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Classification of global solutions of a free boundary problem in the plane

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Abstract. We classify non-trivial, non-negative, positively homogeneous solutions of the equation

$$\Delta u = \gamma u^{\gamma-1}$$

in the plane. The problem is motivated by the analysis of the classical Alt–Phillips free boundary problem, but considered here with negative exponents γ. The proof relies on several bespoke results for ordinary differential equations.

1. Introduction

Several problems of interest in the calculus of variations can be reduced to the study of critical points of an energy functional of the type

$$\int \frac{|\nabla u|^2}{2} + F(u),$$

where, up to normalization, $F(r) \geqslant 0$ for all $r \in \mathbb{R}$ and $F(r) = 0$ for all $r \in (-\infty, 0]$.

An archetypal example of the potential F is given by power-like functions such as

$$F(r) := r^\gamma \chi_{(0,+\infty)}(r), \tag{1.1}$$

for a given $\gamma \in \mathbb{R}$. In this case, non-negative critical points of the energy functional formally correspond to solutions of the equation

$$\Delta u = \gamma u^{\gamma-1} \tag{1.2}$$

in $\{u > 0\}$.

When $\gamma \geqslant 2$, we have that $F \in C^{1,1}(\mathbb{R})$ and the right-hand side of (1.2) is Lipschitz continuous in u. In particular, in this case one can define $c := -\gamma u^{\gamma-2}$ and deduce that c

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is continuous if u is: in this setting, the Strong Maximum Principle (see, e.g., [7, Theorem 1.7]) yields that non-negative solutions of (1.2) are actually strictly positive inside the domain in which the equation takes place.

When $\gamma \in (0, 2)$, the situation changes significantly: for instance, it is readily checked that

$$u(x) := \left(\frac{(2-\gamma)^2 \, (x_n)_+^2}{2} \right)^{\frac{1}{2-\gamma}} \tag{1.3}$$

is in this case Lipschitz continuous and, for every $\phi \in C_0^\infty(B_1)$, the partial integration yields the identity

$$\begin{aligned}
&\int_{B_1} \nabla u(x) \cdot \nabla \phi(x) + \gamma u^{\gamma-1}(x)\phi(x)\, dx \\
&\quad = \int_{B_1 \cap \{x_n > 0\}} \nabla u(x) \cdot \nabla \phi(x) + \gamma u^{\gamma-1}(x)\phi(x)\, dx \\
&\quad = \int_{B_1 \cap \{x_n > 0\}} \frac{(2-\gamma)^{\frac{\gamma}{2-\gamma}}}{2^{\frac{\gamma-1}{2-\gamma}}} x_n^{\frac{\gamma}{2-\gamma}} \, \partial_n \phi(x) + \gamma \left(\frac{(2-\gamma)^2 \, x_n^2}{2} \right)^{\frac{\gamma-1}{2-\gamma}} \phi(x)\, dx \\
&\quad = \int_{B_1 \cap \{x_n > 0\}} \partial_n \left(\frac{(2-\gamma)^{\frac{\gamma}{2-\gamma}}}{2^{\frac{\gamma-1}{2-\gamma}}} x_n^{\frac{\gamma}{2-\gamma}} \, \phi(x) \right) dx = 0,
\end{aligned}$$

providing an example of a weak solution[1] of (1.2) with a vanishing point (actually, a vanishing region) in the interior of the domain.

For this reason, equation (1.2) when $\gamma \in (0, 2)$ has been widely investigated in the context of free boundary problems and it is indeed the main topic of a classical article by H. W. Alt and D. Phillips; see [2].

From the point of view of applications, equation (1.2) also models a reaction–diffusion problem of gas distribution in a porous catalyst pellet (see, e.g., [9]). To understand the regularity of the minimizers of the associated energy functional and the way in which the free boundary separates the zero set of the solution from the positive region, one of the main tools relies on the blow-up analysis of the problem, as well as on the understanding of the corresponding homogeneous solutions (see, e.g., [2, Sections 1.15 and 1.16]; see also [11, Theorem 5.1] for the range $\gamma \in (1, 2)$).

The case $\gamma = 1$ in (1.1) corresponds to an obstacle problem and is covered by the classical work in [3]. Similarly, the case $\gamma = 0$ in (1.1) produces the seminal case studied in [1]. The case $\gamma \in (0, 1)$ has also been considered in [10].

The case of negative exponents γ appears to have been studied much less in the literature. Once we have completed this paper, the preprint [5] will become available online, where the case $\gamma \in (-2, 0)$ has been taken into account (our perspective here is, however,

[1] We stress that, at this level, the solution considered in (1.3) is a weak solution. The setting will be different in the forthcoming Theorems 1.1 and 1.2, where classical solutions will be taken into account without integrability assumptions (the function in (1.3) will, however, appear in (1.9)).

quite different from that in [5], since we do not focus our attention on the regularity of the local minimizers of the energy functional but rather on classification results for global solutions, without energy constraints, concentrating on the case of classical solutions).

The main goal of this paper is indeed to consider all possible ranges of γ, addressing in particular the two-dimensional case.

Specifically, we focus on homogeneous solutions, which play a special role in free boundary problems, since this kind of function appears as limits of blow-ups and their classification is thereby an essential ingredient toward a free boundary regularity theory.

A natural assumption for us, in view of the degree a of homogeneity of the solution, is to consider the case in which $u^{\frac{1}{a}}$ meets the zero set[2] in a suitably regular fashion. In this situation, as expected, one obtains positive and rotationally invariant solutions, as well as "one-dimensional" one phase solutions whose positivity set is a halfplane. But, perhaps more surprisingly, when $a = 1/2$, one also detects a "resonance" which produces new solutions whose positivity set is a non-trivial cone (and even the union of different cones whose opening is an acute angle).

The precise result that we have deals with classical solutions and is the following:

Theorem 1.1. *Let $a > 0$ and $\gamma \neq 0$. Assume that $u \in C(\mathbb{R}^2)$ is a non-trivial, non-negative, positively homogeneous solution of degree a of the equation*

$$\Delta u = \gamma u^{\gamma - 1} \quad \text{in a connected component of } \mathbb{R}^2 \cap \{u > 0\}. \tag{1.4}$$

Then,

$$\gamma < 2 \quad \text{and} \quad a = \frac{2}{2-\gamma}. \tag{1.5}$$

If $a \neq \frac{1}{2}$, suppose additionally[3] *that, for each connected component S of $(B_2 \setminus B_{1/2}) \cap \{u > 0\}$,*

$$u^{\frac{1}{a}} \in C^{\xi}(\overline{S}) \quad \text{for some } \xi > \begin{cases} 3 - 2a & \text{if } a \in (0,1), \\ \dfrac{1}{a} & \text{if } a > 1. \end{cases} \tag{1.6}$$

[2] For instance, in [8, Theorem 1.2] it is proved that when $\gamma = 1$, under zero Dirichlet boundary data, the free boundary meets the fixed boundary in a C^1 way and without density assumptions (and the result holds also in the fully non-linear case).

[3] As is customary, when $\xi \in (0, +\infty) \setminus \mathbb{N}$, we can write $\xi = \xi_1 + \xi_2$, with $\xi_1 \in \mathbb{N}$ and $\xi_2 \in (0,1)$. In this setting, C^{ξ} is a short notation for C^{ξ_1,ξ_2}, that is, having derivatives up to order ξ_1, with the derivatives of order ξ_1 satisfying a Hölder condition with exponent ξ_2.

Notice that in condition (1.6) a neighborhood of the origin is removed: the intuitive idea for it is that, for a "typical" situation in the plane arising from homogeneous solutions, the positivity set of the solution is given by some cone and condition (1.6) aims at detecting the way in which the solution meets the free boundary at the regular points (and not at the origin, where the free boundary may display a singularity).

We also point out that the equation $\Delta u = \eta u^{\gamma-1}$ for any $\eta \in \mathbb{R}$ such that $\eta\gamma \in (0, +\infty)$ can be reduced to (1.4) by setting $v := (\frac{\gamma}{\eta})^{\frac{1}{2-\gamma}} u$.

Then, only the following possible, non-exclusive scenarios can happen:

(1) *We have*

$$\gamma \in (0, 2) \tag{1.7}$$

and

$$u(x) = C_a\,|x|^a \quad \textit{with} \quad C_a := \frac{(2(a-1))^{\frac{a}{2}}}{a^{\frac{3a}{2}}}. \tag{1.8}$$

(2) *Up to a rotation,*

$$u(x) = \frac{2^{\frac{a}{2}}}{a^a}\,(x_2)_+^a. \tag{1.9}$$

(3) *Up to a rotation,*

$$u(x) = \frac{2^{\frac{a}{2}}}{a^a}\,|x_2|^a. \tag{1.10}$$

(4) *The following situation occurs:*

- $a = 1/2$,
- *given* $c \in \mathbb{R} \setminus \{0\}$, *up to a rotation and a reflection, the positivity set of* u *contains the cone*

$$\mathcal{C}_c := \big\{(r\cos\theta, r\sin\theta) \mid r > 0, \theta \in (0, T_c)\big\}, \tag{1.11}$$

with

$$T_c := \begin{cases} 2\pi - 2\arctan(1/c) & \textit{when } c > 0, \\ -2\arctan(1/c) & \textit{when } c < 0, \end{cases} \tag{1.12}$$

- $u = 0$ *on* $\partial\mathcal{C}_c$,
- *for every* $x \in \mathcal{C}_c$,

$$u(x) = 2^{\frac{3}{4}}\,\sqrt{x_2 - cx_1 + c|x|}. \tag{1.13}$$

We stress that the scenarios (1), (2), (3), and (4) described in Theorem 1.1 are non-exclusive, that is, when $a = 1/2$, the solution u can take any of the forms in (1.9), (1.10), and (1.13) (but not the form in (1.8), since this requires $\gamma > 0$, that is, $a > 1$).

Similarly, when $\gamma \in (0, 2)$, the solution can take the form of (1.8), (1.9), and (1.10).

Another interesting feature of Theorem 1.1 is that the "degenerate" case in which the free boundary reduces to a single point, as described by (1.8), can only occur when $\gamma \in (0, 2)$, as detailed in (1.7). Instead, the case $\gamma < 0$ only produces a "flat free boundary", as given in (1.9), with the only possible exception of $\gamma = -2$, in which a resonance can produce the situation described in (1.13).

The solution in (1.9) also coincides with that pointed out below [5, (2.3)].

Some of the solutions described in Theorem 1.1 are depicted in Figures 1, 2, 3, and 4; see also Table 1 for a summary of all these solutions.

In relation to (1.12), we also remark that $T_c \in (\pi, 2\pi)$ when $c > 0$, and $T_c \in (0, \pi)$ when $c < 0$. In particular, the case $c < 0$ produces acute cones in (1.11): in this scenario, the solutions in (1.13) can be rotated and glued to form solutions with positive sets in

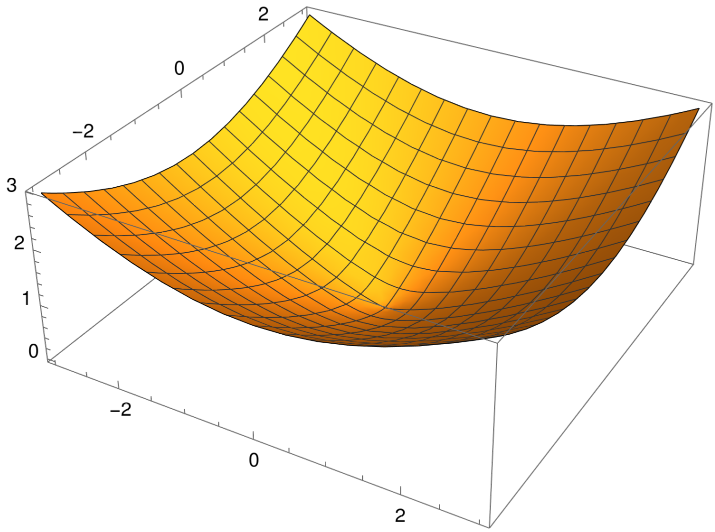

Figure 1. The solution in (1.8) with $a := 4/3$.

multi-flap cones; see, for example, Figure 5 (and, as a matter of fact, these superpositions can be iterated, thus also producing solutions whose positive sets are cones with countably many disjoint flaps).

