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On derivations of evolving surface Navier–Stokes equations

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Abstract. In recent literature several derivations of incompressible Navier–Stokes-type equations that model the dynamics of an evolving fluidic surface have been presented. These derivations differ in the physical principles used in the modeling approach and in the coordinate systems in which the resulting equations are represented. This is an overview paper in the sense that we put five different derivations of surface Navier–Stokes equations into one framework. This then allows a systematic comparison of the resulting surface Navier–Stokes equations and shows that some, but not all, of the resulting models are the same. Furthermore, based on a natural splitting approach in tangential and normal components of the velocity, we show that all five derivations that we consider yield the same tangential surface Navier–Stokes equations.

1. Introduction

Navier–Stokes-type equations posed on manifolds is a classical topic in analysis—for example, see [1, 7, 14, 26, 27]. In recent years there has been a strongly growing interest in surface Navier–Stokes equations, particularly in physical principles related to these equations and to tailor-made numerical discretization methods (see [3, 8–13, 15, 17, 17, 19, 20, 25]). One reason for this recent growing interest lies in the fact that these equations are used in the modeling of biological interfaces; see the overview paper [28] and the references therein.

In this paper, we focus on *derivations* of surface Navier–Stokes equations for *evolving* surfaces. In the past few years, several derivations have been presented in the literature [10–12,15,17] which differ in the physical principles used in the modeling approach and in the coordinate systems in which the resulting equations are represented. In [10, 11], mass and momentum conservation laws for material *surfaces* are used as basic physical principles, whereas in [15, 17] similar conservation laws of mass and momentum for a material *volume* are used and combined with a thin film technique. In [12], the derivation is based on energy minimization principles. Besides these differences in physical principles, there

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is also a difference in the representation of the resulting flow equations. In some papers, for example, [2, 10, 17, 18], local coordinate systems (curvilinear coordinates) are used, whereas in other literature [11, 12, 15] the standard Euclidean basis of $\mathbb{R}^3$, in which the evolving surface is embedded, is used. Such different coordinate systems lead to different representations of surface differential operators such as a covariant derivative or a surface divergence, and one has to be careful when comparing equations formulated in such different coordinate systems. Both the local curvilinear and the global Cartesian coordinate system have attractive properties. The local coordinate system can be very useful for modeling of more complex fluid properties, for example, in certain classes of fluid membranes [10,28] or in flows of liquid crystals [17,18]. The representation in global Cartesian coordinates is very convenient for the development of numerical simulation methods for these flow equations.

This is an overview paper in the sense that we put the different derivations of surface Navier–Stokes equations presented in [10–12, 15, 17] into one framework. Besides the unified survey of derivations, we also present the following (new) results:

(1) Precise relations of certain relevant differential operators, such as covariant derivatives and surface divergence operators, in different coordinate systems are given. Most of these can be found or are (implicitly) used at different places in the literature. Here, we put this into one framework and derive precise relations, for example, as in Theorem 3.8 and Lemma 3.11.

(2) The presentation in a unified framework allows a systematic comparison of the resulting surface Navier–Stokes equations. We will conclude that some of these are identical but also some are different.

(3) A splitting approach in tangential and normal components of the velocity is presented, which shows that all five derivations that we consider yield *the same tangential* surface Navier–Stokes equations.

Since the (incompressible) surface Navier–Stokes equations play a fundamental role in the modeling of interfaces or surfaces with fluidic behavior, we consider a good understanding of several known surface Navier–Stokes systems to be of major importance.

The remainder of this paper is organized as follows: In Section 2, we define evolving material surfaces. In Section 3, surface differential operators in different coordinate systems are defined and compared. Five derivations of surface Navier–Stokes equations, known from the literature, that differ in the underlying physical principles and in the coordinate systems used, are treated in Section 4. In Section 5, we discuss and compare these equations. In particular, a splitting of these equations in the tangential and the normal components is derived and it is shown that all five derivations result in the same tangential surface Navier–Stokes system.

2. Evolving material surfaces

We outline how evolving material surfaces are defined. A more precise formal description of the notion "material" is given in, for example, [16]. Let $\Gamma = \Gamma(0)$ be a smooth (at least C^2) connected surface embedded in $\mathbb{R}^3$. A material point $\mathbf{z} \in \Gamma(0)$ moves in time along a trajectory with coordinates $\mathbf{x}(\mathbf{z}, t) \in \mathbb{R}^3$ and a smooth velocity field $\mathbf{v}(\mathbf{x}(\mathbf{z}, t), t) \in \mathbb{R}^3$. For all $\mathbf{z} \in \Gamma(0)$, the solutions of the initial value problem

$$\begin{cases} \mathbf{x}(\mathbf{z}, 0) = \mathbf{z}, \\ \frac{d}{dt}\mathbf{x}(\mathbf{z}, t) = \mathbf{v}(\mathbf{x}(\mathbf{z}, t), t) \end{cases} \tag{2.1}$$

define the evolving surface

$$\Gamma(t) = \left\{ \mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = \mathbf{x}(\mathbf{z}, t),\ \mathbf{z} \in \Gamma(0) \right\}. \tag{2.2}$$

The flow map $\Phi_t : \Gamma(0) \to \Gamma(t)$, $0 \le t \le T$ is defined by $\Phi_t(\mathbf{z}) = \mathbf{x}(\mathbf{z}, t)$. Let $\Phi_U : \mathbb{R}^2 \supset U \to \Gamma(0)$ be a local parametrization. We assume that the mapping $\Phi_U : U \to \Phi_U(U)$ is a diffeomorphism. The coordinates in U are denoted by $\boldsymbol{\xi} = (\xi_1, \xi_2)$. Composition of Φ_U and Φ_t yields the mapping

$$R(\boldsymbol{\xi}, t) := \Phi_t(\Phi_U(\boldsymbol{\xi})), \tag{2.3}$$

which gives the position of the material point $\mathbf{y} = \Phi_U(\boldsymbol{\xi}) \in \Gamma(t) \subset \mathbb{R}^3$ at time t. In Section 3.1, we use $\boldsymbol{\xi} \to R(\boldsymbol{\xi}, t)$ as a (local) parametrization of $\Gamma(t)$. Note that if the flow field $\mathbf{v}$ is not identically zero, this parametrization is *non*-constant as a function of t, even if $\Gamma(t) = \Gamma(0)$ for all t.

The outward pointing normal vector on $\Gamma(t)$ is denoted by $\mathbf{n} = \mathbf{n}(\mathbf{y}, t)$, and $\mathbf{P} = \mathbf{P}(\mathbf{y}, t) = \mathbf{I} - \mathbf{n}\mathbf{n}^T$ is the projection on the tangential plane at $\mathbf{y} \in \Gamma(t)$. Throughout this paper, we often delete the argument $(\mathbf{y}, t)$ in the notation. For a vector field $\mathbf{u}$ on $\Gamma(t)$ we shall use throughout this paper the notation $\mathbf{u}_T = \mathbf{P}\mathbf{u}$ for the tangential component and $u_N = \mathbf{u} \cdot \mathbf{n}$ for the coordinate in normal direction, so that

$$\mathbf{u} = \mathbf{u}_T + u_N \mathbf{n} \quad \text{on } \Gamma(t). \tag{2.4}$$

If in the particle velocity $\mathbf{v}(\cdot, t) = \mathbf{v}_T(\cdot, t) + v_N(\cdot, t)\mathbf{n}(\cdot, t)$ we have $v_N(\cdot, t) = 0$ on $\Gamma(t)$, there is no normal velocity of the surface, which means that the geometry of $\Gamma(t)$ is stationary and there is only a tangential particle flow field.

We assume that on $\Gamma(t)$ there is a continuous strictly positive particle density distribution denoted by $\rho(\mathbf{y}, t)$, $\mathbf{y} \in \Gamma(t)$.

In Section 4, based on certain physical principles we derive Navier–Stokes-type equations that determine the particle velocity field $\mathbf{v}$ and the density distribution ρ. As discussed in the introduction we will compare derivations in different coordinate systems. Therefore, in the next section we collect results concerning representations of surface differential operators in different coordinate systems, which will be used in Section 4.

3. Coordinate systems and surface differential operators

In this section, we introduce surface differential operators in two different coordinate systems.

3.1. Coordinate systems

We treat representations of vector fields $\mathbf{u} : \Gamma \to \mathbb{R}^3$ and of operator-valued mappings $\mathbf{T} : \Gamma \to L(\mathbb{R}^3, \mathbb{R}^3)$, where $L(\mathbb{R}^3, \mathbb{R}^3)$ denotes the space of linear mappings $\mathbb{R}^3 \to \mathbb{R}^3$, in two different coordinate systems. The first one is the *Cartesian coordinate* system corresponding to the standard Euclidean basis in $\mathbb{R}^3$, denoted by $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$. The second one is a *curvilinear coordinate system*, that we introduce below. Most of the results presented in this section are standard material that can be found in many textbooks, such as [5,23]. We use tensor notation and the Einstein summation convention in the following way: using Latin indices $(i, j, k, \ldots)$ we sum over $1, 2, 3$, and using Greek indices $(\alpha, \beta, \gamma, \ldots)$ we sum up over $1, 2$. Partial derivatives with respect to the Cartesian coordinates ξ_α in the standard basis of $\mathbb{R}^2$ are denoted by $\partial_\alpha = \frac{\partial}{\partial \xi_\alpha}$.

In the remainder of this section we take a fixed t. The local parametrization of $\Gamma = \Gamma(t)$ is given by $R(\boldsymbol{\xi}) = R(\boldsymbol{\xi}, t)$, $\boldsymbol{\xi} \in U$. Hence, $R(U) \subset \Gamma(t)$. We assume that this parametrization is an immersion, hence the matrix

$$\begin{pmatrix} \partial_1 R(\boldsymbol{\xi}) & \partial_2 R(\boldsymbol{\xi}) \end{pmatrix} \in \mathbb{R}^{3\times 2}$$

has rank two for each $\boldsymbol{\xi} \in U$. Each point $\mathbf{y} \in R(U)$ can be unambiguously written as $\mathbf{y} = R(\boldsymbol{\xi})$ with $\boldsymbol{\xi} \in U$. The two coordinates ξ_α of $\boldsymbol{\xi}$ are called *curvilinear* or *local coordinates* of $\mathbf{y} = R(\boldsymbol{\xi})$. We introduce the covariant basis of the tangent space at $\mathbf{y} = R(\boldsymbol{\xi}) \in \Gamma$ given by $\mathbf{g}_\alpha = \mathbf{g}_\alpha(\boldsymbol{\xi}) := \partial_\alpha R(\boldsymbol{\xi}) \in \mathbb{R}^3$. The components of the metric tensor (or first fundamental form) are defined by

$$g_{\alpha\beta}(\boldsymbol{\xi}) := \mathbf{g}_\alpha(\boldsymbol{\xi}) \cdot \mathbf{g}_\beta(\boldsymbol{\xi}). \tag{3.1}$$

The metric tensor is symmetric positive definite. The contravariant basis of the tangent plane $\mathbf{g}^\beta$ is defined by $\mathbf{g}_\alpha \cdot \mathbf{g}^\beta = \delta_\alpha^\beta$. Here, δ_α^β denotes the Kronecker symbol. The contravariant components of the metric tensor are defined by $g^{\alpha\beta}(\boldsymbol{\xi}) := \mathbf{g}^\alpha(\boldsymbol{\xi}) \cdot \mathbf{g}^\beta(\boldsymbol{\xi})$. The following relations hold:

$$\mathbf{g}_\alpha = g_{\alpha\beta}\mathbf{g}^\beta, \quad \mathbf{g}^\alpha = g^{\alpha\beta}\mathbf{g}_\beta, \quad g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta.$$

In order to have a basis of $\mathbb{R}^3$, we add to the covariant and contravariant basis a third vector, namely the normal vector (at $\mathbf{y} = R(\boldsymbol{\xi})$):

$$\mathbf{g}_3 = \mathbf{g}^3 := \mathbf{n} = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{\|\mathbf{g}_1 \times \mathbf{g}_2\|} = \frac{\mathbf{g}^1 \times \mathbf{g}^2}{\|\mathbf{g}^1 \times \mathbf{g}^2\|}.$$

Note that, given the first fundamental form, this determines the choice of the orientation of the normal vector $\mathbf{n}$. The vectors $\mathbf{g}_i$ and $\mathbf{g}^i$ for $i = 1, 2, 3$ each form a basis of $\mathbb{R}^3$.

We can (locally) interpret the basis functions $\mathbf{g}_i$ and $\mathbf{g}^i$ as functions defined on the surface: $\mathbf{g}_i(\mathbf{y}) := \mathbf{g}_i(R(\boldsymbol{\xi}))$, $\boldsymbol{\xi} \in U$. For presentation purposes, it is convenient to identify the (contravariant) Euclidean basis in $\mathbb{R}^3$ with its covariant one, that is, $\widehat{\mathbf{e}}^i := \widehat{\mathbf{e}}_i$, $i = 1, 2, 3$.

For a vector field $\mathbf{u} : \Gamma \to \mathbb{R}^3$ we introduce the representations

$$\mathbf{u} = u^i \mathbf{g}_i = u_i \mathbf{g}^i = \widehat{u}_i \widehat{\mathbf{e}}^i.$$

Note that $u_i = \mathbf{u} \cdot \mathbf{g}_i$, $u^i = \mathbf{u} \cdot \mathbf{g}^i$ and $\widehat{u}_i = \mathbf{u} \cdot \widehat{\mathbf{e}}_i$ hold. The u_i (u^i) are called covariant (contravariant) components or also local coordinates. The $\widehat{u}_i$ are the Cartesian coordinates.

For a representation of an operator-valued mapping $\mathbf{T} : \Gamma \to L(\mathbb{R}^3, \mathbb{R}^3)$, we use the tensor calculus format (cf. [5, Section 8.4]):

$$\mathbf{T} = T_{ij}(\mathbf{g}^i \otimes \mathbf{g}^j) = T^{ij}(\mathbf{g}_i \otimes \mathbf{g}_j) = \widehat{T}_{ij}(\widehat{\mathbf{e}^i} \otimes \widehat{\mathbf{e}}^j),$$

with the outer product given by $(\mathbf{u} \otimes \mathbf{v})\mathbf{w} = (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. These representations define corresponding matrices that are representations of the same linear operator in different bases. The matrix entries satisfy identities $T_{ij} = \mathbf{g}_i \cdot (\mathbf{T}\mathbf{g}_j)$, $T^{ij} = \mathbf{g}^i \cdot (\mathbf{T}\mathbf{g}^j)$, $\widehat{T}_{ij} = \widehat{\mathbf{e}}_i \cdot (\mathbf{T}\,\widehat{\mathbf{e}}_j)$, which are called covariant, contravariant and Cartesian components, respectively. We define the transposed linear operator $\mathbf{T}^T$ by the relation $\mathbf{T}\mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{T}^T \mathbf{w}$, where $\cdot$ denotes the Euclidean scalar product in $\mathbb{R}^3$. Tensors can also be represented using *mixed* components (cf. [5, Section 8.4]). For a *symmetric* linear operator $\mathbf{T}$ (i.e., one such that $\mathbf{T} = \mathbf{T}^T$), we introduce the mixed (between covariant and contravariant) matrix representation

$$\mathbf{T} = T^i_j(\mathbf{g}_i \otimes \mathbf{g}^j) = T^i_j(\mathbf{g}^j \otimes \mathbf{g}_i).$$

The relations $T^i_j = \mathbf{g}^i \cdot (\mathbf{T}\mathbf{g}_j) = \mathbf{g}_j \cdot (\mathbf{T}\mathbf{g}^i)$ hold.

For a symmetric linear operator $\mathbf{T}$, the sum of its eigenvalues is denoted by $\mathrm{tr}(\mathbf{T})$. Since eigenvalues are invariant under basis transformations, we have $\mathrm{tr}(\mathbf{T}) = T_{ii} = \widehat{T}_{ii} = T^i_i$.

The projection operator given by $\mathbf{P} = \mathbf{I} - \mathbf{n}\mathbf{n}^T$ is defined in local coordinates by $\mathbf{P}(c^i \mathbf{g}_i) := c^\alpha \mathbf{g}_\alpha$. In local coordinates, the splitting given in (2.4) takes the form $\mathbf{u}_T := \mathbf{P}\mathbf{u} = u^\alpha \mathbf{g}_\alpha$ and $u_N = \mathbf{u} \cdot \mathbf{n} = u^3 = u_3$.

