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The Ginzburg–Landau energy with a pinning term oscillating faster than the coherence length

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Abstract. The aim of this article is to study the magnetic Ginzburg–Landau functional with an oscillating pinning term. We consider here oscillations of the pinning term that are much faster than the coherence length $\varepsilon > 0$, which is also the inverse of the Ginzburg–Landau parameter. We study both the case of a periodic potential and of a random stationary ergodic one. We prove that we can reduce the study of the problem to the case where the pinning term is replaced by its average in the periodic case and by its expectation with respect to the random parameter in the random case. In order to do that, we use a decoupling of the energy (detailed in Lassoued and Mironescu's 1999 paper) that leads us to the study of the convergence of a scalar positive minimizer of the Ginzburg–Landau energy with pinning term and with homogeneous Neumann boundary conditions. We prove uniform convergence of this minimizer towards the mean value of the pinning term by using a blow-up argument and a Liouville-type result for non-vanishing entire solutions of the real Ginzburg–Landau/ Allen–Cahn equation, due to the results of Farina (2003).

1. Introduction

Let $G \subset \mathbb{R}^2$ be a smooth bounded domain. The main goal of this article is to study the following pinned Ginzburg–Landau energy:

$$GL_\varepsilon^{\text{pin}}(u, A) = \frac{1}{2}\int_G |(\nabla - iA)u|^2 + \frac{1}{4\varepsilon^2}\int_G (a_\varepsilon(x) - |u|^2)^2 + \frac{1}{2}\int_G |\operatorname{curl} A - h_{\text{ex}}|^2, \tag{1.1}$$

where $\varepsilon > 0, h_{\text{ex}} \geq 0$ are parameters (here, ε is a small parameter: $\varepsilon \to 0$), $u \in H^1(G, \mathbb{C})$, $A \in H^1(G, \mathbb{R}^2)$, $\operatorname{curl} A = \partial_1 A_2 - \partial_2 A_1$, $(\nabla - iA)u$ is the covariant gradient of u (i.e., the vector with complex components $(\partial_x u - iA_1 u, \partial_y u - iA_2 u)^T$), and a_ε is a function oscillating at a rate $\delta = \delta_\varepsilon \ll \varepsilon$. More precisely, we will study the case where $a_\varepsilon(x) = a_0(\frac{x}{\delta})$, with $a_0 : \mathbb{R}^2 \to \mathbb{R}$ a 1-periodic function and the case where $a_\varepsilon(x) = a_1(T(\frac{x}{\delta})\omega)$, where $a_1 : \Omega \to \mathbb{R}$ is a random variable defined on a probability space Ω and T denotes

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an action of $\mathbb{R}^2$ on Ω which is stationary and ergodic. Although we do not specify it in our notation, the parameter δ depends on ε, hence the notation a_ε. The functional $GL_\varepsilon^{\mathrm{pin}}$ is used to describe the behavior of type-II superconductors in presence of impurities. In this model u is the complex order parameter, with $|u|^2$ representing a normalized density of Cooper pairs of electrons in the sample G, and $h := \operatorname{curl} A$ represents the magnetic field inside the sample. When the sample is a homogeneous material (i.e., $a_\varepsilon \equiv 1$), and in the absence of magnetic field (i.e., when $A = h_{\mathrm{ex}} = 0$), functional (1.1) has been studied in the pioneering work of Bethuel–Brezis–Hélein [10]. For the study of the functional with magnetic field, we refer to [25] and references therein. In order to describe heterogeneous materials, various authors have considered a modified Ginzburg–Landau energy where various fixed weights appear [3–5, 8, 22]. Oscillating pinning terms were also studied in [1, 15, 16]. Here, our setting is close to the one in [1], except that the assumptions on a_ε are different: in [1] the pinning term a_ε oscillates slower than ε (with our notation, their assumption would correspond to $\delta \gg |\log \varepsilon|^{-1}$). The study of (1.1) combines the difficulties of concentration phenomena in phase transition theory and of homogenization effects due to oscillations. This is also the case in the recent paper [2] where the authors study the homogenization of an oscillating Ginzburg–Landau energy where the oscillating term occurs in the gradient. Oscillations in phase transition problems were also studied in the context of the Allen–Cahn/ Modica–Mortola functional. On this subject, we refer to [6, 12–14, 19]. We note that the oscillating weight in the energies studied in those works is different from the one studied here. Of particular interest for us in this article are [6, 19], where the case when the oscillations are much faster than the phase transition parameter ε is considered. We note that, in these references, the hypothesis that $\delta = o_\varepsilon(1)$ is not sufficient to obtain a homogenization result and the authors assume instead that $\delta = o_\varepsilon(\varepsilon^{3/2})$ in both papers, even though the techniques used in these papers are different.

We will describe the asymptotic behavior of minimizers of (1.1), but also of a similar pinned Ginzburg–Landau functional in three dimensions and a pinned Allen–Cahn functional in d dimensions, for arbitrary d.

Let $Q = (0,1)^d$ be the unit cube in $\mathbb{R}^d$. We consider a function $a_0 \in L^\infty(Q, \mathbb{R})$ which satisfies the following condition:

$$\text{there exist } 0 < m < M \text{ such that } m < a_0(x) < M \text{ a.e. in } Q. \tag{1.2}$$

Without loss of generality, we can assume that

$$\int_Q a_0(y)dy = 1 \quad \text{and} \quad m < 1 < M,$$

and we will indicate how to adapt the arguments to the case $\mathcal{M} = \sqrt{\int_Q a_0} \neq 1$.

We can see a_0 as a 1-periodic function (still denoted by a_0) in $\mathbb{R}^d$ by setting

$$a_0(x) = a_0(x_1 - \lfloor x_1 \rfloor, \dots, x_d - \lfloor x_d \rfloor) \quad \text{for } x = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

where $\lfloor\cdot\rfloor$ denotes the integer part of a real number. We first consider the case where the pinning term is defined by

$$a_\varepsilon(x) := a_0\Big(\frac{x}{\delta}\Big) \quad \text{with } \delta = \delta_\varepsilon \ll \varepsilon. \tag{1.3}$$

We also consider the case where the pinning term oscillates randomly. Let (Ω, Σ, μ) be a probability space. We assume that $\mathbb{R}^d$ acts on Ω by measurable isomorphisms and we denote this action by T. More precisely, this means that for every $x \in \mathbb{R}^d$, we have an application $T(x) : \Omega \to \Omega$ such that $\mu[T(x)(A)] = \mu(A)$ for every set A in the σ-algebra Σ, and we have that $T(x+y) = T(x) \circ T(y)$ for every x, y in $\mathbb{R}^d$.

We recall that a function $a : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is said to be *stationary* with respect to the action T if $a(\omega, x+y) = a(T(y)\omega, x)$ for every $x, y \in \mathbb{R}^d$ and for almost every $\omega \in \Omega$. A typical example of a stationary process is given by

$$\widetilde{a}_0(\omega, x) = a_1(T(x)\omega) \quad \text{with } a_1 : \Omega \to \mathbb{R} \text{ a measurable function.} \tag{1.4}$$

We also recall that a function $f : \Omega \to \mathbb{R}$ is T-invariant if $f(T(x)\omega) = f(\omega)$ for every $x \in \mathbb{R}^d$ and a.e. $\omega \in \Omega$. The action T is *ergodic* if every function that is invariant with respect to T on Ω is constant almost everywhere on Ω.

When considering a stationary-ergodic pinning term, we will assume that $\widetilde{a}_0$ is given by (1.4) with $a_1 \in L^\infty(\Omega, \mathbb{R})$ which satisfies

$$m < a_1 < M \quad \text{for some } 0 < m < M. \tag{1.5}$$

Without loss of generality, we will assume that

$$\mathbb{E}(a_1) = 1 \quad \text{and} \quad 0 < m < 1 < M,$$

and we will indicate briefly how to adapt the argument to the case $\mathbb{E}(a_1) \neq 1$.

Then, the pinning term will take the form

$$a_\varepsilon(\omega, x) = \widetilde{a}_0\Big(\omega, \frac{x}{\delta}\Big) = a_1\Big(T\Big(\frac{x}{\delta}\Big)\omega\Big) \quad \text{with } \delta = \delta_\varepsilon \ll \varepsilon. \tag{1.6}$$

Given a smooth bounded domain G in $\mathbb{R}^2$, we define the (unpinned) Ginzburg–Landau energy of $(u, A) \in \mathscr{H} := H^1(G, \mathbb{C}) \times H^1(G, \mathbb{R}^2)$ by

$$GL_\varepsilon(u, A) = \frac{1}{2}\int_G |\nabla u - iAu|^2 + \frac{1}{4\varepsilon^2}\int_G (1 - |u|^2)^2 + \frac{1}{2}\int_G |\operatorname{curl} A - h_{\mathrm{ex}}|^2. \tag{1.7}$$

For functions u in $H^1(G, \mathbb{C})$ we also define the pinned energy without magnetic field:

$$E_\varepsilon^{\mathrm{pin}}(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G (a_\varepsilon(x) - |u|^2)^2. \tag{1.8}$$

We define the denoised energy as

$$\widetilde{GL}_\varepsilon^{\mathrm{pin}}(u, A) := GL_\varepsilon^{\mathrm{pin}}(u, A) - \min_{U \in H^1(G)} E_\varepsilon^{\mathrm{pin}}(U). \tag{1.9}$$

We will see in Corollary 2.3 and Definition 2.4 below that $E_\varepsilon^{\text{pin}}$ has a unique positive minimizer U_ε in $H^1(G,\mathbb{R})$ and that U_ε is still a minimizer of $E_\varepsilon^{\text{pin}}$ in $H^1(G,\mathbb{C})$.

Our main result is the following:

Theorem 1.1. *Assume that $\delta = o_\varepsilon(\varepsilon)$ and that a_ε is given by (1.3) (respectively (1.6)). Then, U_ε, the unique positive minimizer of $E_\varepsilon^{\text{pin}}$ in $H^1(G,\mathbb{C})$, satisfies (respectively satisfies almost surely)*

$$\lim_{\varepsilon\to 0} \|U_\varepsilon - 1\|_{L^\infty(G)} = 0.$$

Also, given $(u_\varepsilon, A_\varepsilon)$, we have

$$\widetilde{GL}_\varepsilon^{\text{pin}}(u_\varepsilon, A_\varepsilon) = GL_\varepsilon(v_\varepsilon, A_\varepsilon)(1 + O_\varepsilon(\|U_\varepsilon - 1\|_{L^\infty(G)})), \tag{1.10}$$

where $v_\varepsilon := u_\varepsilon / U_\varepsilon$.

In particular, $(u_\varepsilon, A_\varepsilon)$ is a family of quasi-minimizers of $GL_\varepsilon^{\text{pin}}$ and $\widetilde{GL}_\varepsilon^{\text{pin}}$ in $\mathcal{H}$ if and only if $(v_\varepsilon, A_\varepsilon)$ is a family of quasi-minimizers of the unpinned energy GL_ε in $\mathcal{H}$. This equivalence holds only almost surely in the stationary ergodic case.

By a family (x_ε) of quasi-minimizers for some family of functionals (F_ε) we mean a family which satisfies $F_\varepsilon(x_\varepsilon) = (1 + o_\varepsilon(1)) \inf(F_\varepsilon)$ as $\varepsilon \to 0$.