We also stress that condition (1.6) cannot be removed, otherwise a family of new solutions arises, as detailed in the following result (in which condition (1.6) is not assumed):

Theorem 1.2. *Let $a > 0$ and $\gamma \neq 0$. Assume that $u \in C(\mathbb{R}^2)$ is a non-trivial, non-negative, positively homogeneous solution of degree a of the equation*

$$\Delta u = \gamma u^{\gamma-1} \quad \text{in a connected component of } \mathbb{R}^2 \cap \{u > 0\}.$$

Then,

$$\gamma < 2 \quad \text{and} \quad a = \frac{2}{2-\gamma}.$$

Also, either u is one of the solutions listed in Theorem 1.1 or $a \neq \frac{1}{2}$ and, up to a rotation,

$$u(r,\theta) = \frac{2^{\frac{a}{2}}}{a^a} r^a \, y^a(\theta),$$

where the function y is defined implicitly by

$$\theta = \int_0^{y(\theta)} \frac{dY}{\sqrt{1 + m\,Y^{2(1-a)} - Y^2}}$$

for some $m \in \mathbb{R}$, with $m \geqslant 0$ if $a > 1$.

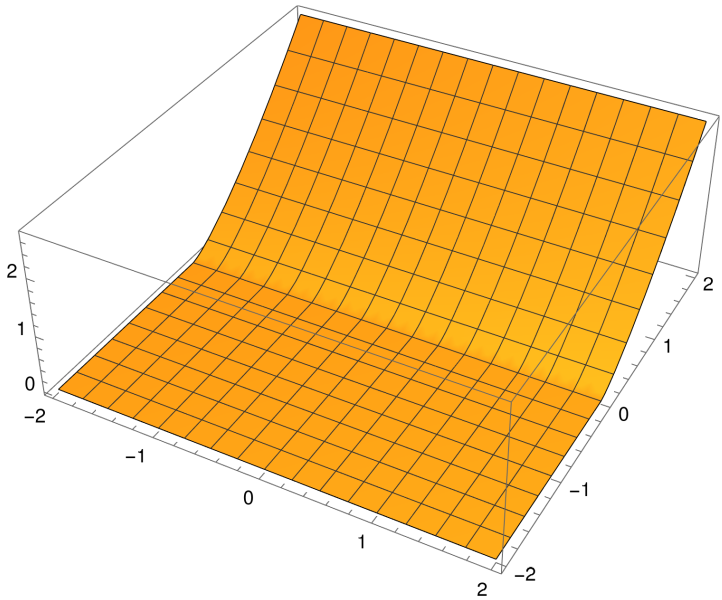

Figure 2. The solution in (1.9) with $a := 4/3$.

A particular explicit solution of the family listed in Theorem 1.2 is given by

$$u(x) = \frac{x_2^2 + 2x_1x_2}{2}. \tag{1.14}$$

This is a solution of $\Delta u = 1$ which is positive in the cone $\{x_2(x_2 + 2x_1) > 0\}$, corresponding to a solution of $\Delta u = \gamma u^{\gamma-1}$ with $\gamma = 1$; see Figure 6 for a diagram of this function (and Remark 7.2 for its explicit link to the family of solutions presented in Theorem 1.2).

The paper is organized as follows: in Section 2 we present a brief heuristic discussion of the ODE analysis performed in this paper and on the difficulties related to the singularity of the associated Cauchy problem. The rigorous analysis begins in Section 3, where we reduce the PDE problem to a non-standard ODE problem. Besides a family of explicit solutions, the ODE analysis will leverage a special function of improper integral-type and its inverse; these additional functions will be introduced and studied in Sections 4, 5, and 6. In Section 7 we present a series of tailored results on ODEs which will lead to the proof of Theorems 1.1 and 1.2, as given in Section 8.

Finally, in Section 9 we remark that the implicit solutions presented in Theorem 1.2, when extended by zero outside their positivity cone, are actually *not* weak solutions of $\Delta u = \gamma u^{\gamma-1}\chi_{\{u>0\}}$.

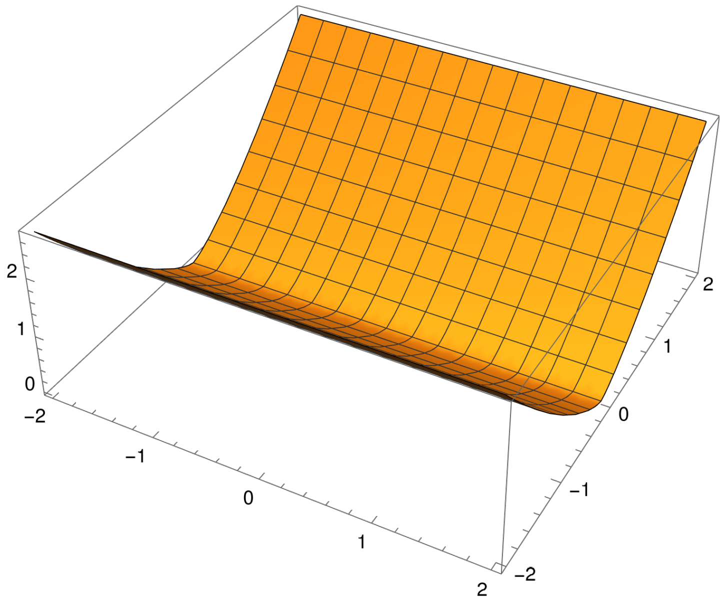

Figure 3. The solution in (1.10) with $a := 4/3$.

2. A heuristic discussion

We give here a sketchy description of the ODE analysis related to our problem. The classification of homogeneous solutions $u = r^a g(\theta)$ leads, with the substitution $y := \frac{a}{\sqrt{2}} g^{\frac{1}{a}}$, to[4] the ODE

$$y^2(\theta) + y(\theta)\, y''(\theta) + (a-1)(y^2(\theta) + (y'(\theta))^2 - 1) = 0 \tag{2.1}$$

or, equivalently,

$$ay^2(\theta) + y(\theta)\, y''(\theta) + (a-1)(y'(\theta))^2 + (1-a) = 0.$$

[4]The choice of working with y instead of g presents advantages and disadvantages. On the one hand, the ODE for g is $a^2 g + g'' = \frac{2(a-1)}{a} g^{\frac{a-2}{a}}$, which is more standard than (2.1). On the other hand, the ODE in (2.1) has the advantages of placing the dependence on the exponent γ (i.e., on the parameter a) only in the coefficients and of presenting useful algebraic properties in terms of factorization and reduction.

In a sense, the convenience of working with y instead of g is hinted by the "one-dimensional" situation described by the solution in (1.9), namely

$$u(x) = \frac{2^{\frac{a}{2}}}{a^a}(x_2)_+^a = \frac{2^{\frac{a}{2}} r^a}{a^a}(\sin\theta)_+^a.$$

In this case, in its positivity set $g(\theta)$ would be $\sin^a\theta$, while $y(\theta)$ would have the simpler expression $\sin\theta$.

The structural simplification in the one-dimensional case is also our motivation to write the regularity assumption in (1.6) in terms of powers of $\frac{1}{a}$.

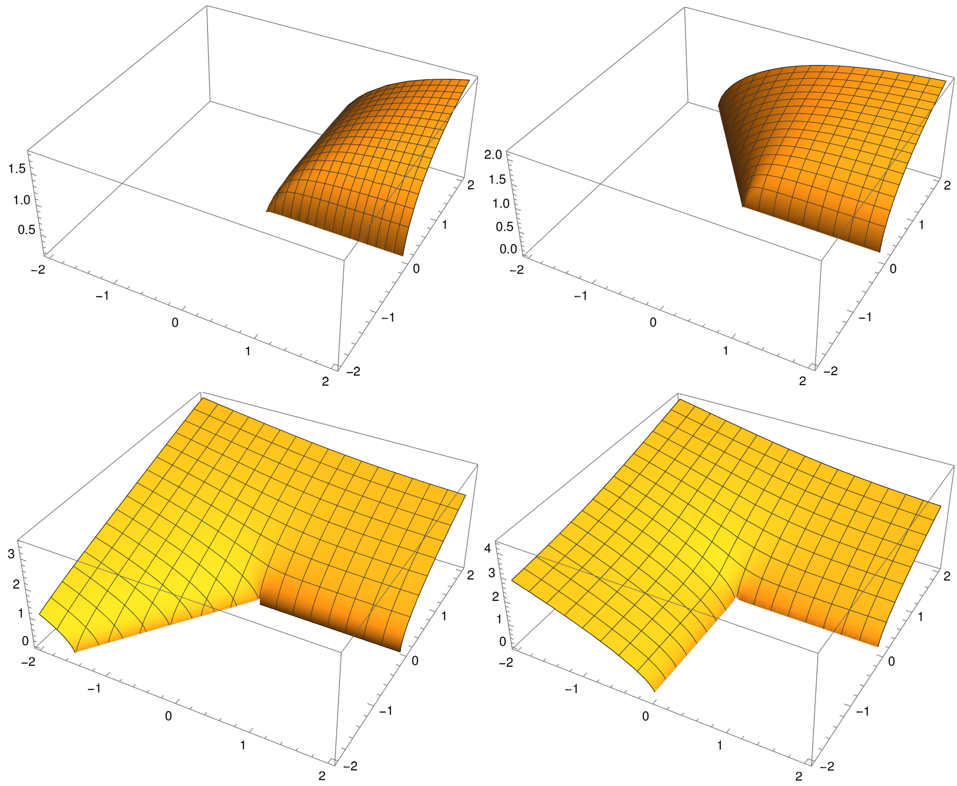

Figure 4. The solution in (1.13) with $c = -1,\ -1/2,\ 1/2,\ 1$.

This equation can be reduced to a first-order ODE by the substitution

$$y' = u(y), \tag{2.2}$$

arriving at

$$ay^2(\theta) + yu'(y)y' + (a-1)u^2(y) + (1-a) = 0,$$

and using the new unknown function u to substitute y' in the last equation yields

$$ay^2 + (a-1)u^2 + \frac{y}{2}(u^2)' + (1-a) = 0.$$

Finally, taking $U := u^2(y) - 1$, we have the linear first-order ODE

$$\frac{y}{2}U' + (a-1)U + ay^2 = 0.$$

The explicit solution of this ODE is given by

$$(y')^2 = 1 + my^{2(1-a)} - y^2, \tag{2.3}$$

Ranges of γ	Solution u	Free boundary $\partial\{u > 0\}$
$\gamma \in (0,2)$	$u(x) = \frac{(2(a-1))^{\frac{a}{2}}}{a^{\frac{3a}{2}}} \lvert x\rvert^a$	$\{0\}$
$\gamma < 2$	$u(x) = \frac{2^{\frac{a}{2}}}{a^a} (x_2)_+^a$	$\{x \in \mathbb{R}^2, x_2 = 0\}$
$\gamma < 2$	$u(x) = \frac{2^{\frac{a}{2}}}{a^a} \lvert x_2\rvert^a$	$\{x \in \mathbb{R}^2, x_2 = 0\}$
$\gamma = -2$	$u(x) = 2^{\frac{3}{4}} \sqrt{x_2 - cx_1 + c\lvert x\rvert}$	$\{re^{i\theta}, r > 0,\ \theta \in (0, T_c)\}$, with T_c given in (1.12)

Table 1. Display of the solutions detected in Theorem 1.1.

where m is the integration constant.

Note that (2.3) gives an implicit relation between θ and y and some extra care is needed to choose a branch of inverse function that produces the desired solution of our problem. Similarly, the change of independent variable from x to y utilized in (2.2) needs to be justified; for example, by showing that $y' \neq 0$ in the region of interest. We also remark that, since the ODE in (2.1) is singular at the origin, it is not sufficient in our framework just to "exhibit" a solution to complete a classification result, since in principle other solutions may arise due to a lack of uniqueness for a non-standard Cauchy problem.

Moreover, we observe that for one special case $a = 2$ and $m > 0$, the implicit relation for y takes the form of an elliptic integral

$$t = \int_0^{y(t)} \frac{Y\,dY}{\sqrt{Y^2 + m - Y^4}},$$

which can be solved explicitly; see Sections 3, 4, and 5. However, in general, explicit representation of solutions is impossible: instead, the presence of unusual integral equations describing solutions in an implicit way is actually the content of Theorem 1.2.

3. Reduction to ODEs

Here, we point out that, for a homogeneous function, satisfying the partial differential equation in (1.4) is equivalent to having an appropriate power of the angular component satisfying a suitable ordinary differential equation. The proof is a direct computation, though some care is needed, since the ordinary differential equation obtained is not a standard one.

