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Fully nonlinear free transmission problems

Edgard A. Pimentel and Makson S. Santos

Abstract. We examine a free transmission problem driven by fully nonlinear elliptic operators. Since the transmission interface is determined endogenously, our analysis regards this object as a free boundary. We start by relating our problem with a pair of viscosity inequalities. Then, approximation methods ensure that strong solutions are of class $C^{1,\text{Log-Lip}}$, locally. In addition, under further conditions on the problem, we prove quadratic growth of the solutions away from branch points.

1. Introduction

We consider a fully nonlinear transmission problem of the form

$$\begin{aligned} F_1(D^2u) &= 1 \quad \text{in } \Omega^+(u) \cap B_1, \\ F_2(D^2u) &= 1 \quad \text{in } \Omega^-(u) \cap B_1, \end{aligned} \tag{1.1}$$

where $F_1, F_2 : S(d) \to \mathbb{R}$ are (λ, Λ)-elliptic operators, $\Omega^-(u) := \{x \in B_1 \mid u < 0\}$, and $\Omega^+(u) := \{x \in B_1 \mid u > 0\}$. We examine the local regularity of strong solutions to (1.1) and study their growth regime at branch points. In particular, we prove that solutions are locally $C^{1,\text{Log-Lip}}$-regular, with estimates. Under further conditions, we prove quadratic growth of the solutions away from branch points.

We emphasize the operators F_1 and F_2 are comparable only locally in $S(d)$. As a consequence, (1.1) differs from the usual obstacle problem. We also stress that discontinuities arise as solutions change sign.

Transmission problems comprise a class of models aimed at examining a variety of phenomena in heterogeneous media. The problems under the scope of this formulation include thermal and electromagnetic conductivity, composite materials, and other diffusion processes driven by discontinuous laws.

A given domain $\Omega \subset \mathbb{R}^d$ gets split into mutually disjoint subregions $\Omega_i \Subset \Omega$ for $i = 1, \ldots, k$, for some $k \in \mathbb{N}$. The operator governing the problem is smooth within Ω_i, though discontinuous across $\partial\Omega_i$. A paramount, subtle aspect of the theory concerns the nature of those subregions.

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Either one prescribes $(\Omega_i)_{i=1}^k$ and the geometry of $\partial\Omega_i$ a priori, or those structures are endogenously determined. The latter setting frames the theory in the context of free boundary problems. Both cases differ substantially; consequently, their analysis also requires distinct techniques. Most former studies on transmission problems presuppose a priori knowledge of the subregions Ω_i and their geometric properties. A workhorse of the theory is the divergence-form equation

$$\operatorname{div}(a(x)Du) = 0 \quad \text{in } \Omega, \tag{1.2}$$

where the matrix-valued function $a(\cdot)$ is defined as

$$a(x) := a_i \quad \text{for } x \in \Omega_i,$$

for constant matrices a_i and $i = 1, \ldots, k$. Though smooth within every Ω_i, the coefficients of (1.2) can be discontinuous across $\partial\Omega_i$. This feature introduces genuine difficulties in the analysis.

The first formulation of a transmission problem appeared in [31] and addressed a topic in material sciences, namely, elasticity theory. In that paper, the author proves the uniqueness of solutions for a model consisting of two subregions known a priori. Although not examined in detail, the existence of solutions is the subject of [31]; see also [30].

The formulation in [31] motivated many subsequent studies [6, 12–14, 19, 27, 29, 35–37]. Those papers present a wide range of developments. They include the existence of solutions for the transmission problem in [31] and the analysis of several variants. We refer the reader to [5] for an account of those results and methods.

Estimates and regularity results for the solutions to transmission problems have also been treated in the literature. In [26], the authors consider a bounded subdomain $\Omega \subset \mathbb{R}^d$, split into a finite number of subregions $\Omega_1, \Omega_2, \ldots, \Omega_k$, known a priori. The motivation is in the study of composite materials with closely spaced inclusions. The cross-section of a fiber-reinforced material is an example in dimension $d = 2$. The mathematical analysis amounts to the study of

$$\frac{\partial}{\partial x_i}\left(a(x)\frac{\partial}{\partial x_j}u\right) = f \quad \text{in } \Omega, \tag{1.3}$$

where

$$a(x) := \begin{cases} a_i(x) & \text{for } x \in \Omega_i, \quad i = 1, \ldots, k, \\ a_{k+1}(x) & \text{for } x \in \Omega \setminus \bigcup_{i=1}^k \Omega_i. \end{cases}$$

Under natural assumptions on the data, the authors establish local Hölder continuity for the gradient of the solutions. From the applied perspective, the gradient encodes information on the stress of the material. Their findings imply bounds on the gradient *independent of the location of the fibers* (c.f. [3]).

The vectorial setting is the subject of [25]. The authors extend the developments reported in [26] to systems in that paper. Moreover, they produce bounds for higher derivatives of the solutions.

In [1], the authors consider a domain with two subregions, which are supposed to be ε apart, for some $\varepsilon > 0$. Within each subregion, the divergence-form equation is governed by a constant coefficient k. Conversely, outside those subregions, the diffusivity coefficient is 1. By setting $k = +\infty$, the authors frame the problem in the context of perfect conductivity.

