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Higher integrability of the gradient for the thermal insulation problem

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Abstract. We prove the higher integrability of the gradient for local minimizers of the thermal insulation problem: an analogue of De Giorgi's conjecture for the Mumford–Shah functional. We deduce that the singular part of the free boundary has Hausdorff dimension strictly less than $n-1$.

1. Introduction

We fix a bounded connected set $\Omega \subset \mathbb{R}^n$. The thermal insulation problem consists in minimizing the functional

$$\mathcal{I}(A,u) := \int_A |\nabla u|^2 \,\mathrm{d}\mathcal{L}^n + \int_{\partial A} |u^*|^2 \,\mathrm{d}\mathcal{H}^{n-1} + \mathcal{L}^n(A) \tag{1}$$

among all pairs (A,u) where $A \subset \mathbb{R}^n$ is an admissible domain and $u \in W^{1,2}(A)$ is a function such that $u = 1$ for $\mathcal{L}^n$-a.e. on Ω. Here, u^* is the trace of u on ∂A.

The problem has been studied by Caffarelli and Kriventsov in [5, 11] and Bucur, Giacomini and Luckhaus in [3, 4]. The authors transpose the problem to a slightly different setting in order to apply the direct method of the calculus of variations. The authors represent a pair (A,u) by the function $u\mathbf{1}_A$ and relax the functional on SBV. The new problem consists in minimizing the functional

$$\mathcal{F}(u) := \int_{\mathbb{R}^n} |\nabla u|^2 \,\mathrm{d}\mathcal{L}^n + \int_{J_u} (\overline{u}^2 + \underline{u}^2) \,\mathrm{d}\mathcal{H}^{n-1} + \mathcal{L}^n(\{u > 0\}) \tag{2}$$

among all functions $u \in SBV(\mathbb{R}^n)$ such that $u = 1$ $\mathcal{L}^n$-a.e. on Ω. This new setting is more suited to a direct minimization since it enjoys the compactness and closure properties of SBV. In short, there always exist functions $u \in SBV(\mathbb{R}^n)$ such that $u = 1$ $\mathcal{L}^n$-a.e. on Ω and $\mathcal{F}(u) < \infty$. For example, $u := \mathbf{1}_B$ where B is an open ball containing Ω. In [5, Theorem 4.2], Caffarelli and Kriventsov prove that the SBV problem has a solution u. A key property of this solution is that there exists $0 < \delta < 1$ (depending on n, Ω) such that $\mathrm{spt}(u) \subset B(0,\delta^{-1})$ and

$$u \in \{0\} \cup [\delta, 1] \quad \mathcal{L}^n\text{-a.e. on } \mathbb{R}^n. \tag{3}$$

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This property has also been proved in [4]. On another note, some minimality criteria have been proved by calibrations in [12].

The main goal of the present article is to prove that there exists $p > 1$ such that $|\nabla u|^2 \in L^p_{\mathrm{loc}}(\mathbb{R}^n \setminus \overline{\Omega})$ (Theorem 4.1). A parallel property was conjectured by De Giorgi for minimizers of the Mumford–Shah functional and solved by De Lellis and Focardi in the planar case [7] and then De Philippis and Figalli [8] in the general case. Our proof is inspired by the technique of [8] and it relies on three key properties: the Ahlfors-regularity of the free boundary, the uniform rectifiability of the free boundary and the ε-regularity theorem. In particular, this implies a porosity property which means that the singular part Σ of the free boundary has many holes in a quantified way. In contrast to the Mumford–Shah situation, the ε-regularity theorem describes a regular part of the boundary as a pair of graphs rather than just one graph. The minimizer satisfies an elliptic equation with a Robin boundary condition at the boundary rather than a Neumann boundary condition. We present the technique of [8] in a different way by singling out a higher integrability lemma and a covering lemma and by removing the need for [8, Lemma 3.2] (the existence of good radii). Once we establish the higher integrability of the gradient, we are also able to conclude that the dimension of Σ is strictly less than $n-1$ (Theorem 5.1). The link between the higher integrability of the gradient and the dimension of the singular part has been observed first for the Mumford–Shah functional by Ambrosio, Fusco, Hutchinson in [1]. An open question from Caffarelli and Kriventsov hints that for all minimizers in the planar case, Σ is empty and the optimal exponent is $p = \infty$ (see also Remark 5.2).

2. Generalities about minimizers

2.1. Definition

Notation. Our ambient space is an open set X of $\mathbb{R}^n$. One can think of X as $\mathbb{R}^n \setminus \overline{\Omega}$. For $x \in \mathbb{R}^n$ and $r > 0$, $B(x,r)$ is the *open ball* centered in x and of radius r. If there is no ambiguity, it is simply denoted by B_r. Given an open ball $B := B(x,r)$ and a scalar $t > 0$, the notation tB means $B(x,tr)$. Given a set $A \subset \mathbb{R}^n$, the *indicator function* of A is denoted by $\mathbf{1}_A$. Given two sets $A, B \subset \mathbb{R}^n$, the notation $A \subset\subset B$ means that there exists a compact set $K \subset \mathbb{R}^n$ such that $A \subset K \subset B$.

Given $u \in SBV_{\mathrm{loc}}(X)$, we denote by K the support of the singular part of Du:

$$K := \mathrm{spt}(|\overline{u} - \underline{u}|\mathcal{H}^{n-1} \llcorner J_u) \tag{4a}$$

$$:= \mathrm{spt}(\mathcal{H}^{n-1} \llcorner J_u). \tag{4b}$$

For $x \in K$ and $r > 0$ such that $\overline{B}(x,r) \subset X$, we define

$$\omega_2(x,r) := r^{-(n-1)} \int_{B(x,r)} |\nabla u|^2 \,\mathrm{d}\mathcal{L}^n, \tag{5a}$$

$$\beta_2(x,r) := \Big(r^{-(n+1)} \inf_V \int_{K \cap \overline{B}(x,r)} \mathrm{d}(y,V)^2 \,\mathrm{d}\mathcal{H}^{n-1}(y)\Big)^{\frac{1}{2}}, \tag{5b}$$

where V runs among $n-1$ planes $V \subset \mathbb{R}^n$ passing through x. When there is ambiguity, we will write $\beta_{K,2}$ instead of β_2.

For any open ball B such that $\overline{B} \subset X$, we define a *competitor of* u *in* B as a function $v \in SBV_{\mathrm{loc}}(X)$ such that $v = u$ $\mathcal{L}^n$-a.e. on $X \setminus \overline{B}$. We fix a constant $\delta \in]0,1[$ throughout the paper.

Definition 2.1. We say that $u \in SBV_{\mathrm{loc}}(X)$ is a *local minimizer* if

(1) for $\mathcal{L}^n$-a.e. $x \in X$, we have $u \in \{0\} \cup [\delta, \delta^{-1}]$;

(2) for all open balls B such that $\overline{B} \subset X$ and for all competitors v of u in B,

$$\int_B |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n + \int_{J_u \cap \overline{B}} (\overline{u}^2 + \underline{u}^2) \, \mathrm{d}\mathcal{H}^{n-1} + \mathcal{L}^n(\{u > 0\} \cap B)$$
$$\leq \int_B |\nabla v|^2 \, \mathrm{d}\mathcal{L}^n + \int_{J_v \cap \overline{B}} (\overline{v}^2 + \underline{v}^2) \, \mathrm{d}\mathcal{H}^{n-1} + \mathcal{L}^n(\{v > 0\} \cap B). \quad (6)$$

As a first consequence, we have that $\overline{u}, \underline{u} \in \{0\} \cup [\delta, \delta^{-1}]$ *everywhere* in X. In particular, $\overline{u} \geq \delta$ everywhere on S_u. For all open balls B such that $\overline{B} \subset X$, we have

$$\int_B |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n + \int_{J_u \cap \overline{B}} (\overline{u}^2 + \underline{u}^2) \, \mathrm{d}\mathcal{H}^{n-1} < \infty. \quad (7)$$

This shows that $|\nabla u|^2 \in L^1_{\mathrm{loc}}(X)$ and that S_u is $\mathcal{H}^{n-1}$-locally finite in X. In $X \setminus \overline{S_u}$, the function u belongs to $W^{1,2}_{\mathrm{loc}}$ and locally minimizes its Dirichlet energy. Therefore, u is harmonic (and thus continuous) in $X \setminus \overline{S_u}$. We conclude that in each connected component of $X \setminus \overline{S_u}$, we have either $u > \delta$ everywhere or $u = 0$ everywhere.

2.2. Properties

The next results (Ahlfors-regularity, uniform rectifiability and the ε-regularity theorem) also hold true for the almost-minimizers of [11, Definition 2.1]. We are going to cite [5, Corollary 3.3 and Theorem 5.1].