Theorem 1.1 allows us to describe the behavior of u_ε as $\varepsilon \to 0$, and in particular the behavior of the vortices of u_ε, by using the literature concerning the minimizers of GL_ε. We refer to Theorems 5.1, 5.3, 5.5, and 5.8 for precise statements.

Remark 1.2. If we do not assume that $\mathcal{M} := \sqrt{\int_Q a_0(y)dy} = 1$ in the periodic case, or that $\mathcal{M} := \sqrt{\mathbb{E}(a)} = 1$ in the random case, then Theorem 1.1 has to be modified as follows: with the same assumptions and notations, we have that $(v_\varepsilon, A_\varepsilon)$ is a family of quasi-minimizers of $GL_\varepsilon^{\mathcal{M}}$ (instead of GL_ε), where for $(v, A) \in \mathcal{H}$

$$\begin{aligned}
GL_\varepsilon^{\mathcal{M}}(v, A) &= GL^{\mathcal{M}}_{\varepsilon,h_{\text{ex}},G}(v, A) \\
&:= \frac{\mathcal{M}^2}{2}\int_G |\nabla v - iAv|^2 + \frac{\mathcal{M}^4}{4\varepsilon^2}\int_G (1-|v|^2)^2 + \frac{1}{2}\int_G |\operatorname{curl} A - h_{\text{ex}}|^2 \\
&= \frac{\mathcal{M}^2}{2}\int_{\mathcal{M}\cdot G} |\nabla v' - iA'v'|^2 + \frac{1}{2\varepsilon^2}(1-|v'|^2)^2 + |\operatorname{curl} A' - h_{\text{ex}}/\mathcal{M}^2|^2 \\
&= \mathcal{M}^2 GL_{\varepsilon,h_{\text{ex}}/\mathcal{M}^2,\mathcal{M}\cdot G}(v, A),
\end{aligned}$$

where $A'(\cdot) = A(\cdot/\mathcal{M})/\mathcal{M}$, $v'(\cdot) = v(\cdot/\mathcal{M})$ and expansion (1.10) holds. The study of $GL^{\mathcal{M}}_{\varepsilon,h_{\text{ex}},G}$ reduces to the study of $GL_{\varepsilon,h_{\text{ex}}/\mathcal{M}^2,\mathcal{M}\cdot G}$ in the dilated domain $\mathcal{M}\cdot G$ by a change of variable and a dilation of the unknowns. This is similar to the operations made in the non-dimensionalizing process of the Ginzburg–Landau functional (see, e.g., [25, Section 2.1.1]).

This article is organized as follows: in Section 2 we recall the decomposition lemma from [22] and show how it reduces the study of the problem to the convergence of a minimizer of $E_\varepsilon^{\text{pin}}$ in $H^1(G,\mathbb{R})$, with $E_\varepsilon^{\text{pin}}$ being defined in (1.8). In particular, we show that there exists a unique positive minimizer U_ε of $E_\varepsilon^{\text{pin}}$ in $H^1(G,\mathbb{C})$. In Section 3 we prove the convergence, in the L^∞ norm, of this minimizer to the square root of the average of a_0 in the periodic case and to the square root of the expectation of a_1 in the random case. The proofs of both results make use of a blow-up argument. This convergence is sufficient to prove Theorem 1.1. We also give another proof of the convergence of U_ε, which has the advantage of working in Lipschitz bounded domains and giving explicit rates of convergence and the disadvantage of requiring $\delta = O(\varepsilon^2)$. Then, we use Theorem 1.1 and known results in the literature to describe the behavior of minimizers of $GL_\varepsilon^{\text{pin}}$ in the two-dimensional case in Section 5. We also use analogous results to Theorem 1.1 in three dimensions in Section 6 and also for the Allen–Cahn problem with prescribed mass in Section 7.

2. The decomposition lemma

In the framework of pinned Ginzburg–Landau-type energies, a useful decomposition method is described in [22]. This can be expressed by the following lemma:

Lemma 2.1 (Decomposition lemma). *Let G be a Lipschitz domain of $\mathbb{R}^d$ with $d \geq 1$. Let $p \in L^\infty(G,\mathbb{R}^+)$ and let us assume that $U \in H^1(G,\mathbb{R})$ is a solution of*

$$\begin{cases} -\Delta U = \dfrac{1}{\varepsilon^2} U(p - |U|^2) & \text{in } G, \\ \partial_\nu U = 0 & \text{on } \partial G, \end{cases} \tag{2.1}$$

which satisfies $U \geq m > 0$ in G for some $m > 0$. We consider

$$E_\varepsilon^p(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G (p - |u|^2)^2,$$

and for $d = 2$ we define

$$GL_\varepsilon^p(u,A) = \frac{1}{2}\int_G |\nabla u - iAu|^2 + \frac{1}{4\varepsilon^2}\int_G (p(x) - |u|^2)^2 + \frac{1}{2}\int_G |\operatorname{curl} A - h_{\text{ex}}|^2.$$

Then, for every $d \geq 1$, if we set $u = Uv$, we obtain

$$E_\varepsilon^p(u) = E_\varepsilon^p(U) + \frac{1}{2}\int_G U^2|\nabla v|^2 + \frac{1}{4\varepsilon^2}\int_G U^4(1 - |v|^2)^2. \tag{2.2}$$

For $d = 2$, with $u = Uv$ we find

$$GL_\varepsilon^p(u,A) = E_\varepsilon^p(U) + \frac{1}{2}\int_G U^2|\nabla v - iAv|^2 + \frac{1}{4\varepsilon^2}\int_G U^4(1 - |v|^2)^2 + \frac{1}{2}\int_G |\operatorname{curl} A - h_{\text{ex}}|^2. \tag{2.3}$$

Remark 2.2. In [22] the decomposition lemma was proved in the context of a planar pinned Ginzburg–Landau-type energy without magnetic field (given by (1.8), where a_ε is independent of ε). In this context, the Ginzburg–Landau-type energy studied is minimized under a Dirichlet boundary condition $g \in \mathcal{C}^\infty(\partial G, \mathbb{S}^1)$, with G a smooth bounded domain. The authors of [22] proved the decoupling given in (2.2) with $U_{\mathrm{Dir}} \in H^1(G, \mathbb{R}^+)$ instead of U, where U_{Dir} is the unique minimizer of E_ε^p submitted to the boundary condition $U_{\mathrm{Dir}} = 1$ on ∂G (they also need that $p = 1$ on ∂G in the sense of traces and $p \geq m > 0$ in G). If we look at their proof, we can see that the minimality of U_{Dir} is only used through the validity of the Euler–Lagrange equation $-\Delta U_{\mathrm{Dir}} = \varepsilon^{-2} U_{\mathrm{Dir}}(p - U_{\mathrm{Dir}}^2)$, while the boundary condition $U_{\mathrm{Dir}} = 1$ on ∂G makes the boundary terms vanish since $U_{\mathrm{Dir}} - p = 0$ on ∂G. These boundary terms may also be canceled with a homogeneous Neumann boundary condition as in (2.1). Hence, one may follow (in arbitrary dimension) the argument of [22] to prove (2.2) and also (2.3).

Corollary 2.3. *Let G be a Lipschitz domain of $\mathbb{R}^d$ with $d \geq 1$, $p \in L^\infty(G, \mathbb{R})$ and $\varepsilon > 0$. Assume that $p \geq m > 0$. Then, there exists a unique minimizer U_ε of E_ε^p in $H^1(G, \mathbb{C})$ which is non-negative. It satisfies*

$$\|p\|_{L^\infty(G)} \geq U_\varepsilon^2 \geq m.$$

Any other minimizer is of the form αU_ε for some $\alpha \in \mathbb{S}^1$. Moreover, U_ε is the only positive solution of (2.1).

Proof. Let $\varepsilon > 0$, $p \in L^\infty(G, \mathbb{R})$ be such that $p \geq m$ for some $m > 0$. Let U be a minimizer of E_ε^p in $H^1(G, \mathbb{C})$. It is clear that $E_\varepsilon^p(|U|) \leq E_\varepsilon^p(U)$. Moreover, $V = \max(|U|, \sqrt{m})$ satisfies $E_\varepsilon^p(V) \leq E_\varepsilon^p(|U|)$. Thus, by minimality of $|U|$, $E_\varepsilon^p(V) = E_\varepsilon^p(|U|)$. This implies

$$\int_{\{|U|<\sqrt{m}\}} |\nabla |U||^2 = \frac{1}{4\varepsilon^2} \int_{\{|U|<\sqrt{m}\}} ((p - m^2)^2 - (p - |U|^2)^2) \leq 0,$$

since in $\{|U| < \sqrt{m}\}$ we have $p - |U|^2 > p - m \geq 0$. This implies that $\{|U| < m\} = \emptyset$. By considering $V' = \min(|U|, \sqrt{\|p\|_{L^\infty(G)}})$, we find $|U| \leq \sqrt{\|p\|_{L^\infty(G)}}$.

Assume U' is another minimizer of E_ε^p in $H^1(G, \mathbb{C})$. Then, by definition $E_\varepsilon^p(U') = E_\varepsilon^p(U)$, but from Lemma 2.1, letting $v = U'/U$ we have

$$0 = E_\varepsilon^p(U') - E_\varepsilon^p(U) = \frac{1}{2} \int_G |U|^2 |\nabla v|^2 + \frac{|U|^4}{2\varepsilon^2}(1 - |v|^2)^2.$$

Hence, v is a constant and $v \in \mathbb{S}^1$.

To prove the last statement, we consider V a positive solution of (2.1); by the minimizing property of U, we can say that $E_\varepsilon^p(U) - E_\varepsilon^p(V) \leq 0$. There exists $m' > 0$ such that $V \geq m'$. Using Lemma 2.1 again, letting $v = U/V$ we have

$$0 \geq E_\varepsilon^p(U) - E_\varepsilon^p(V) = \frac{1}{2} \int_G V^2 |\nabla v|^2 + \frac{V^4}{2\varepsilon^2}(1 - |v|^2)^2.$$

Since $v \geq 0$ and $V \geq m'$ on $\overline{G}$, we deduce $v = 1$, that is, $V = |U| = U$. ∎

From Corollary 2.3, one may state the following definition:

Definition 2.4. Let $G \subset \mathbb{R}^d$ be a Lipschitz bounded domain. For $p \in L^\infty(G,\mathbb{R})$ such that $p \geq m$ for some $m > 0$ and for $\varepsilon > 0$, we let U_ε^p be the unique positive minimizer of E_ε^p in $H^1(G,\mathbb{C})$. It satisfies $m \leq |U_\varepsilon^p|^2 \leq \|p\|_{L^\infty}$ in G. Moreover, if G is a $\mathscr{C}^2$ bounded domain, then by elliptic regularity, $U_\varepsilon^p \in W^{2,q}(G,\mathbb{R})$ for every $1 \leq q < +\infty$. If $p = a_\varepsilon$ is given by (1.3) or (1.6), we simply write U_ε.

Next, we give a Lipschitz estimate on U_ε^p in the case where G is a $\mathscr{C}^1$ domain. The proof follows the argument of [9, Lemma A.2].

