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On the wave equation on moving domains: Regularity, energy balance and application to dynamic debonding

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Abstract. We revisit some issues about existence and regularity for the wave equation in noncylindrical domains. Using a method of diffeomorphisms we show, through increasing regularity assumptions, how the existence of weak solutions, their improved regularity and an energy balance can be derived. As an application, we give a rigorous definition of dynamic energy release rate density for some problems of debonding, and we formulate a proper notion of solutions for such problems. We compare the consistence of such a formulation with that of previous ones, given in the literature for particular cases.

1. Introduction

In this paper, we revisit some issues about existence and regularity for the wave equation in noncylindrical domains, that is, in domains evolving in time. Our main motivation comes from an elastodynamic model for a thin film, initially glued onto a rigid substrate and progressively debonded by applying an external load. As a result of this process, the debonded region is deformed according to the law of elastodynamics; moreover, its oscillations influence the evolution of the debonding front, that is, the interface between the debonded region and the part of film still attached onto the substrate. It is then natural to parametrize the debonded region by means of a time-dependent, growing domain, where the (transverse component of the) displacement satisfies the classical wave equation.

On the other hand, the evolution of the domain is also an unknown of the model and is governed by the physical principle of stability of the internal energy (kinetic and potential) of the body. Rigorously writing the precise form of this energetic criterion requires some technical work, particularly to characterize the energy release rate, which measures, loosely speaking, the amount of energy dissipated during an infinitesimal growth of the debonded region. In fact, in the literature the energy release rate and the propagation criterion were identified only in special cases, such as the one-dimensional setting [11, 32] and the case with radial solutions [23], where some explicit formulas can be used.

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The main scope of this paper is to define the energy release rate for the dynamic debonding problem in a general setting, removing restrictive assumptions on the shape of the growing domains; this will lead to the flow rule governing the evolution of the domain. We show an integral formula for the dynamic energy release rate naturally arising from the energy balance for the wave equation and extending what was previously found in special cases. To obtain this, we have to revisit the problem of the wave equation in time-dependent domains (with initial and boundary conditions) and provide a set of assumptions ensuring existence, uniqueness and regularity.

In the literature there are several results on the wave equation in noncylindrical domains, obtained with various methods and under different assumptions on the evolution of the domains. In [2] an abstract formulation is proposed, as well as a regularization procedure for the operators involved in the problem. In [35] the author uses the Galerkin method combined with a suitable penalization on the boundary. In [8–10, 28, 33, 34] the authors employ changes of variables in order to recast the problem into a fixed domain and then apply abstract results on hyperbolic equations via semigroup theory. We also mention, for example, [1, 3, 4, 17, 19] and references therein for various approaches on different evolution equations (parabolic, Schrödinger, Navier-Stokes, etc.) in noncylindrical domains.

In this paper we collect some results on existence, uniqueness and regularity for the wave equation and present them in a unitary perspective. Some results, already known in the literature, are provided here with different proofs or under slightly different assumptions. In Section 2 we introduce a family Ω_t of Lipschitz domains in $\mathbb{R}^N$ depending on time $t \in [0, T]$. We define weak solutions of the wave equation in Ω_t, complemented by natural initial conditions on Ω_0 and a homogeneous Dirichlet condition on the boundary $\partial\Omega_t$. The results are then extended to nonhomogeneous Dirichlet conditions in Section 5.1. We show an energy balance formula that holds true if the solution has a certain regularity in time and space (Theorem 2.4). The majority of the paper is then devoted to the rigorous proof of such a regularity property.

To this end, in Section 3 we follow the technique of changes of variables, assuming a certain time regularity of the family Ω_t. Specifically, we require that there is a diffeomorphism $\Phi : [0, T] \times \overline{\Omega}_0 \to \mathbb{R}^N$ such that $\Phi(t, \Omega_0) = \Omega_t$ for every t. This leads us to a hyperbolic problem in the fixed domain Ω_0, with coefficients depending on time and space. We compare two notions of solutions for such a problem, called weak solutions and strong-weak solutions, respectively, and we prove they are equivalent (Proposition 3.8). Existence, uniqueness and regularity of solutions to the hyperbolic problem in the cylindrical domain $[0, T] \times \Omega_0$ are proved in Section 4, by means of the Galerkin method (Theorem 4.10). The corresponding results in the noncylindrical domain follow under suitable assumptions on the diffeomorphisms (Theorems 3.9 and 5.4).

As a technical remark, in our results the regularity required on such diffeomorphisms is different if compared to the assumptions of other works in literature. For instance, in [10] and [34] the authors consider changes of variables of class $C^2([0, T]; C^2(\overline{\Omega}_0))$ and $C^3([0, T] \times \overline{\Omega}_0)$, respectively; in contrast, we require diffeomorphisms of the class $C^{1,1}([0, T]\times\overline{\Omega}_0)$ for existence of solutions (Sections 3 and 4.2), while we need diffeomorphisms of class $C^{2,1}([0,T]\times\overline{\Omega}_0)$ for uniqueness, regularity and energy balance (Sections 4.3 and 5). Moreover, for the regularity result we need to assume that Ω_0 is convex or of class C^2. We stress in particular that, differently from previous works, we actually allow for a wider choice of the reference configuration Ω_0, including some nonsmooth cases.

In Section 5 we combine the results of Sections 3 and 4, thus providing a full statement of the energy balance (Theorem 5.4). Moreover, we show how our results may be applied to different settings: the case of dimension one, extensively analyzed in, for instance, [11, 14, 22, 24–26, 30–32]; the case where each domain Ω_t is homothetic to Ω_0; and the case where each domain is the sublevel set of a smooth function, which includes the case of radial solutions investigated in [23].

Our approach is deeply related to some works dealing with dynamic models for crack propagation in brittle fracture by means of the same "method of diffeomorphisms" (see [5–7, 12]). Indeed, the formulation of dynamic fracture also relies on the wave equation in a time-dependent domain—in this case, a domain with a growing crack. However, since such a domain is not Lipschitz, our results do not apply to this case. On the other hand, in a dynamic fracture problem the domains only differ by a set of codimension one.

The main similarity between the dynamic models for fracture and debonding is that the wave equation is coupled with a flow rule governing the evolution of the domain. The latter arises from an energetic criterion [16] which may be stated as a maximum dissipation principle [21], or equivalently as a Griffith-type criterion involving the dynamic energy release rate. In particular, it turns out that the flow rule implicitly depends on the solution of the wave equation.

In Section 6 we show how the results of the previous sections allow us to rigorously define the energy release rate (and thus the propagation criterion) for the dynamic debonding model in a quite general setting. More precisely, we introduce the *density* of the dynamic energy release rate, which is obtained by a localization procedure; and a corresponding local version of Griffith's criterion, satisfied at each point of the debonding front.

Our main achievement in this respect is a proper formulation of the coupled problem of dynamic debonding (wave equation together with local Griffith criterion) which includes the special cases analyzed in previous papers and may be applied without assuming a special form of the domains. We indeed show how the solutions found in the one-dimensional [11, 32] and radial [23] setting fulfill the formulation proposed here (Theorems 6.11 and 6.12). The well-posedness in the general framework still remains an open question, due to the high complexity arising from the coupling between the wave equation and Griffith criterion.

Notation

Throughout the paper, the set of $M\times M$ matrices with real entries is denoted by $\mathbb{R}^{M\times M}$, and the subset of symmetric matrices is $\mathbb{R}^{M\times M}_{\mathrm{sym}}$. The identity matrix is denoted by I. For the transpose of a matrix A we adopt the symbol A^T. The scalar product between two vectors $w, v \in \mathbb{R}^M$ is indicated by $w \cdot v$.

By $\dot{f}$ we mean the time derivative of a function $f = f(t,x)$. If f is scalar-valued we write ∇f for its gradient with respect to spatial components, represented as a column vector. If f is vector-valued, we instead write Df for its Jacobian matrix with respect to spatial components; as usual, each row of Df is the gradient of the corresponding component of f.

Given an open set $E \subseteq \mathbb{R}^M$ with Lipschitz boundary ∂E, we denote by ν_E the outward unit normal to E. If $E \subseteq \mathbb{R} \times \mathbb{R}^N$, as is often the case throughout the paper, we write ν_E^t and ν_E^x for the time- and space-component of the outward normal, respectively, that is, $\nu_E = (\nu_E^t, \nu_E^x) \in \mathbb{R} \times \mathbb{R}^N$.

The integration with respect to the Lebesgue and to the M-dimensional Hausdorff measure is denoted respectively by $\mathrm{d}x$ (or $\mathrm{d}y$) and by $\mathrm{d}\mathcal{H}^M$. We adopt standard notations for Lebesgue and Sobolev spaces and for Bochner spaces. Given a Banach space X, we denote by $\langle w, v\rangle_X$ the duality product between $w \in X^*$ and $v \in X$. If $X = L^2(E)$, we identify it with its dual and, with a slight abuse of notation, we mean by $\langle w, v\rangle_{L^2(E)}$ the scalar product between w and v. In the case of $X = H_0^1(E)$, we instead adopt the convention of rigged Hilbert spaces, that is, for $w \in L^2(E)$ and $v \in H_0^1(E)$ one has $\langle w, v\rangle_{H_0^1(E)} = \langle w, v\rangle_{L^2(E)}$.

2. The wave equation on moving domains

For $T > 0$, we consider a family $\{\Omega_t\}_{t\in[0,T]}$ of domains in $\mathbb{R}^N$, with $N \in \mathbb{N}$, that is,

$$\text{for every } t \in [0,T] \text{ the set } \Omega_t \subseteq \mathbb{R}^N \text{ is nonempty, open, bounded and Lipschitz.} \tag{2.1a}$$

In some of the results, in view of the applications to debonding models (see Section 6), we shall also assume that the family $\{\Omega_t\}_{t\in[0,T]}$ is nondecreasing with respect to inclusion:

$$\Omega_s \subseteq \Omega_t \quad \text{for every } 0 \le s \le t \le T. \tag{2.1b}$$

We denote the complement of Ω_t by

$$\Omega_t^c := \mathbb{R}^N \setminus \Omega_t.$$

Furthermore, we introduce the "space-time" domain $\mathcal{O}$ and its parabolic boundary Γ by

$$\mathcal{O} := \bigcup_{t\in(0,T)} \{t\} \times \Omega_t \quad \text{and} \quad \Gamma := \bigcup_{t\in(0,T)} \{t\} \times \partial\Omega_t. \tag{2.2}$$

Let us consider the following formal problem for a function $u : \overline{\mathcal{O}} \to \mathbb{R}$:

$$\begin{cases} \ddot{u}(t,x) - \Delta u(t,x) = f(t,x), & (t,x) \in \mathcal{O}, \\ u(t,x) = 0, & (t,x) \in \Gamma, \\ u(0,x) = u_0(x), & x \in \Omega_0, \\ \dot{u}(0,x) = u_1(x), & x \in \Omega_0. \end{cases} \tag{2.3}$$

The system above consists of a wave equation in the noncylindrical domain $\mathcal{O}$ with forcing term

$$f \in L^2(\mathcal{O}), \tag{2.4a}$$

complemented by initial conditions

$$u_0 \in H^1_0(\Omega_0), \quad u_1 \in L^2(\Omega_0), \tag{2.4b}$$

and a homogeneous Dirichlet boundary condition on Γ.

Remark 2.1. All the results within the paper may also be adapted to more general hyperbolic equations of the form

$$\ddot{u}(t,x) - \operatorname{div}(A(t,x)\nabla u(t,x)) = f(t,x), \quad (t,x) \in \mathcal{O}, \tag{2.5}$$

which model, for instance, nonhomogeneous materials. The minimal assumptions on the matrix $A(t,x)$ needed to perform all the arguments are contained in the regularity property

$$A \in C^{1,1}(\overline{\mathcal{O}}; \mathbb{R}^{N\times N}_{\mathrm{sym}}), \tag{2.6}$$

and in the uniform ellipticity condition

$$(A(t,x)w)\cdot w \geq c_A|w|^2 \quad \text{for all } w \in \mathbb{R}^N, \tag{2.7}$$

which must hold for all $(t,x) \in \overline{\mathcal{O}}$ with a positive constant $c_A > 0$.

Since the proofs remain basically unchanged, in order to avoid heavy notations, throughout the paper we prefer to focus our attention on problem (2.3), that is, with $A(t,x) = I$. The main changes in the statements are instead highlighted throughout the paper (see Remarks 2.7, 3.2, 3.4, 5.6, 5.8 and 6.7).

For the sake of clarity, we will also adopt the following convention:

$$|w|_{A(t,x)} := \sqrt{(A(t,x)w)\cdot w} \quad \text{for all } w \in \mathbb{R}^N.$$

Notice that by (2.6) and (2.7), the function $|\cdot|_{A(t,x)}$ defines a norm on $\mathbb{R}^N$ for every fixed $(t,x) \in \overline{\mathcal{O}}$.

Since we are working in time-dependent domains it is useful to introduce time-dependent Bochner spaces. Given a family of normed spaces $\{X_t\}_{t\in[0,T]}$, with a slight abuse of notation we say that a function v belongs to $L^p(0,T;X_t)$, with $p \in [1,+\infty]$, if $v(t) \in X_t$ for a.e. $t \in (0,T)$ and the map $t \mapsto \|v(t)\|_{X_t}$ is in $L^p(0,T)$. Notice that $L^2(\mathcal{O}) = L^2(0,T;L^2(\Omega_t))$ by Fubini's theorem. Using a similar convention for Sobolev spaces, whenever $\mathcal{O}$ is open one may write

$$H^1(\mathcal{O}) = L^2(0,T;H^1(\Omega_t)) \cap H^1(0,T;L^2(\Omega_t))$$

and

$$H^2(\mathcal{O}) = L^2(0,T;H^2(\Omega_t)) \cap H^1(0,T;H^1(\Omega_t)) \cap H^2(0,T;L^2(\Omega_t)).$$

However, we prefer to employ the notation of time-dependent Bochner spaces only when necessary, and only for spaces of the type $L^p(0,T;X_t)$.

We can now give the definition of weak solution to problem (2.3). Notice that property (i) of the definition below involves the time-dependent spaces $H^1(\Omega_t)$ and $L^2(\Omega_t)$, while property (ii) features usual Bochner spaces of continuous functions with fixed target. In property (ii) it is understood that we consider the restriction of u to $[0,T]\times\Omega_0$ and of $\dot u$ to $[0,\delta]\times\Omega'$. Here and henceforth, all solutions u to equation (2.3) will be extended to 0 outside $\mathcal{O}$.

Definition 2.2. We say that $u\colon\overline{\mathcal{O}}\to\mathbb{R}$ is a weak solution to problem (2.3) with data (2.4) if

(i) $u\in L^2(0,T;H^1_0(\Omega_t))$ and $\dot u\in L^2(0,T;L^2(\Omega_t))$;

(ii) $u\in C^0([0,T];L^2(\Omega_0))$ and $\dot u\in C^0([0,\delta];H^{-1}(\Omega'))$ for every $\delta>0$ and $\Omega'\subseteq\Omega_0$ open such that $[0,\delta]\times\Omega'\subseteq\mathcal{O}$; in addition, the initial conditions $u(0)=u_0$ and $\dot u(0)=u_1$ hold;

(iii) u satisfies

$$-\int_0^T\langle\dot u(t),\dot\eta(t)\rangle_{L^2(\Omega_t)}\,\mathrm{d}t+\int_0^T\langle\nabla u(t),\nabla\eta(t)\rangle_{L^2(\Omega_t)}\,\mathrm{d}t$$
$$=\int_0^T\langle f(t),\eta(t)\rangle_{L^2(\Omega_t)}\,\mathrm{d}t,\tag{2.8}$$

for every $\eta\in L^2(0,T;H^1_0(\Omega_t))$ with $\dot\eta\in L^2(0,T;L^2(\Omega_t))$ and $\eta(T)=\eta(0)=0$.

Remark 2.3. The regularity assumptions of property (ii), which allow one to state the initial conditions on position and velocity, are actually consequences of (i) and (iii) under additional hypotheses on $t\mapsto\Omega_t$.

Assume, for instance, that (2.1b) holds, as in Section 2.1. Then, (i) implies that u and $\dot u$ belong to $L^2(0,T;L^2(\Omega_0))$, hence $u\in C^0([0,T];L^2(\Omega_0))$; moreover, the wave equation in (iii) implies that $\ddot u=\Delta u+f\in L^2(0,T;H^{-1}(\Omega_0))$, hence $\dot u\in C^0([0,T];H^{-1}(\Omega_0))$.

Without assuming monotonicity, a similar argument holds provided there exist diffeomorphisms as in (3.1), satisfying (H1); such properties are assumed from Section 3 onward.

In the paper we show how to obtain existence, uniqueness and an energy balance for solutions u of problem (2.3), in particular to explicitly derive an expression for the energy variation due to the evolution of the domain (which will be interpreted as the energy spent during the debonding process). Such energy balance will be crucial in Section 6, where we present applications to dynamic debonding models. If one assumes existence of a regular solution as in (2.10), deriving the energy balance (2.11) is a direct computation: we present it in Theorem 2.4 for the reader's convenience. A more delicate issue is the rigorous proof of regularity property (2.10): this will be the aim of Sections 3 and 4. In Section 5 we will then obtain a more precise statement of the energy balance (see Theorem 5.4).

Theorem 2.4. *Assume* (2.1a)*, assume that* $\mathcal{O}$ *is open with Lipschitz boundary and that*

$$\partial\mathcal{O} = \Gamma \cup (\{T\} \times \overline{\Omega}_T) \cup (\{0\} \times \overline{\Omega}_0). \tag{2.9}$$

Let u *be a weak solution to problem* (2.3) *satisfying the following regularity property:*

$$u \in L^2(0,T; H^2(\Omega_t) \cap H^1_0(\Omega_t)), \quad \dot{u} \in L^2(0,T; H^1(\Omega_t)), \quad \ddot{u} \in L^2(0,T; L^2(\Omega_t)). \tag{2.10}$$

Then, for every $t \in [0,T]$ *the following energy balance holds true:*

$$\begin{aligned}\frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega_t)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega_t)} - \int_{\Gamma_t} \frac{\nu^t_{\mathcal{O}}}{2}\Big[1 - \Big(\frac{\nu^t_{\mathcal{O}}}{|\nu^x_{\mathcal{O}}|}\Big)^2\Big]|\nabla u|^2 \,\mathrm{d}\mathcal{H}^N \\ = \frac{1}{2}\|u_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega_0)} + \int_0^t \langle f(s), \dot{u}(s)\rangle_{L^2(\Omega_s)} \,\mathrm{d}s,\end{aligned} \tag{2.11}$$

where $\Gamma_t := \{(s,x) \in \Gamma : s \in (0,t)\}$.

In energy balance (2.11), we recognize in the left-hand side the kinetic energy, the potential energy and a term corresponding to the evolution of the domain, while in the right-hand side we see the initial energy and the work of external forces. The integral over Γ_t may be positive or negative, according to the geometry of $\mathcal{O}$. If monotonicity condition (2.1b) is in force one has $\nu^t_{\mathcal{O}} \leq 0$; moreover, a typical assumption is that the growth of the domains is subsonic, that is, $|\nu^t_{\mathcal{O}}| \leq |\nu^x_{\mathcal{O}}|$ (in this situation, the set $\mathcal{O}$ is usually called time-like; see [8–10, 28, 33, 35]). In applications to debonding models, where both conditions hold, the integral over Γ_t can be thus interpreted as energy dissipated in the debonding process.

Remark 2.5. We point out that condition (2.9) is a weak regularity assumption on the time-dependence of the domain, and it does not hold in general for sets satisfying (2.1). Indeed, if there is a time-discontinuity at time t_0, the boundary of $\mathcal{O}$ will contain a set of the form $\{t_0\} \times D$, for some set D. The reader may think, for instance, to the simple one-dimensional example

$$\Omega_t = \begin{cases} (0,1) & \text{if } t \in [0,1), \\ (0,2) & \text{if } t \in [1,2], \end{cases}$$

in which the set $\{1\} \times (1,2)$ is contained in $\partial\mathcal{O}$. In Section 3 we will assume a stronger regularity condition, which will imply (2.9) (see Remark 3.1).

Remark 2.6. The request of higher regularity (2.10) is needed to give a meaning to the term in (2.11) representing the energy variation due to the evolution of the domain, where ∇u has to be integrated along the lateral boundary Γ. A reformulation of this term, in such a way that the energy balance may be written for genuine weak solutions (i.e., just satisfying property (i) in Definition 2.2), would certainly be desirable; unfortunately, to our better knowledge, a suitable rewriting of such term is still not available.