Lemma 3.1. *Let $a > 0$ and $\gamma \neq 0$. Let $u : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ be a homogeneous function of degree a expressed in polar coordinates $(r, \theta) \in (0, +\infty) \times \mathbb{S}^1$ as*

$$u(r, \theta) = r^a\, g(\theta). \tag{3.1}$$

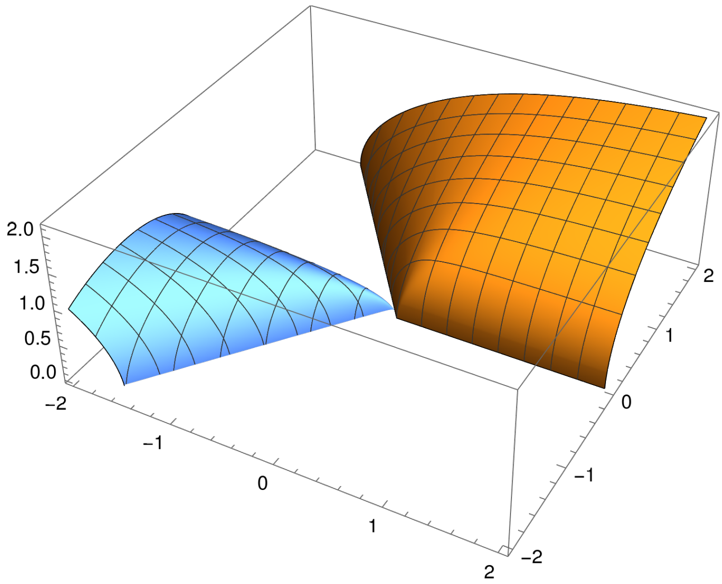

Figure 5. Superposition of the solutions in (1.13) with $c = -1/2$ and $c = -2$.

Let $S \subseteq \mathbb{S}^1$, assume that $g > 0$ in S, and set

$$y(\theta) := \frac{a}{\sqrt{2}}\, g^{\frac{1}{a}}(\theta).$$

If u is a solution of

$$\Delta u = \gamma u^{\gamma-1} \quad \text{in } (0, +\infty) \times S, \tag{3.2}$$

then

$$\gamma < 2, \quad a = \frac{2}{2-\gamma} \tag{3.3}$$

and y is a solution of

$$y^2(\theta) + y(\theta)\, y''(\theta) + (a-1)(y^2(\theta) + (y'(\theta))^2 - 1) = 0 \quad \text{for } \theta \in S. \tag{3.4}$$

Conversely, if (3.3) holds true and y is a solution of (3.4), then u is a solution of (3.2).

Proof. We use the polar representation of the Laplace operator

$$\begin{aligned}
\Delta u(x) &= \partial_r^2 (r^a\, g(\theta)) + \frac{1}{r}\partial_r (r^a\, g(\theta)) + \frac{1}{r^2}\partial_\theta^2 (r^a\, g(\theta)) \\
&= a(a-1) r^{a-2} g(\theta) + a r^{a-2} g(\theta) + r^{a-2} g''(\theta) \\
&= a^2 r^{a-2} g(\theta) + r^{a-2} g''(\theta)
\end{aligned}$$

$$
\begin{aligned}
&= a^2 r^{a-2}\Big(\frac{\sqrt{2}\,y(\theta)}{a}\Big)^a + r^{a-2}\partial_\theta^2\Big[\Big(\frac{\sqrt{2}\,y(\theta)}{a}\Big)^a\Big]\\
&= 2^{\frac{a}{2}} a^{2-a} r^{a-2} y^a(\theta) + \frac{2^{\frac{a}{2}} r^{a-2}}{a^a}(a(a-1)y^{a-2}(\theta)(y'(\theta))^2 + a y^{a-1}(\theta) y''(\theta))\\
&= 2^{\frac{a}{2}} a^{1-a} r^{a-2} y^{a-2}(\theta)(a y^2(\theta) + (a-1)(y'(\theta))^2 + y(\theta)y''(\theta)).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Delta u - \gamma u^{\gamma-1} &= 2^{\frac{a}{2}} a^{1-a} r^{a-2} y^{a-2}(a y^2 + (a-1)(y')^2 + y y'') - \gamma (r^a\, g)^{\gamma-1}\\
&= r^{a-2}\Big[2^{\frac{a}{2}} a^{1-a} y^{a-2}(a y^2 + (a-1)(y')^2 + y y'') - \gamma r^{a(\gamma-1)-a+2}\Big(\frac{\sqrt{2}}{a} y\Big)^{a(\gamma-1)}\Big]\\
&= r^{a-2}\Big[2^{\frac{a}{2}} a^{1-a} y^{a-2}(a y^2 + (a-1)(y')^2 + y y'')\\
&\qquad - \frac{2^{\frac{a(\gamma-1)}{2}}\gamma}{a^{a(\gamma-1)}} r^{a(\gamma-2)+2} y^{a(\gamma-1)}\Big].
\end{aligned}
\tag{3.5}
$$

For this reason, if u is a solution of (3.2), then

$$
2^{\frac{a}{2}} a^{1-a} y^{a-2}(a y^2 + (a-1)(y')^2 + y y'') - \frac{2^{\frac{a(\gamma-1)}{2}}\gamma}{a^{a(\gamma-1)}} r^{a(\gamma-2)+2} y^{a(\gamma-1)} = 0. \tag{3.6}
$$

Now, to prove (3.3), we suppose by contradiction that $a \neq \frac{2}{2-\gamma}$ (note that if we reach a contradiction, then (3.3) is established, since we assumed $a > 0$). We thus write (3.6) as

$$
2^{\frac{a}{2}} a^{1-a} y^{a-2}(a y^2 + (a-1)(y')^2 + y y'') - \frac{2^{\frac{a(\gamma-1)}{2}}\gamma}{a^{a(\gamma-1)}} r^{\delta} y^{a(\gamma-1)} = 0
$$

with $\delta \neq 0$. But, it cannot be that $\delta > 0$, otherwise we would reach a contradiction by sending $r \to +\infty$; nor can it be that $\delta < 0$, otherwise we would reach a contradiction by sending $r \searrow 0$. The proof of (3.3) is thereby complete.

In the light of (3.3), equation (3.6) reduces to

$$
2^{\frac{a}{2}} a^{1-a} y^{a-2}(a y^2 + (a-1)(y')^2 + y y'') - 2^{\frac{a}{2}} a^{1-a}(a-1) y^{a-2} = 0,
$$

whence

$$
(a y^2 + (a-1)(y')^2 + y y'') - (a-1) = 0,
$$

which is (3.4), as desired.

Now we assume that y satisfies (3.4) and (3.3) holds true. Then, we infer from (3.5) that

$$
\frac{a^{a-1}\, r^{2-a}\, y^{2-a}}{2^{\frac{a}{2}}}(\Delta u - \gamma u^{\gamma-1}) = (a y^2 + (a-1)(y')^2 + y y'') - (a-1) = 0,
$$

showing that (3.2) holds true. ∎

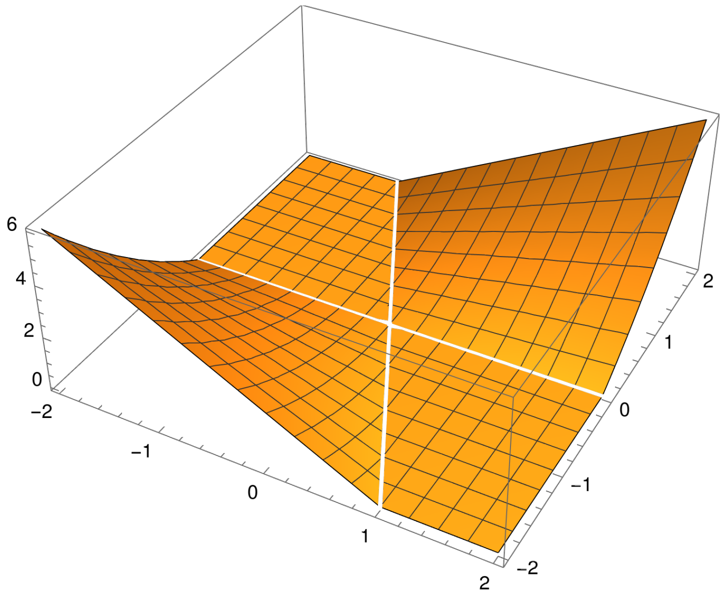

Figure 6. The solution in (1.14).

4. Two special functions $\psi(y)$ and $\Psi(y)$

In this section, we study a special function built from an improper integral.

Let $m \in \mathbb{R}$ and $a \in (0,1) \cup (1,+\infty)$. If $a > 1$, assume additionally that

$$m \geqslant 0. \tag{4.1}$$

For all $y > 0$, we let

$$\psi(y) := 1 + m\,y^{2(1-a)} - y^2. \tag{4.2}$$

We observe that, by (4.1),

$$\lim_{y\searrow 0} \psi(y) = \begin{cases} 1 & \text{if } a \in (0,1), \\ 1 & \text{if } a > 1 \text{ and } m = 0, \\ +\infty & \text{if } a > 1 \text{ and } m > 0. \end{cases}$$

As a result,

$$\lim_{y\searrow 0} \psi(y) \geqslant 1 > -\infty = \lim_{y\to+\infty} \psi(y),$$

and therefore, there exists a unique $y_* > 0$ such that

$$\psi(y) > 0 \quad \text{for all } y \in (0, y_*) \qquad \text{and} \qquad \psi(y_*) = 0. \tag{4.3}$$

Lemma 4.1. *We have that*

$$\psi'(y_*) < 0. \tag{4.4}$$

Proof. By (4.3), for all small $\varepsilon > 0$,

$$0 \leqslant \frac{\psi(y_* - \varepsilon)}{\varepsilon} = \frac{\psi(y_*) - \psi'(y_*)\varepsilon + O(\varepsilon^2)}{\varepsilon} = -\psi'(y_*) + O(\varepsilon);$$

hence, sending $\varepsilon \searrow 0$, we find that $\psi'(y_*) \leqslant 0$.

Consequently, to establish (4.4), it suffices to check that

$$\psi'(y_*) \neq 0. \tag{4.5}$$

To this end, suppose, by contradiction, that $\psi'(y_*) = 0$. Then,

$$0 = \psi'(y_*) = 2(1-a)m\, y_*^{1-2a} - 2y_*,$$

whence $y_*^{2a} = (1-a)m$.

This gives that

$$(1-a)m > 0 \tag{4.6}$$

and

$$y_* = ((1-a)m)^{\frac{1}{2a}}.$$

Accordingly, if $\psi'(y_*) = 0$, we deduce that

$$\begin{aligned} 0 = \psi(y_*) &= \psi(((1-a)m)^{\frac{1}{2a}}) = 1 + m\,((1-a)m)^{\frac{1-a}{a}} - ((1-a)m)^{\frac{1}{a}} \\ &= 1 + ((1-a)m)^{\frac{1}{a}}\Big(\frac{1}{1-a} - 1\Big) = 1 + ((1-a)m)^{\frac{1}{a}}\frac{a}{1-a}. \end{aligned}$$

Necessarily, this gives that $\frac{a}{1-a} < 0$, and so $a > 1$. Combined with (4.6), this establishes that $m < 0$, but this is against our assumption in (4.1). The proof of (4.5) is thereby complete. ∎

Now, for all $y \in [0, y_*)$, we define

$$\Psi(y) := \int_0^y \frac{dY}{\sqrt{\psi(Y)}} = \int_0^y \frac{dY}{\sqrt{1 + m\,Y^{2(1-a)} - Y^2}}. \tag{4.7}$$

Note that $\Psi(0) = 0$. Moreover,

$$\Psi \text{ is a strictly increasing function of } y \in [0, y_*) \tag{4.8}$$

and we may define

$$t_* := \lim_{y \nearrow y_*} \Psi(y) \in (0, +\infty]. \tag{4.9}$$

Now we show that t_* is always finite, according to the next observation:

Lemma 4.2. *We have that $t_* < +\infty$.*

Proof. We let $\varepsilon > 0$, to be taken suitably small. Then,

$$\psi(y_* - \varepsilon) = -\psi'(y_*)\varepsilon + O(\varepsilon^2) = |\psi'(y_*)|\varepsilon + O(\varepsilon^2)$$

and therefore, for all $\delta > \eta > 0$ suitably small,

$$\int_{y_*-\delta}^{y_*-\eta} \frac{dY}{\sqrt{\psi(Y)}} = \int_{\eta}^{\delta} \frac{d\varepsilon}{\sqrt{\psi(y_* - \varepsilon)}} = \int_{\eta}^{\delta} \frac{d\varepsilon}{\sqrt{\varepsilon}\,\sqrt{|\psi'(y_*)| + O(\varepsilon)}}.$$

As a result, using (4.4), we find for small δ that

$$\int_{y_*-\delta}^{y_*-\eta} \frac{dY}{\sqrt{\psi(Y)}} \leqslant \int_{\eta}^{\delta} \frac{d\varepsilon}{\sqrt{\varepsilon}\,\sqrt{|\psi'(y_*)|/2}} \leqslant 2\sqrt{\frac{2\delta}{|\psi'(y_*)|}}.$$

Therefore, for all $y \in [y_* - \delta, y_*)$,

$$\Psi(y) = \Psi(y_* - \delta) + \int_{y^*-\delta}^{y} \frac{dY}{\sqrt{\psi(Y)}} \leqslant \Psi(y_* - \delta) + 2\sqrt{\frac{2\delta}{|\psi'(y_*)|}},$$

and therefore, we can send $y \nearrow y_*$ and obtain that

$$t_* \leqslant \Psi(y_* - \delta) + 2\sqrt{\frac{2\delta}{|\psi'(y_*)|}} < +\infty.$$ ∎

5. The inverse function Υ of Ψ

Now we aim at inverting the special function constructed in the previous section. This method of implicitly inverting an integral equation is somewhat inspired by that used in the study of cnoidal wave solutions to the Korteweg–de Vries equation; see, for example, [6].