Proposition 2.2 (Ahlfors-regularity). *Let $u \in SBV_{\mathrm{loc}}(X)$ be a local minimizer. There exist constants $0 < r_0 \leq 1$ and $C \geq 1$ (both depending on n, δ) such that the following holds true:*

(1) *For all $x \in X$ and for all $0 \leq r \leq r_0$ such that $B(x,r) \subset X$,*

$$\int_{B(x,r)} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n + \mathcal{H}^{n-1}(K \cap B(x,r)) \leq C r^{n-1}. \quad (8)$$

(2) *For all $x \in \overline{S_u}$ and for all $0 \leq r \leq r_0$ such that $B(x,r) \subset X$,*

$$\mathcal{H}^{n-1}(K \cap B(x,r)) \geq C^{-1} r^{n-1}. \quad (9)$$

Corollary 2.3. *Let $u \in SBV_{\mathrm{loc}}(X)$ be a local minimizer. The following statements hold:*

(i) *We have $K = \overline{S_u} = \overline{J_u}$ and $\mathcal{H}^{n-1}(K \setminus J_u) = 0$.*

(ii) *The set $A_u := \{\overline{u} > 0\} \setminus K$ is open and $\partial A_u = K$.*

Proof. It is straightforward to see by definition that $K \subset \overline{J_u} \subset \overline{S_u}$. On the other hand, property (9) shows that $\overline{S_u} \subset K$. We shall show that $\mathcal{H}^{n-1}(K \setminus J_u) = 0$. The jump set J_u is Borel and $\mathcal{H}^{n-1}$-locally finite in X, so for $\mathcal{H}^{n-1}$-a.e. $x \in X \setminus J_u$,

$$\lim_{r \to 0} \frac{\mathcal{H}^{n-1}(J_u \cap B(x,r))}{r^{n-1}} = 0 \tag{10}$$

(see [14, Theorem 6.2]). We draw our claim from the observation that this limit contradicts (9).

Consider the set A_u. We recall that the function $\overline{u}$ is continuous in $X \setminus K$ (since it coincides with u outside S_u) and $\overline{u} \in \{0\} \cup [\delta, 1]$ *everywhere* in $X \setminus K$. As a consequence, the sets

$$A_u := \left\{\overline{u} > 0\right\} \setminus K, \tag{11}$$

$$B_u := \left\{\overline{u} = 0\right\} \setminus K \tag{12}$$

are open subsets of $X \setminus K$ and thus of X. The space X is the disjoint union

$$X = K \cup A_u \cup B_u, \tag{13}$$

where A_u and B_u are open and K is relatively closed, so $\overline{A_u} \subset A_u \cup K$.

We show that $S_u \subset \overline{A_u}$. Let us suppose that there exist $x \in S_u$ and $r > 0$ such that $A_u \cap B(x,r) = \emptyset$. Then, $B(x,r) \setminus K \subset \{\overline{u} = 0\}$, so we have $u = 0$ $\mathcal{L}^n$-a.e. on $B(x,r)$ and thus x is a Lebesgue point of u, which is a contradiction. We conclude that $S_u \subset \overline{A_u}$ and in turn, $K \subset \overline{A_u}$ so $\overline{A_u} = A_u \cup K$. ■

Now, we are going to apply [6] to show that K is locally contained in a uniformly rectifiable set. We underline that our local minimizers are not quasiminimizers as in [6, Definition 7.21]. We show in Appendix B that the results of [6] also apply to our local quasiminimizers (see Remark B.4).

Proposition 2.4 (Uniform rectifiability). *Let $u \in SBV_{\mathrm{loc}}(X)$ be a local minimizer. There exists $0 < r_0 \leq 1$ (depending on n, δ) such that the following holds true: For all $x \in K$ and $0 < r \leq r_0$ such that $B(x,r) \subset X$, there is a closed, Ahlfors-regular, uniformly rectifiable set E of dimension $n-1$ such that $K \cap \frac{1}{2}B(x,r) \subset E$. The constants for the Ahlfors-regularity and uniform rectifiability depend on n, δ.*

Proof. We want to show that (u, K) satisfies Definition B.1, or rather the alternative Definition given in Remark B.4. Then, the Proposition will follow from Theorem B.3. First, it is clear that (u, K) is an admissible pair. Let B be an open ball of radius $r > 0$ such that $\overline{B} \subset X$. Let an admissible pair (v, L) be a competitor of (u, K) in B. We can assume without loss of generality that L is $\mathcal{H}^{n-1}$-locally finite. Therefore, $v \in SBV_{\mathrm{loc}}(X)$ and $\mathcal{H}^{n-1}(J_v \setminus L) = 0$. We can now apply the minimality inequality. We have

$$\begin{aligned}\int_B |\nabla u|^2 \,\mathrm{d}\mathcal{L}^n + \int_{J_u \cap \overline{B}} (\overline{u}^2 + \underline{u}^2) \,\mathrm{d}\mathcal{H}^{n-1} + \mathcal{L}^n(\{u > 0\} \cap B)\\ \leq \int_B |\nabla v|^2 \,\mathrm{d}\mathcal{L}^n + \int_{J_v \cap \overline{B}} (\overline{v}^2 + \underline{v}^2) \,\mathrm{d}\mathcal{H}^{n-1} + \mathcal{L}^n(\{v > 0\} \cap B),\end{aligned} \tag{14}$$

so

$$\begin{aligned}\int_B |\nabla u|^2 \,\mathrm{d}\mathcal{L}^n + \delta^2 \mathcal{H}^{n-1}(J_u \cap \overline{B}) + \mathcal{L}^n(\{u > 0\} \cap B)\\ \leq \int_B |\nabla v|^2 \,\mathrm{d}\mathcal{L}^n + \delta^{-2} \mathcal{H}^{n-1}(J_v \cap \overline{B}) + \mathcal{L}^n(\{v > 0\} \cap B).\end{aligned} \tag{15}$$

We omit the term $\mathcal{L}^n(\{u > 0\} \cap B)$ on the left and we bound the term $\mathcal{L}^n(\{v > 0\} \cap B)$ on the right by $\omega_n r^n$, where ω_n is the Lebesgue volume of the unit ball. We can replace J_u by K on the left, since $\mathcal{H}^{n-1}(K \setminus J_u) = 0$. We can replace J_v by L on the right, since $\mathcal{H}^{n-1}(J_v \setminus L) = 0$. It follows that

$$\mathcal{H}^{n-1}(K \cap \overline{B}) \leq \delta^{-4} \mathcal{H}^{n-1}(L \cap \overline{B}) + \delta^{-2} \Delta E + \delta^{-2} \omega_n r^n \tag{16}$$

where

$$\Delta E := \int_B |\nabla v|^2 - \int_B |\nabla u|^2 \,\mathrm{d}\mathcal{L}^n. \tag{17}$$

This completes the proof. ∎

We are going to cite the ε-regularity theorem for our problem [11, Theorem 14.1]. Contrary to the ε-regularity theorem for the Mumford–Shah problem, it does not require $\omega_2(x, r)$ to be small. It says that when K is very close to a plane, K is given by a pair of smooth graphs. We describe this situation in the next definition.

Given a point $x \in \mathbb{R}^n$ and a vector $e_n \in \mathbf{S}^{n-1}$, we can decompose each point $y \in \mathbb{R}^n$ under the form $y = x + (y' + y_n e_n)$, where $y' \in e_n^\perp$ and $y_n \in \mathbb{R}$. Then, for all functions $f : e_n^\perp \to \mathbb{R}$, we define the graph of f in the coordinate system (x, e_n) as

$$\Gamma_{(x,e_n)}(f) := \{y \in \mathbb{R}^n \mid y_n = f(y')\}. \tag{18}$$

Definition 2.5. Let $u \in SBV_{\mathrm{loc}}(X)$ be a local minimizer. Let $x \in K$ and $R > 0$ be such that $B(x, R) \subset X$. Let $0 < \alpha \leq 1$. We say that K is $C^{1,\alpha}$*-regular* in $B := B(x, R) \subset X$ if it satisfies the following three conditions:

(i) There exist a vector $e_n \in \mathbf{S}^{n-1}$ and two functions $f_i : e_n^\perp \to \mathbb{R}$ $(i = 1, 2)$ such that $f_1 \leq f_2$ and

$$K \cap B = \Big(\bigcup_{i=1,2} \Gamma_{(x,e_n)}(f_i)\Big) \cap B. \tag{19}$$

The functions f_1, f_2 are $C^{1,\alpha}$, and

$$R^{-1} |f_i|_\infty + |\nabla f_i|_\infty + R^\alpha \, [\nabla f_i]_\alpha \leq \frac{1}{4}. \tag{20}$$

(ii) There are two possible cases. The first case is

$$\begin{cases} u > 0 & \text{in } \{y \in B \mid y_n < f_1(y') \text{ or } y_n > f_2(y')\}, \\ u = 0 & \text{in } \{y \in B \mid f_1(y') < y_n < f_2(y')\}. \end{cases} \tag{21}$$

The second case is $f_1 = f_2$ and

$$\begin{cases} u > 0 & \text{in } \{y \in B \mid y_n > f_1(y')\}, \\ u = 0 & \text{in } \{y \in B \mid y_n < f_1(y')\}, \end{cases} \tag{22}$$

or the inverted versions of either of the above.