Remark 2.7. Under the additional assumptions of Remark 2.1, the energy balance (2.11) changes to

$$
\begin{aligned}
&\frac{1}{2}\|\dot u(t)\|^2_{L^2(\Omega_t)}+\frac{1}{2}\langle A(t)\nabla u(t),\nabla u(t)\rangle_{L^2(\Omega_t)}-\int_{\Gamma_t}\frac{\nu^t_{\mathcal{O}}}{2}\Big[|\nabla u|^2_A-\Big(\frac{\nu^t_{\mathcal{O}}}{|\nu^x_{\mathcal{O}}|}\Big)^2|\nabla u|^2\Big]\mathrm{d}\mathcal{H}^N\\
&\quad=\frac{1}{2}\|u_1\|^2_{L^2(\Omega_0)}+\frac{1}{2}\langle A(0)\nabla u_0,\nabla u_0\rangle_{L^2(\Omega_0)}+\int_0^t\langle f(s),\dot u(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s\\
&\qquad+\frac{1}{2}\int_0^t\langle \dot A(s)\nabla u(s),\nabla u(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s.
\end{aligned}
$$

Proof of Theorem 2.4. By exploiting (2.10), we deduce that u belongs to $H^2(\mathcal{O})$ and thus satisfies

$$\ddot u(t,x)-\Delta u(t,x)=f(t,x)\quad \text{for a.e. } (t,x)\in\mathcal{O}.$$

Multiplying the previous equation by a function $\varphi\in L^2(0,T;H^1(\Omega_t))$ such that $\dot\varphi\in L^2(0,T;L^2(\Omega_t))$ and integrating by parts in the "space-time" domain $\mathcal{O}$, for all $t\in[0,T]$ we obtain

$$
\begin{aligned}
&\langle \dot u(t),\varphi(t)\rangle_{L^2(\Omega_t)}-\langle u_1,\varphi(0)\rangle_{L^2(\Omega_0)}\\
&\qquad-\int_0^t\langle \dot u(s),\dot\varphi(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s+\int_0^t\langle \nabla u(s),\nabla\varphi(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s\\
&\quad=\int_0^t\langle f(s),\varphi(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s-\int_{\Gamma_t}\big(\dot u\,\nu^t_{\mathcal{O}}-\nabla u\cdot\nu^x_{\mathcal{O}}\big)\varphi\,\mathrm{d}\mathcal{H}^N.
\end{aligned}
$$

Thanks to the regularity provided by (2.10), we can choose as the test function $\varphi=\dot u$. We thus obtain

$$
\begin{aligned}
&\|\dot u(t)\|^2_{L^2(\Omega_t)}-\|u_1\|^2_{L^2(\Omega_0)}-\int_0^t\langle \dot u(s),\ddot u(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s+\int_0^t\langle \nabla u(s),\nabla\dot u(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s\\
&\quad=\int_0^t\langle f(s),\dot u(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s-\int_{\Gamma_t}(\dot u)^2\nu^t_{\mathcal{O}}\,\mathrm{d}\mathcal{H}^N+\int_{\Gamma_t}\big(\nabla u\cdot\nu^x_{\mathcal{O}}\big)\dot u\,\mathrm{d}\mathcal{H}^N.
\end{aligned}\tag{2.12}
$$

Integrating by parts in time the integral terms in the first line, we get

$$
\begin{aligned}
\int_0^t\langle \dot u(s),\ddot u(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s&=\frac{1}{2}\|\dot u(t)\|^2_{L^2(\Omega_t)}-\frac{1}{2}\|u_1\|^2_{L^2(\Omega_0)}\\
&\quad+\frac{1}{2}\int_{\Gamma_t}(\dot u)^2\nu^t_{\mathcal{O}}\,\mathrm{d}\mathcal{H}^N,
\end{aligned}\tag{2.13a}
$$

$$
\begin{aligned}
\int_0^t\langle \nabla u(s),\nabla\dot u(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s&=\frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega_t)}-\frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega_0)}\\
&\quad+\frac{1}{2}\int_{\Gamma_t}|\nabla u|^2\nu^t_{\mathcal{O}}\,\mathrm{d}\mathcal{H}^N.
\end{aligned}\tag{2.13b}
$$

We now notice that, since $u \equiv 0$ on Γ, it must hold that

$$\dot u\, \nu_{\mathcal{O}}^{x} = \nu_{\mathcal{O}}^{t} \nabla u, \quad \mathcal{H}^N\text{-a.e. on } \Gamma,$$

which in particular implies the relations

$$\dot u(\nabla u \cdot \nu_{\mathcal{O}}^{x}) = \nu_{\mathcal{O}}^{t} |\nabla u|^2, \qquad \mathcal{H}^N\text{-a.e. on } \Gamma, \tag{2.14a}$$

$$(\dot u)^2 |\nu_{\mathcal{O}}^{x}|^2 = (\nu_{\mathcal{O}}^{t})^2 |\nabla u|^2, \quad \mathcal{H}^N\text{-a.e. on } \Gamma. \tag{2.14b}$$

By plugging (2.13) and (2.14) into (2.12), we finally conclude that

$$\begin{aligned} &\frac{1}{2}\|\dot u(t)\|^2_{L^2(\Omega_t)} - \frac{1}{2}\|u_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega_t)} - \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega_0)} \\ &\quad = \int_0^t \langle f(s), \dot u(s)\rangle_{L^2(\Omega_s)} \,\mathrm{d}s + \int_{\Gamma_t} \frac{\nu_{\mathcal{O}}^{t}}{2}\Big[1 - \Big(\frac{\nu_{\mathcal{O}}^{t}}{|\nu_{\mathcal{O}}^{x}|}\Big)^2\Big] |\nabla u|^2 \,\mathrm{d}\mathcal{H}^N, \end{aligned}$$

and thus, the statement is proved. ∎

2.1. An existence result

We conclude this section by proposing a novel strategy to prove existence of solutions to the wave equations in noncylindrical domains; see [4] for a similar approach in the context of parabolic equations. It is based on time-discretization and it does not require any time-regularity on the growth of the sets Ω_t. However, it is now crucial to require that the family $\{\Omega_t\}_{t\in[0,T]}$ is nondecreasing, that is, (2.1b) holds. Under these assumptions, we also obtain an energy inequality (in contrast, the energy balance was obtained before under stronger regularity hypotheses on the solution).

We stress that our discretization procedure is substantially different (and from our point of view, simpler and more intuitive) with respect to the (variant of) the classical minimizing movements approach used for hyperbolic problems, applied, for instance, in the context of dynamic fracture mechanics in [13]. Indeed, the latter relies on an iterative minimization of a suitable energy, followed by the construction of a piecewise affine interpolant. In our approach, instead, after the discretization of the time interval $[0, T]$ we consider the related piecewise constant evolution of the domains Ω_t, and in each discrete-time interval we pick the solution of the wave equation in the corresponding cylindrical domain (see (2.17)). This allows us to employ well-known results for the wave equation in cylindrical domains.

Before stating the result we recall that, given a Banach space X, the set $C^0_{\mathrm{w}}([0, T]; X)$ denotes the space of functions $u : [0, T] \to X$ which are continuous with respect to the weak topology of X. Notice that here X is independent of time; in fact, we adopt the convention that the solutions of (2.3) are extended to zero outside $\mathcal{O}$.

Theorem 2.8. *Assume* (2.1) *and* (2.4)*. Then, there exists a weak solution u of problem* (2.3) *in the sense of Definition* 2.2*. Moreover,*

$$
\begin{aligned}
&u \in C^0_{\mathrm{w}}([0,T]; H^1_0(\Omega_T)),\\
&\dot u \in L^\infty(0,T; L^2(\Omega_t)) \cap C^0_{\mathrm{w}}([\bar t, T]; L^2(\Omega_{\bar t})) \quad \textit{for all } \bar t \in [0,T).
\end{aligned}
$$

Furthermore, the following energy inequality holds true for every $t \in [0,T]$:

$$
\begin{aligned}
\frac{1}{2}\|\dot u(t)\|^2_{L^2(\Omega_t)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega_t)} \le \frac{1}{2}\|u_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega_0)}&\\
+ \int_0^t \langle f(s), \dot u(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s.&
\end{aligned}
\tag{2.15}
$$

Proof. We adopt a time discretisation argument: we consider a sequence of partitions of $[0,T]$ with vanishing size, that is, for every $n \in \mathbb{N}$ we take $0 = t^n_0 < t^n_1 < \cdots < t^n_{k(n)} = T$ satisfying

$$
\lim_{n\to+\infty} \sup_{k=1,\dots,k(n)} |t^n_k - t^n_{k-1}| = 0. \tag{2.16}
$$

For every $k = 1,\dots,k(n)$, we then take u^n_k as the unique weak solution of the wave equation in the cylinder $(t^n_{k-1}, t^n_k) \times \Omega_{t^n_{k-1}}$ with initial data $u^n_{k-1}(t^n_{k-1})$ and $\dot u^n_{k-1}(t^n_{k-1})$ (with the convention $u^n_0(0) = u_0$ and $\dot u^n_0(0) = u_1$), that is,

$$
\begin{cases}
\ddot u^n_k - \Delta u^n_k = f & \text{in } (t^n_{k-1}, t^n_k) \times \Omega_{t^n_{k-1}},\\
u^n_k = 0 & \text{in } (t^n_{k-1}, t^n_k) \times \partial\Omega_{t^n_{k-1}},\\
u^n_k(t^n_{k-1}) = u^n_{k-1}(t^n_{k-1}), &\\
\dot u^n_k(t^n_{k-1}) = \dot u_{k-1}(t^n_{k-1}). &
\end{cases}
\tag{2.17}
$$

We adopt the usual convention that u^n_k is extended to 0 in $\Omega^c_{t^n_{k-1}}$. Standard arguments show the following properties:

(i) u^n_k belongs to $C^0([t^n_{k-1}, t^n_k]; H^1_0(\Omega_T)) \cap C^1([t^n_{k-1}, t^n_k]; L^2(\Omega_T))$;

(ii) $u^n_k(t^n_{k-1}) = u^n_{k-1}(t^n_{k-1})$ in the sense of the space $C^0([t^n_{k-1}, t^n_k]; H^1_0(\Omega_T))$ and $\dot u^n_k(t^n_{k-1}) = \dot u^n_{k-1}(t^n_{k-1})$ in the sense of the space $C^0([t^n_{k-1}, t^n_k]; L^2(\Omega_T))$;

(iii) for every $\eta \in L^2(t^n_{k-1}, t^n_k; H^1_0(\Omega_T)) \cap H^1(t^n_{k-1}, t^n_k; L^2(\Omega_T))$ such that $\eta(t) = 0$ in $\Omega^c_{t^n_{k-1}}$ and for a.e. $t \in (t^n_{k-1}, t^n_k)$ it holds that

$$
\begin{aligned}
-\int_{t^n_{k-1}}^{t^n_k} \langle \dot u^n_k(s), \dot\eta(s)\rangle_{L^2(\Omega_T)}\,\mathrm{d}s + \int_{t^n_{k-1}}^{t^n_k} \langle \nabla u^n_k(s), \nabla\eta(s)\rangle_{L^2(\Omega_T)}\,\mathrm{d}s&\\
= \int_{t^n_{k-1}}^{t^n_k} \langle f(s), \eta(s)\rangle_{L^2(\Omega_T)}\,\mathrm{d}s&\\
+ \langle \dot u^n_{k-1}(t^n_{k-1}), \eta(t^n_{k-1})\rangle_{L^2(\Omega_T)} - \langle \dot u^n_k(t^n_k), \eta(t^n_k)\rangle_{L^2(\Omega_T)}.&
\end{aligned}
$$

Furthermore, we have the following energy balance:

(iv) for every $t \in [t^n_{k-1}, t^n_k]$ it holds that

$$
\begin{aligned}
&\frac{1}{2}\|\dot u^n_k(t)\|^2_{L^2(\Omega_{t^n_{k-1}})} + \frac{1}{2}\|\nabla u^n_k(t)\|^2_{L^2(\Omega_{t^n_{k-1}})} \\
&\quad = \frac{1}{2}\|\dot u^n_{k-1}(t_{k-1})\|^2_{L^2(\Omega_{t^n_{k-2}})} + \frac{1}{2}\|\nabla u^n_{k-1}(t_{k-1})\|^2_{L^2(\Omega_{t^n_{k-2}})} \\
&\qquad + \int_{t^n_{k-1}}^t \langle f(s), \dot u^n_k(s)\rangle_{L^2(\Omega_{t^n_{k-1}})}\,\mathrm{d}s,
\end{aligned} \tag{2.18}
$$

where we also extended f in the whole of $(0,T)\times\Omega_T$ by setting $f \equiv 0$ outside $\mathcal{O}$. In particular, by recalling again that $u^n_k(t)$ vanishes in $\Omega^c_{t^n_{k-1}}$ and by summing (2.18) for $j = 2, \dots, k$, we deduce for every $t \in [t^n_{k-1}, t^n_k]$

$$
\begin{aligned}
&\frac{1}{2}\|\dot u^n_k(t)\|^2_{L^2(\Omega_T)} + \frac{1}{2}\|\nabla u^n_k(t)\|^2_{L^2(\Omega_T)} \\
&\quad = \frac{1}{2}\|u_1\|^2_{L^2(\Omega_T)} + \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega_T)} + \int_{t^n_{k-1}}^t \langle f(s), \dot u^n_k(s)\rangle_{L^2(\Omega_T)}\,\mathrm{d}s \\
&\qquad + \sum_{j=2}^k \int_{t^n_{j-2}}^{t^n_{j-1}} \langle f(s), \dot u^n_{j-1}(s)\rangle_{L^2(\Omega_T)}\,\mathrm{d}s.
\end{aligned} \tag{2.19}
$$

For every $n \in \mathbb{N}$, we now define

$$
u^n(t) := \begin{cases} u^n_k(t) & \text{if } t \in [t^n_{k-1}, t^n_k) \text{ for some } k = 1, \dots, k(n), \\ u^n_{k(n)}(T) & \text{if } t = T. \end{cases}
$$

By construction, u^n belongs to $C^0([0,T]; H^1_0(\Omega_T)) \cap C^1([0,T]; L^2(\Omega_T))$ and satisfies the following properties:

(i') $u^n = 0$ in $\bigcup_{k=1}^{k(n)} [t^n_{k-1}, t^n_k] \times \Omega^c_{t^n_{k-1}} \supseteq \bigcup_{t\in[0,T]} \{t\} \times \Omega^c_t$;

(ii') $u^n(0) = u_0$ in the sense of $C^0([0,T]; H^1_0(\Omega_T))$ and $\dot u^n(0) = u_1$ in the sense of $C^0([0,T]; L^2(\Omega_T))$;

(iii') for every $\eta \in C^\infty_c((0,T)\times\Omega_T)$ such that $\operatorname{supp}\eta \subseteq \bigcup_{k=1}^{k(n)} [t^n_{k-1}, t^n_k) \times \Omega_{t^n_{k-1}}$, it holds that

$$
\begin{aligned}
&-\int_0^T \langle \dot u^n(s), \dot\eta(s)\rangle_{L^2(\Omega_T)}\,\mathrm{d}s + \int_0^T \langle \nabla u^n(s), \nabla\eta(s)\rangle_{L^2(\Omega_T)}\,\mathrm{d}s \\
&\quad = \int_0^T \langle f(s), \eta(s)\rangle_{L^2(\Omega_T)}\,\mathrm{d}s.
\end{aligned} \tag{2.20}
$$

Furthermore, energy balance (2.19) reads as follows:

(iv') for every $t \in [0, T]$ it holds that

$$\frac{1}{2}\|\dot{u}^n(t)\|^2_{L^2(\Omega_T)}+\frac{1}{2}\|\nabla u^n(t)\|^2_{L^2(\Omega_T)} \\ = \frac{1}{2}\|u_1\|^2_{L^2(\Omega_T)}+\frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega_T)}+\int_0^t \langle f(s), \dot{u}^n(s)\rangle_{L^2(\Omega_T)}\,\mathrm{d}s. \tag{2.21}$$

By a classical Grönwall argument, since the forcing term f is in $L^2((0,T)\times\Omega_T)$, we thus deduce

$$\max_{t\in[0,T]}\Big(\frac{1}{2}\|\dot{u}^n(t)\|^2_{L^2(\Omega_T)} + \frac{1}{2}\|\nabla u^n(t)\|^2_{L^2(\Omega_T)}\Big) \le C.$$

This implies the existence of $u \in L^\infty(0,T;H^1_0(\Omega_T))$ and of $u^* \in L^\infty(0,T;L^2(\Omega_T))$ such that, up to subsequences (not relabeled), we have

$$\begin{aligned} u^n &\rightharpoonup u \quad \text{weakly in } L^2(0,T;H^1_0(\Omega_T)),\\ \dot{u}^n &\rightharpoonup u^* \quad \text{weakly in } L^2(0,T;L^2(\Omega_T)). \end{aligned}$$

It is standard to show that $u^* = \dot{u}$. Thus, we deduce the existence of a function $u \in L^\infty(0,T;H^1_0(\Omega_T))$ with $\dot{u} \in L^\infty(0,T;L^2(\Omega_T))$ such that

$$u^n \rightharpoonup u \quad \text{weakly in } L^2(0,T;H^1_0(\Omega_T)) \cap H^1(0,T;L^2(\Omega_T)). \tag{2.22}$$

Notice that in (2.22) the target spaces are independent of time; however, by property (i') we easily deduce that $u \equiv 0$ outside $\mathcal{O}$, so we get the stronger conditions $u \in L^\infty(0,T; H^1_0(\Omega_t))$ and $\dot{u} \in L^\infty(0,T;L^2(\Omega_t))$. By the continuous embedding

$$L^\infty(0,T;H^1_0(\Omega_T)) \cap H^1(0,T;L^2(\Omega_T)) \subseteq C^0_{\mathrm{w}}([0,T];H^1_0(\Omega_T)),$$

we also obtain $u \in C^0_{\mathrm{w}}([0,T];H^1_0(\Omega_T))$.

To complete the proof that u satisfies Definition 2.2, we prove (2.8) by passing to the limit in (2.20) by means of (2.22). Here, a technical issue is that the spaces of test functions in (2.8) and in (2.20) are different. However, given a function $\eta \in L^2(0,T;H^1_0(\Omega_t))$ with $\dot{\eta} \in L^2(0,T;L^2(\Omega_t))$ and $\eta(T) = \eta(0) = 0$, we can approximate it by a sequence of smooth functions η^n as in property (iii'): this readily follows thanks to (2.16) and concludes the proof of (2.8). We finally observe that for every $\bar{t} \in [0,T)$ the function u is in particular a weak solution of the wave equation in the cylinder $(\bar{t},T)\times\Omega_{\bar{t}}$, and thus it belongs to $C^1_w([\bar{t},T];L^2(\Omega_{\bar{t}}))$.

We are only left to prove energy inequality (2.15). We integrate (2.21) between arbitrary times α, β with $0 \le \alpha \le \beta \le T$. By (2.22) and standard lower semicontinuity arguments, as $n \to +\infty$ we obtain

$$\int_\alpha^\beta \frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega_T)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega_T)}\,\mathrm{d}t \\ \le (\beta-\alpha)\Big(\frac{1}{2}\|u_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega_0)}\Big) + \int_\alpha^\beta\int_0^t \langle f(s), \dot{u}(s)\rangle_{L^2(\Omega_T)}\,\mathrm{d}s\,\mathrm{d}t.$$

By the arbitrariness of α and β, for a.e. $t \in [0, T]$ the above inequality yields

$$\frac{1}{2}\|\dot u(t)\|^2_{L^2(\Omega_T)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega_T)} \leq \frac{1}{2}\|u_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega_0)} + \int_0^t \langle f(s), \dot u(s)\rangle_{L^2(\Omega_T)}\,\mathrm{d}s. \tag{2.23}$$

We will now improve (2.23) by providing an energy inequality valid for every time. We fix $\bar t \in [0, T)$ and consider a sequence $t_k \searrow \bar t$ along which (2.23) is satisfied. Since $u \in C^0_{\mathrm{w}}([0, T]; H^1_0(\Omega_T))$, $\dot u \in C^0_{\mathrm{w}}([\bar t, T]; L^2(\Omega_{\bar t}))$ and $\Omega_{\bar t} \subseteq \Omega_T$, again by weak lower semicontinuity we deduce

$$\begin{aligned}
&\frac{1}{2}\|\dot u(\bar t)\|^2_{L^2(\Omega_{\bar t})} + \frac{1}{2}\|\nabla u(\bar t)\|^2_{L^2(\Omega_{\bar t})}\\
&\quad\leq \liminf_{k\to+\infty}\Big(\frac{1}{2}\|\dot u(t_k)\|^2_{L^2(\Omega_T)} + \frac{1}{2}\|\nabla u(t_k)\|^2_{L^2(\Omega_T)}\Big)\\
&\quad\leq \frac{1}{2}\|u_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega_0)} + \int_0^{\bar t} \langle f(s), \dot u(s)\rangle_{L^2(\Omega_T)}\,\mathrm{d}s\\
&\quad= \frac{1}{2}\|u_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega_0)} + \int_0^{\bar t} \langle f(s), \dot u(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s.
\end{aligned}$$

Hence, (2.15) is satisfied for all $\bar t \in [0, T)$. Its validity also in $\bar t = T$ follows by taking a larger final time $\widetilde T > T$, defining, for instance, $\Omega_t := \Omega_T$ for $t \in (T, \widetilde T]$ and arguing in the same way. ∎

3. Equivalent reformulation on a fixed domain

In this section we recast problem (2.3) into a hyperbolic problem in a fixed domain (see (3.5) below). To this end, we adapt the method of diffeomorphisms developed in [8–10] which was employed more recently, for example, in [5, 12, 34].

We thus assume (2.1a) and the existence of two functions

$$\Phi : [0, T] \times \overline{\Omega}_0 \to \mathbb{R}^N, \quad \Psi : \overline{\mathcal{O}} \to \overline{\Omega}_0,$$

satisfying

$$\Phi(t, \Omega_0) = \Omega_t,\ \Psi(t, \Omega_t) = \Omega_0 \quad \text{for all } t \in [0, T], \tag{3.1a}$$

$$\Phi(t, \Psi(t, x)) = x \quad \text{for all } (t, x) \in \overline{\mathcal{O}}, \tag{3.1b}$$

$$\Psi(t, \Phi(t, y)) = y \quad \text{for all } (t, y) \in [0, T] \times \overline{\Omega}_0, \tag{3.1c}$$

$$\Phi(0, y) = y \quad \text{for all } y \in \overline{\Omega}_0. \tag{3.1d}$$

We also assume that they fulfill the following assumptions:

(H1) Φ, Ψ are of class $C^{1,1}$ on their domains of definition;

(H2) $|\dot{\Phi}(t,y)| < 1$ for every $(t,y) \in [0,T] \times \overline{\Omega}_0$.

Condition (H2) ensures that the growth speed of the sets Ω_t is always strictly less than the speed of the traveling waves of problem (2.3); it is crucial in order to guarantee that the transformed problem (see (3.5)) is still hyperbolic (see (3.8)).

Remark 3.1. We notice that the existence of such diffeomorphisms automatically implies that the set $\mathcal{O}$ introduced in (2.2) is open and with Lipschitz boundary; furthermore, (2.9) is valid. See also Lemma 5.1 and Corollary 5.2.

Remark 3.2. In the nonhomogeneous case depicted in Remark 2.1, the wave speed is no longer always equal to one. In this situation condition (H2) can be rewritten as

(H2A) $|\dot{\Phi}(t,y)| < \sqrt{c_A}$ for every $(t,y) \in [0,T] \times \overline{\Omega}_0$,

where $c_A > 0$ is the positive constant appearing in (2.7).