In this setting, estimates on the gradient deteriorate as the two subregions approach each other. The analysis in [1] yields blow-up rates for the gradient bounds as $\varepsilon \to 0$. The case of multiple inclusions, covering perfect conductivity and insulation ($k = 0$), is discussed in [2]; see also [7].

Recently, new developments have been obtained under minimal regularity requirements for the transmission interfaces. In [11], the authors consider a smooth and bounded domain Ω and fix $\Omega_1 \Subset \Omega$, defining $\Omega_2 := \Omega \setminus \overline{\Omega}_1$. They suppose the boundary of the transmission interface $\partial\Omega_1$ to be of class $C^{1,\alpha}$ and prove existence, uniqueness, and $C^{1,\alpha}(\overline{\Omega}_i)$-regularity of the solutions to the problem, for $i = 1, 2$. Their argument imports regularity from flat problems through a new stability result; see [11, Theorem 4.2].

Another class of transmission problems concerns models where the subregions of interest are determined endogenously. For example, given $\Omega \subset \mathbb{R}^d$, one would consider

$$\Omega_1 : \left\{x \in \Omega \mid u(x) < 0\right\} \quad \text{and} \quad \Omega_2 : \left\{x \in \Omega \mid u(x) > 0\right\},$$

where $u : \Omega \to \mathbb{R}$ solves a prescribed equation. Roughly speaking, knowledge of the solution is required to determine the subregions of the domain where distinct diffusion phenomena occur. In this context, a further structure arises, namely, the free interface or free boundary. Here, in addition to the analysis of the solutions, properties of the free boundary are also of central interest.

In [15] the authors examine the (p, q)-functional

$$J_{p,q}(v) := \int_\Omega (|Dv^+|^p + |Dv^-|^q)\mathrm{d}x. \tag{1.4}$$

Heuristically, in the region where v is positive, the functional satisfies a p-growth regime. In the region where v is negative, a q-growth regime is in force. Though the functional in (1.4) is discontinuous, and distinct regimes drive the process in distinct subregions of the domain, such discontinuities depend on the sign of the argument v.

Among the findings in [15], we mention the existence of minimizers for $J_{p,q}$ and their Hölder continuity. In addition, the authors prove the free boundary is of class $C^{1,\alpha}$ with respect to the p-harmonic measure $\Delta_p u^+$. Finally, they conclude that $\Delta_p u^+$ is supported on a set of σ-finite $(d-1)$-dimensional Hausdorff measure.

We remark that (1.1) relates to (and is very much inspired by) the fully nonlinear obstacle problem literature. To the best of our knowledge, the fully nonlinear obstacle problem was first examined in [23]; see also [24]. In [16], the authors introduced the unconstrained free boundary problems. This class of fully nonlinear models accommodates a variety of distinct formulations, unifying the approach to regularity of the solutions and the analysis of the free boundary; see also [17, 22].

Also, in the context of the obstacle problem governed by fully nonlinear operators, we mention the issue of non-transversality; see, for instance, [20, 21]. By examining the intersection of the fixed and the free boundaries, one can extract geometrical information on the latter. In addition, the techniques involved in this analysis have important spillovers on the classification of blow-up limits.

In the present paper, we study $W^{2,d}$-strong solutions to (1.1). We start by noticing that a $W^{2,d}$-solution to (1.1) is a continuous viscosity solution to

$$\min(F_1(D^2u), F_2(D^2u)) \leq 1 \quad \text{in } B_1 \tag{1.5}$$

and

$$\max(F_1(D^2u), F_2(D^2u)) \geq -1 \quad \text{in } B_1. \tag{1.6}$$

We emphasize the importance of (1.5)–(1.6), even in the context of $W^{2,d}$-strong solutions. Although it is clear that a solution $u \in W^{2,d}_{\text{loc}}(B_1)$ is α-Hölder continuous for every $\alpha \in (0,1)$, this inclusion does not ensure *universal estimates* for u. Because our analysis relies on the precompactness of strong solutions to (1.1), such estimates are critical. By noticing that strong solutions to (1.1) are viscosity solutions to (1.5)–(1.6), we access a maximum principle, stability results, and a Krylov–Safonov theory.

By requiring F_1 and F_2 to satisfy a near convexity condition, we prove that solutions to (1.1) are locally of class $C^{1,\text{Log-Lip}}$, with the appropriate estimates. It follows from approximation methods; see [8, 9]. Our first main result reads as follows:

Theorem 1.1 (Local $C^{1,\text{Log-Lip}}$-regularity). *Let $u \in W^{2,d}_{\text{loc}}(B_1)$ be a strong solution to (1.1). Suppose Assumptions 1–2 are in force. Then, $u \in C^{1,\text{Log-Lip}}_{\text{loc}}(B_1)$ and there exists $C > 0$ such that*

$$\sup_{x \in B_r(x_0)} |u(x) - u(x_0) - Du(x_0) \cdot (x - x_0)| \leq C r^2 \ln \frac{1}{r},$$

for every $x_0 \in B_{1/2}$ and $r \in (0, 1/4)$. In addition, $C = C(d, \lambda, \Lambda, \|u\|_{L^\infty(B_1)})$.

Remark 1. We notice the optimal regularity of the solutions to (1.1) is unknown. Of particular interest is whether or not the end-point $W^{2,\infty}$-regularity is available for strong solutions to this problem.