The special function that we obtain reconstructs all the solutions of the ODE presented in Section 3 and therefore, in light of Lemma 3.1, all the suitable powers of the angular components of the solutions of the PDE in (1.4).

Let us now present the analytical details of this construction. In view of Lemma 4.2, we can extend Ψ continuously at the point y_* by setting

$$\Psi(y_*) := t_* \in (0, +\infty). \tag{5.1}$$

Thus, by (4.8), we can define the inverse function of $\Psi : [0, y_*] \to [0, t_*]$, that is, we denote by $\Upsilon : [0, t_*] \to [0, y_*]$ the unique function such that $\Psi(\Upsilon(t)) = t$ (i.e., $\Upsilon(t)$ is the unique solution y of the equation $t = \Psi(y)$; note also that $\Upsilon > 0$ in $(0, t_*]$). We then evenly extend Υ across $t = t_*$ by setting, for all $t \in (t_*, 2t_*]$,

$$\Upsilon(t) := \Upsilon(2t_* - t). \tag{5.2}$$

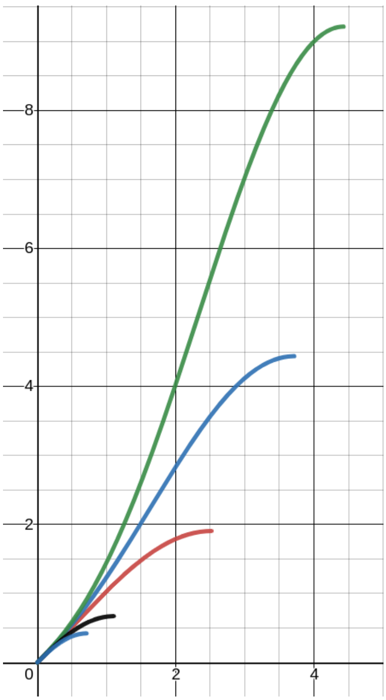

Figure 7. Numerical plot of Υ for $t \in [0, t_*]$ when $a = \frac{1}{4}$ and $m \in \{-3, -1, 1, 2, 3\}$.

In this way, $\Upsilon \in C([0, 2t_*])$.

See Figures 7 and 8 for some numerical plots of Υ for $t \in [0, t_*]$. The basic properties of this function are listed below.

Proposition 5.1. *We have that* $\Upsilon \in C^2((0, 2t_*))$. *Moreover,* $\Upsilon(0) = 0 = \Upsilon(2t_*)$ *and, for all* $t \in (0, 2t_*)$, *we have that*

$$\Upsilon^2(t) + \Upsilon(t)\,\Upsilon''(t) + (a-1)(\Upsilon^2(t) + (\Upsilon'(t))^2 - 1) = 0. \tag{5.3}$$

Proof. Since $\Psi(\Upsilon(0)) = 0 = \Psi(0)$, we obtain that $\Upsilon(0) = 0$. As a result, $\Upsilon(2t_*) = \Upsilon(2t_* - 2t_*) = \Upsilon(0) = 0$.

We also observe that $\Upsilon \in C^2((0, t_*))$ and, for all $t \in (0, t_*)$,

$$1 = \frac{d}{dt}(t) = \frac{d}{dt}(\Psi(\Upsilon(t))) = \Psi'(\Upsilon(t))\,\Upsilon'(t) = \frac{\Upsilon'(t)}{\sqrt{1 + m\,\Upsilon^{2(1-a)}(t) - \Upsilon^2(t)}},$$

that is,

$$\Upsilon'(t) = \sqrt{1 + m\,\Upsilon^{2(1-a)}(t) - \Upsilon^2(t)}. \tag{5.4}$$

Thus,

$$\Upsilon''(t) = \frac{d}{dt}\Big(\sqrt{1 + m\,\Upsilon^{2(1-a)}(t) - \Upsilon^2(t)}\Big)$$

$$= \frac{(m(1-a)\,\Upsilon^{1-2a}(t) - \Upsilon(t))\,\Upsilon'(t)}{\sqrt{1+m\,\Upsilon^{2(1-a)}(t) - \Upsilon^2(t)}}$$
$$= m(1-a)\,\Upsilon^{1-2a}(t) - \Upsilon(t). \tag{5.5}$$

By even symmetry, this also gives that $\Upsilon \in C^2((0,t_*) \cup (t_*, 2t_*))$.

Additionally, again by even symmetry,

$$\Upsilon'(2t_* - t) = -\Upsilon'(t) \quad \text{and} \quad \Upsilon''(2t_* - t) = \Upsilon''(t).$$

We also observe that $\Psi(\Upsilon(t_*)) = t_* = \Psi(y_*)$, whence $\Upsilon(t_*) = y_*$, and therefore,

$$1 + m\,\Upsilon^{2(1-a)}(t_*) - \Upsilon^2(t_*) = 1 + m\,y_*^{2(1-a)} - y_*^2 = \psi(y_*) = 0.$$

From the observations above, we infer that

$$\lim_{t\nearrow t_*} \Upsilon'(t) - \lim_{t\searrow t_*} \Upsilon'(t) = \lim_{t\nearrow t_*} \Upsilon'(t) + \lim_{t\searrow t_*} \Upsilon'(2t_* - t) = 2\lim_{t\nearrow t_*} \Upsilon'(t)$$
$$= 2\sqrt{1 + m\,\Upsilon^{2(1-a)}(t_*) - \Upsilon^2(t_*)} = 0,$$

and therefore, $\Upsilon \in C^1((0, 2t_*))$.

Also,

$$\lim_{t\nearrow t_*} \Upsilon''(t) - \lim_{t\searrow t_*} \Upsilon''(t) = \lim_{t\nearrow t_*} \Upsilon''(t) - \lim_{t\searrow t_*} \Upsilon''(2t_* - t)$$
$$= \lim_{t\nearrow t_*} \Upsilon''(t) - \lim_{t\nearrow t_*} \Upsilon''(t) = 0,$$

and therefore $\Upsilon \in C^2((0, 2t_*))$, as desired.

It remains to check (5.3). For this, we observe that, if $t \in (t_*, 2t_*)$,

$$\Upsilon^2(2t_* - t) + \Upsilon(2t_* - t)\,\Upsilon''(2t_* - t) + (a-1)(\Upsilon^2(2t_* - t) + (\Upsilon'(2t_* - t))^2 - 1)$$
$$= \Upsilon^2(t) + \Upsilon(t)\,\Upsilon''(t) + (a-1)(\Upsilon^2(t) + (-\Upsilon'(t))^2 - 1)$$
$$= \Upsilon^2(t) + \Upsilon(t)\,\Upsilon''(t) + (a-1)(\Upsilon^2(t) + (\Upsilon'(t))^2 - 1),$$

hence, it suffices to check (5.3) for $t \in (0, t_*]$ (or, actually, for $t \in (0, t_*)$ since the values at t_* can be reached by continuity).

To this end, in $(0, t_*)$, we recall (5.4) and (5.5) and we compute that

$$\Upsilon^2 + \Upsilon\,\Upsilon'' + (a-1)(\Upsilon^2 + (\Upsilon')^2 - 1)$$
$$= a\Upsilon^2 + \Upsilon\,(m(1-a)\,\Upsilon^{1-2a} - \Upsilon) + (a-1)(1 + m\,\Upsilon^{2(1-a)} - \Upsilon^2 - 1) = 0,$$

thus completing the proof of the desired result. ∎

We now study the dependence of the above quantities with respect to the parameter m (considering a as given). For this, we use the notations $\psi(y,m)$, $\Psi(y,m)$, $y_*(m)$, $t_*(m)$, and $\Upsilon(t,m)$ to emphasize their dependence upon m.

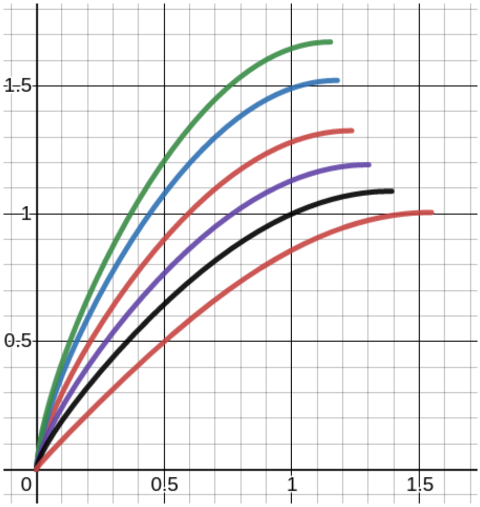

Figure 8. Numerical plot of Υ for $t \in [0, t_*]$ when $a = \frac{3}{2}$ and $m \in \{0.01, 0.2, 0.5, 1, 2, 3\}$.

Lemma 5.2. *We have that*

$$y_*(0) = 1 \tag{5.6}$$

and

$$t_*(0) = \frac{\pi}{2}. \tag{5.7}$$

Proof. By (4.2) and (4.3),

$$0 = \psi(y_*(0), 0) = 1 - (y_*(0))^2,$$

leading to (5.6).

Moreover, by (4.7), (4.9), and (5.6),

$$t_*(0) = \lim_{y \nearrow y_*(0)} \Psi(y, 0) = \lim_{y \nearrow 1} \int_0^y \frac{dY}{\sqrt{1 - Y^2}} = \int_0^1 \frac{dY}{\sqrt{1 - Y^2}} = \frac{\pi}{2},$$

which demonstrates (5.7). ∎

Corollary 5.3. *There exists $\mathcal{M} \subseteq \mathbb{R}$ such that $\mathcal{M} \neq \emptyset$ and, if $m \in \mathcal{M}$, the following holds true:*

There exist $t_* \in (0, \pi]$ and a function $\Upsilon \in C^2((0, 2t_*)) \cap C([0, 2t_*])$ such that $\Upsilon(0) = 0 = \Upsilon(2t_*)$ and, for all $t \in (0, 2t_*)$, we have that $\Upsilon(t) > 0$ and

$$\Upsilon^2(t) + \Upsilon(t)\,\Upsilon''(t) + (a - 1)(\Upsilon^2(t) + (\Upsilon'(t))^2 - 1) = 0. \tag{5.8}$$

Also, for all $t \in [0, 2t_*]$, we have that $\Upsilon(2t_* - t) = \Upsilon(t)$ and

$$t = \int_0^{\Upsilon(t)} \frac{dY}{\sqrt{1 + m\,Y^{2(1-a)} - Y^2}}.$$

Proof. The existence and basic properties of Υ follow from Proposition 5.1. The additional ingredient here is that we can find $\mathcal{M} \neq \emptyset$ such that when $m \in \mathcal{M}$ it holds that $t_*(m) \in (0, \pi]$, which is warranted by Lemma 5.2. ■

The importance of having that $t_* \in (0, \pi]$ in Corollary 5.3 consists in being able to use the function Υ as a suitable power of the angular component of a solution of the PDE in (1.4) (indeed, for this scope, one wants that $[0, 2t_*] \subseteq [0, 2\pi]$).