In Section 5, when we study higher regularity of solutions to problem (2.3), we will additionally require:

(H1') Φ, Ψ are of class $C^{2,1}$ on their domains of definition.

In the following lemma we summarize some properties of the diffeomorphisms Φ and Ψ needed in Theorem 3.9 below:

Lemma 3.3. *Let Φ, Ψ be as in (3.1) and satisfy (H1). Then, for almost every $(t,y) \in [0,T] \times \overline{\Omega}_0$ the following relations hold:*

- $$D\Psi(t,\Phi(t,y))D\Phi(t,y) = I, \tag{3.2a}$$
- $$\det D\Psi(t,\Phi(t,y)) \det D\Phi(t,y) = 1, \tag{3.2b}$$
- $$\dot{\Psi}(t,\Phi(t,y)) = -D\Psi(t,\Phi(t,y))\dot{\Phi}(t,y), \tag{3.2c}$$
- $$\begin{aligned}&\nabla[\det D\Psi(t,\cdot)](\Phi(t,y)) \det D\Phi(t,y)\\ &\quad = -\det D\Psi(t,\Phi(t,y)) D\Psi(t,\Phi(t,y))^T \nabla \det D\Phi(t,y),\end{aligned} \tag{3.2d}$$
- $$\begin{aligned}&\big(\partial_t[\det D\Psi(\cdot,\Phi(t,y))](t) + \nabla[\det D\Psi(t,\cdot)](\Phi(t,y)) \cdot \dot{\Phi}(t,y)\big) \det D\Phi(t,y)\\ &\quad = -\det D\Psi(t,\Phi(t,y))\, \partial_t \det D\Phi(t,y),\end{aligned} \tag{3.2e}$$
- $$\begin{aligned}&\nabla[\det D\Psi(t,\cdot)](\Phi(t,y)) \cdot \dot{\Phi}(t,y) \det D\Phi(t,y)\\ &\quad = \dot{\Psi}(t,\Phi(t,y)) \cdot \nabla \det D\Phi(t,y) \det D\Psi(t,\Phi(t,y)),\end{aligned} \tag{3.2f}$$
- $$\partial_t \det D\Phi(t,y) + \operatorname{div}\big(\dot{\Psi}(t,\Phi(t,y)) \det D\Phi(t,y)\big) = 0. \tag{3.2g}$$

In particular, we notice that

$$\det D\Phi(t,y) > 0 \quad \textit{for every } (t,y) \in [0,T] \times \overline{\Omega}_0. \tag{3.3}$$

Proof. Relations (3.2a) and (3.2c) simply follow by differentiating (3.1c) with respect to y and t, respectively. Then, (3.2a) easily implies (3.2b) and (3.3) by (3.1d). Moreover, differentiating identity (3.2b) with respect to y and using (3.2a), one obtains (3.2d). Similarly, differentiating identity (3.2b) with respect to t, one gets (3.2e). Multiplying both sides of (3.2d) by $\dot{\Phi}(t, y)$ and using (3.2c), one also deduces (3.2f). Finally, we prove (3.2g): by (3.1b), for a.e. $t \in [0, T]$ and $x \in \overline{\Omega}_t$ it holds that

$$\mathrm{tr}\Big[\frac{\mathrm{d}}{\mathrm{d}t}(D\Phi(t, \Psi(t, x)))\Big] = 0.$$

The above identity can be written in components as

$$\sum_{i,j=1}^{N} \partial_j \dot{\Phi}_i(t, \Psi(t, x))\partial_i \Psi_j(t, x) + \sum_{i,j=1}^{N} \partial_j \Phi_i(t, \Psi(t, x))\partial_i \dot{\Psi}_j(t, x) \\ + \sum_{i,j,k=1}^{N} \partial_k \partial_j \Phi_i(t, \Psi(t, x))\dot{\Psi}_k(t, x)\partial_i \Psi_j(t, \Psi(t, x)) = 0.$$

Setting $x = \Phi(t, y)$, we now get

$$\sum_{i,j=1}^{N} \partial_j \dot{\Phi}_i(t, y)\partial_i \Psi_j(t, \Phi(t, y)) + \sum_{i,j=1}^{N} \partial_j \Phi_i(t, y)\partial_i \dot{\Psi}_j(t, \Phi(t, y)) \\ + \sum_{i,j,k=1}^{N} \partial_k \partial_j \Phi_i(t, y)\dot{\Psi}_k(t, \Phi(t, y))\partial_i \Psi_j(t, y) = 0.$$

Finally, we multiply the previous equality by $\det D\Phi(t, y)$ and apply the Jacobi identity

$$\partial_t \det M(t) = \det M(t)\, \mathrm{tr}\big[M(t)^{-1}\partial_t M(t)\big],$$

with $M(t) = D\Phi(t, y)$. Thus, we deduce (3.2g). ∎

Given a weak solution u of problem (2.3), we now consider the auxiliary function

$$v(t, y) := u(t, \Phi(t, y)) \quad \text{for all } (t, x) \in [0, T] \times \overline{\Omega}_0. \tag{3.4a}$$

Equivalently,

$$u(t, x) = v(t, \Psi(t, x)) \quad \text{for all } (t, x) \in \overline{\mathcal{O}}. \tag{3.4b}$$

This change of variables yields the following problem with fixed domain:

$$\begin{cases} \ddot{v} - \mathrm{div}(B\nabla v) + a \cdot \nabla v - 2b \cdot \nabla \dot{v} = g & \text{in } (0, T) \times \Omega_0, \\ v = 0 & \text{in } (0, T) \times \partial\Omega_0, \\ v(0) = v_0, \\ \dot{v}(0) = v_1, \end{cases} \tag{3.5}$$

whose coefficients are given by

$$B(t,y) := D\Psi(t,\Phi(t,y))D\Psi(t,\Phi(t,y))^T - \dot{\Psi}(t,\Phi(t,y)) \otimes \dot{\Psi}(t,\Phi(t,y)), \tag{3.6a}$$

$$a(t,y) := -\big\{B(t,y)^T \nabla \det D\Phi(t,y) + \partial_t\big[b(t,y) \det D\Phi(t,y)\big]\big\} \det D\Psi(t,\Phi(t,y)), \tag{3.6b}$$

$$b(t,y) := -\dot{\Psi}(t,\Phi(t,y)); \tag{3.6c}$$

the forcing term is

$$g(t,y) := f(t,\Phi(t,y)); \tag{3.6d}$$

and the initial data are defined by

$$v_0 := u_0, \quad v_1 := u_1 + \dot{\Phi}(0,\cdot) \cdot \nabla u_0. \tag{3.6e}$$

Remark 3.4. If the equation under study is (2.5), the only change in the new coefficients is given by

$$B(t,y) = D\Psi(t,\Phi(t,y))A(t,\Phi(t,y))D\Psi(t,\Phi(t,y))^T - \dot{\Psi}(t,\Phi(t,y)) \otimes \dot{\Psi}(t,\Phi(t,y)).$$

We also refer to [12, (2.29)] for a comparison.

The following proposition shows the regularity of the new data:

Proposition 3.5. *Assume (2.1a) and (2.4) and let Φ, Ψ be as in (3.1) and satisfy (H1). Let relations (3.6) hold. Then,*

$$B \in C^{0,1}([0,T] \times \overline{\Omega}_0; \mathbb{R}^{N\times N}_{\mathrm{sym}}), \tag{3.7a}$$

$$a \in L^\infty([0,T] \times \overline{\Omega}_0; \mathbb{R}^N), \tag{3.7b}$$

$$b \in C^{0,1}([0,T] \times \overline{\Omega}_0; \mathbb{R}^N), \tag{3.7c}$$

$$g \in L^2((0,T) \times \Omega_0), \tag{3.7d}$$

$$v_0 \in H^1_0(\Omega_0), \quad v_1 \in L^2(\Omega_0). \tag{3.7e}$$

Moreover, if (H2) is also satisfied, then B is uniformly elliptic, that is, there exists a positive constant $c_B > 0$ such that for every $(t,y) \in [0,T] \times \overline{\Omega}_0$ one has

$$(B(t,y)w) \cdot w \geq c_B |w|^2 \quad \textit{for all } w \in \mathbb{R}^N. \tag{3.8}$$

If in addition $f \in H^1(\mathcal{O})$ and (H1') is fulfilled, then it holds that

$$B \in C^{1,1}([0,T] \times \overline{\Omega}_0; \mathbb{R}^{N\times N}_{\mathrm{sym}}), \tag{3.9a}$$

$$a \in C^{0,1}([0,T] \times \overline{\Omega}_0; \mathbb{R}^N), \tag{3.9b}$$

$$b \in C^{1,1}([0,T] \times \overline{\Omega}_0; \mathbb{R}^N), \tag{3.9c}$$

$$g \in H^1(0,T; L^2(\Omega_0)). \tag{3.9d}$$

Proof. Regularity properties (3.7) and (3.9) directly follow from the explicit expressions in (3.6) together with (H1) and (H1'), respectively. For ellipticity property (3.8) we refer to [23, Lemma B.3]. ∎

To deal with problem (3.5) we introduce two equivalent notions of solution (see Proposition (3.8)), whose terminologies are consistent with the one introduced in [13]. Definition 3.6 is the analogue of Definition 2.2 in the current context, while Definition 3.7 does not involve integration by parts in time (as is classical in the analysis of hyperbolic problems; see, for instance, the textbooks [15, 27]). The first notion is useful to show the equivalence between problem (2.3) in a moving domain and problem (3.5) in a fixed domain. The second notion is more suited to the Galerkin method and will be used in Section 4 to obtain higher regularity.

Definition 3.6. We say that $v : [0,T] \times \overline{\Omega}_0 \to \mathbb{R}$ is a weak solution of problem (3.5) with data (3.7) if

(i) $v \in L^2(0,T;H_0^1(\Omega_0))$ and $\dot{v} \in L^2(0,T;L^2(\Omega_0))$;

(ii) $v(0) = v_0$ in the sense of $C^0([0,T];L^2(\Omega_0))$ and $\dot{v}(0) = v_1$ in the sense of $C^0([0,T];H^{-1}(\Omega_0))$;

(iii) v satisfies

$$\begin{aligned} -\int_0^T \langle \dot{v}(t), \dot{\xi}(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t + \int_0^T \langle B(t)\nabla v(t), \nabla\xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ + \int_0^T \langle a(t)\cdot\nabla v(t), \xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t + 2\int_0^T \langle \dot{v}(t), \operatorname{div}(b(t)\xi(t))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ = \int_0^T \langle g(t), \xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t, \end{aligned} \tag{3.10}$$

for every $\xi \in L^2(0,T;H_0^1(\Omega_0)) \cap H_0^1(0,T;L^2(\Omega_0))$.

Definition 3.7. We say that $v : [0,T] \times \overline{\Omega}_0 \to \mathbb{R}$ is a strong-weak solution of problem (3.5) with data (3.7) if

(i) $v \in L^2(0,T;H_0^1(\Omega_0))$, $\dot{v} \in L^2(0,T;L^2(\Omega_0))$, and $\ddot{v} \in L^2(0,T;H^{-1}(\Omega_0))$;

(ii) $v(0) = v_0$ in the sense of $C^0([0,T];L^2(\Omega_0))$ and $\dot{v}(0) = v_1$ in the sense of $C^0([0,T];H^{-1}(\Omega_0))$;

(iii) v satisfies

$$\begin{aligned} \langle \ddot{v}(t), \phi\rangle_{H_0^1(\Omega_0)} + \langle B(t)\nabla v(t), \nabla\phi\rangle_{L^2(\Omega_0)} + \langle a(t)\cdot\nabla v(t), \phi\rangle_{L^2(\Omega_0)} \\ + 2\langle \dot{v}(t), \operatorname{div}(b(t)\phi)\rangle_{L^2(\Omega_0)} = \langle g(t), \phi\rangle_{L^2(\Omega_0)}, \end{aligned} \tag{3.11}$$

for a.e. $t \in [0,T]$ and for every $\phi \in H_0^1(\Omega_0)$.

The next proposition shows the equivalence of the two notions of solution just introduced.

Proposition 3.8. *Assume* (3.7)*. Then, a function* v *is a weak solution of problem* (3.5) *in the sense of Definition* 3.6 *if and only if it is a strong-weak solution in the sense of Definition* 3.7*.*

Proof. First, assume that v is a strong-weak solution and fix a function

$$\xi \in L^2(0,T;H_0^1(\Omega_0)) \cap H_0^1(0,T;L^2(\Omega_0)).$$

Then, $\xi(t) \in H_0^1(\Omega_0)$ for almost every $t \in [0,T]$ and (3.11) holds with $\phi = \xi(t)$. Integrating by parts in time, we obtain that

$$\begin{aligned}\int_0^T \langle g(t), \xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t &= -\int_0^T \langle \dot v(t), \dot\xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t + \int_0^T \langle B(t)\nabla v(t), \nabla\xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ &+ \int_0^T \langle a(t)\cdot\nabla v(t), \xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t + 2\int_0^T \langle \dot v(t), \operatorname{div}(b(t)\xi(t))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t,\end{aligned}$$

and so we conclude that v is a weak solution.

We now prove the reverse implication. Let v be a weak solution. We first prove that $\ddot v$ belongs to $L^2(0,T;H^{-1}(\Omega_0))$. Since $\dot v \in L^2(0,T;L^2(\Omega_0))$, a priori we know that $\ddot v \in H^{-1}(0,T;L^2(\Omega_0))$ as a distributional derivative. By definition, it acts in the following way:

$$\langle \ddot v, \xi\rangle_{H_0^1(0,T;L^2(\Omega_0))} = -\int_0^T \langle \dot v(t), \dot\xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \quad \text{for all } \xi \in H_0^1(0,T;L^2(\Omega_0)).$$

We now fix $\xi \in L^2(0,T;H_0^1(\Omega_0)) \cap H_0^1(0,T;L^2(\Omega_0))$. Hence, (3.10) states that

$$\begin{aligned}&-\int_0^T \langle \dot v(t), \dot\xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ &\quad= -\int_0^T \langle B(t)\nabla v(t), \nabla\xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t - \int_0^T \langle a(t)\cdot\nabla v(t), \xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ &\qquad -2\int_0^T \langle \dot v(t), \operatorname{div}(b(t)\xi(t))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t + \int_0^T \langle g(t), \xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t.\end{aligned} \tag{3.12}$$

Due to (3.7), by developing the divergence term

$$\operatorname{div}(b(t)\xi(t)) = \operatorname{div}(b(t))\,\xi(t) + b(t)\cdot\nabla\xi(t),$$

we conclude that the following inequalities hold:

$$\begin{aligned}&\left|\int_0^T \langle B(t)\nabla v(t), \nabla\xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t\right| \\ &\qquad\le \|B\|_{L^\infty((0,T)\times\Omega_0)}\|\nabla v\|_{L^2(0,T;L^2(\Omega_0))}\|\nabla\xi\|_{L^2(0,T;L^2(\Omega_0))},\end{aligned} \tag{3.13a}$$

$$\left|\int_0^T \langle a(t)\cdot\nabla v(t), \xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t\right|$$
$$\le \|a\|_{L^\infty((0,T)\times\Omega_0)}\|\nabla v\|_{L^2(0,T;L^2(\Omega_0))}\|\xi\|_{L^2(0,T;L^2(\Omega_0))}, \tag{3.13b}$$
$$\left|\int_0^T \langle \dot v(t), \operatorname{div}(b(t)\xi(t))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t\right| \le \big(\|\operatorname{div}(b)\|_{L^\infty((0,T)\times\Omega_0)} + \|b\|_{L^\infty((0,T)\times\Omega_0)}\big)$$
$$\times \|\dot v\|_{L^2(0,T;L^2(\Omega_0))}\|\xi\|_{L^2(0,T;H^1_0(\Omega_0))}, \tag{3.13c}$$
$$\left|\int_0^T \langle g(t), \xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t\right| \le \|g\|_{L^2(0,T;L^2(\Omega_0))}\|\xi\|_{L^2(0,T;L^2(\Omega_0))}. \tag{3.13d}$$

Hence, there exists a constant $C > 0$ such that

$$|\langle \ddot v, \xi\rangle_{H^1_0(0,T;L^2(\Omega_0))}| \le C\|\xi\|_{L^2(0,T;H^1_0(\Omega_0))}.$$

By the density of $L^2(0,T;H^1_0(\Omega_0)) \cap H^1_0(0,T;L^2(\Omega_0))$ in $L^2(0,T;H^1_0(\Omega_0))$, we conclude that $\ddot v$ is in $L^2(0,T;H^{-1}(\Omega_0))$.

Employing an integration by parts in time in (3.12), which now is allowed, we then deduce that

$$\int_0^T \langle g(t), \xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t = \int_0^T \langle \ddot v(t), \xi(t)\rangle_{H^1_0(\Omega_0)}\,\mathrm{d}t + \int_0^T \langle B(t)\nabla v(t), \nabla\xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t$$
$$+ \int_0^T \langle a(t)\cdot\nabla v(t), \xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t$$
$$+ 2\int_0^T \langle \dot v(t), \operatorname{div}(b(t)\xi(t))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t, \tag{3.14}$$

for every $\xi \in L^2(0,T;H^1_0(\Omega_0))$. Now let $\{\phi_n\}_{n\in\mathbb{N}} \subseteq H^1_0(\Omega_0)$ be a countable dense subset of $H^1_0(\Omega_0)$ and consider, for $n \in \mathbb{N}$ and $t \in [0,T]$, the following properties:

$$\lim_{h\to 0^+} \int_t^{t+h} \langle \ddot v(s), \phi_n\rangle_{H^1_0(\Omega_0)}\,\mathrm{d}s = \langle \ddot v(t), \phi_n\rangle_{H^1_0(\Omega_0)}, \tag{3.15a}$$
$$\lim_{h\to 0^+} \int_t^{t+h} \langle B(s)\nabla v(s), \nabla\phi_n\rangle_{L^2(\Omega_0)}\,\mathrm{d}s = \langle B(t)\nabla v(t), \nabla\phi_n\rangle_{L^2(\Omega_0)}, \tag{3.15b}$$
$$\lim_{h\to 0^+} \int_t^{t+h} \langle a(s)\cdot\nabla v(s), \phi_n\rangle_{L^2(\Omega_0)}\,\mathrm{d}s = \langle a(t)\cdot\nabla v(t), \phi_n\rangle_{L^2(\Omega_0)}, \tag{3.15c}$$
$$\lim_{h\to 0^+} \int_t^{t+h} \langle \dot v(s), \operatorname{div}(b(s)\phi_n)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s = \langle \dot v(t), \operatorname{div}(b(t)\phi_n)\rangle_{L^2(\Omega_0)}, \tag{3.15d}$$
$$\lim_{h\to 0^+} \int_t^{t+h} \langle g(s), \phi_n\rangle_{L^2(\Omega_0)}\,\mathrm{d}s = \langle g(t), \phi_n\rangle_{L^2(\Omega_0)}. \tag{3.15e}$$

For every $n \in \mathbb{N}$, we now define the following set:

$$A_n := \big\{t \in [0,T] : \text{relations (3.15) hold}\big\}.$$

By the regularity of v and of the data, we have that A_n has full measure; hence, the set

$$Z := \bigcup_{n\in\mathbb{N}} Z_n \quad \text{with} \quad Z_n := [0,T]\setminus A_n$$

has null measure. By considering the functions

$$\psi_n^h(s) := \frac{1}{h}\,\phi_n\,\chi_{[t,t+h]}(s) \quad \text{for all } h>0,\ n\in\mathbb{N},\ t\in[0,T]\setminus Z$$

and testing (3.14) by $\xi = \psi_n^h \in L^2(0,T;H_0^1(\Omega_0))$, we thus obtain

$$\begin{aligned}
&\frac{1}{h}\int_t^{t+h}\langle \ddot v(s),\phi_n\rangle_{H_0^1(\Omega_0)}\,\mathrm{d}s + \frac{1}{h}\int_t^{t+h}\langle B(s)\nabla v(s),\nabla\phi_n\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\\
&\qquad + \frac{1}{h}\int_t^{t+h}\langle a(s)\cdot\nabla v(s),\phi_n\rangle_{L^2(\Omega_0)}\,\mathrm{d}s + \frac{2}{h}\int_t^{t+h}\langle \dot v(s),\operatorname{div}(b(s)\phi_n)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\\
&= \frac{1}{h}\int_t^{t+h}\langle g(s),\phi_n\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \quad \text{for all } h>0,\ n\in\mathbb{N},\ t\in[0,T]\setminus Z.
\end{aligned}$$

Letting $h\to 0^+$, for $t\in[0,T]\setminus Z$ we now get

$$\begin{aligned}
&\langle \ddot v(t),\phi_n\rangle_{H_0^1(\Omega_0)} + \langle B(t)\nabla v(t),\nabla\phi_n\rangle_{L^2(\Omega_0)} + \langle a(t)\cdot\nabla v(t),\phi_n\rangle_{L^2(\Omega_0)}\\
&\qquad + 2\langle \dot v(t),\operatorname{div}(b(t)\phi_n)\rangle_{L^2(\Omega_0)} = \langle g(t),\phi_n\rangle_{L^2(\Omega_0)},
\end{aligned}$$

for every $n\in\mathbb{N}$. By the density of $\{\phi_n\}_{n\in\mathbb{N}}$ in $H_0^1(\Omega_0)$, we finally conclude that v is a strong-weak solution. Hence, the proof is complete. ∎

The main result of the section is contained in the next theorem, which states that problems (2.3) and (3.5) are actually equivalent.

Theorem 3.9. *Assume* (2.1a) *and* (2.4) *and let* Φ,Ψ *be as in* (3.1) *and satisfy* (H1)*. Then,* u *is a weak solution of problem* (2.3) *in the sense of Definition* 2.2 *if and only if the corresponding function* v *defined as in* (3.4a) *is a weak solution of problem* (3.5) *with data* (3.6) *in the sense of Definition* 3.6.