Theorem 2.6 (ε-regularity theorem). *Let $u \in SBV_{\text{loc}}(X)$ be a local minimizer and let $x \in K$. Then, the following holds:*

(i) *For all $\varepsilon > 0$, there exists $\varepsilon_1 > 0$ (depending on n, δ, β) such that the following holds true: For $r > 0$ such that $\overline{B}(x,r) \subset X$ and $\beta_2(x,r) + r \leq \varepsilon_1$, we have $\omega_2(x, \frac{r}{2}) \leq \varepsilon$.*

(ii) *There exist $\varepsilon > 0$, $C \geq 1$ and $0 < \alpha < 1$ (both depending on n, δ) such that the following holds true: For $r > 0$ such that $\overline{B}(x,r) \subset X$ and $\beta_2(x,r) + r \leq \varepsilon$, the set K is $C^{1,\alpha}$-regular in $B(x, C^{-1}R)$.*

This last result is specific to local minimizers and does not hold true for the general almost-minimizers of [11, Definition 2.1].

Proposition 2.7. *Let $u \in SBV_{\text{loc}}(X)$ be a local minimizer. Let $x \in K$ and $R > 0$ be such that $B(x,R) \subset X$. We assume that K is regular in $B := B(x,R)$ (Definition 2.5) and we denote by Γ_i the graph of f_i in B ($i = 1,2$). Then, for each $i = 1,2$, $u_{|A_i}$ solves the Robin problem*

$$\begin{cases} \Delta u = 0 & \text{in } A_i, \\ \partial_{\nu_i} u - u_i = 0 & \text{in } \Gamma_i, \end{cases} \tag{23}$$

where

$$A_1 := \{y \in B \mid y_n < f_1(y')\}, \tag{24}$$

$$A_2 := \{y \in B \mid y_n > f_2(y')\}, \tag{25}$$

and ν_i is the inner normal vector to A_i.

Proof. We only provide details for case (21) of Definition 2.5 and we prove the proposition for $i = 2$. We partition B into three sets (modulo $\mathcal{L}^n$):

$$A_1 := \{z \in B \mid y_n < f_1(y')\}, \tag{26}$$

$$A_2 := \{z \in B \mid y_n > f_2(y')\}, \tag{27}$$

$$A_3 := \{z \in B \mid f_1(y') < x_n < f_2(y')\}. \tag{28}$$

The first paragraph is devoted to detailing a few generalities about traces and upper/lower limits. We consider a general $v \in L^\infty(B) \cap W^{1,2}_{\mathrm{loc}}(B \setminus K)$ such that $v = 0$ in A_3. For each $i = 1, 2$, there exists $v_i^* \in L^1(\Gamma_i)$ such that for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma_i$,

$$\lim_{r \to 0} r^{-n} \int_{A_i \cap B(x,r)} |v(y) - v_i^*(x)| \, \mathrm{d}\mathcal{L}^n(y) = 0. \tag{29}$$

The boundary Γ_i is C^1, so for all $x \in \Gamma_i$, there is a vector $\nu_i(x) \in \mathbf{S}^{n-1}$ such that

$$\lim_{r \to 0} r^{-n} \mathcal{L}^n((A_i \Delta H_i^+(x)) \cap B(x,r)) = 0 \tag{30}$$

where

$$H_i^+(x) := \{y \in \mathbb{R}^n \mid (y - x) \cdot \nu_i(x) > 0\}. \tag{31}$$

Therefore, (29) is equivalent to

$$\lim_{r \to 0} r^{-n} \int_{H_i^+(x) \cap B(x,r)} |v(y) - v_i^*(x)| \, \mathrm{d}\mathcal{L}^n(y) = 0. \tag{32}$$

For $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma_2$, we describe the relationship between $\overline{v}(x)^2 + \underline{v}(x)^2$ and $v_i^*(x)$. We fix $x \in \Gamma_2 \setminus \Gamma_1$ such that (32) is satisfied for $i = 2$. We have $v = 0$ on A_3, and $B(x,r)$ is disjoint from A_1 for small $r > 0$, so

$$\lim_{r \to 0} r^{-n} \int_{(X \setminus A_2) \cap B(x,r)} |v| \, \mathrm{d}\mathcal{L}^n = 0, \tag{33}$$

which is equivalent to

$$\lim_{r \to 0} r^{-n} \int_{H_2^-(x) \cap B(x,r)} |v| \, \mathrm{d}\mathcal{L}^n = 0. \tag{34}$$

Combining (32) for $i = 2$ and (34), we deduce

$$\overline{v}(x)^2 + \underline{v}(x)^2 = v_2^*(x)^2. \tag{35}$$

Next, we fix $x \in \Gamma_1 \cap \Gamma_2$ such that (32) holds true for $i = 1$ and $i = 2$. The surfaces Γ_1 and Γ_2 necessarily have the same tangent plane at x and the vectors ν_i are opposed. Combining (32) for $i = 1$ and $i = 2$, we deduce

$$\overline{v}^2 + \underline{v}^2 = (v_1^*)^2 + (v_2^*)^2. \tag{36}$$

We come back to our local minimizer $u \in SBV_{\mathrm{loc}}(X)$. We fix $\varphi \in C_c^1(B)$. For $\varepsilon \in \mathbb{R}$, we define $v : X \to \mathbb{R}$ by

$$v := \begin{cases} u + \varepsilon\varphi & \text{in } A_2, \\ u & \text{in } X \setminus A_2. \end{cases} \tag{37}$$

It is clear that $\{v \neq u\} \subset\subset B$ and that v is C^1 in $X \setminus K$. As K is $\mathcal{H}^{n-1}$-locally finite in X, we have $v \in SBV_{\mathrm{loc}}(X)$ and $S_v \subset K$. Remember that $u \geq \delta$ in $A_1 \cup A_2$, and $u = 0$

in A_3. We take ε small enough so that $\varepsilon|\varphi|_\infty < \delta$. As a consequence, $v > 0$ in $A_1 \cup A_2$ and $v = 0$ in A_3. Let us check the multiplicities on the discontinuity set. As we have seen before, $J_v \cap B \subset \Gamma_1 \cup \Gamma_2$. We observe that for $x \in \Gamma_2$ such that the trace $u_2^*(x)$ exists, we have

$$v_2^*(x) = u_2^*(x) + \varepsilon\varphi(x), \tag{38}$$

and for $x \in \Gamma_1$ such the trace $u_1^*(x)$ exists, we have

$$v_1^*(x) = u_1^*(x). \tag{39}$$

Using the previous observations, we deduce that for $\mathcal{H}^{n-1}$-a.e. on $\Gamma_2 \setminus \Gamma_1$,

$$\begin{aligned} \overline{v}^2 + \underline{v}^2 &= (u_2^* + \varepsilon\varphi)^2 \\ &= (\underline{u}^2 + \overline{u}^2) + 2\varepsilon\varphi u_2^* + \varepsilon^2|\varphi|^2; \end{aligned} \tag{40}$$

for $\mathcal{H}^{n-1}$-a.e. on $\Gamma_2 \cap \Gamma_1$,

$$\begin{aligned} \overline{v}^2 + \underline{v}^2 &= (u_2^* + \varepsilon\varphi)^2 + (u_1^*)^2 \\ &= (\underline{u}^2 + \overline{u}^2) + 2\varepsilon\varphi u_2^* + \varepsilon^2|\varphi|^2; \end{aligned} \tag{41}$$

and that for $\mathcal{H}^{n-1}$-a.e. on $\Gamma_1 \setminus \Gamma_2$,

$$\overline{v}^2 + \underline{v}^2 = (u_1^*)^2 = \underline{u}^2 + \overline{u}^2. \tag{42}$$

Finally, it is clear that

$$\int_B |\nabla v|^2 \,\mathrm{d}\mathcal{L}^n = \int_B |\nabla u|^2 \,\mathrm{d}\mathcal{L}^n + 2\varepsilon \int_{A_2} \langle \nabla u, \nabla\varphi \rangle \,\mathrm{d}\mathcal{L}^n + \varepsilon^2 \int_{A_2} |\nabla\varphi|^2 \,\mathrm{d}\mathcal{L}^n. \tag{43}$$

We then plug all this information into the minimality inequality and obtain that

$$0 \leq 2\varepsilon \int_{A_2} \langle \nabla u, \nabla\varphi \rangle \,\mathrm{d}\mathcal{L}^n + 2\varepsilon \int_{\Gamma_2} \varphi u_2^* \,\mathrm{d}\mathcal{H}^{n-1} + C(\varphi)\varepsilon^2. \tag{44}$$

As this holds true for all small ε (positive or negative), we conclude that

$$\int_{A_2} \langle \nabla u, \nabla\varphi \rangle \,\mathrm{d}\mathcal{L}^n + \int_{\Gamma_2} \varphi u_2^* \,\mathrm{d}\mathcal{H}^{n-1} = 0. \tag{45}$$

This completes the proof. ■

3. Porosity of the singular part

The next result says that the part where K is not regular has many holes in a quantified way. It also holds true for the almost-minimizers of [11, Definition 2.1]. This result is simpler to obtain than its Mumford–Shah counterpart (see [15]) because the ε-regularity theorem of [11] only requires us to control the flatness.