Lemma 2.5. *Let us assume that G is a $\mathscr{C}^1$ bounded domain. Let $\varepsilon > 0$ and let $p \in L^\infty(G,\mathbb{R})$ be such that $p \geq m$ for some $m > 0$. Let $U_\varepsilon^p \in H^1(G,\mathbb{R})$ be the unique positive minimizer of E_ε^p in $H^1(G,\mathbb{C})$ from the previous definition. There is $C > 1$ depending only on G and $\|p\|_{L^\infty}$ such that*

$$\|\nabla U_\varepsilon^p\|_{L^\infty(G)} \leq \frac{C}{\varepsilon}. \tag{2.4}$$

Proof. Following step by step the proof of [9, Lemma A.2], we obtain that if $u \in H^1(G,\mathbb{C})$ and $f \in L^\infty(G,\mathbb{C})$ with $\int_G f(x)\mathrm{d}x = 0$ satisfy

$$\begin{cases} -\Delta u = f & \text{in } G, \\ \partial_\nu u = 0 & \text{on } \partial G, \end{cases}$$

then there exists $C > 0$ which depends only on G such that

$$\|\nabla u\|_{L^\infty(G)} \leq C\|f\|_{L^\infty(G)}\|u\|_{L^\infty(G)}.$$

The only modification with respect to the proof of [9, Lemma A.2] is the use of the following Neumann elliptic estimate in place of its Dirichlet counterpart: let $A \in \mathscr{C}^1(B_1^+, \mathscr{M}_d(\mathbb{R}))$, which is bounded and uniformly elliptic, with $B_1^+ = \{x \in B_1(0) \mid x_d > 0\}$; $g \in L^\infty(B_1^+,\mathbb{R})$; and $v \in H^1(B_1^+,\mathbb{R})$ satisfy

$$\begin{cases} -\operatorname{div}(A(x)\nabla v) = g & \text{in } B_1^+, \\ \partial_\nu v = 0 & \text{on } B_1 \cap \{x \in B_1^+ \mid x_d = 0\}; \end{cases}$$

then $\|\nabla v\|_{L^\infty(B_{1/2}^+)} \leq C(\|g\|_{L^\infty(B_1^+)} + \|v\|_{L^\infty(B_1^+)})$ for some $C > 0$ depending on the ellipticity constant of A and on $\|A\|_{\mathscr{C}^1(B_1^+)}$. ∎

3. Convergence of the free minimizer

3.1. The periodic case

Let $E_\varepsilon^{\text{pin}}$ be defined by (1.8) with a_ε being defined by (1.3).

Theorem 3.1. *Let $G \subset \mathbb{R}^d$ be a $\mathcal{C}^1$ bounded domain. Let U_ε be the minimizer of E_ε in $H^1(G, \mathbb{C})$ given by Definition 2.4. Then,*

$$\lim_{\varepsilon \to 0} \|U_\varepsilon - \mathcal{M}\|_{L^\infty(G)} = 0. \tag{3.1}$$

We recall that, for simplicity, we assumed that $\mathcal{M} = \sqrt{\fint_Q a_0(x)\mathrm{d}x} = 1$.

Proof. By contradiction, we assume that (3.1) is not true. Then, there exist $\eta > 0$ and a sequence of points $(x_\varepsilon)_{\varepsilon>0}$ such that $|U_\varepsilon(x_\varepsilon) - 1| \geq \eta$ for all $\varepsilon > 0$ small enough.

We first assume that $\rho_\varepsilon := \operatorname{dist}(x_\varepsilon, \partial G) \gg \varepsilon$. We then consider the blow-up function $V_\varepsilon(y) = U_\varepsilon(x_\varepsilon + \varepsilon y)$ defined for $y \in B(0, \rho_\varepsilon/\varepsilon)$. This function satisfies

$$-\Delta V_\varepsilon = V_\varepsilon(b_\varepsilon - V_\varepsilon^2) \quad \text{in } B(0, \rho_\varepsilon/\varepsilon), \tag{3.2}$$

with $b_\varepsilon(y) := a_\varepsilon(x_\varepsilon + \varepsilon y) = a_0(\frac{x_\varepsilon + \varepsilon y}{\delta})$ for $y \in B(0, \rho_\varepsilon/\varepsilon)$.

Claim: After extraction, the functions $\{b_\varepsilon\}_\varepsilon$ converge to 1 as $\varepsilon \to 0$ in the L^∞-weak star topology. Indeed, since $\{b_\varepsilon\}_\varepsilon$ is bounded in L^∞, it converges after extraction to some b_0 in the L^∞-weak star topology. But, since the function b_ε is periodic with period δ/ε tending to 0 as $\varepsilon \to 0$, the function b_0 is constant and equal to the average of b_ε over a period, that is, $b_0 = \fint_Q a_0 = 1$. This can be seen by observing that $\int_{\mathbb{R}^d} b_\varepsilon(x)\mathbf{1}_D(x)\mathrm{d}x \to \fint_A a_0(x)\mathrm{d}x|D|$, where D is any measurable set in $\mathbb{R}^d$. This latter fact can be proved by dividing D into small cubes of size ε and using the periodicity of b_ε as in the proof of Proposition 3.3 below.

Therefore, after extraction and for any $\{f_\varepsilon\}_\varepsilon$ converging strongly to f in L^1, and supported in a fixed compact set, we have

$$\int_{\mathbb{R}^d} f_\varepsilon b_\varepsilon \to \int_{\mathbb{R}^d} f. \tag{3.3}$$

Besides, $m \leq V_\varepsilon \leq \|U_\varepsilon\|_{L^\infty(G)} \leq M$ and, by Lipschitz estimate (2.4), we have that V_ε satisfies $\|\nabla V_\varepsilon\|_{L^\infty(B(0,\rho_\varepsilon/\varepsilon))} \leq C$. This implies by the Arzelà–Ascoli theorem that, up to passing to a subsequence, $V_\varepsilon \to V_0$ locally uniformly in $\mathbb{R}^d$ for some continuous $V_0 : \mathbb{R}^d \to [m, M]$. It then follows from (3.3), choosing $f_\varepsilon = V_\varepsilon \varphi$ for any $\mathcal{C}^\infty$ compactly supported φ, that $V_\varepsilon b_\varepsilon \rightharpoonup V_0$ in $\mathcal{D}'(\mathbb{R}^d)$. On the other hand, $V_\varepsilon^3 \to V_0^3$ locally uniformly in $\mathbb{R}^d$, and hence in the sense of distributions as well.

Passing to the limit in the sense of distributions in (3.2), we find that the limit V_0 satisfies

$$-\Delta V_0 = V_0(1 - V_0^2) \quad \text{in } \mathbb{R}^d. \tag{3.4}$$

But, since $m \leq V_0 \leq M$, by using [18, Theorem 2.1] we conclude that $V_0 \equiv 1$. Thus, $V_\varepsilon(0) = U_\varepsilon(x_\varepsilon) \to 1$, which is a contradiction.

Now we assume that, up to passing to a subsequence, $\operatorname{dist}(x_\varepsilon, \partial G) = O(\varepsilon)$. Thus, we may define $y_\varepsilon := \Pi_{\partial G}(x_\varepsilon)$, the orthogonal projection of x_ε on ∂G. We then have

$|x_\varepsilon - y_\varepsilon| = O(\varepsilon)$. Up to passing to a further subsequence, we may assume that $y_\varepsilon \to y_0 \in \partial G$. We let $\rho_\varepsilon := 2\max(|y_\varepsilon - y_0|, \sqrt{\varepsilon})$ and we set $V_\varepsilon(y) := U_\varepsilon(y_\varepsilon + \varepsilon y)$ for

$$y \in B_\varepsilon^+ = \frac{B(y_0 - y_\varepsilon, \rho_\varepsilon) \cap (G - y_\varepsilon)}{\varepsilon}.$$

By using that G is $\mathscr{C}^1$, up to passing to a subsequence and up to considering a vectorial rotation, we may assume that for all $x \in \mathbb{R}^d_+$ there exists $\varepsilon_0 > 0$ such that $x \in B_\varepsilon^+$ for all $\varepsilon < \varepsilon_0$. Then, as in the first case, we can obtain the existence of $V_0 : \mathbb{R}^d_+ \to \mathbb{R}$ such that, up to a subsequence, $V_\varepsilon \to V_0$ locally uniformly in $\mathbb{R}^d_+$. Passing to the limit in (3.2), we find that V_0 satisfies

$$\begin{cases} -\Delta V_0 = V_0(1 - V_0^2) & \text{in } \mathbb{R}^d_+, \\ \partial_\nu V_0 = 0 & \text{on } \partial\mathbb{R}^d_+. \end{cases} \tag{3.5}$$

We can consider a new function defined for $y = (y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ by

$$\widetilde{V}_0(y) = \begin{cases} V_0(y) & \text{if } y_d \geq 0, \\ V_0(y', -y_d) & \text{if } y_d < 0. \end{cases}$$

We can check that $\widetilde{V}_0$ satisfies $-\Delta \widetilde{V}_0 = \widetilde{V}_0(1 - \widetilde{V}_0^2)$ in $\mathbb{R}^d$ and we conclude as before that $\widetilde{V}_0 \equiv 1$. On the other hand, since $|x_\varepsilon - y_\varepsilon| = O(\varepsilon)$, up to passing to a subsequence, there exists $y_\star$ in $\mathbb{R}^d_+$ such that $y_\star := \lim_{\varepsilon\to 0} \dfrac{x_\varepsilon - y_\varepsilon}{\varepsilon}$ and $|\widetilde{V}_0(y_\star) - 1| = \lim_{\varepsilon \to 0} |V_\varepsilon(y_\varepsilon) - 1| \geq \eta > 0$. This is a contradiction and this concludes the proof of the theorem. ∎

Remark 3.2. We recall that, for simplicity, we assumed that $\mathcal{M} = \sqrt{\frac{1}{|Q|}\int_Q a_0(x)\mathrm{d}x} = 1$. If $\mathcal{M} \neq 1$, then we change the definitions of V_ε and b_ε by letting

$$V_\varepsilon(y) = \begin{cases} U_\varepsilon(x_\varepsilon + \varepsilon y/\mathcal{M}^2)/\mathcal{M} & \text{if } \operatorname{dist}(x_\varepsilon, \partial\Omega) \gg \varepsilon, \\ U_\varepsilon(y_\varepsilon + \varepsilon y/\mathcal{M}^2)/\mathcal{M} & \text{if } \operatorname{dist}(x_\varepsilon, \partial\Omega) = \mathcal{O}(\varepsilon) \end{cases}$$

and

$$b_\varepsilon(y) = \begin{cases} a_\varepsilon(x_\varepsilon + \varepsilon y/\mathcal{M}^2)/\mathcal{M}^2 & \text{if } \operatorname{dist}(x_\varepsilon, \partial\Omega) \gg \varepsilon, \\ a_\varepsilon(y_\varepsilon + \varepsilon y/\mathcal{M}^2)/\mathcal{M}^2 & \text{if } \operatorname{dist}(x_\varepsilon, \partial\Omega) = \mathcal{O}(\varepsilon). \end{cases}$$

Convergence (3.3) reads as $b_\varepsilon \rightharpoonup 1$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and thus (3.4) and (3.5) still hold. The rest of the proof is unchanged.