After examining the local regularity of solutions, we turn our attention to the so-called branch points. In brief, such points lie at the interface of $\Omega^+(u)$, $\Omega^-(u)$, and $\{u = 0\}$. More rigorously, a point on the free boundary $\Gamma(u) := (\partial\Omega^+(u) \cup \partial\Omega^-(u)) \cap B_1$ can be of three different types.

First, $x_0 \in \Gamma(u)$ is a one-phase point if

$$x_0 \in (\partial\Omega^\pm(u) \setminus \partial\Omega^\mp(u)) \cap B_1.$$

If this is the case, the local regularity of the solutions and properties of the free boundary follow from the fully nonlinear obstacle problem [23]. Alternatively, $x_0 \in \Gamma(u)$ may behave as a two-phase point; that is,

$$x_0 \in (\partial\Omega^+(u) \cap \partial\Omega^-(u)) \cap B_1.$$

Among two-phase points, branch points are of particular interest, as they are at the interface of the positive and the negative phases with the region where the solutions vanish. Formally, we say that $x^* \in \Gamma(u)$ is a branch point if

$$|B_r(x^*) \cap \{u = 0\}| > 0$$

for every $0 < r \ll 1$. We denote by $\Gamma_{\mathrm{BR}}(u) \subset \Gamma(u)$ the set of branch points. Under a small-density condition for the negative phase, we prove a result on the quadratic growth of the solutions away from branch points.

Our argument requires both F_1 and F_2 to be convex and supposes they are positively homogeneous of degree 1. Here, a dyadic analysis builds upon the maximum principle and a scaling strategy, using the L^∞-norms of the solutions as a normalization factor. This machinery first appeared in [10] in the context of an obstacle problem driven by the Laplacian. In [24], the authors studied the fully nonlinear setting and developed a fairly complete analysis of the obstacle problem governed by fully nonlinear operators. We also refer the reader to [23].

We consider the quantity

$$V_r(x^*, u) := \frac{\mathrm{vol}(B_r(x^*) \cap \Omega^-(u))}{r^d};$$

by supposing that $V_r(x^*, u)$ is controlled for a branch point $x^* \in \Gamma_{\mathrm{BR}}(u)$, we are capable of proving quadratic growth for the solutions, away from x_0. We state our second main result in the following:

Theorem 1.2 (Quadratic growth away from branch points). *Let $u \in W^{2,d}_{\mathrm{loc}}(B_1)$ be a strong solution to (1.1). Suppose Assumptions 1, 3, and 4, to be detailed further, are in force. Let $x^* \in \Gamma_{\mathrm{BR}}(u)$ be such that Assumption 5, yet to be presented, holds at x^*. Then, there exists a universal constant $C > 0$ such that*

$$\sup_{x \in B_r(x^*)} |u(x)| \leq C r^2$$

for every $0 < r \ll 1$.

We note Theorem 1.2 does not require F_1 and F_2 to be close, or even comparable, in any topology.

Remark 2. The small density of the negative phase is critical in establishing Theorem 1.2. Were it reasonable to suppose it holds for every $x_0 \in \Gamma(u) \cap B_{1/2}$, the conclusion of Theorem 1.2 would hold for every such point. Then, a clever scaling argument, as in [10, 24], would produce local $C^{1,1}$-regularity estimates for the solutions. However, to impose a small-density condition for every free boundary point $x_0 \in \Gamma(u) \cap B_{1/2}$ implies the negative phase does not affect the problem. Ultimately, it turns (1.1) into a one-phase problem whose theory is currently well understood and documented.

Remark 3. We notice the formulation in (1.1) includes the free transmission obstacle problem

$$F_1(D^2u)\chi_{\{u>0\}} + F_2(D^2u)\chi_{\{u<0\}} = \chi_{\{u\neq 0\}} \quad \text{in } B_1, \tag{1.7}$$

in the sense that solutions to (1.7) also solve (1.1).

The remainder of this paper is organized as follows: Section 2 gathers elementary results and details the main assumptions under which we work. In Section 3, we study the regularity of the strong solutions to (1.1) and present the proof of Theorem 1.1. Section 4 examines the growth regime of the solutions away from branch points and puts forward the proof of Theorem 1.2.

2. Preliminaries

This section presents some preliminary material and the main hypotheses we use in the paper. By $S(d)$, we denote the space of symmetric matrices of order d; when convenient, we identify $S(d) \sim \mathbb{R}^{\frac{d(d+1)}{2}}$. We start with the uniform ellipticity of the operators F_i.

Assumption 1 (Uniform ellipticity). For $i = 1, 2$, we suppose the operator $F_i : S(d) \to \mathbb{R}$ is (λ, Λ)-uniformly elliptic, that is, for $0 < \lambda \leq \Lambda$, it holds that

$$\lambda \|N\| \leq F_i(M + N) - F_i(M) \leq \Lambda \|N\|,$$

for every $M, N \in S(d)$, $N \geq 0$, and $i = 1, 2$. We also suppose $F_i(0) = 0$.