Proposition 3.1 (Porosity). *Let $u \in SBV_{\mathrm{loc}}(X)$ be a local minimizer. There exist constants $0 < r_0 \leq 1$, $C \geq 2$ and $0 < \alpha < 1$ (all depending on n,δ) for which the following holds true: For all $x \in K$ and all $0 < r \leq r_0$ such that $B(x,r) \subset X$, there exists a smaller ball $B(y, C^{-1}r) \subset B(x,r)$ in which K is $C^{1,\alpha}$-regular.*

Proof. The letter C is a constant greater than or equal to 1 that depends on n, δ. The letter α is the constant from Theorem 2.6. For $y \in K$ and $t > 0$ such that $\overline{B}(y,t) \subset X$, we define the L^∞ flatness as

$$\beta_K(y,t) := \inf_V \sup_{z \in K \cap \overline{B}(y,t)} t^{-1}\,\mathrm{d}(z,V), \tag{46}$$

where the infimum is taken over the affine hyperplanes V of $\mathbb{R}^n$ passing through y. Note that in [6, (41.2)], the infimum is taken over all affine hyperplanes V of $\mathbb{R}^n$ (not necessarily passing through y); this would decrease our number β, but by no more than a factor of $\frac{1}{2}$. Indeed, if V is any hyperplane of $\mathbb{R}^n$ and y' is the orthogonal projection of y onto V, then the hyperplane $V - (y' - y)$ is passing through y, so we have

$$\begin{aligned}\beta_K(y,t) &\leq \sup_{z \in K \cap \overline{B}(y,t)} \mathrm{d}(z, V - (y'-y)) \\ &\leq |y' - y| + \sup_{z \in K \cap \overline{B}(y,t)} \mathrm{d}(z,V) \\ &\leq 2 \sup_{z \in K \cap \overline{B}(y,t)} \mathrm{d}(z,V).\end{aligned} \tag{47}$$

We also observe that

$$\beta_{K,2}(y,t)^2 \leq t^{-(n-1)}\mathcal{H}^{n-1}(K \cap \overline{B}(y,t))\beta_K(y,t)^2, \tag{48}$$

so as soon as t is small enough for the Ahlfors-regularity to hold, the inequality $\beta_{K,2}(y,t) \leq C\beta_K(y,t)$ is satisfied.

Let r_0 be the minimum between the radius of Proposition 2.2 (Ahlfors-regularity) and the radius of Proposition 2.4 (uniform rectifiability). We fix $x \in K$ and $0 < r \leq r_0$ such that $B(x,r) \subset X$. According to Proposition 2.4, there exists an Ahlfors-regular and uniformly rectifiable set E such that $K \cap \frac{1}{2}B(x,r) \subset E$. Moreover, the constants for the Ahlfors-regularity and uniform rectifiability depend on n, δ. For $y \in E$ and $t > 0$, we define as before

$$\beta_E(y,t) := \inf_V \sup_{z \in E \cap \overline{B}(y,t)} t^{-1}\,\mathrm{d}(z,V), \tag{49}$$

where the infimum is taken on the set of all affine hyperplanes V of $\mathbb{R}^n$ passing through x. As E is Ahlfors-regular and uniformly rectifiable, the Weak Geometric Lemma (see [6, (73.13)]) states that for all $\varepsilon > 0$, the set

$$\left\{(y,t) \mid y \in E,\ 0 < t < \mathrm{diam}(E),\ \beta_E(y,t) > \varepsilon\right\} \tag{50}$$

is a Carleson set. This means that for all $\varepsilon > 0$, there exists $C_0(\varepsilon) \geq 1$ (depending on n, δ, ε) such that for all $y \in E$ and all $0 < t < \mathrm{diam}(E)$,

$$\int_0^t \int_{E \cap B(y,t)} \mathbf{1}_{\{\beta_E(z,s) > \varepsilon\}}(z) \, \mathrm{d}\mathcal{H}^{n-1}(z) \frac{\mathrm{d}s}{s} \leq C_0(\varepsilon) t^{n-1}. \tag{51}$$

We only apply this property with $y := x$. We observe that for all $z \in K \cap B(x, \frac{1}{4}r)$ and for all $0 < s \leq \frac{1}{4}r$, we have $K \cap \overline{B}(z,s) \subset E \cap \overline{B}(z,s)$, so $\beta_K(z,s) \leq \beta_E(z,s)$. Thus, for all $0 < t < \mathrm{diam}(K \cap \frac{1}{2}B(x,r))$ such that $t \leq \frac{1}{4}r$, we have

$$\int_0^t \int_{K \cap B(x,t)} \mathbf{1}_{\{\beta_K(z,s) > \varepsilon\}}(z) \, \mathrm{d}\mathcal{H}^{n-1}(z) \frac{\mathrm{d}s}{s} \leq C_0(\varepsilon) t^{n-1}. \tag{52}$$

We only apply this property with $t := \frac{1}{4}\mathrm{diam}(K \cap B(x,r))$. Note that $C^{-1}r \leq t \leq \frac{1}{4}r$, where the first inequality comes from the Ahlfors-regularity of K. We are going to deduce from (52) that for all $\varepsilon > 0$, there exist $C(\varepsilon) \geq 1$, a point $z \in K \cap B(x,t)$ and a radius s such that $C(\varepsilon)^{-1} t \leq s \leq t$ and $\beta_K(z,s) \leq \varepsilon$. We proceed by contradiction for some $C(\varepsilon)$ to be determined. We therefore have

$$\begin{aligned} \int_0^t \int_{K \cap B(x,t)} \mathbf{1}_{\{\beta_K(z,s) > \varepsilon\}}(z) \, \mathrm{d}\mathcal{H}^{n-1}(z) \frac{\mathrm{d}s}{s} &\geq \mathcal{H}^{n-1}(K \cap B(x,t)) \int_{C(\varepsilon)^{-1}t}^t \frac{\mathrm{d}s}{s} \\ &\geq \mathcal{H}^{n-1}(K \cap B(x,t)) \ln(C(\varepsilon)) \\ &\geq C^{-1} t^{n-1} \ln(C(\varepsilon)). \end{aligned} \tag{53}$$

This contradicts (52) if $C(\varepsilon)$ is too big compared to $C_0(\varepsilon)$.

We fix $\varepsilon > 0$ (to be determined soon) and we assume that we have a corresponding pair (z,s) as above. In particular, $\beta_{K,2}(z,s) \leq C\beta_K(z,s) \leq C\varepsilon$. According to the second statement of Theorem 2.6, we can fix ε (depending on n, δ) so that if $r_0 \leq \varepsilon$, then K is $C^{1,\alpha}$-regular in $B(z, C^{-1}s)$. ■

4. Higher integrability of the gradient

Theorem 4.1. *Let $u \in SBV_{\mathrm{loc}}(X)$ be minimal. There exist $0 < r_0 \leq 1$, $C \geq 1$ and $p > 1$ (depending on n, δ) such that the following holds true: For all $x \in X$ and all $0 \leq r \leq r_0$ such that $B(x,r) \subset X$,*

$$\int_{\frac{1}{2}B(x,r)} |\nabla u|^{2p} \, \mathrm{d}\mathcal{L}^n \leq C r^{n-p}. \tag{54}$$

The higher integrability is well known for weak solutions of elliptic systems ([9, Theorem 2.1]). In this case, the proof consists in combining the Caccioppoli–Leray inequality and the Sobolev–Poincaré inequality to deduce that $|\nabla u|^{\frac{2n}{n+2}}$ satisfies a reverse Hölder inequality. The higher integrability is then an immediate consequence of the Gehring

Lemma. In our case, u is still a weak solution of an elliptic system, but we lack information about the regularity of K to carry out this method.

We draw inspiration from [8], but we simplify the proof by singling out a higher integrability lemma (Lemma 4.2 below) and a covering lemma (Lemma 4.3 below) and by removing the need of [8, Lemma 3.2] (the existence of good radii).

Proof of Theorem 4.1. There exists $0 < r_0 \leq 1$ such that for all $x \in X$ and for all $0 \leq R \leq r_0$ such that $B(x,R) \subset X$, one can apply Lemma 4.2 below in the ball $B(x,R)$ to the function $v := R|\nabla u|^2$. Assumption (i) follows from the Ahlfors-regularity of K (Proposition 2.2). Assumption (ii) follows from the porosity (Proposition 3.1). Assumption (iii) follows from interior/boundary gradient estimates for the Robin problem and from the Ahlfors-regularity. In particular, the interior estimate can be derived from the subharmonicity of $|\nabla u|^2$ in $X \setminus K$ and the boundary estimate is detailed in Lemma A.1 in Appendix A. ■

Lemma 4.2. *We fix a radius $R > 0$ and an open ball B_R of radius R. Let K be a closed subset of B_R and $v : B_R \to \mathbb{R}^+$ be a non-negative Borel function. We assume that there exist $C_0 \geq 1$ and $0 < \alpha \leq 1$ such that the following holds true:*

(i) *For all balls $B(x,r) \subset B_R$,*

$$C_0 r^{n-1} \leq \mathcal{H}^{n-1}(K \cap B(x,r)) \leq C_0 r^{n-1}. \tag{55}$$

(ii) *For all balls $B(x,r) \subset B_R$ centered in K, there exists a smaller ball $B(y, C_0^{-1} r) \subset B(x,r)$ in which K is $C^{1,\alpha}$-regular (Definition 2.5).*

(iii) *For all balls $B(x,r) \subset B_R$ such that K is disjoint from $B(x,r)$ or K is $C^{1,\alpha}$-regular in $B(x,r)$ (Definition 2.5), we have*

$$\sup_{\frac{1}{2}B(x,r)} v(x) \leq C_0 \left(\frac{R}{r}\right). \tag{56}$$

Then, there exist $p > 1$ and $C \geq 1$ (depending on n, C_0) such that

$$⨍_{\frac{1}{2}B_R} v^p \leq C. \tag{57}$$

The proof of Lemma 4.2 takes advantage of a covering lemma, which we discuss next. We use the notation $\Gamma_{(x,e_n)}$ defined by (18). Assumption (ii) says that in each double ball $2B_k$, the set E is a union of Lipschitz graphs which are close to a hyperplane.