We can now give a proof of Theorem 1.1 in the periodic case.

Proof of Theorem 1.1 in the periodic case. Let $(u, A) \in \mathcal{H}$ and $\varepsilon > 0$. Letting $v = u/U_\varepsilon$, with (2.2) we have

$$\begin{aligned} \widetilde{GL}_\varepsilon^{\mathrm{pin}}(u, A) &= \frac{1}{2}\int_G U_\varepsilon^2 |\nabla v - iAv|^2 + \frac{1}{4\varepsilon^2}\int_G U_\varepsilon^4 (1 - |v|^2)^2 + \frac{1}{2}\int_G |\operatorname{curl} A - h_{\mathrm{ex}}|^2 \\ &=: GL_\varepsilon^{\mathrm{weight}}(v, A). \end{aligned}$$

By using that $0 < m \le 1 \le M$, we obtain:

$$m^4 \times GL_\varepsilon(v, A) \le GL_\varepsilon^{\text{weight}}(v, A) \le M^4 \times GL_\varepsilon(v, A).$$

Therefore,

$$m^4 \times \inf_{\mathcal{H}} GL_\varepsilon \le \inf_{\mathcal{H}} GL_\varepsilon^{\text{weight}} \le M^4 \times \inf_{\mathcal{H}} GL_\varepsilon.$$

Moreover, if $(v, A) \in \mathcal{H}$ is such that $GL_\varepsilon(v, A) \le 2M^4 \times \inf_{\mathcal{H}} GL_\varepsilon$, then

$$|GL_\varepsilon^{\text{weight}}(v, A) - GL_\varepsilon(v, A)| \le 2M^4 \times \max(\|U_\varepsilon^2 - 1\|_{L^\infty}, \|U_\varepsilon^4 - 1\|_{L^\infty}) \inf_{\mathcal{H}} GL_\varepsilon. \quad (3.6)$$

In particular, with Theorem 3.1, we get

$$\inf_{\mathcal{H}} GL_\varepsilon = (1 + o_\varepsilon(1)) \inf_{\mathcal{H}} GL_\varepsilon^{\text{weight}} \quad (3.7)$$

when $\varepsilon \to 0$. Now let $\varepsilon \to 0$, let $(u_\varepsilon, A_\varepsilon) \in \mathcal{H}$ be a family of configuration, and write $v_\varepsilon = u_\varepsilon / U_\varepsilon$. We have

$$\begin{aligned} & (u_\varepsilon, A_\varepsilon) \text{ is a family of quasi-minimizers of } \widetilde{GL}_\varepsilon^{\text{pin}} \\ \iff & \widetilde{GL}_\varepsilon^{\text{pin}}(u_\varepsilon, A_\varepsilon) = (1 + o_\varepsilon(1)) \times \inf_{\mathcal{H}} \widetilde{GL}_\varepsilon^{\text{pin}} \\ \iff & GL_\varepsilon^{\text{weight}}(v_\varepsilon, A_\varepsilon) = (1 + o_\varepsilon(1)) \times \inf_{\mathcal{H}} GL_\varepsilon^{\text{weight}} \\ \underbrace{\iff}_{\text{with (3.6)\&(3.7)}} & GL_\varepsilon(v_\varepsilon, A_\varepsilon) = (1 + o_\varepsilon(1)) \times \inf_{\mathcal{H}} GL_\varepsilon \\ \iff & (v_\varepsilon, A_\varepsilon) \text{ is a family of quasi-minimizers of } GL_\varepsilon. \end{aligned}$$

■

3.2. Rate of convergence in special cases

Although Theorem 3.1 has the advantage of working in every dimension and its proof can be extended to a random stationary ergodic pinning term, it has the disadvantage of requiring $\mathcal{C}^1$ regularity for ∂G and of not giving a rate of convergence, which could be useful in some regime of applied magnetic fields h_{ex}.

In what follows, we give rates of convergence under additional assumptions. We work in dimension 2 and the assumptions take two forms: either an assumption on $\delta = \delta(\varepsilon)$ or a symmetry assumption on a_0.

Proposition 3.3. *Let $G \subset \mathbb{R}^2$ be a Lipschitz bounded domain. Let U_ε be the minimizer of $E_\varepsilon^{\text{pin}}$ given by Definition 2.4. There exists $C > 0$ (independent of δ and ε) such that*

$$\|U_\varepsilon - 1\|_{L^2(G)} \le C\Big(\frac{\delta}{\varepsilon} + \sqrt{\delta}\Big). \quad (3.8)$$

If we assume furthermore that

$$\delta = O_\varepsilon(\varepsilon^2), \quad (3.9)$$

then

$$\|U_\varepsilon - 1\|_{L^\infty(G)} = O_\varepsilon\Big[\Big(\frac{\delta}{\varepsilon^2}\Big)^{1/4}\Big]. \quad (3.10)$$

Note that if we assume that $\delta = O_\varepsilon(\varepsilon^2)$, then (3.8) becomes $\|U_\varepsilon - 1\|_{L^2(G)} = O_\varepsilon(\sqrt{\delta})$. Before proving Proposition 3.3, we present an estimate which may be obtained with a weaker assumption.

Remark 3.4. We may get an explicit speed of convergence with assumptions weaker than $\delta = O_\varepsilon(\varepsilon^2)$. Specifically, let $\chi := \delta/\varepsilon$ and consider the assumption

$$\chi^{1/8} \times \mathrm{e}^{\chi^{1/8}|\ln\delta|} \to 0. \tag{3.11}$$

It is clear that $\delta = O_\varepsilon(\varepsilon^2)$ (which reads as $\chi = O_\varepsilon(\varepsilon)$) implies (3.11). One may prove that if (3.11) holds we have

$$\|U_\varepsilon - 1\|_{L^\infty(G)} < 4\chi^{1/8} = 4\Big(\frac{\delta}{\varepsilon^2}\Big)^{1/8}. \tag{3.12}$$

The proof of (3.12) is quite long and thus it is omitted here.

Proof of Proposition 3.3. We let $\chi := \frac{\delta}{\varepsilon}$. For $0 < \varepsilon < 1$, we consider the energy

$$\widehat{E}_\varepsilon(u) = \frac{1}{2}\int_Q |\nabla u|^2 + \frac{\chi^2}{4}\int_Q (a_0 - |u|^2)^2 \tag{3.13}$$

defined for $u \in H^1(Q, \mathbb{C})$. Recall that $Q = (0,1)^2$ is the unit square. We write $H^1(Q) = H^1(Q, \mathbb{C})$.

From Corollary 2.3, there exists a unique positive minimizer $\widehat{U}_\varepsilon$ of $\widehat{E}_\varepsilon$ in $H^1(Q)$ and $m \le \widehat{U}_\varepsilon \le M$. This minimizer satisfies

$$\begin{cases} -\Delta\widehat{U}_\varepsilon = \chi^2\widehat{U}_\varepsilon(a_0 - \widehat{U}_\varepsilon^2) & \text{in } Q, \\ \partial_\nu\widehat{U}_\varepsilon = 0 & \text{on } \partial Q. \end{cases} \tag{3.14}$$

We set $\ell_\varepsilon = \ell := \int_Q \widehat{U}_\varepsilon$. By using the homogeneous Neumann boundary condition to extend $\widehat{U}_\varepsilon$ in the square $(-1,2)^2$, we can use interior elliptic estimates in Q and obtain that, for all $2 \le p < +\infty$,

$$\|\widehat{U}_\varepsilon - \ell\|_{W^{1,p}(Q)} = O_\varepsilon(\chi^2). \tag{3.15}$$

By multiplying (3.14) by $\widehat{U}_\varepsilon$, integrating by parts, and using (3.15), we find

$$\chi^2\int_Q \widehat{U}_\varepsilon^2(a_0 - \widehat{U}_\varepsilon^2) = \int_Q |\nabla\widehat{U}_\varepsilon|^2 = O_\varepsilon(\chi^4). \tag{3.16}$$

We infer that

$$\int_Q (\ell^2 + O_\varepsilon(\chi^2))(a_0 - \ell^2 + O(\chi^2)) = O_\varepsilon(\chi^2).$$

Since $\ell \ge m > 0$, we obtain that $\ell^2 = \int_Q a_0(x)\mathrm{d}x + O_\varepsilon(\chi^2) = 1 + O_\varepsilon(\chi^2)$. Thus, we can reformulate (3.15) as: for all $2 \le p < +\infty$,

$$\|\widehat{U}_\varepsilon - 1\|_{W^{1,p}(Q)} = O_\varepsilon(\chi^2). \tag{3.17}$$

We thus obtain from (3.16) that

$$\widehat{E}_\varepsilon(\widehat{U}_\varepsilon, Q) = \widehat{E}_\varepsilon(1, Q) + O_\varepsilon(\chi^4). \tag{3.18}$$

Now let U_ε be the positive minimizer of E_ε in $H^1(G, \mathbb{C})$ given by Definition 2.4. For $k, l \in \mathbb{Z}$ such that $Q_{k,l} := \delta(k, l) + \delta Q \subset G$, we define $\widetilde{U}_\varepsilon^{k,l} : Q \to \mathbb{R}$ by $\widetilde{U}_\varepsilon^{k,l}(y) = U_\varepsilon(\delta(y + (k, l)))$. By minimality of $\widehat{U}_\varepsilon$ in $H^1(Q)$, we have

$$\widehat{E}_\varepsilon(\widetilde{U}_\varepsilon^{k,l}, Q) \geq \widehat{E}_\varepsilon(\widehat{U}_\varepsilon, Q) = \widehat{E}_\varepsilon(1, Q) + O_\varepsilon(\chi^4). \tag{3.19}$$

We can then decompose G in cells $Q_{k,l}$. We denote by N_δ the number of cells $Q_{k,l}$ included in G. We have

$$N_\delta = \frac{|G|}{\delta^2} + O_\varepsilon\Big(\frac{1}{\delta}\Big).$$

We also denote $G_\delta := G \setminus \bigcup_{Q_{k,l} \subset G} Q_{k,l}$ and we can see that $|G_\delta| = O_\varepsilon(\delta)$. We have

$$\begin{aligned}
E_\varepsilon^{\text{pin}}(U_\varepsilon, G) &\geq \sum_{Q_{k,l} \subset G} E_\varepsilon^{\text{pin}}(U_\varepsilon, Q_{k,l}) \geq \sum_{Q_{k,l} \subset G} \widehat{E}_\varepsilon(\widetilde{U}_\varepsilon^{k,l}, Q) \\
&\geq \sum_{Q_{k,l} \subset G} \Big(\widehat{E}_\varepsilon(1, Q) + O_\varepsilon\Big(\frac{\delta^4}{\varepsilon^4}\Big)\Big) \\
&= O_\varepsilon\Big(\frac{\delta^2}{\varepsilon^4}\Big) + \sum_{Q_{k,l} \subset G} E_\varepsilon^{\text{pin}}(1, Q_{k,l}).
\end{aligned}$$

But, we observe that

$$\sum_{Q_{k,l} \cap G_\delta \neq \emptyset} E_\varepsilon^{\text{pin}}(1, Q_{k,l}) = O_\varepsilon\Big(\frac{\delta}{\varepsilon^2}\Big).$$

