Definition 4 from the auxiliary weights $\widehat{\vartheta}_i$ of the previous step, as can be seen from the definitions. ■

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Sebastian Hensel
Institute of Science and Technology Austria, Am Campus 1, 3400 Klosterneuburg, Austria; and Hausdorff Center for Mathematics, Universität Bonn, Endenicher Allee 62, 53115 Bonn, Germany; sebastian.hensel@ist.ac.at; sebastian.hensel@hcm.uni-bonn.de

Tim Laux
Hausdorff Center for Mathematics, Universität Bonn, Endenicher Allee 62, 53115 Bonn, Germany; tim.laux@hcm.uni-bonn.de