Lemma 4.3 (Covering lemma). *Let $E \subset \mathbb{R}^n$ be a bounded set. Let (B_k) be a family of open balls with center $x_k \in \mathbb{R}^n$ and radius $R_k > 0$. We assume that*

(i) *for all $k \neq l$, $2B_k \cap B_l = \emptyset$;*

(ii) *for all k and all $x \in E \cap 2B_k$, there exist a vector $e_n \in \mathbf{S}^{n-1}$ and a $\frac{1}{2}$-Lipschitz function $f : e_n^\perp \to \mathbb{R}$ such that $|f| \leq \frac{1}{2} R_k$ and*

$$x \in \Gamma_{(x_k, e_n)}(f) \cap 2B_k \subset E. \tag{58}$$

Let $0 < r \leq \inf_k R_k$. There exists a sequence of open balls $(D_i)_{i \in I}$ of radius r and centered in $E \setminus \bigcup_k B_k$ such that

$$E \setminus \bigcup_k B_k \subset \bigcup_{i \in I} D_i \tag{59}$$

and the balls $(20^{-1} D_i)_{i \in I}$ are pairwise disjoint and disjoint from $\bigcup_k B_k$.

Proof. Let $0 < r_0 \leq \inf_k R_k$. We introduce the set

$$F := E \setminus \bigcup_k B_k. \tag{60}$$

The goal is to cover F with a controlled number of balls of radius r_0. Let r be a radius $0 < r \leq r_0$ which will be determined during the proof. As F is bounded, there exists a maximal sequence of points $(x_i) \in F$ such that $B(x_i, r) \subset \mathbb{R}^n \setminus \bigcup_k B_k$ and $|x_i - x_j| \geq r$. For $i \neq j$, we have $|x_i - x_j| \geq r$, so the balls $(B(x_i, \frac{1}{2} r))_i$ are disjoint. Next, we show that

$$F \subset \bigcup_i B(x_i, 10r). \tag{61}$$

Let $x \in F$. If $B(x, r) \subset \mathbb{R}^n \setminus \bigcup_k B_k$, then by maximality of (x_i), there exists i such that $x \in B(x_i, r) \subset B(x_i, 10r)$. Now we focus on the case where there exists an index k_0 such that $B(x, r) \cap B_{k_0} \neq \emptyset$. The radius of B_{k_0} is denoted by R and we assume without loss of generality that its center is 0. As $x \in F = E \setminus \bigcup_k B_k$ and $B(x, r) \cap B(0, R) \neq \emptyset$, we have $R < |x| < R + r$. We are going to build a point $y \in E$ such that $R + r < |y| < R + 7r$ and $|x - y| < 9r$.

Since $r \leq R$, we observe that $x \in B(0, 2R)$. According to the assumptions of the lemma, there exist two scalars $0 < \varepsilon, L \leq \frac{1}{2}$, a vector $e_n \in \mathbf{S}^{n-1}$ and an L-Lipschitz function $f : e_n^\perp \to R$ such that $|f| \leq \varepsilon R$ and

$$x \in \left\{ y \in B(0, 2R) \mid y_n = f(y') \right\} \subset E. \tag{62}$$

Here, we have decomposed each point $y \in \mathbb{R}^n$ under the form $y = y' + y_n e_n$, where $y' \in n^\perp$ and $y_n \in \mathbb{R}$. The estimate $R < |x| < R + r$ can be rewritten as

$$R < |x' + f(x') e_n| < R + r. \tag{63}$$

We consider $t \geq 1$ such that $|tx' + f(x')e_n| = R + 4r$ and we estimate how close tx' is to x'. We have

$$|x'| \geq \sqrt{R^2 - |f(x')|^2}, \tag{64}$$
$$|tx'| \leq \sqrt{(R+4r)^2 - |f(x')|^2}, \tag{65}$$

so

$$\begin{aligned} |tx' - x'| &\leq \sqrt{(R+4r)^2 - |f(x')|^2} - \sqrt{R^2 - |f(x')|^2} \\ &\leq \frac{4Rr + 8r^2}{\sqrt{R^2 - |f(x')|^2}}. \end{aligned} \tag{66}$$

We assume $r \leq \frac{1}{8}R$ and we recall that $|f(x')| \leq \varepsilon R$ with $\varepsilon \leq \frac{1}{2}$, so this simplifies to

$$|tx' - x'| \leq \frac{5r}{\sqrt{1-\varepsilon^2}} < 6r. \tag{67}$$

Next, we define $y := tx' + f(tx')e_n$ and we recall that f is L-Lipschitz with $L \leq \frac{1}{2}$ to estimate

$$|y - [tx' + f(x')e_n]| = |f(tx') - f(x')| < 3r. \tag{68}$$

Since $|tx' + f(x')e_n| = R + 4r$, this yields

$$R + r < |y| < R + 7r. \tag{69}$$

We also estimate

$$|y - x| \leq |tx' - x'| + |f(tx') - f(x')| < 9r. \tag{70}$$

As $r \leq \frac{1}{8}R$, inequalities (69) imply $y \in B(0, 2R)$ and thus $y \in E$. We are going to show that $B(y, r) \subset \mathbb{R}^n \setminus \bigcup_k B_k$. We recall that $B(0, 2R)$ is disjoint from all the other balls of the family (B_k). By (69), we observe that

$$\begin{aligned} B(y, r) &\subset B(0, R + 8r) \setminus B(0, R) \\ &\subset B(0, 2R) \setminus B(0, R), \end{aligned} \tag{71}$$

and our claim follows. By maximality of the family (x_i), there exists i such that the inequality $|y - x_i| < r$ holds, and in turn by (70), $|x - x_i| < 10r$. We finally choose $r := \frac{1}{10}r_0$. The balls (D_i) are given by $D_i := B(x_i, 10r) = B(x_i, r_0)$. ■

Proof of Lemma 4.2. We observe that for any ball $B \subset B_R$ and for $p \geq 1$,

$$\begin{aligned} \int_B v^p \, \mathrm{d}\mathscr{L}^n &= \int_0^\infty \mathscr{L}^n(B \cap \{v^p > t\}) \, \mathrm{d}t \\ &= p \int_0^\infty s^{p-1} \mathscr{L}^n(B \cap \{v > s\}) \, \mathrm{d}s \end{aligned} \tag{72}$$

and for $M \geq 1$,

$$\begin{aligned} p \int_1^{\infty} s^{p-1} \mathcal{L}^n(B \cap \{v > s\}) \, \mathrm{d}s &\leq p \sum_{h=0}^{\infty} \int_{M^h}^{M^{h+1}} s^{p-1} \mathcal{L}^n(B \cap \{v > s\}) \, \mathrm{d}s \\ &\leq p \sum_{h=0}^{\infty} \Big(\int_{M^h}^{M^{h+1}} s^{p-1} \, \mathrm{d}s \Big) \mathcal{L}^n(B \cap \{v > M^h\}) \\ &\leq (M^p - 1) \sum_{h=h}^{\infty} M^{hp} \mathcal{L}^n(B \cap \{v > M^h\}). \end{aligned} \tag{73}$$

Thus, it suffices to prove that there exist $N > M \geq 1$, $C \geq 1$ (depending on n, C_0) such that for all $h \geq 0$,

$$\mathcal{L}^n(\tfrac{1}{2} B_R \cap \{v > M^h\}) \leq C R^n N^{-h}, \tag{74}$$

and then take $p > 1$ such that $M^p N^{-1} < 1$.

To simplify the notation, we change the constant C_0 so that property (ii) yields that $40B(y, C_0^{-1} r) \subset B(x, r)$ and that K is $C^{1,\alpha}$-regular in $4B(y, C_0^{-1} r)$.