Proof. Let u be a weak solution of problem (2.3) and let v be defined by the change of variables Φ as in (3.4a). Let

$$\xi \in L^2(0,T;H_0^1(\Omega_0))\cap H_0^1(0,T;L^2(\Omega_0)).$$

Then, consider the test function defined by

$$\eta(t,x) = \xi(t,\Psi(t,x))\det D\Psi(t,x) \quad \text{for a.e. } (t,x)\in\mathcal{O}.$$

Observe that by hypothesis (H1), clearly $\eta\in L^2(0,T;H_0^1(\Omega_t))$ with $\dot\eta\in L^2(0,T;L^2(\Omega_t))$ and $\eta(T)=\eta(0)=0$. By (3.4b), the following relations hold for a.e. $(t,x)\in\mathcal{O}$:

$$\begin{aligned}
\dot u(t,x) &= \dot v(t,\Psi(t,x)) + \nabla v(t,\Psi(t,x))\cdot\dot\Psi(t,x),\\
\nabla u(t,x) &= D\Psi(t,x)^T\,\nabla v(t,\Psi(t,x)).
\end{aligned}$$

Hence, by (2.8) we get

$$\underbrace{-\int_0^T \Big\langle \dot v(t,\Psi(t,\cdot)) + \nabla v(t,\Psi(t,\cdot))\cdot\dot\Psi(t,\cdot)),\ \frac{\mathrm d}{\mathrm dt}\Big[\xi(t,\Psi(t,\cdot))\det D\Psi(t,\cdot)\Big]\Big\rangle_{L^2(\Omega_t)}\,\mathrm dt}_{J_1}$$

$$+\underbrace{\int_0^T \Big\langle D\Psi(t,\cdot)^T\,\nabla v(t,\Psi(t,\cdot)),\ \nabla\Big[\xi(t,\Psi(t,\cdot))\det D\Psi(t,\cdot)\Big]\Big\rangle_{L^2(\Omega_t)}\,\mathrm dt}_{J_2}$$

$$=\int_0^T \langle f(t,\cdot),\xi(t,\Psi(t,\cdot))\det D\Psi(t,\cdot)\rangle_{L^2(\Omega_t)}\,\mathrm dt. \tag{3.16}$$

Then, by the change of variables $x=\Phi(t,y)$ and using identity (3.2b) of Lemma 3.3, we get

$$\int_0^T \langle f(t,\cdot),\xi(t,\Psi(t,\cdot))\det D\Psi(t,\cdot)\rangle_{L^2(\Omega_t)}\,\mathrm dt=\int_0^T \langle g(t,\cdot),\xi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm dt. \tag{3.17}$$

In order to conclude the proof, we shall prove that J_1+J_2 coincides with the left-hand side of (3.10). We start by considering the term J_2. Expanding the term

$$\nabla\big[\xi(t,\Psi(t,\cdot))\det D\Psi(t,\cdot)\big]$$

yields

$$\begin{aligned}&\nabla\big[\xi(t,\Psi(t,\cdot))\det D\Psi(t,\cdot)\big]\\&\quad= D\Psi(t,\cdot)^T\nabla\xi(t,\Psi(t,\cdot))\det D\Psi(t,\cdot)+\xi(t,\Psi(t,\cdot))\,\nabla\det D\Psi(t,\cdot).\end{aligned}$$

Performing again the change of variables $x=\Phi(t,y)$, by (3.2b), we obtain

$$\begin{aligned}J_2&=\int_0^T\langle D\Psi(t,\Phi(t,\cdot))^T\nabla v(t,\cdot),D\Psi(t,\Phi(t,\cdot))^T\nabla\xi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm dt\\&\quad+\int_0^T\langle D\Psi(t,\Phi(t,\cdot))^T\nabla v(t,\cdot),\xi(t,\cdot)\,\nabla\det D\Psi(t,\Phi(t,\cdot))\det D\Phi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm dt.\end{aligned}$$

Then, by (3.2d) in Lemma 3.3, we deduce that

$$\begin{aligned}J_2&=\int_0^T\langle D\Psi(t,\Phi(t,\cdot))D\Psi(t,\Phi(t,\cdot))^T\nabla v(t,\cdot),\nabla\xi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm dt\\&\quad-\int_0^T\langle D\Psi(t,\Phi(t,\cdot))D\Psi(t,\Phi(t,\cdot))^T\nabla v(t,\cdot),\\&\qquad\qquad\xi(t,\cdot)\,\nabla\det D\Phi(t,\cdot)\det D\Psi(t,\Phi(t,\cdot))\rangle_{L^2(\Omega_0)}\,\mathrm dt.\end{aligned}$$

We now consider the term J_1. Expanding the term $\frac{\mathrm d}{\mathrm dt}[\xi(t,\Psi(t,\cdot))\det D\Psi(t,\cdot)]$ yields

$$\begin{aligned}\frac{\mathrm d}{\mathrm dt}\big[\xi(t,\Psi(t,\cdot))\det D\Psi(t,\cdot)\big]&=\dot\xi(t,\Psi(t,\cdot))\det D\Psi(t,\cdot)\\&\quad+\nabla\xi(t,\Psi(t,\cdot))\cdot\dot\Psi(t,\cdot)\det D\Psi(t,\cdot)+\xi(t,\Psi(t,\cdot))\,\partial_t\det D\Psi(t,\cdot).\end{aligned}$$

Arguing as before, we obtain

$$\begin{aligned} J_1 = &- \int_0^T \langle \dot v(t,\cdot), \dot\xi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t - \int_0^T \langle \dot v(t,\cdot), \nabla\xi(t,\cdot)\cdot\dot\Psi(t,\Phi(t,\cdot))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ &- \int_0^T \langle \dot v(t,\cdot), \xi(t,\cdot)\,\partial_t[\det D\Psi(t,\cdot)](\Phi(t,\cdot))\det D\Phi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ &- \int_0^T \langle \nabla v(t,\cdot)\cdot\dot\Psi(t,\Phi(t,\cdot)), \dot\xi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ &- \int_0^T \langle \nabla v(t,\cdot)\cdot\dot\Psi(t,\Phi(t,\cdot)), \nabla\xi(t,\cdot)\cdot\dot\Psi(t,\Phi(t,\cdot))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ &- \int_0^T \langle \nabla v(t,\cdot)\cdot\dot\Psi(t,\Phi(t,\cdot)), \xi(t,\cdot)\,\partial_t[\det D\Psi(t,\cdot)](\Phi(t,\cdot))\det D\Phi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t. \end{aligned}$$

Then, by identity (3.2e) of Lemma 3.3, we deduce that

$$\begin{aligned} J_1 = &- \int_0^T \langle \dot v(t,\cdot), \dot\xi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t - \int_0^T \langle \dot v(t,\cdot), \nabla\xi(t,\cdot)\cdot\dot\Psi(t,\Phi(t,\cdot))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ &+ \int_0^T \langle \dot v(t,\cdot), \xi(t,\cdot)\,\partial_t \det D\Phi(t,\cdot)\det D\Psi(t,\Phi(t,\cdot))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ &+ \int_0^T \langle \dot v(t,\cdot), \xi(t,\cdot)\,\nabla[\det D\Psi(t,\cdot)](\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot)\det D\Phi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ &- \int_0^T \langle \nabla v(t,\cdot)\cdot\dot\Psi(t,\Phi(t,\cdot)), \dot\xi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ &- \int_0^T \langle \dot\Psi(t,\Phi(t,\cdot))\otimes\dot\Psi(t,\Phi(t,\cdot))\nabla v(t,\cdot), \nabla\xi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ &+ \int_0^T \langle \nabla v(t,\cdot)\cdot\dot\Psi(t,\Phi(t,\cdot)), \xi(t,\cdot)\,\partial_t \det D\Phi(t,\cdot)\det D\Psi(t,\Phi(t,\cdot))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ &+ \int_0^T \langle \nabla v(t,\cdot)\cdot\dot\Psi(t,\Phi(t,\cdot)), \\ &\qquad \xi(t,\cdot)\,\nabla[\det D\Psi(t,\cdot)](\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot)\det D\Phi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t. \end{aligned} \tag{3.18}$$

We now notice that, in light of relation (3.2f) of Lemma 3.3, we can rewrite the last summand of (3.18) as follows:

$$\begin{aligned} &\int_0^T \langle \nabla v(t,\cdot)\cdot\dot\Psi(t,\Phi(t,\cdot)), \xi(t,\cdot)\,\nabla[\det D\Psi(t,\cdot)](\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot)\det D\Phi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\ &\quad = \int_0^T \langle \dot\Psi(t,\Phi(t,\cdot))\otimes\dot\Psi(t,\Phi(t,\cdot))\nabla v(t,\cdot), \\ &\qquad\qquad \xi(t,\cdot)\,\nabla\det D\Phi(t,\cdot)\det D\Psi(t,\Phi(t,\cdot))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t. \end{aligned}$$

The fourth summand of (3.18) can instead be rewritten as

$$\begin{aligned}
&\int_0^T \langle \dot v(t,\cdot), \xi(t,\cdot)\, \nabla[\det D\Psi(t,\cdot)](\Phi(t,\cdot)) \cdot \dot\Phi(t,\cdot) \det D\Phi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\
&\qquad = \int_0^T \langle \dot v(t,\cdot), \xi(t,\cdot)\, \dot\Psi(t,\Phi(t,\cdot)) \cdot \nabla \det D\Phi(t,\cdot) \det D\Psi(t,\Phi(t,\cdot))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t.
\end{aligned}$$

By using (3.2g) in Lemma 3.3 (keeping in mind also (3.2b)), the sum of the third and the fourth summand in (3.18) gives

$$\begin{aligned}
&\int_0^T \langle \dot v(t,\cdot), \xi(t,\cdot)\, \partial_t \det D\Phi(t,\cdot) \det D\Psi(t,\Phi(t,\cdot))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\
&\qquad + \int_0^T \langle \dot v(t,\cdot), \xi(t,\cdot)\, \dot\Psi(t,\Phi(t,\cdot)) \cdot \nabla \det D\Phi(t,\cdot) \det D\Psi(t,\Phi(t,\cdot))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\
&= -\int_0^T \big\langle \dot v(t,\cdot), \xi(t,\cdot) \operatorname{div}\big(\dot\Psi(t,\Phi(t,\cdot))\big)\big\rangle_{L^2(\Omega_0)}\,\mathrm{d}t.
\end{aligned} \tag{3.19}$$

We finally consider the second and the fifth summand of (3.18). We split the term $\dot\Psi(t,\Phi(t,y))\dot\xi(t,y)$ into

$$\dot\Psi(t,\Phi(t,y))\dot\xi(t,y) = \frac{\mathrm{d}}{\mathrm{d}t}(\dot\Psi(t,\Phi(t,y))\xi(t,y)) - \Big(\frac{\mathrm{d}}{\mathrm{d}t}\dot\Psi(t,\Phi(t,y))\Big)\xi(t,y)$$

and rewrite the term $\xi(t,y)\operatorname{div}(\dot\Psi(t,\Phi(t,y))) + \nabla\xi(t,y)\cdot\dot\Psi(t,\Phi(t,y))$ (cf. the second summand of (3.18) and right-hand side of (3.19)) as

$$\xi(t,y)\operatorname{div}(\dot\Psi(t,\Phi(t,y))) + \nabla\xi(t,y)\cdot\dot\Psi(t,\Phi(t,y)) = \operatorname{div}(\dot\Psi(t,\Phi(t,y))\xi(t,y)).$$

Finally, integrating by parts in space and in time, we conclude that

$$\begin{aligned}
&-\int_0^T \langle \dot v(t,\cdot), \nabla\xi(t,\cdot)\cdot\dot\Psi(t,\Phi(t,\cdot))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\
&\qquad -\int_0^T \langle \dot v(t,\cdot), \xi(t,\cdot)\operatorname{div}(\dot\Psi(t,\Phi(t,\cdot)))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\
&\qquad -\int_0^T \langle \nabla v(t,\cdot)\cdot\dot\Psi(t,\Phi(t,\cdot)), \dot\xi(t,\cdot)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\
&= -2\int_0^T \big\langle \dot v(t,\cdot), \operatorname{div}\big(\dot\Psi(t,\Phi(t,\cdot))\xi(t,\cdot)\big)\big\rangle_{L^2(\Omega_0)}\,\mathrm{d}t \\
&\qquad + \int_0^T \Big\langle \nabla v(t,\cdot)\cdot\Big(\frac{\mathrm{d}}{\mathrm{d}t}\dot\Psi(t,\Phi(t,\cdot))\Big), \xi(t,\cdot)\Big\rangle_{L^2(\Omega_0)}\,\mathrm{d}t.
\end{aligned}$$

Hence, recalling expressions (3.6a), (3.6b) and (3.6c), we deduce that

$$J_1 + J_2 = -\int_0^T \langle \dot v(t), \dot\xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t + \int_0^T \langle B(t)\nabla v(t), \nabla\xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t$$

$$+\int_0^T \langle a(t)\cdot\nabla v(t),\xi(t)\rangle_{L^2(\Omega_0)}\,\mathrm{d}t+2\int_0^T \langle \dot v(t),\operatorname{div}(b(t)\xi(t))\rangle_{L^2(\Omega_0)}\,\mathrm{d}t,$$

which, due to (3.16) and (3.17), yields the conclusion.

Using the very same argument, one can prove the reverse implication, thus the proof is complete. ■

4. Nonautonomous hyperbolic equations

In this section we focus on hyperbolic problem (3.5), independently of its relation to the original problem given by the explicit formulas in (3.6). We employ the classical Galerkin method in order to obtain the higher regularity we are looking for. In Section 4.2 we show the first basic estimates, which also provide a way to prove existence of strong-weak solutions to the problem under consideration. In Section 4.3 we refine such estimates, strengthening the assumptions on the data and finally deducing more regularity for the solutions previously obtained.

We thus consider problem (3.5) with nonautonomous coefficients B, a, b, forcing term g and initial data v_0, v_1 satisfying (3.7) and (3.8). We tacitly assume throughout the whole section that

$$\text{the set } \Omega_0\subseteq\mathbb{R}^N \text{ is nonempty, open, bounded and with Lipschitz boundary.} \tag{4.1}$$

Moreover, the definitions of weak solutions and strong-weak solutions are the ones given in Definitions 3.6 and 3.7, respectively.

4.1. Galerkin approximation

The Galerkin method consists in projecting problem (3.5) onto finite-dimensional spaces, and in finding uniform estimates on the lower-dimensional problems which allow us to retrieve information on the infinite-dimensional one.

To this end, let $\{w_k\}_{k\in\mathbb{N}}\subseteq H^2(\Omega_0)\cap H^1_0(\Omega_0)$ be the set of eigenfunctions of $-\Delta$ in $H^1_0(\Omega_0)$ normalized in $L^2(\Omega_0)$. It is a standard fact that they form an orthogonal basis of $H^1_0(\Omega_0)$ and an orthonormal basis of $L^2(\Omega_0)$. Furthermore, by their very definition, for every $k\in\mathbb{N}$ they fulfill

$$\langle\phi,w_k\rangle_{L^2(\Omega_0)}=\frac{\langle\nabla\phi,\nabla w_k\rangle_{L^2(\Omega_0)}}{\|\nabla w_k\|^2_{L^2(\Omega_0)}}\quad\text{for all } \phi\in H^1_0(\Omega_0). \tag{4.2}$$

For every $m\in\mathbb{N}$, we seek functions $d_k^m\in H^2(0,T)$ such that the function defined by

$$v^m(t):=\sum_{k=1}^m d_k^m(t)w_k\in H^2(0,T;H^2(\Omega_0)\cap H^1_0(\Omega_0)) \tag{4.3}$$

satisfies for every $k = 1, \dots, m$ and for almost every $t \in [0, T]$ the finite-dimensional version of problem (3.5), namely,

$$\begin{aligned}\langle \ddot{v}^m(t), w_k\rangle_{H_0^1(\Omega_0)} + \langle B(t)\nabla v^m(t), \nabla w_k\rangle_{L^2(\Omega_0)} + \langle a(t)\cdot\nabla v^m(t), w_k\rangle_{L^2(\Omega_0)}\\ - 2\langle b(t)\cdot\nabla\dot{v}^m(t), w_k\rangle_{L^2(\Omega_0)} = \langle g(t), w_k\rangle_{L^2(\Omega_0)},\end{aligned} \tag{4.4}$$

with initial conditions

$$d_k^m(0) = \langle v_0, w_k\rangle_{L^2(\Omega_0)}, \tag{4.5a}$$

$$\dot{d}_k^m(0) = \langle v_1, w_k\rangle_{L^2(\Omega_0)}. \tag{4.5b}$$

Comparing (4.3) and (4.4), we obtain an auxiliary ordinary differential equation whose (time-dependent) coefficients are given by

$$\begin{aligned} B_{lk}(t) &:= \langle B(t)\nabla w_l, \nabla w_k\rangle_{L^2(\Omega_0)},\\ a_{lk}(t) &:= \langle a(t)\cdot\nabla w_l, w_k\rangle_{L^2(\Omega_0)},\end{aligned}$$

and

$$\begin{aligned} b_{lk}(t) &:= \langle b(t)\cdot\nabla w_l, w_k\rangle_{L^2(\Omega_0)}\\ g_k(t) &:= \langle g(t), w_k\rangle_{L^2(\Omega_0)}\end{aligned} \quad \text{for a.e. } t \in [0,T] \text{ and all } l, k \in \mathbb{N}.$$

Then, by classical results on linear ordinary differential equations, for every $m \in \mathbb{N}$ there exists a unique m-tuple of functions

$$d^m = (d_1^m, \dots, d_m^m) \in (H^2(0,T))^m$$

satisfying

$$\begin{cases} \ddot{d}_k^m(t) - 2\sum_{l=1}^m b_{lk}(t)\,\dot{d}_l^m(t)\\ \qquad + \sum_{l=1}^m (B_{lk}(t) + a_{lk}(t)) d_l^m(t) = g_k(t) \quad \text{for a.e. } t \in [0,T],\\ d_k^m(0) = \langle v_0, w_k\rangle_{L^2(\Omega_0)},\\ \dot{d}_k^m(0) = \langle v_1, w_k\rangle_{L^2(\Omega_0)},\end{cases} \tag{4.6}$$

so that the corresponding function v^m defined by (4.3) satisfies equation (4.4) for almost every $t \in [0, T]$.

4.2. First estimates and existence

The goal now is finding suitable uniform estimates on the functions v^m. We start with the following proposition:

Proposition 4.1. *Assume* (3.7) *and* (3.8)*. Then, there exists a constant* $C > 0$ *(independent of* $m \in \mathbb{N}$*) such that*

$$\sup_{0\le t\le T}\left(\|\dot{v}^m(t)\|^2_{L^2(\Omega_0)} + \|v^m(t)\|^2_{H_0^1(\Omega_0)}\right) \le C. \tag{4.7}$$

Moreover,

$$\ddot{v}^m \textit{ is uniformly bounded in } L^2(0,T;H^{-1}(\Omega_0)). \tag{4.8}$$

Proof. The proof is rather standard; see [15], for instance. We however present it in detail, since similar estimates will be employed in the (more involved) proof of Proposition 4.9.

Consider equation (4.4) satisfied by v^m, then multiply it by $\dot{d}_k^m(t)$ and sum from $k = 1,\dots,m$. Fixing $t \in [0,T]$ and integrating with respect to time over $(0,t)$, we thus obtain

$$\begin{aligned}
&\underbrace{\int_0^t \langle \ddot{v}^m(s), \dot{v}^m(s)\rangle_{H_0^1(\Omega_0)}\,\mathrm{d}s}_{J_1} + \underbrace{\int_0^t \langle B(s)\nabla v^m(s), \nabla \dot{v}^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s}_{J_2} \\
&\qquad + \underbrace{\int_0^t \langle a(s)\cdot\nabla v^m(s), \dot{v}^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s}_{J_3} - \underbrace{2\int_0^t \langle b(s)\cdot\nabla \dot{v}^m(s), \dot{v}^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s}_{J_4} \\
&\quad = \underbrace{\int_0^t \langle g(s), \dot{v}^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s}_{J_5}.
\end{aligned}$$

Let us consider each of the terms $J_1,\dots,J_5$ defined above. For the term J_1, we have

$$\int_0^t \langle \ddot{v}^m(s), \dot{v}^m(s)\rangle_{H_0^1(\Omega_0)}\,\mathrm{d}s = \frac{1}{2}\|\dot{v}^m(t)\|^2_{L^2(\Omega_0)} - \frac{1}{2}\|\dot{v}^m(0)\|^2_{L^2(\Omega_0)}.$$

We notice, moreover, that $\|\dot{v}^m(0)\|^2_{L^2(\Omega_0)} \le \|v_1\|^2_{L^2(\Omega_0)}$ by (4.5b).

As for the term J_2, by the symmetry of the matrix B and integration by parts in time, we have

$$\begin{aligned}
\int_0^t \langle B(s)\nabla v^m(s), \nabla \dot{v}^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s &= \frac{1}{2}\langle B(t)\nabla v^m(t), \nabla v^m(t)\rangle_{L^2(\Omega_0)} \\
&\quad - \frac{1}{2}\langle B(0)\nabla v^m(0), \nabla v^m(0)\rangle_{L^2(\Omega_0)} \\
&\quad - \frac{1}{2}\int_0^t \langle \dot{B}(s)\nabla v^m(s), \nabla v^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s.
\end{aligned}$$

Moreover, by (3.8) and (3.7a), we deduce that

$$\begin{aligned}
\frac{1}{2}\langle B(t)\nabla v^m(t), \nabla v^m(t)\rangle_{L^2(\Omega_0)} &\ge \frac{c_B}{2}\|\nabla v^m(t)\|^2_{L^2(\Omega_0)}, \\
\frac{1}{2}\big|\langle B(0)\nabla v^m(0), \nabla v^m(0)\rangle_{L^2(\Omega_0)}\big| &\le \frac{1}{2}\|B(0)\|_{L^\infty(\Omega_0)}\|\nabla v^m(0)\|^2_{L^2(\Omega_0)}, \\
\left|\frac{1}{2}\int_0^t \langle \dot{B}(s)\nabla v^m(s), \nabla v^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\right| &\le \frac{1}{2}\|\dot{B}\|_{L^\infty((0,T)\times\Omega_0)}\int_0^t \|\nabla v^m(s)\|^2_{L^2(\Omega_0)}\,\mathrm{d}s.
\end{aligned}$$

We notice, in particular, that $\|\nabla v^m(0)\|^2_{L^2(\Omega_0)} \le \|\nabla v_0\|^2_{L^2(\Omega_0)}$ by (4.2) and (4.5a).