Uniform ellipticity relates closely to extremal operators

$$\mathcal{M}^+_{\lambda,\Lambda}(M) := \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i$$

and

$$\mathcal{M}^-_{\lambda,\Lambda}(M) := \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i,$$

where $(e_1, \ldots, e_d)$ are the eigenvalues of the matrix M. In fact, Assumption 1 can be rephrased as

$$\mathcal{M}^-_{\lambda,\Lambda}(M - N) \leq F_i(M) - F_i(N) \leq \mathcal{M}^+_{\lambda,\Lambda}(M - N),$$

for every $M, N \in S(d)$ and $i = 1, 2$. For completeness, we recall the definition of viscosity solutions; next, we denote by $\mathrm{USC}(B_1)$ the set of upper semi-continuous functions defined on B_1. Similarly, $\mathrm{LSC}(B_1)$ denotes the set of lower semi-continuous functions on B_1.

Definition 1 (C-viscosity solution). Let $G : S(d) \to \mathbb{R}$ be a (λ, Λ)-elliptic operator. We say that $u \in \mathrm{USC}(B_1)$ is a C-viscosity subsolution to

$$G(D^2u) = 0 \quad \text{in } B_1 \tag{2.1}$$

if, for every $\varphi \in C^2_{\text{loc}}(B_1)$ and $x_0 \in B_1$ such that $u - \varphi$ attains a local maximum at x_0, we have

$$G(D^2\varphi(x_0)) \leq 0.$$

Similarly, we say that $u \in \text{LSC}(B_1)$ is a C-viscosity supersolution to (2.1) if, for every $\varphi \in C^2_{\text{loc}}(B_1)$ and $x_0 \in B_1$ such that $u - \varphi$ attains a local minimum at x_0, we have

$$G(D^2\varphi(x_0)) \geq 0.$$

If $u \in C(B_1)$ is simultaneously a subsolution and a supersolution to (2.1), we say it is a viscosity solution to the equation.

For $0 < \lambda \leq \Lambda$ and $f \in C(B_1)$, we define $\overline{S}(\lambda, \Lambda, f)$ as the set of functions $u \in C(B_1)$ satisfying

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq f$$

in B_1, in the viscosity sense. Similarly, $\underline{S}(\lambda, \Lambda, f)$ is the set of functions $u \in C(B_1)$ satisfying

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \geq f.$$

Finally, we set

$$S(\lambda, \Lambda, f) := \overline{S}(\lambda, \Lambda, f) \cap \underline{S}(\lambda, \Lambda, f)$$

and

$$S^*(\lambda, \Lambda, f) := \overline{S}(\lambda, \Lambda, -|f|) \cap \underline{S}(\lambda, \Lambda, |f|).$$

For a comprehensive account of the theory of C-viscosity solutions, we refer the reader to [9]. We proceed with the definition of a $W^{2,d}$-strong solution.

Definition 2 ($W^{2,d}$-strong solution). We say that $u \in W^{2,d}_{\text{loc}}(B_1)$ is a strong solution to

$$G(D^2u(x)) = 0 \quad \text{in } B_1$$

if u satisfies the equation at almost every $x \in B_1$.

We refer the reader to [18, Chapter 9] for further details on this class of solutions and their properties. In the remainder of the paper, we put forward two assumptions concerning the convexity of the operators F_1 and F_2. We start with a near-convexity condition used in local $C^{1,\text{Log-Lip}}$-regularity.

Assumption 2 (Near-convexity condition). For $i = 1, 2$, we suppose that the operator $\overline{F}_i : S(d) \to \mathbb{R}$ satisfies a near-convexity condition, that is, there exists a convex (λ, Λ)-elliptic operator $F : S(d) \to \mathbb{R}$ such that

$$|F_i(M) - \overline{F}(M)| \leq \tau(1 + \|M\|),$$

for some small constant $\tau > 0$, yet to be determined.

For a class of operators satisfying Assumption 2, one may consider *small perturbations* of a convex operator $\overline{F}$. A source of such examples is the class of Isaacs operators. Indeed, if the matrix governing the equation is uniformly close to a constant matrix, Assumption 2 is satisfied (see, for instance, [32]). For the use of this assumption in the study of strong solutions for (1.1), see [33].

When it comes to the analysis of branching points, we require F_1 and F_2 to be convex operators.

Assumption 3 (Convexity). For $i = 1, 2$, we suppose the operator $F_i : S(d) \to \mathbb{R}$ is convex.

The next assumption concerns homogeneity of degree 1 and plays a major role in the quadratic growth of the solutions. The argument towards quadratic growth in [10] uses the linearity of the Laplacian operator. In [24], the authors notice that, in the fully nonlinear case, the condition that parallels linearity is the homogeneity of degree 1.

Assumption 4 (Homogeneity of degree 1). We suppose F_1 and F_2 to be homogeneous of degree 1, that is, for every $\tau \in \mathbb{R}$ and $M \in S(d)$, we have

$$F_i(\tau M) = \tau F_i(M)$$

for every $i = 1, 2$.