Let $M := \max\{4C_0, \frac{1}{4} C_0^2\} \geq 4$. We define for $h \geq 1$

$$A_h := \big\{ x \in \tfrac{1}{2} B_R \setminus K \mid v > M^h \big\}. \tag{75}$$

The proof is based on the fact that A_h is at distance $\sim M^{-h} R$ from K and has many holes of size $\sim M^{-h} R$ near K. We explain more precisely these observations. For the first one, let $h \geq 1$, let $x \in A_h$ and assume that $B(x, C_0 M^{-h} R)$ is disjoint from K. Then, we use property (ii) to estimate

$$v(x) \leq M^h. \tag{76}$$

This contradicts the definition of A_h. We deduce that there exists $y \in K$ such that the inequality $|x - y| < C_0 M^{-h} R$ holds. For the second observation, let $h \geq 2$, let $x \in K \cap \frac{15}{16} B_R$ and apply the porosity property to the ball $B(x, M^{-h} R)$.

We obtain an open ball $B \subset X$ centered in K, of radius $C_0^{-1} M^{-h} R$ and such that K is $C^{1,\alpha}$-regular in $4B$. Then, by property (iii) and since $M \geq \frac{1}{4} C_0^2$,

$$\sup_{2B} v \leq \tfrac{1}{4} C_0^2 M^h \leq M^{h+1}. \tag{77}$$

In particular, $2B$ is disjoint from A_{h+1}.

We start the proof by defining for $h \geq 1$,

$$r(h) := M^{-h} R, \tag{78}$$

$$R(h) := \Big(\frac{3}{4} + M^{-h+1} \Big) R. \tag{79}$$

The sequence $(R(h))$ is decreasing, $\lim_{h \to \infty} R(h) = \frac{3}{4} R$ and $R(h+1) + r(h) \leq R(h)$. For each $h \geq 1$, we build an index set $I(h)$ and a family of balls $(B_i)_{i \in I(h)}$ as follows: First we define $I(1) := \emptyset$, $(B_i)_{i\in I(1)} := \emptyset$. Let $h \geq 2$ be such that $(B_i)_{i\in I(1)}, \dots, (B_i)_{i\in I(h-1)}$ have been built. We assume that the index sets $I(g)$, where $g = 1, \dots, h-1$, are pairwise disjoint. We assume that for all $i \in I_g$, the balls B_i have radius $C_0^{-1} r(g) = C_0^{-1} M^{-g} R$. We assume that for all indices $i, j \in \bigcup_{g=1}^{h-1} I(g)$ with $i \neq j$, we have that $2B_i \cap B_j = \emptyset$ and that K is $C^{1,\alpha}$-regular in $2B_i$. Then, we introduce the sets

$$K_h := K \cap B_{R(h)} \setminus \bigcup_{g=1}^{h-1} \bigcup_{i\in I(g)} B_i, \tag{80}$$

$$K_h^* := K \cap B_{R(h+1)} \setminus \bigcup_{g=1}^{h-1} \bigcup_{i\in I(g)} B_i. \tag{81}$$

According to Lemma 4.3, there exists a sequence of open balls $(D_i)_{i\in I(h)}$ centered in K_h^* of radius $r(h) = M^{-h} R$ such that

$$K_h^* \subset \bigcup_{i\in I(h)} D_i, \tag{82}$$

and such that the balls $(20^{-1} D_i)$ are pairwise disjoint and disjoint from $\bigcup_{g=1}^{h-1} \bigcup_{i\in I(g)} B_i$. We can assume that the index set $I(h)$ is disjoint from the sets $I(g)$, $g = 1, \dots, h-1$. Since $R(h+1) + r(h) \leq R(h)$, we observe that the balls $(20^{-1} D_i)$ are included in

$$B_{R(h)} \setminus \bigcup_{g=1}^{h-1} \bigcup_{i\in I(g)} B_i. \tag{83}$$

Next, we apply the porosity to the balls (D_i). For each $i \in I(h)$, there exists an open ball B_i centered in K, of radius $C_0^{-1} M^{-h} R$ such that $B_i \subset 40^{-1} D_i$, K is $C^{1,\alpha}$-regular in 4 B_i and by property (iii),

$$\sup_{2B_i} v \leq \tfrac{1}{4} C_0^2 M^h \leq M^{h+1}. \tag{84}$$

We should not forget to mention that for all $i \in I(h)$, we have $2B_i \subset 20^{-1} D_i$, so $2B_i$ is disjoint from all the other balls we have built so far.

Now, we estimate $\mathcal{L}^n(A_h)$ for $h \geq 1$. We show first that the points of A_h cannot be too far from K_h^*. Let $x \in A_h$. We have seen earlier that there exists $y \in K$ such that $|x - y| < C_0 M^{-h} R$. We are going to show that $y \in K_h^*$. Since $|x| \leq \frac{1}{2} R$ and $M \geq 4C_0$, we have

$$|y| \leq \tfrac{1}{2} R + C_0 M^{-h} R \leq \tfrac{3}{4} R. \tag{85}$$

Let us assume that there exist $g = 1, \dots, h-1$ and $i \in I(g)$ such that $y \in B_i$. The radius of B_i is $C_0^{-1} M^{-(h-1)} R$ and since $|x - y| < C_0 M^{-h} R$, we have $x \in 2B_i$. However $2B_i$ is disjoint from A_h by construction. We have shown that $y \in K_h^*$. As a consequence, there exists $i \in I(h)$ such that $y \in D_i$. The radius of D_i is $r(h) = M^{-h} R$ and we have that $|x-y| < C_0 M^{-h} R$, so

$$A_h \subset \bigcup_{i \in I(h)} (1+C_0) D_i. \tag{86}$$

This allows us to estimate

$$\mathcal{L}^n(A_h) \leq \omega_n (1+C_0)^n |I(h)| r(h)^n, \tag{87}$$

where ω_n is the Lebesgue measure of the unit ball.

Next, we want to control $|I(h)|$. The balls $(20^{-1} D_i)_{i \in I(h)}$ are disjoint and included in the set $B(R(h)) \setminus \bigcup_{g=2}^{h-1} \bigcup_{i \in I(g)} B_i$, so by Ahlfors-regularity,

$$\begin{aligned} C_0^{-1} 20^{-(n-1)} r(h)^{(n-1)} |I(h)| &\leq \sum_{i \in I(h)} \mathcal{H}^{n-1}(K \cap 12^{-1} D_i) \\ &\leq \mathcal{H}^{n-1}(K_h). \end{aligned} \tag{88}$$

We are going to see that $\mathcal{H}^{n-1}(K_h)$ is bounded from above by a decreasing geometric sequence. We have

$$\begin{aligned} \mathcal{H}^{n-1}(K_h^*) &\leq \sum_{i \in I(h)} \mathcal{H}^{n-1}(K \cap D_i) \\ &\leq C_0 \sum_{i \in I(h)} r(h)^{n-1} \\ &\leq C_0^{n+1} \sum_{i \in I(h)} (C_0^{-1} r(h))^{n-1} \\ &\leq C_0^{n+1} \sum_{i \in I(h)} \mathcal{H}^{n-1}(K \cap B_i) \\ &\leq C_0^{n+1} \mathcal{H}^{n-1}(K_h \setminus K_{h+1}). \end{aligned} \tag{89}$$

We deduce

$$\mathcal{H}^{n-1}(K_h) \leq C_0^{n+1} \mathcal{H}^{n-1}(K_h \setminus K_{h+1}) + \mathcal{H}^{n-1}(K \cap B_{R(h)} \setminus B_{R(h+1)}). \tag{90}$$

We rewrite this inequality as

$$\mathcal{H}^{n-1}(K_{h+1}) \leq \lambda^{-1} \mathcal{H}^{n-1}(K_h) + C_0^{-(n+1)} \mathcal{H}^{n-1}(K \cap B_{R(h)} \setminus B_{R(h+1)}), \tag{91}$$

where $\lambda := C_0^{n+1}(C_0^{n+1}-1)^{-1} > 1$. Then, we multiply both sides of the inequality by λ^{h+1}:

$$\begin{aligned} &\lambda^{h+1} \mathcal{H}^{n-1}(K_{h+1}) \\ &\quad \leq \lambda^h \mathcal{H}^{n-1}(K_h) + C_0^{-(n+1)} \lambda^{-h} \mathcal{H}^{n-1}(K \cap B_{R(h)} \setminus B_{R(h+1)}) \\ &\quad \leq \lambda^h \mathcal{H}^{n-1}(K_h) + \mathcal{H}^{n-1}(K \cap B_{R(h)} \setminus B_{R(h+1)}). \end{aligned} \tag{92}$$

Summing this telescopic inequality, we obtain that for all $h \geq 1$,

$$\lambda^h \mathcal{H}^{n-1}(K_h) \leq 2\mathcal{H}^{n-1}(K \cap B_R) \leq 2C_0 R^{n-1}. \tag{93}$$

In summary, we have proved that for some constant $C \geq 1$, $\lambda > 1$ (depending on n, C_0) and for $h \geq 1$,

$$\mathcal{L}^n(A_h) \leq CR^n(\lambda M)^{-h}. \tag{94}$$

This concludes the proof. ∎

5. Dimension of the singular part

Notation. The Hausdorff dimension of a set $A \subset \mathbb{R}^n$ is defined by

$$\dim_{\mathcal{H}}(A) := \inf\{s \geq 0 \mid H^s(A) = 0\}. \tag{95}$$

We take the convention that for $s < 0$, the term $\mathcal{H}^s$-*a-e.* means *everywhere* and the inequality $\dim_{\mathcal{H}}(A) < 0$ means $A = \emptyset$.