As for J_3, by (3.7b), we have

$$\begin{aligned}&\Big|\int_0^t \langle a(s)\cdot \nabla v^m(s), \dot v^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\Big|\\&\qquad\le \|a\|_{L^\infty((0,T)\times\Omega)}\int_0^t \Big(\|\dot v^m(s)\|^2_{L^2(\Omega_0)} + \|\nabla v^m(s)\|^2_{L^2(\Omega_0)}\Big)\,\mathrm{d}s.\end{aligned}$$

Regarding J_4, we observe that for each $s \in (0,t)$ one has $\dot v^m(s) \in H^1_0(\Omega_0)$, and hence, integrating by parts in space and exploiting (3.7c), we get

$$\begin{aligned}2\Big|\int_0^t \langle b(s)\cdot \nabla \dot v^m(s), \dot v^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\Big| &= \Big|\int_0^t \langle b(s), \nabla|\dot v^m(s)|^2\rangle_{L^1(\Omega_0)}\,\mathrm{d}s\Big|\\&= \Big|-\int_0^t \langle \mathrm{div}(b(s)), |\dot v^m(s)|^2\rangle_{L^1(\Omega_0)}\,\mathrm{d}s\Big|\\&\le \|\mathrm{div}(b)\|_{L^\infty((0,T)\times\Omega_0)}\int_0^t \|\dot v^m(s)\|^2_{L^2(\Omega_0)}\,\mathrm{d}s.\end{aligned}$$

Finally, for the term J_5, by Young's inequality we readily deduce that

$$\Big|\int_0^t \langle g(s), \dot v^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\Big| \le \frac12\|g\|^2_{L^2(0,T;L^2(\Omega_0))} + \frac12\int_0^t \|\dot v^m(s)\|^2_{L^2(\Omega_0)}\,\mathrm{d}s.$$

By the estimates obtained for $J_1,\dots,J_5$, we deduce that there exist positive constants c_1 and c_2 such that for every $t \in (0,T)$, it holds that

$$\|\dot v^m(t)\|^2_{L^2(\Omega_0)} + \frac{c_B}{2}\|\nabla v^m(t)\|^2_{L^2(\Omega_0)} \le c_1 + c_2\int_0^t \Big(\|\dot v^m(s)\|^2_{L^2(\Omega_0)} + \|\nabla v^m(s)\|^2_{L^2(\Omega_0)}\Big)\,\mathrm{d}s.$$

By a classical Grönwall argument together with Poincaré inequality, we deduce the existence of a constant $C > 0$ such that (4.7) holds true.

In order to conclude the proof, it remains to prove (4.8). Fix $w \in H^1_0(\Omega_0)$ such that $\|w\|_{H^1_0(\Omega_0)} \le 1$. Now write $w = w^m_{(1)} + w^m_{(2)}$, where $w^m_{(1)} \in \mathrm{span}\{w_k\}_{k=1}^m$ and where $\langle w^m_{(2)}, w_k\rangle_{L^2(\Omega_0)} = 0$ for every $k \in \{1,\dots,m\}$. Due to (4.2), we have $\|w^m_{(1)}\|_{H^1_0(\Omega_0)} \le 1$. Then, by (4.3) and (4.4), we obtain

$$\begin{aligned}\langle \ddot v^m(t), w\rangle_{H^1_0(\Omega_0)} &= \langle \ddot v^m(t), w^m_{(1)}\rangle_{L^2(\Omega_0)}\\&= -\langle B(t)\nabla v^m(t), \nabla w^m_{(1)}\rangle_{L^2(\Omega_0)} - \langle a(t)\cdot\nabla v^m(t), w^m_{(1)}\rangle_{L^2(\Omega_0)}\\&\quad + 2\langle b(t)\cdot\nabla \dot v^m(t), w^m_{(1)}\rangle_{L^2(\Omega_0)} + \langle g(t), w^m_{(1)}\rangle_{L^2(\Omega_0)}\\&= -\langle B(t)\nabla v^m(t), \nabla w^m_{(1)}\rangle_{L^2(\Omega_0)} - \langle a(t)\cdot\nabla v^m(t), w^m_{(1)}\rangle_{L^2(\Omega_0)}\\&\quad - 2\langle \dot v^m(t), \mathrm{div}(b(t)w^m_{(1)})\rangle_{L^2(\Omega_0)} + \langle g(t), w^m_{(1)}\rangle_{L^2(\Omega_0)}.\end{aligned}$$

Then, using again (3.7), exploiting (4.7) and recalling that $\|w^m_{(1)}\|_{H^1_0(\Omega_0)} \le 1$, the previous equation yields

$$\int_0^T \|\ddot{v}^m(s)\|^2_{H^{-1}(\Omega_0)}\,\mathrm{d}s \le C,$$

for a suitable positive constant C. Here, we do not detail the estimates, as they are similar to (3.13) in Proposition 3.8. ∎

As a simple corollary we deduce the following result:

Theorem 4.2. *Assume* (3.7) *and* (3.8). *Then, there exists a strong-weak solution v for problem* (3.5) *in the sense of Definition* 3.7 *which fulfills*

$$\begin{aligned} &v \in C^0_{\mathrm{w}}([0,T]; H^1_0(\Omega_0)),\\ &\dot{v} \in C^0_{\mathrm{w}}([0,T]; L^2(\Omega_0)). \end{aligned} \tag{4.9}$$

Proof. By the uniform bounds obtained in Proposition 4.1, one deduces the existence of a subsequence (not relabeled) such that

$$v^m \rightharpoonup v \quad \text{weakly in } L^2(0,T;H^1_0(\Omega_0)) \cap H^1(0,T;L^2(\Omega_0)) \cap H^2(0,T;H^{-1}(\Omega_0)).$$

By integrating (4.4) with respect to time, letting $m \to +\infty$ and recalling that $\{w_k\}_{k\in\mathbb{N}}$ is a basis of $H^1_0(\Omega_0)$, it is easy to conclude that the limit function v is a strong-weak solution to problem (3.5). Regularity property (4.9) follows by classical embeddings, as in the proof of Theorem 2.8. ∎

By slightly strengthening the assumptions on the coefficients, a uniqueness result is also available. Notice that (4.10) below is implied by (3.9b) and (3.9c).

Proposition 4.3. *In addition to* (3.7) *and* (3.8)*, assume that*

$$a \in C^{0,1}([0,T] \times \overline{\Omega}_0; \mathbb{R}^N) \quad \textit{and} \quad \operatorname{div} b \in C^{0,1}([0,T]; L^\infty(\Omega_0)). \tag{4.10}$$

Then, the strong-weak solution to problem (3.5) *is unique.*

Proof. The proof is based on an argument introduced by Ladyzenskaya in [20]. For details we refer to [12, Theorem 3.10]. ∎

For the sake of completeness, we also present a result regarding the energy balance for weak solutions to problem (3.5), which can be proved by following [12, Proposition 3.11]. Unfortunately, this energy balance is not easily transferable to problem (2.3) with moving domains.

Proposition 4.4. *Assume* (3.7) *and* (3.8) *and let v be a weak solution of problem* (3.5) *satisfying* (4.9). *Then, for every $t \in [0,T]$ we have*

$$\begin{aligned} &\frac{1}{2}\|\dot{v}(t)\|^2_{L^2(\Omega_0)} + \frac{1}{2}\langle B(t)\nabla v(t), \nabla v(t)\rangle_{L^2(\Omega_0)}\\ &\qquad = \frac{1}{2}\|v_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\langle B(0)\nabla v_0, \nabla v_0\rangle_{L^2(\Omega_0)} + \mathcal{R}[v](t), \end{aligned} \tag{4.11}$$

where the remainder is given by

$$\mathcal{R}[v](t) = \int_0^t \frac{1}{2}\langle \dot{B}(s)\nabla v(s), \nabla v(s)\rangle_{L^2(\Omega_0)} - \langle a(s)\cdot\nabla v(s), \dot{v}(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s$$
$$+ \int_0^t (-\langle \operatorname{div} b(s), |\dot{v}(s)|^2\rangle_{L^1(\Omega_0)} + \langle g(s), \dot{v}(s)\rangle_{L^2(\Omega_0)})\,\mathrm{d}s.$$

Remark 4.5. The energy balance given in (4.11) allows us to slightly increase the regularity of the solution v obtained in Theorem 4.2, which actually fulfills

$$v \in C^0([0,T]; H_0^1(\Omega_0)),$$
$$\dot{v} \in C^0([0,T]; L^2(\Omega_0)).$$

This easily follows by the weak lower semicontinuity (with respect to t) of the left-hand side of (4.11) (we recall (3.7a) and (3.8)) combined with the continuity of the right-hand side, which indeed yields convergence of the norms.

4.3. Further estimates and higher regularity

We now improve previous estimates, assuming that the data of the problem satisfy the stronger assumptions (3.9). As a byproduct, we deduce more regularity for strong-weak solutions to problem (3.5). As for the initial data, we require:

$$v_0 \in H^2(\Omega_0)\cap H_0^1(\Omega_0) \quad \text{and} \quad v_1 \in H_0^1(\Omega_0). \tag{4.12}$$

We also need to assume that

$$\Omega_0 \text{ is convex or of class } C^2. \tag{4.13}$$

The latter assumption is used in Lemma 4.8 below.

Remark 4.6. In the next section we will apply our results to a family of moving domains Ω_t satisfying (2.1a) and (4.13). From this point of view, it may be surprising to notice that the convexity assumption on Ω_0 allows one to circumvent other regularity assumptions on the domains. In particular, one may consider a family Ω_t of merely Lipschitz, nonconvex domains, such that they can be mapped into a single convex domain through changes of variables as in (3.1), satisfying (H1) and (H2). Unfortunately, it is difficult to characterize the class of sets fulfilling such properties, however, this method gives in principle the possibility to deal with irregular domains in concrete cases.

Remark 4.7. Under these stronger assumptions, the functions d_k^m solving (4.6) are of class $H^3(0,T)$, so v^m belongs to $H^3(0,T; H^2(\Omega_0)\cap H_0^1(\Omega_0))$.

We start by stating a lemma on elliptic regularity which will be used in Proposition 4.9 below.

Lemma 4.8. *Assume* (4.13) *and let* $\widetilde{B} \in C^{1,1}(\overline{\Omega}_0; \mathbb{R}^{N\times N}_{\mathrm{sym}})$ *be elliptic with ellipticity constant* $c_{\widetilde{B}}$. *Then, there exists a positive constant* $\widetilde{D}$, *depending only on* $\Omega_0, c_{\widetilde{B}}$, *and* $\|\widetilde{B}\|_{C^{1,1}(\overline{\Omega}_0;\mathbb{R}^{N\times N}_{\mathrm{sym}})}$, *such that*

$$\widetilde{D}\|\mathrm{div}(\widetilde{B}\nabla z)\|_{L^2(\Omega_0)} \geq \|z\|_{H^2(\Omega_0)} \quad \textit{for all } z \in H^2(\Omega_0) \cap H^1_0(\Omega_0).$$

Proof. This is a classical result on elliptic regularity which can be proved through the difference quotient technique, as in, for example, [18]. In particular, the case of Ω_0 of class C^2 is contained in [18, Sect. 2.3–2.4]. Furthermore, if Ω_0 is convex *and* of class C^2, we stress that the constant $\widetilde{D}$ may be chosen in such a way that the dependence on Ω_0 only involves its diameter (this can be found in [18, Sect. 3.1]). The dependence just on the diameter allows one to extend the result to the case of Ω_0 merely convex without further regularity of the boundary, by a standard method of approximation by convex C^2-subdomains (see [18, Sect. 3.2]). ∎

With this tool at our disposal, we are now in a position to deduce higher uniform estimates for the functions v^m.

Proposition 4.9. *Assume* (3.8), (3.9), (4.13) *and* (4.12). *Then, there exists a constant* $D > 0$ *(independent of* $m \in \mathbb{N}$*) such that*

$$\sup_{0\leq t\leq T}\left(\|\ddot{v}^m(t)\|^2_{L^2(\Omega_0)} + \|\dot{v}^m(t)\|^2_{H^1_0(\Omega_0)} + \|v^m(t)\|^2_{H^2(\Omega_0)}\right) \leq D.$$

Proof. Defining $V^m := \dot{v}^m$ and recalling Remark 4.7, we know that

$$V^m \in H^2(0,T; H^2(\Omega_0) \cap H^1_0(\Omega_0)).$$

By differentiating (4.4) with respect to time, we obtain

$$\begin{aligned}
\langle \ddot{V}^m(t), w_k\rangle_{H^1_0(\Omega_0)} &+ \langle \dot{B}(t)\nabla v^m(t), \nabla w_k\rangle_{L^2(\Omega_0)} + \langle B(t)\nabla V^m(t), \nabla w_k\rangle_{L^2(\Omega_0)} \\
&+ \langle \dot{a}(t)\cdot\nabla v^m(t), w_k\rangle_{L^2(\Omega_0)} + \langle a(t)\cdot\nabla V^m(t), w_k\rangle_{L^2(\Omega_0)} \\
&- 2\langle \dot{b}(t)\cdot\nabla V^m(t), w_k\rangle_{L^2(\Omega_0)} - 2\langle b(t)\cdot\nabla \dot{V}^m(t), w_k\rangle_{L^2(\Omega_0)} \\
= \langle \dot{g}(t), w_k\rangle_{L^2(\Omega_0)}. &
\end{aligned} \tag{4.14}$$

Now multiply (4.14) by $\ddot{d}^m_k(t)$ and sum from $k = 1,\dots,m$. Fixing $t \in [0,T]$ and integrating with respect to time over $(0,t)$, we obtain

$$\begin{aligned}
&\underbrace{\int_0^t \langle \ddot{V}^m(s), \dot{V}^m(s)\rangle_{H^1_0(\Omega_0)}\,\mathrm{d}s}_{J_1} + \underbrace{\int_0^t \langle \dot{B}(s)\nabla v^m(s), \nabla \dot{V}^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s}_{J_2} \\
&\quad + \underbrace{\int_0^t \langle B(s)\nabla V^m(s), \nabla \dot{V}^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s}_{J_3} + \underbrace{\int_0^t \langle \dot{a}(s)\cdot\nabla v^m(s), \dot{V}^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s}_{J_4}
\end{aligned}$$

$$+\underbrace{\int_0^t \langle a(s)\cdot\nabla V^m(s),\dot V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s}_{J_5}-\underbrace{2\int_0^t \langle \dot b(s)\cdot\nabla V^m(s),\dot V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s}_{J_6}$$
$$\underbrace{-2\int_0^t \langle b(s)\cdot\nabla \dot V^m(s),\dot V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s}_{J_7}=\underbrace{\int_0^t \langle \dot g(s),\dot V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s}_{J_8}.$$

Next we estimate each of the terms in the previous identity. For J_1, the following holds:

$$\int_0^t \langle \ddot V^m(s),\dot V^m(s)\rangle_{H_0^1(\Omega_0)}\,\mathrm{d}s=\frac12\|\dot V^m(t)\|^2_{L^2(\Omega_0)}-\frac12\|\dot V^m(0)\|^2_{L^2(\Omega_0)}.$$

We claim that

$$\begin{aligned}\|\dot V^m(0)\|_{L^2(\Omega_0)}\le \|g(0)\|_{L^2(\Omega_0)}&+\|a(0)\|_{L^\infty(\Omega_0)}\|\nabla v_0\|_{L^2(\Omega_0)}\\&+2\|b(0)\|_{L^\infty(\Omega_0)}\|\nabla v_1\|_{L^2(\Omega_0)}+\|DB(0)\|_{L^\infty(\Omega_0)}\|\nabla v_0\|_{L^2(\Omega_0)}\\&+C\|B(0)\|_{L^\infty(\Omega_0)}\|v_0\|_{H^2(\Omega_0)}.\end{aligned}\tag{4.15}$$

We assume for the moment that the claim is true, and continue by estimating J_2: integrating by parts in time, we have

$$\begin{aligned}&\int_0^t \langle \dot B(s)\nabla v^m(s),\nabla\dot V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\\&\quad=\langle \dot B(t)\nabla v^m(t),\nabla V^m(t)\rangle_{L^2(\Omega_0)}-\langle \dot B(0)\nabla v^m(0),\nabla V^m(0)\rangle_{L^2(\Omega_0)}\\&\qquad-\int_0^t \langle \ddot B(s)\nabla v^m(s),\nabla V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s-\int_0^t \langle \dot B(s)\nabla V^m(s),\nabla V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s.\end{aligned}$$

Furthermore, by the uniform bound for ∇v^m provided in Proposition 4.1, by (3.9a) and by Young's weighted inequality, the estimates

$$\begin{aligned}&|\langle \dot B(t)\nabla v^m(t),\nabla V^m(t)\rangle_{L^2(\Omega_0)}|\\&\qquad\le\|\dot B\|_{L^\infty((0,T)\times\Omega_0)}\|\nabla v^m(t)\|_{L^2(\Omega_0)}\|\nabla V^m(t)\|_{L^2(\Omega_0)}\\&\qquad\le\frac C2\|\dot B\|_{L^\infty((0,T)\times\Omega_0)}\Big(\frac1\varepsilon+\varepsilon\|\nabla V^m(t)\|^2_{L^2(\Omega_0)}\Big)\quad\text{for all }\varepsilon>0,\end{aligned}\tag{4.16}$$

$$\begin{aligned}|\langle \dot B(0)\nabla v^m(0),\nabla V^m(0)\rangle_{L^2(\Omega_0)}|&\le\|\dot B(0)\|_{L^\infty(\Omega_0)}\|\nabla v^m(0)\|_{L^2(\Omega_0)}\|\nabla V^m(0)\|_{L^2(\Omega_0)}\\&\le\|\dot B(0)\|_{L^\infty(\Omega_0)}\|\nabla v_0\|_{L^2(\Omega_0)}\|\nabla v_1\|_{L^2(\Omega_0)},\end{aligned}$$

$$\begin{aligned}&\Big|\int_0^t \langle \ddot B(s)\nabla v^m(s),\nabla V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\Big|\\&\qquad\le\frac12\|\ddot B\|_{L^\infty((0,T)\times\Omega_0)}\Big(CT+\int_0^t\|\nabla V^m(s)\|^2_{L^2(\Omega_0)}\,\mathrm{d}s\Big),\end{aligned}$$

$$\Big|\int_0^t \langle \dot B(s)\nabla V^m(s),\nabla V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\Big|\le\|\dot B\|_{L^\infty((0,T)\times\Omega_0)}\int_0^t\|\nabla V^m(s)\|^2_{L^2(\Omega_0)}\,\mathrm{d}s.$$

hold. As for the term J_3, by the symmetry of the matrix B and integrating by parts in time, we have

$$\begin{aligned}\int_0^t &\langle B(s)\nabla V^m(s), \nabla \dot V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\\ &= \frac12\langle B(t)\nabla V^m(t), \nabla V^m(t)\rangle_{L^2(\Omega_0)} - \frac12\langle B(0)\nabla V^m(0), \nabla V^m(0)\rangle_{L^2(\Omega_0)}\\ &\quad - \frac12\int_0^t \langle \dot B(s)\nabla V^m(s), \nabla V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s.\end{aligned}$$

Moreover, by the ellipticity of B and again by (3.9a), we deduce that

$$\begin{aligned}\frac12\langle B(t)\nabla V^m(t), \nabla V^m(t)\rangle_{L^2(\Omega_0)} &\ge \frac{c_B}{2}\|\nabla V^m(t)\|^2_{L^2(\Omega_0)},\\ \frac12\big|\langle B(0)\nabla V^m(0), \nabla V^m(0)\rangle_{L^2(\Omega_0)}\big| &\le \frac12\|B(0)\|_{L^\infty(\Omega_0)}\|\nabla V^m(0)\|^2_{L^2(\Omega_0)}\\ &\le \|B(0)\|_{L^\infty(\Omega_0)}\|\nabla v_1\|^2_{L^2(\Omega_0)},\\ \left|\frac12\int_0^t \langle \dot B(s)\nabla V^m(s), \nabla V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\right| &\le \frac12\|\dot B\|_{L^\infty((0,T)\times\Omega_0)}\int_0^t \|\nabla V^m(s)\|^2_{L^2(\Omega_0)}\,\mathrm{d}s.\end{aligned}$$

We now focus on J_4 and J_5: by using (3.9b) and by the uniform bound for ∇v^m provided by Proposition 4.1, we obtain that

$$\begin{aligned}\Big|\int_0^t &\langle \dot a(s)\cdot\nabla v^m(s), \dot V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\Big|\\ &\le \frac12\|\dot a\|_{L^\infty((0,T)\times\Omega_0)}\Big(CT + \int_0^t \|\dot V^m(s)\|^2_{L^2(\Omega_0)}\,\mathrm{d}s\Big),\\ \Big|\int_0^t &\langle a(s)\cdot\nabla V^m(s), \dot V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\Big|\\ &\le \frac12\|a\|_{L^\infty([0,T]\times\Omega_0)}\int_0^t (\|\dot V^m(s)\|^2_{L^2(\Omega_0)} + \|\nabla V^m(s)\|^2_{L^2(\Omega_0)})\,\mathrm{d}s.\end{aligned}$$

As for the terms J_6 and J_7, due to (3.9c) and by observing that for each $s \in (0,t)$ one has $\dot V^m(s) \in H^1_0(\Omega_0)$, we obtain

$$\begin{aligned}\Big|\int_0^t &\langle \dot b(s)\cdot\nabla V^m(s), \dot V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\Big|\\ &\le \frac12\|\dot b\|_{L^\infty((0,T)\times\Omega_0)}\int_0^t \Big(\|\dot V^m(s)\|^2_{L^2(\Omega_0)} + \|\nabla V^m(s)\|^2_{L^2(\Omega_0)}\Big)\,\mathrm{d}s,\\ \Big|\int_0^t &\langle b(s)\cdot\nabla \dot V^m(s), \dot V^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\Big| = \Big|\int_0^t \langle b(s), \nabla|\dot V^m(s)|^2\rangle_{L^1(\Omega_0)}\,\mathrm{d}s\Big|\\ &= \Big|\int_0^t \langle \mathrm{div}(b(s)), |\dot V^m(s)|^2\rangle_{L^1(\Omega_0)}\,\mathrm{d}s\Big|\\ &\le \|\mathrm{div}(b)\|_{L^\infty((0,T)\times\Omega_0)}\int_0^t \|\dot V^m(s)\|^2_{L^2(\Omega_0)}\,\mathrm{d}s.\end{aligned}$$