Before further assumptions, we gather some notation used throughout the paper. We denote by $\Omega^+(u)$ the subset of the unit ball where $u > 0$, whereas $\Omega^-(u)$ stands for the set where $u < 0$, that is,

$$\Omega^+(u) := \left\{x \in B_1 \mid u(x) > 0\right\} \quad \text{and} \quad \Omega^-(u) := \left\{x \in B_1 \mid u(x) < 0\right\}.$$

When referring to the set where $u \neq 0$, it is convenient to use the notation $\Omega(u) := \Omega^+(u) \cup \Omega^-(u)$. By $\Gamma(u)$, we denote the union of the topological boundaries of Ω^+ and Ω^-, that is,

$$\Gamma(u) := (\partial\Omega^+(u) \cup \partial\Omega^-(u)) \cap B_1.$$

We say that $x^* \in \Gamma(u)$ is a branch point if

$$|B_r(x^*) \cap \Sigma(u)| > 0$$

for every $0 < r < 1$. We denote the set of branch points by $\Gamma_{\mathrm{BR}}(u)$. In addition, we denote by $\Sigma(u)$ the set where u vanishes:

$$\Sigma(u) = \left\{x \in B_1 \mid u(x) = 0\right\}.$$

A further condition regards the subregion $\Omega^-(u)$; it is critical to prove the quadratic growth of the solutions through the set of methods used in the paper. For $x^* \in \partial\Omega$ and

$0 < r \ll 1$, we consider the quantity

$$V_r(x^*, u) := \frac{\mathrm{vol}(B_r(x^*) \cap \Omega^-(u))}{r^d}. \tag{2.2}$$

For ease of notation, we set $V_r(0, u) =: V_r(u)$.

Assumption 5 (Normalized volume of $\Omega^-(u)$). Let $x^* \in \Gamma_{\mathrm{BR}}(u)$ be fixed. We suppose there exists $C_0 > 0$, to be determined later, such that

$$V_r(x^*, u) \le C_0$$

for every $r \in (0, 1/2)$.

The former assumption imposes a control on the size of the subregion where u is negative, in a vicinity of $x^* \in \Gamma_{\mathrm{BR}}(u)$. It resonates with the geometry of the free boundary. In the next section, we examine the regularity of strong solutions to (1.1). In particular, we present the proof of Theorem 1.1.

3. Local regularity of solutions

In this section, we detail the proof of Theorem 1.1. We start by relating (1.1) with viscosity inequalities of the form

$$\min(F_1(D^2u), F_2(D^2u)) \le 1 \quad \text{in } B_1 \tag{3.1}$$

and

$$\max(F_1(D^2u), F_2(D^2u)) \ge -1 \quad \text{in } B_1. \tag{3.2}$$

Lemma 1. *Let $u \in W^{2,d}_{\mathrm{loc}}(B_1)$ be a strong solution to (1.1). Suppose Assumption 1 holds true. Then, u is a C-viscosity solution to inequalities (3.1)–(3.2).*

The proof of Lemma 1 follows from standard computations and the maximum principle for $W^{2,d}$ functions; see [28, Corollary 3] and [4]. In addition, if u is a continuous viscosity solution to (3.1)–(3.2) we also have $u \in S^*(\lambda, \Lambda, 1)$. In fact, because

$$\mathcal{M}^-_{\lambda,\Lambda}(M) \le F_i(M) \le \mathcal{M}^+_{\lambda,\Lambda}(M)$$

holds for $i = 1, 2$, we have

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \le \min(F_1(D^2u), F_2(D^2u)) \le 1$$

and

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \ge \max(F_1(D^2u), F_2(D^2u)) \ge -1.$$

As a consequence of $u \in S^*(\lambda, \Lambda, 1)$, we derive the Hölder continuity for the strong solutions to (1.1), with universal estimates.

Lemma 2 (Hölder continuity). *Let $u \in W^{2,d}_{\mathrm{loc}}(B_1)$ be a strong solution to (1.1) and suppose Assumption 1 holds. Then, $u \in C^{\alpha}_{\mathrm{loc}}(B_1)$, for some $\alpha \in (0, 1)$, and there exists $C > 0$*

such that

$$\|u\|_{C^\alpha(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^d(B_1)}).$$

In addition, $\alpha = \alpha(\lambda, \Lambda, d)$ *and* $C = C(\lambda, \Lambda, d)$.

For a proof of Lemma 2, see [9, Lemma 4.10]. In the following, we prove that solutions to (1.1) satisfy a quadratic growth *away from branch points*:

3.1. Proof of Theorem 1.1

We continue with an approximation lemma.

Proposition 1. *Let* $u \in W^{2,d}_{\text{loc}}(B_1)$ *be a* $W^{2,d}$*-strong solution to* (1.1)*. Suppose Assumptions* 1 *and* 2 *hold true. Given* $\delta > 0$*, there exists* $0 < \tau_0 \ll 1$ *such that, if the parameter* $\tau > 0$ *in Assumption* 2 *satisfies* $\tau < \tau_0$*, there exists* $h \in C^{2,\alpha}_{\text{loc}}(B_{9/10})$ *with*

$$\|u - h\|_{L^\infty(B_{8/9})} \leq \delta$$

and

$$\|h\|_{C^{2,\alpha}(B_{8/9})} \leq C,$$

for some universal constant $C > 0$ *and some universal exponent* $\alpha \in (0, 1)$.

Proof. For ease of presentation, we split the proof into three main steps.