The goal of this section is to explain the link between the integrability exponent of the gradient and the dimension of the singular part. It has been first observed for the Mumford–Shah functional by Ambrosio, Fusco and Hutchinson in [1].

Theorem 5.1. *Let $u \in SBV_{\mathrm{loc}}(X)$ be a local minimizer. We define*

$$\Sigma := \{x \in K \mid K \text{ is not regular at } x\}. \tag{96}$$

For $p > 1$ such that $|\nabla u|^2 \in L^p_{\mathrm{loc}}(X)$, we have

$$\dim_{\mathcal{H}}(\Sigma) \leq \max\{n - p, n - 8\} < n - 1. \tag{97}$$

Remark 5.2. In dimension $n \leq 7$, Caffarelli and Kriventsov have shown that if a point $x \in K$ is at the boundary of two local connected components where $u > 0$ or if it is a 0-density point of $\{u = 0\}$, then x is a regular point ([5, Theorem 8.2]). In dimension $n = 2$, they show furthermore that if x is at the boundary of a connected component of $\{u = 0\}$, then it is a regular point ([5, Corollary 9.2]). Thus, in the planar case, a point of Σ must be an accumulation point of connected components of $\{u = 0\}$. There is, however, no known example of such a situation.

Theorem 5.1 will be proved very easily with the help of [5, Theorem 8.2] and the following well-known result:

Lemma 5.3. *Let $v \in L^p_{\mathrm{loc}}(X)$ for some $p \geq 1$ and let $s < n$. Then, for $\mathcal{H}^{n-p(n-s)}$-a.e. $x \in X$,*

$$\lim_{r \to 0} r^{-s} \int_{B(x,r)} v \, d\mathcal{L}^n = 0. \tag{98}$$

Proof. Without loss of generality, we assume $v \geq 0$. We start with the case $p = 1$. We define μ as the measure $v\mathcal{L}^n$ and we want to show that for $\mathcal{H}^s$-a.e. $x \in X$, we have

$$\lim_{r\to 0} r^{-s}\mu(B(x,r)) = 0. \tag{99}$$

If $s < 0$, the limit is indeed 0 for every $x \in X$. In the case $0 \leq s < n$, we fix a closed ball $\overline{B} \subset X$, a scalar $\lambda > 0$ and a set

$$A := \left\{x \in \overline{B} \mid \limsup_{r\to 0} r^{-s}\mu(B(x,r)) > \lambda\right\}. \tag{100}$$

According to [2, Theorem 2.56],

$$\mu(A) \geq \lambda H^s(A). \tag{101}$$

As $A \subset \overline{B}$ and μ is a Radon measure, we have $\mu(A) < \infty$. Then, (101) gives $H^s(A) < \infty$ and since $s < n$, $\mathcal{L}^n(A) = 0$. The measure μ is dominated by $\mathcal{L}^n$, so $\mu(A) = 0$ and now (101) gives $H^s(A) = 0$. We can take a sequence of scalars $\lambda_k \to 0$ to deduce

$$\mathcal{H}^s\left(\left\{x \in \overline{B} \mid \limsup_{r\to 0} r^{-s}\mu(B(x,r)) > 0\right\}\right) = 0. \tag{102}$$

We can then conclude that

$$\mathcal{H}^s\left(\left\{x \in X \mid \limsup_{r\to 0} r^{-s}\mu(B(x,r)) > 0\right\}\right) = 0, \tag{103}$$

by covering X with a sequence of closed balls $\overline{B_k} \subset X$.

Now we come to the general case $p \geq 1$. Let us fix $t < n$. For $x \in X$ and for $r > 0$, the Hölder inequality shows that

$$r^{-(n-\frac{n}{p})}\int_{B(x,r)} v \,\mathrm{d}\mathcal{L}^n \leq \left(\int_{B(x,r)} v^p \,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p}}, \tag{104}$$

so

$$r^{-(n+\frac{t}{p}-\frac{n}{p})}\int_{B(x,r)} v \,\mathrm{d}\mathcal{L}^n \leq \left(r^{-t}\int_{B(x,r)} v^p \,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p}}. \tag{105}$$

We apply the first part to see that for $\mathcal{H}^t$-a.e. $x \in X$,

$$\lim_{r\to 0} r^{-(n+\frac{t}{p}-\frac{n}{p})}\int_{B(x,r)} v \,\mathrm{d}\mathcal{L}^n = 0. \tag{106}$$

The scalar t such that $s = n + \frac{t}{p} - \frac{n}{p}$ is given by $t := n - p(n-s) < n$. ■

Proof of Theorem 5.1. According to Lemma 5.3, we have for $\mathcal{H}^{n-p}$-a.e. $x \in X$,

$$\lim_{r\to 0} \omega_2(x,r) = 0 \tag{107}$$

and according to [5, Theorem 8.2], the set

$$\left\{x \in X \cap \Sigma \mid \lim_{r\to 0} \omega_2(x,r) = 0\right\} \tag{108}$$

has Hausdorff dimension less than or equal to $n - 8$. ■

A. A Robin problem

A.1. Statement

We work in the Euclidean space $\mathbb{R}^n$ $(n \geq 2)$. For $r > 0$, B_r denotes the ball of radius r centered at 0. We fix a radius $0 < R \leq 1$, an exponent $0 < \alpha \leq 1$, a constant $A > 0$ and a $C^{1,\alpha}$ function $f : \mathbb{R}^{n-1} \cap B_R \to \mathbb{R}$ such that $f(0) = 0$, $\nabla f(0) = 0$ and $R^\alpha \, [\nabla f]_\alpha \leq A$. We introduce

$$V_R := \{x \in B_R \mid x_n > f(x')\}, \tag{109}$$

$$\Gamma_R := \{x \in B_R \mid x_n = f(x')\}. \tag{110}$$

We denote by ν the normal vector field to Γ_R going upward. For $0 < t \leq 1$, we write tV_R for $V_R \cap B_t$ and $t\Gamma_R$ for $\Gamma_R \cap B_t$. For $u \in W^{1,2}(V_R)$, we denote by u^* the trace of u in $L^1(\partial V_R)$. It is characterized by the property that for $\mathcal{H}^{n-1}$-a.e. $x \in \partial V_R$,

$$\lim_{r \to 0} r^{-n} \int_{V_R \cap B(x,r)} |u - u^*(x)| \, \mathrm{d}\mathcal{L}^n = 0. \tag{111}$$

We denote by $W_0^{1,2}(V_R \cup \Gamma_R)$ the space of functions $v \in W^{1,2}(V_R)$ such that $v^* = 0$ on $\partial V_R \setminus \Gamma_R$. Our objects of study are the functions $u \in W^{1,2}(V_R) \cap L^\infty(V_R)$ which are weak solutions of

$$\begin{cases} \Delta u = 0 & \text{in } V_R, \\ \partial_\nu u - u = 0 & \text{in } \Gamma_R, \end{cases} \tag{112}$$

that is, for all $v \in W_0^{1,2}(V_R \cup \Gamma_R)$,

$$\int_{V_R} \langle \nabla u, \nabla v \rangle \, \mathrm{d}\mathcal{L}^n + \int_{\Gamma_R} u^* v^* \, \mathrm{d}\mathcal{H}^{n-1} = 0. \tag{113}$$

According to Weyl's lemma, u coincides almost-everywhere in V_R with harmonic functions. We replace u by this harmonic representative so that u is pointwise defined and smooth in V_R. Our goal is to prove the following estimate:

Lemma A.1. *There exists $C \geq 1$ (depending on n, α, A) such that*

$$|\nabla u|_\infty \leq C \left(\fint_{V_R} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{2}} + C |u|_\infty, \tag{114}$$

where the left-hand side is computed on $\frac{1}{2} V_R$.

Although well known to experts, we present the steps of the proof because we have not found a satisfactory reference with the estimate as asserted. Lemma A.1 will be an immediate consequence of Lemma A.5 and Lemma A.6.

Viscosity solutions of such problems have been studied in [13]. The viscosity approach is based on the maximum principle, but in our case, we also have a maximum principle for weak solutions. Thus, we can follow the ideas of [13]. Once we get that u is C^α up to the boundary, the Robin boundary condition can be written as a Neumann boundary condition with a C^α right-hand side and we will apply the usual estimates for Neumann problems ([10]). We are going to write the key lemmas and steps below and make short comments for the proof.

Lemma A.2 (Maximum principle). *Let $u \in W^{1,2}(V_R)$ be a weak solution of*

$$\begin{cases} \Delta u \geq 0 & \text{in } V_R, \\ \partial_\nu u - u \geq 0 & \text{in } \Gamma_R, \end{cases} \tag{115}$$

that is, for all non-negative functions $v \in W_0^{1,2}(V_R \cup \Gamma_R)$,

$$\int_{V_R} \langle \nabla u, \nabla v \rangle \, \mathrm{d}\mathcal{L}^n + \int_{\Gamma_R} u^* v^* \, \mathrm{d}\mathcal{H}^{n-1} \leq 0. \tag{116}$$

If $u^* \leq 0$ on $\partial V_R \setminus \Gamma_R$, then $u \leq 0$ on V_R.