Finally, for the term J_8, by the regularity of $\dot{g}$ and by Young's inequality, we readily deduce that

$$\Big|\int_0^t \langle \dot{g}(s), \dot{V}^m(s)\rangle_{L^2(\Omega_0)}\,\mathrm{d}s\Big| \le \frac{1}{2}\|\dot{g}\|^2_{L^2(0,T;L^2(\Omega_0))} + \frac{1}{2}\int_0^t \|\dot{V}^m(s)\|^2_{L^2(\Omega_0)}\,\mathrm{d}s.$$

By the estimates obtained for $J_1,\dots,J_8$, by (4.15) and by a suitable choice of ε in (4.16), we deduce that there exist positive constants c_1, c_2 and c such that

$$\begin{aligned}&\|\dot{V}^m(t)\|^2_{L^2(\Omega_0)} + c\|\nabla V^m(t)\|^2_{L^2(\Omega_0)}\\ &\qquad \le c_1 + c_2\int_0^t \Big(\|\dot{V}^m(s)\|^2_{L^2(\Omega_0)} + \|\nabla V^m(s)\|^2_{L^2(\Omega_0)}\Big)\,\mathrm{d}s.\end{aligned}$$

By using Grönwall's inequality together with Poincaré inequality, we thus deduce the existence of a constant $C > 0$ such that

$$\sup_{0\le t\le T}\Big(\|\dot{V}^m(t)\|^2_{L^2(\Omega_0)} + \|V^m(t)\|^2_{H^1_0(\Omega_0)}\Big) \le C,$$

which yields, by the definition of V^m,

$$\sup_{0\le t\le T}\Big(\|\ddot{v}^m(t)\|^2_{L^2(\Omega_0)} + \|\dot{v}^m(t)\|^2_{H^1_0(\Omega_0)}\Big) \le C. \tag{4.17}$$

In order to conclude the proof, we have to prove claim (4.15) and provide a bound on the H^2-norm of $v^m(t)$ uniformly with respect to t. Let us fix $t \in [0,T]$. We define by $g^m(t)$ the L^2 projection of the function $g(t)$ on the finite-dimensional space spanned by $w_1,\dots,w_m$. Then, by (4.4) and by using the fact that $\{w_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega_0)$ and an orthogonal basis of $H^1_0(\Omega_0)$, we deduce that

$$\begin{aligned}&-\langle \operatorname{div}(B(t)\nabla v^m(t)), \varphi\rangle_{L^2(\Omega_0)}\\ &\qquad = \langle g^m(t) - \ddot{v}^m(t) - a(t)\cdot\nabla v^m(t) + 2b(t)\cdot\nabla\dot{v}^m(t), \varphi\rangle_{L^2(\Omega_0)},\end{aligned}$$

for every $t \in [0,T]$ and $\varphi \in H^1_0(\Omega_0)$. In particular, for every $t \in [0,T]$ we have

$$-\operatorname{div}(B(t)\nabla v^m(t)) = g^m(t) - \ddot{v}^m(t) - a(t)\cdot\nabla v^m(t) + 2b(t)\cdot\nabla\dot{v}^m(t), \tag{4.18}$$

in the sense of distributions. By choosing $t = 0$ in (4.18) and recalling that $\dot{V}^m(0) = \ddot{v}^m(0)$, we can thus estimate

$$\begin{aligned}\|\dot{V}^m(0)\|_{L^2(\Omega_0)} &\le \|g^m(0)\|_{L^2(\Omega_0)} + \|a(0)\|_{L^\infty(\Omega_0)}\|\nabla v^m(0)\|_{L^2(\Omega_0)}\\ &\qquad + 2\|b(0)\|_{L^\infty(\Omega_0)}\|\nabla\dot{v}^m(0)\|_{L^2(\Omega_0)} + \|\operatorname{div}(B(0)\nabla v^m(0))\|_{L^2(\Omega_0)}\\ &\le \|g(0)\|_{L^2(\Omega_0)} + \|a(0)\|_{L^\infty(\Omega_0)}\|\nabla v_0\|_{L^2(\Omega_0)}\\ &\qquad + 2\|b(0)\|_{L^\infty(\Omega_0)}\|\nabla v_1\|_{L^2(\Omega_0)} + \|DB(0)\|_{L^\infty(\Omega_0)}\|\nabla v_0\|_{L^2(\Omega_0)}\\ &\qquad + \|B(0)\|_{L^\infty(\Omega_0)}\|v^m(0)\|_{H^2(\Omega_0)}.\end{aligned}$$

By arguing as in [15, Chapter 7.1, Theorem 5], one can deduce that $\|v^m(0)\|_{H^2(\Omega_0)} \leq C\|v_0\|_{H^2(\Omega_0)}$ for a suitable constant $C > 0$, and so claim (4.15) is proved.

We now observe that, by Lemma 4.8, there exists a positive constant $\widetilde{D}$ (depending on Ω_0, on the uniform constant of ellipticity of B, which is independent of time, and of $\|B\|_{C^{1,1}([0,T]\times\overline{\Omega}_0)}$) such that

$$\|v^m(t)\|_{H^2(\Omega_0)} \leq \widetilde{D}\|\mathrm{div}(B(t)\nabla v^m(t))\|_{L^2(\Omega_0)} \quad \text{for all } t \in [0,T]. \tag{4.19}$$

Then, by (4.18) and (4.19) and since $\|g^m(t)\|_{L^2(\Omega_0)} \leq \|g(t)\|_{L^2(\Omega_0)}$, we deduce that

$$\begin{aligned}
&\|v^m(t)\|_{H^2(\Omega_0)} \\
&\quad \leq \widetilde{D}(\|g^m(t)\|_{L^2(\Omega_0)} + \|\ddot{v}^m(t)\|_{L^2(\Omega_0)} + \|a(t)\cdot\nabla v^m(t)\|_{L^2(\Omega_0)} \\
&\qquad + \|2b(t)\cdot\nabla\dot{v}^m(t)\|_{L^2(\Omega_0)}) \\
&\quad \leq \widetilde{D}(\|g(t)\|_{L^2(\Omega_0)} + \|\ddot{v}^m(t)\|_{L^2(\Omega_0)} + \|a\|_{L^\infty((0,T)\times\Omega_0)}\|\nabla v^m(t)\|_{L^2(\Omega_0)}) \\
&\qquad + 2\widetilde{D}(\|b\|_{L^\infty((0,T)\times\Omega_0)}\|\nabla\dot{v}^m(t)\|_{L^2(\Omega_0)}).
\end{aligned}$$

We now use (4.17), (3.9b), (3.9c), (3.9d) and the uniform bounds provided by Proposition 4.1 to conclude that

$$\sup_{0\leq t\leq T}\|v^m(t)\|_{H^2(\Omega_0)} \leq C$$

for some constant $C > 0$, and so, the statement is proved. ∎

As a corollary, we now obtain the main result of the section:

Theorem 4.10. *Assume* (3.8), (3.9), (4.13) *and* (4.12). *Then, there exists a unique strong-weak solution v of problem* (3.5) *which satisfies:*

$$\begin{aligned}
&v \in L^\infty(0,T;H^2(\Omega_0)\cap H_0^1(\Omega_0)), \\
&\dot{v} \in L^\infty(0,T;H_0^1(\Omega_0)), \\
&\ddot{v} \in L^\infty(0,T;L^2(\Omega_0)).
\end{aligned} \tag{4.20}$$

Proof. The estimates provided by Proposition 4.9 allow us to conclude that there exists a function v satisfying (4.20) such that, up to a subsequence which we still denote by v^m, the following convergences hold:

$$\begin{aligned}
v^m &\rightharpoonup v \quad \text{weakly in } L^2(0,T;H^2(\Omega_0)\cap H_0^1(\Omega_0)), \\
\dot{v}^m &\rightharpoonup \dot{v} \quad \text{weakly in } L^2(0,T;H_0^1(\Omega_0)), \\
\ddot{v}^m &\rightharpoonup \ddot{v} \quad \text{weakly in } L^2(0,T;L^2(\Omega_0)).
\end{aligned}$$

The fact that v is a strong-weak solution follows by arguing as in Theorem 4.2. Uniqueness is instead ensured by Proposition 4.3. ∎

Remark 4.11. By standard results on interpolation of spaces, property (4.20) easily yields $v \in C^0([0,T]; H_0^1(\Omega_0))$ and $\dot{v} \in C^0([0,T]; L^2(\Omega_0))$. We point out that actually (4.20) may also be improved by replacing L^∞ with C^0 (as in Remark 4.5); this can be seen, for example, by exploiting the semigroup approach (see [10] or [7]). However, this (slightly) stronger regularity will not be needed for the purposes of the present paper.

5. Existence, uniqueness and energy balance

In this section we come back to the original problem given in (2.3), still employing the approach of diffeomorphisms. In this way we are able to improve Theorem 2.4, by getting existence and uniqueness of weak solutions and by showing a more precise version of the energy balance (see Theorem 5.4). To this end, we need a lemma, the details of which can be found in [29, Proposition 17.1].

Lemma 5.1. *Let $E \subseteq \mathbb{R}^M$ be an open set with Lipschitz boundary and let $\mathfrak{f} : \overline{E} \to \mathbb{R}^M$ be a diffeomorphism of class C^1 with inverse $\mathfrak{g} \in C^1$. Then, $\mathfrak{f}(E)$ is an open set with Lipschitz boundary, and $\partial(\mathfrak{f}(E)) = \mathfrak{f}(\partial E)$. Moreover,*

$$\nu_{\mathfrak{f}(E)}(z) = \frac{D\mathfrak{g}(z)^T \nu_E(\mathfrak{g}(z))}{|D\mathfrak{g}(z)^T \nu_E(\mathfrak{g}(z))|} \quad \text{for } \mathcal{H}^{M-1}\text{-a.e. } z \in \partial(\mathfrak{f}(E)), \tag{5.1}$$

and

$$\int_{\partial\mathfrak{f}(E)} \mathfrak{h}\, \nu_{\mathfrak{f}(E)} \, \mathrm{d}\mathcal{H}^{M-1} = \int_{\partial E} (\mathfrak{h} \circ \mathfrak{f}) |\det D\mathfrak{f}| (D\mathfrak{g} \circ \mathfrak{f})^T \nu_E \, \mathrm{d}\mathcal{H}^{M-1} \tag{5.2}$$

for all $\mathfrak{h} \in L^1(\mathfrak{f}(E))$.

Corollary 5.2. *Assume (2.1a) and let Φ, Ψ be as in (3.1) and satisfy (H1). Then, the set $\mathcal{O}$ introduced in (2.2) is open with Lipschitz boundary, and (2.9) is satisfied.*

Furthermore, one has

$$\begin{aligned} \nu_{\mathcal{O}}(t,x) &= (\nu^t_{\mathcal{O}}(t,x), \nu^x_{\mathcal{O}}(t,x)) \\ &= \frac{(-\omega(t,x), \nu_{\Omega_t}(x))}{\sqrt{1+\omega(t,x)^2}} \quad \text{for all } t \in [0,T] \text{ and } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t, \end{aligned} \tag{5.3}$$

where we introduced the scalar normal velocity

$$\omega(t,x) := \dot{\Phi}(t, \Psi(t,x)) \cdot \nu_{\Omega_t}(x) \quad \text{for all } t \in [0,T] \text{ and } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t. \tag{5.4}$$

Moreover, the following identity holds true for every $\mathfrak{h} \in L^1(\Gamma)$:

$$\int_\Gamma \mathfrak{h}\, \nu_{\mathcal{O}} \, \mathrm{d}\mathcal{H}^N = \int_0^T \int_{\partial\Omega_t} \mathfrak{h}(t,x) \begin{pmatrix} -\omega(t,x) \\ \nu_{\Omega_t}(x) \end{pmatrix} \mathrm{d}\mathcal{H}^{N-1}(x) \, \mathrm{d}t. \tag{5.5}$$

Remark 5.3. We point out that actually the scalar normal velocity ω does not depend on the choice of the diffeomorphisms Φ and Ψ, but it is intrinsically related to the set $\mathcal{O}$ (and so to the family $\{\Omega_t\}_{t\in[0,T]}$). Indeed, by (5.3) we have

$$\omega(t,x) = -\frac{\nu^t_{\mathcal{O}}(t,x)}{|\nu^x_{\mathcal{O}}(t,x)|} \quad \text{for all } t \in [0,T] \text{ and } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t. \tag{5.6}$$

Moreover, it is immediate to check that the following statements hold:

- if (H2) is in force, then $\|\omega\|_{L^\infty(\Gamma)} < 1$;
- if (2.1b) is in force, then $\omega(t,x) \geq 0$ for all $t \in [0,T]$ and $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega_t$.

Indeed, the former fact is a direct consequence of the explicit form given in (5.4), while the latter can be inferred by (5.6), since under (2.1b) one has $\nu^t_{\mathcal{O}}(t,x) \geq 0$.

Proof of Corollary 5.2. We introduce maps $\widetilde{\Phi} : [0,T]\times\overline{\Omega}_0 \to \overline{\mathcal{O}}$ and $\widetilde{\Psi} : \overline{\mathcal{O}} \to [0,T]\times\overline{\Omega}_0$ defined as

$$\widetilde{\Phi}(t,y) := (t, \Phi(t,y)) \quad \text{and} \quad \widetilde{\Psi}(t,x) := (t, \Psi(t,x)).$$

It is immediate to check that $\overline{\mathcal{O}} = \widetilde{\Phi}([0,T]\times\overline{\Omega}_0)$ and that, due to (H1), $\widetilde{\Phi}$ is a diffeomorphism of class $C^{1,1}$ with inverse $\widetilde{\Psi} \in C^{1,1}$. By Lemma 5.1, the statement regarding $\mathcal{O}$ is hence verified.

By using (5.1) with the function $\Phi(t,\cdot)$, we now observe that for all $t \in [0,T]$ it holds that

$$\nu_{\Omega_t}(x) = \frac{D\Psi(t,x)^T \nu_{\Omega_0}(\Psi(t,x))}{|D\Psi(t,x)^T \nu_{\Omega_0}(\Psi(t,x))|} \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t. \tag{5.7}$$

By using the same formula with $\widetilde{\Phi}$, we instead obtain

$$\nu_{\mathcal{O}}(t,x) = \frac{D_{(t,x)}\widetilde{\Psi}(t,x)^T \nu_{(0,T)\times\Omega_0}(t,\Psi(t,x))}{|D_{(t,x)}\widetilde{\Psi}(t,x)^T \nu_{(0,T)\times\Omega_0}(t,\Psi(t,x))|} \quad \text{for } \mathcal{H}^{N}\text{-a.e. } (t,x) \in \Gamma. \tag{5.8}$$

By recalling (3.2c), we now compute

$$\begin{aligned} D_{(t,x)}\widetilde{\Psi}(t,x)^T \nu_{(0,T)\times\Omega_0}(t,\Psi(t,x)) &= \left[\begin{array}{c|c} 1 & \dot{\Psi}(t,x) \\ \hline 0 & D\Psi(t,x)^T \end{array}\right] \begin{pmatrix} 0 \\ \nu_{\Omega_0}(\Psi(t,x)) \end{pmatrix} \\ &= \begin{pmatrix} \dot{\Psi}(t,x)\cdot\nu_{\Omega_0}(\Psi(t,x)) \\ D\Psi(t,x)^T\nu_{\Omega_0}(\Psi(t,x)) \end{pmatrix} \\ &= \begin{pmatrix} -\dot{\Phi}(t,\Psi(t,x))\cdot D\Psi(t,x)^T\nu_{\Omega_0}(\Psi(t,x)) \\ D\Psi(t,x)^T\nu_{\Omega_0}(\Psi(t,x)) \end{pmatrix}. \end{aligned} \tag{5.9}$$

Since $D_{(t,x)}\widetilde{\Psi}(t,x)^T \in \mathbb{R}^{(N+1)\times(N+1)}$, in the first equality above we have gathered its components using the row vector $\dot{\Psi}(t,x) \in \mathbb{R}^N$ and the matrix $D\Psi(t,x)^T \in \mathbb{R}^{N\times N}$. By plugging the last equality into (5.8) and recalling (5.7) and (5.4), we get (5.3).

In order to prove (5.5), we exploit (5.2) for the function $\widetilde{\Phi}$, deducing

$$
\begin{aligned}
&\int_\Gamma \mathfrak{h}\, \nu_{\mathcal{O}}\, \mathrm{d}\mathcal{H}^N \\
&=\int_{(0,T)\times\partial\Omega_0} \mathfrak{h}(\widetilde{\Phi}(t,y))|\det D_{(t,y)}\widetilde{\Phi}(t,y)|\, D_{(t,x)}\widetilde{\Psi}(\widetilde{\Phi}(t,y))^T \nu_{(0,T)\times\Omega_0}(t,y)\, \mathrm{d}\mathcal{H}^N(t,y) \\
&= \int_0^T \int_{\partial\Omega_0} \mathfrak{h}(t,\Phi(t,y)) \det D\Phi(t,y) \begin{pmatrix} -\dot{\Phi}(t,y)\cdot D\Psi(t,\Phi(t,y))^T \nu_{\Omega_0}(y) \\ D\Psi(t,\Phi(t,y))^T \nu_{\Omega_0}(y) \end{pmatrix} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\mathcal{H}^{N-1}(y)\, \mathrm{d}t,
\end{aligned}
$$

where we used the fact that $|\det D_{(t,y)}\widetilde{\Phi}(t,y)| = \det D\Phi(t,y)$ and (5.9) with $x = \Phi(t,y)$. By using again (5.2) with $\Phi(t,\cdot)$ for the integral over $\partial\Omega_0$, we conclude by recalling formula (5.4) for ω. ∎

Combining the results of Sections 3 and 4 and exploiting the previous corollary, we can now state the following theorem, which rigorously extends Theorem 2.4:

Theorem 5.4. *Assume* (2.1a) *and* (4.13) *and let* Φ, Ψ *be as in* (3.1) *and satisfy* (H1') *and* (H2). *Let the forcing term* f *be in* $H^1(\mathcal{O})$ *and assume the initial data satisfy*

$$
u_0 \in H^2(\Omega_0)\cap H_0^1(\Omega_0) \quad \textit{and} \quad u_1 + \dot{\Phi}(0,\cdot)\cdot\nabla u_0 \in H_0^1(\Omega_0). \tag{5.10}
$$

Then, there exists a unique weak solution u *of problem* (2.3) *in the sense of Definition* 2.2 *which satisfies*

$$
\begin{aligned}
&u \in L^\infty(0,T; H^2(\Omega_t)\cap H_0^1(\Omega_t)),\\
&\dot{u} \in L^\infty(0,T; H^1(\Omega_t)),\\
&\ddot{u} \in L^\infty(0,T; L^2(\Omega_t)).
\end{aligned}
$$

Moreover, for every $t \in [0,T]$ *the following energy balance holds true:*

$$
\begin{aligned}
&\frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega_t)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega_t)} + \int_0^t\int_{\partial\Omega_s} \frac{\omega(s,x)}{2}(1-\omega(s,x)^2)\Big(\frac{\partial u}{\partial\nu_{\Omega_s}}(s,x)\Big)^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}s \\
&\quad = \frac{1}{2}\|u_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega_0)} + \int_0^t \langle f(s),\dot{u}(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s,
\end{aligned}
\tag{5.11}
$$

where the scalar normal velocity ω *was introduced in* (5.4).

Remark 5.5. We point out that since $u_0 \in H^2(\Omega_0)\cap H_0^1(\Omega_0)$, it must hold that $\nabla u_0 = \frac{\partial u_0}{\partial\nu_{\Omega_0}}\nu_{\Omega_0}$ on $\partial\Omega_0$. So, the second compatibility condition in (5.10) is actually equivalent to

$$
u_1 \in H^1(\Omega_0) \quad \text{and} \quad u_1 + \omega(0,\cdot)\frac{\partial u_0}{\partial\nu_{\Omega_0}} = 0 \quad \text{on } \partial\Omega_0.
$$

For the same reason we deduce that

$$\nabla u(t,x) = \frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\nu_{\Omega_t}(x) \quad \text{for a.e. } t \in [0,T] \text{ and } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t. \tag{5.12}$$

Remark 5.6. Under the additional assumptions of Remark 2.1 and (H2A), energy balance (5.11) now reads as in Remark 2.7, with the boundary term taking the form

$$\int_0^t \int_{\partial\Omega_s} \frac{\omega(s,x)}{2}\big(|\nu_{\Omega_s}(x)|^2_{A(s,x)} - \omega(s,x)^2\big)\Big(\frac{\partial u}{\partial \nu_{\Omega_s}}(s,x)\Big)^2 \, \mathrm{d}\mathcal{H}^{N-1}(x)\, \mathrm{d}s.$$

Proof of Theorem 5.4. Existence and uniqueness of the weak solution u satisfying the regularity properties in the statement follow by combining Propositions 3.5 and 3.8, and Theorems 3.9 and 4.10.