Hence, we obtain

$$\begin{aligned}
E_\varepsilon^{\text{pin}}(U_\varepsilon, G) &\geq E_\varepsilon^{\text{pin}}(1, G) - \sum_{Q_{k,l} \cap G_\delta \neq \emptyset} E_\varepsilon^{\text{pin}}(1, Q_{k,l}) + O_\varepsilon\Big(\frac{\delta^2}{\varepsilon^4}\Big) \\
&\geq E_\varepsilon^{\text{pin}}(1, G) + O_\varepsilon\Big(\frac{\delta^2}{\varepsilon^4} + \frac{\delta}{\varepsilon^2}\Big).
\end{aligned}$$

Thus, since $E_\varepsilon^{\text{pin}}(1, G) \geq E_\varepsilon^{\text{pin}}(U_\varepsilon, G)$, we find that

$$E_\varepsilon^{\text{pin}}(1) - E_\varepsilon^{\text{pin}}(U_\varepsilon) = O_\varepsilon\Big(\frac{\delta^2}{\varepsilon^4} + \frac{\delta}{\varepsilon^2}\Big). \tag{3.20}$$

Now we use Lemma 2.1, writing $1 = U_\varepsilon v$, to get

$$E_\varepsilon^{\text{pin}}(1) = E_\varepsilon^{\text{pin}}(U_\varepsilon) + \frac{1}{2}\int_G U_\varepsilon^2 |\nabla v|^2 + \frac{1}{4\varepsilon^2}\int_G U_\varepsilon^4 (1 - v^2)^2. \tag{3.21}$$

We deduce from (3.21), (3.20), and the fact that $U_\varepsilon \geq m > 0$ that

$$\frac{1}{2}\int_G |\nabla v|^2 + \frac{1}{4\varepsilon^2}\int_G (1-v^2)^2 = O_\varepsilon\Big(\frac{\delta^2}{\varepsilon^4} + \frac{\delta}{\varepsilon^2}\Big). \tag{3.22}$$

Hence, we find that $\int_G (1-v^2)^2 = O_\varepsilon(\frac{\delta^2}{\varepsilon^2} + \delta)$, which implies (3.8). Now we assume that $\delta = O_\varepsilon(\varepsilon^2)$. This implies $\|U_\varepsilon - 1\|_{L^2(G)} = O_\varepsilon(\sqrt{\delta})$. To prove the L^∞ estimate, we argue by contradiction. We assume that there exist two sequences $\varepsilon_n \to 0$ and $(x_n)_n \subset G$ such that

$$1 - v(x_n)^2 \geq (n+1)\Big(\frac{\delta_n}{\varepsilon_n^2}\Big)^{1/4}. \tag{3.23}$$

We set $\rho_n := \delta_n/\varepsilon_n^2$. Since $\|\nabla v\|_{L^\infty(G)} \leq \frac{M}{m}\|\nabla U_\varepsilon\|_{L^\infty(G)} = O_\varepsilon(\frac{1}{\varepsilon})$, we find that there exists $c > 0$ independent of ε such that

$$1 - v^2(x) \geq n\rho_n^{1/4} \quad \text{for every } x \in B(x_n, c\varepsilon_n\rho_n^{1/4}) \cap G. \tag{3.24}$$

By Lipschitz regularity of G, we can assume that $c > 0$ is small enough (independent of ε_n) so that $|B(x_n, c\varepsilon_n\rho_n^{1/4}) \cap G| \geq c^3\varepsilon_n^2\rho_n^{1/2}$. We then have

$$\int_{B(x_n, c\varepsilon_n\rho_n^{1/4})\cap G} (1-v^2)^2 \geq c^3 n^2 \varepsilon_n^2 \rho_n.$$

By using that $\int_G (1-v^2)^2 = O_\varepsilon(\delta)$ (from (3.22) and (3.9)), we arrive at $\delta_n \geq cn^2\varepsilon_n^2(\frac{\delta_n}{\varepsilon_n^2})$ $= cn^2\delta_n$ for some $c > 0$ sufficiently small (independent of ε) and for all $n \in \mathbb{N}$. This is a contradiction; and then we find that $\|1 - v\|_{L^\infty(G)} = O_{\varepsilon_n}((\frac{\delta_n}{\varepsilon_n^2})^{1/4})$, which implies the second part of (3.10). ∎

In some cases we can improve the L^∞ bound obtained in the previous proposition. For example, we make the following symmetry assumption on $a_0 : Q \to \mathbb{R}$:

$$\begin{cases} a_0(\frac{1}{2} - x_1, x_2) = a_0(x_1, x_2), & \forall\,(x_1, x_2) \in (0, \frac{1}{2}) \times (0,1), \\ a_0(x_1, \frac{1}{2} - x_2) = a_0(x_1 x_2), & \forall\,(x_1, x_2) \in (0,1) \times (0, \frac{1}{2}). \end{cases} \tag{3.25}$$

Proposition 3.5. *Assume that G is a square in $\mathbb{R}^2$ of size L. Let $\delta_n := \frac{L}{n} \to 0$ and $\varepsilon_n \to 0$ be such that $\delta_n = o_{\varepsilon_n}(\varepsilon_n)$. Let a_{ε_n} be defined by (1.3) on a $\delta_n \times \delta_n$ grid matching with G and assume that (3.25) holds.*

Let U_{ε_n} be the positive minimizer of $E^{\text{pin}}_{\varepsilon_n}$ given by Definition 2.4. Then, there exists $C > 0$ such that

$$\|U_{\varepsilon_n} - 1\|_{L^\infty(G)} \leq C\frac{\delta_n^2}{\varepsilon_n^2}. \tag{3.26}$$

Remark 3.6. Proposition 3.5 is still valid for a polygonal domain G such that G matches with the union of cells of $\delta_n \times \delta_n$ grids with $\delta_n \to 0$.

Proof of Proposition 3.5. We drop the subscript n for simplicity. We decompose the domain G in small regular cells of size δ which we denote by $Q_{k,l}$ for $k,l \in \mathbb{Z}$. Let $\widehat{U}_\varepsilon$ be the positive minimizer of $\widehat{E}_\varepsilon(u) = \frac{1}{2}\int_Q |\nabla u|^2 + \frac{\delta^2}{4\varepsilon^2}\int_Q (a_0(x) - |u|^2)^2$ in $H^1(Q)$. Note that $\widehat{U}_\varepsilon$ satisfies (3.14). We claim that

$$\mathrm{tr}_{|\{0\}\times(0,1)}\,\widehat{U}_\varepsilon = \mathrm{tr}_{|\{1\}\times(0,1)}\,\widehat{U}_\varepsilon \quad \text{and} \quad \mathrm{tr}_{|(0,1)\times\{0\}}\,\widehat{U}_\varepsilon = \mathrm{tr}_{|(0,1)\times\{1\}}\,\widehat{U}_\varepsilon. \tag{3.27}$$

Indeed, we can check that, thanks to the symmetry assumption on a_0,

$$U_\varepsilon^{(1)} : (x_1, x_2) \mapsto \widehat{U}_\varepsilon\Big(\frac{1}{2} - x_1, x_2\Big) \quad \text{and} \quad U_\varepsilon^{(2)} : (x_1, x_2) \mapsto \widehat{U}_\varepsilon\Big(x_1, \frac{1}{2} - x_2\Big)$$

satisfy the same equation as $\widehat{U}_\varepsilon$ in Q with the same boundary condition. By the uniqueness result given in Corollary 2.3, we obtain $\widehat{U}_\varepsilon^{(1)} = \widehat{U}_\varepsilon^{(2)} = \widehat{U}_\varepsilon$ and hence the equality of the traces on opposite faces of the square Q.

Now we set

$$U_\varepsilon(x) = \widehat{U}_\varepsilon(\tilde{x}_1, \tilde{x}_2) \tag{3.28}$$

if $x \in G$ can be written as $x = (k\delta + \tilde{x}_1\delta, l\delta + \tilde{x}_2\delta)$ for $(\tilde{x}_1, \tilde{x}_2) \in Q$. Thanks to the homogeneous Neumann boundary condition satisfied by $\widehat{U}_\varepsilon$ on Q and because the traces of $\widehat{U}_\varepsilon$ are equal on opposite faces, we can prove that U_ε satisfies

$$\begin{cases} -\Delta U_\varepsilon = \dfrac{U_\varepsilon}{\varepsilon^2}(a_0(x/\delta) - U_\varepsilon^2) & \text{in } G, \\ \partial_\nu U_\varepsilon = 0 & \text{on } \partial G. \end{cases} \tag{3.29}$$

We can then apply the uniqueness result of Corollary 2.3 to obtain that U_ε is the positive minimizer of $E_\varepsilon^{\mathrm{pin}}$ in $H^1(G)$. We then obtain that

$$\|U_\varepsilon - 1\|_{L^\infty(G)} = \|\widehat{U}_\varepsilon - 1\|_{L^\infty(Q)}, \quad \|\nabla U_\varepsilon\|_{L^\infty(G)} = \delta\|\nabla \widehat{U}_\varepsilon\|_{L^\infty(Q)}.$$

The conclusion follows from the bound on the L^∞ norm of $\widehat{U}_\varepsilon$ and of its gradient which satisfies (3.14). Note that the estimate on $\nabla\widehat{U}_\varepsilon$ can be obtained as an interior estimate after extending $\widehat{U}_\varepsilon$ in a bigger square thanks to the homogeneous Neumann boundary condition. ∎

3.3. The stationary ergodic case

In this section we consider the case of a random stationary ergodic pinning term. More precisely, we assume that a_ε is given by (1.6). We will use the Birkhoff ergodic theorem:

Theorem 3.7 ([21, Theorem 7.2] and [17, Section VIII.7]). *Let (Ω, Σ, μ) be a probability space and $T = (T(x))_{x\in\mathbb{R}^d}$ be an action of $\mathbb{R}^d$ on Ω by measurable isomorphisms. Assume that $a_1 \in L^p(\Omega)$ for some $1 \le p < +\infty$. Then, for a.e. $\omega \in \Omega$, the function $a_1(T(\frac{\cdot}{\eta})\omega) : \mathbb{R}^d \to \mathbb{R}$ weakly converges in $L^p(\mathbb{R}^d)$ when $\eta \to 0$. We denote by $\mathcal{N}(a_1(T(x)\omega))$ its weak limit in $L^p(\mathbb{R}^d)$. Then, as a function of ω, $\mathcal{N}(a_1(T(x)\omega))$ is* *invariant under T and we have*

$$\int_{\Omega} \mathcal{N}(a_1(T(x)\omega))\mathrm{d}\mu = \mathbb{E}(a_1).$$

Besides, if T is ergodic, then $\mathcal{N}(a_1(T(x)\omega)) = \mathbb{E}(a_1)$ for a.e. $\omega \in \Omega$.