6. Behavior of Υ near boundary points

Now we address the boundary regularity properties of the function Υ introduced in Section 5:

Proposition 6.1. *If $m = 0$, then $\Upsilon(t) = \sin t$. If instead $m \neq 0$, the following claims hold true:*

(1) *If $a = \frac{1}{2}$, then*

$$\Upsilon(t) = \sin t + \frac{m}{2}(1 - \cos t). \tag{6.1}$$

(2) *If $a \in (0, 1)$ then $\Upsilon \in C^1([0, 2t_*])$,*

$$\Upsilon'(0) = 1 = -\Upsilon'(2t_*), \tag{6.2}$$

and

$$\lim_{t \searrow 0} \frac{\Upsilon'(t) - \Upsilon'(0)}{t^{2(1-a)}} = \frac{m}{2}. \tag{6.3}$$

(3) *More precisely, if $a \in (0, \frac{1}{2})$, then*

$$\begin{aligned} &\Upsilon \in C^{2,1-2a}([0, 2t_*]) \text{ with } \Upsilon''(0) = 0 \text{ and} \\ &\lim_{t \searrow 0} \frac{\Upsilon''(t) - \Upsilon''(0)}{t^{1-2a}} = (1 - a)m, \text{ but} \\ &\Upsilon \notin C^{2,\xi}([0, 2t_*]) \text{ if } \xi > 1 - 2a, \end{aligned} \tag{6.4}$$

while if $a \in (\frac{1}{2}, 1)$, then

$$\Upsilon \in C^{1,2(1-a)}([0, 2t_*]), \text{ but } \Upsilon \notin C^{1,\xi}([0, 2t_*]) \text{ if } \xi > 2(1 - a). \tag{6.5}$$

(4) *If $a > 1$, then*

$$\begin{aligned} &\lim_{t \searrow 0} \frac{\Upsilon(t) - \Upsilon(0)}{t^{\frac{1}{a}}} = a^{\frac{1}{a}} m^{\frac{1}{2a}}; \\ &\Upsilon \in C^{\frac{1}{a}}([0, 2t_*]), \text{ but } \Upsilon \notin C^{\xi}([0, 2t_*]) \text{ when } \xi > \frac{1}{a}. \end{aligned} \tag{6.6}$$

Proof. We focus on the regularity theory at $t = 0$, since the one at $t = 2t_*$ can be inferred by symmetry, owing to (5.2).

To this end, when $m = 0$ we have that $t_* = \frac{\pi}{2}$ thanks to (5.7), whence, for all $t \in (0, \frac{\pi}{2})$,

$$t = \Psi(\Upsilon(t)) = \int_0^{\Upsilon(t)} \frac{dY}{\sqrt{1-Y^2}} = \arcsin \Upsilon(t),$$

therefore $\Upsilon(t) = \sin t$, and this holds for all $t \in (0, \pi)$, using the parity of Ψ across $t = t_* = \frac{\pi}{2}$, as claimed.

Hence, we now suppose that $m \neq 0$. When $a = \frac{1}{2}$, we have that

$$0 = \psi(y_*) = 1 + my_* - y_*^2,$$

and therefore,

$$y_* = \frac{m + \sqrt{m^2+4}}{2}.$$

Consequently,

$$\begin{aligned} t_* = \Psi(y_*) &= \int_0^{y_*} \frac{dY}{\sqrt{1+mY-Y^2}} \\ &= \arctan\frac{m}{2} - \arctan\frac{m-2y_*}{2\sqrt{1+my_*-y_*^2}} \\ &= \arctan\frac{m}{2} + \arctan\frac{\sqrt{m^2+4}}{0^+} \\ &= \arctan\frac{m}{2} + \arctan(+\infty) = \arctan\frac{m}{2} + \frac{\pi}{2} \in (0,\pi). \end{aligned}$$

Furthermore, for all $t \in (0, t_*)$,

$$t = \Psi(\Upsilon(t)) = \int_0^{\Upsilon(t)} \frac{dY}{\sqrt{1+mY-Y^2}} = \arctan\frac{m}{2} - \arctan\frac{m-2\Upsilon(t)}{2\sqrt{1+m\Upsilon(t)-\Upsilon^2(t)}},$$

and therefore,

$$\frac{m-2\Upsilon(t)}{2\sqrt{1+m\Upsilon(t)-\Upsilon^2(t)}} = \tan T,$$

where

$$T := \arctan\frac{m}{2} - t \in \left(\arctan\frac{m}{2} - t_*, \arctan\frac{m}{2}\right) = \left(-\frac{\pi}{2}, \arctan\frac{m}{2}\right) \subseteq \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

This gives

$$\frac{m^2 - 4m\Upsilon(t) + 4\Upsilon^2(t)}{4\tan^2 T} = 1 + m\Upsilon(t) - \Upsilon^2(t),$$

and, as a result,

$$0 = \Upsilon^2(t) - m\Upsilon(t) + \frac{m^2}{4}\cos^2 T - \sin^2 T$$

$$
\begin{aligned}
&= \left(\Upsilon(t) - \frac{m}{2}\right)^2 - \frac{m^2+4}{4}\sin^2 T \\
&= \left(\Upsilon(t) - \frac{m}{2}\right)^2 - \frac{(m\cos t - 2\sin t)^2}{4} \\
&= \left(\Upsilon(t) - \frac{m}{2}\right)^2 - \left(\frac{m}{2}\cos t - \sin t\right)^2 \\
&= \left(\Upsilon(t) - \frac{m}{2} - \frac{m}{2}\cos t + \sin t\right)\left(\Upsilon(t) - \frac{m}{2} + \frac{m}{2}\cos t - \sin t\right) \\
&= \left(\Upsilon(t) - \frac{m}{2}(1+\cos t) + \sin t\right)\left(\Upsilon(t) - \frac{m}{2}(1-\cos t) - \sin t\right). \qquad (6.7)
\end{aligned}
$$

This yields that Υ has the form stated in (6.1). To check this, let us argue by contradiction and suppose that Υ does not agree with the form in (6.1) in some interval (t_1, t_2), with $0 \leqslant t_1 < t_2 \leqslant t_*$, and let us suppose that this interval is as large as possible. Then, in view of (6.7), we know that for every $t \in (t_1, t_2)$,

$$
\Upsilon(t) = \frac{m}{2}(1+\cos t) - \sin t,
$$

and in particular, $\Upsilon(t_1) = \frac{m}{2}(1+\cos t_1) - \sin t_1$.

Hence, $t_1 > 0$ necessarily, otherwise we would have that

$$
0 = \Upsilon(0) = \frac{m}{2}(1+\cos 0) - \sin 0 = m,
$$

which contradicts our assumptions.

This observation and (6.7) give that

$$
\Upsilon(t) = \begin{cases} \dfrac{m}{2}(1-\cos t) + \sin t & \text{if } t \in (0, t_1), \\[2ex] \dfrac{m}{2}(1+\cos t) - \sin t & \text{if } t \in (t_1, t_2). \end{cases}
$$

Since, by (5.8), we know that Υ is C^1 in a neighborhood of t_1, we thereby find that

$$
\frac{m}{2}\sin t_1 + \cos t_1 = \lim_{t \nearrow t_1} \Upsilon'(t) = \lim_{t \searrow t_1} \Upsilon'(t) = -\frac{m}{2}\sin t_1 - \cos t_1,
$$

leading to

$$
0 = \frac{m}{2}\sin t_1 + \cos t_1 = \frac{m}{2}\cos\left(t_1 - \frac{\pi}{2}\right) - \sin\left(t_1 - \frac{\pi}{2}\right),
$$

and thus, $t_1 = \frac{\pi}{2} + \arctan\frac{m}{2} = t_*$. This also contradicts our assumptions. This gives (6.1), as desired.

Now we point out that, for $t \in (0, t_*)$,

$$
\begin{aligned}
1 = \frac{d}{dt}t = \frac{d}{dt}(\Psi(\Upsilon(t))) = \frac{d}{dt}\left(\int_0^{\Upsilon(t)} \frac{dY}{\sqrt{1 + m\,Y^{2(1-a)} - Y^2}}\right) \\
= \frac{\Upsilon'(t)}{\sqrt{1 + m\,\Upsilon^{2(1-a)}(t) - \Upsilon^2(t)}},
\end{aligned}
$$

whence

$$\Upsilon'(t) = \sqrt{1 + m\,\Upsilon^{2(1-a)}(t) - \Upsilon^2(t)}. \tag{6.8}$$

Therefore, if $a \in (0, 1)$, it follows that Υ' is continuous at $t = 0$ and $\Upsilon'(0) = 1$, giving (6.2).

As a result, $\Upsilon(t) = t + o(t)$ as $t \searrow 0$, whence, by (6.8),

$$\begin{aligned}\Upsilon'(t) &= \sqrt{1 + m\,(t + o(t))^{2(1-a)} - (t + o(t))^2} \\ &= 1 + \frac{m\,t^{2(1-a)}(1 + o(1))^{2(1-a)} - t^2(1 + o(1))^2}{2},\end{aligned} \tag{6.9}$$

leading to (6.3).

We also deduce from (6.9) that as $t \searrow 0$,

$$\Upsilon'(t) = 1 + \frac{m\,t^{2(1-a)}}{2} + o(t^{2(1-a)}),$$

and thus,

$$(\Upsilon'(t))^2 = 1 + m\,t^{2(1-a)} + o(t^{2(1-a)}).$$

As a consequence, by (5.3),

$$\begin{aligned}0 &= \frac{\Upsilon^2(t) + \Upsilon(t)\Upsilon''(t) + (a-1)(\Upsilon^2(t) + (\Upsilon'(t))^2 - 1)}{t} \\ &= (1 + o(1))\Upsilon''(t) + (a-1)(m\,t^{1-2a} + o(t^{1-2a})) + O(t) \\ &= (1 + o(1))\Upsilon''(t) + (a-1)m\,t^{1-2a} + o(t^{1-2a}).\end{aligned} \tag{6.10}$$

Let us now assume that $a \in (0, \frac{1}{2})$. Then, the asymptotics in (6.10) shows that $\Upsilon \in C^2([0, 2t_*])$, with $\Upsilon''(0) = 0$ and

$$\lim_{t \searrow 0} \frac{\Upsilon''(t) - \Upsilon''(0)}{t^{1-2a}} = (1-a)m. \tag{6.11}$$

Furthermore, employing (5.3) for taking one more derivative, for small $t > 0$ we have that

$$\begin{aligned}\Upsilon'''(t) &= \frac{d}{dt}(\Upsilon''(t)) \\ &= -\frac{d}{dt}\Big(a\Upsilon(t) + \frac{(a-1)((\Upsilon'(t))^2 - 1)}{\Upsilon(t)}\Big) \\ &= -a\Upsilon'(t) + \frac{2(1-a)\Upsilon'(t)\Upsilon''(t)}{\Upsilon(t)} + \frac{(a-1)((\Upsilon'(t))^2 - 1)\Upsilon'(t)}{\Upsilon^2(t)} \\ &= -a(1 + o(1)) + \frac{2(1-a)(1 + o(1))((1-a)m\,t^{1-2a} + o(t^{1-2a}))}{t + o(t)} \\ &\quad + \frac{(a-1)(m\,t^{2(1-a)} + o(t^{2(1-a)}))(1 + o(t))}{t^2 + o(t^2)}\end{aligned}$$

$$= -a + (a-1)(2a-1)mt^{-2a} + o(t^{-2a}).$$

For this reason, if $\hat{t} > 0$ is sufficiently small and $0 < t_1 < t_2 < \hat{t}$, we infer that

$$\begin{aligned}
|\Upsilon''(t_2) - \Upsilon''(t_1)| &\leqslant \int_{t_1}^{t_2} |\Upsilon'''(t)|\,dt \\
&\leqslant a\int_{t_1}^{t_2} dt + 2(1-a)(1-2a)m\int_{t_1}^{t_2} t^{-2a}\,dt \\
&= a(t_2 - t_1) + \frac{2(1-a)(1-2a)m}{1-2a}(t_2^{1-2a} - t_1^{1-2a}) \\
&\leqslant C(t_2 - t_1)^{1-2a}
\end{aligned}$$

for some $C > 0$ depending only on a and m, which shows that $\Upsilon \in C^{2,1-2a}([0, 2t_*])$.

Additionally, if $\xi > 1 - 2a$,

$$\lim_{t\searrow 0} \frac{\Upsilon''(t) - \Upsilon''(0)}{t^{\xi}} = (1-a)m \lim_{t\searrow 0} t^{1-2a-\xi} = +\infty,$$

due to (6.11). Hence, $\Upsilon \notin C^{2,\xi}([0, 2t_*])$. The proof of (6.4) is thereby complete.

Let us now deal with the case $a \in (\frac{1}{2}, 1)$. In this situation, we deduce from (6.10) that

$$\Upsilon''(t) = (1-a)m\,t^{1-2a} + o(t^{1-2a}),$$

and consequently, if $\hat{t} > 0$ is sufficiently small and $0 < t_1 < t_2 < \hat{t}$,

$$\begin{aligned}
|\Upsilon'(t_2) - \Upsilon'(t_1)| &\leqslant \int_{t_1}^{t_2} |\Upsilon''(t)|\,dt \leqslant 2(1-a)m\int_{t_1}^{t_2} t^{1-2a}\,dt \\
&= 2(1-a)m(t_2^{2(1-a)} - t_1^{2(1-a)}) \leqslant C(t_2 - t_1)^{2(1-a)},
\end{aligned}$$

which shows that $\Upsilon \in C^{1,2(1-a)}([0, 2t_*])$.