We recall the second fundamental form $\mathbf{B} = \mathbf{B}(\mathbf{y})$, $\mathbf{y} \in \Gamma$, also called Weingarten mapping or shape operator, which in terms of the covariant components is defined by (cf. [5, Theorems 8.13-1 and 8.14-1])

$$b_{\alpha\beta} = \mathbf{g}_3 \cdot \partial_\alpha \mathbf{g}_\beta = -\partial_\alpha \mathbf{g}_3 \cdot \mathbf{g}_\beta = b_{\beta\alpha}. \tag{3.2}$$

For this symmetric linear operator we have $\mathbf{B} = \mathbf{P}\mathbf{B}\mathbf{P}$. Hence, it follows that the equation $\mathbf{B} = b_{ij}(\mathbf{g}^i \otimes \mathbf{g}^j) = b_{\alpha\beta}(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta)$ holds. For the mixed components of the second fundamental form, the relation $b^\beta_\alpha = g^{\beta\sigma} b_{\sigma\alpha}$ holds. Let κ_1, κ_2 and 0 be the eigenvalues of $\mathbf{B}$. We introduce the (doubled) mean curvature $\kappa = \mathrm{tr}(\mathbf{B}) = \kappa_1 + \kappa_2$ and the Gaussian curvature $K = \kappa_1 \kappa_2$. The mean curvature can be represented in mixed components by $\kappa = b^\alpha_\alpha$.

3.2. Surface differential operators

In this section, we recall several surface differential operators. For a given t, we let $\phi : \Gamma = \Gamma(t) \to \mathbb{R}$ be a scalar function, $\mathbf{u} : \Gamma \to \mathbb{R}^3$ be a (not necessarily tangential) vector field and $\mathbf{T} : \Gamma \to L(\mathbb{R}^3, \mathbb{R}^3)$ an operator valued mapping. All are assumed to be at least C^1-smooth. We will study partial derivatives and gradients of ϕ, $\mathbf{u}$ and $\mathbf{T}$ and divergence operators for $\mathbf{u}$ and $\mathbf{T}$. Representations in different bases of $\mathbf{u} = \mathbf{u}(\mathbf{y})$ and $\mathbf{T} = \mathbf{T}(\mathbf{y})$, $\mathbf{y} \in \Gamma$ are considered. First, in Section 3.2.1, we recall standard definitions and results for derivatives in their local coordinates representation. Note that in this case the basis used in $\mathbb{R}^3$ depends on the (base) point $\mathbf{y}$. In Section 3.2.2, we list (standard) definitions for analogous gradient and divergence operators in case of representation in Cartesian coordinates in $\mathbb{R}^3$. In Section 3.2.3, we then derive relations between the corresponding operators in the different representations. In the last part of this section we introduce the material derivative in a direction along the moving surface, which can also be formulated both in curvilinear and Cartesian coordinates.

3.2.1. Surface differential operators in curvilinear coordinates. We recall some basic differential geometry concepts (e.g., [5, Chapter 8]). Note that we have the following representations in curvilinear coordinates: $\mathbf{u} = u^i \mathbf{g}_i = u_i \mathbf{g}^i$ and $\mathbf{T} = T^{ij}(\mathbf{g}_i \otimes \mathbf{g}_j) = T_{ij}(\mathbf{g}^i \otimes \mathbf{g}^j)$. All component functions are differentiable because the basis vectors $\mathbf{g}_i$ and $\mathbf{g}^i$ are smooth. Because R is an immersion, there exist uniquely defined functions $\overline{\phi} : U \to \mathbb{R}$, $\overline{\mathbf{u}} : U \to \mathbb{R}^3$ and $\overline{\mathbf{T}} : U \to L(\mathbb{R}^3, \mathbb{R}^3)$ such that the identities $\overline{\phi}(\boldsymbol{\xi}) = \phi(R(\boldsymbol{\xi}))$, $\overline{\mathbf{u}}(\boldsymbol{\xi}) = \mathbf{u}(R(\boldsymbol{\xi}))$ and $\overline{\mathbf{T}}(\boldsymbol{\xi}) = \mathbf{T}(R(\boldsymbol{\xi}))$ hold.

Definition 3.1. The *partial* derivatives ∂_α of the scalar function ϕ, the vector field $\mathbf{u}$ and the linear operator $\mathbf{T}$ are defined in terms of the corresponding functions $\overline{\phi}$, $\overline{\mathbf{u}}$ and $\overline{\mathbf{T}}$ by

$$\partial_\alpha \phi(\mathbf{y}) := \partial_\alpha \overline{\phi}(\boldsymbol{\xi}), \quad \partial_\alpha \mathbf{u}(\mathbf{y}) := \partial_\alpha \overline{\mathbf{u}}(\boldsymbol{\xi}), \quad \partial_\alpha \mathbf{T}(\mathbf{y}) := \partial_\alpha \overline{\mathbf{T}}(\boldsymbol{\xi}) \qquad \text{with } \mathbf{y} = R(\boldsymbol{\xi}).$$

We now derive representations of these partial derivatives in terms of a curvilinear coordinate system, which are used at several places in the remainder of this paper. For this we use the Christoffel symbols (cf. [5, Theorem 8.13-1])

$$\Gamma^\sigma_{\alpha\beta} := \mathbf{g}^\sigma \cdot \partial_\alpha \mathbf{g}_\beta = \Gamma^\sigma_{\beta\alpha}.$$

These symbols can also be formulated in terms of the metric tensor (cf. [5, Theorems 8.13-1 and 8.14-1]):

$$\Gamma^\sigma_{\alpha\beta} = \frac{1}{2} g^{\sigma\tau}(\partial_\beta g_{\alpha\tau} + \partial_\alpha g_{\beta\tau} - \partial_\tau g_{\alpha\beta}).$$

Representations of partial derivatives of vector fields in terms of a curvilinear coordinate system are given in the next theorem, taken from [5, Theorem 8.13-1]. We extend this theorem with an analogous result (see (3.4)) for partial derivatives of operator-valued functions $\mathbf{T} : \Gamma \to L(\mathbb{R}^3, T\Gamma)$, where $T\Gamma$ denotes the tangent bundle of Γ. A proof of result (3.4) is given in Appendix A.

Theorem 3.2. *For a vector field* $\mathbf{u}$, *the partial derivatives have the following representations:*

$$\begin{aligned}\partial_\alpha \mathbf{u} = \partial_\alpha(u_i \mathbf{g}^i) &= (\partial_\alpha u_\beta - \Gamma^\gamma_{\alpha\beta} u_\gamma - b_{\alpha\beta} u_3)\mathbf{g}^\beta + (\partial_\alpha u_3 + b^\beta_\alpha u_\beta)\mathbf{g}^3 \\ &= (u_{\beta|\alpha} - b_{\alpha\beta} u_3)\mathbf{g}^\beta + (u_{3|\alpha} + b^\beta_\alpha u_\beta)\mathbf{g}^3 \\ = \partial_\alpha(u^i \mathbf{g}_i) &= (\partial_\alpha u^\beta + \Gamma^\beta_{\alpha\gamma} u^\gamma - b^\beta_\alpha u^3)\mathbf{g}_\beta + (\partial_\alpha u^3 + b_{\alpha\beta} u^\beta)\mathbf{g}_3 \\ &= (u^\beta_{\ |\alpha} - b^\beta_\alpha u^3)\mathbf{g}_\beta + (u^3_{\ |\alpha} + b_{\beta\alpha} u^\beta)\mathbf{g}_3,\end{aligned} \tag{3.3}$$

where we use the abbreviations

$$u_{\beta|\alpha} := \partial_\alpha u_\beta - \Gamma^\gamma_{\alpha\beta} u_\gamma, \quad u^\beta_{\ |\alpha} := \partial_\alpha u^\beta + \Gamma^\beta_{\gamma\alpha} u^\gamma, \quad u_{3|\alpha} = u^3_{\ |\alpha} := \partial_\alpha u_3.$$

Let $\mathbf{T} = T^{\alpha\beta}(\mathbf{g}_\alpha \otimes \mathbf{g}_\beta) = T_{\alpha\beta}(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta)$ *be a function with values in* $L(\mathbb{R}^3, T\Gamma)$. *For the partial derivatives we have the representations*

$$\begin{aligned}\partial_\gamma \mathbf{T} &= T^{\alpha\beta}_{\ \ |\gamma}(\mathbf{g}_\alpha \otimes \mathbf{g}_\beta) + T^{\alpha\beta} b_{\gamma\alpha}(\mathbf{g}_3 \otimes \mathbf{g}_\beta) + T^{\alpha\beta} b_{\gamma\beta}(\mathbf{g}_\alpha \otimes \mathbf{g}_3) \\ &= T_{\alpha\beta|\gamma}(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) + T_{\alpha\beta} b^\alpha_\gamma(\mathbf{g}^3 \otimes \mathbf{g}^\beta) + T_{\alpha\beta} b^\beta_\gamma(\mathbf{g}^\alpha \otimes \mathbf{g}^3),\end{aligned} \tag{3.4}$$

where we use the abbreviations

$$T^{\alpha\beta}_{\ \ |\gamma} := \partial_\gamma T^{\alpha\beta} + \Gamma^\alpha_{\mu\gamma} T^{\mu\beta} + \Gamma^\beta_{\mu\gamma} T^{\alpha\mu}, \quad T_{\alpha\beta|\gamma} := \partial_\gamma T_{\alpha\beta} - \Gamma^\mu_{\alpha\gamma} T_{\mu\beta} - \Gamma^\mu_{\beta\gamma} T_{\alpha\mu}.$$

The relation $\mathbf{P}\partial_\alpha \mathbf{u} = u_{\beta|\alpha}\mathbf{g}^\beta$ for tangential vector fields $\mathbf{u}$ motivates the notation $u_{\beta|\alpha}$.

We now recall standard definitions of surface differential operators in curvilinear coordinates [23, 24].

Definition 3.3. For a scalar function $\phi \in C^1(\Gamma, \mathbb{R})$, the surface gradient is defined by

$$\nabla_\Gamma \phi := \partial_\alpha \phi \, \mathbf{g}^\alpha.$$

For a vector field $\mathbf{u} \in C^1(\Gamma, \mathbb{R}^3)$ we define the α-th partial covariant derivative $\nabla_\alpha \mathbf{u}$ and the covariant derivative $\nabla_\Gamma \mathbf{u}$ by

$$\nabla_\alpha \mathbf{u} := \mathbf{P}\partial_\alpha \mathbf{u}, \quad \nabla_\Gamma \mathbf{u} := \nabla_\alpha \mathbf{u} \otimes \mathbf{g}^\alpha.$$

The surface divergence of $\mathbf{u} \in C^1(\Gamma, \mathbb{R}^3)$ and $\mathbf{T} \in C^1(\Gamma, L(\mathbb{R}^3, \mathbb{R}^3))$ are defined by

$$\operatorname{div}_\Gamma \mathbf{u} := \partial_\alpha \mathbf{u} \cdot \mathbf{g}^\alpha, \quad \operatorname{div}_\Gamma \mathbf{T} := (\partial_\alpha \mathbf{T})^T \mathbf{g}^\alpha. \tag{3.5}$$

Note that there is a transpose in the definition of $\operatorname{div}_\Gamma \mathbf{T}$. The definitions of the surface gradient, covariant derivative and surface divergence operators above do not depend on the choice of the parametrization (cf. [23]).

Remark 3.4. Another surface differential operator for a vector field $\mathbf{u} \in C^1(\Gamma, \mathbb{R}^3)$ that plays a natural role in this setting is the *surface gradient* of $\mathbf{u}$, which is defined by $\nabla_S \mathbf{u} := \mathbf{g}^\alpha \otimes \partial_\alpha \mathbf{u}$ (cf. [23]). Note that it maps into the tangent bundle. It is related to the covariant derivative via $\nabla_\Gamma \mathbf{u} = \mathbf{P}\nabla_S^T \mathbf{u}$ (we use the notation $\nabla_S^T \mathbf{u} = (\nabla_S \mathbf{u})^T$). We use this surface gradient only in the proof of Theorem 3.8.

Remark 3.5. In [23, 24], the covariant derivative of *tangential* vector functions $\mathbf{u}$ is defined by $\nabla_\Gamma \mathbf{u} := \nabla_\alpha \mathbf{u} \otimes \mathbf{g}^\alpha$. In Definition 3.3 we extended this to general (not necessary tangential) vector fields.

Using results from Theorem 3.2 one obtains representations of the α-th partial covariant derivative in the covariant basis $\mathbf{g}_\alpha$ in terms of (derivatives of) the contravariant components u^i in $\mathbf{u} = u^i \mathbf{g}_i$:

$$\nabla_\alpha \mathbf{u} = \mathbf{P}\partial_\alpha \mathbf{u} = (\partial_\alpha u^\beta + \Gamma^\beta_{\alpha\gamma} u^\gamma - b^\beta_\alpha u^3)\mathbf{g}_\beta = (u^\beta_{\ |\alpha} - b^\beta_\alpha u^3)\mathbf{g}_\beta. \tag{3.6}$$

This result shows that the notation $u^\beta_{\ |\alpha}$ introduced in Theorem 3.2 is natural, in the sense that for tangential $\mathbf{u}$ we have $\nabla_\alpha \mathbf{u} = u^\beta_{\ |\alpha} \mathbf{g}_\beta$.

If $\mathbf{u}$ is tangential, the relation

$$\nabla_\Gamma \mathbf{u} = u_{\alpha|\beta} (\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) \tag{3.7}$$

holds, that is, the covariant components of $\nabla_\Gamma \mathbf{u}$ are given by $u_{\alpha|\beta}$. This induces an equivalent alternative definition of the covariant derivative that is sometimes used in the literature. An alternative definition of surface divergence of a vector field can be based on the relation

$$\mathrm{div}_\Gamma \mathbf{u} = u^\alpha_{\ |\alpha}. \tag{3.8}$$

In the next lemma, we present an analogous representation result for the divergence of an operator-valued function. A proof is given in Appendix A.

Lemma 3.6. *For* $\mathbf{T} = T^{\alpha\beta}(\mathbf{g}_\alpha \otimes \mathbf{g}_\beta)$*, the following holds:*

$$\mathrm{div}_\Gamma \mathbf{T} = T^{\alpha\beta}_{\ \ |\alpha} \mathbf{g}_\beta + T^{\alpha\beta} b_{\alpha\beta} \mathbf{g}_3.$$

3.2.2. Surface differential operators in Cartesian coordinates. We recall definitions of surface differential operators in terms of representations in Cartesian coordinates as in [11]. The partial derivatives with respect to the standard basis $\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3$ in $\mathbb{R}^3$, that is, $\mathbf{y} = y_i \widehat{\mathbf{e}}_i$, are denoted by $\widehat{\partial}_k := \frac{\partial}{\partial y_k}$. The gradient of a scalar function f with respect to the Cartesian coordinates is given by the vector $\widehat{\nabla} f := (\widehat{\partial}_i f)\widehat{\mathbf{e}}_i$. The gradient (Jacobian) of a vector-valued function $\mathbf{u}$ is given by $\widehat{\nabla}\mathbf{u} := \widehat{\partial}_k \mathbf{u} \otimes \widehat{\mathbf{e}}_k$, or in matrix notation $(\widehat{\nabla}\mathbf{u})_{ij} = \widehat{\partial}_j \mathbf{u}_i$. Note the structural analogy between $\widehat{\nabla}\mathbf{u} = \widehat{\partial}_k \mathbf{u} \otimes \widehat{\mathbf{e}}_k$ and the definition of the covariant derivative $\nabla_\Gamma = \mathbf{P}\partial_\alpha \mathbf{u} \otimes \mathbf{g}^\alpha$ (cf. Definition 3.3).