A.2. Hölder continuity up to the boundary

Lemma A.3 (Hölder continuity). *Let $u \in W^{1,2}(V_R) \cap L^\infty(V_R)$ be a weak solution of (112). There exist constants $C \geq 1$ (depending on n, α, A) and $0 < \sigma < 1$ (depending on n) such that for all $x, y \in V_R$,*

$$|u(x) - u(y)| \leq C|u|_\infty \left(\frac{|x-y|}{r}\right)^\sigma, \tag{117}$$

where $r := \max\{\mathrm{d}(x, \mathbb{R}^n \setminus B_R), \mathrm{d}(y, \mathbb{R}^n \setminus B_R)\}$.

Lemma A.3 is a standard consequence of a weak Harnack inequality at the boundary. We temporarily redefine the notation V_R, Γ_R in the next statement because it is more convenient to work with cylinders than with balls.

Lemma A.4 (Weak Harnack inequality). *We fix a radius $0 < R \leq 1$. We fix a vector $e_n \in \mathbf{S}^{n-1}$ and we decompose each point $x \in \mathbb{R}^n$ as $x = x' + x_n e_n$, where $x' \in e_n^\perp$ and $x_n \in \mathbb{R}$. We fix a 1-Lipschitz function $f : e_n^\perp \to \mathbb{R}$ and assume that $0 \leq f \leq \delta R$ for a certain $0 < \delta \leq \frac{1}{2}$ small enough (depending on n). Finally, we define*

$$V_R := \left\{x \in \mathbb{R}^n \mid |x'| < R,\ f(x') < x_n < 2R\right\}, \tag{118}$$

$$\Gamma_R := \left\{x \in \mathbb{R}^n \mid |x'| < R,\ x_n = f(x')\right\}. \tag{119}$$

Let $u \in W^{1,2}(V_R)$ be a non-negative weak solution of

$$\begin{cases} \Delta u = 0 & \text{in } V_R, \\ \partial_\nu u - u = 0 & \text{in } \Gamma_R. \end{cases} \tag{120}$$

Then, there exists a constant $C \geq 1$ (depending on n) such that

$$\begin{aligned} &\sup\left\{u(x) \mid |x'| \leq \tfrac{1}{2}R,\ 2\delta R \leq x_n \leq \tfrac{3}{2}R\right\} \\ &\quad \leq C \inf\left\{u(x) \mid |x'| \leq \tfrac{1}{2}R,\ f(x') < x_n \leq \tfrac{3}{2}R\right\}. \end{aligned} \tag{121}$$

Lemma A.4 is proved by building an appropriate barrier function as in [13, Theorem 2.2].

A.3. Gradient estimates

Theorem 1.2 in [13] states that viscosity solutions are pointwise $C^{1,\alpha}$ up to the boundary with Schauder estimates. Although we use a weak formulation, their proof applies in our case because it relies on the maximum principle (Lemma A.2), the Hölder continuity (Lemma A.3) and regularity results for solutions of the Neumann problem in a spherical cap. We derive the following estimate:

Lemma A.5 (Schauder estimate). *Let $u \in W^{1,2}(V_R) \cap L^\infty(V_R)$ be a weak solution of (112). Then, there exist $C \geq 1$ and $0 < \sigma < 1$ (depending on n, α, A) such that*

$$|\nabla u|_\infty + R^\sigma\, [\nabla u]_\sigma \leq C R^{-1} \operatorname{osc}(u) + C|u|_\infty, \tag{122}$$

where the left-hand side is computed on $\frac{1}{2}V_R$ and the symbol $\operatorname{osc}(u)$ *is given by the value* $\sup\{|u(x) - u(y)| \mid x, y \in V_R\}$.

Finally, we control the oscillations of u using a local boundedness estimate for weak solutions of Neumann problems [10, Theorem 1.6 and Remark 1.12].

Lemma A.6 (Oscillations estimate). *Let $u \in W^{1,2}(V_R) \cap L^\infty(V_R)$ be a weak solution of (112). Then, there exists $C \geq 1$ (depending on n, α, A) such that*

$$\operatorname{osc}(u) \leq CR\Big(\fint_{V_R} |\nabla u|^2 \,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{2}} + CR|u|_\infty, \tag{123}$$

where $\operatorname{osc}(u) := \sup\{|u(x) - u(y)| \mid x, y \in \frac{1}{2}V_R\}$.

B. Uniform rectifiability of quasiminimizers

In this section, we recall the definition of quasiminimizers in [6] and their uniform rectifiability property. Our local minimizers are not quasiminimizers as in [6], but we will show that the proof from [6] works in our case.

We work in an open set X of the Euclidean space $\mathbb{R}^n$ ($n \geq 2$) and we fix a triple of parameters $\mathcal{P} := (r_0, a, M)$ composed of $r_0 > 0$, $a \geq 0$ and $M \geq 1$.

Definition B.1. The set of *admissible pairs* $\mathcal{A}$ is the set of all pairs (u, K) where $K \subset X$ is relatively closed in X and $u \in W^{1,2}_{\mathrm{loc}}(X \setminus K)$. Let (u, K) be an admissible pair and let B be an open ball such that $\overline{B} \subset X$. A *competitor* of (u, K) in B is a pair $(v, L) \in \mathcal{A}$ such that $K \setminus \overline{B} = L \setminus \overline{B}$ and $u = v$ $\mathcal{L}^n$-a.e. on $X \setminus (K \cup \overline{B})$. In this case, we set

$$E(u) := \int_B |\nabla u|^2 \,\mathrm{d}\mathcal{L}^n, \quad E(v) := \int_B |\nabla v|^2 \tag{124}$$

and

$$\Delta E := \max\{E(v) - E(u), M(E(v) - E(u))\}. \tag{125}$$

We say that (u, K) is a *local $\mathcal{P}$-quasiminimizer in X* if for all open balls B of radius $0 < r \leq r_0$ such that $\overline{B} \subset X$, and for all competitors (v, L) of (u, K) in B, we have

$$\mathcal{H}^{n-1}(K \setminus L) \leq M\mathcal{H}^{n-1}(L \setminus K) + \Delta E + ar^{n-1}. \tag{126}$$

In addition, we say that (u, K) is *coral* if $K = \mathrm{spt}(\mathcal{H}^{n-1} \llcorner K)$ in X. This means that for all $x \in K$ and all $r > 0$, $\mathcal{H}^{n-1}(K \cap B(x, r)) > 0$.

We don't give a definition of uniform rectifiability because there are too many, but we underline that they are equivalent for closed, Ahlfors-regular sets. The reader can find a survey of uniform rectifiability in [6, Section 73] and also on Guy David's webpage (Notes-Parkcity.dvi).

Definition B.2 (Ahlfors-regularity). A closed set $E \subset \mathbb{R}^n$ is Ahlfors-regular of dimension $n - 1$ if there exists a constant $C \geq 1$ such that for all $x \in E$ and for all $0 < r < \mathrm{diam}(E)$,

$$C^{-1}r^{n-1} \leq \mathcal{H}^{n-1}(E \cap B(x, r)) \leq Cr^{n-1}. \tag{127}$$

Theorem B.3. *Let $\mathcal{P} := (r_0, a, M)$ be a triple of parameters composed of $r_0 > 0$, $a \geq 0$ and $M \geq 1$. Assume that a (depending on n, M) is small enough. Let (u, K) be a coral and local $\mathcal{P}$-quasiminimizer in X. For all $x \in K$ and $0 < r < r_0$ such that $B(x, r) \subset X$, there is a closed, Ahlfors-regular, uniformly rectifiable set E of dimension $n - 1$ such that $K \cap \frac{1}{2}B(x, r) \subset E$. The constants for the Ahlfors-regularity and uniform rectifiability depend on n, M and a.*

Remark B.4. One can observe that (126) implies

$$\mathcal{H}^{n-1}(K \cap \overline{B}) \leq M\mathcal{H}^{n-1}(L \cap \overline{B}) + \Delta E + ar^{n-1}. \tag{128}$$

This is equivalent when $M = 1$, but is strictly weaker when $M > 1$.

We claim that Theorem B.3 still holds true with (128) in place of (126). In [6, Section 74], David builds a suitable competitor (w, G) of (u, K) in a ball B. The set G is of the form $G = (K \setminus B) \cup Z$, where Z is a special subset of ∂B containing $K \cap \partial B$. The quasi-minimality condition (126) is used only once at [6, Section 74, line 22]. Then, David uses the inequalities

$$\mathcal{H}^{n-1}(K \setminus G) \geq \mathcal{H}^{n-1}(K \cap B), \tag{129}$$

$$\mathcal{H}^{n-1}(G \setminus K) \leq \mathcal{H}^{n-1}(Z), \tag{130}$$

but we also have

$$\mathcal{H}^{n-1}(K \cap \overline{B}) \geq \mathcal{H}^{n-1}(K \cap B), \tag{131}$$

$$\mathcal{H}^{n-1}(G \cap \overline{B}) \leq \mathcal{H}^{n-1}(Z). \tag{132}$$

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