However, if $\xi > 2(1-a)$,

$$\lim_{t\searrow 0} \frac{\Upsilon'(t) - \Upsilon'(0)}{t^{\xi}} = +\infty,$$

due to (6.3). Hence, $\Upsilon \notin C^{1,\xi}([0, 2t_*])$. We have therefore completed the proof of (6.5).

Now we assume that $a > 1$. In this case, we have that

$$\lim_{t\searrow 0} \Upsilon^{1-a}(t) = +\infty.$$

Therefore, for small $t > 0$, it is convenient to write (6.8) in the form

$$\Upsilon'(t) = \Upsilon^{1-a}(t)\sqrt{m + \Upsilon^{2(a-1)}(t) - \Upsilon^{2a}(t)} = (\sqrt{m} + o(1))\Upsilon^{1-a}(t), \tag{6.12}$$

and accordingly,

$$\frac{d}{dt}(\Upsilon^{a}(t)) = a\Upsilon^{a-1}(t)\Upsilon'(t) = a\sqrt{m} + o(1).$$

This entails that

$$\Upsilon^a(t) = a\sqrt{m}\,t + o(t),$$

and therefore,

$$\Upsilon(t) = (a\sqrt{m}\,t + o(t))^{\frac{1}{a}} = (a^{\frac{1}{a}} m^{\frac{1}{2a}} + o(1))\, t^{\frac{1}{a}}. \tag{6.13}$$

This shows that $\Upsilon \notin C^{\xi}([0, 2t_*])$ when $\xi > \frac{1}{a}$.

In addition, in light of (6.12) and (6.13),

$$\Upsilon'(t) = (\sqrt{m} + o(1))(a^{\frac{1}{a}} m^{\frac{1}{2a}} + o(1))^{1-a}\, t^{\frac{1-a}{a}} = (a^{\frac{1-a}{a}} m^{\frac{1}{2a}} + o(1))\, t^{\frac{1-a}{a}}.$$

Owing to this, if $\hat{t} > 0$ is sufficiently small and $0 < t_1 < t_2 < \hat{t}$,

$$\begin{aligned} |\Upsilon(t_2) - \Upsilon(t_1)| &\leqslant \int_{t_1}^{t_2} |\Upsilon'(t)|\, dt \leqslant 2a^{\frac{1-a}{a}} m^{\frac{1}{2a}} \int_{t_1}^{t_2} t^{\frac{1-a}{a}}\, dt \\ &= 2a^{\frac{1}{a}} m^{\frac{1}{2a}} (t_2^{\frac{1}{a}} - t_1^{\frac{1}{a}}) \leqslant C(t_2 - t_1)^{\frac{1}{a}} \end{aligned}$$

for some $C > 0$, which demonstrates that $\Upsilon \in C^{\frac{1}{a}}([0, 2t_*])$. This ends the proof of (6.6). ■

7. ODE methods

This section contains some bespoke results on solutions of ordinary differential equations which rely on the preliminary work done in the previous sections and will be used in Section 8 to establish Theorems 1.1 and 1.2.

Lemma 7.1. *Let $a > 0$, with $a \neq 1$. Let $T_0 > 0$ and $y \in C([0, T_0]) \cap C^2((0, T_0))$ be a solution of*

$$\begin{cases} y^2 + yy'' + (a-1)(y^2 + (y')^2 - 1) = 0, \\ y(0) = 0, \\ y(t) > 0 \quad \text{for all } t \in (0, T_0). \end{cases} \tag{7.1}$$

Then, either

$$y(t) = \begin{cases} \sin t & \text{if } a \neq 1/2, \\ \sin t + c(1 - \cos t) & \text{if } a = 1/2 \end{cases} \tag{7.2}$$

with $c \in \mathbb{R}$, or $y(t)$ is implicitly defined by the relation

$$t = \int_0^{y(t)} \frac{dY}{\sqrt{1 + m\,Y^{2(1-a)} - Y^2}} \tag{7.3}$$

for some $m \in \mathbb{R}$, with $m \geqslant 0$ if $a > 1$.

Proof. If y has the form claimed in (7.2), then it solves (7.1) by a direct computation. Furthermore, if y is as in (7.3), then it solves (7.1), due to Proposition 5.1.

Hence, it remains to prove that if y solves (7.1), then it is of the form claimed in either (7.2) or (7.3). To establish this, we first observe that if $\tilde{y} \in C([0,T_0]) \cap C^2((0,T_0))$ is a solution of (7.1); then, by the uniqueness result for regular Cauchy problems, we deduce that

$$\begin{aligned} &\text{if } y(t) = \tilde{y}(t) \text{ for all } t \text{ in an interval } I \subsetneqq [0,T_0], \\ &\text{then } y(t) = \tilde{y}(t) \text{ for all } t \in [0,T_0]. \end{aligned} \tag{7.4}$$

As a consequence, in light of (7.4), it is sufficient to prove that if y solves (7.1), then it is of the form claimed in either (7.2) or (7.3) for all t in a suitable interval.

To this end, we observe that

$$y \text{ cannot be constant in an open interval.} \tag{7.5}$$

Indeed, suppose by contradiction that $y(t) = c_0$ for all t in an open interval I. Then, by (7.1), for all $t \in I$,

$$0 = c_0^2 + 0 + (a-1)(c_0^2 + 0 - 1) = ac_0^2 - a + 1.$$

In particular, $a \neq 0$ necessarily, and then $c_0^2 = \frac{a-1}{a} \neq 0$. This says that y is equal to $c_0 \neq 0$ in an open interval and we can therefore divide by y in the ordinary differential equation in (7.1) and extend the solution. But then, using the initial value in (7.1), we see that $0 = y(0) = c_0 \neq 0$, which is a contradiction and hence, (7.5) is proved.

Now, we define

$$w(t) := y^2(t) + (y'(t))^2 - 1. \tag{7.6}$$

We first suppose that w vanishes identically in an open interval I. In this case, for all $t \in I$,

$$(y'(t))^2 = 1 - y^2(t).$$

Also, by (7.5), we can find an interval $I' \subseteq I$ in which $y' \neq 0$. Thus, we conclude that for every $t \in I'$,

$$by'(t) = \sqrt{1 - y^2(t)}$$

with $b \in \{-1, 1\}$, and accordingly,

$$\frac{d}{dt}(b \arcsin y(t) - t) = \frac{by'(t)}{\sqrt{1 - y^2(t)}} - 1 = 0.$$

From this, we arrive at

$$y(t) = \sin \frac{t}{b} + \bar{c} = b \sin t + \bar{c} \qquad \text{for all } t \in I', \tag{7.7}$$

where $\bar{c} \in \mathbb{R}$. Consequently, by (7.1), for all $t \in I'$,

$$\begin{aligned} 0 &= (b \sin t + \bar{c})^2 - b(b \sin t + \bar{c}) \sin t + (a-1)((b \sin t + \bar{c})^2 + \cos^2 t - 1) \\ &= \bar{c}(a\bar{c} + b(2a-1) \sin t). \end{aligned} \tag{7.8}$$

This gives that

$$\bar{c} = 0. \tag{7.9}$$

Because, if not, we deduce from (7.8) that the real analytic function $a\bar{c} + b(2a-1)\sin t$ vanishes for all $t \in I'$ and so, by analytic continuation, for all $t \in \mathbb{R}$. Hence, taking $t \in \{0, \frac{\pi}{2}\}$,

$$a\bar{c} = 0 \quad \text{and} \quad a\bar{c} + b(2a-1) = 0,$$

yielding that $a = \frac{1}{2}$ and then (7.9), as desired.

In light of (7.9), we deduce that (7.7) boils down to $y(t) = b\sin t$ for all $t \in I'$. Actually, by (7.4), we have that $y(t) = b\sin t$ for all $t \in [0, T_0]$. Now, if $b = -1$, then we obtain a contradiction with the assumption that $y(t) > 0$. Therefore, we conclude that $b = 1$, and accordingly, $y(t) = \sin t$ for all $t \in [0, T_0]$, which is of the form claimed in (7.2).

Thus, from now on, we can assume that

$$w \text{ cannot be identically zero in an open interval.} \tag{7.10}$$

In this setting, we recall (7.5) and we deduce that y' cannot be identically zero in an open interval. Furthermore, since y is analytic in $(0, T_0)$ (being the solution of an analytic Cauchy problem; see, e.g., [4, page 124]), we have that y' is analytic in $(0, T_0)$ as well, and therefore, the set $\{y' = 0\}$ cannot have accumulation points in $(0, T_0)$.

As a consequence of this observation, we have that there exists $\mathcal{I} \subset \mathbb{N}$ such that

$$\{y' \neq 0\} = \bigcup_{i \in \mathcal{I}} (\theta_{i+1}, \theta_i), \tag{7.11}$$

where $\theta_0 = T_0$ and $\theta_{i+1} \in [0, \theta_i)$.

We now claim that

$$(y'(t))^2 = 1 + m\, y^{2(1-a)}(t) - y^2(t) \quad \text{for all } t \in (0, T_0), \tag{7.12}$$

for some $m \in \mathbb{R} \setminus \{0\}$.

To prove it, we observe that

$$w' = 2yy' + 2y'y'' = 2y'(y + y'').$$

Hence, in every interval of the form (θ_{i+1}, θ_i), we can divide by $2y'$ and find that

$$\frac{w'}{2y'} = y + y''.$$

This and (7.1) give that

$$0 = y^2 + yy'' + (a-1)w = y(y + y'') + (a-1)w = \frac{w'y}{2y'} + (a-1)w,$$

which produces

$$\frac{d}{dt}(\log|w|) = \frac{w'}{w} = -2(a-1)\frac{y'}{y} = 2(1-a)\frac{d}{dt}(\log|y|).$$

As a result, for every $\varepsilon_i \in (\theta_{i+1}, \theta_i)$ such that $w(\varepsilon_i) \neq 0$ (whose existence is warranted by (7.10)), we have that

$$\log\frac{|w(t)|}{|w(\varepsilon_i)|} = 2(1-a)\log\frac{|y(t)|}{|y(\varepsilon_i)|}.$$

Hence, since $y(t) > 0$,

$$|w(t)| = |w(\varepsilon_i)|\Big(\frac{|y(t)|}{|y(\varepsilon_i)|}\Big)^{2(1-a)} = \frac{|w(\varepsilon_i)|}{y^{2(1-a)}(\varepsilon_i)}\, y^{2(1-a)}(t). \tag{7.13}$$

Without loss of generality, we can assume that

$$w \text{ has a strict sign in the interval } (\theta_{i+1}, \theta_i), \tag{7.14}$$

otherwise we can pick a sequence of points $\varepsilon_k \in (\theta_{i+1}, \theta_i)$ such that $w(\varepsilon_k) \neq 0$, $\varepsilon_k \to \overline{\varepsilon}_i \in (\theta_{i+1}, \theta_i)$ as $k \to +\infty$ and $w(\overline{\varepsilon}_i) = 0$. This and (7.13) give that, for every $t \in (\theta_{i+1}, \theta_i)$,

$$|w(t)| = \lim_{k\to+\infty}\frac{|w(\varepsilon_k)|}{y^{2(1-a)}(\varepsilon_k)}\, y^{2(1-a)}(t) = \frac{|w(\overline{\varepsilon}_i)|}{y^{2(1-a)}(\overline{\varepsilon}_i)}\, y^{2(1-a)}(t) = 0,$$

which contradicts our statement in (7.10). This proves (7.14).

Also, by (7.6), (7.13), and (7.14), for every $t \in (\theta_{i+1}, \theta_i)$,

$$y^2(t) + (y'(t))^2 - 1 = w(t) = \frac{w(\varepsilon_i)}{y^{2(1-a)}(\varepsilon_i)}\, y^{2(1-a)}(t) = m_i\, y^{2(1-a)}(t), \tag{7.15}$$

where

$$m_i := \frac{w(\varepsilon_i)}{y^{2(1-a)}(\varepsilon_i)} \in \mathbb{R}\setminus\{0\}.$$

We claim that

$$m_{i+1} = m_i \quad \text{for all } i \in \mathcal{I}. \tag{7.16}$$

Indeed, since (7.15) holds true in (θ_{i+1}, θ_i) with coefficient m_i and in $(\theta_{i+2}, \theta_{i+1})$ with coefficient m_{i+1}, we have that

$$\begin{aligned} & y^2(t) + (y'(t))^2 - 1 - m_i\, y^{2(1-a)}(t) = 0 \quad \text{for all } t \in (\theta_{i+1}, \theta_i) \\ \text{and} \quad & y^2(t) + (y'(t))^2 - 1 - m_{i+1}\, y^{2(1-a)}(t) = 0 \quad \text{for all } t \in (\theta_{i+2}, \theta_{i+1}). \end{aligned}$$

For this reason,

$$y^2(\theta_{i+1}) + (y'(\theta_{i+1}))^2 - 1 - m_i\, y^{2(1-a)}(\theta_{i+1})$$

$$= 0 = y^2(\theta_{i+1}) + (y'(\theta_{i+1}))^2 - 1 - m_{i+1}\, y^{2(1-a)}(\theta_{i+1}),$$

which gives (7.16).