To define Cartesian *surface* differential operators based on Cartesian representations, we extend functions defined on the surface to a small open neighborhood $G_\delta(\Gamma) := G_\delta := \{\mathbf{x} \in \mathbb{R}^3 \mid \mathrm{dist}(\mathbf{x}, \Gamma) < \delta\}$ with some sufficiently small $\delta > 0$. For a given scalar function ϕ on Γ, a smooth extension to a function defined on G_δ is denoted by ϕ^e. Similar notation is used for vector fields and operator-valued functions on Γ. The specific choice of the extension is not essential; one may use a constant extension along normals. We now introduce surface differential operators based on the "Cartesian gradient" $\widehat{\nabla}$, applied to the extended quantities.

Definition 3.7. For a scalar function $\phi \in C^1(\Gamma, \mathbb{R})$, the surface gradient is defined by

$$\widehat{\nabla}_\Gamma \phi := \mathbf{P}\widehat{\nabla}\phi^e.$$

For a vector field $\mathbf{u} \in C^1(\Gamma, \mathbb{R}^3)$, we define the covariant derivative $\widehat{\nabla}_\Gamma \mathbf{u}$ by

$$\widehat{\nabla}_\Gamma \mathbf{u} := \mathbf{P}\widehat{\nabla}\mathbf{u}^e\mathbf{P}.$$

The surface divergence of $\mathbf{u} \in C^1(\Gamma, \mathbb{R}^3)$ and $\mathbf{T} \in C^1(\Gamma, L(\mathbb{R}^3, \mathbb{R}^3))$ is defined by

$$\widehat{\operatorname{div}}_\Gamma \mathbf{u} := \operatorname{tr}(\mathbf{P}\widehat{\nabla}\mathbf{u}^e\mathbf{P}), \quad \widehat{\operatorname{div}}_\Gamma \mathbf{T} := \widehat{\operatorname{div}}_\Gamma\big(\mathbf{T}^T\widehat{\mathbf{e}}_i\big)\widehat{\mathbf{e}}_i.$$

These definitions of the surface differential operators in Cartesian coordinates are independent of the choice of the extension and only depend on the function values on the surface. Note that the definition of the surface divergence of the operator-valued function $\mathbf{T}$ in Cartesian coordinates is based on the surface divergence of the vector field $\mathbf{T}^T\widehat{\mathbf{e}}_i$. In matrix notation, this means that we take the surface divergence of $\mathbf{T}$ row-wise, which agrees with the usual definition in the literature (cf. [3, 8, 11]).

3.2.3. Relations between the surface differential operators in different coordinate systems. In this section, we derive relations between surface differential operators given in Definitions 3.3 and 3.7. The results are as expected and have been used (implicitly) at several places in the literature. We did not find, however, proofs of all these basic results in the literature. Therefore, we include elementary proofs here.

Theorem 3.8. *Let* $\phi \in C^1(\Gamma, \mathbb{R})$, $\mathbf{u} \in C^1(\Gamma, \mathbb{R}^3)$ *and* $\mathbf{T} \in C^1(\Gamma, L(\mathbb{R}^3, \mathbb{R}^3))$. *For the surface gradients, covariant derivatives and surface divergence operators defined in Definitions* 3.3 *and* 3.7, *the following relations hold on* Γ*:*

$$\nabla_\Gamma \phi = \widehat{\nabla}_\Gamma \phi, \quad \nabla_\Gamma \mathbf{u} = \widehat{\nabla}_\Gamma \mathbf{u}, \quad \operatorname{div}_\Gamma \mathbf{u} = \widehat{\operatorname{div}}_\Gamma \mathbf{u}, \quad \operatorname{div}_\Gamma \mathbf{T} = \widehat{\operatorname{div}}_\Gamma(\mathbf{T}^T). \tag{3.9}$$

Proof. A proof of the first equality can be found, for example, in [6,23]. For completeness, we include an elementary proof. Using the chain rule we get, with $\mathbf{y} = R(\boldsymbol{\xi}) \in \Gamma$,

$$\begin{aligned}\partial_\alpha \phi(\mathbf{y}) &= \partial_\alpha(\phi \circ R)(\boldsymbol{\xi}) = \partial_\alpha(\phi^e \circ R)(\boldsymbol{\xi}) = \widehat{\partial}_k \phi^e(R(\boldsymbol{\xi}))(\partial_\alpha R(\boldsymbol{\xi}) \cdot \widehat{\mathbf{e}}_k) \\ &= \widehat{\partial}_k \phi^e(\mathbf{y})(\mathbf{g}_\alpha \cdot \widehat{\mathbf{e}}_k).\end{aligned}$$

Thus, we get

$$\begin{aligned}\nabla_\Gamma \phi(\mathbf{y}) = \partial_\alpha \phi(\mathbf{y})\mathbf{g}^\alpha &= \big[\widehat{\partial}_k \phi^e(\mathbf{y})(\mathbf{g}_\alpha \cdot \widehat{\mathbf{e}}_k)\big]\mathbf{g}^\alpha \\ &= \widehat{\partial}_k \phi^e(\mathbf{y})\big[\underbrace{(\mathbf{g}_\alpha \cdot \widehat{\mathbf{e}}_k)\mathbf{g}^\alpha}_{\mathbf{P}\widehat{\mathbf{e}}_k}\big] \\ &= \widehat{\partial}_k \phi^e(\mathbf{y})\mathbf{P}\widehat{\mathbf{e}}_k = \mathbf{P}\big[\widehat{\partial}_k \phi^e(\mathbf{y})\widehat{\mathbf{e}}_k\big] = \mathbf{P}\widehat{\nabla}\phi^e(\mathbf{y}).\end{aligned}$$

For vector fields, the transposed Jacobian is given by

$$\widehat{\nabla}^T \mathbf{u}^e = \widehat{\mathbf{e}}_k \otimes \widehat{\partial}_k \mathbf{u}^e. \tag{3.10}$$

Using the chain rule we get, with $\mathbf{y} = R(\boldsymbol{\xi}) \in \Gamma$,

$$\begin{aligned}\partial_\alpha \mathbf{u}(\mathbf{y}) \cdot \hat{\mathbf{e}}_i &= \partial_\alpha (\mathbf{u} \circ R)(\boldsymbol{\xi}) \cdot \hat{\mathbf{e}}_i \\ &= \big[\hat{\partial}_k \mathbf{u}^e(R(\boldsymbol{\xi})) \cdot \hat{\mathbf{e}}_i\big]\big[\partial_\alpha R(\boldsymbol{\xi}) \cdot \hat{\mathbf{e}}_k\big] = \big[\hat{\partial}_k \mathbf{u}^e(\mathbf{y}) \cdot \hat{\mathbf{e}}_i\big]\big[\mathbf{g}_\alpha \cdot \hat{\mathbf{e}}_k\big].\end{aligned}$$

Combining this with (3.10) and using the surface gradient $\nabla_S \mathbf{u} = \mathbf{g}^\alpha \otimes \partial_\alpha \mathbf{u}$ (cf. Remark 3.4), we obtain

$$\begin{aligned}\nabla_S \mathbf{u}\,\hat{\mathbf{e}}_i &= (\mathbf{g}^\alpha \otimes \partial_\alpha \mathbf{u})\hat{\mathbf{e}}_i = \mathbf{g}^\alpha(\partial_\alpha \mathbf{u} \cdot \hat{\mathbf{e}}_i) = \mathbf{g}^\alpha\big[(\hat{\partial}_k \mathbf{u}^e \cdot \hat{\mathbf{e}}_i)(\mathbf{g}_\alpha \cdot \hat{\mathbf{e}}_k)\big] \\ &= (\hat{\partial}_k \mathbf{u}^e \cdot \hat{\mathbf{e}}_i)\mathbf{P}\hat{\mathbf{e}}_k = \mathbf{P}(\hat{\mathbf{e}}_k \otimes \hat{\partial}_k \mathbf{u}^e)\hat{\mathbf{e}}_i = \mathbf{P}\hat{\nabla}^T \mathbf{u}^e \hat{\mathbf{e}}_i. \end{aligned} \tag{3.11}$$

Using $\nabla_\Gamma \mathbf{u} = \mathbf{P}\nabla_S^T \mathbf{u}$ completes the proof of the relation for the covariant derivative of $\mathbf{u}$. For the surface divergence of a vector function, we get

$$\begin{aligned}\mathrm{div}_\Gamma \mathbf{u} &= \partial_\alpha \mathbf{u} \cdot \mathbf{g}^\alpha = \delta^\alpha_\gamma(\partial_\alpha \mathbf{u} \cdot \mathbf{g}^\gamma) = (\mathbf{g}^\alpha \cdot \mathbf{g}_\gamma)(\partial_\alpha \mathbf{u} \cdot \mathbf{g}^\gamma) = [(\mathbf{g}^\alpha \otimes \partial_\alpha \mathbf{u})\mathbf{g}^\gamma] \cdot \mathbf{g}_\gamma \\ &= (\nabla_S \mathbf{u}\, \mathbf{g}^\gamma) \cdot \mathbf{g}_\gamma \overset{(3.11)}{=} (\mathbf{P}\hat{\nabla}^T \mathbf{u}^e \mathbf{g}^\gamma) \cdot \mathbf{g}_\gamma = (\mathbf{P}\hat{\nabla}^T \mathbf{u}^e \mathbf{P}\mathbf{g}^i) \cdot \mathbf{g}_i = \mathrm{tr}(\mathbf{P}\hat{\nabla}\mathbf{u}^e\mathbf{P}) \\ &= \widehat{\mathrm{div}}_\Gamma \mathbf{u}.\end{aligned}$$

For the surface divergence of an operator-valued function $\mathbf{T}$, we have

$$\begin{aligned}\mathrm{div}_\Gamma \mathbf{T} \cdot \hat{\mathbf{e}}_i &= ((\partial_\alpha \mathbf{T})^T \mathbf{g}^\alpha) \cdot \hat{\mathbf{e}}_i = \delta^\alpha_\beta \hat{\mathbf{e}}_i \cdot ((\partial_\alpha \mathbf{T})^T \mathbf{g}^\beta) = (\mathbf{g}^\alpha \cdot \mathbf{g}_\beta)(\partial_\alpha \mathbf{T}\,\hat{\mathbf{e}}_i) \cdot \mathbf{g}^\beta \\ &= ((\mathbf{g}^\alpha \otimes \partial_\alpha \mathbf{T}\,\hat{\mathbf{e}}_i)\mathbf{g}^\beta) \cdot \mathbf{g}_\beta = ((\mathbf{g}^\alpha \otimes \partial_\alpha \mathbf{T}\,\hat{\mathbf{e}}_i)\mathbf{P}\mathbf{g}^i) \cdot \mathbf{P}\mathbf{g}_i \\ &= \mathrm{tr}\big[\mathbf{P}(\mathbf{g}^\alpha \otimes \partial_\alpha \mathbf{T}\,\hat{\mathbf{e}}_i)\mathbf{P}\big] = \mathrm{tr}\big[\mathbf{P}\nabla_S(\mathbf{T}\,\hat{\mathbf{e}}_i)\mathbf{P}\big] \\ &\overset{(3.11)}{=} \mathrm{tr}\big[\mathbf{P}\hat{\nabla}^T(\mathbf{T}^e\hat{\mathbf{e}}_i)\mathbf{P}\big] = \mathrm{tr}\big[\mathbf{P}\hat{\nabla}(\mathbf{T}^e\hat{\mathbf{e}}_i)\mathbf{P}\big] \\ &= \widehat{\mathrm{div}}_\Gamma(\mathbf{T}\,\hat{\mathbf{e}}_i) = \widehat{\mathrm{div}}_\Gamma(\mathbf{T}^T) \cdot \hat{\mathbf{e}}_i. \qquad \blacksquare\end{aligned}$$

Note that in the relation for the surface divergence of $\mathbf{T}$ in (3.9), a *transpose* is needed. This would vanish if either in Definition 3.3 or in Definition 3.7 one deletes the transpose in the definition of the surface divergence of $\mathbf{T}$. The results in Theorem 3.8 confirm that the operators defined in Definition 3.3 indeed do not depend on the parametrization.

The shape operator, given in curvilinear coordinates in (3.2), can be represented in the Cartesian coordinate system as $\mathbf{B} = -\hat{\nabla}_\Gamma \mathbf{n}^e$ (the proof of this is given in Appendix A).

3.2.4. The material derivative. We introduce a derivative in which the time dependence of the parametrization $R(\boldsymbol{\xi}, t)$, $\boldsymbol{\xi} \in U$, is used. Let $I = (0, T)$ be a time interval with $T > 0$ small enough so that for all $\mathbf{z} \in \Phi_U(U) \subset \Gamma(0)$ the ordinary differential equation given in (2.1) has a unique solution for $t \in I$. We define the (local) evolving surface $\Gamma_U(t) = \{\mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = R(\boldsymbol{\xi}, t),\ \boldsymbol{\xi} \in U\}$, $t \in I$. The corresponding space-time manifold is given by

$$\mathcal{S} = \mathcal{S}(U, I) := \bigcup_{t \in I} \Gamma_U(t) \times \{t\} \subset \mathbb{R}^4.$$

Note that $\mathcal{S}$ is parametrized by $\mathcal{R} : U \times I \to \mathcal{S}$, $\mathcal{R}(\boldsymbol{\xi}, t) = (R(\boldsymbol{\xi}, t), t)$. Given the velocity $\mathbf{v}(\mathbf{y}, t)$, $\mathbf{y} \in \Gamma_U(t)$ from (2.1) we define $\overline{\mathbf{v}}(\boldsymbol{\xi}, t) := \mathbf{v}(R(\boldsymbol{\xi}, t), t)$, $(\boldsymbol{\xi}, t) \in U \times I$. Thus, we have the relation

$$\overline{\mathbf{v}}(\boldsymbol{\xi}, t) = \frac{\partial}{\partial t} R(\boldsymbol{\xi}, t) \quad \text{on } U \times I. \tag{3.12}$$

Definition 3.9. Let $f \in C^1(\mathcal{S})$ be a scalar or vector function and $\overline{f} \in C^1(U \times I)$ be the function defined by $\overline{f}(\boldsymbol{\xi}, t) = f(R(\boldsymbol{\xi}, t), t)$ for $(\boldsymbol{\xi}, t) \in U \times I$. The *material* derivative of f on $\mathcal{S}$ is defined by

$$\dot{f}(\mathbf{y}, t) := \partial_t \overline{f}(\boldsymbol{\xi}, t), \quad \mathbf{y} = R(\boldsymbol{\xi}, t).$$

Clearly, this is a definition in terms of the local coordinates $\boldsymbol{\xi}$ of the surface $\Gamma(0)$.

To obtain a Cartesian representation of the material derivative, we use the same approach as in the previous section and extend the functions defined on the space-time manifold $\mathcal{S}$ to an open neighborhood $\mathcal{G}_\delta = \mathcal{G}_\delta(\mathcal{S})$, given by $\mathcal{G}_\delta = \bigcup_{t \in I} G_\delta(\Gamma_U(t)) \times \{t\}$. The neighborhood $G_\delta(\Gamma_U(t))$ of $\Gamma_U(t)$ is as defined in the previous section. The next lemma yields a representation of the material derivative defined above in terms of derivatives with respect to Cartesian coordinates in $\mathbb{R}^3 \times \mathbb{R}$. The result is well known and easy to prove, based on application of the chain rule. For completeness, we include an elementary proof.