Step 1 - We argue by contradiction; suppose the statement of the proposition is false. Then, there exist sequences $(u_n)_{n\in\mathbb{N}}$, $(F_1^n)_{n\in\mathbb{N}}$, and $(F_2^n)_{n\in\mathbb{N}}$ such that:

(1) F_i^n satisfies Assumption 1 for $i = 1, 2$ and every $n \in \mathbb{N}$. Moreover,

$$|F_i^n(M) - \overline{F}(M)| \leq \frac{1}{n}(1 + \|M\|) \tag{3.3}$$

for every $i = 1, 2$, $n \in \mathbb{N}$, and $M \in S(d)$;

(2) u_n is a viscosity solution to

$$\min(F_1^n(D^2u_n), F_2^n(D^2u_n)) \leq \frac{1}{n} \tag{3.4}$$

and

$$\max(F_1^n(D^2u_n), F_2^n(D^2u_n)) \geq -\frac{1}{n} \tag{3.5}$$

in $B_{9/10}$, with $u_n = u$ on $\partial B_{9/10}$, for every $n \in \mathbb{N}$;

(3) there exists $\delta_0 > 0$ for which

$$\|u_n - h\|_{L^\infty(B_{8/9})} > \delta_0$$

for every $h \in C^{2,\alpha}(B_{9/10})$ with $\|h\|_{C^{2,\alpha}(B_{8/9})} \leq C$, and every $n \in \mathbb{N}$.

Step 2 - Because of (3.4)–(3.5) and Lemma 2, we learn that $\|u_n\|_{C^\beta(B_{9/10})} \leq C$ for every $n \in \mathbb{N}$, for some universal constant $C > 0$. Consequently, it converges locally uniformly, through a subsequence if necessary, to a function $u_\infty \in C^{\beta/2}_{\text{loc}}(B_{9/10})$. Also, (3.3) ensures that F_1^n and F_2^n converge locally uniformly on $S(d)$ to the convex operator $\overline{F}$ from Assumption 2. The stability of viscosity subsolutions and supersolutions implies

$$\overline{F}(D^2 u_\infty) = 0 \quad \text{in } B_{9/10}.$$

Step 3 - Because $\overline{F}$ is convex, we infer $u_\infty \in C^{2,\alpha}_{\text{loc}}(B_{9/10})$, with $\|u_\infty\|_{C^{2,\alpha}(B_{8/9})} \leq C$, for some universal constant $C > 0$ and some universal exponent $\alpha \in (0, 1)$. Set $h := u_\infty$ to get a contradiction and thus complete the proof. ∎

Proposition 2. *Let $u \in W^{2,d}_{\text{loc}}(B_1)$ be a $W^{2,d}$-strong solution to (1.1). Suppose Assumptions 1 and 2 hold true. There exists $0 < \tau_0 \ll 1$ such that, if the parameter $\tau > 0$ in Assumption 2 satisfies $\tau < \tau_0$, one can find $0 < \rho \ll 1$ and a sequence of quadratic polynomials $(P_n)_{n\in\mathbb{N}}$ with*

$$P_n(x) := a_n + \mathrm{b}_n \cdot x + \frac{x \cdot C_n x}{2}$$

such that

$$\|u - P_n\|_{L^\infty(B_{\rho^n})} \leq \rho^{2n}, \tag{3.6}$$

$$\overline{F}(C_n) = 0, \tag{3.7}$$

and

$$|a_n - a_{n-1}| + \rho^{n-1}|\mathrm{b}_n - \mathrm{b}_{n-1}| + \rho^{2(n-1)}|C_n - C_{n-1}| \leq C\rho^{2(n-1)}, \tag{3.8}$$

for every $n \in \mathbb{N}$.

Proof. We resort to an induction argument; we split the proof into four steps for the reader's convenience.

Step 1 - We consider the base case. Set $P_0 := 0$; let $h \in C^{2,\alpha}_{\text{loc}}(B_{9/10})$ be the δ-approximating function whose existence follows from Proposition 1 and define

$$P_1(x) := h(0) + Dh(0) \cdot x + \frac{x \cdot D^2h(0)x}{2}.$$

We verify (3.6)–(3.8) for the case $n = 1$. Notice that

$$\sup_{x\in B_\rho} |u(x) - P_1(x)| \leq \sup_{x\in B_{\rho^n}} |u(x) - h(x)| + \sup_{x\in B_{\rho^n}} |h(x) - P_1(x)| \leq \delta + C\rho^{2+\alpha}.$$

By choosing

$$\delta := \frac{\rho}{2} \quad \text{and} \quad \rho := \left(\frac{1}{2C}\right)^{\frac{1}{\alpha}},$$

one ensures (3.6) holds. Because h is the approximating function from Proposition 1, we have (3.7). Finally, (3.8) follows from the $C^{2,\alpha}$-estimates available for h.

Step 2 - Now, we formulate the induction hypothesis: suppose (3.6)–(3.8) have been verified for $n = k$. We examine the case $n = k + 1$. Let $v_k : B_1 \to \mathbb{R}$ be defined as

$$v_k(x) := \frac{u(\rho^k x) - P_k(\rho^k x)}{\rho^{2k}}.$$

It is clear from the induction hypothesis that v_k is a normalized viscosity solution to

$$\min(F_1(D^2 v_k + C_k), F_2(D^2 v_k + C_k)) \leq 1 \quad \text{in } B_1$$

and

$$\max(F_1(D^2 v_k + C_k), F_2(D^2 v_k + C_k)) \geq -1 \quad \text{in } B_1.$$

Also, Assumption 2 implies

$$\sup_{M \in S(d)} |F_i(M + C_k) - \overline{F}_k(M)| \leq \tau(1 + \|M\|),$$

where $\overline{F}_k$ is the convex operator defined as $\overline{F}_k(M) := \overline{F}(M + C_k)$. Because of the induction hypothesis, $\overline{F}(C_k) = 0$; hence, $\overline{F}(D^2 w) = 0$ and $\overline{F}_k(D^2 w) = 0$ have the same estimates.