As a consequence of (7.16), we can set $m := m_i$, recall (7.11), and obtain that the equation in (7.12) is satisfied in $\{y' \neq 0\}$, and thus in $\overline{\{y' \neq 0\}} \cap (0, T_0) = (0, T_0)$. This completes the proof of (7.12).

Now we claim that

$$\begin{aligned} &\text{if } a \in (0,1), \text{ then } \lim_{t \searrow 0} y'(t) = 1; \\ &\text{and if } a > 1, \text{ then } m > 0 \text{ and } \lim_{t \searrow 0} y'(t) = +\infty. \end{aligned} \tag{7.17}$$

To check this, let us first suppose that $a \in (0,1)$. Then, by (7.12),

$$\lim_{t \searrow 0} (y'(t))^2 = \lim_{t \searrow 0} (1 + m\, y^{2(1-a)}(t) - y^2(t)) = 1.$$

Since y is positive for small t, this gives (7.17) in this case.

Let us now suppose that $a > 1$. Thus, using (7.12) we obtain that

$$\lim_{t \searrow 0} (y'(t))^2 = \lim_{t \searrow 0} (1 + m\, y^{2(1-a)}(t) - y^2(t)) = 1 + m\, \infty; \tag{7.18}$$

therefore, in this case, since the left-hand side is non-negative, we have that $m \in (0, +\infty)$. Hence, we obtain from (7.18) that

$$\lim_{t \searrow 0} (y'(t))^2 = +\infty$$

and the claim in (7.17) follows, since y is positive for small t.

As a consequence of (7.17) we obtain that, if $\eta > 0$ is chosen appropriately small, then for all $t \in (0, \eta)$,

$$y'(t) \in (0, +\infty]. \tag{7.19}$$

Thus, exploiting (7.12) and (7.19), we find that for every $t \in (0, \eta)$,

$$y'(t) = \sqrt{1 + m\, y^{2(1-a)}(t) - y^2(t)}. \tag{7.20}$$

Now we recall the function Ψ introduced in (4.7) and we claim that, for every $t \in [0, \eta)$,

$$\Psi(y(t)) = t. \tag{7.21}$$

Indeed,

$$\lim_{t \searrow 0} (\Psi(y(t)) - t) = \lim_{r \searrow 0} \int_0^r \frac{dY}{\sqrt{1 + m\, Y^{2(1-a)} - Y^2}} = 0$$

and, in view of (7.20),

$$\frac{d}{dt}(\Psi(y(t)) - t) = \Psi'(y(t))\, y'(t) - 1 = \frac{y'(t)}{\sqrt{1 + m\, y^{2(1-a)}(t) - y^2(t)}} - 1 = 0.$$

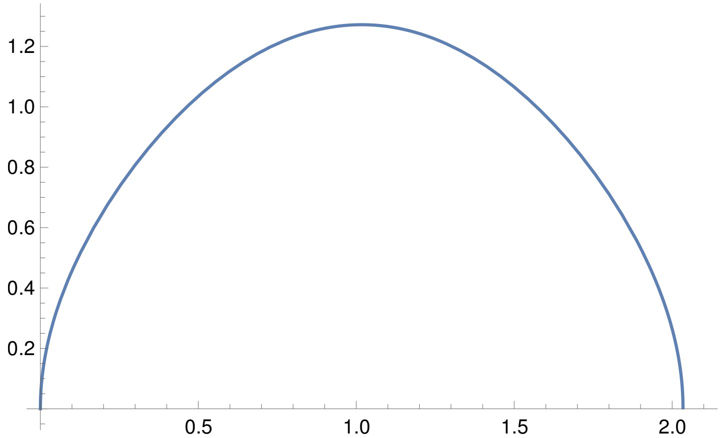

Figure 9. The function in (7.22) for $m := 1$.

These observations establish (7.21), as desired.

From (7.21), we deduce that, for every $t \in [0, \eta)$, the solution $y(t)$ must coincide with the inverse function $\Upsilon(t)$ of Ψ, as detailed in (5.1). This gives that y is as in (7.3). In particular, if $a = 1/2$, $y(t)$ is as in (7.2), thanks to (6.1) in Proposition 6.1. ■

Remark 7.2. We stress that some explicit solutions can be found among those presented in (7.3); for example, if $a := 2$ and $m > 0$, then (7.3) reads

$$t = \int_0^{y(t)} \frac{Y\,dY}{\sqrt{Y^2 + m - Y^4}}.$$

Since a primitive of $\frac{Y}{\sqrt{Y^2+m-Y^4}}$ is given by $-\frac{1}{2}\arctan\frac{1-2Y^2}{2\sqrt{m+Y^2-Y^4}}$, we find that

$$t = \frac{1}{2}\left(\arctan\frac{1}{2\sqrt{m}} - \arctan\frac{1-2y^2(t)}{2\sqrt{m+y^2(t)-y^4(t)}}\right),$$

and therefore,

$$\frac{1-2y^2(t)}{2\sqrt{m+y^2(t)-y^4(t)}} = \tan\left(\arctan\frac{1}{2\sqrt{m}} - 2t\right).$$

Hence, using the trigonometric formula

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta},$$

we obtain that

$$\frac{1-2y^2(t)}{2\sqrt{m+y^2(t)-y^4(t)}} = \frac{\frac{1}{2\sqrt{m}} - \tan(2t)}{1+\frac{1}{2\sqrt{m}}\tan(2t)},$$

which gives

$$y^2(t) = \frac{1}{2}(1 \pm (2\sqrt{m}\sin(2t) - \cos(2t))).$$

Noticing that

$$\lim_{t \searrow 0} \frac{d}{dt}(y^2(t)) = \lim_{t \searrow 0} \pm(2\sqrt{m}\cos(2t) + \sin(2t)) = \pm 2\sqrt{m},$$

we find that $0 \leqslant y^2(t) = \pm 2\sqrt{m}t + o(t)$ for small t. This solves the sign ambiguity, leading to

$$y^2(t) = \frac{1}{2}(1 + (2\sqrt{m}\sin(2t) - \cos(2t))) = \frac{1 - \cos(2t)}{2} + \sqrt{m}\sin(2t),$$

and therefore,

$$y(t) = \sqrt{\frac{1 - \cos(2t)}{2} + \sqrt{m}\sin(2t)}; \tag{7.22}$$

see Figure 9 for a diagram of this function when $m := 1$.

Recalling Lemma 3.1, we infer from this example that the function

$$u = \frac{r^2(1 - \cos(2\theta) + 2\sqrt{m}\sin(2\theta))}{4} = \frac{x_2^2 + 2\sqrt{m}\,x_1 x_2}{2}$$

is a solution of $\Delta u = 1$ (which can also be checked by a direct calculation). This observation is related to (1.14).

The counterpart of Lemma 7.1 for the non-singular equations is given by the following result:

Lemma 7.3. *Let $a > 0$. Assume that there exists a periodic solution $y \in C^2(\mathbb{R})$ of the problem*

$$\begin{cases} y^2 + yy'' + (a-1)(y^2 + (y')^2 - 1) = 0, \\ \min_{\mathbb{R}} y > 0. \end{cases} \tag{7.23}$$

Then,

$$a > 1 \tag{7.24}$$

and

$$y \text{ is constantly equal to } \sqrt{\frac{a-1}{a}}. \tag{7.25}$$

Proof. Let $t_0 \in \mathbb{R}$ be such that

$$y(t_0) = \min_{[0,2\pi]} y > 0.$$

Then, we have that $y'(t_0) = 0$ and $y''(t_0) \geqslant 0$. This and the equation in (7.23) give that

$$\begin{aligned} 0 &= y^2(t_0) + y(t_0)y''(t_0) + (a-1)(y^2(t_0) - 1) \\ &= ay^2(t_0) + y(t_0)y''(t_0) - a + 1 > -a + 1, \end{aligned} \tag{7.26}$$

which yields (7.24), as desired.

It is also useful to remark that, in view of (7.26),

$$0 = ay^2(t_0) + y(t_0)y''(t_0) - a + 1 \geqslant ay^2(t_0) - a + 1,$$

and therefore,

$$y(t_0) \leqslant \sqrt{\frac{a-1}{a}}. \tag{7.27}$$

Similarly, if t_1 is such that

$$y(t_1) = \max_{[0,2\pi]} y > 0, \tag{7.28}$$

we have that $y'(t_1) = 0$ and $y''(t_1) \leqslant 0$, whence the equation in (7.23) gives that

$$0 = y^2(t_1) + y(t_1)y''(t_1) + (a-1)(y^2(t_1) - 1) \leqslant ay^2(t_1) - a + 1,$$

and accordingly,

$$y(t_1) \geqslant \sqrt{\frac{a-1}{a}}.$$

We claim that

$$y(t_1) = \sqrt{\frac{a-1}{a}}. \tag{7.29}$$

For this, we argue by contradiction, supposing that

$$y(t_1) > \sqrt{\frac{a-1}{a}}. \tag{7.30}$$

We define

$$W(t) := 1 - y^2(t) - (y'(t))^2$$

and we observe that, in light of (7.27),

$$W(t_0) = 1 - y^2(t_0) \geqslant 1 - \frac{a-1}{a} = \frac{1}{a} > 0. \tag{7.31}$$

Therefore, W is strictly positive in some interval $I := (t_0 - \delta, t_0 + \delta)$, for a suitable $\delta > 0$. As a consequence, we can consider the logarithm of W in I and exploit the equation in (7.23) to see that

$$\frac{d}{dt} \log W = \frac{W'}{W} = \frac{-2yy' - 2y'y''}{1 - y^2 - (y')^2} = \frac{-2y'(y + y'')}{1 - y^2 - (y')^2} = \frac{-2y'(y^2 + yy'')}{y(1 - y^2 - (y')^2)}$$

$$= \frac{2(a-1)y'(y^2+(y')^2-1)}{y(1-y^2-(y')^2)} = \frac{-2(a-1)y'}{y} = -2(a-1)\frac{d}{dt}\log y.$$

As a result, for all $t \in I$,

$$\log\frac{W(t)}{W(t_0)} = -2(a-1)\log\frac{y(t)}{y(t_0)} = \log\left(\frac{y(t)}{y(t_0)}\right)^{2(1-a)}.$$

Therefore, setting

$$\kappa := \frac{W(t_0)}{(y(t_0))^{2(1-a)}}, \tag{7.32}$$

we find that for all $t \in I$,

$$1 - y^2(t) - (y'(t))^2 = W(t) = \kappa(y(t))^{2(1-a)}. \tag{7.33}$$

We also remark that y is an analytic function, since it is a solution of an analytic Cauchy problem (the sign condition in (7.23) ensuring that the source term of the differential equation is non-singular, after a division by y); see, for example, [4, page 124]. Consequently, the relation in (7.33) is globally valid, namely

$$(y'(t))^2 = 1 - y^2(t) - \kappa(y(t))^{2(1-a)} \quad \text{for all } t \in \mathbb{R}. \tag{7.34}$$

Moreover, recalling (7.27), (7.31), and (7.32),

$$\kappa \geqslant \frac{1/a}{((a-1)/a)^{1-a}} = \frac{1}{a}\left(\frac{a}{a-1}\right)^{1-a}.$$

For this reason and (7.34), we have that

$$0 \leqslant 1 - y^2(t) - \kappa(y(t))^{2(1-a)} \leqslant 1 - y^2(t) - \frac{1}{a}\left(\frac{a}{a-1}\right)^{1-a}(y(t))^{2(1-a)} \quad \text{for all } t \in \mathbb{R}.$$

From this and (7.30), we find that

$$0 < 1 - \frac{a-1}{a} - \frac{1}{a}\left(\frac{a}{a-1}\right)^{1-a}\left(\frac{a-1}{a}\right)^{1-a} = 0.$$

This is a contradiction, and thus (7.29) is established.