From this theorem we obtain, writing $\mathcal{M} := \sqrt{\mathbb{E}(a_1)}$, the following result:

Theorem 3.8. *Let G be a bounded $\mathcal{C}^1$ domain of $\mathbb{R}^d$. Let U_ε be the minimizer of $E_\varepsilon^{\mathrm{pin}}$ in $H^1(G, \mathbb{C})$ given by Definition 2.4, where a_ε is defined by (1.6). Then,*

$$\lim_{\varepsilon \to 0} \|U_\varepsilon - \mathcal{M}\|_{L^\infty(G)} = 0 \quad \textit{for a.e. } \omega \in \Omega. \tag{3.30}$$

Proof. Recall that, without loss of generality, we can assume that $\mathbb{E}(a_1) = 1$. By contradiction, we assume that (3.30) is not true. Then, there exists a set $O \subset \Omega$ with $\mu(O) > 0$ such that for every $\omega \in O$, there exist $\eta^\omega > 0$ and a sequence of points $(x_\varepsilon^\omega)_{\varepsilon>0} = (x_\varepsilon)_{\varepsilon>0}$ such that $|U_\varepsilon(x_\varepsilon, \omega) - 1| \geq \eta$ for all $\varepsilon > 0$ small enough. In what follows, we fix ω and drop the subscript ω. We first assume that $\rho_\varepsilon := \mathrm{dist}(x_\varepsilon, \partial G) \gg \varepsilon$. We then consider the blow-up function $V_\varepsilon(y, \omega) = U_\varepsilon(x_\varepsilon + \varepsilon y, \omega)$ defined for $y \in B(0, \rho_\varepsilon/\varepsilon) \subset G$. This function satisfies

$$-\Delta V_\varepsilon = V_\varepsilon(b_\varepsilon - V_\varepsilon^2) \quad \text{in } B(0, \rho_\varepsilon/\varepsilon), \tag{3.31}$$

with $b_\varepsilon(y) := a_\varepsilon(x_\varepsilon + \varepsilon y, \omega) = a_1(T(\frac{\varepsilon x + x_\varepsilon}{\delta})\omega)$.

Claim: Almost surely, after extraction the functions $\{b_\varepsilon\}_\varepsilon$ converge to 1 as $\varepsilon \to 0$ in the L^∞-weak star topology. Consider the random variable $X_\varepsilon = |B_r|^{-1} \int_{B_r(x_0)} b_\varepsilon$, for an arbitrarily chosen $x_0 \in G$ and $r > 0$. From the definition of b_ε, we have that $X_\varepsilon = |B_{\alpha_\varepsilon r}|^{-1} \int a_0(T(y - y_\varepsilon)\omega)\, dy$, where $y_\varepsilon = \alpha_\varepsilon x_0$ and $\alpha_\varepsilon = \delta/\varepsilon \to +\infty$ as $\varepsilon \to 0$. Then, from the hypothesis of stationarity, X_ε has the same law as $Y_\varepsilon = |B_{\alpha_\varepsilon r}|^{-1} \int a_0(T(y)\omega)\, dy$ which, from the ergodic theorem above, converges to 1 almost surely. Therefore, for any x_0, r, the random variable X_ε converges in law to the constant 1, hence in probability as well (since the limit is constant).

It follows that after extraction, X_ε converges almost surely to 1. By a diagonal extraction process, we deduce that there exists a subsequence of $\{b_\varepsilon\}_\varepsilon$ such that $|B_r|^{-1} \int_{B_r(x_0)} b_\varepsilon$ converges to 1 almost surely for any x_0 in a countable dense set in G and r belonging to, for instance, the set $\{1/n \mid n \in \mathbb{N}^*\}$. This implies that, almost surely, this subsequence of $\{b_\varepsilon\}_\varepsilon$ converges to 1 in the L^∞-weak star topology, thus proving the claim.

As in the proof of Theorem 3.1, it follows from the claim that after extraction and almost surely, $V_\varepsilon \to 1$ locally uniformly, which contradicts the fact that $|U_\varepsilon(x_\varepsilon) - 1| \geq \eta$ with positive probability if ε is small enough, since $U_\varepsilon(x_\varepsilon) = V_\varepsilon(0)$.

The case where $\mathrm{dist}(x_\varepsilon, \partial G) = O(\varepsilon)$ also follows from the claim using the same arguments as in the proof of Theorem 3.1. ■

Theorem 1.1 in the random case follows from Lemma 2.1 and Theorem 3.8.

Remark 3.9. As in Remark 1.2, we may adapt the proof to prove $U_\varepsilon \to \mathcal{M}$ in $L^\infty(G)$ for a.e. $\omega \in \Omega$ when $\mathcal{M} = \sqrt{\mathbb{E}(a_1)} \neq 1$.

4. Γ-convergence and quasi-minimizers

In this section we recall the definition of Γ-convergence of functionals and show that it allows us to describe the asymptotic behavior of quasi-minimizers of a family of functionals.

Definition 4.1. For $\varepsilon \in (0,1]$, we consider a family of functionals

$$F_\varepsilon : \mathcal{I}_\varepsilon \to (-\infty, +\infty] \quad \text{for a topological space } \mathcal{I}_\varepsilon$$

and

$$F : \mathcal{I} \to (-\infty, +\infty] \quad \text{for a topological space } \mathcal{I}.$$

We define

$$\mathcal{I}_0 := \{x \in \mathcal{I} \mid F(x) < +\infty\}.$$

We say that F_ε Γ-converges to F as $\varepsilon \to 0$ if for every $\varepsilon \in (0,1]$ there exists $P_\varepsilon : \mathcal{I}_\varepsilon \to \mathcal{I}$ such that:

Lower bound: If $x \in \mathcal{I}_0$ and $x_\varepsilon \in \mathcal{I}_\varepsilon$ is a sequence such that $P_\varepsilon(x_\varepsilon) \to x$ (for the topology of $\mathcal{I}$) as $\varepsilon \to 0$, then

$$\liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) \geq F(x).$$

Upper bound: For every $x \in \mathcal{I}_0$, for every $\varepsilon \in (0,1]$, there exists $x_\varepsilon \in \mathcal{I}_\varepsilon$ such that $P_\varepsilon(x_\varepsilon) \to x$ in $\mathcal{I}$ and

$$\limsup_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) \leq F(x).$$

The first two properties (Lower and Upper bounds) in the above definition are taken from [20, Section 3.1] and are adapted from the original definition given by De Giorgi. The adaptation comes from the fact that in Ginzburg–Landau theory, the limiting space on which the Γ-limit is defined is not the same as the original space on which the Ginzburg–Landau functional is defined.

In addition to these two properties, the supplementary compactness property is added:

$$\begin{cases} \textbf{Compactness: } \text{If, for some } \varepsilon_0 \in (0,1], \sup_{\varepsilon \in (0,\varepsilon_0]} F_\varepsilon(x_\varepsilon) < +\infty, \text{ then} \\ \text{for a sequence } \varepsilon = \varepsilon_n \downarrow 0, \text{ there exist } x \in \mathcal{I}_0 \text{ and a subsequence (still} \\ \text{denoted by } x_\varepsilon) \text{ such that } P_\varepsilon(x_\varepsilon) \to x \text{ in } \mathcal{I}. \end{cases} \tag{4.1}$$

The notion of Γ-convergence has been conceived so that the *infima* of F_ε converge to the infimum of F and a family of minimizers of F_ε converges to a minimizer of F. This property remains true for a family of quasi-minimizers. Indeed, we have the following proposition:

Proposition 4.2. *Let $F_\varepsilon : \mathcal{I}_\varepsilon \to (-\infty, +\infty]$ be a family of functionals defined on topological spaces $\mathcal{I}_\varepsilon$ and $F : \mathcal{I} \to (-\infty, +\infty]$ be a functional defined on a topological space $\mathcal{I}$.*

Assume that F_ε Γ-converges towards F as $\varepsilon \to 0$ and that compactness property (4.1) *holds. Let $(x_\varepsilon)_\varepsilon$ be a family of quasi-minimizers of F_ε.*

If $F \not\equiv +\infty$, then there exists $x \in \mathcal{I}$ such that, up to a subsequence, $P_\varepsilon(x_\varepsilon) \to x$ in $\mathcal{I}$ and

$$F(x) = \inf_{y \in \mathcal{I}} F(y).$$

In other words, a family of quasi-minimizers also converges (up to a subsequence) towards a minimizer of the Γ-limit. The proof of this proposition is an adaptation of [11, Theorem 1.21].

Hence, using Proposition 4.2 and Theorem 3.1, we are able to understand the asymptotic behavior of minimizers of $GL_\varepsilon^{\text{pin}}$ thanks to existing Γ-convergence results on GL_ε. These asymptotics are the subject of the remaining sections.

5. Asymptotics for the pinned 2D Ginzburg–Landau energy

In this section we deduce from Theorem 1.1 results on the asymptotic behavior of minimizers of $GL_\varepsilon^{\text{pin}}$ given by (1.1) with a_ε either given by (1.3) or by (1.6). The main ingredient to pass from Theorem 1.1 to the description of minimizers of $GL_\varepsilon^{\text{pin}}$ is Proposition 4.2. In this section G is a smooth bounded domain of $\mathbb{R}^2$.

We first introduce some notations. For $(u, A) \in H^1(G, \mathbb{C}) \times H^1(G, \mathbb{R}^2)$, we recall that $\nabla_A u = (\nabla - iA)u$ and we define

$$j(u) = (iu, \nabla_A u), \quad \mu(u, A) = \operatorname{curl} j(u) + \operatorname{curl} A. \tag{5.1}$$

Here, $(iu, \nabla_A u) = \frac{i}{2}(u\overline{\nabla_A u} - \overline{u}\nabla_A u)$. We let $\mathbb{M}(G)$ be the set of Radon measures. For $\lambda > 0$, we define $E_\lambda : \mathbb{M}(G) \to (-\infty, +\infty]$ in the following way: for $\mu \in \mathbb{M}(G) \cap H^{-1}(G)$, we consider the solution h_μ of

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } G, \\ h_\mu = 1 & \text{on } \partial G. \end{cases} \tag{5.2}$$

We then set

$$E_\lambda(\mu) = \begin{cases} \frac{\|\mu\|}{2\lambda} + \frac{1}{2}\int_G (|\nabla h_\mu|^2 + |h_\mu - 1|^2) & \text{if } \mu \in \mathbb{M}(G) \cap H^{-1}(G), \\ +\infty & \text{otherwise.} \end{cases} \tag{5.3}$$

Theorem 5.1. *Assume that $G \subset \mathbb{R}^2$ is a smooth simply connected bounded domain. Assume that $\frac{h_{\text{ex}}}{|\log \varepsilon|} \to \lambda > 0$ when $\varepsilon \to 0$. We consider $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$, a family of minimizers of $G_\varepsilon^{\text{pin}}$. If we write $u_\varepsilon = U_\varepsilon v_\varepsilon$ where U_ε is given by Definition 2.4, then as $\varepsilon \to 0$,*

$$\frac{\mu(v_\varepsilon, A_\varepsilon)}{h_{\text{ex}}} \to \mu_* \quad \text{in } (\mathcal{C}^{0,\gamma}(G))^* \text{ for every } \gamma \in (0,1), \tag{5.4}$$

$$\frac{h_\varepsilon}{h_{\text{ex}}} \to h_{\mu_*} \quad \text{weakly in } H_1^1(G) \text{ and strongly in } W^{1,p}(G),\ \forall p < 2, \tag{5.5}$$

where μ_* *is the unique minimizer of* E_λ *given by* (5.3), *and*

$$\frac{GL_\varepsilon^{\text{pin}}(u_\varepsilon, A_\varepsilon) - E_\varepsilon^{\text{pin}}(U_\varepsilon)}{h_{\text{ex}}^2} \to E_\lambda(\mu_*). \tag{5.6}$$

Moreover,

$$\frac{g_\varepsilon(v_\varepsilon, A_\varepsilon)}{h_{\text{ex}}} \to \frac{1}{2\lambda}|\mu_*| + \frac{1}{2}(|\nabla h_{\mu_*}|^2 + |h_{\mu_*} - 1|^2) \tag{5.7}$$

and

$$\left|\nabla\left(\frac{h_\varepsilon}{h_{\text{ex}}}\right)\right| \to \frac{1}{\lambda}\mu_* \tag{5.8}$$

in the weak sense of measures.