Lemma 3.10. *For $\phi \in C^1(\mathcal{S}, \mathbb{R})$ and $\mathbf{u} \in C^1(\mathcal{S}, \mathbb{R}^3)$, let $\phi^e \in C^1(\mathcal{G}_\delta, \mathbb{R})$ and $\mathbf{u}^e \in C^1(\mathcal{G}_\delta, \mathbb{R}^3)$ be corresponding smooth extensions. For the material derivatives of ϕ and $\mathbf{u}$, the following holds:*

$$\dot{\phi}(\mathbf{y}, t) = \partial_t \phi^e(\mathbf{y}, t) + \widehat{\nabla} \phi^e(\mathbf{y}, t) \cdot \mathbf{v}(\mathbf{y}, t), \quad \dot{\mathbf{u}}(\mathbf{y}, t) = \partial_t \mathbf{u}^e(\mathbf{y}, t) + \widehat{\nabla} \mathbf{u}^e(\mathbf{y}, t) \mathbf{v}(\mathbf{y}, t).$$

Proof. For $(\mathbf{y}, t) \in \mathcal{S}$, we write $\mathbf{y} = R(\boldsymbol{\xi}, t)$. We use the chain rule for the function $\phi^e(R(\boldsymbol{\xi}, t), t) = (\phi^e \circ F)(t)$ with the auxiliary function $F : I \to \mathbb{R}^4$, $t \mapsto (R(\boldsymbol{\xi}, t), t)$ and get

$$\begin{aligned}
\dot{\phi}(\mathbf{y}, t) = \partial_t \overline{\phi}(\boldsymbol{\xi}, t) &= \frac{d}{dt} \phi(R(\boldsymbol{\xi}, t), t) \\
&= \sum_{k=1}^{3} \widehat{\partial}_k \phi^e(R(\boldsymbol{\xi}, t), t) \Big(\frac{\partial}{\partial t} R(\boldsymbol{\xi}, t) \cdot \hat{\mathbf{e}}_k \Big) + \partial_t \phi^e(R(\boldsymbol{\xi}, t), t) \cdot 1 \\
&= \partial_t \phi^e(R(\boldsymbol{\xi}, t), t) + \widehat{\nabla} \phi^e(R(\boldsymbol{\xi}, t), t) \cdot \frac{\partial}{\partial t} R(\boldsymbol{\xi}, t).
\end{aligned} \tag{3.13}$$

Using $\mathbf{y} = R(\boldsymbol{\xi}, t)$ and relation (3.12), we obtain

$$\dot{\phi}(\mathbf{y}, t) = \partial_t \phi^e(\mathbf{y}, t) + \widehat{\nabla} \phi^e(\mathbf{y}, t) \cdot \hat{\mathbf{v}}(\mathbf{y}, t).$$

The same arguments can be used to derive the relation for $\mathbf{u}$. ■

The material derivative is used, for example, in the Leibniz rule or transport theorem for an arbitrary material subdomain $\gamma(t) \subset \Gamma_U(t)$:

$$\frac{d}{dt}\int_{\gamma(t)} f \; ds = \int_{\gamma(t)} \dot{f} + f \operatorname{div}_\Gamma \mathbf{v} \; ds, \tag{3.14}$$

for $f \in C^1(\mathcal{S},\mathbb{R})$.

3.3. Time derivative of first fundamental form

In this section, we consider a time derivative of the first fundamental form, which will be used in the remainder. The local coordinate system introduced in Section 3.1 depends on the time variable t (cf. (2.3) and Section 3). In particular, for the covariant basis $\mathbf{g}_\alpha = \partial_\alpha R$ we have $\mathbf{g}_\alpha = \mathbf{g}_\alpha(\boldsymbol{\xi},t)$, $\boldsymbol{\xi} \in U$, $t \in I$. Hence, the first fundamental form (cf. (3.1)), depends not only on $\boldsymbol{\xi}$ but also on the time variable, $g_{\alpha\beta} = g_{\alpha\beta}(\boldsymbol{\xi},t)$. The change (as function of time) of the metric tensor is determined by the velocity field $\mathbf{v}$, which determines the time dependence of the parametrization $R = R(\boldsymbol{\xi},t) = \Phi_t(\Phi_U(\boldsymbol{\xi}))$ via the flow map Φ_t. Using Theorem 3.2, the following relation for the time derivative of the metric tensor is derived (recall $\mathbf{v} = v_i \mathbf{g}^i = v^i \mathbf{g}_i$):

$$\begin{aligned}
\frac{\partial}{\partial t} g_{\alpha\beta} &= \partial_t \mathbf{g}_\alpha \cdot \mathbf{g}_\beta + \mathbf{g}_\alpha \cdot \partial_t \mathbf{g}_\beta = \partial_t \partial_\alpha R \cdot \mathbf{g}_\beta + \mathbf{g}_\alpha \cdot \partial_t \partial_\beta R = \partial_\alpha \mathbf{v} \cdot \mathbf{g}_\beta + \partial_\beta \mathbf{v} \cdot \mathbf{g}_\alpha \\
&= \big((v_{\gamma|\alpha} - b_{\alpha\gamma} v_3)\mathbf{g}^\gamma + (v_{3|\alpha} + b_\alpha^\gamma v_\gamma)\mathbf{g}^3\big) \cdot \mathbf{g}_\beta \\
&\quad + \big((v_{\gamma|\beta} - b_{\beta\gamma} v_3)\mathbf{g}^\gamma + (v_{3|\beta} + b_\beta^\gamma v_\gamma)\mathbf{g}^3\big) \cdot \mathbf{g}_\alpha \\
&= v_{\beta|\alpha} + v_{\alpha|\beta} - 2 v_3 b_{\alpha\beta}.
\end{aligned} \tag{3.15}$$

For this time derivative of the metric tensor, scaled with a factor $\frac{1}{2}$, we introduce the notation

$$E_{\alpha\beta} := \frac{1}{2}\frac{\partial}{\partial t} g_{\alpha\beta}. \tag{3.16}$$

For a given $(\boldsymbol{\xi},t) \in \mathcal{S}$, a corresponding linear operator $\mathbf{E} = \mathbf{E}(\boldsymbol{\xi},t) : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\mathbf{E} := E_{\alpha\beta}(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) = E^{\alpha\beta}(\mathbf{g}_\alpha \otimes \mathbf{g}_\beta)$. This operator can also be expressed in terms of the covariant derivatives introduced in Definitions 3.3 and 3.7, as shown in the next lemma. A proof of this lemma is given in Appendix A.

Lemma 3.11. *The following relations hold:*

$$\mathbf{E} = \frac{1}{2}(\nabla_\Gamma \mathbf{v} + \nabla_\Gamma^T \mathbf{v}) = \frac{1}{2}(\widehat{\nabla}_\Gamma \mathbf{v} + \widehat{\nabla}_\Gamma^T \mathbf{v}). \tag{3.17}$$

4. Derivations of surface Navier–Stokes equations

In this section, we outline five different derivations of surface Navier–Stokes equations known from the literature [10–12, 15, 17], which use both different physical principles and representations in different coordinate systems. In the five subsections below we present, in a unified framework, the following derivations:

(1) In [10], the conservation laws of *surface* mass and momentum quantities are used as physical principles. Surface Navier–Stokes equations in *curvilinear coordinates* are derived.

(2) In [11], the same conservation laws of *surface* mass and momentum quantities as in [10] are used and surface Navier–Stokes equations in *Cartesian coordinates* in $\mathbb{R}^3$ are derived.

(3) In [12], the same surface mass conservation law as in [10, 11] is used. Instead of a surface momentum conservation principle, a *variational energy principle* is used. The equations are derived in *Cartesian coordinates* in $\mathbb{R}^3$.

(4) In [15], the conservation laws of *volume* mass and momentum quantities are used as physical principles in a thin tubular neighborhood of the (evolving) surface. Combined with a thin film limit procedure, surface Navier–Stokes equations are derived in *Cartesian coordinates*.

(5) In [17], the same physical principles of *volume* mass and momentum conservation in a thin tubular neighborhood as in [15] are used. The resulting volume Navier–Stokes equations are represented in a thin film *curvilinear* local coordinate system. A thin film limit procedure is applied to derive *tangential* surface Navier–Stokes equations in curvilinear coordinates.

In the approaches (1), (2), (4), and (5), one uses an ansatz for the viscous stress tensor, namely, the standard Newtonian tensor in (4) and (5) and the Boussinesq–Scriven tensor in (1) and (2). In (3) an ansatz for the viscous surface dissipation energy is used. Below we outline only the key ideas of the derivations and refer to the corresponding papers for more details.

4.1. Surface mass and momentum conservation in curvilinear coordinates

In this section, a derivation of surface Navier–Stokes equations along the same lines as in [10] is presented. In that paper, the resulting surface Navier–Stokes equations are formulated in tensor calculus *without* using surface differential operators like ∇_Γ and div_Γ. To be able to compare the resulting equations with those obtained in the other approaches, we rewrite these using the differential operators introduced in Section 3.2.1 and results derived in Section 3.3.

In the approach outlined in this section, the unknowns are the evolving surface $\Gamma(t)$, the surface (tangential and normal) velocity $\mathbf{v}$ and the surface pressure p.

The derivation is based on conservation laws of mass and momentum. We assume the surface to be inextensible, that is, $\frac{d}{dt}\int_{\gamma(t)} 1\, ds = 0$ holds for an arbitrary material subdomain $\gamma(t) \subset \Gamma(t)$. The Leibniz rule (3.14) and the arbitrariness of $\gamma(t)$ yield

$$\mathrm{div}_\Gamma \mathbf{v} = 0. \tag{4.1}$$

Let ρ denote the surface mass density. Conservation of mass, the Leibniz rule and $\mathrm{div}_\Gamma \mathbf{v} = 0$ lead to

$$0 = \frac{d}{dt} \int_{\gamma(t)} \rho \, ds = \int_{\gamma(t)} \dot{\rho} \, ds.$$

Arbitrariness of $\gamma(t)$ and a smoothness assumption on ρ imply $\dot{\rho} = 0$. Hence, if ρ is constant on $\Gamma(0)$, which we assume here, it follows that the surface mass density ρ is constant on the evolving surface $\Gamma(t)$.

As an ansatz for surface momentum conservation, the equation

$$\frac{d}{dt} \int_{\gamma(t)} \rho \mathbf{v} \, ds = \mathbf{F}(\gamma(t)) \tag{4.2}$$

is used, with a force $\mathbf{F}$ decomposed into external area forces acting on $\gamma(t)$ and internal forces acting on the boundary $\partial\gamma(t)$.

Remark 4.1. The (surface) integral of a vector-valued function $\int_{\gamma(t)} \mathbf{u} \, ds$ (cf. (4.2)) is defined in the usual way. We choose a fixed (not necessarily orthogonal) basis of $\mathbb{R}^3$, say $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. For $\mathbf{u}(s) = u^i(s)\mathbf{w}_i$, we then define $\int_{\gamma(t)} \mathbf{u}(s) \, ds = \mathbf{w}_i \int_{\gamma(t)} u^i(s) \, ds$. The results derived below are independent of the choice of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.

We collect the external forces, consisting of normal and shear stresses, in the force term $\mathbf{f} = f^\alpha \mathbf{g}_\alpha + f_N \mathbf{n}$. For the internal forces the Cauchy ansatz is made, that is, we assume that these forces are of the form $\mathbf{T}\boldsymbol{\nu}$ with a stress tensor $\mathbf{T}$ and $\boldsymbol{\nu}$ the in-plane unit normal on $\partial\gamma(t)$. Using $\mathbf{T}\boldsymbol{\nu} = T^{\alpha\beta} \nu_\alpha \mathbf{g}_\beta$, the total net force on $\gamma(t)$ can be written as

$$\mathbf{F}(\gamma(t)) = \int_{\gamma(t)} \mathbf{f} \, ds + \int_{\partial\gamma(t)} T^{\alpha\beta} \nu_\alpha \mathbf{g}_\beta \, ds. \tag{4.3}$$

As in [10], we apply the Leibniz rule on the left-hand side of (4.2) and Green's formula on the boundary integral of the right-hand side of (4.3). Using $\mathrm{div}_\Gamma \mathbf{v} = 0$, this yields

$$\int_{\gamma(t)} \rho \dot{\mathbf{v}} \, ds = \int_{\gamma(t)} \mathbf{f} + T^{\beta\alpha}_{|\beta} \mathbf{g}_\alpha + T^{\alpha\beta} b_{\alpha\beta} \mathbf{n} \, ds.$$

Due to the arbitrariness of $\gamma(t)$, we obtain the following system of surface partial differential equations (cf. [10, (31)]):

$$\rho(\dot{\mathbf{v}} \cdot \mathbf{g}^\alpha) = f^\alpha + T^{\beta\alpha}_{|\beta}, \quad \rho(\dot{\mathbf{v}} \cdot \mathbf{n}) = f_N + T^{\alpha\beta} b_{\alpha\beta}, \tag{4.4}$$

which consists of two equations for tangential velocity change $\dot{\mathbf{v}} \cdot \mathbf{g}^\alpha$ and one equation for velocity change in normal direction $\dot{\mathbf{v}} \cdot \mathbf{n}$. As an ansatz for the stress tensor $\mathbf{T}$, the Boussinesq–Scriven form (in curvilinear coordinates)

$$T^{\alpha\beta} = -p g^{\alpha\beta} + 2\mu_0 E^{\alpha\beta} \tag{4.5}$$

is used, which involves the surface pressure p, the viscosity coefficient μ_0 and the time derivative of the metric tensor $E^{\alpha\beta}$ (cf. (3.16)). Equations (4.1), (4.4) and (4.5) form the surface Navier–Stokes system derived in [10]. Note that the evolution of the surface depends on the unknown velocity $\mathbf{v}$ as described in (2.1)–(2.2).

To be able to compare this surface Navier–Stokes system, which is formulated in terms of curvilinear coordinates, to equations derived in the sections below, we rewrite these equations using surface differential operators (cf. Definition 3.3). From Lemma 3.11, we obtain for the rate of strain tensor $\mathbf{E} = E_{\alpha\beta}(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta)$ the representation

$$\mathbf{E} = \mathbf{E}(\mathbf{v}) = \frac{1}{2}(\nabla_\Gamma \mathbf{v} + \nabla_\Gamma^T \mathbf{v}),$$

and thus the operator representation

$$\mathbf{T} = -p\mathbf{P} + 2\mu_0 \mathbf{E} \tag{4.6}$$

for the stress tensor. From Lemma 3.6, we get that the equations in (4.4) can be rewritten as

$$\rho\dot{\mathbf{v}} = \mathbf{f} + \mathrm{div}_\Gamma(\mathbf{T}).$$

Using this and the identity $\mathrm{div}_\Gamma(p\mathbf{P}) = \nabla_\Gamma p + p\kappa\mathbf{n}$, we obtain the following representation of the surface Navier–Stokes system (4.1), (4.4)–(4.5) in terms of the surface differential operators as in Definition 3.3. For a given initial surface $\Gamma(0)$, viscosity μ_0, force term $\mathbf{f}$ and a constant surface mass density ρ, find $\mathbf{v}$, p and $\Gamma(t)$, parametrized by $\mathbf{x}(\mathbf{z}, t)$ (cf. (2.2)), such that

$$\begin{cases} \rho\dot{\mathbf{v}} = \mathbf{f} - \nabla_\Gamma p - p\kappa\mathbf{n} + 2\mu_0\, \mathrm{div}_\Gamma \mathbf{E}(\mathbf{v}) & \text{on } \Gamma(t), \\ \mathrm{div}_\Gamma \mathbf{v} = 0 & \text{on } \Gamma(t), \\ \dfrac{d}{dt}\mathbf{x}(\mathbf{z}, t) = \mathbf{v}(\mathbf{x}(\mathbf{z}, t), t), \quad \mathbf{x}(\mathbf{z}, 0) = \mathbf{z} \in \Gamma(0). \end{cases} \tag{4.7}$$

4.2. Surface mass and momentum conservation in Cartesian coordinates

We recall the model derived in [11]. It is based on the same fundamental laws of surface continuum mechanics as in the previous section. The formulation of the equations, however, is in Cartesian coordinates in $\mathbb{R}^3$. Hence, the surface differential operators ($\widehat{\nabla}_\Gamma$ and $\widehat{\mathrm{div}}_\Gamma$) used are as in Definition 3.7. The material derivative $\dot{\mathbf{v}}$ is defined in Cartesian coordinates as formulated in Lemma 3.10.

As in the previous section, the unknowns are the evolving surface $\Gamma(t)$, the surface (tangential and normal) velocity $\mathbf{v}$ and the surface pressure p.