As a consequence of (7.28) and (7.29), we have that

$$y(t_1) = \sqrt{\frac{a-1}{a}} \quad \text{and} \quad y'(t_1) = 0.$$

Since, by inspection, the function $y_\star$ which is constantly equal to $\sqrt{\frac{a-1}{a}}$ is also a solution of (7.23), by the uniqueness result of the standard Cauchy problem we infer that $y(t) = y_\star(t)$ for every $t \in \mathbb{R}$, and this proves the desired claim in (7.25). ∎

8. Proof of Theorems 1.1 and 1.2

In light of Lemma 3.1, we can express u in the polar form $u(r,\theta) = r^a\, g(\theta)$ and we know that $\gamma < 2$ and $a = \frac{2}{2-\gamma}$. Also thanks to Lemma 3.1, when setting $y(\theta) := \frac{a}{\sqrt{2}}\, g^{\frac{1}{a}}(\theta)$, we obtain that

$$y^2(\theta) + y(\theta)\, y''(\theta) + (a-1)(y^2(\theta) + (y'(\theta))^2 - 1) = 0 \quad \text{for all } \theta \in S,$$

where S is an open subset of $\mathbb{S}^1$ (or simply of $[0, 2\pi]$ under periodicity assumptions).

Our goal is now to use the ODE analysis carried out in Section 7. For this, to distinguish between the settings in (7.1) and (7.23), we recall that y is non-negative; hence, two cases may hold:

$$\text{either} \quad \inf_{[0,2\pi]} y > 0, \tag{8.1}$$

$$\text{or} \quad y \text{ vanishes somewhere.} \tag{8.2}$$

Assume first that (8.1) holds true. Then, y is as in (7.23), whence we can apply Lemma 7.3 and infer that

$$a > 1 \tag{8.3}$$

and, for all $\theta \in [0, 2\pi]$,

$$\sqrt{\frac{a-1}{a}} = y(\theta) = \frac{a}{\sqrt{2}}\, g^{\frac{1}{a}}(\theta).$$

This and (3.1) give that

$$u = \frac{(2(a-1))^{a/2}}{a^{3a/2}}\, r^a,$$

hence (1.8) is established.

We also remark that the function in (1.8) is indeed a solution of (1.4) since

$$\begin{aligned}
&\frac{(2(a-1))^{a/2} a(a-1)}{a^{3a/2}}\, r^{a-2} + \frac{(2(a-1))^{a/2} a}{a^{3a/2}}\, r^{a-2} - \gamma \Big(\frac{(2(a-1))^{a/2}}{a^{3a/2}}\, r^a \Big)^{\gamma-1} \\
&= \frac{(2(a-1))^{a/2} a^2}{a^{3a/2}}\, r^{a-2} - \frac{2(a-1)}{a} \Big(\frac{(2(a-1))^{a/2}}{a^{3a/2}}\, r^a \Big)^{(a-2)/a} \\
&= \Big(\frac{(2(a-1))^{a/2}}{a^{(3a-4)/2}} - \frac{2(a-1)}{a}\, \frac{(2(a-1))^{(a-2)/2}}{a^{3(a-2)/2}} \Big)\, r^{a-2} \\
&= 0.
\end{aligned}$$

Finally, (1.7) follows from (3.3) and (8.3).

Now, we can focus on the case in which (8.2) is satisfied. Up to a rotation, we can suppose that $y > 0$ in $(0, T)$, with $y(0) = y(T) = 0$ for some $T \in (0, 2\pi]$. We then make use of Lemma 7.1 (and note that $a \neq 1$, owing to (3.3) and the assumption that $\gamma \neq 0$).

As a consequence, we find that, for every $\theta \in (0, T)$, either

$$y(\theta) = \sin\theta + c(1 - \cos\theta), \tag{8.4}$$

with c an arbitrary real constant when $a = 1/2$ and $c = 0$ when $a \neq 1/2$, or $y(\theta)$ is implicitly defined by the relation

$$\theta = \int_0^{y(\theta)} \frac{dY}{\sqrt{1 + mY^{2(1-a)} - Y^2}} \tag{8.5}$$

for some $m \in \mathbb{R}$, with $m \geqslant 0$ if $a > 1$.

The expression in (8.5) is precisely the one proposed in Theorem 1.2. We also stress that such an expression is excluded in Theorem 1.1, thanks to assumption (1.6). More precisely, we know from (6.4), (6.5), and (6.6) that, if (8.5) holds true, then:

- if $a \in (0, \frac{1}{2})$ and $\xi > 1 - 2a$, then $y \notin C^{2,\xi}$,
- if $a \in (\frac{1}{2}, 1)$ and $\xi > 2(1-a)$, then $y \notin C^{1,\xi}$,
- if $a > 1$ and $\xi > \frac{1}{a}$, then $y \notin C^{\xi}$,

and therefore, assumption (1.6) excludes the appearance of solutions described by (8.5) in Theorem 1.1.

Therefore, it remains to check that (8.4) provides all the possible solutions classified in the statement of Theorem 1.1.

To this end, notice that if $a \neq 1/2$, then $y(\theta) = \sin\theta$ and $T = \pi$. This gives that, for every $x = (x_1, x_2)$ with $x_2 > 0$,

$$u = r^a g = \frac{2^{\frac{a}{2}}}{a^a} r^a y^a = \frac{2^{\frac{a}{2}}}{a^a}(r\sin\theta)^a = \frac{2^{\frac{a}{2}}}{a^a} x_2^a.$$

This gives two possibilities:

$$\text{either} \quad u(x) = \frac{2^{\frac{a}{2}}}{a^a}(x_2)_+^a$$
$$\text{or} \quad u(x) = \frac{2^{\frac{a}{2}}}{a^a}|x_2|^a$$

for all $x \in \mathbb{R}^2$, therefore (1.9) and (1.10) are established in this case.

If instead $a = 1/2$, we have that for every $\theta \in (0, T)$,

$$y(\theta) = \sin\theta + c(1 - \cos\theta),$$

with $c \in \mathbb{R}$, and the case $c = 0$ reduces to the previous situation. Hence, we can suppose that $c \neq 0$ and we use the formulas

$$\cos\theta = \frac{1 - \tau^2}{1 + \tau^2} \quad \text{and} \quad \sin\theta = \frac{2\tau}{1 + \tau^2}, \qquad \text{where} \quad \tau := \tan\frac{\theta}{2}.$$

In this way, we have that

$$y = \frac{2\tau(1+c\tau)}{1+\tau^2},$$

which is positive when $\tau \in (-\infty, -1/c) \cup (0, +\infty)$ if $c > 0$, and when $\tau \in (0, -1/c)$ if $c < 0$.

In other words, $y(\theta)$ is positive when $\theta \in (0, 2\pi - 2\arctan(1/c))$ if $c > 0$, and when $\theta \in (0, -2\arctan(1/c))$ if $c < 0$. This gives that $T = 2\pi - 2\arctan(1/c) \in (\pi, 2\pi)$ when $c > 0$, and that $T = -2\arctan(1/c) \in (0, \pi)$ when $c < 0$.

Hence, in the cone $\mathcal{C}_c$ introduced in (1.11), we have that

$$u = r^a g = \frac{2^{\frac{a}{2}}}{a^a} r^a y^a = \frac{2^{\frac{a}{2}}}{a^a} r^a (\sin\theta + c(1-\cos\theta))^a = \frac{2^{\frac{a}{2}}}{a^a}(x_2 - cx_1 + c|x|)^a,$$

and this is the setting described in (1.13).

We stress that the function in (1.9) satisfies (1.6) and is also a solution of (1.4), since, in this setting,

$$\begin{aligned}\Delta u - \gamma u^{\gamma-1} &= \frac{2^{\frac{a}{2}} a(a-1)}{a^a} x_2^{a-2} - \gamma\Big(\frac{2^{\frac{a}{2}}}{a^a} x_2^{a}\Big)^{\gamma-1} \\ &= \frac{2^{\frac{a}{2}} a(a-1)}{a^a} x_2^{a-2} - \frac{2a-2}{a}\,\frac{2^{\frac{a-2}{2}}}{a^{a-2}} x_2^{a-2} = 0\end{aligned}$$

when $x_2 > 0$, thanks to (1.5).

We also observe that the function in (1.13) satisfies (1.6) and is a solution of (1.4), since

$$\begin{aligned}&\Delta u - \gamma u^{\gamma-1} \\ &= \frac{2^{\frac{a}{2}}}{a^{a-1}}(x_2 - cx_1 + c|x|)^{a-2}\Big[(a-1)\Big(\Big(\frac{cx_1}{|x|} - c\Big)^2 + \Big(\frac{cx_2}{|x|} + 1\Big)^2\Big) \\ &\quad + \frac{c}{|x|}(x_2 - cx_1 + c|x|)\Big] - \gamma\Big(\frac{2^{\frac{a}{2}}}{a^a}(x_2 - cx_1 + c|x|)^a\Big)^{\gamma-1} \\ &= \frac{2^{\frac{a}{2}}}{a^{a-1}}(x_2 - cx_1 + c|x|)^{a-2}\Big[(a-1)\Big(2c^2 + 1 + \frac{2c}{|x|}(x_2 - cx_1)\Big) \\ &\quad + \frac{c}{|x|}(x_2 - cx_1 + c|x|)\Big] - \frac{2a-2}{a}\,\frac{2^{\frac{a-2}{2}}}{a^{a-2}}(x_2 - cx_1 + c|x|)^{a-2} \\ &= \frac{1}{2^{1/4}}(x_2 - cx_1 + c|x|)^{-3/2}\Big[-\frac{1}{2}\Big(2c^2 + 1 + \frac{2c}{|x|}(x_2 - cx_1)\Big) \\ &\quad + \frac{c}{|x|}(x_2 - cx_1 + c|x|)\Big] + \frac{1}{2^{5/4}}(x_2 - cx_1 + c|x|)^{-3/2} \\ &= \frac{1}{2^{1/4}}(x_2 - cx_1 + c|x|)^{-3/2}\Big[-c^2 - \frac{1}{2} + c^2\Big] + \frac{1}{2^{5/4}}(x_2 - cx_1 + c|x|)^{-3/2} \\ &= 0.\end{aligned}$$

9. A comment about weak solutions

We point out that none of the implicit solutions presented in Theorem 1.2, when extended by zero outside their positivity cone, are weak solutions of $\Delta u = \gamma u^{\gamma-1}\chi_{\{u>0\}}$. Indeed, suppose that one of these functions is a weak solution and that its positivity cone is given by the set $\{(r,\theta) \in \mathbb{R} \times (0,\varphi)\}$, for some $\varphi \in (0,2\pi)$.

Consider a test function ϕ supported in a small ball B around $e_1 = (1,0)$. Then,

$$\int_B \nabla u(x) \cdot \nabla\phi(x)\, dx = -\gamma \int_B u^{\gamma-1}(x)\chi_{\{u>0\}}(x)\phi(x)\, dx = -\gamma \int_{B^+} u^{\gamma-1}(x)\phi(x)\, dx,$$

where $B^+ := B \cap \{x_2 > 0\}$.

But,

$$\begin{aligned}\int_B \nabla u(x) \cdot \nabla\phi(x)\, dx &= \int_{B\cap\{u>0\}} \nabla u(x) \cdot \nabla\phi(x)\, dx = \int_{B^+} \nabla u(x) \cdot \nabla\phi(x)\, dx \\ &= \int_{B^+} \operatorname{div}(\phi(x)\, \nabla u(x))\, dx - \int_{B^+} \Delta u(x)\phi(x)\, dx \\ &= -\int_H \phi(x_1,0^+)\partial_2 u(x_1,0^+)\, dx_1 - \gamma \int_{B^+} u^{\gamma-1}(x)\phi(x)\, dx,\end{aligned}$$

where $H := B \cap \{x_2 = 0\}$.

Therefore, $\partial_2 u(x_1,0^+) = 0$ along the x_1-axis.

But, the implicit solutions constructed in Theorem 1.2 do not satisfy this condition, since (up to multiplicative constants that we omit for simplicity):

- if $a \in (0,1)$, then $y(\theta) = \theta\,(1+o(1))$, due to (6.2),
- if $a > 1$, then $y(\theta) = \theta^{\frac{1}{a}}(1+o(1))$, due to (6.6).

Therefore,

$$g(\theta) = \begin{cases} y^a(\theta) = \theta^a(1+o(1)) & \text{when } a \in (0,1), \\ \theta(1+o(1)) & \text{when } a > 1, \end{cases}$$

whence

$$\partial_2 u(1,0^+) = g'(0^+) = \begin{cases} +\infty & \text{when } a \in (0,1), \\ 1 & \text{when } a > 1. \end{cases}$$

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