Here,

$$g_\varepsilon(u, A) = \frac{|(\nabla - iA)u|^2}{2} + \frac{(1-|u|^2)^2}{4\varepsilon^2} + \frac{|\operatorname{curl} A - h_{\text{ex}}|^2}{2}.$$

Remark 5.2. We have that $j(u_\varepsilon) = U_\varepsilon^2 j(v_\varepsilon)$ and that $\mu(u_\varepsilon, A_\varepsilon) = U_\varepsilon^2 \mu(v_\varepsilon, A) + 2U_\varepsilon \nabla^\perp U_\varepsilon \cdot j(v_\varepsilon) + \operatorname{curl} A(1 - U_\varepsilon^2)$.

Proof. We use Theorem 1.1, Proposition 4.2, and the Γ-convergence result on $GL_\varepsilon/h_{\text{ex}}^2$ in this regime of the applied magnetic field (cf. [25, Theorem 7.1]) to deduce (5.4), (5.5), and (5.6). Note that in [25, Theorem 7.1], the Γ-convergence result is obtained with

$$\mathcal{I}_\varepsilon := H^1(G, \mathbb{C}) \times H^1(G, \mathbb{R}^2), \quad \mathcal{I} := (\mathcal{C}^{0,\gamma}(G))^* \times L^2(G, \mathbb{R}^2)$$

for any $\gamma \in (0,1)$ where $\mathcal{I}$ is endowed with the product topology, $(\mathcal{C}^{0,\gamma}(G))^*$ is endowed with the weak-$*$ topology, and $L^2(G, \mathbb{R}^2)$ with the weak topology. Furthermore, with the notations of Definition 4.1, we have

$$\begin{aligned} P_\varepsilon : H^1(G, \mathbb{C}) \times H^1(G, \mathbb{R}^2) &\to (\mathcal{C}^{0,\gamma}(G))^* \times L^2(G, \mathbb{R}^2), \\ (u_\varepsilon, A_\varepsilon) &\mapsto (\mu(u_\varepsilon, A_\varepsilon), \operatorname{curl} A_\varepsilon). \end{aligned}$$

Statements (5.7) and (5.8) follow exactly as in the proof of [25, Theorem 7.2]. ■

Theorem 5.3. *Assume that* $|\log \varepsilon| \ll h_{\text{ex}} \ll 1/\varepsilon^2$ *as* $\varepsilon \to 0$. *Let* $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ *be a family of minimizers of* $GL_\varepsilon^{\text{pin}}$ *in* $\mathcal{H}$. *We set* $u_\varepsilon = U_\varepsilon v_\varepsilon$ *where* U_ε *is given by Definition* 2.4. *Then,*

$$\frac{2g_\varepsilon(v_\varepsilon, A_\varepsilon)}{h_{\text{ex}}|\log \varepsilon\sqrt{h_{\text{ex}}}|} \rightharpoonup dx \quad \text{as } \varepsilon \to 0$$

in the weak sense of measures and

$$\min_{(u,A)\in\mathcal{H}} G_\varepsilon(u, A) \simeq \frac{|G|}{2} h_{\text{ex}} |\log \varepsilon \sqrt{h_{\text{ex}}}| \quad \text{as } \varepsilon \to 0.$$

Besides,

$$\frac{h_\varepsilon}{h_{\text{ex}}} \to 1 \quad \text{in } H^1(G) \qquad \text{and} \qquad \frac{\mu(v_\varepsilon, A_\varepsilon)}{h_{\text{ex}}} \to dx \quad \text{in } H^{-1}(G).$$

Proof. This follows from [25, Theorem 8.1 and Corollary 8.1] along with Theorem 1.1 and Proposition 4.2. ■

Unfortunately, Theorem 1.1 is not sufficient to describe the behavior of minimizers of $G_\varepsilon^{\text{pin}}$ near the so-called first critical field, or more generally, when there is a number of vortices much smaller than the applied magnetic field h_{ex}. This is because the leading-order term in the asymptotic expansion of $GL_\varepsilon(v_\varepsilon, A_\varepsilon)$ is independent of the position of the vortices. In the so-called intermediate regime it is also independent of the number of vortices and is of order h_{ex}. However, with an explicit rate of convergence of U_ε, the positive minimizer of $E_\varepsilon^{\text{pin}}$ in $H^1(G, \mathbb{C})$, we can give a condition on this rate such that results of [25, Chapters 9–11] can be applied to describe the asymptotic behavior of minimizers near the first critical field.

We first introduce some notations: We define h_0 to be the solution of

$$\begin{cases} -\Delta h_0 + h_0 = 0 & \text{in } G, \\ h_0 = 1 & \text{on } \partial G \end{cases}$$

and

$$\xi_0 := h_0 - 1 \quad \text{and} \quad \underline{\xi_0} = \min_G \xi_0.$$

We suppose that ξ_0 has a unique minimizer p in G. We set

$$Q(x) := D^2(\xi_0)(p)(x, x)$$

and we assume that Q is a positive definite quadratic form. We set

$$J_0 = \frac{1}{2}\int_G |\nabla h_0|^2 + |h_0 - 1|^2 = \frac{1}{2}\|\xi_0\|^2_{H^1(G)}.$$

We also set

$$H^0_{c_1} := \frac{1}{2|\underline{\xi_0}|}|\log\varepsilon|.$$

We denote by $\mathcal{G}$ the modified Green function, solution to

$$\begin{cases} -\Delta_x \mathcal{G}(x, y) + \mathcal{G}(x, y) = \delta_y & \text{in } G, \\ \mathcal{G}(x, y) = 0 & \text{on } \partial G, \end{cases}$$

and we set

$$S_G(x, y) = 2\pi\mathcal{G}(x, y) + \log|x - y|.$$

For $n \in \mathbb{N}$, we set $\ell := \sqrt{\frac{n}{h_{\text{ex}}}}$. We denote by φ the blow-up centered at p for the scale ℓ defined by

$$\varphi(x) = \frac{x - p}{\ell}.$$

If μ is a measure, we will denote by $\widetilde{\mu}$ its push-forward by the mapping φ (i.e., $\widetilde{\mu}(U) = \mu(\varphi^{-1}(U))$ for every U measurable subset of $\mathbb{R}^2$). If x is a point, then we let $\widetilde{x} = \varphi(x)$. Now, we define a functional on the space of probability measures on $\mathbb{R}^2$ denoted by $\mathscr{P}$:

$$I(\mu) = -\pi \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| \mathrm{d}\mu(x)\mathrm{d}\mu(y) + \pi \int_{\mathbb{R}^2} Q(x)\mathrm{d}\mu(x) \quad \text{for } \mu \in \mathscr{P}.$$

It is known that the infimum $\inf_{\mu\in\mathscr{P}} I(\mu)$ is uniquely achieved (see, e.g., [24]). We denote by μ_0 the minimizer and we let

$$I_0 := I(\mu_0) = \inf_{\mu\in\mathscr{P}} I(\mu).$$

For $n \in \mathbb{N}$, we define

$$\begin{aligned} g_\varepsilon(n) := h_{\mathrm{ex}}^2 J_0 + \pi n|\log\varepsilon| - 2\pi n h_{\mathrm{ex}}|\underline{\xi_0}| + \pi(n^2-n)\log\frac{1}{\ell} \\ + \pi n^2 S_G(p,p) + n^2 I_0. \end{aligned} \tag{5.9}$$

We recall the following from [25, Lemma 9.5]:

Lemma 5.4. *There exist constants $\alpha, \varepsilon_0 > 0$ and for each $0 < \varepsilon < \varepsilon_0$ an increasing sequence $(H_n)_n$ defined for integers $0 \le n \le \alpha|\log\varepsilon|$, such that if $h_{\mathrm{ex}} > H_{c_1}^0/2$, then n minimizes g_ε over the integers in the interval $[0, \alpha|\log\varepsilon|]$ if and only if*

$$h_{\mathrm{ex}} \in [H_n, H_{n+1}].$$

We can now state the following result:

Theorem 5.5. *Assume that h_{ex} is such that*

$$|\log|\log\varepsilon|| \ll h_{\mathrm{ex}}(\varepsilon) - H_{c_1}^0 \ll |\log\varepsilon|$$

and let N_ε be a corresponding minimizer of $g_\varepsilon(\cdot)$ over $[0, \alpha|\log\varepsilon|]$. Also let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $GL_\varepsilon^{\mathrm{pin}}$. We write $u_\varepsilon = U_\varepsilon v_\varepsilon$ where U_ε is given by Definition 2.4. Assume that

$$\|U_\varepsilon - 1\|_{L^\infty(G)} \times g_\varepsilon(N_\varepsilon) = o_\varepsilon(N_\varepsilon^2). \tag{5.10}$$

Then, for any $\gamma \in (0,1)$,

$$\frac{\widetilde{\mu}(u_\varepsilon, A_\varepsilon)}{2\pi N_\varepsilon} \rightharpoonup \mu_0 \quad \text{in } (\mathscr{C}_c^{0,\gamma}(\mathbb{R}^2))^*,$$

where μ_0 is the unique minimizer of I and

$$GL_\varepsilon(v_\varepsilon, A_\varepsilon) = g_\varepsilon(N_\varepsilon) + o_\varepsilon(N_\varepsilon^2), \quad GL_\varepsilon^{\mathrm{pin}}(u_\varepsilon, A_\varepsilon) = E_\varepsilon^{\mathrm{pin}}(U_\varepsilon) + g_\varepsilon(N_\varepsilon) + o_\varepsilon(N_\varepsilon^2).$$

Proof. Again, we deduce this theorem from Theorem 1.1, Proposition 4.2, and existing results in the literature. Here, the results used are [25, Theorems 9.1 and 9.2]. Assumption (5.10) is used to guarantee that $g_\varepsilon(N_\varepsilon) \times \|U_\varepsilon - 1\|_{L^\infty(G)}$ is negligible compared to all the terms of g_ε. ∎

Remark 5.6. From Proposition 3.3, assumption (5.10) is satisfied, for example, when $\delta = O_\varepsilon(\varepsilon^2)$ and $\frac{\delta^{1/4}}{\varepsilon^{1/2}}h_{\mathrm{ex}}^2 = o_\varepsilon(1)$. This means $\delta = o_\varepsilon(\frac{\varepsilon^2}{h_{\mathrm{ex}}^8})$.