Using the Leibniz rule, the inextensibility condition $\frac{d}{dt}\int_{\gamma(t)} 1\, ds = 0$ yields

$$\widehat{\mathrm{div}}_\Gamma \mathbf{v} = 0. \tag{4.8}$$

From mass conservation $\frac{d}{dt}\int_{\gamma(t)} \rho\, ds = 0$ we obtain, with the same arguments as in the previous section, that ρ remains constant on $\Gamma(t)$ if it is constant on $\Gamma(0)$. We now consider the conservation of surface momentum, expressed by the equation

$$\frac{d}{dt}\int_{\gamma(t)} \rho\mathbf{v}\,ds = \int_{\gamma(t)} \mathbf{f}\,ds + \int_{\partial\gamma(t)} \mathbf{f}_{\boldsymbol{\nu}}\,ds, \tag{4.9}$$

with a contact force term $\mathbf{f}_{\boldsymbol{\nu}}$ on $\partial\gamma(t)$ and an area force term $\mathbf{f}$. The integrals are defined as in Remark 4.1. As in the previous section, for the contact force term we use a Cauchy ansatz and Boussinesq–Scriven ansatz:

$$\mathbf{f}_{\boldsymbol{\nu}} = \mathbf{T}\boldsymbol{\nu}, \quad \mathbf{T} = -p\mathbf{P} + 2\mu_0\mathbf{E}(\mathbf{v}), \quad \mathbf{E}(\mathbf{v}) = \frac{1}{2}(\widehat{\nabla}_\Gamma\mathbf{v} + \widehat{\nabla}_\Gamma^T\mathbf{v}). \tag{4.10}$$

Lemma 3.11 shows that the definition of the rate of strain tensor $\mathbf{E}$ equals the one from [10] (cf. equation (4.6)). From Stokes' theorem and the identity $\widehat{\operatorname{div}}_\Gamma(p\mathbf{P}) = \widehat{\nabla}_\Gamma p + p\kappa\mathbf{n}$, we obtain the momentum balance for $\gamma(t)$:

$$\frac{d}{dt}\int_{\gamma(t)} \rho\mathbf{v}\,ds = \int_{\gamma(t)} \mathbf{f} - \widehat{\nabla}_\Gamma p - p\kappa\mathbf{n} + 2\mu_0\,\widehat{\operatorname{div}}_\Gamma\mathbf{E}(\mathbf{v})\,ds.$$

Using the Leibniz rule and combining the result with (4.8), we obtain the following surface Navier–Stokes system: For a given initial surface $\Gamma(0)$, viscosity μ_0, force term $\mathbf{f}$ and a constant surface mass density ρ, find $\mathbf{v}$, p and $\Gamma(t)$, parametrized by $\mathbf{x}(\mathbf{z},t)$ (cf. (2.2)) such that

$$\begin{cases} \rho\dot{\mathbf{v}} = \mathbf{f} - \widehat{\nabla}_\Gamma p - p\kappa\mathbf{n} + 2\mu_0\,\widehat{\operatorname{div}}_\Gamma\mathbf{E}(\mathbf{v}) & \text{on } \Gamma(t), \\ \widehat{\operatorname{div}}_\Gamma\mathbf{v} = 0 & \text{on } \Gamma(t), \\ \frac{d}{dt}\mathbf{x}(\mathbf{z},t) = \mathbf{v}(\mathbf{x}(\mathbf{z},t),t), \quad \mathbf{x}(\mathbf{z},0) = \mathbf{z} \in \Gamma(0). \end{cases} \tag{4.11}$$

Based on Theorem 3.8, we conclude that this PDE system is exactly the same as in (4.7). This is not surprising, since the derivations of the two systems start from exactly the same physical principles.

4.3. Energetic variational principle in Cartesian coordinates

In this section, we summarize the variational approach presented in [12] to derive a surface Navier–Stokes system. This derivation is performed in Cartesian coordinates in $\mathbb{R}^3$.

It is assumed that $\Gamma(t)$ is a closed surface and that *the geometric evolution in terms of the normal velocity* V_Γ *of* $\Gamma(t)$ *is given*, that is,

$$\mathbf{v}\cdot\mathbf{n} = V_\Gamma. \tag{4.12}$$

Hence, in this approach the unknowns are the *tangential component* of the velocity $\mathbf{v}$ and the surface pressure p.

First, in exactly the same way as in the sections above, inextensibility and mass conservation lead to the equation

$$\widehat{\operatorname{div}}_\Gamma\mathbf{v} = 0 \tag{4.13}$$

(in Cartesian coordinates) and the fact that ρ is constant on $\mathcal{S}$. Instead of a momentum conservation ansatz as in [10, 11] (cf. equations (4.2) and (4.9)), an energetic variational approach based on the so-called Least Action and Minimum Dissipation Principles is used. We outline the key steps.

The so-called action integral ("kinetic energy") is defined by

$$A(\mathbf{x}) := \int_0^T \int_{\Gamma(t)} \frac{1}{2}\rho|\mathbf{v}(\mathbf{x},t)|^2 \, d\mathbf{x}\, dt.$$

Recall that $\mathbf{x} = \mathbf{x}(\mathbf{z},t) \in \Gamma(t)$, $\mathbf{z} \in \Gamma(0)$, are the particle trajectories and $\mathbf{v}(\mathbf{x},t)$ the corresponding velocity fields (cf. (2.1)). Note that $\Gamma(t)$ and $\mathbf{v}$ are uniquely determined by the trajectories $\mathbf{x}(\mathbf{z},t)$. The variation of the action integral with respect to $\mathbf{x}(\mathbf{z},t)$ can be formally written as

$$D_{\mathbf{x}}A(\mathbf{x})(\mathbf{w}) = \int_0^T \int_{\Gamma(t)} F_{\text{cons}} \cdot \mathbf{w} \, ds\, dt =: \langle F_{\text{cons}}, \mathbf{w}\rangle$$

for a "suitable" class of admissible velocities $\mathbf{w}$. This relation defines a conservative force F_{cons} (cf. [29]). In [12, Theorem 1.5], it is shown that under reasonable assumptions,

$$F_{\text{cons}} = -\rho\dot{\mathbf{v}} \tag{4.14}$$

holds. Another force, the so-called dissipation force, is derived from variation of "surface viscosity" energy, which is modeled by the functional

$$E_{\text{diss}}(\mathbf{v}) = -\int_0^T \int_{\Gamma(t)} \mu_0 |\mathbf{E}(\mathbf{v})|^2 \, ds\, dt, \tag{4.15}$$

with viscosity coefficient μ_0 and a rate of strain tensor $\mathbf{E}$ as in (4.10) (cf. [12]). Variation with respect to the velocity field $\mathbf{v}$ leads to the dissipation force

$$D_{\mathbf{v}}E_{\text{diss}}(\mathbf{v})(\mathbf{w}) = \langle F_{\text{diss}}, \mathbf{w}\rangle,$$

for a "suitable" class of admissible velocities $\mathbf{w}$ (cf. [29]). In [12, Theorem 1.6], the relation

$$F_{\text{diss}} = 2\mu_0 \,\widehat{\operatorname{div}}_\Gamma \mathbf{E}(\mathbf{v}) \tag{4.16}$$

is derived. The Onsager principle (cf. [21, 22, 29]) states that the dynamic of a system is determined by a competition between internal energy (here, the kinetic energy) and dissipation. In our setting, the corresponding equation is formally given by

$$D_{\mathbf{x}}A = -D_{\mathbf{v}}E_{\text{diss}} \tag{4.17}$$

(cf. [12, p. 385]). This implies $\langle F_{\text{cons}} + F_{\text{diss}}, \mathbf{w}\rangle = 0$ for all admissible velocity fields $\mathbf{w}$. Due to the fact that we consider incompressible surface flows, we restrict to velocity fields $\mathbf{w}$ with $\widehat{\operatorname{div}}_\Gamma \mathbf{w} = 0$. The following corollary is based on [12, Lemma 2.7]:

Corollary 4.2. *Let* $\mathbf{g} \in C(\mathcal{S})^3$ *be such that* $\langle \mathbf{g}, \mathbf{w} \rangle = 0$ *for all* $\mathbf{w} \in C^\infty(\mathcal{S})$ *with* $\widehat{\mathrm{div}}_\Gamma \mathbf{w} = 0$. *Then, there exists* $p \in C^1(\mathcal{S})$ *such that*

$$\mathbf{g} = \widehat{\nabla}_\Gamma p + p\kappa \mathbf{n}.$$

Applying this corollary, we obtain $F_{\text{cons}} + F_{\text{diss}} = \widehat{\nabla}_\Gamma p + p\kappa\mathbf{n}$ for a suitable (pressure) function p. By combining this with (4.12), (4.13), (4.14) and (4.16), one obtains the following surface Navier–Stokes equations (with *given* normal velocity V_Γ, viscosity μ_0 and a constant surface mass density ρ):

$$\begin{cases} \mathbf{v} \cdot \mathbf{n} = V_\Gamma & \text{on } \Gamma(t), \\ \rho\dot{\mathbf{v}} - 2\mu_0 \,\widehat{\mathrm{div}}_\Gamma \mathbf{E}(\mathbf{v}) = -\widehat{\nabla}_\Gamma p - p\kappa\mathbf{n} & \text{on } \Gamma(t), \\ \widehat{\mathrm{div}}_\Gamma \mathbf{v} = 0 & \text{on } \Gamma(t). \end{cases} \tag{4.18}$$

There is the following subtle issue, also discussed in [12]: Due to the fact that the surface normal velocity is given, system (4.18) in general is an overdetermined system. In fact, in (4.18) there are four unknowns: the velocity (having essentially three unknowns) and the pressure. There are, however, five equations including incompressibility and the redundant first equation. To obtain a closed system, the equations can be "projected" to get a system for the tangential velocity and the pressure. This is further explained in Section 5.

4.4. Thin film approach in Cartesian coordinates

A different approach for deriving surface Navier–Stokes equations, based on a thin film limit procedure, is introduced in [15].

As in the previous section, it is assumed that $\Gamma(t)$ is a closed surface and that the *geometric evolution in terms of the normal velocity* V_Γ *of* $\Gamma(t)$ *is given*. Hence, in this approach the unknowns are the *tangential component* of the velocity and the surface pressure.

Around this surface, a thin film domain $\Omega_\varepsilon(t) := \{\mathbf{x} \in \mathbb{R}^3 \mid \mathrm{dist}(\mathbf{x}, \Gamma(t)) < \varepsilon\}$ with a sufficiently small $\varepsilon > 0$ is defined, which evolves with constant thickness and such that the surface remains located in the middle of this domain. In this evolving thin film, the incompressible three-dimensional Navier–Stokes equations with appropriate boundary conditions on $\partial\Omega_\varepsilon(t)$ are given. These equations describe mass and momentum conservation in the volume domain $\Omega_\varepsilon(t)$. One then studies the limit of the thickness going to zero and the resulting surface equations. In [15], these limit equations are derived using formal asymptotic expansions (in the parameter ε). We outline a few key steps in the derivation and refer to [15] for further explanations.

The signed distance function to $\Gamma(t)$ is denoted by $d(\cdot, t)$. For ε sufficiently small, the closest point projection of $\mathbf{x} \in \Omega_\varepsilon(t)$ is given by $\pi(\mathbf{x}, t) = \mathbf{x} - d(\mathbf{x}, t)\mathbf{n}(\mathbf{x}, t)$. We define the space-time domain $Q_{\varepsilon,I}$ and its boundary $\partial Q_{\varepsilon,I}$ by

$$Q_{\varepsilon,I} := \bigcup_{t\in I} \Omega_\varepsilon(t) \times \{t\}, \quad \partial Q_{\varepsilon,I} := \bigcup_{t\in I} \partial\Omega_\varepsilon(t) \times \{t\}. \tag{4.19}$$

The unit outward normal vector $\mathbf{n}_\varepsilon(\mathbf{x},t)$ and outward normal velocity $V_\varepsilon(\mathbf{x},t)$ on $\partial\Omega_\varepsilon$ are given by

$$\mathbf{n}_\varepsilon(\mathbf{x},t) = \begin{cases} \mathbf{n}(\pi,t), & \text{if } d(\mathbf{x},t)=\varepsilon, \\ -\mathbf{n}(\pi,t), & \text{if } d(\mathbf{x},t)=-\varepsilon, \end{cases} \qquad V_\varepsilon(\mathbf{x},t) = \begin{cases} V_\Gamma(\pi,t), & \text{if } d(\mathbf{x},t)=\varepsilon, \\ -V_\Gamma(\pi,t), & \text{if } d(\mathbf{x},t)=-\varepsilon, \end{cases}$$

with $\pi = \pi(\mathbf{x},t)$ and where V_Γ is the normal velocity of the surface $\Gamma(t)$.

We consider an incompressible Navier–Stokes system in $Q_{\varepsilon,I}$ with (perfect slip) Navier boundary conditions:

$$\begin{aligned} \partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \widehat{\nabla})\mathbf{v}_\varepsilon + \widehat{\nabla} p_\varepsilon &= \mu_0 \widehat{\operatorname{div}}(\widehat{\nabla}\mathbf{v}_\varepsilon) && \text{in } Q_{\varepsilon,I}, \\ \widehat{\operatorname{div}}\, \mathbf{v}_\varepsilon &= 0 && \text{in } Q_{\varepsilon,I}, \\ \mathbf{v}_\varepsilon \cdot \mathbf{n}_\varepsilon &= V_\varepsilon && \text{on } \partial Q_{\varepsilon,I}, \\ [\mathbf{E}_3(\mathbf{v}_\varepsilon)\mathbf{n}_\varepsilon]_{\tan} &= 0 && \text{on } \partial Q_{\varepsilon,I}, \end{aligned} \tag{4.20}$$

where $[\mathbf{a}]_{\tan}$ denotes the tangential component to $\partial\Omega_\varepsilon(t)$ of a vector $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{E}_3(\mathbf{v}) := \frac{1}{2}(\widehat{\nabla}\mathbf{v} + \widehat{\nabla}^T\mathbf{v})$ the rate of strain tensor. We use the notation $\mathbf{E}_3(\cdot)$ to distinguish this *three*-dimensional rate of strain tensor from the surface rate of strain tensor $\mathbf{E}(\cdot)$ used in the previous sections. The differential operators $\widehat{\operatorname{div}}$, $\widehat{\nabla}$ (cf. Section 3.2.2) are the usual ones in $\mathbb{R}^3$ and ∂_t is the usual time derivative. Note that here (following the presentation in [15]) the density is scaled to $\rho = 1$, but this is not essential.

The system defines the velocity $\mathbf{v}_\varepsilon$ and pressure p_ε of a fluid in $Q_{\varepsilon,I}$. To derive equations defining the velocity of the fluid on $\Gamma(t)$ only, consistent with (4.20) and depending only on values of functions on $\Gamma(t)$, formal asymptotic expansions are *assumed*. More precisely, it is assumed that for the solution pair $(\mathbf{v}_\varepsilon, p_\varepsilon)$, there exist vector fields $\mathbf{v}$, $\mathbf{v}^1$, $\mathbf{v}^2$ and scalar functions p, p^1 such that

$$\mathbf{v}_\varepsilon(\mathbf{x},t) = \mathbf{v}(\pi,t) + d(\mathbf{x},t)\mathbf{v}^1(\pi,t) + d(\mathbf{x},t)^2\mathbf{v}^2(\pi,t) + r(d^3), \tag{4.21a}$$

$$p_\varepsilon(\mathbf{x},t) = p(\pi,t) + d(\mathbf{x},t)p^1(\pi,t) + r(d^2). \tag{4.21b}$$

Here, $r(d^k) = r(d(\mathbf{x},t)^k)$ denotes a higher-order term (cf. [15]). The analysis in [15] is not rigorous in the sense that it is not clear under which assumptions, if any, such expansions exist. These expansions are substituted in (4.20) to derive equations for the zero-order terms $\mathbf{v}$ and p.