Consequently, if $0 < \tau < \tau_0$, Proposition 1 ensures the existence of $\widetilde{h} \in C^{2,\alpha}_{\text{loc}}(B_{9/10})$, with $\|\widetilde{h}\|_{C^{2,\alpha}(B_{8/9})} \leq C$ satisfying

$$\|v_k - \widetilde{h}\|_{L^\infty(B_{8/9})} \leq \delta.$$

Arguing as in the former step, one concludes with the existence of

$$\widetilde{P}(x) := \widetilde{a} + \widetilde{\mathrm{b}} \cdot x + \frac{x \cdot \widetilde{C} x}{2}$$

such that

$$\sup_{x \in B_\rho} |v_k(x) - \widetilde{P}(x)| \leq \rho^2.$$

The induction assumption and the definition of v_k yield

$$\sup_{x \in B_{\rho^{k+1}}} |u(x) - P_{k+1}(x)| \leq \rho^{2(k+1)},$$

where P_{k+1} is given by

$$P_{k+1}(x) := a_k + \rho^{2k}\widetilde{a} + (\mathrm{b}_k + \rho^k \widetilde{\mathrm{b}}) \cdot x + \frac{x \cdot (C_k + \widetilde{C})x}{2}. \tag{3.9}$$

Because $\widetilde{C} = D^2 \widetilde{h}(0)$, it follows that $\overline{F}(C_{k+1}) = 0$. Defining a_{k+1}, b_{k+1}, and C_{k+1} as in (3.9), one ensures that (3.8) is also satisfied at the $(k + 1)$-level, and thus the proof is complete. ■

Proof of Theorem 1.1. Once Proposition 2 is available, the proof of Theorem 1.1 follows from (by now) standard computations (see, e.g., [34, proof of Theorem 2.6, p. 1398]). ■

4. Quadratic growth away from branch points

Let $x^* \in \Gamma_{\rm BR}(u) \cap B_1$ be fixed. Consider the maximal subset of $\mathbb{N}$ whose elements j are such that

$$\sup_{x \in B_{2^{-j-1}}(x^*)} |u(x)| \geq \frac{1}{16} \sup_{x \in B_{2^{-j}}(x^*)} |u(x)|; \tag{4.1}$$

we denote this set by $\mathcal{M}(x^*, u)$.

Proposition 3. *Let $u \in W^{2,d}_{\rm loc}(B_1)$ be a strong solution to (1.1). Suppose Assumptions 1, 3, and 4 hold true. Let $x^* \in \Gamma_{\rm BR}(u)$ and suppose Assumption 5 holds at x^*. There exists a choice of $C_0 > 0$ in Assumption 5 such that, if*

$$V_{2^{-j}}(x^*, u) < C_0 \tag{4.2}$$

for every $j \in \mathcal{M}(x^*, u)$, then

$$\sup_{x \in B_{2^{-j}}(x^*)} |u(x)| \leq \frac{1}{C_0} 2^{-2j}, \quad \forall j \in \mathcal{M}(x^*, u).$$

Proof. For ease of presentation, we split the proof into three steps.

Step 1 - Set $x^* = 0$ and $\mathcal{M}(u) := \mathcal{M}(0, u)$. We resort to a contradiction argument; suppose the statement of the proposition is false. Then, there exist sequences $(u_n)_{n \in \mathbb{N}}$ and $(j_n)_{n \in \mathbb{N}}$ such that u_n is a normalized strong solution to (1.1) such that

$$V_{\frac{1}{2^n}}(u_n) < \frac{1}{n}, \tag{4.3}$$

with

$$\sup_{x \in B_{2^{-j_n}}} |u_n(x)| > \frac{n}{2^{2j_n}}, \tag{4.4}$$

for every $j_n \in \mathcal{M}(u_n)$ and $n \in \mathbb{N}$. Because $\|u_n\|_{L^\infty(B_1)}$ is uniformly bounded, it follows from (4.4) that $j_n \to \infty$. In particular, we re-write (4.3) as

$$V_{\frac{1}{2^{j_n}}}(u_n) < \frac{1}{j_n}. \tag{4.5}$$

Step 2 - Now, we introduce an auxiliary function $v_n : B_1 \to \mathbb{R}$, given by

$$v_n(x) := \frac{u_n(2^{-j_n} x)}{\|u_n\|_{L^\infty(B_{2^{-(j_n+1)}})}}.$$

Clearly, $v_n(0) = 0$. In addition, $V_1(v_n) \to 0$. Moreover, it follows from the definition of v_n that

$$\sup_{B_{1/2}} |v_n(x)| = 1 \tag{4.6}$$

and

$$\sup_{B_1} |v_n(x)| \leq 16.$$

We notice that

$$D^2 v_n(x) = \frac{D^2 u_n(2^{-j_n} x)}{2^{2j_n} \|u_n\|_{L^\infty(B_{2^{-(j_n+1)}})}}.$$