Finally, it remains to examine the case of a bounded number of vortices. We let

$$f_\varepsilon(n) = h_{\mathrm{ex}}^2 J_0 + \pi n \log\frac{\ell}{\varepsilon} - 2\pi n h_{\mathrm{ex}}|\underline{\xi_0}| + \pi n^2 S_G(p,p) + \pi n^2 \log\frac{1}{\ell}. \tag{5.11}$$

We recall the following from [25, Lemma 12.1]:

Lemma 5.7. *For every $\varepsilon > 0$, there exists an increasing sequence $(H_n(\varepsilon))_n$, $H_0 = 0$, such that the following holds: Given $n \geq 0$ independent of ε, if $h_{\mathrm{ex}}(\varepsilon) \gg 1$ is such that*

$$g_\varepsilon(n) \leq \min(g_\varepsilon(n-1), g_\varepsilon(n+1)) + o_\varepsilon(1),$$

then

$$H_n - o_\varepsilon(1) \leq h_{\mathrm{ex}} \leq H_{n+1} + o_\varepsilon(1).$$

Moreover, the following asymptotic expansion holds as $\varepsilon \to 0$:

$$H_n = \frac{1}{2|\underline{\xi_0}|}\Big[|\log\varepsilon| + (n-1)\log\frac{|\log\varepsilon|}{2|\underline{\xi_0}|} + K_n\Big] + o_\varepsilon(1),$$

where

$$K_n = (n-1)\log\frac{1}{n} + \frac{n^2-3n+2}{2}\log\frac{n-1}{n}$$
$$+ \frac{1}{\pi}\Big(\min_{(\mathbb{R}^2)^n} w_n - \min_{(\mathbb{R}^2)^{n-1}} w_{n-1} + \gamma + (2n-1)\pi S_G(p,p)\Big).$$

Here, γ is a universal constant and

$$w_n(x_1,\ldots,x_n) = -\pi\sum_{i\neq j}\log|x_i - x_j| + \pi n\sum_{i=1}^n Q(x_i). \tag{5.12}$$

We can now state the following result:

Theorem 5.8. *Assume that $N \in \mathbb{N}$. There exists $c_\varepsilon \to 0$ as $\varepsilon \to 0$ such that if $\varepsilon < \varepsilon_0(N)$ and*

$$H_N + c_\varepsilon \leq h_{\mathrm{ex}} \leq H_{N+1} - c_\varepsilon,$$

and if $(u_\varepsilon, A_\varepsilon)$ is a minimizer of $GL_\varepsilon^{\mathrm{pin}}$, then writing $u_\varepsilon = U_\varepsilon v_\varepsilon$ where U_ε is the minimizer of $E_\varepsilon^{\mathrm{pin}}$ given by Definition 2.4, then the following holds. If

$$\|U_\varepsilon - 1\|_{L^\infty(G)} \times f_\varepsilon(N) = o_\varepsilon(1), \tag{5.13}$$

then v_ε has N vortices $a_1^\varepsilon, \ldots, a_N^\varepsilon$ and, possibly after extraction and letting $\tilde{a}_i^\varepsilon := (a_i^\varepsilon - p)/\ell$, the N-tuple $(\tilde{a}_1^\varepsilon, \ldots, \tilde{a}_N^\varepsilon)$ converges as $\varepsilon \to 0$ to a minimizer of w_N given by (5.12) and

$$GL_\varepsilon(v_\varepsilon, A_\varepsilon) = f_\varepsilon(N) + \min_{(\mathbb{R}^2)^N} w_N + N\gamma + o_\varepsilon(1) \text{ as } \varepsilon \to 0.$$

Remark 5.9. Assumption (5.13) is satisfied for example when $\delta = O_\varepsilon(\varepsilon^2)$ and

$$\frac{\delta^{1/4}}{\varepsilon^{1/2}} \times h_{\text{ex}}^2 = o_\varepsilon(1),$$

leading to $\delta = o_\varepsilon(\frac{\varepsilon^2}{h_{\text{ex}}^8})$.

Proof of Theorem 5.8. Here, we use Theorem 1.1, Proposition 4.2, and [25, Theorem 12.1]. Hypothesis (5.13) is here to guarantee that $GL_\varepsilon(v_\varepsilon, A_\varepsilon) \times \|U_\varepsilon - 1\|_{L^\infty(G)}$ is much smaller than all the terms in the asymptotic expansion of $\inf_{(v_\varepsilon, A_\varepsilon)} GL_\varepsilon(v_\varepsilon, A_\varepsilon)$. ∎

6. Asymptotics for the pinned 3D Ginzburg–Landau energy

Let $G \subset \mathbb{R}^3$ be a smooth bounded domain. In this section we consider a 3D variant of energy (1.1). Here, we use differential forms formalism. We define

$$\mathcal{F}_\varepsilon^{\text{pin}}(u, A) = \frac{1}{2}\int_G |du - iAu|^2 + \frac{1}{4\varepsilon^2}\int_G (a_\varepsilon(x) - |u|^2)^2 + \frac{1}{2}\int_{\mathbb{R}^3} |dA - h_{\text{ex}}|^2; \quad (6.1)$$

here, $u \in H^1(G, \mathbb{C})$, du is a 1-form, $h_{\text{ex}} \in L^2_{\text{loc}}(\Lambda^2\mathbb{R}^3)$ is a 2-form, $A \in H^1(\Lambda^1\mathbb{R}^3)$ is a 1-form, and a_ε is defined by (1.3) or by (1.6). We define

$$\dot{H}^1_*(\Lambda^1\mathbb{R}^3) = \{A \in \dot{H}^1(\Lambda^1\mathbb{R}^3) \mid d^*A = 0\} \quad (6.2)$$

and we endow this space with the inner product

$$(A, B)_{\dot{H}^1_*(\Lambda^1\mathbb{R}^3)} := (dA, dB)_{L^2(\Lambda^2\mathbb{R}^3)}, \quad (6.3)$$

for which $\dot{H}^1_*(\Lambda^1\mathbb{R}^3)$ is a Hilbert space. For $u \in H^1(G, \mathbb{C})$, we define (writing $u = u^1 + iu^2, u^1, u^2 \in H^1(G, \mathbb{R})$)

$$ju := (iu, du) = u^1du^2 - u^2du^1, \quad Ju = du^1 \wedge du^2 = \frac{1}{2}d(ju). \quad (6.4)$$

Theorem 6.1. *Assume that* $h_{\text{ex}} = dA_{ex,\varepsilon}$ *and that there exists* $A_{ex,0} \in H^1_{\text{loc}}(\Lambda^1\mathbb{R}^3)$ *such that*

$$\frac{A_{ex,\varepsilon}}{|\log\varepsilon|} - A_{ex,0} \to 0 \quad \text{in } \dot{H}^1_*(\Lambda^1\mathbb{R}^3).$$

Let $(u_\varepsilon, A_\varepsilon) \in H^1(G, \mathbb{C}) \times [A_{ex,0} + \dot{H}^1_*(\Lambda^1\mathbb{R}^3)]$ *be a family of minimizers of* $\mathcal{F}_\varepsilon^{\text{pin}}$. *We write* $u_\varepsilon = U_\varepsilon v_\varepsilon$ *where* U_ε *is the minimizer of* $E_\varepsilon^{\text{pin}}$ *given by Definition* 2.4. *Then, up to a subsequence, we have*

$$\frac{A_\varepsilon}{|\log\varepsilon|} \rightharpoonup A_* \quad \text{in } \dot{H}^1_*(\Lambda^1\mathbb{R}^3)$$

for some $A_* \in A_{ex,0} + \dot{H}^1_*(\Lambda^1\mathbb{R}^3)$, *and*

$$\frac{jv_\varepsilon}{|\log\varepsilon|} \rightharpoonup w_* \quad in\ L^{\frac{8}{6}}(\Lambda^1 G),$$
$$\frac{jv_\varepsilon}{|v_\varepsilon||\log\varepsilon|} \rightharpoonup w_* \quad in\ L^2(\Lambda^1 G),$$
$$\frac{Jv_\varepsilon}{|\log\varepsilon|} = \frac{d(jv_\varepsilon)}{2|\log\varepsilon|} \to J_* \quad in\ W^{-1,p}(\Lambda^2 G)\ \forall p < 3/2$$

for some $(J_*, w_*) \in \mathcal{A}_0 := \{(J, w) \mid J$ *is an exact measure-valued* 2*-form in* G, $v \in L^2(\Lambda^1 G)\}$ *and* $J_* = \frac{dw_*}{2} \in H^{-1}(\Lambda^2 G)$. *Besides,* (w_*, A_*) *is a minimizer of the functional defined for* $(v, A) \in L^2(\Lambda^1 G) \times [A_{ex,0} + \dot{H}^1_*(\Lambda^1\mathbb{R}^3)]$ *by*

$$\mathcal{F}(v, A) = \begin{cases} \frac{1}{2}\|dv\| + \frac{1}{2}\|v - A\|^2_{L^2(\Lambda^1 G)} & \\ \qquad + \frac{1}{2}\|dA - dA_{ex,0}\|^2_{L^2(\Lambda^2\mathbb{R}^3)} & \text{if } \|dv\| = |dv|(\Omega) < +\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that an analog of Lemma 2.1 holds for the 3D magnetic Ginzburg–Landau energy. With Propositions 3.1 and 3.8, we find that the analog of Theorem 1.1 is true for the 3D Ginzburg–Landau energy. We conclude by using Proposition 4.2 and [7, Theorem 4]. ∎

7. Asymptotics for the pinned Allen–Cahn energy

In this section G is a $\mathcal{C}^1$ bounded open set of $\mathbb{R}^d$, $d \geq 1$. By taking $A = 0$ and $h_{\text{ex}} = 0$, we are able to describe the asymptotic behavior of a pinned Allen–Cahn functional. For $u \in H^1(G, \mathbb{R})$, we define

$$AC_\varepsilon^{\text{pin}}(u) = \varepsilon \int_G |\nabla u|^2 + \frac{1}{\varepsilon}\int_G (a_\varepsilon(x) - u^2)^2, \tag{7.1}$$

where a_ε is given by (1.3) or (1.6).

Theorem 7.1. *Let* $0 < \beta < 1$ *and* $(u_\varepsilon)_\varepsilon \subset H^1(G, \mathbb{R})$ *be a family of minimizers of the pinned Allen–Cahn energy given in* (7.1) *under the constraint* $\frac{1}{|G|}\int_G u_\varepsilon = \beta$. *Then, we can write* $u_\varepsilon = U_\varepsilon v_\varepsilon$ *with* U_ε *given by Definition* 2.4, *and we have that there exists* $v \in BV(G, \{\pm 1\})$ *such that*

$$v_\varepsilon \to v \quad in\ L^1(G)$$

and v *minimizes*

$$A(w) = \frac{4}{3}\int_G |Dw|$$

for $w \in BV(G, \{\pm 1\})$ *under the constraint* $\frac{1}{|G|}\int_G w = \beta$.

Remark 7.2. Recall that we normalized the average of a_0 and a_1 such that these quantities are equal to 1.

Proof of Theorem 7.1. This follows from an analog of Theorem 1.1 which we know to be true thanks to Lemma 2.1 and Theorems 3.1 and 3.8. We conclude with Proposition 4.2 and the Γ-convergence results in [23]. ∎

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