To prove (5.11) we just need to show that

$$\begin{aligned} &-\int_{\Gamma_t} \frac{\nu^t_{\mathcal{O}}}{2}\Big[1 - \Big(\frac{\nu^t_{\mathcal{O}}}{|\nu^x_{\mathcal{O}}|}\Big)^2\Big]|\nabla u|^2 \, \mathrm{d}\mathcal{H}^N \\ &\quad = \int_0^t \int_{\partial\Omega_s} \frac{\omega(s,x)}{2}(1 - \omega(s,x)^2)\Big(\frac{\partial u}{\partial \nu_{\Omega_s}}(s,x)\Big)^2 \, \mathrm{d}\mathcal{H}^{N-1}(x)\, \mathrm{d}s, \end{aligned}$$

and then exploit (2.11). The above equality easily follows by applying (5.5) together with (5.6) and (5.12). ■

5.1. Moving boundary conditions

We now show how our result can be applied to problems driven by time-dependent boundary conditions, which often appear in mechanical models of debonding. To this end, we ask that the boundary of the set Ω_t is composed of a fixed part Λ^1 and a moving one, Λ^2_t, that is, we require that

$$\begin{aligned} &\text{for all } t \in [0,T] \text{ it holds that } \partial\Omega_t = \Lambda^1 \cup \Lambda^2_t, \text{ where } \Lambda^1 \text{ and } \Lambda^2_t \text{ are } \mathcal{H}^{N-1}\text{-measurable} \\ &\qquad \text{sets, with } \Lambda^1 \text{ independent of time } t, \text{ and satisfying } \mathcal{H}^{N-1}(\Lambda^1 \cap \Lambda^2_t) = 0. \end{aligned} \tag{5.13}$$

Consistently with the previous notation, we define

$$\Gamma^1 := (0,T) \times \Lambda^1 \quad \text{and} \quad \Gamma^2 := \bigcup_{t \in (0,T)} \{t\} \times \Lambda^2_t,$$

so that $\Gamma = \Gamma^1 \cup \Gamma^2$. For the sake of clarity, we instead denote by ν_{Λ^1} and $\nu_{\Lambda^2_t}$ the outward unit normal to Ω_t restricted to Λ^1 and Λ^2_t, respectively.

From a mechanical point of view, it is meaningful to prescribe a time-dependent external loading W on the fixed boundary Γ^1, while homogeneous Dirichlet boundary conditions are assumed on the moving boundary Γ^2. We thus consider the problem

$$\begin{cases} \ddot{U}(t,x) - \Delta U(t,x) = 0, & (t,x) \in \mathcal{O}, \\ U(t,x) = W(t,x), & (t,x) \in \Gamma^1, \\ U(t,x) = 0, & (t,x) \in \Gamma^2, \\ U(0,x) = U_0(x), & x \in \Omega_0, \\ \dot{U}(0,x) = U_1(x), & x \in \Omega_0. \end{cases} \tag{5.14}$$

The notion of weak solution for problem (5.14) is given by Definition 2.2, with the obvious changes regarding the boundary conditions. Such a problem can be tackled by assuming that W is the trace on Γ^1 of a regular function, still denoted by W, which is everywhere defined and vanishes on Γ^2 (as stated in Theorem 5.7). Indeed, by considering

$$u(t,x) := U(t,x) - W(t,x), \tag{5.15}$$

one can resort to Theorem 5.4.

Theorem 5.7. *Assume* (2.1a), (4.13), (5.13) *and let* Φ, Ψ *be as in* (3.1) *and satisfy* (H1') *and* (H2). *Let the external loading* W *and the initial data satisfy:*

- $W \in H^3(0,T; L^2_{\mathrm{loc}}(\mathbb{R}^N)) \cap H^2(0,T; H^1_{\mathrm{loc}}(\mathbb{R}^N)) \cap H^1(0,T; H^2_{\mathrm{loc}}(\mathbb{R}^N)) \cap L^2(0,T; H^3_{\mathrm{loc}}(\mathbb{R}^N))$ *such that* $W = 0$ *on* Γ^2;
- $U_0 \in H^2(\Omega_0)$ *such that* $U_0 = W(0,\cdot)$ *on* Λ^1 *and* $U_0 = 0$ *on* Λ^2_0;
- $U_1 \in H^1(\Omega_0)$, *such that* $U_1 = \dot{W}(0,\cdot)$ *on* Λ^1 *and* $U_1 + \omega(0,\cdot)\frac{\partial U_0}{\partial \nu_{\Lambda^2_0}} = 0$ *on* Λ^2_0.

Then, there exists a unique weak solution U *of problem* (5.14) *which satisfies*

$$\begin{aligned} &U \in L^\infty(0,T; H^2(\Omega_t)), \\ &\dot{U} \in L^\infty(0,T; H^1(\Omega_t)), \\ &\ddot{U} \in L^\infty(0,T; L^2(\Omega_t)). \end{aligned}$$

Moreover, for every $t \in [0,T]$ *the following energy balance holds true:*

$$\begin{aligned} &\frac{1}{2}\|\dot{U}(t)\|^2_{L^2(\Omega_t)} + \frac{1}{2}\|\nabla U(t)\|^2_{L^2(\Omega_t)} + \int_0^t \int_{\Lambda^2_s} \frac{\omega(s,x)}{2}(1-\omega(s,x)^2)\Big(\frac{\partial U}{\partial \nu_{\Lambda^2_s}}(s,x)\Big)^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}s \\ &\quad = \frac{1}{2}\|U_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\|\nabla U_0\|^2_{L^2(\Omega_0)} + \int_0^t \int_{\Lambda^1} \dot{W}(s,x)\frac{\partial U}{\partial \nu_{\Lambda^1}}(s,x)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}s. \end{aligned}$$

Remark 5.8. Under the additional assumptions of Remark 2.1 and (H2A), in the energy balance above, besides the usual changes on the potential energy and the presence of the forcing term due to $\dot{A}$, the boundary terms related to Λ^1 and Λ^2_t need to be replaced by

$$\int_0^t \int_{\Lambda^1} \dot{W}(s,x)|\nu_{\Lambda^1}(x)|^2_{A(s,x)}\frac{\partial U}{\partial \nu_{\Lambda^1}}(s,x)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}s$$

and

$$\int_0^t \int_{\Lambda_s^2} \frac{\omega(s,x)}{2} \big(|\nu_{\Lambda_s^2}(x)|^2_{A(s,x)} - \omega(s,x)^2\big) \Big(\frac{\partial U}{\partial \nu_{\Lambda_s^2}}(s,x)\Big)^2 \, \mathrm{d}\mathcal{H}^{N-1}(x) \, \mathrm{d}s.$$

Proof of Theorem 5.7. By considering the function u defined in (5.15), it is easy to see that, from the point of view of weak solutions, problem (5.14) is equivalent to problem (2.3) with data

$$\begin{gathered} f(t,x) = \Delta W(t,x) - \ddot{W}(t,x), \\ u_0(x) = U_0(x) - W(0,x) \quad \text{and} \quad u_1(x) = U_1(x) - \dot{W}(0,x). \end{gathered} \tag{5.16}$$

By the assumptions on W, U_0, U_1 we infer that $f \in H^1(\mathcal{O})$ and that (5.10) holds (see also Remark 5.5), and so Theorem 5.4 yields existence, uniqueness and regularity of the weak solution U.

The energy balance instead follows from (5.11) by exploiting (5.15) and (5.16) and after some simple (but tedious) manipulation via integration by parts. ∎

5.2. Examples of moving domains

We finally collect some examples of sets Ω_t satisfying (2.1a) (and sometimes also (2.1b)) for which we can explicitly construct diffeomorphisms Φ and Ψ satisfying (3.1) and the regularity conditions (H1') and (H2).

5.2.1. One-dimensional setting. Let $\ell \in C^{2,1}([0,T])$ satisfy

$$\ell(t) > 0 \quad \text{and} \quad |\dot{\ell}(t)| < 1 \qquad \text{for all } t \in [0,T],$$

and consider the sets $\Omega_t := (0, \ell(t))$. By defining

$$\Phi(t,y) := \frac{\ell(t)}{\ell(0)} y, \quad \Psi(t,x) := \frac{\ell(0)}{\ell(t)} x,$$

it is elementary to verify the validity of all the assumptions of Theorem 5.4 and also of (5.13). If in addition ℓ is nondecreasing, then (2.1b) is also satisfied. In this situation it holds that

$$\omega(t,0) = 0 \quad \text{and} \quad \omega(t,\ell(t)) = \dot{\ell}(t) \qquad \text{for every } t \in [0,T]. \tag{5.17}$$

This setting has been analyzed in [11, 24–26, 30–32], where ℓ is just required to be Lipschitz. Their argument strongly relies on an explicit representation of solutions of the wave equation provided by d'Alembert's formula, which holds true only in dimension one.

5.2.2. Homothetic transformations. Let $\Omega_0 \subseteq \mathbb{R}^N$ satisfy (4.1) and (4.13) and let $\lambda \in C^{2,1}([0,T])$ satisfy $\lambda(0) = 1$ and

$$\lambda(t) > 0 \quad \text{and} \quad |\dot{\lambda}(t)| \max_{y \in \overline{\Omega}_0} |y| < 1 \qquad \text{for all } t \in [0,T].$$

Consider the sets

$$\Omega_t := \lambda(t)\Omega_0 = \big\{x \in \mathbb{R}^N : x = \lambda(t)y \text{ for some } y \in \Omega_0\big\},$$

and the diffeomorphisms

$$\Phi(t, y) := \lambda(t)y \quad \text{and} \quad \Psi(t, x) := \frac{x}{\lambda(t)}.$$

It is again immediate to check all the assumptions of Theorem 5.4. If, moreover, λ is nondecreasing and Ω_0 is positively balanced, meaning that

$$\varepsilon\Omega_0 \subseteq \Omega_0 \quad \text{for every } \varepsilon \in (0, 1), \tag{5.18}$$

then (2.1b) is also satisfied. Notice that (5.18) is related to the position of Ω_0 with respect to the origin. It is, for instance, fulfilled by star-shaped sets at the origin.

In this setting we have

$$\omega(t, x) = \dot\lambda(t)\frac{x}{\lambda(t)} \cdot \nu_{\Omega_0}\Big(\frac{x}{\lambda(t)}\Big) \quad \text{for all } t \in [0, T] \text{ and for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t,$$

where $\partial\Omega_t = \lambda(t)\partial\Omega_0$. In the particular case $\Omega_0 = B_1(0)$, we infer $\omega(t, x) = \dot\lambda(t)$ and we get a radial symmetry similar to the one studied in [23].

If instead Ω_0 is a tetrahedron of the form

$$\Omega_0 = \big\{y \in \mathbb{R}^N : y \cdot n < 1 \text{ and } y_i > 0 \text{ for all } i = 1, \dots, N\big\}, \tag{5.19}$$

where $n \in \mathbb{R}^N$ is a unit vector with positive components, then

$$\omega(t, x) = \begin{cases} 0 & \text{if } x_i = 0 \text{ for some } i = 1, \dots, N, \\ \dot\lambda(t) & \text{if } x \cdot n = \lambda(t), \end{cases} \quad \text{for every } t \in [0, T].$$

We may also treat the case of an octahedron

$$\Omega_0 = \big\{y \in \mathbb{R}^N : y \cdot n_i < 1, \text{ for } i = 1, \dots, 2^N\big\},$$

where $n_i \in \mathbb{R}^N$ is a unit vector belonging to the i−th orthant for all $i = 1, \dots, 2^N$; in this situation we have $\omega(t, x) = \dot\lambda(t)$.

5.2.3. Sublevel sets. Let $g \in C^0(\mathbb{R}^N) \cap C^{3,1}(\mathbb{R}^N \setminus \{g = 0\})$ be a nonnegative function satisfying

$$\nabla g(x) \neq 0 \quad \text{for every } x \in \mathbb{R}^N \setminus \{g = 0\},$$

and assume there exists $R > 0$ such that the sublevel set $\{g < R\}$ is bounded; then, consider a nondecreasing function $\rho \in C^{2,1}([0, T])$ such that

$$0 < \rho(0) \leq \rho(T) < R.$$

We now define the sets

$$\Omega_t := \{R - \rho(t) < g < R\},$$

which fulfill assumptions (2.1), (4.13) and (5.13). A similar choice, with the obvious changes, is $\Omega_t = \{r < g < \rho(t)\}$ with $r \in (0, \rho(0))$.

We thus need to build the diffeomorphisms Φ, Ψ satisfying (3.1), (H1') and (H2). To this end, let us consider the vector field $X : [0,T] \times (\mathbb{R}^N \setminus \{g = 0\}) \to \mathbb{R}^N$ defined by

$$X(t,x) := \frac{\dot\rho(t)}{\rho(t)}(g(x) - R)\frac{\nabla g(x)}{|\nabla g(x)|^2},$$

and notice that X is of class $C^{1,1}([0,T]; C^{2,1}(\mathbb{R}^N \setminus \{g < \varepsilon\}))$ for all $\varepsilon > 0$. The map Φ is now defined as follows: $\Phi(t, y)$ is the evolution at time t of the point $y \in \overline{\Omega}_0$ through the flow

$$\begin{cases} \dot\Phi(s,y) = X(s, \Phi(s,y)), & s \in (0,T), \\ \Phi(0,y) = y. \end{cases}$$

By standard results on ODEs, for all $y \in \overline{\Omega}_0$ there exists a time $T_y \in (0, T]$ such that there exists a unique local solution $\Phi(\cdot, y) \in C^{2,1}([0, T_y])$ of the Cauchy problem above. We now show that actually the solution is global (i.e., $T_y = T$) by proving that $\Phi(t, \Omega_0) = \Omega_t$ for all $t \in [0, T]$. Indeed, this ensures that the flow is always well contained in the region where g does not vanish, and hence where the vector field X does not blow up.

To this end, we notice that for all $t \in [0, T_y]$ we have

$$\begin{aligned} g(\Phi(t,y)) &= g(y) + \int_0^t \nabla g(\Phi(s,y)) \cdot \dot\Phi(s,y)\,\mathrm{d}s \\ &= g(y) + \int_0^t \frac{\dot\rho(s)}{\rho(s)}(g(\Phi(s,y)) - R)\,\mathrm{d}s. \end{aligned}$$

By setting $F_y(t) := g(\Phi(t,y)) - R$ and by differentiating the above equality, we get

$$\begin{cases} \dot F_y(t) = \frac{\dot\rho(t)}{\rho(t)} F_y(t), & t \in [0, T_y], \\ F_y(0) = g(y) - R. \end{cases} \tag{5.20}$$

The only solution to (5.20) is given by $F_y(t) = \frac{\rho(t)}{\rho(0)} F_y(0)$, which finally implies

$$g(\Phi(t,y)) - R = \frac{\rho(t)}{\rho(0)}(g(y) - R) \quad \text{for every } t \in [0, T_y]. \tag{5.21}$$

This implies that $\Phi(t, \cdot)$ maps level sets of g in level sets of g and hence that $\Phi(t, \Omega_0) = \Omega_t$; recalling that $R > \rho(T)$, we now infer that Φ is well-defined on the whole of $[0, T] \times \overline{\Omega}_0$ and it belongs to $C^{2,1}([0,T]; C^{2,1}(\overline{\Omega}_0))$.

We can now introduce the function $\Psi : \overline{\mathcal{O}} \to [0,T] \times \overline{\Omega}_0$ as the "space"-inverse of Φ, namely $\Psi(t,x) := [\Phi(t,\cdot)]^{-1}(x)$, so that (3.1) is fulfilled by construction. By the Inverse Function Theorem and the Implicit Function Theorem, from the regularity of Φ we also deduce (H1').

To finally have (H2), we require in addition that

$$\dot\rho(t) < \min_{x \in \overline{\Omega}_t} |\nabla g(x)| \quad \text{for every } t \in [0,T]; \tag{5.22}$$

indeed, by also exploiting (5.21), it yields

$$|\dot\Phi(t,y)| = \frac{\dot\rho(t)}{\rho(t)} \frac{(R - g(\Phi(t,y)))}{|\nabla g(\Phi(t,y))|} = \frac{\dot\rho(t)}{|\nabla g(\Phi(t,y))|} \frac{R - g(y)}{\rho(0)} \le \frac{\dot\rho(t)}{|\nabla g(\Phi(t,y))|} < 1.$$

The scalar normal velocity can be finally computed as follows:

$$\omega(t,x) = X(t,x) \cdot \nu_{\Omega_t}(x) = \begin{cases} \frac{\dot\rho(t)}{|\nabla g(x)|} & \text{if } g(x) = R - \rho(t), \\ 0 & \text{if } g(x) = R, \end{cases} \quad \text{for every } t \in [0,T].$$

With the particular choice $g(x) = |x|$ the sets Ω_t are annuli, so we recover the radial case already mentioned above. Since now $|\nabla g(x)| = 1$, condition (5.22) reads as $\dot\rho(t) < 1$ and the scalar normal velocity is

$$\omega(t,x) = \begin{cases} \dot\rho(t) & \text{if } |x| = R - \rho(t), \\ 0 & \text{if } |x| = R \end{cases} \quad \text{for every } t \in [0,T]. \tag{5.23}$$

The radial case was analyzed in [23] with slightly lower regularity assumptions, by reducing the problem to a one-dimensional one, as in the example given in Section 5.2.1.

The situations described in the examples given in Sections 5.2.2 and 5.2.3 above refer to debonding models where one assumes to know a priori the possible shapes of the debonding front and of the debonded set. The unknown to be determined is the evolution law providing the time when a certain debonded front is reached. This is analogous to models with a prescribed crack in fracture mechanics.

A simple concrete application is the following: in dimension two, the possible debonded regions are triangles

$$\Omega_t = \{x \in \mathbb{R}^2 : x \cdot n < \lambda(t) \text{ and } x_i > 0 \text{ for } i = 1,2\},$$

with $n \in \mathbb{R}^N$ a unit vector with positive components and $\lambda \in C^{2,1}([0,T];[1,+\infty))$ nondecreasing (cf. (5.19)). Boundary conditions are given, for example, on the segments $\{x = (x_1, 0) : 0 < x \cdot n < \lambda(0)/2\}$ and $\{x = (0, x_2) : 0 < x \cdot n < \lambda(0)/2\}$. (This represents an external load pulling the film from a region close to the vertex $(0,0)$.) Such examples can be treated with both methods shown in the examples in Sections 5.2.2 and 5.2.3. As in the one-dimensional case of the example in Section 5.2.1, the possible debonding fronts are parallel lines (modeling a material that can be detached only in a certain direction). However, this example is genuinely two-dimensional, since the debonding front is not parallel to the edges where the boundary condition is imposed, which results in nontrivial reflections in the propagation of waves in the debonded region.

6. Application to dynamic debonding

In this final section we propose a proper formulation of a dynamic debonding model. Different from the previous sections, in this setting the evolution of the sets $t \mapsto \Omega_t$ is also unknown, and it has to be recovered by means of energetic considerations which involve the solution u of problem (2.3) in an implicit and complex way. Here we rigorously define the dynamic energy release rate in a general framework, that is, without any ansatz on the shape of the domains. This allows us to state the energetic principle governing the evolution, called the (dynamic) Griffith criterion.

We assume that the energy needed to debond a portion of film parametrized on a (measurable) set $E \subseteq \mathbb{R}^N$ is given by

$$\int_E \kappa(x)\,\mathrm{d}x, \tag{6.1}$$

where $\kappa \in C^0(\mathbb{R}^N)$ is a positive function representing the *toughness* of the glue between the film and the substrate.

The next lemma shows how the integral in (6.1) varies when the domain of integration depends on time, and its evolution is given.

Lemma 6.1. *Assume* (2.1a) *and let* Φ, Ψ *be as in* (3.1) *and satisfy* (H1). *Given any function* $\kappa \in C^0(\mathbb{R}^N)$, *for every* $t \in [0, T]$ *it holds that*

$$\int_{\Omega_t} \kappa(x)\,\mathrm{d}x = \int_{\Omega_0} \kappa(x)\,\mathrm{d}x + \int_0^t \int_{\partial\Omega_s} \omega(s,x)\kappa(x)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}s. \tag{6.2a}$$

In particular, we have

$$|\Omega_t| = |\Omega_0| + \int_0^t \int_{\partial\Omega_s} \omega(s,x)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}s. \tag{6.2b}$$

Proof. We recall that by Corollary 5.2 the set $\mathcal{O}$ is open, with Lipschitz boundary, and satisfies (2.9). Setting $\mathcal{O}_t := \{(s,x) \in \mathcal{O} : s \in (0,t)\}$, by an application of the fundamental theorem of calculus in $\mathbb{R}^{N+1}$ we thus deduce that

$$0 = \int_{\mathcal{O}_t} \frac{\partial}{\partial s}\kappa(x)\,\mathrm{d}s\,\mathrm{d}x = \int_{\partial O_t} \kappa\,\nu^t_{\mathcal{O}}\,\mathrm{d}\mathcal{H}^N \int_{\Omega_t} \kappa(x)\,\mathrm{d}x - \int_{\Omega_0} \kappa(x)\,\mathrm{d}x + \int_{\Gamma_t} \kappa\,\nu^t_{\mathcal{O}}\,\mathrm{d}\mathcal{H}^N.$$

We now conclude by using (5.5). ∎

6.1. The dynamic energy release rate, maximum dissipation principle and Griffith criterion

Now, let u be the weak solution found in Theorem 5.4 for a given nondecreasing family Ω_t (i.e., also satisfying (2.1b)). The *dynamic energy release rate* [16] is the opposite of the (infinitesimal) energy variation due to the change in time of the domain, without accounting for the energy variation due to the evolution of the external forces. Recalling energy balance (5.11), we thus consider the internal energy (kinetic and potential) subtracted by the work of external forces:

$$\mathcal{E}(t) := \frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega_t)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega_t)} - \int_0^t \langle f(s), \dot{u}(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s. \tag{6.3}$$

We now provide the definition of the dynamic energy release rate.

Definition 6.2. For $t \in [0, T]$, we define the *dynamic energy release rate* of the debonding model as

$$\mathcal{G}(t) := \lim_{h\to 0^+} -\frac{\mathcal{E}(t+h) - \mathcal{E}(t)}{|\Omega_{t+h} \setminus \Omega_t|},$$

whenever such limit exists.