A key ingredient to obtain surface Navier–Stokes equations from the Navier–Stokes system in the thin film domain $Q_{\varepsilon,I}$ is the following lemma (cf. [15, Lemma 2.7]):

Lemma 4.3. *Let ϕ be a scalar and $\mathbf{u}$ a vector-valued function on $\mathcal{S}$. The derivatives of the composite functions $\phi(\pi(\mathbf{x},t),t)$ and $\mathbf{u}(\pi(\mathbf{x},t),t)$ with respect to $\mathbf{x}$ and t are of the form*

$$\widehat{\nabla}\phi(\pi,t) = (\widehat{\nabla}_\Gamma\phi)(\pi,t) + d(\mathbf{x},t)[\mathbf{B}\widehat{\nabla}_\Gamma\phi](\pi,t) + r(d(\mathbf{x},t)^2),$$

$$\partial_t\phi(\pi,t) = \frac{d}{dt}\phi(\pi(\mathbf{x},t),t) + d(\mathbf{x},t)(\widehat{\nabla}_\Gamma V_\Gamma \cdot \widehat{\nabla}_\Gamma\phi)(\pi,t) + r(d(\mathbf{x},t)^2),$$

and

$$\widehat{\nabla}\mathbf{u}(\pi,t) = (\widehat{\nabla}\mathbf{u}\mathbf{P})(\pi,t) + d(\mathbf{x},t)[\widehat{\nabla}\mathbf{u}\mathbf{B}](\pi,t) + r(d(\mathbf{x},t)^2),$$
$$\partial_t\mathbf{u}(\pi,t) = \frac{d}{dt}\mathbf{u}(\pi(\mathbf{x},t),t) + d(\mathbf{x},t)[\widehat{\nabla}\mathbf{u}\widehat{\nabla}_\Gamma V_\Gamma](\pi,t) + r(d(\mathbf{x},t)^2),$$

for $(\mathbf{x},t) \in Q_{\varepsilon,I}$ *and with* $\pi = \pi(\mathbf{x},t)$.

Substituting expansions (4.21) into the Navier–Stokes equations, collecting zero- and first-order (in ε) terms and using Lemma 4.3, the following result is derived in [15, Section 4]:

Theorem 4.4. *Let* $\mathbf{v}_\varepsilon$ *and* p_ε *satisfy the Navier–Stokes equations (see* (4.20)*) in the moving domain* $\Omega_\varepsilon(t)$ *with given normal velocity* V_Γ*. Then, the zeroth-order velocity field* $\mathbf{v}$ *and the zeroth- and first-order terms* p *and* p^1 *satisfy the following equations on* $\Gamma(t)$*:*

$$\begin{cases} \mathbf{v}\cdot\mathbf{n} = V_\Gamma, \\ \dot{\mathbf{v}} = -\widehat{\nabla}_\Gamma p - p^1\mathbf{n} + 2\mu_0\,\widehat{\mathrm{div}}_\Gamma\mathbf{E}(\mathbf{v}), \\ \widehat{\mathrm{div}}_\Gamma\mathbf{v} = 0. \end{cases} \tag{4.22}$$

Here, the surface rate of strain tensor $\mathbf{E}$ *is as in* (4.10).

We briefly discuss the result of Theorem 4.4. Comparing (4.22) with (4.18), we see that instead of $-p\kappa\mathbf{n}$ in (4.18) we now have $p^1\mathbf{n}$ and an additional equation $\mathbf{v}\cdot\mathbf{n} = V_\Gamma$, with a given V_Γ. In (4.22) we then have a closed system for the unknowns $\mathbf{v}$, p, p^1. The redundant equation $\mathbf{v}\cdot\mathbf{n} = V_\Gamma$ can be eliminated and a "projected" system for the tangential velocity and surface pressure p can be derived. This is further discussed in Section 5.

We now indicate why in (4.22) the first-order term p^1 arises. From differentiation of expansion (4.21b), we get for a fixed $t \in I$:

$$\begin{aligned} \widehat{\nabla} p_\varepsilon(\mathbf{x},t) &= \widehat{\nabla}[p(\pi,t)] + \widehat{\nabla}[d(\mathbf{x},t)p^1(\pi,t)] + r(d) \\ &= \widehat{\nabla}_\Gamma p^e + \widehat{\nabla} d(\mathbf{x},t)\; p^1(\pi,t) + d(\mathbf{x},t)\widehat{\nabla}[p^1(\pi,t)] + r(d) \\ &= \widehat{\nabla}_\Gamma p^e + \mathbf{n}(\mathbf{x},t)p^1(\pi,t) + r(d), \end{aligned} \tag{4.23}$$

with $\pi = \pi(\mathbf{x},t)$ and $d = d(\mathbf{x},t)$. For $\varepsilon \to 0$ we obtain the relation $\widehat{\nabla} p_\varepsilon = \widehat{\nabla}_\Gamma p + p^1\mathbf{n}$. Hence, we expect $\widehat{\nabla}_\Gamma p + p^1\mathbf{n}$, and not only $\widehat{\nabla}_\Gamma p$, to occur in (4.22). Analogous to (4.23) we get, for $\varepsilon \to 0$, expressions for $\widehat{\nabla}\mathbf{v}_\varepsilon$ and $\widehat{\mathrm{div}}\,(\widehat{\nabla}\mathbf{v}_\varepsilon)$ that contain$\mathbf{v}^1$ and $\mathbf{v}^2$. The functions $\mathbf{v}^1$ and $\mathbf{v}^2$, however, do *not* occur in the surface Navier–Stokes system (4.22). This is based on the relations

$$\mathbf{v}^1 = -(\nabla_\Gamma\mathbf{v})\mathbf{n}, \quad \mathbf{v}^2 = -\frac{1}{2}\big(\mathbf{B}\nabla_\Gamma\mathbf{v} + \nabla_\Gamma\mathbf{v}^1\big)\mathbf{n} = 0$$

(cf. [15, Remark 4.5]). Therefore, the resulting system of equations (see (4.22)) has (only) the unknowns $\mathbf{v}$, p and p^1.

4.5. Thin film approach in curvilinear coordinates

Similar to the modeling approach outlined in the previous subsection, the authors of [17] derive *tangential* surface Navier–Stokes equations based on a thin film limit procedure. Instead of using Cartesian coordinates, a *three-dimensional curvilinear* thin film coordinate system is used. In the subsections below, we outline this approach and the resulting surface Navier–Stokes equations.

As in the previous section, it is assumed that the evolution of $\Gamma(t)$ is known a priori through a *given* normal velocity V_Γ. Furthermore, an evolving thin film domain $\Omega_\varepsilon(t)$ is given, which has constant thickness with the surface located in the middle of this domain.

4.5.1. Thin film curvilinear coordinate system. We introduce a surface parametrization, different from the one in (2.1)–(2.2), based on the normal velocity field [4]. More precisely, we consider the initial value problems given in (2.1) with the velocity field $\mathbf{v}$ replaced by $V_\Gamma \mathbf{n}$ and a corresponding flow map (cf. Section 2) denoted by Φ_t^n. Instead of the parametrization in (2.3), we use $R_n(\boldsymbol{\xi}, t) := \Phi_t^n(\Phi_U(\boldsymbol{\xi}))$. A natural parametrization of the thin film domain $\Omega_\varepsilon(t)$ is given by

$$\widetilde{R}_n(\boldsymbol{\xi}, \zeta, t) := R_n(\boldsymbol{\xi}, t) + \zeta \mathbf{n}(\boldsymbol{\xi}, t), \tag{4.24}$$

with $\boldsymbol{\xi} \in U$, $\zeta \in (-\varepsilon, \varepsilon)$. Based on this thin film parametrization, we introduce—analogous to Section 3—curvilinear coordinates and representations of differential operators in these coordinates. Note that in Section 3 we used a *two*-dimensional surface parametrization with first fundamental form denoted by $g_{\alpha\beta}$, whereas in this section we have a *three*-dimensional parametrization of the tubular domain $\Omega_\varepsilon(t)$. As in the previous sections, we use Greek letters to sum over $1, 2$ and Latin letters to sum over $1, 2, 3$. Partial derivatives are denoted by ∂_i, that is, $\partial_i = \frac{\partial}{\partial \xi_i}$, $i = 1, 2$, $\partial_3 = \frac{\partial}{\partial \zeta}$. We introduce the covariant basis $\mathbf{G}_i = \partial_i \widetilde{R}_n$, the corresponding contravariant basis $\mathbf{G}^i$, the metric tensor $G_{ij} := \mathbf{G}_i \cdot \mathbf{G}_j$ and the Christoffel symbols $\boldsymbol{\Gamma}_{ij}^k := \frac{1}{2} G^{kl}(\partial_i G_{jl} + \partial_j G_{il} - \partial_l G_{ij})$. Derivatives in curvilinear coordinates $(\boldsymbol{\xi}, \zeta)$ can be defined completely by analogy with Section 3.2.1. For a scalar function ϕ, we define the gradient $\nabla \phi := \partial_i \phi \mathbf{G}^i$; for a vector field $\mathbf{u}$, we define the (covariant) derivative $\nabla \mathbf{u} = \partial_i \mathbf{u} \otimes \mathbf{G}^i$ and the divergence $\operatorname{div} \mathbf{u} := \partial_i \mathbf{u} \cdot \mathbf{G}^i$; and for an operator-valued function $\mathbf{T}$, the divergence $\operatorname{div} \mathbf{T} = (\partial_i \mathbf{T})^T \mathbf{G}^i$. Using the fact that these operators do not depend on the choice of the parametrization (cf. [5]), one obtains the following relations with differential operators in Euclidean three-dimensional space, for which we used the $\widehat{\ }$ notation (cf. Section 3.2.2):

$$\nabla \phi = \widehat{\nabla} \phi, \quad \nabla \mathbf{u} = \widehat{\nabla} \mathbf{u}, \quad \operatorname{div} \mathbf{u} = \widehat{\operatorname{div}} \mathbf{u}, \quad \operatorname{div} \mathbf{T} = \widehat{\operatorname{div}} (\mathbf{T}^T), \tag{4.25}$$

where $\widehat{\operatorname{div}} \mathbf{T} := \widehat{\operatorname{div}} (\mathbf{T}^T \widehat{\mathbf{e}}_i) \widehat{\mathbf{e}}_i$ is the usual row-wise divergence of a tensor. Below, we use the notation without $\widehat{\ }$. Analogous to Theorem 3.2 (cf. also equations (3.6)–(3.8)), one can represent these operators in terms of local components, for example:

$$(\nabla \mathbf{u})_{ij} = u_{i|j} (\mathbf{G}^i \otimes \mathbf{G}^j), \quad \operatorname{div} \mathbf{u} = u^i{}_{|i}, \quad \operatorname{div} \mathbf{T} = T_i{}^j{}_{|j} \mathbf{G}^i,$$

with

$$u_{i|j} := \partial_j u_i - \mathbf{\Gamma}^k_{ij} u_k, \quad u^j_{\;|i} := \partial_i u^j + \mathbf{\Gamma}^j_{ki} u^k, \quad T_i^{\;j}{}_{|k} := G^{jl} T_{il|k},$$
$$T^{ij}_{\;|k} := \partial_k T^{ij} + T^{lj}\mathbf{\Gamma}^i_{lk} + T^{il}\mathbf{\Gamma}^j_{lk}, \quad T_{ij|k} := \partial_k T_{ij} - T_{lj}\mathbf{\Gamma}^l_{ik} - T_{il}\mathbf{\Gamma}^l_{jk}.$$

When deriving a limit equation in Section 4.5.3 below, it is convenient to relate the three-dimensional metric tensor G_{ij} to a suitable surface metric on $\Gamma(t)$. For the latter, we use the one induced by the parametrization R_n. With a slight abuse of notation, we use the same symbols as in Section 3.1; for example, $g_{\alpha\beta}$ for the metric tensor induced by R_n. One can derive the following useful results for these metric tensors [17]:

$$\begin{aligned} &G_{\alpha\beta} = g_{\alpha\beta} - 2\zeta b_{\alpha\beta} + \zeta^2 b_{\alpha\gamma} b^\gamma_\beta, \quad G_{\zeta\zeta} = 1, \quad G_{\zeta\alpha} = G_{\alpha\zeta} = 0, \\ &G^{\alpha\beta} = g^{\alpha\beta} + \mathcal{O}(\zeta), \quad G^{\zeta\zeta} = 1, \quad G^{\zeta\alpha} = G^{\alpha\zeta} = 0, \quad \mathbf{\Gamma}^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} + \mathcal{O}(\zeta), \\ &\mathbf{\Gamma}^\zeta_{\alpha\beta} = b_{\alpha\beta} + \mathcal{O}(\zeta), \quad \mathbf{\Gamma}^\beta_{\alpha\zeta} = \mathbf{\Gamma}^\beta_{\zeta\alpha} = -b^\beta_\alpha + \mathcal{O}(\zeta), \quad \mathbf{\Gamma}^\zeta_{i\zeta} = \mathbf{\Gamma}^\zeta_{\zeta i} = \mathbf{\Gamma}^j_{\zeta\zeta} = 0. \end{aligned} \tag{4.26}$$

The material derivative is defined as in Section 3.2.4, but now with respect to the parametrization $\widetilde{R}_n(\boldsymbol{\xi}, \zeta, t)$:

$$\dot{f}(\mathbf{y}, t) := \partial_t \bar{f}(\boldsymbol{\xi}, \zeta, t) = \partial_t f(\widetilde{R}_n(\boldsymbol{\xi}, \zeta, t), t), \quad \mathbf{y} = \widetilde{R}_n(\boldsymbol{\xi}, \zeta, t) \in \Omega_\varepsilon(t). \tag{4.27}$$

For the velocity field corresponding to the parametrization $\widetilde{R}_n$, we use the notation

$$\mathbf{w}_R(\mathbf{y}, t) := \frac{\partial}{\partial t}\widetilde{R}_n(\boldsymbol{\xi}, \zeta, t), \quad \mathbf{y} = \widetilde{R}_n(\boldsymbol{\xi}, \zeta, t) \in \Omega_\varepsilon(t). \tag{4.28}$$

Using $\frac{\partial}{\partial t} R_n(\boldsymbol{\xi}, t) = (V_\Gamma \mathbf{n})(R_n(\boldsymbol{\xi}, t), t)$, it follows that $\mathbf{w}_R = V_\Gamma \mathbf{n} + \mathcal{O}(\zeta)$ holds. The material derivative can be reformulated in Cartesian coordinates as

$$\dot{f}(\mathbf{y}, t) = \partial_t f(\mathbf{y}, t) + \nabla f(\mathbf{y}, t) \cdot \mathbf{w}_R(\mathbf{y}, t), \quad \mathbf{y} \in \Omega_\varepsilon(t). \tag{4.29}$$

4.5.2. Navier–Stokes equation in thin film. In [17], the authors derive a surface Ericksen–Leslie model, starting from a simplified local three-dimensional Ericksen–Leslie model (cf. [17, (B1)–(B3)]) in the given evolving thin film domain $\Omega_\varepsilon(t)$. We simplify these equations by taking $\lambda = 0$ in equation (B1). The resulting surface Navier–Stokes equations are similar to the ones in Section 4.4. Note, however, that in that section we used Cartesian coordinates, whereas in this section curvilinear thin film coordinates are used. The space-time domain is as defined in (4.19). The thin film Navier–Stokes system from [17] is given by

$$\begin{aligned} \partial_t \overline{\mathbf{v}}_\varepsilon + \nabla^{\mathbf{u}} \mathbf{v}_\varepsilon &= -\nabla p_\varepsilon + \mu_0 \Delta \mathbf{v}_\varepsilon && \text{in } Q_{\varepsilon,I}, \\ \operatorname{div} \mathbf{v}_\varepsilon &= 0 && \text{in } Q_{\varepsilon,I}, \\ \mathbf{v}_\varepsilon \cdot \mathbf{n}_\varepsilon &= \pm V_\Gamma && \text{on } \partial Q_{\varepsilon,I}, \\ [\mathbf{E}_3(\mathbf{v}_\varepsilon)\mathbf{n}_\varepsilon]_{\tan} &= 0 && \text{on } \partial Q_{\varepsilon,I}, \end{aligned} \tag{4.30}$$

with V_Γ the *given* normal velocity of $\Gamma(t)$, the Laplace operator $\Delta \mathbf{v}_\varepsilon := \operatorname{div} \nabla \mathbf{v}_\varepsilon + \nabla \operatorname{div} \mathbf{v}_\varepsilon$, the tangential component to $\partial\Omega_\varepsilon(t)$ denoted by $[\cdot]_{\tan}$ as in (4.20), and the rate of strain tensor $\mathbf{E}_3(\mathbf{v}_\varepsilon) := \frac{1}{2}(\nabla \mathbf{v}_\varepsilon + \nabla^T \mathbf{v}_\varepsilon)$. In [17], this rate of strain tensor is denoted by the Lie derivative of the metric tensor, $\mathcal{L}_{\mathbf{v}_\varepsilon}\mathbf{G} = \nabla \mathbf{v}_\varepsilon + \nabla^T \mathbf{v}_\varepsilon$. The time derivative $\partial_t \overline{\mathbf{v}}_\varepsilon$ is defined in curvilinear coordinates as in (4.27). The direction $\mathbf{u}$ used in the directional derivative $\nabla^{\mathbf{u}} \mathbf{v}_\varepsilon = \nabla \mathbf{v}_\varepsilon \, \mathbf{u}$ is the relative fluid velocity defined by $\mathbf{u} := \mathbf{v}_\varepsilon - \mathbf{w}_R$, with $\mathbf{w}_R$ as in (4.28). Using (4.29) and (4.25), we obtain

$$\partial_t \overline{\mathbf{v}}_\varepsilon + \nabla^{\mathbf{u}} \mathbf{v}_\varepsilon = \partial_t \mathbf{v}_\varepsilon + \nabla \mathbf{v}_\varepsilon \, \mathbf{w}_R + \nabla \mathbf{v}_\varepsilon (\mathbf{v}_\varepsilon - \mathbf{w}_R) = \partial_t \mathbf{v}_\varepsilon + \nabla \mathbf{v}_\varepsilon \mathbf{v}_\varepsilon,$$

and thus, this is the usual material derivative in Cartesian coordinates; in particular, it is the same as in (4.20). We conclude that the two volume Navier–Stokes systems, (4.30) and (4.20), are equal.