Hence, the homogeneity of F_1 and F_2 yields

$$\min(F_1(D^2 v_n(x)), F_2(D^2 v_n(x))) \leq \frac{\min(F_1(D^2 u_n(2^{-j_n} x)), F_2(D^2 u_n(2^{-j_n} x)))}{2^{2j_n} \|u_n\|_{L^\infty(B_{2^{-(j_n+1)}})}};$$

see Assumption 4. Therefore,

$$\min(F_1(D^2 v_n), F_2(D^2 v_n)) \leq \frac{1}{n} \frac{C \|u_n\|_{L^\infty(B_{2^{-j_n}})}}{\|u_n\|_{L^\infty(B_{2^{-(j_n+1)}})}} \leq \frac{C}{n} \leq C_0, \tag{4.7}$$

for some $C_0 > 0$ and $n \gg 1$. On the other hand,

$$\begin{aligned} \max(F_1(D^2 v_n), F_2(D^2 v_n)) &\geq \frac{\max\left(F_1\left(D^2 u_n\left(\frac{x}{2^{j_n}}\right)\right), F_2\left(D^2 u_n\left(\frac{x}{2^{j_n}}\right)\right)\right)}{2^{2j_n} \|u_n\|_{L^\infty(B_{2^{-(j_n+1)}})}} \\ &\geq -C_0. \end{aligned} \tag{4.8}$$

It follows from (4.7)–(4.8) that $(v_n)_{n\in\mathbb{N}} \subset S^*(\lambda, \Lambda, C_0)$. As a consequence, $v_n \in C^\alpha_{\mathrm{loc}}(B_1)$ for every $n \in \mathbb{N}$, for some unknown $\alpha \in (0, 1)$, with uniform estimates; see [9, Proposition 4.10]. Therefore, there exists v_∞ such that $v_n \to v_\infty$ in $C^\beta_{\mathrm{loc}}(B_1)$, for every $0 < \beta < \alpha$. Since $v_n(0) = 0$ for every $n \in \mathbb{N}$, we infer that $v_\infty(0) = 0$, whereas (4.6) leads to $\|v_\infty\|_{L^\infty(B_{1/2})} = 1$. Because $V_1(v_n) \to 0$, we conclude that $v_\infty \geq 0$ in B_1.

Step 3 - Standard stability results for viscosity solutions build upon (4.7) to ensure

$$\min(F_1(D^2 v_\infty), F_2(D^2 v_\infty)) \leq 0 \quad \text{in } B_1.$$

We conclude that $v_\infty \in \overline{S}(\lambda, \Lambda, 0)$ attains an interior local minimum at the origin. It leads to a contradiction and finishes the proof; see [9, Proposition 4.9]. ∎

In Proposition 3 the constant $C_0 > 0$ informing Assumption 5 is determined. This quantity remains unchanged henceforth. The following result extrapolates the former analysis from $\mathcal{M}(x^*, u)$ to the entire set of natural numbers:

Proposition 4. *Let $u \in W^{2,d}_{\mathrm{loc}}(B_1)$ be a strong solution to (1.1). Suppose Assumptions 1, 3, and 4 hold true. Let $x^* \in \Gamma_{\mathrm{BR}}(u)$ and suppose Assumption 5 holds at x^*. Finally, suppose that for every $j \in \mathcal{M}(x^*, u)$ we have*

$$V_{2^{-j}}(x^*, u) < C_0,$$

for $C_0 > 0$ fixed in (4.2). Then,

$$\sup_{x \in B_{2^{-j}}(x^*)} |u(x)| \leq \frac{4}{C_0} 2^{-2j}, \quad \forall j \in \mathbb{N}.$$

Proof. As before, we set $x^* = 0$ and argue through a contradiction argument; suppose the proposition is false. Let $m \in \mathbb{N}$ be the smallest natural number such that

$$\sup_{B_{2^{-m}}} |u(x)| > \frac{4}{C_0} 2^{-2m}. \tag{4.9}$$

We claim that $m - 1 \in \mathcal{M}(u)$. Indeed,

$$\sup_{B_{2^{1-m}}} |u(x)| \leq \frac{4}{C_0} 2^{-2(m-1)} = \frac{16}{C_0} 2^{-2m} < 4 \sup_{B_{2^{-m}}} |u(x)|.$$

We conclude that

$$\sup_{B_{2^{-m}}} |u(x)| \leq \sup_{B_{2^{1-m}}} |u(x)| \leq \frac{1}{C_0} 2^{-2(m-1)} = \frac{4}{C_0} 2^{-2m},$$

which contradicts (4.9) and thus completes the proof. ∎

Consequential to Proposition 4 is the quadratic growth of u away from the branch point x^*. We detail this argument in the proof of Theorem 1.2.

Proof of Theorem 1.2. Find $j \in \mathbb{N}$ satisfying $2^{-(j+1)} \leq r < 2^{-j}$. It is straightforward to notice that

$$\sup_{B_r} |u(x)| \leq \sup_{B_{2^{-j}}} |u(x)| \leq C \left[\left(\frac{1}{2} \right)^{j+1-1} \right]^2 \leq C r^2,$$

which ends the proof. ∎

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Edgard A. Pimentel
Department of Mathematics, Universidade de Coimbra, Largo D. Dinis, 3004-531 Coimbra, Portugal; edgard.pimentel@mat.uc.pt

Makson S. Santos
Department of Mathematics, Instituto Superior Técnico, Avenida Rovisco Pais 1, 1049-001 Lisbon, Portugal; makson.santos@tecnico.ulisboa.pt