Due to energy balance (5.11) together with (6.2b), by means of (6.3), we thus infer that the dynamic energy release rate can be computed as follows:

$$\begin{aligned}\mathcal{G}(t) &= -\frac{\dot{\mathcal{E}}(t)}{\frac{\mathrm{d}}{\mathrm{d}t}|\Omega_{(\cdot)}|(t)}\\ &= \frac{\int_{\partial\Omega_t} \frac{\omega(t,x)}{2}(1-\omega(t,x)^2)\big(\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\big)^2 \mathrm{d}\mathcal{H}^{N-1}(x)}{\int_{\partial\Omega_t} \omega(t,x)\,\mathrm{d}\mathcal{H}^{N-1}(x)} \quad \text{if} \int_{\partial\Omega_t} \omega(t,x)\,\mathrm{d}\mathcal{H}^{N-1}(x) > 0.\end{aligned}$$

In the one-dimensional setting and in the radial case (see Sections 5.2.1 and 5.2.3), analyzed in [11, 32] and [23] respectively, the integrands in the latter formula actually do not depend on x due to the symmetry of the problem; in other words, in those cases the released energy is the same at each point of the boundary $\partial\Omega_t$. In contrast, in the general situation here depicted the released energy may be different from point to point. It is thus convenient to introduce the *density* of the dynamic energy release rate, which is obtained by localizing the above formula around a point $x \in \partial\Omega_t$ as in the following definition:

Definition 6.3. Given $t \in [0, T]$ and $x \in \partial\Omega_t$ for which $\alpha := \omega(t,x) > 0$, the *dynamic energy release rate density* at the point (t,x) with speed $\alpha \in (0,1)$ is defined by

$$\begin{aligned}G_\alpha(t,x) &:= \lim_{r\to 0^+} \frac{\int_{\partial\Omega_t\cap B_r(x)} \frac{\omega(t)}{2}(1-\omega(t)^2)\big(\frac{\partial u}{\partial\nu_{\Omega_t}}(t)\big)^2\,\mathrm{d}\mathcal{H}^{N-1}}{\int_{\partial\Omega_t\cap B_r(x)} \omega(t)\,\mathrm{d}\mathcal{H}^{N-1}}\\ &= \frac{1}{2}(1-\alpha^2)\Big(\frac{\partial u}{\partial\nu_{\Omega_t}}(t,x)\Big)^2.\end{aligned} \tag{6.4}$$

If $\alpha = 0$, the dynamic energy release rate density is extended by continuity, setting

$$G_0(t,x) := \frac{1}{2}\Big(\frac{\partial u}{\partial\nu_{\Omega_t}}(t,x)\Big)^2.$$

Remark 6.4. We now provide the explicit expression of the dynamic energy release rate in the one-dimensional setting and in the radial one, recalling that in those cases the dynamic energy release rate coincides with its density and recovering the formulas obtained in [11, 23, 32]. In the one-dimensional situation, by means of (5.17), it is easy to check that

$$\mathcal{G}(t) = G_{\dot\ell(t)}(t, \ell(t)) = \frac{1}{2}(1 - \dot\ell(t)^2) u_x(t, \ell(t))^2 \quad \text{for a.e. } t \in [0, T].$$

In the radial case the weak solution $u(t, \cdot)$ has radial symmetry [23], hence the dynamic energy release rate density $G_\alpha(t, \cdot)$ is radial as well. Thus, by considering functions $u^{\mathrm{rad}}(t, r) := u(t, x)$ and $G_\alpha^{\mathrm{rad}}(t, r) := G_\alpha(t, x)$ for $r = R - |x|$ and by using (5.23), we obtain

$$\mathcal{G}(t) = G^{\mathrm{rad}}_{\dot\rho(t)}(t, \rho(t)) = \frac{1}{2}(1 - \dot\rho(t)^2) u_r^{\mathrm{rad}}(t, \rho(t))^2 \quad \text{for a.e. } t \in [0, T].$$

Remark 6.5. We notice that the dynamic energy release rate density can be written in an equivalent way by using the relation

$$\dot u(t, x) + \omega(t, x) \frac{\partial u}{\partial \nu_{\Omega_t}}(t, x) = 0 \qquad \text{for a.e. } t \in [0, T] \text{ and } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t,$$

which follows since $u \equiv 0$ on Γ. Indeed, from the above equality we deduce

$$\begin{aligned} G_{\omega(t,x)}(t, x) &= \frac{1}{2}(1 - \omega(t, x)^2)\Big(\frac{\partial u}{\partial \nu_{\Omega_t}}(t, x)\Big)^2 \\ &= \frac{1}{2}\frac{1 - \omega(t, x)}{1 + \omega(t, x)}\Big[(1 + \omega(t, x))\frac{\partial u}{\partial \nu_{\Omega_t}}(t, x)\Big]^2 \\ &= \frac{1}{2}\frac{1 - \omega(t, x)}{1 + \omega(t, x)}\Big[\frac{\partial u}{\partial \nu_{\Omega_t}}(t, x) - \dot u(t, x)\Big]^2. \end{aligned} \tag{6.5}$$

This will be used in Proposition 6.6.

Given a positive toughness $\kappa \in C^0(\mathbb{R}^N)$, we now postulate that during the evolution process the following energy balance is satisfied:

$$\mathcal{E}(t) + \int_{\Omega_t \setminus \Omega_0} \kappa(x)\, \mathrm{d}x = \mathcal{E}(0) \qquad \text{for every } t \in [0, T]. \tag{6.6}$$

By comparing (6.6), (5.11), (6.2a) and (6.4), we observe that the energy is conserved if one requires

$$\omega(t, x)\kappa(x) = \omega(t, x) G_{\omega(t,x)}(t, x) \quad \text{for a.e. } t \in [0, T] \text{ and for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t.$$

However, the above condition is not sufficient to determine a proper evolution of the sets Ω_t; indeed, $\omega \equiv 0$ (i.e., $\Omega_t \equiv \Omega_0$) is always an admissible choice.

A stronger requirement is the following *local maximum dissipation principle*, which essentially says that Ω_t grows whenever it is possible, while preserving the energy balance:

$$\begin{gathered} \omega(t, x) = \max\big\{\alpha \in [0, 1) : \alpha\, \kappa(x) = \alpha G_\alpha(t, x)\big\}, \\ \text{for a.e. } t \in [0, T] \text{ and for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t. \end{gathered} \tag{6.7}$$

We refer to [11, 23, 32] for a discussion. The next proposition states two equivalent forms of the local maximum dissipation principle: the first one is the (local) dynamic Griffith criterion, the second one consists of two equivalent equations for the scalar normal velocity, involving the normal derivative of the displacement u. Note that the condition $\omega < 1$ in (6.8) corresponds to the physical requirement that the speed of growth of the domain is subsonic.

Proposition 6.6. *Let $\kappa \in C^0(\mathbb{R}^N)$ be positive. Assume (2.1b) and the hypotheses of Theorem 5.4, and let u be the unique weak solution of problem (2.3). Then, the following three conditions are equivalent:*

- *the local maximum dissipation principle (see (6.7)) holds true;*
- *the* local dynamic Griffith criterion*, namely*

$$\begin{cases} 0 \le \omega(t,x) < 1, \\ G_{\omega(t,x)}(t,x) \le \kappa(x), \\ \omega(t,x)[G_{\omega(t,x)}(t,x) - \kappa(x)] = 0 \end{cases} \qquad \text{for a.e. } t \in [0,T] \text{ and } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t \tag{6.8}$$

  holds true;
- *for a.e. $t \in [0,T]$ and for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega_t$ the scalar normal velocity ω is given by*

$$\omega(t,x) = \begin{cases} \sqrt{1 - \dfrac{2\kappa(x)}{\left(\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\right)^2}} & \text{if } \left(\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\right)^2 > 2\kappa(x), \\ 0 & \text{otherwise,} \end{cases} \tag{6.9a}$$

  or equivalently,

$$\omega(t,x) = \max\left\{ \frac{\left[\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x) - \dot{u}(t,x)\right]^2 - 2\kappa(x)}{\left[\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x) - \dot{u}(t,x)\right]^2 + 2\kappa(x)}, 0 \right\}. \tag{6.9b}$$

Proof. Let us first fix a pair (t,x). We then observe that the set

$$A(t,x) := \left\{\alpha \in [0,1) : \alpha\kappa(x) = \alpha G_\alpha(t,x)\right\}$$

appearing in (6.7) consists of at most two elements; indeed, by recalling (6.4) and since $\kappa(x) > 0$, it is easy to check that

$$A(t,x) = \begin{cases} \left\{0, \sqrt{1 - \dfrac{2\kappa(x)}{\left(\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\right)^2}}\right\} & \text{if } \left(\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\right)^2 > 2\kappa(x), \\ \{0\} & \text{otherwise.} \end{cases}$$

Hence, the equivalence between (6.7) and (6.9a) is proved. Analogously, (6.7) and (6.9b) turn out to be equivalent by employing (6.5). The equivalence between (6.9a) and (6.8) is shown straightforwardly by exploiting the explicit form given in (6.4), and so we conclude. ∎

Remark 6.7. We conclude this section by listing the changes that must be taken into account in the case of the hyperbolic equation (see (2.5)). The energy $\mathcal{E}$ in (6.3) now takes the form

$$\begin{aligned}\mathcal{E}(t) = \frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega_t)} + \frac{1}{2}\langle A(t)\nabla u(t), \nabla u(t)\rangle_{L^2(\Omega_t)} - \int_0^t \langle f(s), \dot{u}(s)\rangle_{L^2(\Omega_s)}\, \mathrm{d}s \\ - \frac{1}{2}\int_0^t \langle \dot{A}(s)\nabla u(s), \nabla u(s)\rangle_{L^2(\Omega_s)}\, \mathrm{d}s.\end{aligned}$$

As a consequence, the dynamic energy release rate density becomes

$$G_\alpha(t,x) = \frac{1}{2}(|\nu_{\Omega_t}(x)|^2_{A(t,x)} - \alpha^2)\Big(\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\Big)^2,$$

and so

$$\begin{aligned}G_{\omega(t,x)}(t,x) &= \frac{1}{2}(|\nu_{\Omega_t}(x)|^2_{A(t,x)} - \omega(t,x)^2)\Big(\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\Big)^2 \\ &= \frac{1}{2}\frac{|\nu_{\Omega_t}(x)|_{A(t,x)} - \omega(t,x)}{|\nu_{\Omega_t}(x)|_{A(t,x)} + \omega(t,x)}\Big[|\nu_{\Omega_t}(x)|_{A(t,x)}\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x) - \dot{u}(t,x)\Big]^2.\end{aligned}$$

This implies that equations (6.9) have to be rewritten as

$$\begin{aligned}\omega(t,x) &= \begin{cases}\sqrt{|\nu_{\Omega_t}(x)|^2_{A(t,x)} - \frac{2\kappa(x)}{\left(\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\right)^2}} & \text{if } \left(|\nu_{\Omega_t}(x)|_{A(t,x)}\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\right)^2 > 2\kappa(x), \\ 0 & \text{otherwise}\end{cases} \\ &= \max\left\{|\nu_{\Omega_t}(x)|_{A(t,x)}\frac{\left[|\nu_{\Omega_t}(x)|_{A(t,x)}\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x) - \dot{u}(t,x)\right]^2 - 2\kappa(x)}{\left[|\nu_{\Omega_t}(x)|_{A(t,x)}\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x) - \dot{u}(t,x)\right]^2 + 2\kappa(x)},\ 0\right\},\end{aligned}$$

and also that the first line in the local dynamic Griffith criterion becomes

$$0 \le \omega(t,x) < |\nu_{\Omega_t}(x)|_{A(t,x)}.$$

The resulting changes in Definition 6.8 below are straightforward.

6.2. Formulation of the coupled problem

We are now in the position to provide a proper formulation of a dynamic debonding model, by combining the wave equation in (2.3) with the local maximum dissipation principle in (6.7) (or, equivalently, with the local dynamic Griffith criterion in (6.8), or with (6.9)). We point out that the resulting system features a strong coupling: indeed, the evolution of the domain of the wave equation is governed by (6.9), which in turn depends on the solution u to the wave equation itself.

A solution to the dynamic debonding model is defined as follows:

Definition 6.8. Given the data

- $\Omega_0 \subseteq \mathbb{R}^N$ satisfying (4.1) and (4.13),
- $\kappa \in C^0(\mathbb{R}^N)$ satisfying $\kappa(x) > 0$ for all $x \in \mathbb{R}^N$;
- $f \in H^1(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^N)) \cap L^2(0,T;H^1_{\mathrm{loc}}(\mathbb{R}^N))$;
- $u_0 \in H^2(\Omega_0) \cap H^1_0(\Omega_0)$ and $u_1 \in H^1(\Omega_0)$ satisfying

$$\begin{aligned} &\text{either} \quad u_1(x) = 0 \quad \text{and} \quad \Big(\frac{\partial u_0}{\partial \nu_{\Omega_0}}(x)\Big)^2 \le 2\kappa(x), \\ &\text{or} \quad u_1(x) \neq 0, \Big(\frac{\partial u_0}{\partial \nu_{\Omega_0}}(x)\Big)^2 - u_1(x)^2 = 2\kappa(x) \quad \text{and} \quad \frac{\frac{\partial u_0}{\partial \nu_{\Omega_0}}(x)}{u_1(x)} < -1 \end{aligned} \tag{6.10}$$

for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega_0$,

we say that an evolution $[0,T] \ni t \mapsto (u(t), \Omega_t)$ is a weak solution of the coupled problem ((2.3),(6.7)) if the following conditions are satisfied:

(i) there exists a map $\Phi : [0,T] \times \overline{\Omega}_0 \to \mathbb{R}^N$ with “space”-inverse $\Psi(t,\cdot)$ satisfying (3.1), (H1’) and (H2), for which

$$\Omega_t = \Phi(t, \Omega_0) \quad \text{for every } t \in [0,T];$$

(ii) u is the weak solution to problem (2.3) with forcing term f and initial data u_0, u_1;

(iii) the local maximum dissipation principle in (6.7) is satisfied, or equivalently the scalar normal velocity $\omega(t,x) = \dot{\Phi}(t,\Psi(t,x)) \cdot \nu_{\Omega_t}(x)$ fulfills one of the two (equivalent) equations in (6.9) for a.e. $t \in [0,T]$ and for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega_t$.

Remark 6.9. We point out that the definition makes sense. Indeed, condition (i) ensures the regularity (2.1a) of the family $\{\Omega_t\}_{t\in[0,T]}$. Furthermore, by simple computations, one can check that compatibility condition (6.10) is actually equivalent to

$$u_1 + \omega(0,\cdot)\frac{\partial u_0}{\partial \nu_{\Omega_0}} = 0 \quad \text{on } \partial\Omega_0,$$

where $\omega(0,\cdot)$ is defined by (6.9). Hence, by Remark 5.5 one can apply Theorem 5.4, concluding that the wave equation (see (2.3)) has a unique weak solution u, whose regularity allows one to give a meaning to the normal derivative $\frac{\partial u}{\partial \nu_{\Omega_t}}$ at the boundary, appearing in (6.9). Finally, notice that (6.9) also implies monotonicity property (2.1b).

Remark 6.10. Definition 6.8 can be adapted to the case of moving boundary conditions described in Section 5.1, with minor modifications. In this setting Ω_0 also satisfies (5.13) (at $t = 0$), and the external loading fulfills $W \equiv 0$ in a neighborhood of $(0,T) \times \Lambda^2_0$. The compatibility conditions on U_0 (and on U_1 on Λ^1) are those of Theorem 5.7, while (6.10) has to be valid for U_0 and U_1 on Λ^2_0. Finally, condition (ii) is prescribed only on Λ^2_t, while on Λ^1 we must have $\omega(t,x) \equiv 0$.

We conclude the paper by showing how Definition 6.8 covers the particular cases of the one-dimensional and radial models already analyzed in [11, 32] and [23], respectively (see also Sections 5.2.1 and 5.2.3). In fact, in those papers the notion of solution to the coupled problem is given in a slightly different form, and the existence is obtained by exploiting d'Alembert's formula. We prove that, if the initial data are well-prepared, the solution found in the aforementioned works fulfills Definition 6.8, at least for short times.

Theorem 6.11. *Let $\ell_0 > 0$ and let $\kappa \in C^{1,1}_{\mathrm{loc}}([\ell_0, +\infty))$ satisfy $\kappa(x) > 0$ for all $x \in [\ell_0, +\infty)$. Assume that $f \in C^{0,1}([0,T] \times [0,+\infty))$ and that $u_0 \in C^{2,1}([0,\ell_0])$ and $u_1 \in C^{1,1}([0,\ell_0])$ satisfy*

$$u_0(0) = 0, \quad u_0(\ell_0) = 0 \quad \textit{and} \quad u_1(0) = 0 \tag{6.11a}$$

and

$$u_1(\ell_0) \neq 0, \quad u_0'(\ell_0)^2 - u_1(\ell_0)^2 = 2\kappa(\ell_0), \quad \frac{u_0'(\ell_0)}{u_1(\ell_0)} < -1. \tag{6.11b}$$

Then, there exist $T^* \in (0, T]$ and a unique weak solution $t \mapsto (u(t), (0, \ell(t)))$ to the coupled problem ((2.3),(6.7)) in $[0, T^*]$ in the sense of Definition 6.8.

Proof. By [32, Theorem 4.6], there exists a unique pair (u, ℓ) such that:

(i) u is a weak solution to the wave equation with forcing term f and initial data u_0 and u_1 in the moving domain $\bigcup_{t\in(0,T)}\{t\} \times (0, \ell(t))$;

(ii) $\ell(0) = \ell_0$ and in a right neighborhood of 0, the function ℓ is a solution of the following ODE:

$$\dot\ell(t) = \max\left\{ \frac{\big[u_0'(\ell(t)-t) - u_1(\ell(t)-t) - \int_0^t f(\tau, \tau - t + \ell(t))d\tau\big]^2 - 2\kappa(\ell(t))}{\big[u_0'(\ell(t)-t) - u_1(\ell(t)-t) - \int_0^t f(\tau, \tau - t + \ell(t))d\tau\big]^2 + 2\kappa(\ell(t))}, 0 \right\}. \tag{6.12}$$

We observe that the equation solved by ℓ is the analogue of (6.9b) in the one-dimensional setting (see (5.17)), by means of d'Alembert's formula. Hence, conditions (ii) and (iii) of Definition 6.8 are satisfied by the function u and the sets $(0, \ell(t))$. To conclude, we need to check also the validity of (i).

To this end, we notice that (6.11b) implies

$$[u_0'(\ell_0) - u_1(\ell_0)]^2 > 2\kappa(\ell_0).$$

As a consequence, from (6.12) one obtains that $\dot\ell(0) > 0$ and so, by continuity, there exists $T^* \in (0, T]$ such that ℓ solves

$$\dot\ell(t) = \frac{\big[u_0'(\ell(t)-t) - u_1(\ell(t)-t) - \int_0^t f(\tau, \tau - t + \ell(t))\, d\tau\big]^2 - 2\kappa(\ell(t))}{\big[u_0'(\ell(t)-t) - u_1(\ell(t)-t) - \int_0^t f(\tau, \tau - t + \ell(t))\, d\tau\big]^2 + 2\kappa(\ell(t))}, \quad \text{in } [0, T^*].$$

In particular, as pointed out in [32, Remarks 4.9 and 4.12], by a classical bootstrap argument the regularity assumptions on f, u_0 and u_1 and compatibility conditions (6.11) ensure that $\ell \in C^{2,1}([0,T^*])$. Moreover, (6.12) directly yields $0 \le \dot{\ell}(t) < 1$ for all $t \in [0,T^*]$. Hence, the construction presented in Section 5.2.1 provides the existence of the diffeomorphisms required in (i) and we can conclude. ∎

The following result deals with the radial case in dimension 2; the extension to arbitrary dimension is straightforward. In order to state it we introduce the following notation. Given a ball or an annulus $A \subseteq \mathbb{R}^2$, we denote by $C^{k,\alpha}_{\mathrm{rad}}(\overline{A})$ the space of functions $h \in C^{k,\alpha}(\overline{A})$ which are radial, meaning that there exists a function $h^{\mathrm{rad}} : \mathbb{R} \to \mathbb{R}$ such that $h(x) = h^{\mathrm{rad}}(|x|)$ for all $x \in \overline{A}$.

Theorem 6.12. *Let $R > \rho_0 > 0$ and let $\kappa \in C^{1,1}_{\mathrm{rad}}(\overline{B_R(0)})$ satisfy $\kappa(x) > 0$ for all $x \in \overline{B_R(0)}$. Setting $\Omega_0 := \{x \in \mathbb{R}^2 : R - \rho_0 < |x| < R\}$, assume that $f \in C^{0,1}([0,T]; C^{0,1}_{\mathrm{rad}}(\overline{B_R(0)}))$, and that $u_0 \in C^{2,1}_{\mathrm{rad}}(\overline{\Omega_0})$ and $u_1 \in C^{1,1}_{\mathrm{rad}}(\overline{\Omega_0})$ satisfy*

$$\begin{aligned} u_0(x) &= 0 \quad \text{if } |x| = R \text{ or } |x| = R - \rho_0, \\ u_1(x) &= 0 \quad \text{if } |x| = R \end{aligned}$$

and

$$u_1(x) \neq 0, \quad \Big(\frac{\partial u_0}{\partial \nu_{\Omega_0}}(x)\Big)^2 - u_1(x)^2 = 2\kappa(x) \quad \text{and} \quad \frac{\frac{\partial u_0}{\partial \nu_{\Omega_0}}(x)}{u_1(x)} < -1 \qquad \text{if } |x| = R - \rho_0.$$

Then, there exist $T^* \in (0,T]$ and a unique weak solution $t \mapsto (u(t), \Omega_t)$ to the coupled problem ((2.3),(6.7)) in $[0,T^*]$ in the sense of Definition 6.8, where

$$\Omega_t := \big\{x \in \mathbb{R}^2 : R - \rho(t) < |x| < R\big\},$$

for a suitable $\rho \in C^{2,1}([0,T^*])$.

Proof. The proof is analogous to the one of Theorem 6.11, taking into account Section 5.2.3 and (5.23). Here, the existence of the function ρ is guaranteed by [23, Theorem 3.6]. For the regularity of ρ we instead refer to [23, Remarks 3.7 and 3.8]. ∎

We finally stress once again that the well-posedness of Definition 6.8 in the general case seems to be a difficult task, due to the strong coupling between the wave equation and the rule given in (6.9) governing the evolution of the domains. We leave the problem open for future research.

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