4.5.3. Tangential surface Navier–Stokes system. Using the curvilinear coordinate system, a tangential limit system ($\varepsilon \downarrow 0$) of (4.30) is derived in [17]. We sketch the key ingredients of the derivation.

To simplify the notation, we write $\mathbf{v}$ instead of $\mathbf{v}_\varepsilon$ for the velocity in the thin film domain $\Omega_\varepsilon(t)$. Hence, different from the notation used in the previous sections, $\mathbf{v}$ now denotes a velocity defined in the volume instead of on the surface. In this section, we use $\mathbf{v}_T$ to denote a tangential velocity defined only on the surface.

The covariant components of the rate of strain tensor are given by $\frac{1}{2}(v_{j|i} + v_{i|j})$. The homogeneous Navier boundary condition can be rewritten as

$$v_{\alpha|\zeta} + v_{\zeta|\alpha} = 0 \quad \text{on } \partial\Omega_\varepsilon(t). \tag{4.31}$$

Using this, Taylor expansions, and the results in (4.26), the following relations can be derived:

$$\begin{aligned} v_{\zeta|\zeta}|_\Gamma &= \mathcal{O}(\varepsilon^2), \quad (\mathbf{E}_3(\mathbf{v}))_{\alpha\zeta}|_\Gamma = \mathcal{O}(\varepsilon^2), \\ \partial_\zeta(\mathbf{E}_3(\mathbf{v}))_{\alpha\zeta}|_\Gamma &= \mathcal{O}(\varepsilon^2), \quad (\mathbf{E}_3(\mathbf{v}))_{\alpha\zeta|\zeta}|_\Gamma = \mathcal{O}(\varepsilon^2) \end{aligned} \tag{4.32}$$

(cf. [17, (B9)–(B11), (B13)]). On Γ, we denote the tangential component of the velocity by $\mathbf{v}_T$, that is,

$$\mathbf{v}_T = (v_T)_\alpha \mathbf{g}^\alpha = (v_\alpha \mathbf{g}^\alpha)|_\Gamma = \mathbf{P}\mathbf{v}|_\Gamma \in T^1\Gamma.$$

The following identity holds (cf. [17, (B18)]):

$$v_{\alpha|\beta}|_\Gamma = (v_T)_{\alpha|\beta} - v_N b_{\alpha\beta}. \tag{4.33}$$

We aim to derive equations for $\mathbf{v}_T$ and $p = p_\varepsilon|_\Gamma$ on the surface. We first consider the second equation of (4.30). Using (4.32) and (4.33), the following relation can be derived (cf. [17, (B22)]):

$$0 = (\operatorname{div} \mathbf{v})|_\Gamma = \operatorname{div}_\Gamma \mathbf{v}_T - v_N \kappa + \mathcal{O}(\varepsilon^2). \tag{4.34}$$

We now treat the projection of the material derivative in the first equation of (4.30). Using $\mathbf{u}|_\Gamma = \mathbf{P}\mathbf{v}|_\Gamma$, (4.33) and $(v_T)^\beta (v_T)_{\alpha|\beta}\mathbf{g}^\alpha = (\nabla_\Gamma \mathbf{v}_T)\mathbf{v}_T =: \nabla_\Gamma^{\mathbf{v}_T}\mathbf{v}_T$, we obtain for the tangential part of the directional derivative $\nabla^{\mathbf{u}}\mathbf{v}$

$$\begin{aligned}[\nabla^{\mathbf{u}}\mathbf{v}]_\alpha|_\Gamma &= u^i v_{\alpha|i}|_\Gamma = v^\beta v_{\alpha|\beta}|_\Gamma \\ &= (v_T)^\beta\big((v_T)_{\alpha|\beta} - v_N b_{\alpha\beta}\big) = [\nabla_\Gamma^{\mathbf{v}_T}\mathbf{v}_T - v_N \mathbf{B}\mathbf{v}_T]_\alpha. \end{aligned} \tag{4.35}$$

Using $\partial_t \widetilde{R}_n|_\Gamma = v_N \mathbf{n}$ and the splitting $\mathbf{v} = v^\alpha \partial_\alpha \widetilde{R}_n + v_\zeta \mathbf{n}_\varepsilon$, the following relation for the tangential component of the time derivative can be derived (cf. [17, (B24)]):

$$[\partial_t \overline{\mathbf{v}}]_\alpha|_\Gamma = g_{\alpha\beta}\partial_t(\overline{v}_T)^\beta - v_N(b_{\alpha\beta}(v_T)^\beta + \partial_\alpha v_N). \tag{4.36}$$

From (4.35) and (4.36), we obtain (cf. [17, (B26)]):

$$\mathbf{P}(\partial_t \overline{\mathbf{v}} + \nabla^{\mathbf{u}}\mathbf{v})|_\Gamma = (\partial_t(\overline{v}_T)^\alpha)\mathbf{g}_\alpha + \nabla_\Gamma^{\mathbf{v}_T}\mathbf{v}_T - v_N(2\mathbf{B}\mathbf{v}_T + \nabla_\Gamma v_N). \tag{4.37}$$

For the pressure term $p = p_\varepsilon|_\Gamma$ in (4.30), we get:

$$\mathbf{P}(\nabla p_\varepsilon)|_\Gamma = \nabla_\Gamma p. \tag{4.38}$$

Finally, we consider the projection of the Laplacian in the first equation in (4.30). For a solenoidal vector field we have

$$\mathbf{P}(\Delta\mathbf{v})|_\Gamma = ((\Delta\mathbf{v})_\alpha \mathbf{g}^\alpha)|_\Gamma = 2\big((\operatorname{div}\mathbf{E}_3(\mathbf{v}))_\alpha \mathbf{g}^\alpha\big)|_\Gamma.$$

Using (4.32), the following relation can be derived (cf. [17, (B17)]):

$$\begin{aligned}(\operatorname{div}\mathbf{E}_3(\mathbf{v}))_\alpha|_\Gamma &= g^{\beta\gamma}\big(((\mathbf{E}_3(\mathbf{v}))_{\alpha\gamma}|_\Gamma)_{|\beta} - b_{\alpha\beta}(\mathbf{E}_3(\mathbf{v}))_{\zeta\gamma}|_\Gamma\big) + \mathcal{O}(\varepsilon^2) \\ &= g^{\beta\gamma}\big((\mathbf{E}_3(\mathbf{v}))_{\alpha\gamma}|_\Gamma\big)_{|\beta} + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Using $(\mathbf{E}(\mathbf{v}_T))_{\alpha\gamma} = \frac{1}{2}((v_T)_{\alpha|\gamma} + (v_T)_{\gamma|\alpha})$, we get

$$\begin{aligned}(\mathbf{E}_3(\mathbf{v}))_{\alpha\gamma}|_\Gamma = \frac{1}{2}(v_{\alpha|\gamma} + v_{\gamma|\alpha})|_\Gamma &= \frac{1}{2}((v_T)_{\alpha|\gamma} + (v_T)_{\gamma|\alpha}) - v_N b_{\alpha\gamma} \\ &= (\mathbf{E}(\mathbf{v}_T) - v_N\mathbf{B})_{\alpha\gamma}.\end{aligned}$$

Combining these results, we obtain

$$\begin{aligned}\mathbf{P}(\Delta\mathbf{v})|_\Gamma &= 2\big((\operatorname{div}\mathbf{E}_3(\mathbf{v}))_\alpha \mathbf{g}^\alpha\big)|_\Gamma \\ &= 2g^{\beta\gamma}\big((\mathbf{E}_3(\mathbf{v}))_{\alpha\gamma}|_\Gamma\big)_{|\beta}\mathbf{g}^\alpha + \mathcal{O}(\varepsilon^2) \\ &= 2g^{\beta\gamma}(\mathbf{E}(\mathbf{v}_T) - v_N\mathbf{B})_{\alpha\gamma|\beta}\mathbf{g}^\alpha + \mathcal{O}(\varepsilon^2) \\ &= 2(\mathbf{E}(\mathbf{v}_T) - v_N\mathbf{B})^{\beta\mu}_{\ \ |\beta}\, g_{\alpha\mu}\mathbf{g}^\alpha + \mathcal{O}(\varepsilon^2) \\ &= 2(\mathbf{E}(\mathbf{v}_T) - v_N\mathbf{B})^{\beta\mu}_{\ \ |\beta}\,\mathbf{g}_\mu + \mathcal{O}(\varepsilon^2) \\ &= 2\mathbf{P}\operatorname{div}_\Gamma(\mathbf{E}(\mathbf{v}_T) - v_N\mathbf{B}) + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{4.39}$$

Here, we used Lemma 3.6 in the last equation.

Remark 4.5. Note that (4.39) seems to differ from the first equation of (B21) in [17]. However, different definitions of the surface divergence operators for operator-valued functions are involved. Let $\widetilde{\mathrm{div}}_\Gamma$ be the surface divergence operator used in [17]. For an operator-valued function $\mathbf{T}$, the relation

$$\widetilde{\mathrm{div}}_\Gamma \mathbf{T} = \mathbf{P}\,\mathrm{div}_\Gamma \mathbf{T}$$

holds. Using this and $2\mathbf{E}(\mathbf{v}_T) = \nabla_\Gamma \mathbf{v}_T + \nabla_\Gamma^T \mathbf{v}_T$, it follows that the first identity in [17, (B21)] and equation (4.39) coincide.

Combining results (4.37), (4.38), (4.39), (4.34) and considering the thin film limit $\varepsilon \to 0$, we obtain the tangential Navier–Stokes equations on the surface $\Gamma(t)$ in local coordinates (cf. [17, (B27)–(B28)]):

$$\begin{cases} (\partial_t(\overline{v}_T)^\alpha)\mathbf{g}_\alpha + \nabla_\Gamma^{\mathbf{v}_T}\mathbf{v}_T - v_N(2\mathbf{B}\mathbf{v}_T + \nabla_\Gamma v_N) = -\nabla_\Gamma p + 2\mu_0 \mathbf{P}\,\mathrm{div}_\Gamma(\mathbf{E}(\mathbf{v}_T) - v_N\mathbf{B}), \\ \mathrm{div}_\Gamma \mathbf{v}_T = v_N\kappa. \end{cases} \tag{4.40}$$

Using $\partial_t\overline{\mathbf{v}}_T = (\partial_t(\overline{v}_T)^\alpha)\partial_\alpha R_n + (v_T)^\alpha \partial_\alpha\partial_t R_n$, we obtain for the tangential part of the time derivative

$$(\partial_t\overline{\mathbf{v}}_T)_\alpha = \partial_t\overline{\mathbf{v}}_T \cdot \partial_\alpha R_n = g_{\alpha\beta}\partial_t(\overline{v}_T)^\beta - v_N b_{\alpha\beta}(v_T)^\beta.$$

Hence, the tangential surface Navier–Stokes equations in (4.40) posed on the surface $\Gamma(t)$ can be rewritten as (cf. [17, (B30)–(B31)]):

$$\begin{cases} \mathbf{P}\partial_t\overline{\mathbf{v}}_T + \nabla_\Gamma^{\mathbf{v}_T}\mathbf{v}_T - v_N(\mathbf{B}\mathbf{v}_T + \nabla_\Gamma v_N) = -\nabla_\Gamma p + 2\mu_0 \mathbf{P}\,\mathrm{div}_\Gamma(\mathbf{E}(\mathbf{v}_T) - v_N\mathbf{B}), \\ \mathrm{div}_\Gamma \mathbf{v}_T = v_N\kappa. \end{cases} \tag{4.41}$$

5. Discussion of surface Navier–Stokes equations

In this section, we compare the different equations and discuss a directional splitting in tangential and normal components. For the surface differential operators, we use the ones without $\widehat{\ }$, but this is irrelevant (cf. Theorem 3.8). As already mentioned above, approaches (1) and (2) result in the same system of surface Navier–Stokes equations. We recall the resulting equations (cf. (4.7), (4.11)), where for convenience we put $\rho = 1$:

$$\begin{cases} \dot{\mathbf{v}} = \mathbf{f} - \nabla_\Gamma p - p\kappa\mathbf{n} + 2\mu_0\,\mathrm{div}_\Gamma \mathbf{E}(\mathbf{v}) & \text{on } \Gamma(t), \\ \mathrm{div}_\Gamma \mathbf{v} = 0 & \text{on } \Gamma(t), \\ \dfrac{d}{dt}\mathbf{x}(\mathbf{z},t) = \mathbf{v}(\mathbf{x}(\mathbf{z},t),t), \quad \mathbf{x}(\mathbf{z},0) = \mathbf{z} \in \Gamma(0). \end{cases} \tag{5.1}$$

In approaches (3), (4) and (5), the evolution of the evolving surface is assumed to be given, and thus the surface parametrization, which is an unknown in (5.1), is a known quantity. Approach (3) yields (4.18), which coincides with the first two equations in (5.1), with $\mathbf{f} = 0$. Due to the fact that the normal velocity is known, this is an overdetermined system. Below, we derive a (closed) projected system for the tangential velocity (cf. (5.5)).

We recall the system of equations posed on the surface $\Gamma(t)$ resulting from ansatz (4) (cf. (4.22)):

$$\begin{cases} \mathbf{v} \cdot \mathbf{n} = V_\Gamma, \\ \dot{\mathbf{v}} = -\nabla_\Gamma p - p^1 \mathbf{n} + 2\mu_0 \operatorname{div}_\Gamma \mathbf{E}(\mathbf{v}), \\ \operatorname{div}_\Gamma \mathbf{v} = 0. \end{cases} \tag{5.2}$$

In this system an additional unknown scalar function p^1 appears.

In approach (5), (only) a *tangential* surface Navier–Stokes system on the surface $\Gamma(t)$ is derived, given in (4.41), which we repeat here:

$$\begin{cases} \mathbf{P}\partial_t \overline{\mathbf{v}}_T + \nabla_\Gamma \mathbf{v}_T \mathbf{v}_T = -\nabla_\Gamma p + 2\mu_0 \mathbf{P} \operatorname{div}_\Gamma (\mathbf{E}(\mathbf{v}_T) - v_N \mathbf{B}) + v_N (\mathbf{B}\mathbf{v}_T + \nabla_\Gamma v_N), \\ \operatorname{div}_\Gamma \mathbf{v}_T = v_N \kappa. \end{cases} \tag{5.3}$$

In the following, for (5.1) and (5.2) we consider a splitting of the equations for $\mathbf{v} = \mathbf{v}_T + v_N \mathbf{n}$ and p in *coupled* equations for $\mathbf{v}_T$, p ("tangential surface Navier–Stokes") and for v_N (normal velocity); In [11], the relations

$$\begin{aligned} \mathbf{P}\dot{\mathbf{v}} = \dot{\mathbf{v}}_T + (\dot{\mathbf{n}} \cdot \mathbf{v}_T)\mathbf{n} + v_N \dot{\mathbf{n}}, \quad \dot{\mathbf{v}} \cdot \mathbf{n} = \dot{v}_N - \mathbf{v}_T \cdot \dot{\mathbf{n}}, \\ \mathbf{n} \cdot \operatorname{div}_\Gamma \mathbf{E}(\mathbf{v}) = \operatorname{tr}(\mathbf{B}\nabla_\Gamma \mathbf{v}_T) - v_N \operatorname{tr}(\mathbf{B}^2) \end{aligned} \tag{5.4}$$

(cf. [11, Lemma 2.1, (3.9)]) are derived. Using these, we obtain the splitting of the surface Navier–Stokes equations given in (5.1) into (coupled) equations

$$\begin{aligned} \dot{\mathbf{v}}_T &= \mathbf{f}_T - \nabla_\Gamma p + 2\mu_0 \mathbf{P} \operatorname{div}_\Gamma \mathbf{E}(\mathbf{v}) - ((\dot{\mathbf{n}} \cdot \mathbf{v}_T)\mathbf{n} + v_N \dot{\mathbf{n}}), \\ \operatorname{div}_\Gamma \mathbf{v}_T &= v_N \kappa, \end{aligned} \tag{5.5}$$

for the surface pressure p and tangential velocity $\mathbf{v}_T$ and

$$\begin{aligned} \dot{v}_N &= f_N + 2\mu_0 \mathbf{n} \cdot \operatorname{div}_\Gamma \mathbf{E}(\mathbf{v}) - p\kappa + \dot{\mathbf{n}} \cdot \mathbf{v}_T \\ &= f_N + 2\mu_0 \big(\operatorname{tr}(\mathbf{B}\nabla_\Gamma \mathbf{v}_T) - v_N \operatorname{tr}(\mathbf{B}^2)\big) - p\kappa + \dot{\mathbf{n}} \cdot \mathbf{v}_T, \end{aligned} \tag{5.6}$$

for the normal velocity v_N. We used the splitting $\mathbf{f} = \mathbf{f}_T + f_N \mathbf{n}$. Note that $\dot{\mathbf{v}}_T$ denotes the material derivative (along $\mathbf{v}$) of $\mathbf{v}_T$ and not $(\dot{\mathbf{v}})_T$; similarly for $\dot{v}_N$. We call system (5.5) *tangential surface Navier–Stokes equations*. Note that in these equations the normal velocity v_N occurs.

Remark 5.1. The variational principle used in [12] to derive system (4.18) (with $\mathbf{f} = 0$) also directly leads to a tangential surface Navier–Stokes system if the class of "admissible" velocities $\mathbf{w}$ in the defining relations for the force terms F_{cons} and F_{diss} is restricted to tangential ones, that is, $\mathbf{P}\mathbf{w} = \mathbf{w}$. This yields tangential force terms $F_{\text{cons}} = -\rho \mathbf{P}\dot{\mathbf{v}}$ and $F_{\text{diss}} = 2\mu_0 \mathbf{P} \operatorname{div}_\Gamma \mathbf{E}(\mathbf{v})$ and a tangential momentum equation that is the same as the first equation in (5.5) with $\mathbf{f}_T = 0$.

From the relation

$$\dot{\mathbf{n}} = -\mathbf{B}\mathbf{v}_T - \nabla_\Gamma v_N \tag{5.7}$$

(cf. [11, Lemma 2.2]), it follows that no $\frac{\partial}{\partial t}$ is involved in $\dot{\mathbf{n}}$, which indicates that equation (5.6) determines the time dynamics of the normal velocity $v_N(\cdot, t)$, and thus of the surface $\Gamma(t)$, whereas the tangential surface Navier–Stokes equations in (5.5) determine the time dynamics of the tangential velocity $\mathbf{v}_T(\cdot, t)$.

We now consider the splitting of the Navier–Stokes system (5.2). Applying the projection $\mathbf{P}$ to the second equation in (5.2), the term $p^1\mathbf{P}\mathbf{n}$ vanishes and the remaining terms are the same as in the projected version of the first equation in (5.1). This implies that (5.2) results in *the same* tangential surface Navier–Stokes equations as in (5.5) (with $\mathbf{f}_T = 0$). Taking the scalar product of the second equation in (5.2) with $\mathbf{n}$ and using the results in (5.4), one obtains

$$\dot{v}_N = 2\mu_0(\mathrm{tr}(\mathbf{B}\nabla_\Gamma \mathbf{v}_T) - v_N \mathrm{tr}(\mathbf{B}^2)) - p^1 + \dot{\mathbf{n}} \cdot \mathbf{v}_T, \tag{5.8}$$

that is, similar to the normal velocity equation (5.6), but with $p\kappa$ replaced by the first-order unknown pressure function p^1 (and with $f_N = 0$). From the first equation in (5.2), with given V_Γ, one obtains the normal velocity v_N, which can be substituted in the tangential surface Navier–Stokes equations, which then determine $\mathbf{v}_T$ and p. Given v_N and $\mathbf{v}_T$, the unknown p^1 is determined by (5.8).

Finally, we compare the tangential Navier–Stokes equations in (5.5) with the tangential equations in (5.3) that result from ansatz (5). Both systems contain the same equation $\mathrm{div}_\Gamma \mathbf{v}_T = v_N\kappa$, which results from the inextensibility condition. We now show that the two tangential momentum equations in (5.5) and (5.3) are also the same if $\mathbf{f}_T = 0$. This can be done as follows: First, note that the material derivative $\dot{\mathbf{v}}_T$ in (5.5) is in general *not* tangential. Its normal component is balanced by the term $(\dot{\mathbf{n}} \cdot \mathbf{v}_T)\mathbf{n}$ on the right-hand side in (5.5). This normal component can be eliminated by using the relations $\mathbf{n} \cdot \dot{\mathbf{v}}_T = -\dot{\mathbf{n}} \cdot \mathbf{v}_T$, which follows from $\mathbf{n} \cdot \mathbf{v}_T = 0$, and

$$\dot{\mathbf{v}}_T = \mathbf{P}\dot{\mathbf{v}}_T + (\mathbf{n} \cdot \dot{\mathbf{v}}_T)\mathbf{n} = \mathbf{P}\dot{\mathbf{v}}_T - (\dot{\mathbf{n}} \cdot \mathbf{v}_T)\mathbf{n}.$$

Using this, (5.7) and $\mathbf{E}(\mathbf{v}) = \mathbf{E}(\mathbf{v}_T) - v_N\mathbf{B}$, the tangential momentum equation in (5.5), with $\mathbf{f}_T = 0$, can be rewritten as

$$\begin{aligned} \mathbf{P}\dot{\mathbf{v}}_T &= -\nabla_\Gamma p + 2\mu_0 \mathbf{P}\,\mathrm{div}_\Gamma \mathbf{E}(\mathbf{v}) - v_N \dot{\mathbf{n}} \\ &= -\nabla_\Gamma p + 2\mu_0 \mathbf{P}\,\mathrm{div}_\Gamma(\mathbf{E}(\mathbf{v}_T) - v_N\mathbf{B}) + v_N(\mathbf{B}\mathbf{v}_T + \nabla_\Gamma v_N). \end{aligned} \tag{5.9}$$

The right-hand side of this equation is the same as the right-hand side in (5.3). We now compare the material derivatives on the left-hand sides. Applying Lemma 3.10, the left-hand side of (5.9) yields

$$\mathbf{P}\dot{\mathbf{v}}_T = \mathbf{P}(\partial_t \mathbf{v}_T^e + \nabla \mathbf{v}_T^e \mathbf{v}). \tag{5.10}$$

For the left-hand side in (5.3) we obtain, using (4.27) and (4.29),

$$\begin{aligned}\mathbf{P}\partial_t\overline{\mathbf{v}}_T + \nabla_\Gamma \mathbf{v}_T\mathbf{v}_T &= \mathbf{P}(\partial_t\overline{\mathbf{v}}_T + \nabla\mathbf{v}_T^e\mathbf{v}_T)\\ &= \mathbf{P}(\partial_t\mathbf{v}_T^e + v_N\nabla\mathbf{v}_T^e\mathbf{n} + \nabla\mathbf{v}_T^e\mathbf{v}_T)\\ &= \mathbf{P}(\partial_t\mathbf{v}_T^e + \nabla\mathbf{v}_T^e\mathbf{v}),\end{aligned}$$

and comparing this with (5.10), we observe that the material derivatives also coincide. Hence, we conclude that the two tangential momentum equations in (5.5) and (5.3) are the same (for $\mathbf{f}_T = 0$).

In summary, we have shown that all five derivations ((1)–(5)) lead to *the same tangential surface Navier–Stokes equations* (see (5.5)). Derivations (1)–(3) result in the same equation for the normal velocity, namely, the one in (5.6).

A. Appendix

We give a proof of the second equality in (3.4). The first equality can be derived in the same way.

Proof of (3.4). The product rule, (3.3), and the symmetry of the Christoffel symbols yield

$$\begin{aligned}\partial_\gamma\mathbf{T} &= \partial_\gamma T_{\alpha\beta}(\mathbf{g}^\alpha\otimes\mathbf{g}^\beta) + T_{\alpha\beta}\big((\partial_\gamma\mathbf{g}^\alpha\otimes\mathbf{g}^\beta) + (\mathbf{g}^\alpha\otimes\partial_\gamma\mathbf{g}^\beta)\big)\\ &= \partial_\gamma T_{\alpha\beta}(\mathbf{g}^\alpha\otimes\mathbf{g}^\beta) + T_{\alpha\beta}\big((-\Gamma^\alpha_{\gamma\mu}\mathbf{g}^\mu + b^\alpha_\gamma\mathbf{g}^3)\otimes\mathbf{g}^\beta + \mathbf{g}^\alpha\otimes(-\Gamma^\beta_{\gamma\mu}\mathbf{g}^\mu + b^\beta_\gamma\mathbf{g}^3)\big)\\ &= \big(\partial_\gamma T_{\alpha\beta} - \Gamma^\mu_{\gamma\alpha}T_{\mu\beta} - \Gamma^\mu_{\gamma\beta}T_{\alpha\mu}\big)(\mathbf{g}^\alpha\otimes\mathbf{g}^\beta) + T_{\alpha\beta}b^\alpha_\gamma(\mathbf{g}^3\otimes\mathbf{g}^\beta)\\ &\qquad + T_{\alpha\beta}b^\beta_\gamma(\mathbf{g}^\alpha\otimes\mathbf{g}^3)\\ &= T_{\alpha\beta|\gamma}(\mathbf{g}^\alpha\otimes\mathbf{g}^\beta) + T_{\alpha\beta}b^\alpha_\gamma(\mathbf{g}^3\otimes\mathbf{g}^\beta) + T_{\alpha\beta}b^\beta_\gamma(\mathbf{g}^\alpha\otimes\mathbf{g}^3).\end{aligned}$$

■

Proof of Lemma 3.6. We represent $\mathbf{T}$ in local coordinates as $\mathbf{T} = T^{\alpha\beta}(\mathbf{g}_\alpha\otimes\mathbf{g}_\beta)$. From the definition of the divergence and Theorem 3.2, we get

$$\begin{aligned}\mathrm{div}_\Gamma\mathbf{T} &= (\partial_\alpha\mathbf{T})^T\mathbf{g}^\alpha = \big(T^{\gamma\beta}_{\ |\alpha}(\mathbf{g}_\gamma\otimes\mathbf{g}_\beta) + T^{\gamma\beta}b_{\alpha\gamma}(\mathbf{g}_3\otimes\mathbf{g}_\beta) + T^{\gamma\beta}b_{\alpha\beta}(\mathbf{g}_\gamma\otimes\mathbf{g}_3)\big)^T\mathbf{g}^\alpha\\ &= \big(T^{\gamma\beta}_{\ |\alpha}(\mathbf{g}_\beta\otimes\mathbf{g}_\gamma) + T^{\gamma\beta}b_{\alpha\gamma}(\mathbf{g}_\beta\otimes\mathbf{g}_3) + T^{\gamma\beta}b_{\alpha\beta}(\mathbf{g}_3\otimes\mathbf{g}_\gamma)\big)\mathbf{g}^\alpha\\ &= T^{\gamma\beta}_{\ |\alpha}\mathbf{g}_\beta\underbrace{(\mathbf{g}_\gamma\cdot\mathbf{g}^\alpha)}_{\delta^\alpha_\gamma} + T^{\gamma\beta}b_{\alpha\gamma}\mathbf{g}_\beta\underbrace{(\mathbf{g}_3\cdot\mathbf{g}^\alpha)}_{=0} + T^{\gamma\beta}b_{\alpha\beta}\mathbf{g}_3\underbrace{(\mathbf{g}_\gamma\cdot\mathbf{g}^\alpha)}_{=\delta^\alpha_\gamma}\\ &= T^{\alpha\beta}_{\ |\alpha}\mathbf{g}_\beta + T^{\alpha\beta}b_{\alpha\beta}\mathbf{g}_3.\end{aligned}$$

■

Lemma A.1. *The shape operator can be represented in Cartesian coordinates by the matrix* $\mathbf{B} = -\widehat{\nabla}_\Gamma\mathbf{n}^e$.

Proof. We use Theorem 3.8, Theorem 3.2 and the symmetry of $b_{\alpha\beta}$ to derive

$$-\widehat{\nabla}_\Gamma\mathbf{n}^e = -\nabla_\Gamma\mathbf{n} = -\nabla_\alpha\mathbf{n}\otimes\mathbf{g}^\alpha = -(\mathbf{P}\partial_\alpha\mathbf{n})\otimes\mathbf{g}^\alpha = (\mathbf{P}b_{\alpha\beta}\mathbf{g}^\beta)\otimes\mathbf{g}^\alpha = \mathbf{B}.$$

■

Proof of Lemma 3.11. The second equality follows from the equality of the covariant gradients (cf. Theorem 3.8). We prove the first equality. Using (3.7) and (3.3), we get

$$\nabla_\Gamma \mathbf{v}_T + \nabla_\Gamma^T \mathbf{v}_T = (\partial_\alpha v_\beta - \Gamma_{\alpha\beta}^\tau v_\tau)(\mathbf{g}^\beta \otimes \mathbf{g}^\alpha) + (\partial_\alpha v_\beta - \Gamma_{\alpha\beta}^\tau v_\tau)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta).$$

A direct calculation using (3.3) yields

$$\begin{aligned}\frac{1}{2}(\nabla_\Gamma(v_N\mathbf{n}) + \nabla_\Gamma^T(v_N\mathbf{n})) &= \frac{1}{2}(\mathbf{g}^\alpha \otimes \mathbf{P}\partial_\alpha(v_N\mathbf{n}) + \mathbf{P}\partial_\alpha(v_N\mathbf{n}) \otimes \mathbf{g}^\alpha) \\ &= \frac{1}{2}\big(-v_N b_{\alpha\beta}(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) - v_N b_{\alpha\beta}(\mathbf{g}^\beta \otimes \mathbf{g}^\alpha)\big) \\ &= -v_N b_{\alpha\beta}(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta).\end{aligned}$$

Using these results and (3.15), we obtain

$$\begin{aligned}\mathbf{E}(\mathbf{v}) &= E_{\alpha\beta}(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) = \Big(\frac{1}{2}(v_{\alpha|\beta} + v_{\beta|\alpha}) - v_N b_{\alpha\beta}\Big)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) \\ &= \Big(\frac{1}{2}(\partial_\beta v_\alpha - \Gamma_{\alpha\beta}^\tau v_\tau + \partial_\alpha v_\beta - \Gamma_{\alpha\beta}^\tau v_\tau) - v_N b_{\alpha\beta}\Big)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) \\ &= \frac{1}{2}(\partial_\alpha v_\beta - \Gamma_{\alpha\beta}^\tau v_\tau)(\mathbf{g}^\beta \otimes \mathbf{g}^\alpha) + \frac{1}{2}(\partial_\alpha v_\beta - \Gamma_{\alpha\beta}^\tau v_\tau)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) - v_N b_{\alpha\beta}(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) \\ &= \frac{1}{2}(\nabla_\Gamma \mathbf{v}_T + \nabla_\Gamma^T \mathbf{v}_T) + \frac{1}{2}(\nabla_\Gamma(v_N\mathbf{n}) + \nabla_\Gamma^T(v_N\mathbf{n})) = \frac{1}{2}(\nabla_\Gamma \mathbf{v} + \nabla_\Gamma^T \mathbf{v}).\end{aligned}$$

■

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